DEGENERATIONS TO SECANT CUBIC HYPERSURFACES AND LIMITING HODGE STRUCTURE

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Abstract. The secant variety of the Veronese surface is a singular cubic fourfold. The degeneration of Hodge structures of one-parameter degenerations to this secant cubic fourfold is a key ingredient for B. Hassett and R. Laza in studying the moduli space of cubic fourfolds via the period mapping. We generalize some of their results to the cubic hypersurface that is the secant variety of a Severi variety. Specifically, we study the limit mixed Hodge structures associated to one-parameter degenerations to the secant cubic hypersurface. Considering S. Usui's partial compactification of a period domain for Hodge structures of general weights, we apply the limit mixed Hodge structure to characterize a local extension of the period map for the corresponding cubic hypersurfaces.

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Introduction

The secant variety $X_0 := Sec(S)$ of the Veronese surface $S \subset \mathbb{P}^5$ is a cubic fourfold singular along S. It is a semi-stable cubic fourfold that plays an important role in the study of moduli space of cubic fourfolds. In the paper [26], R. Laza showed that the period map for cubic foufolds induces an isomorphism

$$\mathscr{F}^s \xrightarrow{\sim} \Gamma\backslash(\mathscr{D} - \mathscr{H}_2),$$

where $\mathscr{F}^s$ is the moduli space of cubic fourfolds with at worst simple singularities, $\mathscr{D}$ is the period domain for cubic fourfolds, Γ is an arithmetic subgroup of $\mathrm{Aut}(\mathscr{D})$, and $\mathscr{H}_2$ is a Γ-invariant arrangment of hyperplanes of discriminant two. Consider one-parameter families of smooth cubic fourfolds that degenerate to the secant cubic fourfold X_0. According to the theory of limiting Hodge structure by W. Schmid and J. Steenbrink [32, 33], one can associate a canonical mixed Hodge structure to a one-parameter degenertaion to X_0. B. Hassett proved that the limit mixed Hodge structure is pure and a special Hodge structure of discriminant two, see [14, §4.4]. It reflects that the divisor $\Gamma\backslash\mathscr{H}_2$, which parametrizes the Hodge structures of degree two $K3$ surfaces, can be viewed as the limit period points of smooth cubic fourfolds that degenerates to the secant cubic fourfold. In the present paper, we aim to study the degeneration of Hodge structures arising from deformations of a cubic hypersurface obtained as the secant variety of a Severi variety.

The Veronese surface is a so-called *Severi variety*. F. Zak proved that the secant variety $Sec(S)$ of a nondegenerate nonsingular projective variety $S \subset \mathbb{P}^{m+1}$ covers the ambient space $\mathbb{P}^{m+1}$ if $\dim S = d > \frac{2(m-1)}{3}$. The Severi variety is the variety $S^n \subset \mathbb{P}^{m+1}$ having the maximal dimension $\dim S = d = \frac{2(m-1)}{3}$ among those with degenerate secant $Sec(S) \neq \mathbb{P}^{m+1}$. The complete classification of the Severi varieties is also given by F. Zak [36, Thm. 4.7]. Up to projective equivalence, there are only four Severi varities

(1) $d = 2$, $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$, the Veronese surface;
(2) $d = 4$, $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$, the Segre fourfold;
(3) $d = 8$, $Gr(2,6) \hookrightarrow \mathbb{P}^{14}$, the Plücker embedding of the Grassmannian of lines in $\mathbb{P}^5$;
(4) $d = 16$, $E_6 \hookrightarrow \mathbb{P}^{26}$, the Cartan variety (or the Cayley plane $\mathbb{OP}^2$), given by a minimal irreducible representation of the algebraic group E_6.

It is known that, like the case of the Veronese surface, the secant variety of a Severi variety is always a cubic hypersurface in $\mathbb{P}^{m+1}$, which is singular along S, see [36, Thm. 2.4].

Let F be the equation of the secant cubic $Sec(S)$, and G be a nonsingular cubic equation such that $G|_S$ cuts out a smooth hypersurface V in S. The equation $F + tG = 0$ with t the coordinate of the open unit disk Δ presents a one-parameter degeneration to $\mathrm{Sec}(S)$. Following Mumford's semistable reduction theorem, we can associate a specific semistable degeneration $f : \mathfrak{X} \to \Delta$ to the one-parameter degeneration. Denote by H^m_{lim} the middle cohomology of a smooth cubic hypersurface in $\mathbb{P}^{m+1}$. Our main theorem characterizes the limit mixed Hodge structure on H^m_{lim} associated to f.

Theorem 0.1. *Let $S \subset \mathbb{P}^{m+1}$ be a Severi variety with $\dim S > 2$. Then the monodromy transformation of the semistable degeneration f has order two, and the associated monodromy weight filtration is of the form*

$$0 \subset W_{m-1} \subset W_m \subset W_{m+1} = H^m_{\mathrm{lim}}. \tag{1}$$

with $\dim W_{m-1} = 1$. The canonical polarized Hodge structure on the subquotient $Gr^W_m := W_m/W_{m-1}$ determined by the mixed Hodge structure is isomorphic to the middle cohomology of the hypersurface $V \subset S$. The Hodge structure on Gr^W_{m-1} (resp. Gr^W_{m+1}) is a Tate twist of weight $\frac{1-m}{2}$ (resp. $\frac{1+m}{2}$).

For $\dim S > 2$ the mixed Hodge structure on H^m_{lim} is not pure any more, which is different from the case of cubic fourfolds. But still, the degenerating Hodge structure on the main subquotient Gr^W_m is closely related to the hypersurface V in S defined by the degeneration. Such pattern similarly appears in the case of cubic fourfolds.

The main tool to prove Theorem 0.1 is the Clemens-Schmid exact sequence attached to a semistable degeneration. The construction of the specific semistable degeneration $f : \mathfrak{X} \to \Delta$ follows B. Hassett's treatment for the secant cubic fourfold [14, §4.4]. It is essential to compute the cohomology of a quadric fibration as one component in the central fiber of the semistable degeneration. For this purpose we generalize A. Beauville's computation of the cohomology of quadric fibrations over $\mathbb{P}^2$, cf. [2].

The degeneration $f : \mathfrak{X} \to \Delta$ defines a period map on the punctured disk Δ^*. Griffiths generally conjectured the existence of a partial compactification of any period domain such that any period map defined on Δ^* can be continued across the puncture. For the bounded symmetric domain, the Baily-Borel compactification [1, 3] serves as such a role. For general cases there are a series of research [19, 20, 22] considering the construction of such a compactification in terms of mixed Hodge structures or nilpotent orbit cones. In this paper, we employ a particular partial compactification, introduced by S. Usui [35]. Our purpose is using the result of Theorem 0.1 to study local extension property of the period map on the moduli space of cubic hypersurfaces.

Suppose that $\overline{\mathcal{F}}$ is the GIT compactification of the moduli space of smooth cubic hypersurfaces in $\mathbb{P}^{m+1}$, and $\omega \in \overline{\mathcal{F}}$ is the point representing the secant cubic $Sec(S)$. Consider Kirwan's blowing up of $\overline{\mathcal{F}}$ at ω. Let $\mathcal{M}$ be the exceptional divisor. Through Luna's slice theorem [30, p. 198], we can prove

Proposition 0.2. *The exceptional divisor $\mathcal{M}$ can be identified with certain GIT-quotient of hypersurfaces in the Severi variety $S \subset \mathbb{P}^{m+1}$ cut off by cubics in $\mathbb{P}^{m+1}$.*

By the construction of the blowing up, a generic point in $\mathcal{M}$ corresponds to a semistable degeneration of a one-parameter deformation of the secant cubic, also a period map

$$\wp : \Delta^* \to \Gamma\backslash\mathcal{D}$$

where $\Gamma\backslash\mathcal{D}$ is the global period domain for Hodge structures of smooth cubic hypersurfaces in $\mathbb{P}^{m+1}$. Let $\overline{\Gamma\backslash\mathcal{D}}$ denote Usui's partial compactification by adding suitable boundary components $\mathcal{B}(W_*)$ defined by the specific weight filtrations W_* of the form (1). We realize $\mathcal{B}(W_*)$ indeed parametrizes Hodge structures on Gr_m^W of the type $H^{n-1}(V)$ as in Theorem 0.1. Let $\overline{\wp} : \Delta \to \overline{\Gamma\backslash\mathcal{D}}$ be the extended map. Then we show that the limit period point $\overline{\wp}(0) \in \mathcal{B}(W_*)$ is exactly the Hodge structure on Gr_m^W determined by the limit mixed Hodge structure of $\wp$. In conclusion, we have

Theorem 0.3. *Under Kirwan's blowing up, the rational period map $\mathscr{P} : \overline{\mathcal{F}} \dashrightarrow \overline{\Gamma\backslash\mathcal{D}}$ can be (generically) continued across the exceptional divisor $\mathcal{M}$ such that the image of $\mathcal{M}$ is contained in the boundary component $\mathcal{B}(W_*)$. Moreover, the extended map $\overline{\mathscr{P}}|_{\mathcal{M}} : \mathcal{M} \to \mathcal{B}(W_*)$ is exactly the period map for the hypersurfaces that $\mathcal{M}$ parametrizes.*

As for the organization of this paper, the first section is a review of basic notions and properties for limit mixed Hodge structures and the Clemens-Schmid exact sequence. In Section 2, we describe the explicit semistable degeneration of the deformation of secant cubic hypersurface, and prove Theorem 0.1(=Theorem 2.4). The third section is devoted to the cohomology of quadric fibrations with certain mild degenerations. The local extension property of the period map is discussed in Section 4, where the proof of Proposition 0.2(=Corollary 4.5) and Theorem 0.3(=Theorem 4.1) are given. The SL_2-orbit theory is indispensible to study the extension property. For convenience we add the appendix A to collect notions and results of the SL_2-orbit that needed.

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1. Degeneration of Hodge structures

1.1. **Limiting mixed Hodge structures.** Let $\pi : \mathfrak{X}^* \to \Delta^*$ be a family of smooth projective varieties over the punctured disk Δ^*. Fix a non-negative integer m. Let $H_{\mathbb{Q}}$ be the vector space that is isomorphic to the cohomology group $H^m(X, \mathbb{Q})$ of any fiber X of the family π. Let $\mathcal{D}$ be the corresponding classifying space of polarized Hodge structures on H. The *variation of polarized Hodge structures* of the family π defines a *period map*

$$\phi : \Delta^* \to \Gamma \backslash \mathcal{D}$$

where the monodromy group Γ is generated by the *monodromy transformation* $T : H \to H$ of the family π.

Theorem 1.1. [23, Monodromy theorem] *The monodromy transformation T is quasi-unipotent, and the index of the unipotency is at most m, i.e., there exists an integer k such that $(T^k - 1)^{m+1} = 0$.*

Let t be the coordinate on Δ^*. By taking the base change $t \mapsto t^k$, we may assume T is unipotent. Define the nilpotent map N to be

$$N := \log(T) = -\sum_{n=1}^{m} \frac{(I-T)^n}{n}. \tag{2}$$

Note that the logarithm of T is well-defined since the index of the unipotency of T is finite. In particular, the index of T is equal to the index of N. Consider the universal covering

$$e : \mathfrak{h} \to \Delta^*, \ e(z) = e^{2\pi i z}$$

by the upper half plane $\mathfrak{h}$. Since $\mathfrak{h}$ is simply connected, the period map ϕ is lifted to

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "\tilde{\phi}"] \arrow[d, "e"] & \mathcal{D} \arrow[d] \\ \Delta^* \arrow[r, "\phi"] & \Gamma \backslash \mathcal{D} \end{tikzcd}$$

The lifting $\tilde{\phi}$ satisfies $\tilde{\phi}(z+1) = T\tilde{\phi}(z)$ for $z \in \mathfrak{h}$. Set the map $\tilde{\psi} : \mathfrak{h} \to \check{\mathcal{D}}$ by

$$\tilde{\psi}(z) := \exp(-zN)\tilde{\phi}(z)$$

where $\check{\mathcal{D}}$ is the compact dual of $\mathcal{D}$. We have

$$\tilde{\psi}(z+1) = \exp(-(z+1)N)\tilde{\phi}(z+1) = \exp(-zN)T^{-1}T\tilde{\phi}(z) = \tilde{\psi}(z).$$

It implies that $\tilde{\psi}$ descends to a single-valued map $\psi : \Delta^* \to \check{D}$. Cornalba and Griffiths [8] proved that ψ extends across the puncture to a map $\psi : \Delta \to \check{\mathcal{D}}$. The filtration $\psi(0) \in \check{\mathcal{D}}$ is called the *limiting Hodge filtration* and is usually denoted by

$$\{F^p_\infty\} := \lim_{\mathrm{Im}(z) \to \infty} \exp(-zN)\tilde{\phi}(z) \in \check{\mathcal{D}}. \tag{3}$$

The description of the monodromy weight filtration is a matter of linear algebra as follows.

Proposition 1.2 ([32, Lem. 6.4])**.** *Let H be a finite dimensional linear space over a field of characteristic zero. Let $N : H \to H$ be a linear map with $N^{m+1} = 0$. Then there is a unique increasing filtration $W(N)$*

$$0 \subset W_0 \subset \cdots \subset W_{2m} = H$$

such that

1. *$N(W_l) \subset W_{l-2}$,*
2. *the map $N^l : Gr^W_{m+l} \to Gr^W_{m-l}$ is an isomorphism for all $l \geq 0$.*

For any $l \geq 0$, define the primitive part $P_{m+l} \subset Gr^W_{m+l}H$ to be the kernel of

$$N^{l+1} : Gr^W_{m+l}H \to Gr^W_{m-l-2}H,$$

and set $P_{m-l} = 0$. Then there is the decomposition of Lefschetz type

$$Gr^W_k H \cong \bigoplus_i N^i(P_{k+2i}), i \geq max(m-k, 0) \tag{4}$$

If N is an infinitesimal isometry of a nondegenerate form S on H, that is,

$$S(Nu, v) + S(u, Nv) = 0, \ \forall u, v \in H,$$

then $W_l^\perp = W_{2m-l-1}$. Moreover, the bilinear form $S_l := S(\cdot\ , N^l\cdot)$ is nondegenerate on Gr^W_{m+l}, and the bilinear form $S_l := S((N^l)^{-1}\cdot, \cdot)$ is nondegenerate on Gr^W_{m-l}.

Let H be a vector space over $\mathbb{Q}$, let m be an integer, and let S be a nondegenerate bilinear form on H such that $S(u, v) = (-1)^m S(v, u)$.

