Quaternion

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1 Complex Number

1.1 Complex Multiplication

If we have two complex numbers, $z_1 = a + bi, z_2 = c + di$, compute their product:

$$\begin{aligned} z_1 z_2 &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (bc + ad)i \\ &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \end{aligned}$$

If we take $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ as a transformation, there's another expression approach:

$$z_1 z_2 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$$

Note: The multiplication of complex number is commutative.

$$\begin{aligned} z_1 z_2 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} &= \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \\ = z_2 z_1 \end{aligned}$$

1.2 2D rotation

$$\begin{aligned} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} &= \sqrt{a^2 + b^2} \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{-b}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{bmatrix} \\ &= \|z\| \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \end{aligned}$$

This transformation is **the composition of scaling and rotation** which also can be written $\boldsymbol{cos\theta + isin\theta}$ in complex number. Vector rotation:

$$\boldsymbol{v'} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \boldsymbol{v} \tag{1}$$

or in product of two complex number:

$$\begin{aligned} v' &= (cos\theta + isin\theta)v \\ &= e^{i\theta}v \end{aligned} \tag{2}$$

2 Quaternion

$$\begin{aligned} q = a + bi + cj + dk \quad (a, b, c, d \in \mathbb{R}) \\ s.t. \quad i^2 = j^2 = k^2 = ijk = -1 \end{aligned} \tag{3}$$

Similiar with complex number, quaternion can be seen as linear combination of the basis1, i, j, k, written as:

$$q = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Divide the real part from the imaginary part. quaternion is represented as an ordered pair of scalars and vectors.

$$q = [s, \mathbf{v}], \quad \mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, s, x, y, z \in \mathbb{R}$$

Pure quaternion $v = [0, \mathbf{v}]$.

Any 3D vector can be transformed into a pure quaternion.

2.1 Multiplication

$q_1 = a + bi + cj + dk, \quad q_2 = e + fi + gj + hk$

$\times$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

$$
\begin{aligned}
q_1q_2 =& ae + afi + agj + ahk+ \\
& bei - bf + bgk - bhj+ \\
& cej - cfk - cg + chi+ \\
& dek + dfj - dgi - dh \\
=& (ae - bf - cg - dh)+ \\
& (be + af - dg + ch)i \\
& (ce + df + ag - bh)j \\
& (de - cf + bg + ah)k
\end{aligned}
$$

Matrix form:

$$
q_1q_2 = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix}
$$

The multiplication does not comply with the commutative law

$$
q_2q_1 = \begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix} \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix}
$$

Grabmann Product form:

For any quaternion $q_1 = [s, \mathbf{v}]$, $q_2 = [t, \mathbf{u}]$

$$
q_1q_2 = [st - \mathbf{v} \cdot \mathbf{u}, s\mathbf{u} + t\mathbf{v} + \mathbf{v} \times \mathbf{u}] \tag{4}
$$

2.2 Inverse and Conjugate

Inverse:

$$
qq^{-1} = q^{-1}q = 1 \quad (q \neq 0)
$$

Conjugate:

$$
q = [s, \mathbf{v}], \quad q^* = [s, -\mathbf{v}]
$$

read q^* as [q star]

$$
q^*q = qq^* = \|q\|^2
$$

pure quaternion $q^*q = 1$

2.3 3D rotation

3D rotation fomula (quaternion in normal case):
Vector $\mathbf{v}'$ after vector **v** rotating θ degrees around rotation axis **u** defined in terms of unit vector can be got by quaternion product: $v = [0, \mathbf{v}]$, $q = [cos(\frac{1}{2}\theta), sin(\frac{1}{2}\theta)\mathbf{u}]$, then

$$
v' = qvq^* = qvq^{-1} \tag{5}
$$

Note: **u** is unit vector, and $u^2 = [-\mathbf{u} \cdot \mathbf{u}, \mathbf{0}] = -\|\mathbf{u}\|^2 = -1$ which is similiar with i in complex number.

