LinerEquationStability

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1 Condition Number

For the mapping $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$, export the definition **condition number** for the purpose of considering the sensitivity of the output to the input.

condition number is defined as follows:

$$k = \frac{\triangle y/y}{\triangle x/x}, \quad where \triangle y = f(x + \triangle x) - y \tag{1}$$

If the mapping is differentiable, then the derivative approximates the condition number

$$k \approx \left| \frac{f'(x)x}{f(x)} \right| \tag{2}$$

If condition number if large, the problem is called **ill-conditioned** and vice verse **well-conditioned**

Matrix Condition Number

$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, Matrix condition number is defined as:

$$\|A\| = \max_x \frac{\|Ax\|}{\|x\|} \tag{3}$$

And according to the definition above, the norm of A^{-1} can be got. $y = AX, x = A^{-1}y$

$$\|A^{-1}\| = \max_y \frac{\|A^{-1}y\|}{\|y\|} = 1/\min_y \frac{\|y\|}{\|A^{-1}y\|} = 1/\min_x \frac{\|Ax\|}{\|x\|}$$

$$\|A^{-1}\| = 1/\min_x \frac{\|Ax\|}{\|x\|} \tag{4}$$

Finally, the matrix condition number is as follows:

$$k(A) = \|A\| \cdot \|A^{-1}\| \tag{5}$$

The condiction number describes both **the matrix's ability to stretch and compress the vector.** i.e. the ability to deform the vector. The larger condition numbers, the more likely the vector is to change after transformation.

Calculation

Take use of the singular values. the geometric represention of singular value decomposition is to transform a group of orthogonal basis into another group of orthogonal basis. The singular value describes the scaling factor of corresponding basis vectors. The maximum of singular values $\boldsymbol{\sigma_{max}}$ corresponds to the possible maximum magnification after transfomation of the vector and the minimum of sigularvalues $\boldsymbol{\sigma_{min}}$ is analogous to the $\boldsymbol{\sigma_{min}}$.

$$k(A) = \frac{\sigma_{max}}{\sigma_{min}} \tag{6}$$

2 Stability of linear equation

When it comes to stability of linear equation $\boldsymbol{Ax} = \boldsymbol{b}$, how to measure the stability? Before discussion, explain some variables. An equation is said to be stable if a small change in the observation does not drastically change the solution of the equation.

- A – coefficient matrix, generally determined by the model
- x – to be solved.
- b – obseration from senor or something else.

And observation noise $\boldsymbol{\delta b}$ to the right-hand side of the equation.

$$A(x + \delta x) = b + \delta b \tag{7}$$

i.e. $A\delta x = \delta b$ Take the norm of both sides.

$$\|A\delta x\| = \|\delta b\| \tag{8}$$

It can be known from the Equation 3, $\|A\| \geq \frac{\|A\delta x\|}{\|\delta x\|}$

$$\|A\|\|\delta x\| \geq \|A\delta x\| = \|\delta b\| \tag{9}$$

Similarly, from the Equation 4, $\|A^{-1}\| \geq 1/\frac{\|Ax\|}{\|x\|}$ and $\|Ax\| = \|b\|$

$$\|A^{-1}\|\|b\| \geq \|x\| \tag{10}$$

Multiply left side and right side of Equation 9, 10 and arrange them:

$$\frac{\|\delta x\|}{\|x\|} \geq \frac{\|\delta b\|}{\|b\|} \cdot \frac{1}{k(A)} \tag{11}$$

Likesize, combine $\|A\| \geq \frac{\|Ax\|}{\|x\|}$ and $\|A^{-1}\| \geq 1/\frac{\|A\delta x\|}{\|\delta x\|}$ with $\|Ax\| = \|b\|$

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\|\delta b\|}{\|b\|} \cdot k(A) \tag{12}$$

we acquired the the lower bound and upper bound on the solution **x** of the linear equation affected by the observation **b**.

$$\frac{1}{\kappa(A)}\frac{\|\delta b\|}{\|b\|} \leq \frac{\|\delta x\|}{\|x\|} \leq \kappa(A)\frac{\|\delta b\|}{\|b\|} \tag{13}$$

From Equation 13, we can qcquire tow conclusions. First, $k(A)$ uniquely determines the influence degree to which the solution of the linear equation is affected by the noise of the observed value. The larger $k(A)$ is, the more server the influence degree is. The measurement of the stability of the condition number is not affected by the scale, and any proportion of the overall expansion or contraction will not affect the result of the inequality, because the influence degree of noise is measured by the rate of change, which is not concerned about the absolute size.