Definition 1.3. [5, Def. 2.26] *A polarized mixed Hodge structure on H consists of a mixed Hodge structure (W, F) and an infinitesimal isometry N of S such that*

1. *$N^{m+1} = 0$;*
2. *W is the monodromy weight filtration $W(N)$;*
3. *$NF^p \subset F^{p-1}$;*
4. *$S(F^p, F^{m-p+1}) = 0$;*
5. *the Hodge structure on the primitive part P_{m+l} is polarized by the form $S(\cdot, N^l\cdot)$.*

Return back to the family $\pi : \mathfrak{X}^* \to \Delta^*$. Let $H^m_{\lim}$ denote the vector space of the cohomology group of a fiber of π with the natural polarization. Let F_∞ be the limiting Hodge filtration defined by (3). Through Proposition 1.2 and the Monodromy theorem, the nilpotent map $N : H^m_{\lim} \to H^m_{\lim}$ associated to the family π defines the monodromy weight filtration $W(N)$

$$0 \subset W_0 \subset \cdots \subset W_{2m} = H^m_{\lim}. \tag{5}$$

As a consequence of the nilpotent orbit theorem, Schmid showed

Theorem 1.4 ([32, Thm. 6.16])**.** *The two filtrations $W(N)$ and F_∞ determine a polarized mixed Hodge structure on $H^m_{\lim}$. The nilpotent map N is a morphism of mixed Hodge structures of type $(-1, -1)$.*

1.2. **Clemens-Schmid exact sequence.** We say a family $f : \mathfrak{X} \to \Delta$ is a degeneration if $\mathfrak{X}$ is a smooth variety, f is a proper and flat morphism that is smooth over the punctured disk Δ^*. The degeneration f is called *semistable* if the central fiber $\mathfrak{X}_0$ is a reduced divisor in $\mathfrak{X}$ with simple normal crossing. Mumford's semistable reduction theorem [21, §II] states that any degeneration over the unit disk can be brought into a semistable form after a finite base change ramified at the origin and a birational modification on the central fiber.

Let $f : \mathfrak{X} \to \Delta$ be a semistable degeneration. The limit mixed Hodge structure of the smooth family $\pi : \mathfrak{X}^* \to \Delta^*$ can be characterized by the central fiber $\mathfrak{X}_0$ using the Clemens-Schmid exact sequence.

The cohomology of the central fiber $\mathfrak{X}_0$ carries a canonical mixed Hodge structure, which involves combinatorial data of irreducible components of $\mathfrak{X}_0$. Let $\{X_i\}$ be the irreducible components of $\mathfrak{X}_0$. Set

$$\mathfrak{X}^{[p]} := \bigsqcup_{i_0 < \cdots < i_p} X_{i_0} \cap \cdots \cap X_{i_p}$$

to be the disjoint union of the codimension p stratum of $\mathfrak{X}_0$. There is a spectral sequence

$$E_1^{p,q} := H^q(\mathfrak{X}^{[p]}, \mathbb{Q}) \Rightarrow H^m(\mathfrak{X}_0, \mathbb{Q}),\ p+q=m, \tag{6}$$

with the first differential map

$$d_1 : H^q(\mathfrak{X}^{[p]}, \mathbb{Q}) \to H^q(\mathfrak{X}^{[p+1]}, \mathbb{Q})$$

induced by the natural combinatorial boundary map $\iota_p : \mathfrak{X}^{[p+1]} \to \mathfrak{X}^{[p]}$. One can put a weight filtration

$$W_k := \bigoplus_{q \le k} E_1^{*,q}.$$

on the spectral sequence, which induces a weight filtration on $H^m(\mathfrak{X}_0, \mathbb{Q})$:

$$0 \subset W_0 H^m(\mathfrak{X}_0, \mathbb{Q}) \subset \cdots \subset W_m H^m(\mathfrak{X}_0, \mathbb{Q}) = H^m(\mathfrak{X}_0, \mathbb{Q}).$$

Proposition 1.5. *The spectral sequence* (6) *degenerates at E_2. As a result, the k-th subquotient $Gr_k^W H^m(\mathfrak{X}_0, \mathbb{Q})$ is isomorphic to the E_2-term ${E_2}^{m-k,k}$.*

For a simple normal crossing divisor, the strata $\mathfrak{X}^{[p]}$ are smooth. Then the cohomology group of $\mathfrak{X}^{[p]}$ carries a canonical Hodge structure. Through the spectral sequence (6) it induces a decreasing (Hodge) filtration on $H^m(\mathfrak{X}_0, \mathbb{Q})$. Together with the above weight filtration it determines the canonical mixed Hodge structure on $H^m(\mathfrak{X}_0, \mathbb{Q})$. Denote by H^* (resp. H_*) the cohomology (resp. homology) group $H^*(\mathfrak{X}_0, \mathbb{Q})$ (resp. $H_*(\mathfrak{X}_0, \mathbb{Q})$). One can associate a mixed Hodge structure to the homology group H_* by duality. Precisely, the weight filtration on H_m is defined by

$$W_{-k} H_m := \mathrm{Ann}(W_{k-1} H^m) = \{\alpha \in H_m \mid (\alpha, W_{k-1} H^m) = 0\}$$

It is easy to check that $\mathrm{Gr}_{-k}^W H_m \cong (\mathrm{Gr}_k^W H^m)^*$.

Suppose that the semistable degeneration $f : \mathfrak{X} \to \Delta$ is of relative dimension n. Let $\mathfrak{X}_t$ be a smooth fiber of f. Deligne's local invariant cycle theorem asserts any monodromy invariant class on $H^m(\mathfrak{X}_t, \mathbb{Q})$ is a global class on $\mathfrak{X}^*$, i.e., there is the exact sequence

$$H^m(\mathfrak{X}, \mathbb{Q}) \xrightarrow{i^*} H^m(\mathfrak{X}_t, \mathbb{Q}) \xrightarrow{N} H^m(\mathfrak{X}_t, \mathbb{Q})$$

where $i : \mathfrak{X}_t \hookrightarrow \mathfrak{X}$ is the natural inclusion. For a semistable degeneration the total space $\mathfrak{X}$ admits a deformation retraction $r : \mathfrak{X} \to \mathfrak{X}_0$, which induces the isomorphisms

$$r^* : H^*(\mathfrak{X}_0) \xrightarrow{\sim} H^*(\mathfrak{X}),\ r_* : H_*(\mathfrak{X}) \xrightarrow{\sim} H_*(\mathfrak{X}_0).$$

Using the *Wang sequence* for the smooth family $\mathfrak{X}^* \to \Delta^*$, one can extend the above sequence to the Clemens-Schmid exact sequence

$$(7) \qquad \to H_{2n+2-m}(\mathfrak{X}_0) \xrightarrow{\alpha} H^m(\mathfrak{X}_0) \xrightarrow{i^*} H^m(\mathfrak{X}_t) \xrightarrow{N} H^m(\mathfrak{X}_t) \xrightarrow{\beta} H_{2n-m}(\mathfrak{X}_0) \to .$$

The morphisms α and β are induced by the Poincaré duality.

Theorem 1.6 ([29, §3])**.** *The morphism α, i^*, N, β are morphisms of mixed Hodge structures of weight $(n+1, n+1), (0,0), (-1,-1), (-n,-n)$.*

The following result will be useful in the proof of Theorem 2.4.

Corollary 1.7. *Denote by H^*_{lim} (resp. H^*, H_*) the vector space $H^*(\mathfrak{X}_t)$ (resp. $H^*(\mathfrak{X}_0), H_*(\mathfrak{X}_0)$) in the exact sequence (7). Suppose $k > 0$. Then the k-th iterated nilpotent map $N^k : H^m_{\mathrm{lim}} \to H^m_{\mathrm{lim}}$ is zero if and only if $W_{m-k}H^m = 0$.*

Proof. The property (2) in Proposition 1.2 implies that

$$W_{m-k}H^m_{lim} = 0 \text{ if and only if } N^k = 0, \forall\ 0 < k \leq m.$$

Let K be the kernel of the nilpotent map $N : H^m_{\mathrm{lim}} \to H^m_{\mathrm{lim}}$. We claim that

$$W_{m-k}H^m_{lim} = 0 \text{ if and only if } W_{m-k} \cap K = 0.$$

It suffices to show the left hand side follows from the right hand side. Assume that $W_{m-k}H^m_{lim} \neq 0$ and $W_{m-j}H^m_{lim} \subset W_{m-k}H^m_{lim}$ the smallest nonzero weight subspace. Then we have $0 \neq W_{m-j}H^m_{lim} \subset W_{m-k} \cap K$ which is a contradiction.

It remains to prove $W_{m-k} \cap K = W_{m-k}H^m$. By Theorem 1.6 the morphism i^* is strict. Then $i^* : W_{m-k}H^m \twoheadrightarrow W_{m-k} \cap K$ is surjective. Suppose that $x \in W_{m-k}H^m$ such that $i^*(x) = 0$. Again by the strictness of the morphism α, which shifts the weight by $2n+2$, there exists some $y \in W_{-2n-2+m-k}H_{2n+2-m}$ such that $\alpha(y) = x$. However, we note that $W_{-2n-2+m-k}H_{2n+2-m} = 0$ since $2n+2-m+k > 2n+2-m$. As a result, the map i^* is bijective and our assertion follows. □

2. Degenerations to secant cubic hypersurfaces

2.1. **Severi variety and the secant variety.** We start with a brief review of the Severi variety and the secant variety via representations of (semi)simple algebraic groups. Let $S \subset \mathbb{P}^{m+1}$ be a Severi variety of dimension $d = \frac{2}{3}(m-1)$. There is a linear algebraic group H, and an irreducible H-module $W \cong \mathbb{C}^{m+2}$ such that S can be identified with a specific H-orbit in $\mathbb{P}(W)$. In his paper [36, §III, 2.5] Zak described the group H and the representation W for the four Severi varieties

(1) $d = 2$, $H = SL(3,\mathbb{C})$, the space W consists of rank 3 symmetric matrices. The action of H on W is given by $A \mapsto gAg^\intercal$ where $g \in H$ and $A \in W$. The affine cone of the Veronese surface in W is given by the H-orbit consisting of rank one symmetric matrices.
(2) $d = 4$, $H = SL(3,\mathbb{C}) \times SL(3,\mathbb{C})$, the space W consists of rank 3 matrices. The action of H on W is given by $A \mapsto gAh^\intercal$ for any $(g,h) \in H$ and $A \in W$. The affine cone of the Segre fourfold $\mathbb{P}^2 \times \mathbb{P}^2$ in W is the H-orbit consisting of rank one matrices.

(3) $d = 8$, $H = SL(6,\mathbb{C})$, the space W consists of rank 6 skew-symmetric matrices. The action of H on W is given by $A \mapsto gAg^{\intercal}$ where $g \in H$, $A \in W$. The affine cone of the Grassmannian $Gr(2,6)$ in W is given by the H-orbit consisting of rank two skew-symmetric matrices.

(4) $d = 16$, $H = E_6$, let $\mathbb{O}$ be the octonion algebra over $\mathbb{R}$. Let $\mathcal{H}_{\mathbb{O}}$ denote the space of $\mathbb{O}$-Hermitian matrices

$$\left\{ \begin{pmatrix} c_1 & x_1 & x_2 \\ \overline{x_1} & c_2 & x_3 \\ \overline{x_2} & \overline{x_3} & c_3 \end{pmatrix}, c_i \in \mathbb{R}, x_i \in \mathbb{O} \right\}.$$

The space W is the 27-dimensional exceptional Jordan algebra $\mathcal{H}_{\mathbb{O}} \otimes_{\mathbb{R}} \mathbb{C}$ with the Jordan multiplication $A \circ B = \frac{1}{2}(AB+BA)$. The subgroup $SL_3(\mathbb{O})$ of $GL(\mathcal{H}_{\mathbb{O}} \otimes_{\mathbb{R}} \mathbb{C})$ preserving the determinant is isomorphic to the adjoint group E_6, see [24, §3]. The affine cone of the Cayley plane $\mathbb{OP}^2$ is given by the H-orbit consisting of rank one matices in $\mathcal{H}_{\mathbb{O}} \otimes_{\mathbb{R}} \mathbb{C}$.

Moreover, the secant variety $Sec(S)$ of S is given by the cone of degenerate matrices $\{A \in W | \det(A) = 0\}$, which is the reduced hypersurface in $\mathbb{P}(W)$ defined by the determinant equation. For a skew-symmetric matrix A we have $\det(A) = \mathrm{Pf}(A)^2$ where $\mathrm{Pf}(A)$ is the Pfaffian of A. In this case we refer to the Pfaffian as the defining equation of the reduced hypersurface $Sec(S)$. Hence the secant variety of a Severi variety is a cubic hypersurface whose singular locus is S.

2.2. **Semistable degeneration.** Let us consider an one-parameter degeneration $\mathfrak{X} \to \Delta$ to the secant cubic $X_0 := Sec(S)$ defined by the equation

$$F + tG = 0,\ t \in \Delta. \tag{8}$$

Here F is the defining equation of $Sec(S)$, and G is a cubic equation on $\mathbb{P}^{m+1}$ satisfying

- the cubic defined by G transversally intersects along the smooth locus of X_0;
- the hypersurface $V \subset S$ cut out by the equation $G|_S$ is smooth.

Proposition 2.1. *A semistable reduction $\mathfrak{X}' \to \Delta'$ is obtained from the degeneration $\mathfrak{X} \to \Delta$ by the following two steps:* (1) *Taking the base change $\Delta' \to \Delta$ by $t = u^2$;* (2) *Blowing up $\mathfrak{X} \times_{\Delta} \Delta'$ along the subvariety S in the central fiber.*

The central fiber of the semistable reduction $\mathfrak{X}' \to \Delta'$ consists of two irreducible components

- *the blowup $\overline{X_0}$ of the secant cubic X_0 along S;*
- *the exceptional divisor E of the blowup $\mathfrak{X}'$.*

We need the following result on Severi varieties for the degeneration $\mathfrak{X}' \to \Delta'$ being semistable.

Lemma 2.2. *Let $\overline{Sec(S)}$ be the blowup of $Sec(S)$ along the singular locus S, and let E_0 be the exceptional divisor. Then $\pi : E_0 \to S$ is a family of smooth quadrics.*

Proof. The secant variety $Sec(S)$ is double along S, i.e., $mult_x(Sec(S)) = 2$ for $x \in S$. Then $\pi : E_0 \to S$ is a quadric bundle. We aim to show π is a smooth morphism.

For the hypersurface $Sec(S)$ we consider the Gauss map

$$\gamma : Sec(S) \dashrightarrow \check{\mathbb{P}}^{m+1},\ p \mapsto \hat{T}_p Sec(S)$$

where $\hat{T}_p Sec(S)$ is the hyperplane in $\mathbb{P}^{m+1}$ tangent to $Sec(S)$ at a smooth point p. Obviously the rational map γ is not defined on the singular locus S. The conormal variety $\mathcal{C}$ of $Sec(S)$ is the closure of the graph

$$\{(p,[H]) \in (Sec(S) - S) \times \check{\mathbb{P}}^{m+1} \mid \hat{T}_p Sec(S) = H\}.$$

of the map γ. The dual variety $Sec(S)^*$ of $Sec(S)$ is defined to be the image of $\mathcal{C}$ in $\check{\mathbb{P}}^{m+1}$ via the projection $pr : \mathcal{C} \to \check{\mathbb{P}}^{m+1}$. It is well-known that for the Severi variety S the dual variety $Sec(S)^*$ is isomorphic to S in $\check{\mathbb{P}}^{m+1}$, see [36, §III, Thm. 2.4].

