Lie Group And Lie Algebra

Steve Canves

yqykrhf@163.com

1 Group

Group is a kind of set plus an operation. Let's call set A and operation $\cdot$

The group requires the operation to satisfy the following conditions:

1. Closure: $\forall a_1, a_2 \in A, \quad a_1 \cdot a_2 \in A$
2. Associate: $\forall a_1, a_2, a_3 \in A, \quad (a_1 \cdot a_2) \cdot a_3 = a_1 \cdot (a_2 \cdot a_3)$
3. Identity Element: $\exists a_0 \in A, \quad$ s.t. $\quad \forall a \in A, \quad a_0 \cdot a = a \cdot a_0 = a$
4. Inverse: $\forall a \in A, \quad \exists a^{-1} \in A, \quad$ s.t. $\quad a \cdot a^{-1} = a_0$

Common groups are:

- General linear groups GL(n): $n \times n$ invertible matrix plus multiplication.
- Special linear groups SL(n): $n \times n$ matrix with determaint 1 plus multiplication
- Othrhogonal groups O(n): $O(n) = R \in GL(n) | R^T R = I$
- Affine Group A(n): $A \in GL(n)$ and $b \in \mathbb{R}^n$.
- Special orthogonal groups SO(n): rotation matrix plus mulitplication.
- Special Euclidean groups SE(n): euclidean transfomation puls mulitplication.

In summary:

$$SO(n) \subset O(n) \subset GL(n), \quad SE(n) \subset E(n) \subset A(n) \subset GL(n+1)$$

Lie Group

Lie groups are groups with continuous (smooth) properties.

$$SO(3) = \left\{ \boldsymbol{R} \in \mathbb{R}^{3\times 3} \mid \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}, \det(\boldsymbol{R}) = 1 \right\} \tag{1}$$

$$SE(3) = \left\{ \boldsymbol{T} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \mid \boldsymbol{R} \in SO(3), \boldsymbol{t} \in \mathbb{R}^3 \right\} \tag{2}$$

2 Lie Algerbra

Every Lie group has a corresponding Lie algebra which describes the local properties of Lie group.

Lie group consists of a set $\mathbb{V}$, a number field $\mathbb{F}$ and an operation $[,]$. which statisfy the following properties:

1. Closure: $\forall \boldsymbol{X}, \boldsymbol{Y} \in \mathbb{V}, [\boldsymbol{X}, \boldsymbol{Y}] \in \mathbb{V}$
2. Bilinearity: $\forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathbb{V}, a, b \in \mathbb{F},\ then$
   $[a\boldsymbol{X} + b\boldsymbol{Y}, \boldsymbol{Z}] = a[\boldsymbol{X}, \boldsymbol{Z}] + b[\boldsymbol{Y}, \boldsymbol{Z}], \quad [\boldsymbol{Z}, a\boldsymbol{X} + b\boldsymbol{Y}] = a[\boldsymbol{Z}, \boldsymbol{X}] + b[\boldsymbol{Z}, \boldsymbol{Y}]$

3. Alternativity: $\forall \boldsymbol{X} \in \mathbb{V}, [\boldsymbol{X}, \boldsymbol{X}] = \mathbf{0}$
4. Jacobi Identity:
   $\forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathbb{V}, [\boldsymbol{X}, [\boldsymbol{Y}, \boldsymbol{Z}]] + [\boldsymbol{Z}, [\boldsymbol{Y}, \boldsymbol{X}]] + [\boldsymbol{Y}, [\boldsymbol{Z}, \boldsymbol{X}]] = \mathbf{0}$

2.1 SO(3) $\Longleftrightarrow \mathfrak{so}(3)$

The corresponding Lie algebra $\mathfrak{so}(3)$ of $SO(3)$ is a vector defined in $\mathbb{R}^3$, denoted as ϕ. Every 3D vector may generate a antisymmetric matrix:

$$\mathbf{\Phi} = \boldsymbol{\phi}^\wedge = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

Under this defination, two vectors' Lie bracket is defined as:

$$[\phi_1, \phi_2] = (\mathbf{\Phi}_1\mathbf{\Phi}_2 - \mathbf{\Phi}_2\mathbf{\Phi}_1)^\vee \tag{3}$$

ϕ can express the derivative of the rotation matrix R.

