Implicit Function

Steve Canves
yqykrhf@163.com

1 Gradient

If we think of the function f(x, y) as a height field with height = f(x, y),the gradient vector points in the direction ofmaximum upslope, i.e., straight uphill. The gradient vector f(x, y) is given by

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \tag{1}$$

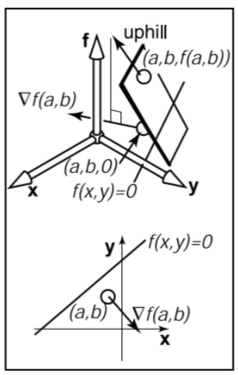

Figure 1: Asurface height = f(x,y) is locally planar near (x,y)=(a,b). The gradient is a projection of the uphill direction onto the height = 0 plane

The gradient vector evaluated at a point on the implicit curve f(x, y)=0 is perpendicular to the tangent vector of the curve at that point. This perpendicular vector is usually called the normal vector to the curve. In addition, since the gradient points uphill, it indicates the direction of the $f(x, y) > 0$ region.

A generalization of this observation is that (x, y) is perpendicular to $k(y, -x)$ where k is any non-zero constant.

2 Implicit 2D Lines

General form is:

$$Ax + By + C = 0 \tag{2}$$

For this line, the gradient vector is (A, B). This vector is perpendicular to the line, and points to the side of the line where $Ax + By + C > 0$

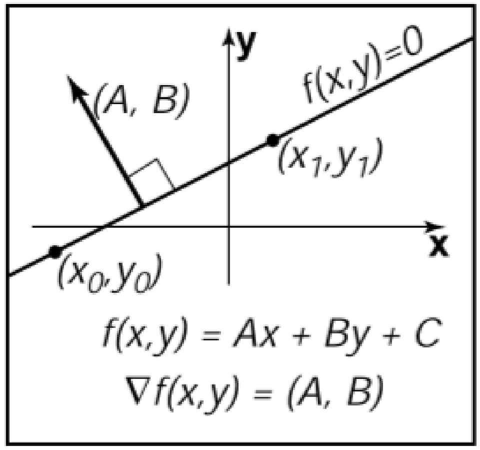

Figure 2: The gradient vector (A, B) is perpendicular to the implicit the Ax + By + C = 0

Assume that we have two points $(x_0, y_0), (x_1, y_1)$, then

$$(y_0 - y_1)\, x + (x_1 - x_0)\, y + C = 0 \tag{3}$$

We can plug either value in and solve for C. Doing this for (x_0, y_0) yields $C = x_0 y_1 - x_1 y_0$, and thus the full equation for the line is

$$(y_0 - y_1)\, x + (x_1 - x_0)\, y + x_0 y_1 - x_1 y_0 = 0 \tag{4}$$

An important property of the implicit line equation is that it can be used to find the **signed distance** from a point to line.

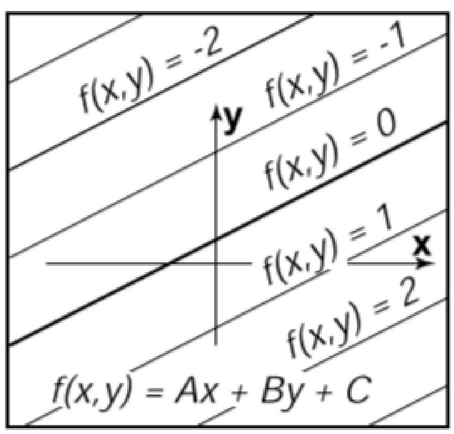

Figure 3: The value of the implicit function f(x,y) = Ax + By + C is a constant times the signed distance from Ax+ By+ C=0

As shown in Figure 4, the distance from a point to the line is the length of the $k(A, B)$, which is

$$distance = k\sqrt{A^2 + B^2} \tag{5}$$

For the point $(x, y) + k(A, B)$, the value of $f(x, y) = Ax + By + C$ is

$$\begin{aligned} f(x + kA, y + kB) &= Ax + kA^2 + By + kB^2 + C \\ &= k\left(A^2 + B^2\right) \end{aligned} \tag{6}$$

Combinate the Equation 5 and Equation 6, we can see that the signed distance from line $Ax + By + C = 0$ to a point (a, b) indicates

$$\text{distance } = \frac{f(a, b)}{\sqrt{A^2 + B^2}} \tag{7}$$

Note that if (A, B) is a unit vector, then $f(a, b)$ is the signed distance.

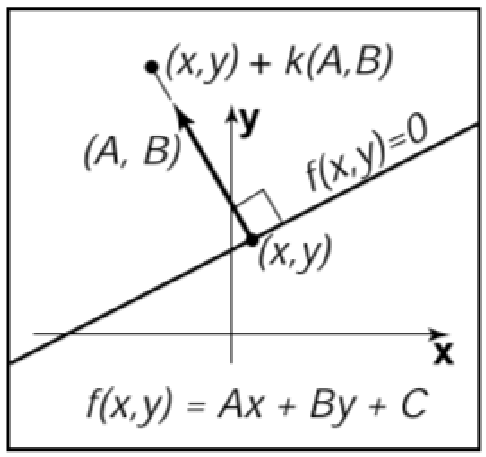

Figure 4: The vector k(A, B) connects a point (x,y) on the line closet to a point not on the line. The distance is proportional to k