For any $[\hat{T}_p Sec(S)] \in Sec(S)^*$, the fiber of the projection $pr : \mathcal{C} \to \check{\mathbb{P}}^{m+1}$ over $[\hat{T}_p Sec(S)]$ is the *secant cone*

$$\Sigma_p := \overline{\{u \in Sec(S) - S \mid \hat{T}_u Sec(S) = \hat{T}_p Sec(S)\}}.$$

For the Severi variety S the secant cone Σ_p is a $(\frac{d}{2}+1)$-dimensional linear subspace in $Sec(S)$, see [27, Thm. 2.1]. Therefore $p : \mathcal{C} \to Sec(S)^*$ is a projective bundle with rank $\frac{d}{2}+2$. As a result, $\mathcal{C}$ is a resolution of the Gauss map obtained by blowing up $Sec(S)$ along the singular locus S.

The exceptional divisor E_0 is the restriction of $\mathcal{C}$ to S. Consider the induced morphism $pr|_{E_0} : E_0 \to Sec(S)^*$. The preimage of a point $[\hat{T}_p Sec(S)] \in Sec(S)^*$ is the intersection

$$Q_p := \Sigma_p \cap S,$$

which is called the *secant locus* of the point p. By [27, §1a], the secant locus Q_p consists of the points $x \in S$ such that the join line $\langle x, p\rangle$ is secant to S. It is a smooth quadric in $\Sigma_p \cong \mathbb{P}^{\frac{d}{2}+1}$. For the quadric bundle $\pi : E_0 \to S$ and any $x \in S$, we have

$$\pi^{-1}(x) = \{(x, [\hat{T}_p Sec(S)]) \mid x \in Q_p\}.$$

In other words, $\pi^{-1}(x)$ parametrizes the secant quadrics in S passing through the point x. It follows from the result [36, §IV, Prop. 3.1] that $\pi^{-1}(x)$ is isomorphic to a smooth quadric with dimension $\frac{d}{2}$. $\square$

Proof of Proposition 2.1. We first prove the total spcae $\mathfrak{X}'$ is smooth, then show the central fiber of $\mathfrak{X}' \to \Delta'$ is a normal crossing divisor.

Under the base change $t = u^2$ the family $\mathfrak{X} \times_\Delta \Delta'$ is defined by the equation $F + u^2 G = 0$. Let $(z_1, \ldots, z_{m+1}, u)$ be a local coordinate of $\mathbb{P}^{m+1}_{\Delta'}$, and f (resp. g) the local equation of F (resp. G). We have the derivatives

(1) $\frac{\partial(f+u^2g)}{\partial z_i} = \frac{\partial f}{\partial z_i} + u^2\frac{\partial g}{\partial z_i}$, $1 \le i \le m+1$;
(2) $\frac{\partial(f+u^2g)}{\partial u} = 2ug$.

We claim the singularity of $\mathfrak{X} \times_\Delta \Delta'$ supports at the central fiber. Otherwise, assume that p is a singular point with $u(p) \neq 0$. We see from the derivative that $g(p) = f(p) = 0$. Hence $p \in X \cap X_0$ where X is the smooth hypersurface defined by G. The derivatives $\frac{\partial g}{\partial z_i}(p)$ are not all equal to zero. Then the equality

$$\frac{\partial f}{\partial z_i}(p) + u(p)^2\frac{\partial g}{\partial z_i}(p) = 0$$

implies that p is a nonsingular point of X_0, and X is tangent to X_0 at p, which contradicts our assumptions on the degeneration (8). Moreover, for $u(p) = 0$, the point p is singular if and only if $\frac{\partial f}{\partial z_i}(p) = 0$ for all i. Therefore the singular locus of $\mathfrak{X} \times_\Delta \Delta'$ is the Severi variety S.

Note that $\mathfrak{X}'$ is the proper transform of $\mathfrak{X} \times_\Delta \Delta'$ in the blow-up of $\mathbb{P}^{m+1}_{\Delta'}$ along the smooth subvariety S. To verify the smoothness of $\mathfrak{X}'$, we look at the blowing up locally. Let $\mathbb{D}_d \subset \mathbb{C}^d$ be a polydisc around a point $p \in S$ with a local chart $(x_1, \ldots, x_d)$. Extend it to a polydisc $\mathbb{D}_{m+2} \subset \mathbb{C}^{m+2}$ around $p \in \mathbb{P}^{m+1}_{\Delta'}$ with a local chart $(x_1, \ldots, x_d, y_1, \ldots, y_{\frac{d}{2}+2}, u)$ such that the local embedding of the Severi variety $S \subset \mathbb{P}^{m+1}$ can be arranged as

$$(x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_d, 0, \ldots, 0).$$

The blow-up of $\mathbb{D}_{m+2}$ along $\mathbb{D}_d$ is the closed subvariety

$$\widetilde{\mathbb{D}_{m+2}} \subset \mathbb{D}_{m+2} \times \mathbb{P}[W_1, \ldots, W_{\frac{d}{2}+2}, T]$$

defined by the following equations

$$y_j W_i - y_i W_j = 0,\ y_i T - u W_i = 0,\ 1 \leq i, j \leq \frac{d}{2} + 2.$$

Denote by f and g the local equations of F and G under the coordinate $(x_1, \ldots, x_d, y_1, \ldots, y_{\frac{d}{2}+2})$. Since $Sec(S)$ is double along the singular locus S the local equation f can be written as

$$f = Q + \sum_{i=1}^{d} x_i Q_i + H$$

where $Q,\ Q_i,\ H$ are homogeneous polynomials in $\mathbb{C}[y_1, \ldots, y_{\frac{d}{2}+2}]$ with $\deg Q = \deg Q_i = 2,\ \deg H = 3$. Under the local chart $(x_1, \ldots, x_d)$, the exceptional divisor E_0 in the blow-up $\overline{X_0}$, as a quadric bundle over S, can be presented by the family of quadratic forms

$$Q + \sum_{i=1}^{d} x_i Q_i.$$

It follows from Lemma 2.2 that the quadratic forms $Q + \sum_{i=1}^{d} a_i Q_i$ are non-degenerate for $\underline{(a_i)} \in \mathbb{D}_d$. In particular, Q is non-degenerate. We may assume that the smooth divisor $V \subset S$ cut out by G is given by the equation $x_d = 0$. Therefore we can write g to be

$$g = x_d + g'$$

where g' has no monomials of $(x_1, \ldots, x_d)$.

Let $U_i \subset \widetilde{\mathbb{D}_{m+2}}$ be the open subset with $W_i \neq 0$. The restriction of $\mathfrak{X}'$ to U_i is read off from the pullback

$$\pi^*(f + u^2 g)|_{U_i} = y_i^2 \tilde{f}|_{U_i}$$

where $\pi : \widetilde{\mathbb{D}_{m+2}} \to \mathbb{D}_{m+2}$ is the natural projection. Here V $(y_i = 0)$ is the exceptional divisor, and $\tilde{f}|_{U_i}$ defines the proper transform $\mathfrak{X}' \cap U_i$. Set $w_j = \frac{W_j}{W_i}, t = \frac{T}{W_i}$. We have

$$\tilde{f}|_{U_i} = Q(\ldots, w_{i-1}, 1, w_{i+1}, \ldots,) + \sum x_i Q_i(\ldots, w_{i-1}, 1, w_{i+1}, \ldots,) + \text{higher terms}.$$

Recall that Q is a non-degenerate quadratic form. Hence $Q(\dots, w_{i-1}, 1, w_{i+1}, \dots,)$ contains a non-trivial linear term, and thus $\mathfrak{X}'$ is smooth in U_i. On the open subset $V \subset \widetilde{\mathbb{D}_{m+2}}$ with $T \neq 0$ we set $w_i = \frac{W_i}{T}$, and have

$$\pi^*(f + u^2 g)|_V = u^2(Q(w_1, \dots, w_d) + x_d + \text{higher terms}).$$

Then $\mathfrak{X}' \cap V$ defined by the equation $x_d + Q + \{\text{higher terms}\} = 0$ is smooth. It concludes the total space $\mathfrak{X}'$ is smooth. The central fiber of $\mathfrak{X}' \to \Delta'$ has two irreducible components:

(1) the proper transform $\overline{X_0}$ of X_0;
(2) the exceptional divisor E of $\mathfrak{X}'$.

Lemma 2.2 shows the proper transform $\overline{X_0}$ is isomorphic to the conormal variety of X_0, which is non-singular. Suppose that $\mathcal{E}$ is the exceptional divisor in the blow-up of $\mathbb{P}^{m+1}_{\Delta'}$. Then E is the intersection of $\mathcal{E}$ and $\mathfrak{X}'$. Both $\mathcal{E}$ and $\mathfrak{X}'$ are smooth divisors of the blow-up. They transversally intersects along E. Hence E is a smooth divisor of $\mathfrak{X}'$. In addition, E intersects with $\overline{X_0}$ along the exceptional divisor E_0 of $\overline{X_0}$. It concludes the central fiber of $\mathfrak{X}'$ is a normal crossing divisor. □

The following corollary is significant to compute the cohomology of the exceptional divisor E in §3.

Corollary 2.3. *The exceptional divisor E is a family of $(\frac{d}{2}+1)$-dimensional quadrics over S. The discriminant locus of the quadric bundle is V. The singular fiber E_s for any $s \in V$ is a quadric cone with a single vertex.*

Proof. The exceptional divisor E is the projective normal cone of S to $\mathfrak{X} \times_\Delta \Delta'$. Applying the analysis of the local equation $f + u^2 g$ in the proof of Proposition 2.1, we can locally present the normal cone by the family of quadratic forms

$$Q + \sum_{i=1}^{d} x_i Q_i + u^2 x_d$$

that parametrized by the coordinate $(x_1, \dots, x_d, u)$ of the polydisc $\mathbb{D}_{d+1}$. By the proof of Proposition 2.1, we conclude that the quadratic forms

$$Q + \sum_{i=1}^{d} a_i Q_i + u^2 a_d$$

is non-degenerate if and only if $a_d \neq 0$. Therefore for $s \in S \setminus V$, i.e., $x_d \neq 0$, the projective normal cone at s is a smooth quadric with the maximal rank $\frac{d}{2}+2$. For $s \in V$, i.e., $x_d = 0$, the projective normal cone at s is a quadric cone with the rank $\frac{d}{2}+1$. □

2.3. **Main statement.** Suppose that $S \subset \mathbb{P}^{m+1}$ is a Severi variety of dimension $d = \frac{2(m-1)}{3}$ with $d > 2$. By the classification of Severi varities we recall that $d = 4, 8, 16$ and $m = 7, 13, 25$ respectively. Let $f : \mathfrak{X}' \to \Delta'$ be the semistable degeneration to the secant cubic $Sec(S)$ discussed in Proposition 2.1, and let $H^m_{\lim}$ denote the $\mathbb{Q}$-coefficients Betti cohomology of a smooth fiber of f with the associated nilpotent map

$$N : H^m_{\lim} \to H^m_{\lim}.$$

Theorem 2.4. *Let V be the hypersurface in S cut off by any smooth fiber of the one-parameter degeneration* (8)*. The nilpotent map N has index 1, i.e., $N \neq 0, N^2 = 0$. Moreover, the limit mixed Hodge structure on H^m_{lim} satisfies*

(1) $Gr^W_{m-k} H^m_{\text{lim}} = 0$ *if* $k > 1$ *or* $k < -1$;
(2) $Gr^W_{m-k} H^m_{\text{lim}}$ *is isomorphic to the Tate twist* $\mathbb{Q}(\frac{k-m}{2})$ *if* $k = \pm 1$;
(3) $Gr^W_m H^m_{\text{lim}} \cong H^{d-1}(V, \mathbb{Q})$ *is an isomorphism of polarized Hodge structures.*

Before giving the proof of the theorem let us look at the following observation on the Segre fourfold, which initiates the work of this article.

Example 2.5. *Let $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$ be the Segre fourfold, and X be a generic cubic sevenfold in $\mathbb{P}^8$. It is known that the Hodge structure of a smooth cubic sevenfold is of Calabi-Yau type, cf.* [18]*. Precisely the Hodge numbers are*

$$h^{7,0} = h^{6,1} = 0, h^{5,2} = 1, h^{4,3} = 84.$$

Consider the $(3,3)$-hypersurface $V := X \cap (\mathbb{P}^2 \times \mathbb{P}^2)$. The adjunction formula

$$K_V \cong (K_{\mathbb{P}^2 \times \mathbb{P}^2} + V)|_V = (\mathcal{O}(-3,-3) + \mathcal{O}(3,3))|_V = \mathcal{O}_V$$

asserts V is Calabi-Yau. By the Lefschetz hyperplane theorem

$$H^i(\mathbb{P}^2 \times \mathbb{P}^2, \mathbb{Z}) \xrightarrow{\sim} H^i(Y, \mathbb{Z}), \ \forall i \leq 2$$

we have $H^i(V, \mathcal{O}_V) = 0$ for $0 \leq i \leq 2$. The Hodge diamond of V turns out to be

$$\begin{array}{lccccccc}
h^0 & & & & 1 & & & \\
h^1 & & & 0 & & 0 & & \\
h^2 & & 0 & & 2 & & 0 & \\
h^3 & 1 & & 83 & & 83 & & 1.
\end{array}$$

The Hodge numbers $h^{1,2} = h^{2,1} = 83$ can be calculated by the Euler characteristic of V which is equal to the top Chern class $c_3(T_V)$ of the tangent bundle T_V. Through the exact sequence

$$0 \to T_V \to T_{\mathbb{P}^2 \times \mathbb{P}^2}|_V \to N_{V/\mathbb{P}^2 \times \mathbb{P}^2} \cong \mathcal{O}(3,3) \to 0$$

we get $c(T_V) \cdot c(N_{V/\mathbb{P}^2 \times \mathbb{P}^2}) = c(T_{\mathbb{P}^2 \times \mathbb{P}^2})|_V$. It follows that

$$\sum_i c_i(V)(1 + 3(H_1 + H_2)) = (1 + p_1^* c_1(\mathbb{P}^2) + p_1^* c_2(\mathbb{P}^2))(1 + p_2^* c_1(\mathbb{P}^2) + p_2^* c_2(\mathbb{P}^2))|_V.$$

where H_i is the pullback of the hyperplane class $H \subset \mathbb{P}^2$ along the i-th projection $p_i : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$. As a result,

$$c_2(V) = c_1(E) \cdot c_1(F) + c_2(E) + c_2(F) = 9H_1H_2 + 3H_1^2 + 3H_2^2;$$
$$c_3(V) = 9(H_1^2 H_2 + H_1 H_2^2) - 3c_2(V)(H_1 + H_2).$$

Then the degree of $c_3(V)$ is equal to -162, and $h^{1,2} = h^{2,1} = 83$. From the numerical analysis, we expect, as well as the case of the Veronse surface and the cubic fourfolds, the limiting Hodge structure of the degeneration of cubic sevenfolds to the secant cubic should be the Hodge structure of the Calabi-Yau threefold V.