$$R(t)R(t)^T = I \tag{4}$$

Take the derivative of both sides with respect to time t,

$$\dot{R}(t)R(t)^T + R(t)\dot{R}(t)^T = 0 \tag{5}$$

which can be rearranged into

$$\dot{R}(t)R(t)^T = -(\dot{R}(t)R(t)^T)^T \tag{6}$$

We see $\dot{R}(t)R(t)^T$ is antisymmetric matrix, denoted as

$$\dot{R}(t)R(t)^T = \phi(t)^\wedge \tag{7}$$

That actually means:

$$\dot{R}(t) = \phi(t)^\wedge R(t) \tag{8}$$

$$\mathfrak{so}(3) = \left\{\phi \in \mathbb{R}^3, \mathbf{\Phi} = \phi^\wedge \in \mathbb{R}^{3\times 3}\right\} \tag{9}$$

Since $R(0) = I$, it follows that $\dot{R}(0) = \phi(0)^\wedge$, then gives the first order approximation of a rotation:

$$R(dt) = R(0) + dR = I + \phi(0)^\wedge dt$$

The projection between $SO(3)$ and $\mathfrak{so}(3)$ is

$$\mathbf{R} = exp(\phi^\wedge) \tag{10}$$

2.2 SE(3) $\Longleftrightarrow \mathfrak{se}(3)$

$$\mathfrak{se}(3) = \left\{\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\rho} \\ \phi \end{bmatrix} \in \mathbb{R}^6, \boldsymbol{\rho} \in \mathbb{R}^3, \boldsymbol{\phi} \in \mathfrak{so}(3), \boldsymbol{\xi}^\wedge = \begin{bmatrix} \phi^\wedge & \boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}\right\} \tag{11}$$

We still use $\wedge$ and $\vee$ to refer to the relationship from vector to matrix and from matrix to vector the Lie bracket is

$$[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2] = (\boldsymbol{\xi}_1^{\wedge}\boldsymbol{\xi}_2^{\wedge} - \boldsymbol{\xi}_2^{\wedge}\boldsymbol{\xi}_1^{\wedge})^{\vee} \tag{12}$$

Note: ρ is not translation itself, which is concerned about translation.

2.3 Exponential map

Any matrix's exponential map can be written as a Taylor expansion in the case of convergence.

$$exp(\mathbf{A}) = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{A}^n \tag{13}$$

2.3.1 SO(3)

$$exp(\phi^{\wedge}) = \sum_{n=0}^{\infty} \frac{1}{n!}\phi^{\wedge n} \tag{14}$$

ϕ is a 3D vector. Define it's norm θ and direction a(unit vector), then $\phi = \theta a$ For a, it statisfies two properties:

$$a^{\wedge}a^{\wedge} = aa^T - \mathbf{I}$$
$$a^{\wedge}a^{\wedge}a^{\wedge} = -a^{\wedge}$$

then, we can get

$$exp(\theta a^{\wedge}) = cos\theta\mathbf{I} + (1 - cos\theta)aa^T + sin\theta a^{\wedge} \tag{15}$$

This is rodrogs equatoion. And for every element in SO(3), it corresponds to a element in $\mathfrak{so}(3)$. However, for every element in $\mathfrak{so}(3)$, it may corresponds to multiple elements in SO(3).

The elements of Lie groups and Lie algebras correspond one to one if rotation angles is in range of $\pm\pi$

$$\mathbf{R} = exp(\theta a^{\wedge}) = cos\theta\mathbf{I} + (1 - cos\theta)aa^T + sin\theta a^{\wedge} \tag{16}$$

then,

$$\begin{aligned} tr(\mathbf{R}) &= cos\theta tr(\mathbf{I}) + (1 - cos\theta)tr(aa^T) + sin\theta tr(a^{\wedge}) \\ &= 3cos\theta + 1 - cos\theta \\ &= 2cos\theta + 1 \end{aligned}$$

which can be simplied into

$$\theta = arccos(\frac{tr(R) - 1}{2}) \tag{17}$$

for hinge vector a, solve this equation and normalize.

$$\mathbf{R}a = a \tag{18}$$

2.3.2 SE(3)

$$
\begin{aligned}
\exp\left(\boldsymbol{\xi}^{\wedge}\right) &= \begin{bmatrix} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\boldsymbol{\phi}^{\wedge}\right)^{n} & \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\left(\boldsymbol{\phi}^{\wedge}\right)^{n} \boldsymbol{\rho} \\ \mathbf{0}^{T} & 1 \end{bmatrix} \\
&\triangleq \begin{bmatrix} \boldsymbol{R} & \boldsymbol{J}\boldsymbol{\rho} \\ \mathbf{0}^{T} & 1 \end{bmatrix} = \boldsymbol{T}
\end{aligned}
\tag{19}
$$

$$
\boldsymbol{J} = \frac{\sin\theta}{\theta}\boldsymbol{I} + \left(1 - \frac{\sin\theta}{\theta}\right) a a^{T} + \frac{1-\cos\theta}{\theta} a^{\wedge} \tag{20}
$$