Derivation

> Lemma 1:
> If q = $[cos(\theta), sin(\theta)\mathbf{u}]$ with unit vector **u**, then $q^2 = qq = [cos(2\theta), sin(2\theta)\mathbf{u}]$

> Lemma 2:
> Assume a pure quaternion $v_{\|}$, then $q = [\alpha, \beta\mathbf{u}]$ in which **u** is a unit vector, $\alpha, \beta \in \mathbb{R}$. In this case, if $\mathbf{v}_{\|}$ is parallel to **u**, then $qv_{\|} = v_{\|}q$

> Lemma 3:
> Assume a pure quaternion $v_{\perp}$, then $q = [\alpha, \beta\mathbf{u}]$ in which **u** is a unit vector, $\alpha, \beta \in \mathbb{R}$. In this case, if $\mathbf{v}_{\perp}$ is orthogonal to **u**, then $qv_{\perp} = v_{\perp}q^*$

Rotation of $v_{\perp}$:

$$\mathbf{v}'_{\perp} = \cos(\theta)\mathbf{v}_{\perp} + \sin(\theta)\,(\mathbf{u} \times \mathbf{v}_{\perp})$$

which can be replace by quaternion form:

$$uv_{\perp} = [-\mathbf{u} \cdot \mathbf{v}_{\perp}, \mathbf{u} \times \mathbf{v}_{\perp}] = [0, \mathbf{u} \times \mathbf{v}_{\perp}] = \mathbf{u} \times \mathbf{v}_{\perp}$$

$$\begin{aligned} v_{\perp}^{'} &= cos(\theta)v_{\perp} + sin(\theta)uv_{\perp} \\ &= (cos(\theta) + sin(\theta)u)v_{\perp} \end{aligned}$$

Take $cos(\theta) + sin(\theta)u$ as quaternion q:

$$\begin{aligned} q &= \cos(\theta) + \sin(\theta)u \\ &= [\cos(\theta), 0] + [0, \sin(\theta)\mathbf{u}] \\ &= [\cos(\theta), \sin(\theta)\mathbf{u}] \end{aligned}$$

> 3D rotation fomula (quaternion in orthogonal case):

Vector $\mathbf{v}'_{\perp}$ after vector $\mathbf{v}_{\perp}$ orthogonal to the u rotating θ degrees around rotation axis **u** defined in terms of unit vector can be got by quaternion product: $v_{\perp} = [0, \mathbf{v}\perp], \quad q = [cos(\theta), sin(\theta)\mathbf{u}]$

$$v_{\perp}^{'} = qv_{\perp} \tag{6}$$

Note: $\|q\| = 1$, which means pure rotation except scaling and $q^{-1} = q^*$

Rotation of v:

$$\begin{aligned} v' &= v'_{\|} + v'_{\perp} \\ &= v_{\|} + qv_{\perp} \end{aligned}$$

$q = pp, \quad p = [cos(\frac{1}{2}\theta), sin(\frac{1}{2}\theta)], \quad$ then

$$\begin{aligned} v' &= v'_{\|} + v'_{\perp} \\ &= v_{\|} + qv_{\perp} \\ &= pp^{-1}v_{\|} + ppv_{\perp} \\ &= pv_{\|}p^{-1} + pv_{\perp}p^* \\ &= p(v_{\|} + v_{\perp})p^* \\ &= pvp^* \end{aligned}$$

The real part of all the rotation quaternion is only the cos of a angle.

Assume unit quaternion $q = [a, \mathbf{b}]$. If we hope to extract the roation angle θ and roation axis:

$$\frac{\theta}{2} = cos^{-1}(a)$$
$$\mathbf{u} = \frac{b}{sin(cos^{-1}(a))}$$

2.3.1 Convert to matrix form

In practical application, maybe we need to combine rotation with translation and scaling. So the matrix form of quaternion rotation is esstential.

The left-hand multiplication of a quaternion q = a + bi + cj + dj is equal to the following matrix:

$$L(q) = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}$$

The right-hand multiplication of a quaternion q = a + bi + cj + dj is equal to the following matrix:

$$R(q) = \begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix}$$

Make use of two matrixs to rewritten $v' = qvq^*$ as matrix form:

Assume $a = cos(\frac{1}{2}\theta)$, $b = sin(\frac{1}{2}\theta)u_x$, $c = sin(\frac{1}{2}\theta)u_y$, $d = sin(\frac{1}{2}\theta)u_z$, p = a + bi + cj + dk:

$$\begin{aligned}
qvq^* &= L(q)R\left(q^*\right)v \qquad (or\ L(q)R(q^*), they\ are\ equivalent) \\
&= \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix} \quad (Note:\ R(q^*) \ = \ R(q)^T) \\
&= \begin{bmatrix} a^2+b^2+c^2+d^2 & ab-ab-cd+cd & ac+bd-ac-bd & ad-bc+bc-ad \\ ab-ab+cd-cd & b^2+a^2-d^2-c^2 & bc-ad-ad+bc & bd+ac+bd+ac \\ ac-bd-ac+bd & bc+ad+ad+bc & c^2-d^2+a^2-b^2 & cd+cd-ab-ab \\ ad+bc-bc-ad & bd-ac+bd-ac & cd+cd+ab+ab & d^2-c^2-b^2+a^2 \end{bmatrix} v
\end{aligned}$$