Proof of Theorem 2.4. Let $\mathfrak{X}_0$ denote the central fiber of the semistable degeneration $f : \mathfrak{X}' \to \Delta'$. Recall that the simple normal crossing divisor $\mathfrak{X}_0$ has two irreducible components $\overline{X_0}$ and E that intersects along E_0, see Proposition 2.1. Consider the Clemens-Schmid exact sequence (cf. (7))

$$\cdots \to H_{m+2}(\mathfrak{X}_0) \xrightarrow{\alpha} H^m(\mathfrak{X}_0) \xrightarrow{i^*} H^m_{lim} \xrightarrow{N} H^m_{lim} \xrightarrow{\beta} H_m(\mathfrak{X}_0) \to \cdots$$

for the semistable degeneration f. Denote by H^* (resp. H_*) the cohomology group $H^*(\mathfrak{X}_0)$ (resp. $H_*(\mathfrak{X}_0)$). To prove $N^2 = 0$, it suffices to show $W_{m-2}H^m = 0$ by Corollary 1.7. Recall the spectral sequence

$$E_1^{p,q} = H^q(\mathfrak{X}^{[p]}, \mathbb{Q}) \Rightarrow H^m(\mathfrak{X}_0).$$

For $p > 1$ the term $E_1^{p,q}$ vanishes since $\mathfrak{X}_0$ has only two components. Note that $\mathrm{Gr}^W_{m-k} H^m \cong E_2^{k,m-k}$. Hence for $k > 1$ we obtain $\mathrm{Gr}^W_{m-k} H^m = 0$ and thus $W_{m-k}H^m = 0$, which asserts $N^2 = 0$.

Since $N^2 = 0$ the subspace $W_{m-2}H^m_{\mathrm{lim}} = 0$, which leads to $Gr_i^W H^m_{\mathrm{lim}} = 0$ for $i < m-1$, as well as $i > m-1$ by symmetry. The strictness of the morphisms in the Clemens-Schmid exact sequence yields the long exact sequence

$$\to Gr^W_{-m-k-2}H_{m+2} \xrightarrow{\alpha} Gr^W_{m-k}H^m \xrightarrow{i^*} Gr^W_{m-k}H^m_{\mathrm{lim}} \xrightarrow{N} Gr^W_{m-k-2}H^m_{\mathrm{lim}} \to$$

on the graded weight spaces. For $k \geq 0$ we have seen $Gr^W_{m-k-2}H^m_{\mathrm{lim}} = 0$. Moreover, we claim that for $k \geq 0$ the map α is zero, which will assert

$$Gr^W_{m-k}H^m \cong Gr^W_{m-k}H^m_{\mathrm{lim}}, \; k \geq 0.$$

It follows from Proposition 1.5 that

$$\mathrm{Gr}^W_{-m-k-2} H_{m+2} \cong (\mathrm{Gr}^W_{m+k+2} H^{m+2})^* \cong (E_2^{-k,m+k+2})^*.$$

It is direct to see $E_2^{-k,m+k+2} = 0$ for $k > 1$ from the definition of the spectral sequence (6). For $k = 0$ we have

$$(E_2^{0,m+2})^* \cong coker(H_{m+2}(E_0) \to H_{m+2}(\overline{X_0}) \oplus H_{m+2}(E)).$$

Recall from Lemma 2.2 that $\overline{X_0}$ is a projective bundle over the dual variety X_0^* of the secant cubic X_0, and E_0 is a smooth family of quadric bundles over X_0^*. Note that the dimension of X_0^* is even since the dual variety X_0^* is isomorphic to the Severi variety in the dual space. Then the odd degree cohomology groups of $\overline{X_0}$ and E_0 vanish. As the integer m is odd in our situation, we get $W_{-m-2}H_{m+2} \cong H_{m+2}(E)$. By Poincaré duality $H_{m+2}(E) \cong H^{m-2}(E)$. In Corollary 3.8 we prove $H^{m-2}(E) = 0$. Therefore the map $\alpha = 0$, and our assertion follows.

When $k = 0$, we have

$$\mathrm{Gr}^W_m H^m \cong E_2^{0,m} = Ker(H^m(\overline{X_0}) \oplus H^m(E) \to H^m(E_0)).$$

By the same reason as above $H^m(\overline{X_0}) = H^m(E_0) = 0$ then $\mathrm{Gr}^W_m H^m = H^m(E)$. We latter prove that

$$H^m(E, \mathbb{Q}) \cong H^{d-1}(V, \mathbb{Q})$$

as an isomorphism of polarized Hodge structures in Proposition 3.4.

The graded piece $Gr^W_{m-1}H^m$ is the cokernel of the following map

$$\rho : H^{m-1}(\overline{X_0}) \oplus H^{m-1}(E) \to H^{m-1}(E_0). \tag{9}$$

For the smooth quadric bundle $\pi : E_0 \to S$, the Leray spectral sequence

$$E_2^{p,q} := H^p(S, R^q\pi_*\mathbb{Q}) \Rightarrow H^{m-1}(E_0), p + q = m - 1$$

degenerates at E_2. Note that the base space S is simply connected. The trivial local system $R^q\pi_*\mathbb{Q}$ is the constant sheaf $H^q(F)$ where F is the fiber of π. Then it gives rise to a (non-canonical) decomposition

$$\bigoplus_{p+q=m-1} H^p(S)\otimes H^q(F)\cong H^{m-1}(E_0).$$

The cohomology classes of S are algebraic since S is a homogeneous space. Hence the cohomology classes of E_0 are algebraic, and the Hodge structure of $H^{m-1}(E_0)$ is isomorphic to $\mathbb{Q}(\frac{1-m}{2})^r$ where r is the rank of $H^{m-1}(E_0)$. Moreover, we show the quotient Hodge structure $\mathrm{Coker}(\rho)$ is isomorphic to $\mathbb{Q}(\frac{1-m}{2})$. The proof is given in the next lemma because it is a bit lengthy to include it here. $\square$

Lemma 2.6. *Let $S\subset\mathbb{P}^{m+1}$ be the Severi variety of dimension $d=4,8,16$. Let F be any fiber of the quadric bundle $\pi:E_0\to S$ in Lemma 2.2. Then the cokernel of the map* (9)

$$\rho: H^{m-1}(\overline{X_0})\oplus H^{m-1}(E)\to H^{m-1}(E_0)$$

is isomorphic to the Tate twist $\mathbb{Q}(\frac{1-m}{2})$ which can be generated by one algebraic class in $H^d(S)\otimes H^{\frac{d}{2}}(F)\subset H^{m-1}(E_0)$.

Proof. The Severi variety S is a homogeneous space. The cohomology groups of S are generated by the Schubert-type classes. The fiber F is a smooth quadric of even dimension $\frac{d}{2}$. The Fano scheme of the $\frac{d}{4}$-planes contained in F has two connected components. Let λ_1,λ_2 be the generators of $H^{\frac{d}{2}}(F)$ represented by two $\frac{d}{4}$-planes in the different components. The goal is to show that the quotient of the map ρ is generated by one class $\sigma\otimes\lambda_i$ where σ represents any $\frac{d}{2}$-dimensional Schubert-type class of S and $i=1$ or 2.

We split the proof into several steps. Based on the decomposition

$$H^{m-1}(E_0,\mathbb{Q})\cong\bigoplus_{p+q=m-1}H^p(S,R^q\pi_*\mathbb{Q})\cong\bigoplus_{p+q=m-1}H^p(S)\otimes H^q(F), \tag{10}$$

the first step is to prove that for $q\neq\frac{d}{2}$ the direct summand $H^p(S)\otimes H^q(F)$ is contained in the image of ρ. The classes $\{\sigma\otimes\lambda_i\}$ form a basis of $H^d(S)\otimes H^{\frac{d}{2}}(F)$, where σ is a Schubert-type class of S, and $\{\lambda_1,\lambda_2\}$ is the two generators of $H^{\frac{d}{2}}(F)$. The second step shows that any two bases are linearly dependent in the quotient $\mathrm{Coker}(\rho)$. The last step assures the class $\sigma\otimes\lambda_i$ is non-trivial in $\mathrm{Coker}(\rho)$.

Step 1. Consider the blowing up diagram

$$\begin{tikzcd} E_0 \arrow[r, hook, "j_{E_0}"] \arrow[d, "\pi"] & \overline{X_0} \arrow[d, "\epsilon"] \\ S \arrow[r, hook, "\iota_S"] & X_0. \end{tikzcd}$$

The map

$$j_{E_0*}+\epsilon^*: H^{m-3}(E_0)\oplus H^{m-1}(X_0)\to H^{m-1}(\overline{X_0}).$$

is surjective. Then the image of $j_{E_0}^*: H^{m-1}(\overline{X_0})\to H^{m-1}(E_0)$ is the image of

$$j_{E_0}^*j_{E_0*}+\pi^*\iota_S^*: H^{m-3}(E_0)\oplus H^{m-1}(X_0)\to H^{m-1}(E_0).$$

The map $\pi^*\iota_S^*$ sends $H^{m-1}(X_0)$ into the component $H^{m-1}(S)\otimes H^0(F)$. The composition $j_{E_0}^* j_{E_0 *}$ is the cup product map

$$\cup[E_0]|_{E_0} : H^{m-3}(E_0) \to H^{m-1}(E_0)$$

where $[E_0]$ denotes the divisor class of E_0 in $\overline{X_0}$. Since $E_0 = E \cap \overline{X_0}$ the line bundle

$$\mathcal{O}_{\overline{X_0}}(E_0)|_{E_0} = \mathcal{O}_E(E)|_{E_0}$$

is isomorphic to the canonical line bundle $\mathcal{O}_\pi(-1)$ for the quadric bundle $\pi : E_0 \to S$. The line bundle $\mathcal{O}_\pi(1)$ corresponds to a section $\eta \in \Gamma(S, R^2\pi_*\mathbb{Q})$. According to the decomposition (10), the cup-product map $\cup[E_0]|_{E_0}$ is induced by the maps of local systems

$$\cup\eta : R^{q-2}\pi_*\mathbb{Q} \to R^q\pi_*\mathbb{Q}.$$

If $i \neq \frac{d}{4}$, the local system $R^{2i}\pi_*\mathbb{Q}$ is of rank one generated by the section η^i. Hence the map $\cup\eta$ is surjective if $q \neq \frac{d}{2}$. As a consequence, the direct summand $H^p(S)\otimes H^q(F)$ is contained in the image of ρ if $q \neq \frac{d}{2}$.

The local system $R^{\frac{d}{2}}\pi_*\mathbb{Q}$ has rank two with the (local) generators λ_1, λ_2. Moreover we have $\eta^{\frac{d}{4}} = \mathbb{Q}\cdot\langle\lambda_1+\lambda_2\rangle$. Therefore the image of

$$\cup\eta : R^{\frac{d}{2}-2}\pi_*\mathbb{Q} \to R^{\frac{d}{2}}\pi_*\mathbb{Q}$$

is generated by the section $\lambda_1+\lambda_2$. Passing to the cohomology, the image of the cup-product map $\cup[E_0]|_{E_0}$ in $H^d(S)\otimes H^{\frac{d}{2}}(F)$ has the form $\sigma\otimes(\lambda_1+\lambda_2), \sigma \in H^d(S)$. Hence $\sigma\otimes\lambda_1 \equiv -\sigma\otimes\lambda_2$ in $\mathrm{Coker}(\rho)$.

Step 2. Now we deal with the map $H^{m-1}(E) \to H^{m-1}(E_0)$. Recall that V is the discriminant locus of f. Let $Y := f^{-1}(V)$ be the family of singular quadrics of f, and let $W := Y \cap E_0$ be the intersection. Consider the Gysin map

$$H^{m-3}(W) \to H^{m-1}(E_0).$$

We claim that the image of $H^{m-3}(W)$ is contained in the image of $H^{m-1}(E)$ in $H^{m-1}(E_0)$.

Let $e : V \hookrightarrow E$ be the closed embedding which assigns every $t \in V$ to the unique singular vertex $e(t)$ in the quadric cone $f^{-1}(t)$. Let $\epsilon : \hat{E} \to E$ be the blowing up along $e(V)$, and let $\hat{Y}$ be the proper transform of Y. We can embed E_0 into $\hat{E}$ since $E_0 \cap e(V) = \emptyset$. Then there is the following cartesian diagram

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d, hook] & E_0 \arrow[d, hook] \\ \hat{Y} \arrow[r, hook] & \hat{E} \end{tikzcd}$$

which induces the commutative homomorphisms

$$\begin{tikzcd} H^{m-3}(\hat{Y}) \arrow[r] \arrow[d] & H^{m-1}(\hat{E}) \arrow[d] \\ H^{m-3}(W) \arrow[r] & H^{m-1}(E_0). \end{tikzcd}$$

Denote by D the exceptional divisor in the blow-up $\hat{E}$. We know that E_0 does not intersect with D in $\hat{E}$. Hence the cohomology class in $H^{m-1}(\hat{E})$ that support on D maps to zero in $H^{m-1}(E_0)$. Therefore the image of $H^{m-1}(\hat{E})$ and $H^{m-1}(E)$ in $H^{m-1}(E_0)$ are the same.

Note that each fiber of the projection $\hat{Y} \to V$ is the blow-up of a quadric cone along the single vertex. The projection factors through a family Q of smooth quadrics over V. Moreover, the subvariety $W \subset \hat{Y}$ is isomorphic to Q via the projection $\hat{Y} \to Q$. It follows that the map $H^{m-3}(\hat{Y}) \to H^{m-3}(W)$ is surjective. By the above diagram the image of $H^{m-3}(W)$ in $H^{m-1}(E_0)$ is contained in the image of $H^{m-1}(\hat{E})$. Our claim thus follows.

The subvariety W is a smooth family of quadrics over V. The Gysin map

$$H^{m-3}(W) \to H^{m-1}(E_0)$$

restricts to the following map

$$\iota_{V*} \otimes id : H^{d-2}(V) \otimes H^{\frac{d}{2}}(F) \to H^d(S) \otimes H^{\frac{d}{2}}(F)$$

where ι_{V*} is the Gysin map for the ample divisor V of S. By the Lefschetz hyperplane theorem, the image of ι_{V*} equals to the image of the cup-product

$$\cup[V] : H^{d-2}(S) \to H^d(S).$$

Let us look into the image case by case.