Thus, translation t admits

$$
t = \boldsymbol{J}\boldsymbol{\rho} \tag{21}
$$

Note: in the process of prove, there are two properties needed:

$$
\xi^{\wedge}\xi^{\wedge} = \phi^{\wedge}\xi^{\wedge} \tag{22}
$$

$$
\xi^{\wedge}\xi^{\wedge}\xi^{\wedge} = (\phi^{\wedge})^{2}\xi^{\wedge} \tag{23}
$$

2.4 Adjoint

2.4.1 SO(3)

For SO(3), then

$$
\mathbf{R} exp(\mathbf{p}^{\wedge})\mathbf{R}^{T} = exp((\mathbf{R}\mathbf{p})^{\wedge}) \tag{24}
$$

At this time said $Ad(\mathbf{R}) = \mathbf{R}$

2.4.2 SE(3)

For SE(3), then

$$
\mathbf{T}\exp\left(\boldsymbol{\xi}^{\wedge}\right)\mathbf{T}^{-1} = \exp\left((\mathrm{Ad}(\mathbf{T})\boldsymbol{\xi})^{\wedge}\right) \tag{25}
$$

then $Ad(\boldsymbol{T})$ is defined as

$$
Ad(\boldsymbol{T}) = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t}^{\wedge}\boldsymbol{R} \\ \mathbf{0} & \boldsymbol{R} \end{bmatrix} \tag{26}
$$

2.5 Derivative

If A,B is scalar, then $e^{A}e^{B} = e^{A+B}$. Unfortunately, when they are matrixes, the equation doesn't work.

BCH formula

Baker-Campbell-Hausdorff formula gives

$$
\ln(\exp(\boldsymbol{A})\exp(\boldsymbol{B})) = \boldsymbol{A} + \boldsymbol{B} + \frac{1}{2}[\boldsymbol{A}, \boldsymbol{B}] + \frac{1}{12}[\boldsymbol{A}, [\boldsymbol{A}, \boldsymbol{B}]] - \frac{1}{12}[\boldsymbol{B}, [\boldsymbol{A}, \boldsymbol{B}]] + \cdots \tag{27}
$$

$\boldsymbol{A}, \boldsymbol{B}$ are matrixes, [] is lie brackes.

When ϕ_1, ϕ_2 is small, small number of quadratic terms can be ignored. Then

$$\ln\left(\exp\left(\phi_1^\wedge\right)\exp\left(\phi_2^\wedge\right)\right)^\vee \approx \begin{cases} \boldsymbol{J}_l\left(\boldsymbol{\phi}_2\right)^{-1}\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 & \text{if } \boldsymbol{\phi}_1 \text{ is small} \\ \boldsymbol{J}_r\left(\boldsymbol{\phi}_1\right)^{-1}\boldsymbol{\phi}_2 + \boldsymbol{\phi}_1 & \text{if } \boldsymbol{\phi}_2 \text{ is small} \end{cases} \tag{28}$$

Left jocobi is

$$\boldsymbol{J_l} = \boldsymbol{J} = \frac{\sin\theta}{\theta}\boldsymbol{I} + \left(1 - \frac{\sin\theta}{\theta}\right)aa^T + \frac{1-\cos\theta}{\theta}a^\wedge \tag{29}$$

whose inverse is

$$\boldsymbol{J}_l^{-1} = \frac{\theta}{2}\cot\frac{\theta}{2}\boldsymbol{I} + \left(1 - \frac{\theta}{2}\cot\frac{\theta}{2}\right)\boldsymbol{a}\boldsymbol{a}^T - \frac{\theta}{2}\boldsymbol{a}^\wedge \tag{30}$$

Right jacobi is

$$\boldsymbol{J_r}(\phi) = \boldsymbol{J_l}(-\phi) \tag{31}$$

SO(3)

