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Synthesis of Face Images Using Generative Algorithms.

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Tensors

1.1 Tensor algebra

A generalization of the product between two matrices are the product of a tensor and a matrix or vector. We can define the n-way tensor product between an nth order tensor $\mathcal{A} \in \mathbb{R}^{I_1 \times \cdots \times I_n \times \cdots \times I_N}$ and a matrix $\mathbf{M} \in R^j$ as[1]

$$\mathcal{B} = \mathcal{A} \times_n \mathbf{M} \tag{1.1}$$

Where the product is tensor $\mathcal{B} \in \mathbb{R}^{i_1 \times \cdots \times j \times \cdots \times i_N}$

We can define mode-n vectors as the column vectors of the matrix $\mathbf{A}_{(n)} \in \mathbb{R}^{I_1 \times \cdots \times J \times \cdots \times I,N}$ [1]

the n-rank R_n is the dimension of the vector space spanned by the mode-n vectors

$$R_n = \mathrm{Rank}(\mathbf{A}_{(n)}) \tag{1.2}$$

We can define en Inner Product of two tensors as

$$\langle \mathcal{Y}, \mathcal{Y} \rangle = x_{i_1, \cdots, i_n} y_{i_1, \cdots, i_n} \tag{1.3}$$

A tensor is *rank one* if it can be written as an outer product of n vectors $\mathbf{a}^(n)$

$$\mathcal{X} = \mathbf{a}^(1) \otimes, \cdots, \otimes, \mathbf{a}^(n) \tag{1.4}$$

1.2 Tensor decomposition

1.3 PCA and SVD

Given a matrix $\mathbf{X} \in \mathbb{R}^{n \times m}$ we can normalize is by subtracting the mean of the column vectors. Let $\mathbf{x}_i \in \mathbb{R}^m$ be the the ith column vector of $\mathbf{X}$. Let us define $\bar{\mathbf{X}}$ to be the matrix with columns

$$\bar{\mathbf{x}}_i = \mathbf{x}_i - \mu \tag{1.5}$$

where

$$\mu = \frac{1}{m} \sum_{j=1}^{m} \mathbf{x}_i \tag{1.6}$$

then the Covariance matrix $\mathbf{C} \in \mathbb{R}^{n \times n}$ can be written as

$$\mathbf{C} = \mathbf{X}\mathbf{X}^T \tag{1.7}$$

Principal component Analysis (PCA) is the eigenvalue decomposition of the Covariance matrix

$$\mathbf{C} = \mathbf{U}\mathbf{S}\mathbf{U}^T \tag{1.8}$$

where $S = \text{Diag} = \{\lambda_1, \cdots, \lambda_n\}$ where λ_n is the nth eigenvalue satisfying $\mathbf{C}\mathbf{u}_n = \lambda_n \mathbf{u}_n$

Any matrix can be decomposed via its Singular Value Composition (SVD) The Rank of a matrix $\mathbf{M}$ is the minimum number of

1.4 HOSVD

Any Tensor $\mathcal{T} \in \mathbb{R}^{m_1 \times m_2 \times \cdots \times n_M}$ of order M can we factorized via it its Tucker decomposition.

$$\mathcal{T} = \mathcal{S} \times_1 \mathbf{U}^{(1)} \cdots \times_n \mathbf{U}^{(1)} \tag{1.9}$$

Where the tensor $\mathcal{S}$ is known as the *core* tensor and is of the same order as $\mathcal{T}$

We follow the convention to denote order 0 tensors (scalars) with lower case letters like $a, b, c \in \mathbb{R}$, we denote order 1 tensors (vectors) with lower case bold letters like $\mathbf{v}, \mathbf{w}, \mathbf{z} \in \mathbb{R}^n$, we use upper case bold for order 2 tensors (matrices) $\mathbf{M}, \mathbf{U}, \mathbf{A} \in \mathbb{R}^{n \times m}$ and finally upper-case caligraphic style letters for any tensor of order larger than 2 which for which we will reserve the word tensor for now on.

Bibliography

[1] M. A. O. Vasilescu and Demetri Terzopoulos. Multilinear analysis of image ensembles: Tensorfaces. In *Proceedings of the 7th European Conference on Computer Vision-Part I*, ECCV '02, page 447–460, Berlin, Heidelberg, 2002. Springer-Verlag.