- $S := \mathbb{P}^2 \times \mathbb{P}^2$, the cohomology $H^2(S)$ has two generators H_1, H_2 where H_i is the hyperplane section on the i-th factor. The image of ι_{V*} is generated by $\langle H_1(H_1 + H_2), H_2(H_1 + H_2)\rangle$. Therefore, for two distinct elements $\sigma, \sigma' \in \{H_1^2, H_2^2, H_1H_2\}$, the classes $\sigma \otimes \lambda_i$ and $\sigma' \otimes \lambda_i$ are linearly dependent in $\mathrm{Coker}(\rho)$.
- $S := Gr(2, 6)$, the cohomology $H^6(S)$ is generated by the Schubert classes $\{\sigma_3, \sigma_{2,1}\}$. The class $[V] = 3\sigma_1$, and the cohomology $H^8(S)$ is generated by the Schubert classes $\{\sigma_4, \sigma_{3,1}, \sigma_{2,2}\}$. By the Pieri's formula [11, §4.3] on the intersection products of Schubert classes, we have
$$\sigma_3 \cdot \sigma_1 = \sigma_4 + \sigma_{3,1},\ \sigma_{2,1} \cdot \sigma_1 = \sigma_{3,1} + \sigma_{2,2}.$$
Then we can see for distinct $\sigma, \sigma' \in \{\sigma_4, \sigma_{3,1}, \sigma_{2,2}\}$ the classes $\sigma \otimes \lambda_i$ and $\sigma' \otimes \lambda_i$ are linearly dependent in $\mathrm{Coker}(\rho)$.
- $S := \mathbb{OP}^2$, we find $\dim H^{14}(S) = 2$ and $\dim H^{16}(S) = 3$. Moreover, the calculus on the Schubert classes of S is the same as the case $Gr(1, 5)$. For the details we refer to [17, §3]

Step 3. It remains to verify any class $\sigma \otimes \lambda_i$ is non-trivial in $\mathrm{Coker}(\rho)$. Through the Step 1, we have seen that $\sigma \otimes \lambda_i$ is not contained in the image of $H^{m-1}(\overline{X_0})$. It is also not contained in the image of $H^{m-1}(E)$. Otherwise, let $U := S \setminus V$ be the open complement of V, and there exsits a class in $H^{m-1}(E|_U)$ that restricts to $\sigma|_U \otimes \lambda_1 \in H^{m-1}(E_0|_U)$. However, it is impossible because $f|_U : E|_U \to U$ is a smooth family and the image of the restriction map $i^* : R^{\frac{d}{2}} f|_{U*}\mathbb{Q} \to R^{\frac{d}{2}}\pi|_{U*}\mathbb{Q}$ is generated by $\lambda_1 + \lambda_2$. □

The Example 2.5 shows that the Hodge structure of a smooth cubic sevenfold and the associated Calabi-Yau threefold are closely related. Now we can see it is a consequence of Theorem 2.4, which deduces analogous results for the higher dimensional Severi varieties as well. To be precise, we have

Corollary 2.7. *Let $\mathcal{O}(1)$ denote the canonical polarizations of the Severi varieties*

$$\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8,\ Gr(2, 6) \subset \mathbb{P}^{14},\ \mathbb{OP}^2 \subset \mathbb{P}^{26}.$$

Let $V \in |\mathcal{O}(3)|$ be a smooth divisor. For $\mathbb{P}^2 \times \mathbb{P}^2$, the Hodge numbers of V are

$$h^{3,0} = 1, h^{2,1} = \binom{9}{3} - 1.$$

For $Gr(2,6)$, the Hodge numders of V are

$$h^{7,0} = 0,\ h^{6,1} = 1,\ h^{5,2} = \binom{15}{3},\ h^{4,3} = \binom{15}{6} - 1.$$

For $\mathbb{OP}^2$, the Hodge numbers of V are

$$h^{15,0} = h^{14,1} = h^{13,2} = 0,$$
$$h^{12,3} = 1,\ h^{11,4} = \binom{27}{3},\ h^{10,5} = \binom{27}{6},\ h^{9,6} = \binom{27}{9},\ h^{8,7} = \binom{27}{12} - 1.$$

Proof. Let $X \subset \mathbb{P}^{m+1}$ be a smooth cubic m-fold that cuts out V. For an odd integer m, we have

$$h^{p,m-p}(X) = \binom{m+2}{2m+1-3p},$$

see [16, Remark 1.17]. More explicitly, when $m = 7, 13, 25$, the non-zero Hodge numbers on $H^m(X)$ are

(1) m=7, $h^{5,2} = h^{2,5} = 1, h^{4,3} = h^{3,4} = 84$;
(2) m=13, $h^{9,4} = h^{4,9} = 1, h^{8,5} = h^{5,8} = \binom{15}{3}, h^{7,6} = h^{6,7} = \binom{15}{6}$;
(3) m=25, $h^{17,8} = h^{8,17} = 1, h^{16,9} = h^{9,16} = \binom{27}{3}, h^{15,10} = h^{10,15} = \binom{27}{6}$, $h^{14,11} = h^{11,14} = \binom{27}{9}, h^{13,12} = h^{12,13} = \binom{27}{12}$.

Consider the degeneration (8) to the secant cubic and the limit mixed Hodge structure on H^m_{lim}. It follows from Theorem 2.4 that $H^{d-1}(V, \mathbb{Q}) \cong Gr^W_m H^m_{\mathrm{lim}}$. Let $F^p \subset H^m(X, \mathbb{C})$ be the Hodge filtration, and let $F^p_\infty \subset H^m_{lim}$ be the limiting Hodge filtration. By (3) we have

$$\dim F^p = \dim F^p_\infty.$$

By abuse of notation we denote by $h^{p,q}, 0 \le p, q \le m$ the virtual Hodge numbers of the mixed Hodge structure on H^m_{lim}. There is

$$\dim H^{p,m-p}(X) = \dim F^p_\infty / F^{p+1}_\infty = \sum_{i=0}^m h^{p,i}.$$

The non-trivial weights of H^m_{lim} are $m-1, m, m+1$, and Gr^W_{m-1} (resp. Gr^W_{m+1}) is isomorphic to the Tate twist $\mathbb{Q}(\frac{1-m}{2})$ (resp. $\mathbb{Q}(\frac{1+m}{2})$). Therefore we have

(1) $\dim F^k_\infty / F^{k+1}_\infty = h^{k,m-k}$, $k \neq \frac{m-1}{2}, \frac{m+1}{2}$,
(2) $\dim F^{\frac{m-1}{2}}_\infty / F^{\frac{m+1}{2}}_\infty = h^{\frac{m-1}{2},\frac{m-1}{2}} + h^{\frac{m-1}{2},\frac{m+1}{2}} = h^{\frac{m-1}{2},\frac{m+1}{2}} + 1$,
(3) $\dim F^{\frac{m+1}{2}}_\infty / F^{\frac{m+3}{2}}_\infty = h^{\frac{m+1}{2},\frac{m-1}{2}} + h^{\frac{m+1}{2},\frac{m+1}{2}} = h^{\frac{m+1}{2},\frac{m-1}{2}} + 1$.

Then the Hodge structure on $Gr^W_m H^m_{\mathrm{lim}}$ follows. $\square$

Remark 2.8. *All four Severi varieties are homogeneous spaces. The Hodge structure of a smooth hypersurface of a homogeneous space may be calculated by the generalized Jacobi-ring* [12].

3. Cohomology of quadric fibrations

The goal of this section is to study the cohomology of the quadric fibrations appeared in Corollary 2.3 and the main theorem. We can work on the quadric fibrations over a general base space in the following set-up.

Let S be a simply connected smooth projective variety with even dimension d, and let V be a smooth ample divisor of S. Suppose that

- $H^*(S,\mathbb{Z})$ are torision-free, and $H^k(X,\mathbb{Z})=0$ for all odd k;
- $f:\mathcal{X}\to S$ is a quadric fibration of relative dimension $2n-1$ whose discriminant divisor is V, and every singular fiber $\mathcal{X}_t$ for $t\in V$ is a quadric cone with corank one.

Notice that the Severi varieties and the quadric fibrations we deal with satisfy the assumption.

Lemma 3.1. *Let (S,V) be the pair of a smooth projective variety and an ample divisor in the above set-up. Denote by $U=S-V$ the open complement. Then we have*

(1) *$H^*(V,\mathbb{Z})$ are torsion-free,*
(2) *$H^k(U,\mathbb{Z})=0$ if $k>d$ or k is odd.*

Proof. The first assertion is deduced from the Lefschetz hyperplane theorem and the universal coefficient theorem for the cohomology of V.

The open subset U is a smooth affine variety with complex dimension d. By Morse theory U is homotopic to a CW-complex of real dimension at most d. Then $H^k(U,\mathbb{Z})=0$ if $k>d$.

To compute the cohomology of odd degree $k<d$, we consider the localization long exact sequence

$$\ldots\to H^k_V(S,\mathbb{Z})\to H^k(S,\mathbb{Z})\to H^k(U,\mathbb{Z})\to H^{k+1}_V(S,\mathbb{Z})\to\ldots.$$

By our assumption $H^k(S,\mathbb{Z})=0$ for odd k. Then for odd k the group $H^k(U,\mathbb{Z})$ is the kernel of the Gysin map

$$H^{k+1}_V(S,\mathbb{Z})\to H^{k+1}(S,\mathbb{Z}).$$

Through the Thom isomorphism $H^{k+1}_V(S,\mathbb{Z})\cong H^{k-1}(V,\mathbb{Z})$ and the isomorphism $H^{k-1}(S,\mathbb{Z})\cong H^{k-1}(V,\mathbb{Z})$ by the Lefschetz hyperplane theorem, we identify the Gysin map to the composition

$$H^{k-1}(S,\mathbb{Z})\xrightarrow{\sim}H^{k-1}(V,\mathbb{Z})\to H^{k+1}(S,\mathbb{Z})$$

which is the cup-product map of the class $[V]$. Since $H^*(S,\mathbb{Z})$ is torsion-free, the hard Lefschetz theorem asserts the cup-product map is injective. Therefore $H^k(U,\mathbb{Z})=0$ for odd $k<d$. $\square$

Corollary 3.2. *Let $f_U:\mathcal{X}_U\to U$ be a smooth family of quadrics of odd dimension. Then the cohomology $H^k(\mathcal{X}_U,\mathbb{Z})$ vanishes if k is odd.*

Proof. Consider the Leray spectral sequence

$$E_2^{p,q}:=H^p(U,R^qf_{U*}\mathbb{Z})\Rightarrow H^{p+q}(\mathcal{X}_U,\mathbb{Z}).$$

If q is odd, the sheaf $R^qf_{U*}\mathbb{Z}$ is zero. If q is even, the sheaf $R^qf_{U*}\mathbb{Z}$ is isomorphic to the constant sheaf $\mathbb{Z}$. It follows from the above Lemma that $H^p(U,R^qf_{U*}\mathbb{Z})=0$ if $p+q$ is odd. Hence our assertion follows. $\square$

For the quadric fibration $f : \mathcal{X} \to S$, there is a closed embedding $e : V \hookrightarrow \mathcal{X}$ such that for each $t \in V$ the image $e(t)$ is the unique singular point of the quadric cone $\mathcal{X}_t := f^{-1}(t)$. Let $\epsilon : \tilde{\mathcal{X}} \to \mathcal{X}$ be the blowing up along the smooth subvariety $e(V)$. Let $\mathcal{Y} := f^{-1}(V)$ be the family of singular quadrics over V, and let $\tilde{\mathcal{Y}}$ be the proper transform of $\mathcal{Y}$ in $\tilde{\mathcal{X}}$. Consider the composition $h : \tilde{\mathcal{Y}} \to \mathcal{Y} \to V$. The fiber $h^{-1}(t)$ for each $t \in V$ is the blow-up of $\mathcal{X}_t$ along $e(t)$. Hence $h^{-1}(t)$ is a $\mathbb{P}^1$-bundle over a $(2n-2)$-dimensional smooth quadric. It follows that h factors through a smooth family $\mathcal{Q}$ of $(2n-2)$-dimensional quadrics over V. Let us write

$$h = q \circ p : \tilde{\mathcal{Y}} \xrightarrow{p} \mathcal{Q} \xrightarrow{q} V$$

where $p : \tilde{\mathcal{Y}} \to \mathcal{Q}$ is a $\mathbb{P}^1$-bundle. We summarize the above operations by the following diagram

$$\begin{tikzcd}
 & \tilde{\mathcal{Y}} \arrow[r, hook, "j"] \arrow[dl, "p"] \arrow[d] & \tilde{\mathcal{X}} \arrow[d, "\epsilon"] \\
\mathcal{Q} \arrow[dr, "q"] & \mathcal{Y} \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\
 & V \arrow[r, hook] & S
\end{tikzcd} \tag{11}$$

Let $\pi_q : \mathcal{F}_{n-1}(\mathcal{Q}/V) \to V$ be the relative Hilbert scheme of $(n-1)$-planes contained in the fibers of q. For a smooth quadric of even dimension, the variety of the maximal linear subspaces contained in the quadric has two disjoint components. Hence every fiber of π_q has two disjoint components. Consider the Stein factorization

$$\mathcal{F}_{n-1}(\mathcal{Q}/V) \to \tilde{V} \xrightarrow{\pi} V.$$

The finite map $\pi : \tilde{V} \to V$ is an étale double covering.

Proposition 3.3. *Keep the above notations. There is an isomorphism*

$$\phi : H^{d-1}(\tilde{V}, \mathbb{Z}) \xrightarrow{\sim} H^{2n+d-3}(\mathcal{Q}, \mathbb{Z})$$

such that

(1) *ϕ is a morphism of Hodge structures of type $(n-1, n-1)$*

$$\phi(H^{p,q}(\tilde{V})) = H^{p+n-1,q+n-1}(\mathcal{Q}), \ \forall\ p+q = d-1;$$

(2) *for any $a, b \in H^{d-1}(\tilde{V}, \mathbb{Z})$, we have*

$$\int_{\mathcal{Q}} \phi(a) \cup \phi(b) = \begin{cases} \langle a, b \rangle, & \text{if } n \text{ is odd;} \\ \langle a, \iota^* b \rangle, & \text{if } n \text{ is even} \end{cases}$$

where ι^* is the natural involution on $\tilde{V}$, and $\langle\ ,\ \rangle$ is the intersection pairing on $\tilde{V}$.