Assume for specified rotation R, whose Lie algera is ϕ. Give it a small rotation, denoted as ΔR, whose Lie algera is $\Delta\phi$

$$\exp\left(\Delta\phi^\wedge\right)\exp\left(\phi^\wedge\right) = \exp\left(\left(\phi + J_l^{-1}(\phi)\Delta\phi\right)^\wedge\right) \tag{32}$$

$$\exp\left((\phi+\Delta\phi)^\wedge\right) = \exp\left(\left(\boldsymbol{J}_l\Delta\boldsymbol{\phi}\right)^\wedge\right)\exp\left(\boldsymbol{\phi}^\wedge\right) = \exp\left(\boldsymbol{\phi}^\wedge\right)\exp\left(\left(\boldsymbol{J}_r\Delta\boldsymbol{\phi}\right)^\wedge\right) \tag{33}$$

There are two models to define derivative:

- Derivative model: according to Lie algebra addition
- Disturbance model: according to Lie group multiplication

Given point p Derivative model:

$$\begin{aligned}
\frac{\partial\left(\exp\left(\boldsymbol{\phi}^\wedge\right)\boldsymbol{p}\right)}{\partial\boldsymbol{\phi}} &= \lim_{\delta\boldsymbol{\phi}\to 0}\frac{\exp\left((\boldsymbol{\phi}+\delta\boldsymbol{\phi})^\wedge\right)\boldsymbol{p} - \exp\left(\boldsymbol{\phi}^\wedge\right)\boldsymbol{p}}{\delta\boldsymbol{\phi}} \\
&= \lim_{\delta\boldsymbol{\phi}\to 0}\frac{\exp\left(\left(\boldsymbol{J}_l\delta\boldsymbol{\phi}\right)^\wedge\right)\exp\left(\boldsymbol{\phi}^\wedge\right)\boldsymbol{p} - \exp\left(\boldsymbol{\phi}^\wedge\right)\boldsymbol{p}}{\delta\boldsymbol{\phi}} \\
&\approx \lim_{\delta\boldsymbol{\phi}\to 0}\frac{\left(\boldsymbol{I} + \left(\boldsymbol{J}_l\delta\boldsymbol{\phi}\right)^\wedge\right)\exp\left(\boldsymbol{\phi}^\wedge\right)\boldsymbol{p} - \exp\left(\boldsymbol{\phi}^\wedge\right)\boldsymbol{p}}{\delta\boldsymbol{\phi}} \\
&= \lim_{\delta\boldsymbol{\phi}\to 0}\frac{\left(\boldsymbol{J}_l\delta\boldsymbol{\phi}\right)^\wedge\exp\left(\boldsymbol{\phi}^\wedge\right)\boldsymbol{p}}{\delta\boldsymbol{\phi}} \\
&= \lim_{\delta\boldsymbol{\phi}\to 0}\frac{-\left(\exp\left(\boldsymbol{\phi}^\wedge\right)\boldsymbol{p}\right)^\wedge\boldsymbol{J}_l\delta\boldsymbol{\phi}}{\delta\boldsymbol{\phi}} = -(\boldsymbol{R}\boldsymbol{p})^\wedge\boldsymbol{J}_l
\end{aligned} \tag{34}$$

Note: $\frac{\partial(\boldsymbol{R}\boldsymbol{p})}{\partial\boldsymbol{R}}$ is not defined in terms of matrix differentiation, just a notation.

Disturbance model:

$$\begin{aligned}
\frac{\partial(\boldsymbol{R}\boldsymbol{p})}{\partial\boldsymbol{\varphi}} &= \lim_{\boldsymbol{\varphi}\to 0}\frac{\exp\left(\boldsymbol{\varphi}^\wedge\right)\exp\left(\boldsymbol{\phi}^\wedge\right)\boldsymbol{p} - \exp\left(\boldsymbol{\phi}^\wedge\right)\boldsymbol{p}}{\boldsymbol{\varphi}} \\
&\approx \lim_{\boldsymbol{\varphi}\to 0}\frac{\left(1+\boldsymbol{\varphi}^\wedge\right)\exp\left(\boldsymbol{\phi}^\wedge\right)\boldsymbol{p} - \exp\left(\boldsymbol{\phi}^\wedge\right)\boldsymbol{p}}{\varphi} \\
&= \lim_{\boldsymbol{\varphi}\to 0}\frac{\varphi^\wedge\boldsymbol{R}\boldsymbol{p}}{\boldsymbol{\varphi}} = \lim_{\boldsymbol{\varphi}\to 0}\frac{-(\boldsymbol{R}\boldsymbol{p})^\wedge\boldsymbol{\varphi}}{\boldsymbol{\varphi}} = -(\boldsymbol{R}\boldsymbol{p})^\wedge
\end{aligned} \tag{35}$$