Because of $a^2 + b^2 + c^2 + d^2 = 1$, which can be simplified into

$$qvq^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-2c^2-2d^2 & 2bc-2ad & 2ac+2bd \\ 0 & 2bc+2ad & 1-2b^2-2d^2 & 2cd-2ab \\ 0 & 2bd-2ac & 2ab+2cd & 1-2b^2-2c^2 \end{bmatrix} v$$

Although the matrix form of 3D rotation is not simple as quaternion form and takes up more space to store, it's more efficient than quaternion multiplication that **pre-calculated matrix** for large number of transformation.

2.3.2 Recombination of rotation

Lemma:
For any quaternion $q_1 = [s, \mathbf{v}],\ q_2 = [t, \mathbf{u}]$:
$q_1^* q_2^* = (q_2 q_1)^*$

$$
\begin{aligned}
v^{''} &= q_2 q_1 v q_1^* q_2^* \\
&= (q_2 q_1) v (q_2 q_1)^*
\end{aligned}
$$

$q_{net} = q_2 q_1$ It should be noted that the equivalent rotation q_{net} of q_1 and q_2 is not two rotations along the two axes of rotation q_1 and q_2 respectively. It is an equivalent rotation along a new axis of rotation,

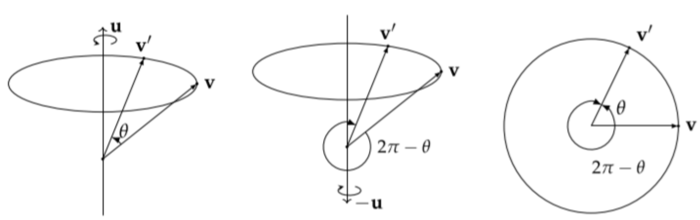
2.3.3 Double cover

The quaternion and 3D rotation are not one-to-one correspondence, which means a 3D rotation will be represented by two different quaternions. **For any unit quaternion $\mathbf{q} = [\mathbf{cos}(\frac{1}{2}\theta), \mathbf{sin}(\frac{1}{2}\theta)\mathbf{u}]$, q and -q represent the same rotation with different rotation axis and angle**.

$$
\begin{aligned}
q &= [cos(\frac{1}{2}\theta), sin(\frac{1}{2}\theta)u] \\
-q &= [-cos(\frac{1}{2}\theta), -sin(\frac{1}{2}\theta)u] \\
&= [cos(\frac{1}{2}(2\pi - \theta)), sin(\frac{1}{2}(2\pi - \theta))(-u)]
\end{aligned}
$$

Correspondences are as follows:

q	$-q$
u	$-u$
θ	$2\pi - \theta$

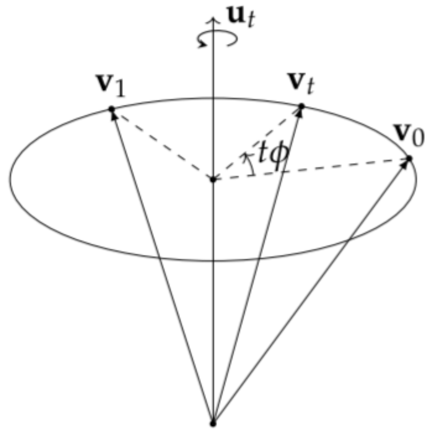


Note: q and -q refers to the same 3D roattion matrix without double cover issue

2.4 Quaternion Interpolation

Since the object of interpolation is two transformation, which can be very difficult to image, let's assume there is an arbitary vector v in 3D space and two transformation q_0, q_1, then

$$
\begin{aligned}
v_0 &= q_0 v q_0^* \\
v_1 &= q_1 v q_1^*
\end{aligned}
$$



we scale the ϕ to achieve the task of interpolaion.

$$\Delta q q_0 = q_1$$

because all the rotaion is the unit quaternion.

$$\Delta q q_0 q_0^* = q_1 q_0^*$$
$$\Delta q = q_1 q_0^*$$

adjust t in $(\Delta q)^t$ in purpose of scaling ϕ to interpolate:

$$q_t = (\Delta q)^t q_0 = (q_1 q_0^*)^t q_0$$
$$s.t\ 0 \leqslant t \leqslant 1$$

t = 0.4 means that perform the q_0 transformation from v to v_0, then rotate it by 40% towards v_1 Actually, this interpolaion is called **slerp(Spherical Linear Interpolation)**.