Proof. We consider the Leray spectral sequence for the smooth family q

$$E_2^{i,j} := H^i(V, R^j q_* \mathbb{Z}) \Rightarrow H^{i+j}(\mathcal{Q}, \mathbb{Z}). \tag{12}$$

The result [10, EXP. XII, Thm. 3.3] asserts an isomorphism

$$u : \pi_* \mathbb{Z} \xrightarrow{\sim} R^{2n-2} q_* \mathbb{Z} \tag{13}$$

which is deduced by the Stein factorization. Then $R^j q_*\mathbb{Z}$ is isomorphic to

$$\begin{cases} 0, & j \text{ odd };\\ \mathbb{Z}, & j \neq 2n-2 \text{ even };\\ \pi_*\mathbb{Z}, & j = 2n-2. \end{cases}$$

Note that d is even. Then $E_2^{d-1,2n-2}$ is the only non-zero term among $\{E_2^{i,j}\}$ for $i+j = 2n+d-3$, which follows from the Lefschetz hyperplane theorem for the pair (S, V). We claim the Leray spectral sequence (12) degenerates at E_2. Therefore

$$H^{d-1}(\tilde{V}, \mathbb{Z}) \cong E_2^{d-1,2n-2} = E_\infty^{d-1,2n-2} = H^{2n+d-3}(\mathcal{Q}, \mathbb{Z}).$$

On one hand, by Deligne's degeneration theorem, the spectral sequence (12) with $\mathbb{Q}$-coefficients degenerates at E_2, i.e., $d_r \otimes \mathbb{Q} = 0, r \geq 2$. On the other hand, it follows from Lemma 3.1 that $E_2^{i,j} = H^i(V, R^j q_*\mathbb{Z})$ are torsion-free for $j \neq 2n-2$. For $j = 2n-2$, we have $E_2^{i,2n-2} = H^i(\tilde{V}, \mathbb{Z})$. The étale cover $\tilde{V} = V \sqcup V$ because V is simply connected. Then the group $H^i(\tilde{V}, \mathbb{Z})$ is torsion-free as well. As a result, we obtain $d_r = 0, r \geq 2$ and (12) degenerates at E_2.

The proof of the rest two properties is faithful to Beauville's strategy in [2, Lem. 2.2]. Since some details of Beauville's arguments are omitted in his paper we sketch the proof here for reader's convenience.

The first property deduces from Borel's result on fiber bundles [15, app. 2, Thm. 2.1]. Let F denote the fiber of q. Let $\mathbf{H}^j(F)$ be the holomorphic vector bundle $R^j q_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_V$ on V, and let $\mathbf{H}_{\overline{\partial}}^{r,s}(F), r+s = j$ be the canonical Hodge subbundles. The complexified Leray spectral sequence of (12) admits a canonical grading

$$\sum_{p+q=i+j, p,q \geq 0} {}^{p,q}E_r^{i,j} = E_r^{i,j}$$

of type (p, q) with the differential $d_r : {}^{p,q}E_r^{i,j} \to {}^{p,q+1}E_r^{i+r,j-r+1}$. Moreover, the grading of type (p, q) converges to the Hodge component $H^{p,q}(\mathcal{Q})$. On the E_2-page, the grading ${}^{p,q}E_2^{i,j}$ has the Künneth decomposition

$${}^{p,q}E_2^{i,j} \cong \sum_{t \geq 0} H_{\overline{\partial}}^{t,i-t}(V, \mathbf{H}_{\overline{\partial}}^{p-t,q-i+t}(F)).$$

In our case, the isomorphism (13) shifts the weights of Hodge structures by $2n-2$. Then one can verify that

$$\phi(H^{p,q}(\tilde{V})) = {}^{p+n-1,q+n-1} E_2^{d-1,2n-2} = H^{p+n-1,q+n-1}(\mathcal{Q}).$$

Now let us prove the second property. Under the isomorphism

$$H^{d-1}(V, R^{2n-2}q_*\mathbb{Z}) \to H^{2n+d-3}(\mathcal{Q}, \mathbb{Z}),$$

the intersection form on $H^{2n+d-3}(\mathcal{Q}, \mathbb{Z})$ corresponds to the cup-product

$$\cup : R^{2n-2}q_*\mathbb{Z} \otimes R^{2n-2}q_*\mathbb{Z} \to R^{4n-4}q_*\mathbb{Z} \xrightarrow{Tr} \mathbb{Z}.$$

The intersection form on $H^{d-1}(\tilde{V}, \mathbb{Z})$ corresponds to the cup-product

$$\langle\ ,\ \rangle : \pi_*\mathbb{Z} \otimes \pi_*\mathbb{Z} \to \pi_*\mathbb{Z} \xrightarrow{Tr} \mathbb{Z}.$$

The relation between the two cup-products $\cup$ and $\langle\ ,\ \rangle$ under the isomorphism (13) can be illustrated as follows (cf. [10, EXP. XII, Thm. 3.3 (iii)]).

Recall the variety of $(n-1)$-planes in the smooth quadric F has two connected components. Any two $(n-1)$-planes $\Lambda, \Lambda' \subset F$ are in the same component if and only if

$$\dim \Lambda \cap \Lambda' \equiv (n-1) \bmod 2. \tag{14}$$

Let $\Lambda_1, \Lambda_2 \subset F$ be two maximal linear subspaces in different components representing the generators of $H^{2n-2}(F)$. The above dimension congruence concludes the intersection relation

$$\begin{cases} \Lambda_1^2 = \Lambda_2^2 = 1, \ \Lambda_1 \cdot \Lambda_2 = 0, & n \text{ odd;} \\ \Lambda_1^2 = \Lambda_2^2 = 0, \ \Lambda_1 \cdot \Lambda_2 = 1, & n \text{ even.} \end{cases}$$

Let U be an étale local chart of V such that $U \times_V \tilde{V} = U \sqcup U$. Write $a, b \in H^{d-1}(U, \pi_* \mathbb{Z})$ by

$$a = \alpha_1 + \alpha_2, b = \beta_1 + \beta_2$$

where α_i, β_i are classes supported on the different pieces of $U \times_V \tilde{V}$. Locally we have

$$u(\alpha_i) = \alpha_i \otimes \Lambda_i, u(\beta_i) = \beta_i \otimes \Lambda_i \in H^{d-1}(U, R^{2n-2} q_* \mathbb{Z}).$$

It follows from the intersection relation on Λ_1 and Λ_2 that

$$u(a) \cup u(b) = \begin{cases} \alpha_1 \cup \beta_1 + \alpha_2 \cup \beta_2, & n \text{ odd;} \\ \alpha_1 \cup \beta_2 + \alpha_2 \cup \beta_1, & n \text{ even.} \end{cases}$$

Note that the involution ι^* on $\tilde{V}$ exchanges two pieces of $U \times_V \tilde{V}$. Therefore

$$u(a) \cup u(b) = \begin{cases} \langle a, b \rangle, & \text{if n is odd;} \\ \langle a, \iota^* b \rangle, & \text{if n is even.} \end{cases}$$

It is done. □

The above proposition and the diagram (11) defines a homomorphism of Hodge structure

$$\tilde{\psi} : H^{d-1}(\tilde{V}, \mathbb{Z}) \to H^{2n+d-1}(\mathcal{X}, \mathbb{Z})$$

via the composition

$$H^{d-1}(\tilde{V}) \xrightarrow{\phi} H^{2n+d-3}(\mathcal{Q}) \xrightarrow{p^*} H^{2n+d-3}(\tilde{\mathcal{Y}}) \xrightarrow{j_*} H^{2n+d-1}(\tilde{\mathcal{X}}) \xrightarrow{\epsilon_*} H^{2n+d-1}(\mathcal{X}). \tag{15}$$

Proposition 3.4. (1) *The homomorphism $\tilde{\psi} : H^{d-1}(\tilde{V}, \mathbb{Z}) \to H^{2n+d-1}(\mathcal{X}, \mathbb{Z})$ is surjective.*

(2) *The intersection form $\langle \ , \ \rangle$ on $H^{d-1}(\tilde{V}, \mathbb{Z})$ satisfies*

$$\int_{\mathcal{X}} \tilde{\psi}(a) \cup \tilde{\psi}(b) = (-1)^n \langle a, b - \iota^* b \rangle, \forall a, b \in H^{d-1}(\tilde{V}, \mathbb{Z})$$

where ι^* is the natural involution on $H^{d-1}(\tilde{V}, \mathbb{Z})$.

Proof. Let $U = S - V$, let $\mathcal{X}_U := f^{-1}(U)$ be the family of smooth quadrics. Consider the localization long exact sequence

$$\to H^{2n+d-1}_{\mathcal{Y}}(\mathcal{X}, \mathbb{Z}) \to H^{2n+d-1}(\mathcal{X}, \mathbb{Z}) \to H^{2n+d-1}(\mathcal{X}_U, \mathbb{Z}) \to .$$

By Lemma 3.2 the cohomology $H^{2n+d-1}(\mathcal{X}_U, \mathbb{Z})$ vanishes. Hence the Gysin map

$$H^{2n+d-3}(\mathcal{Y}, \mathbb{Z}) \cong H^{2n+d-1}_{\mathcal{Y}}(\mathcal{X}, \mathbb{Z}) \to H^{2n+d-1}(\mathcal{X}, \mathbb{Z})$$

is surjective, which implies the composition

$$\epsilon_* j_* : H^{2n+d-3}(\tilde{\mathcal{Y}}, \mathbb{Z}) \to H^{2n+d-1}(\mathcal{X}, \mathbb{Z})$$

is surjective.

Let $\mathcal{E}$ be the exceptional divisor in the blow-up $\tilde{\mathcal{X}}$. Then $\mathcal{E} \cap \tilde{\mathcal{Y}}$ is the exceptional divisor of the proper transform $\tilde{\mathcal{Y}}$ of $\mathcal{Y}$. As shown before, every fiber of $\mathcal{Y} \to V$ is a quadric cone of dimension $2n-1$ with a single vertex, and the proper transform $\tilde{\mathcal{Y}}$ is the blowing up along the locus of the vertices. Therefore $\mathcal{E} \cap \tilde{\mathcal{Y}}$ is a family of smooth quadric of dimension $2n-2$ over V, which is isomorphic to $\mathcal{Q}$ via the projection $p : \tilde{\mathcal{Y}} \to \mathcal{Q}$. The isomorphism gives a section $k : \mathcal{Q} \hookrightarrow \mathcal{Y}$ of p fitting into the following commutative diagram

$$\begin{tikzcd} \mathcal{Q} \arrow[r, hook, "l"] \arrow[d, hook, "k"] & \mathcal{E} \arrow[d, hook, "i"] \\ \tilde{\mathcal{Y}} \arrow[r, hook, "j"] & \tilde{\mathcal{X}}. \end{tikzcd} \tag{16}$$

The canonical line bundle $\mathcal{O}_p(1)$ for the $\mathbb{P}^1$-bundle $p : \tilde{\mathcal{Y}} \to Q$ is isomorphic to $\mathcal{O}(\mathcal{E})|_{\tilde{\mathcal{Y}}}$. Then the above diagram implies the decomposition

$$H^{2n+d-3}(\tilde{\mathcal{Y}}, \mathbb{Z}) \cong p^* H^{2n+d-3}(\mathcal{Q}, \mathbb{Z}) \oplus k_* H^{2n+d-5}(Q, \mathbb{Z}).$$

We claim the direct summand $k_* H^{2n-1}(\mathcal{Q}, \mathbb{Z})$ is annihilated by the map $\epsilon_* j_*$. Then the map

$$\epsilon_* j_* p^* : H^{2n+d-3}(\mathcal{Q}, \mathbb{Z}) \to H^{2n+d-1}(\mathcal{X}, \mathbb{Z})$$

will be surjective. As a consequence, the morphism $\psi : H^{d-1}(\tilde{V}, \mathbb{Z}) \to H^{2n+d-1}(\mathcal{X}, \mathbb{Z})$ is surjective. By the diagram (16) we have $\epsilon_* j_* k_* = \epsilon_* i_* l_*$. The map $\epsilon_* i_* l_*$ factors through

$$\epsilon_* i_* : H^{2n+d-3}(\mathcal{E}, \mathbb{Z}) \to H^{2n+d-1}(\mathcal{X}, \mathbb{Z}).$$

Denote by $h \in H^2(\mathcal{E})$ the divisor class $c_1(\mathcal{O}_p(1))$, and $\pi : \mathcal{E} \to V$ the projective normal bundle. The cohomology of $\mathcal{E}$ has the decomposition

$$H^{2n+d-3}(\mathcal{E}, \mathbb{Z}) \cong \bigoplus_{i=0}^{2n-1} \pi^* H^{2n+d-3-2i}(V, \mathbb{Z}) \cdot [h^i].$$

Note that $\pi_* h^i \neq 0$ unless $i = 2n-1$, and $H^{d-1-2n}(V, \mathbb{Z}) = 0$ by Lemma 3.1. Hence the map $\epsilon_* i_*$ is zero. Our claim holds.

Let $x, y \in H^{2n+d-3}(\mathcal{Q}, \mathbb{Z})$. By the projection formula we have

$$\langle \epsilon_* j_* p^* x, \epsilon_* j_* p^* y \rangle = \langle j_* p^* x, \epsilon^* \epsilon_* j_* p^* y \rangle.$$

Denote by $i : \mathcal{E} \hookrightarrow \tilde{\mathcal{X}}$ the inclusion, N the normal bundle of the smooth subvariety $e(V)$ in $\tilde{\mathcal{X}}$, and $\pi : \mathcal{E} \cong \mathbb{P}(N) \to V$ the projective normal bundle. By Fulton's key formula [13, Prop. 6.7], the cohomology class $\epsilon^* \epsilon_* j_* p^* y$ can be expressed as

$$\epsilon^* \epsilon_* j_* p^* y = j_* p^* y + i_* \Big(\sum_{r=0}^{2n-2} h^r \pi^* \pi_* (\gamma_{2n-2-r} \cdot i^* j_* p^* y) \Big).$$

where

$$\gamma_s = h^s + h^{s-1} \cdot \pi^* c_1(N) + \ldots + \pi^* c_s(N).$$

is a codimension s algebraic cycle in $A^s(\mathcal{E})$. For the first term $j_* p^* y$ we have

$$\langle j_* p^* x, j_* p^* y \rangle = p^* x \cdot p^* y \cdot [\tilde{\mathcal{Y}}]|_{\tilde{\mathcal{Y}}}.$$

The divisor class $[\tilde{\mathcal{Y}}]$ of the proper transform $\tilde{\mathcal{Y}}$ in $\mathrm{Pic}(\tilde{\mathcal{X}})$ is equal to

$$\epsilon^* f^*[V] - 2[\mathcal{E}].$$

It implies $\langle j_* p^* x, j_* p^* y\rangle$ is equal to

$$(p^*(x \cdot y) \cdot (j^* \epsilon^* f^*[V] - 2k_* 1)) = p^*(x \cdot y \cdot q^*[V]|_V) - 2(x \cdot y) = -2(x \cdot y).$$

Now we deal with the second term. Let us set

$$\begin{aligned} P_r &:= \langle j_* p^* x, i_*(h^r \cdot \pi^* \pi_*(\gamma_{2n-2-r} \cdot i^* j_* p^* y))\rangle \\ &= i^* j_* p^* x \cdot h^r \cdot \pi^* \pi_*(\gamma_{2n-2-r} \cdot i^* j_* p^* y). \end{aligned}$$