SE(3)

$\Delta T = exp(\delta\xi^\wedge)$, then $\delta\xi = [\delta\rho, \delta\phi]^T$

$$\begin{aligned}
\frac{\partial(\boldsymbol{T}\boldsymbol{p})}{\partial\delta\boldsymbol{\xi}} &= \lim_{\delta\boldsymbol{\xi}\to 0} \frac{\exp\left(\delta\boldsymbol{\xi}^\wedge\right)\exp\left(\boldsymbol{\xi}^\wedge\right)\boldsymbol{p} - \exp\left(\boldsymbol{\xi}^\wedge\right)\boldsymbol{p}}{\delta\boldsymbol{\xi}} \\
&\approx \lim_{\delta\xi\to 0} \frac{\left(\boldsymbol{I} + \delta\boldsymbol{\xi}^\wedge\right)\exp\left(\boldsymbol{\xi}^\wedge\right)\boldsymbol{p} - \exp\left(\boldsymbol{\xi}^\wedge\right)\boldsymbol{p}}{\delta\boldsymbol{\xi}} \\
&= \lim_{\delta\xi\to 0} \frac{\delta\boldsymbol{\xi}^\wedge\exp\left(\boldsymbol{\xi}^\wedge\right)\boldsymbol{p}}{\delta\boldsymbol{\xi}} \\
&= \lim_{\delta\xi\to 0} \frac{\begin{bmatrix} \delta\phi^\wedge & \delta\boldsymbol{\rho} \\ \boldsymbol{0}^T & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{R}\boldsymbol{p}+\boldsymbol{t} \\ 1 \end{bmatrix}}{\delta\boldsymbol{\xi}} \\
&= \lim_{\delta\xi\to 0} \frac{\begin{bmatrix} \delta\phi^\wedge(\boldsymbol{R}\boldsymbol{p}+\boldsymbol{t}) + \delta\boldsymbol{\rho} \\ 0 \end{bmatrix}}{\delta\boldsymbol{\xi}} = \begin{bmatrix} \boldsymbol{I_{3\times 3}} & -(\boldsymbol{R}\boldsymbol{p}+\boldsymbol{t})^\wedge \\ \boldsymbol{0}^T & \boldsymbol{0}^T \end{bmatrix} \triangleq (\boldsymbol{T}\boldsymbol{p})^\odot
\end{aligned} \tag{36}$$

Transform a point in space with homogeneous coordinates into a 4, 6 matrix

2.6 Sim(3)

In monocular case, we generally express the scale factor explicitly. Given a point, it needs a simliar transfomation rather than Euclidean transfomation.

$$p' = \begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix} p = sRp + t \tag{37}$$

$$\mathrm{Sim}(3) = \left\{ S = \begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \right\} \tag{38}$$

$$\mathrm{sim}(3) = \left\{ \zeta \mid \zeta = \begin{bmatrix} \rho \\ \phi \\ \sigma \end{bmatrix} \in \mathbb{R}^7, \zeta^\wedge = \begin{bmatrix} \sigma I + \phi^\wedge & \rho \\ 0^T & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \right\} \tag{39}$$

$$\exp\left(\boldsymbol{\zeta}^\wedge\right) = \begin{bmatrix} e^\sigma \exp\left(\boldsymbol{\phi}^\wedge\right) & \boldsymbol{J}_s\boldsymbol{\rho} \\ \boldsymbol{0}^T & 1 \end{bmatrix} \tag{40}$$

$$\begin{aligned}
\boldsymbol{J}_s =& \frac{e^\sigma - 1}{\sigma}\boldsymbol{I} + \frac{\sigma e^\sigma \sin\theta + \left(1 - e^\sigma\cos\theta\right)\theta}{\sigma^2 + \theta^2}\boldsymbol{a}^\wedge \\
&+ \left(\frac{e^\sigma - 1}{\sigma} - \frac{\left(e^\sigma\cos\theta - 1\right)\sigma + \left(e^\sigma\sin\theta\right)\theta}{\sigma^2 + \theta^2}\right)\boldsymbol{a}^\wedge\boldsymbol{a}^\wedge
\end{aligned} \tag{41}$$

$$s = e^\sigma, \boldsymbol{R} = \exp\left(\boldsymbol{\phi}^\wedge\right), \boldsymbol{t} = \boldsymbol{J}_s\boldsymbol{\rho} \tag{42}$$