The cartesian diagram (16) deduces that $i^* j_* p^* = l_* k^* p^* = l_*$. It follows that

$$\begin{aligned} P_r &= l_* x \cdot h^r \cdot \pi^* \pi_*(\gamma_{2n-2-r} \cdot l_* y) \\ &= x \cdot l^* h^r \cdot q^* q_*(l^* \gamma_{2n-2-r} \cdot y). \end{aligned}$$

The degree of the cohomology class $q_*(l^* \gamma_{2n-2-r} \cdot y)$ is

$$2(2n-2-r) + 2n + d - 3 - 2(2n-2) = 2n + d - 3 - 2r.$$

Note that the number $2n + d - 3 - 2r$ is odd. The Lefschetz hyperplane theorem asserts $H^{2n+d-1-2r}(V, \mathbb{Z}) = 0$ unless $2n + d - 3 - 2r = d - 1$. Hence P_r is possibly non-zero if and only if $r = n - 1$. Recall

$$\gamma_{n-1} = h^{n-1} + h^{n-2} \cdot \pi^* c_1(N) + \ldots + \pi^* c_{n-1}(N).$$

Then $q_*(l^* \gamma_{n-1} \cdot y) = q_*(l^* h^{n-1} \cdot y)$. Let $a, b \in H^{d-1}(\tilde{V}, \mathbb{Z})$ such that $\phi(a) = x, \phi(b) = y$. Through the above discussion we have

$$\int_{\mathcal{X}} \tilde{\psi}(a) \cup \tilde{\psi}(b) = -2(\phi(a), \phi(b)) + (\phi(a), l^* h^{n-1} \cdot q^* q_*(l^* h^{n-1} \cdot \phi(b)))$$

where $(\ , \ )$ is the intersection pairing on $\mathcal{Q}$. Combining the results of Proposition 3.3 (2) and Lemma 3.5 we obtain

$$\int_{\mathcal{X}} \tilde{\psi}(a) \cup \tilde{\psi}(b) = \begin{cases} \langle a, -b + \iota^* b\rangle, & n \text{ odd;} \\ \langle a, b - \iota^* b\rangle, & n \text{ even.} \end{cases}$$

Therefore the proposition follows. □

Lemma 3.5. *Set $l^* h = \eta \in H^2(\mathcal{Q})$. For all $b \in H^{d-1}(\tilde{V})$, we have*

$$\phi(b + \iota^* b) = \eta^{n-1} \cdot q^* q_*(\eta^{n-1} \cdot \phi(b))$$

Proof. See [2, Lem. 2,4]. □

Corollary 3.6. *The surjective map $\tilde{\psi}$ induces an isomorphism*

$$\psi : H^{d-1}(V, \mathbb{Z}) \xrightarrow{\sim} H^{2n+d-1}(\mathcal{X}, \mathbb{Z})$$

of polarized Hodge structures up to a sign. To be precise, let $(\ , \ )$ denote the intersection form on $H^{d-1}(V)$ and $H^{2n+d-1}(\mathcal{X})$ respectively. For any $x, y \in H^{d-1}(V, \mathbb{Z})$, we have

$$(\psi(x), \psi(y)) = (-1)^n (x, y).$$

Proof. The assertion (2) of Proposition 3.4 implies the following exact sequence

$$0 \to \operatorname{Ker}(1-\iota^*) \to H^{d-1}(\tilde{V}, \mathbb{Z}) \to H^{2n+d-1}(\mathcal{X}, \mathbb{Z}) \to 0.$$

Recall that V is simply connected. Then the étale double cover $\tilde{V}$ is the disjoint union $V \sqcup V$, and the involution ι^* on $\tilde{V}$ exchanges two disjoint pieces. The quotient group $H^{d-1}(\tilde{V}, \mathbb{Z})/\langle \iota^* \rangle$ is isomorphic to $H^{d-1}(V, \mathbb{Z})$ via the map

$$H^{d-1}(\tilde{V}, \mathbb{Z})/\langle \iota^* \rangle \to H^{d-1}(V, \mathbb{Z}),\ (z_1, z_2) \mapsto z_1 - z_2.$$

Therefore it induces an isomorphism $\psi : H^{d-1}(V, \mathbb{Z}) \to H^{2n+d-1}(\mathcal{X}, \mathbb{Z})$.

Let $x, y \in H^{d-1}(V, \mathbb{Z})$, and $a, b \in H^{d-1}(\tilde{V}, \mathbb{Z})$ that map to x, y respectively. We will show

$$(\psi(x), \psi(y)) = (\tilde{\psi}(a), \tilde{\psi}(b)) = (-1)^n (x, y).$$

By Proposition 3.4 (2) it suffices to prove

$$\langle a, b - \iota^* b \rangle = (x, y).$$

Firstly it is independent of the choice of the classes a and b. In fact, suppose any other $b' \in H^{d-1}(\tilde{V}, \mathbb{Z})$ maps to y. Then $b - b'$ is ι^*-invariant, which implies $\langle a, b - \iota^* b \rangle = \langle a, b' - \iota^* b' \rangle$. Similarly it is independent of the choice of a because

$$\langle a, b - \iota^* b \rangle = \langle a - \iota^* a, b \rangle.$$

Hence we can assume $a = (x, 0), b = (y, 0)$. It is direct to verify $\langle a, b - \iota^* b \rangle = (x, y)$. □

For the rest of the section we prove

$$H^{2n+d-3}(\mathcal{X}, \mathbb{Z}) \cong H^{2n+d+1}(\mathcal{X}, \mathbb{Z})^* = 0.$$

Let $f : Q \to S$ be a quadric fibration contained in a projective bundle $\varphi : P \to S$. Denote by $i : Q \hookrightarrow P$ the inclusion over S. We define

$$(R^k f_* \mathbb{Z})_v := \operatorname{coker}(i^* : R^k \varphi_* \mathbb{Z} \to R^k f_* \mathbb{Z}).$$

On the level of cohomology we define

$$H^k(Q, \mathbb{Z})_v := \operatorname{coker}(i^* : H^k(P, \mathbb{Z}) \to H^k(Q, \mathbb{Z})).$$

In the habilitation [31], J. Nagel introduced a modified Leray spectral sequence $E^\bullet(f)_v$ defined as the quotient of the the homomorphism

$$i^* : E^\bullet(\varphi) \to E^\bullet(f)$$

of Leray spectral sequences with respect to φ and f. To be precise, the term $E_r^{p,q}(f)_v$ is defined to be the cokernel of $i^* : E_r^{p,q}(\varphi) \to E_r^{p,q}(f)$. Using the mixed Hodge structure on Leray spectral sequence, Nagel showed that the data $\{E^{p,q}(f)_v\}$ form a spectral sequence that converges to $H^{p+q}(Q, \mathbb{Z})_v$. In particular, on the E_2-page there is the isomorphism

$$E_2^{p,q}(f)_v \cong H^p(S, (R^q f_* \mathbb{Z})_v).$$

Lemma 3.7. *Let $f : \mathcal{X} \to S$ be the quadric fibration of relative dimension $2n-1$ in our set-up, and V be the discriminant divisor of f. Denote by $i : V \hookrightarrow S$ the closed immersion, and by $j : U := S \setminus V \hookrightarrow S$ the open embedding of the open complement. The sheaf $(R^{2n} f_* \mathbb{Z})_v$ fits into the following exact sequence*

$$0 \to j_! \underline{\mathbb{Z}/2\mathbb{Z}} \to (R^{2n} f_* \mathbb{Z})_v \to i_* \mathbb{L} \to 0 \tag{17}$$

where $\mathbb{L}$ is a rank one local system on V. In addition, $(R^q f_*\mathbb{Z})_v = 0$ if $q < 2n$, and $(R^q f_*\mathbb{Z})_v = \underline{\mathbb{Z}/2\mathbb{Z}}$ for even $q > 2n$.

Proof. The sheaf $(R^q f_*\mathbb{Z})_v$ is constructible with respect to the degeneracy loci of the quadric fibration f. The local system on each stratum is fully determined by the corank of the quadrics parametrized by the stratum.

Suppose a quadric $F \subset \mathbb{P}^{2n}$ has corank s. The stalk of the constructible sheaf $(R^q f_*\mathbb{Z})_v$ at the point $[F]$ is the primitive quotient $H^q(F,\mathbb{Z})_v$:

$$H^q(F,\mathbb{Z})_v \cong \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{even } q > 2n-1+s; \\ \mathbb{Z}, & \text{even } q = 2n-1+s; \\ 0, & \text{even } q < 2n-1+s \text{ or odd } q. \end{cases}$$

For the quadric fibration f the possible coranks of fibers are 0 and 1. By the characterization of stalks we immediately conclude $(R^q f_*\mathbb{Z})_v = 0$ if $q < 2n$, and $(R^q f_*\mathbb{Z})_v = \underline{\mathbb{Z}/2\mathbb{Z}}$ for even $q > 2n$.

For $q = 2n$ we consider the natural square diagrams

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook] \arrow[d, "g"] & \mathcal{X} \arrow[d, "f"] & \mathcal{X}_U \arrow[l, hook'] \arrow[d, "f^\circ"] \\ V \arrow[r, hook, "i"] & S & U, \arrow[l, hook', "j"'] \end{tikzcd}$$

and the canonical exact sequence

$$0 \to j_! j^*(R^{2n} f_*\mathbb{Z})_v \to (R^{2n} f_*\mathbb{Z})_v \to i_* i^*(R^{2n} f_*\mathbb{Z})_v \to 0.$$

By the proper base change theorem, it is easy to verify the quotient $(R^k f_*\mathbb{Z})_v$ is invariant under any base change. Namely for any S-scheme $u: T \to S$ the base change map

$$u^*(R^k f_*\mathbb{Z})_v \to (u^* R^k f_*\mathbb{Z})_v$$

is an isomorphism. Therefore we obtain the exact sequence

$$0 \to j_!(R^{2n} f^\circ_*\mathbb{Z})_v \to (R^{2n} f_*\mathbb{Z})_v \to i_*(R^{2n} g_*\mathbb{Z})_v \to 0.$$

Then it suffices to show $(R^{2n} f^\circ_*\mathbb{Z})_v \cong \underline{\mathbb{Z}/2\mathbb{Z}}$ and $(R^{2n} g_*\mathbb{Z})_v \cong \mathbb{L}$.

Note that f° is a smooth family of $(2n-1)$-dimensional quadrics. By passage to the stalk we can see $(R^{2n} f^\circ_*\mathbb{Z})_v$ is a local system of the constant group $\mathbb{Z}/2\mathbb{Z}$. The family g of singular quadrics is not a smooth morphism. Let $\Sigma \subset \mathcal{Y}$ be the locus of the singular vertex in each fiber. Consider the diagram

$$\begin{tikzcd} \mathcal{Y} \setminus \Sigma \arrow[r, hook, "k"] \arrow[dr, "g^\circ"'] & \mathcal{Y} \arrow[d, "g"] & \Sigma \arrow[l, hook', "e"'] \arrow[dl, "h"] \\ & V & \end{tikzcd}.$$

Applying the derived functor Rg_* to the canonical exact sequence

$$0 \to k_!\mathbb{Z} \to \mathbb{Z} \to e_*\mathbb{Z} \to 0$$

yields a triangle

$$Rg^\circ_!\mathbb{Z} \to Rg_*\mathbb{Z} \to Rh_*\mathbb{Z}$$

in the derived category $D^b(V)$. Note that h is an isomorphism. It follows that $R^q g^\circ_!\mathbb{Z} \cong R^q g_*\mathbb{Z}$ for all $q \geq 1$. In particular, $R^q g^\circ_!\mathbb{Z}$ is a local system on V since g° is smooth. By passage to the stalk, we conclude that $(R^{2n} g_*\mathbb{Z})_v$ is a local system of rank one. $\square$

Corollary 3.8. *The cohomology $H^{2n+d-3}(\mathcal{X},\mathbb{Z})$ is zero.*

Proof. In our set-up, the base space S is even dimensional. Then the odd degree cohomology of the projective bundle P vanishes. Hence

$$H^{2n+d-3}(\mathcal{X}) = H^{2n+d-3}(\mathcal{X})_v.$$

We use Nagel's spectral sequence

$$E_2^{p,q}(f)_v := H^p(S,(R^q f_*\mathbb{Z})_v) \Rightarrow H^{2n+d-3}(\mathcal{X})_v.$$

By the description of $(R^q f_*\mathbb{Z})_v$ in the Lemma 3.7, the only non-trivial E_2-term that degenerates to $H^{2n+d-3}(\mathcal{X})_v$ is

$$E_2^{d-3,2n}(f)_v = H^{d-3}(S,(R^{2n} f_*\mathbb{Z})_v).$$

Applying the derived functor $R\Gamma(S,-)$ to the exact sequence (17) we obtain the long exact sequence

$$\cdots \to H_c^{d-3}(U,\mathbb{Z}/2\mathbb{Z}) \to H^{d-3}(S,(R^{2n}f_*\mathbb{Z})_v) \to H^{d-3}(V,\mathbb{L}) \to \cdots.$$

The Poincaré duality asserts $H_c^{d-3}(U,\mathbb{Z}/2\mathbb{Z}) \cong H_{d+3}(U,\mathbb{Z}/2\mathbb{Z})$. Recall that U is a smooth affine variety of dimension d. Then U is homotopic to a CW-complex of real dimension $\leq d$ by the Morse theory. Hence $H_i(U,\mathbb{Z}/2\mathbb{Z}) = 0$ for $i > d$.

The category of local systems on V is equivalent to the category of monodromy representations of $\pi_1(V)$. By our hypothesis V is simply connected. Therefore the rank one local system $\mathbb{L}$ is isomorphic to the constant group $\mathbb{Z}$. Lemma 3.1 asserts $H^{d-3}(V,\mathbb{Z}) = 0$. As a result, we have $H^{d-3}(S,(R^{2n}f_*\mathbb{Z})_v) = 0$. ☐

4. Extension of period mappings

For the integer $N = 8, 14, 26$, we consider the moduli space $\mathcal{F}$ of smooth cubic hypersurfaces in $\mathbb{P}^N$. Let $\overline{\mathcal{F}}$ be the GIT compactification of $\mathcal{F}$, which is obtained as the GIT quotient by the action of the reductive group $SL(N+1,\mathbb{C})$ on the parameter space $\mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^N}(3)))$. Consider Kirwan's blowing up of $\overline{\mathcal{F}}$ at the point $\omega \in \overline{\mathcal{F}}$ which presents the secant variety $Sec(S)$ of the Severi variety $S \subset \mathbb{P}^N$. The moduli space $\mathcal{F}$ is a Zariski open and dense subset in $\overline{\mathcal{F}}^{\mathrm{kir}}$. Let $\mathcal{D}$ be the local period domain for smooth cubic hypersurfaces in $\mathbb{P}^N$, and Γ be the monodromy group. Our goal is to partially extend the period map

$$\mathcal{P}\colon \mathcal{F} \to \Gamma\backslash\mathcal{D}$$

to the exceptional divisor $\mathcal{M}$ in the blow-up $\overline{\mathcal{F}}^{\mathrm{kir}}$ with the target to be Usui's partial compactification $\overline{\Gamma\backslash\mathcal{D}}$. More precisely, we firstly show that $\mathcal{M}$ parametrizes the hypersurfaces in S derived from one-parameter degenerations to the secant cubic. Then we prove

Theorem 4.1. *The period map $\mathcal{P}: \overline{\mathcal{F}}^{kir} \dashrightarrow \overline{\Gamma\backslash\mathcal{D}}$ extends holomorphically over the generic points of the exceptional divisor $\mathcal{M}$. Moreover, the restriction of the extension map $\overline{\mathcal{P}}$ to $\mathcal{M}$ is exactly the period map for the objects that $\mathcal{M}$ parametrizes.*

4.1. **Kirwan's blowing up and the exceptional divisor.** We denote by P the space $\mathbb{P}(\Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(3)))$ and $G := SL(N+1, \mathbb{C})$. Let $x \in P$ represent the secant cubic $Sec(S)$. First of all, we notice that

Lemma 4.2. *$Sec(S)$ is a semistable object in P.*

Proof. The proof is an application of the Hilbert-Mumford criterion that Laza used to prove the secant cubic fourfold is semistable, see [25, Lem. 4.3]. In our case the argument follows the same. □

Using Luna's étale slice theorem [30, p. 198], we can describe Kirwan's blowing up for the GIT quotient $\overline{\mathcal{F}} := P^{ss} /\!\!/ G$ at the point $\omega := [Sec(S)]$ as follows. Let $G_x \subset G$ be the stabilizer of x. The étale local topology around the point ω is isomorphic to the quotient space $\mathcal{N}_x /\!\!/ G_x$. Here $\mathcal{N}_x$ is the normal bundle of the closed orbit Gx in P at the point x. By the étale local structure, the exceptional divisor $\mathcal{M}$ is isomorphic to the GIT quotient $\mathbb{P}(\mathcal{N}_x) /\!\!/ G_x$.

For the case of the secant cubic fourfold, it had been shown in [25, §4.1.1] that the normal bundle $\mathcal{N}_x$ can be identified with the space of plane sextic curves, and the stabilizer group G_x is isomorphic to $SL(3, \mathbb{C})$ which naturally acts on the Veronese surface $S \cong \mathbb{P}^2$. Recall at the beginning of Section 2 we exhibit the irreducible representations of (semi)simple algebraic groups that characterize the Severi varieties. Using the representations and algebraic groups, we can similarly describe the normal bundle $\mathcal{N}_x$ and the stabilizer G_x for higher dimensional Severi varieties.

Let $H \to \mathrm{Aut}(W)$ with $W \cong \mathbb{C}^{N+1}$ be the representation in Section 2 corresponding to the Severi variety $S \subset \mathbb{P}^N$. We have the following lemma.

Lemma 4.3. *Let $x \in P$ represent the secant cubic $Sec(S)$. The stabilizer subgroup $G_x \subset G := SL(N+1, \mathbb{C})$ is isomorphic to H.*

Proof. Recall that $Sec(S)$ is defined by the determinant form of the matrix space W. Then the stabilizer G_x consists of automorphisms in $SL(N+1, \mathbb{C})$ preserving the determinant of any matrix in W. Then Landsberg proved that such G_x is isomorphic to the algebraic group H corresponding to S, see [24, (3.4)] □

Lemma 4.4. *Let $\mathcal{O}_S(1)$ be the induced ample line bundle on the Severi variety $S \subset \mathbb{P}^N$. Then the normal bundle $\mathcal{N}_x$ is isomorphic to the space of global sections of $\mathcal{O}_S(3)$.*

Proof. There is a natural restriction map

$$\mathrm{Sym}^3 W = \Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(3)) \to \Gamma(S, \mathcal{O}_S(3)).$$

Let $T_x Gx$ be the tangent space of the orbit $Gx \subset P$ at the point x. Our assertion will follow if the sequence

$$0 \to T_x Gx \to T_x P = \mathrm{Sym}^3 W / \mathbb{C}x \to \Gamma(\mathcal{O}_S(3))$$

is exact at the middle and the dimension condition

$$(\dim \mathrm{Sym}^3 W - 1) - \dim T_x Gx = \dim \Gamma(\mathcal{O}_S(3)) \tag{18}$$

holds.

Let f be the equation of the secant cubic hypersurface x. The tangent space T_xGx is the subspace of $\mathrm{Sym}^3 W/\mathbb{C}x$ generated by the Jacobian ideal

$$J_f = \langle \frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_N} \rangle$$

of the equation f. It is known that the Severi variety S is cut out by the differentials $\{\frac{\partial f}{\partial x_i}\}_{0 \leq i \leq N}$. Therefore a cubic polynomial q restricts to zero on S if and only if q is generated by the Jacobian ideal J_f, which implies the exactness. Since $Gx \cong G/G_x$, we have

$$\dim T_xGx = \dim G - \dim G_x.$$

It follows from Lemma 4.3 that G_x is the (semi-)simple algebraic group H associated to the Severi variety S. Hence it suffices to verify (18) case by case as follows.

(1) $d = 4, N = 8$. Then $G = SL(9, \mathbb{C})$ and $G_x = SL(3, \mathbb{C}) \times SL(3, \mathbb{C})$. We have

$$\dim \mathrm{Sym}^3 W = \binom{11}{3} = 165, \ \dim G - \dim G_x = 64.$$

It is easy to compute $\dim \Gamma(\mathcal{O}_S(3)) = \dim \Gamma(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(3, 3)) = 100$.

(2) $d = 8, N = 14$. Then $G = SL(15, \mathbb{C})$, $G_x = SL(6, \mathbb{C})$. We have

$$\dim \mathrm{Sym}^3 W = \binom{17}{3} = 680, \ \dim G - \dim G_x = 189.$$

By Borel-Weil-Bott's theory, the vector space $\Gamma(\mathcal{O}_S(3)) = \Gamma(\mathcal{O}_{Gr(2,6)}(3))$ is an irreducible $\mathfrak{sl}_6$-module that corresponds to a fundamental weight of $\mathrm{Sym}^3 W$. Using Weyl's character formula we obtain $\dim \Gamma(\mathcal{O}_{Gr(2,6)}(3)) = 490$.

(3) $d = 16, N = 26$. Then $G = SL(27, \mathbb{C})$ and $G_x = E_6$. We have

$$\dim \mathrm{Sym}^3 W = \binom{29}{3} = 3654, \ \dim G - \dim G_x = 650.$$

Again the space $\Gamma(\mathcal{O}_S(3)) = \Gamma(\mathcal{O}_{\mathbb{OP}^2}(3))$ is an irreducible E_6-module that corresponds to a highest weight of $\mathrm{Sym}^3 W$. Weyl's character formula implies $\dim \Gamma(\mathcal{O}_{\mathbb{OP}^2}(3)) = 3003$, see [28].

We see the dimension condition (18) holds true for each case. □

Corollary 4.5. *Let $S \subset \mathbb{P}^N$ be the Severi variety. Let ω represent the secant cubic $Sec(S)$ in the moduli space $\overline{\mathcal{F}}$, and let $\mathcal{M}$ be the exceptional divisor in Kirwan's blow-up $\overline{\mathcal{F}}^{Kir}$ at the point $\omega \in \overline{\mathcal{F}}$. Then $\mathcal{M}$ is isomorphic to the GIT quotient $\mathbb{P}(\Gamma(\mathcal{O}_S(3)) /\!\!/ H$ for sections of the line bundle $\mathcal{O}_S(3)$ on S, where H is the corresponding algebraic group of S in Section 2.*

When $S = \mathbb{P}^2$, the exceptional divisor $\mathcal{M}$ parametrizes plane sextic curves. When $S = \mathbb{P}^2 \times \mathbb{P}^2$, $\mathcal{M}$ parametrizes Calabi-Yau threefolds shown in Example 2.5. For $S = Gr(2, 6)$ or $\mathbb{OP}^2$, the section of $\mathcal{O}_S(3)$ is a Fano variety.

4.2. **Partial compactification and extension theorem.** To prove Theorem 4.1, it is sufficient to study the extension locally. Let $\Delta^n \subset \overline{\mathcal{F}}^{\mathrm{kir}}$ be an open polycylinder with local coordinate $(z_1, \dots, z_n)$ such that the smooth divisor $\mathcal{M} \cap \Delta^n$ is defined by $z_1 = 0$. Consider the local period map

$$\wp : \Delta^* \times \Delta^{n-1} \to \Gamma \backslash \mathcal{D}.$$

For a generic point $(0,w) \in \mathcal{M} \cap \Delta^n$, the one-variable period map $\wp_w : \Delta^* \to \Gamma\backslash\mathcal{D}$ corresponds to a one-parameter degeneration $F + tG$ as in (8). Here the cubic polynomial G can be chosen such that $G|_S \in \Gamma(\mathcal{O}_S(3))$ represents the point $(0,w) \in \mathcal{M}$. It follows from Theorem 2.4 that the nilpotent map N associated to $\wp_w$ is of order 2, which induces the monodromy weight filtration

$$0 \subset W_{k-1} \subset W_k \subset W_{k+1} = H_{\lim} \tag{19}$$

satisfying

$$\dim \operatorname{Im}(N) = \dim W_{k-1} = 1. \tag{$\star$}$$

We found the extension of period maps on the punctured disk with such specific monodromy weight filtration has been investigated by Usui [35], which is analogous to Cattani and Kaplan's work [7] of the extension of period maps for variations of Hodge structrues of weight two.

Let V be a $\mathbb{Q}$-vector space, k an odd integer, S a non-degenerate anti-symmetric form on V, $\{h^{p,q}\}$ a collection of non-negative integers with $p+q=k$. Let D be the classifying space of polarized Hodge structures of type $(V, h^{p,q}, S, k)$. Suppose that

$$\phi : \Delta^* \to \Gamma\backslash D$$

is any period map whose nilpotent logarithm N has order 2, and satisfies the condition $(\star)$.

Lemma 4.6. *Let W be the monodromy weight filtration on V defined by any rational nilpotent operator N of order 2 that satisfies the condition $(\star)$. Suppose that (W,F) is a mixed Hodge structure on V. Let $\{p_\lambda^{a,b}\}_{a+b=\lambda}$ be the induced primitive Hodge numbers on the primitive part $P_\lambda \subset Gr_\lambda^W, k-1 \le \lambda \le k+1$. Then the integers $p_\lambda^{a,b}$ are independent of the choice of N and F. Precisely, we have*

- $p_\lambda^{a,b} = h^{a,b}, a \neq \frac{k-1}{2}, \frac{k+1}{2}, \lambda = k;$
- $p_\lambda^{a,b} = h^{a,b} - 1, a = \frac{k-1}{2}$ *or* $\frac{k+1}{2}, \lambda = k;$
- $p_\lambda^{a,b} = 1, a = b = \frac{k+1}{2}$ *or* $a = b = \frac{k-1}{2}.$

Proof. Note that in our case $P_\lambda = Gr_\lambda^W$. Since $\dim W_{k-1} = 1$ the induced Hodge structure on $P_{k-1} = W_{k-1}$ must be the Tate twist $\mathbb{Q}(-\frac{k-1}{2})$. Then the Hodge number $p_\lambda^{a,b} = 1$ for $a = b = \frac{k-1}{2}$. By duality the same reason holds as well for P_{k+1} . Note that

$$\dim F^a/F^{a+1} = h^{a,b} = \sum_{j=b-1}^{b+1} p_\lambda^{a,j}.$$

Therefore if $\lambda = k$ the primitive Hodge number $p_\lambda^{a,b} = \begin{cases} h^{a,b}, & a \neq \frac{k-1}{2}, \frac{k+1}{2}; \\ h^{a,b} - 1, & a = \frac{k-1}{2}, \frac{k+1}{2}. \end{cases}$ □

The partial compactification of $\Gamma\backslash D$ is constructed by adding boundary components that are related to weight filtrations of the type (19) and the collection of integers $p := \{p_\lambda^{a,b}\}$. Let $W_{-1} \subset V_{\mathbb{R}}$ be an S-isotropic subspace with $\dim W_{-1} = 1$, $W_0 := W_{-1}^{\perp}$ the annihilator of W_{-1} relative to S. Let $\tilde{S}$ denote the non-degenerate form on W_0/W_{-1} induced by S. Choose a polarizing bilinear form ψ on W_{-1}. Two such forms are considered to be equivalent if they are different up to a positive constant. For convenience we may set $p^{a,b}_{\lambda} := p^{a,b}_{\lambda+k}$. Given the data (W_{-1}, p, ψ), the associated *boundary component* $B(W_{-1}, p, \psi) = B(W_{-1}, p) \times B(W_{-1}, p, \psi)$ is defined by

(1) the classifying space $B(W_{-1}, p)$ of $\tilde{S}$-polarized Hodge structures on the quotient W_0/W_{-1} of type $\{p^{a,b}_0\}$
(2) the classifying space $B(W_{-1}, p, \psi)$ of ψ-polarized Hodge structures on W_{-1} of type $\{p^{a,b}_{-1}\}$.

The *boundary bundle* with respect to (W_{-1}, p) is the disjoint union of boundary components

$$\mathcal{B}(W_{-1}, p) := \bigsqcup_{\psi} B(W_{-1}, p, \psi)$$

where ψ runs over all equivalence classes of polarizing forms on W_{-1}. A boundary bundle is *rational* if the subspace W_{-1} is defined over $\mathbb{Q}$. A boundary component $B(W_{-1}, p, \psi)$ is *rational* if the subspace W_{-1} and the form ψ are defined over $\mathbb{Q}$. In particular, for the monodromy weight filtration $W(N)$ of the nilpotent map N, we have W_{k-1} is a rational S-isotropic subspace, and W_k is the S-annihilator of W_{k-1}. It follows from Proposition 1.2 that $S(N^{-1}\cdot, \cdot)$ is a polarization on W_{k-1}. Therefore it corresponds to a rational boundary component

$$B(W, p, N) := B(W_{k-1}, p, \psi)$$

where $\psi := S(N^{-1}\cdot, \cdot)$.

Denote by $D^{**} \subset D^*$ the union of all rational boundary components and the union of all rational boundary bundles, respectively. Usui described the action of the arithmetic group Γ on the extended set D^* which yields a Satake topology on the arithmetic quotient $\Gamma\backslash D^*$. The partial compactification $\overline{\Gamma\backslash D}$ is set to be the quotient space $\Gamma\backslash D^{**}$ which inherits the Satake topology on $\Gamma\backslash D^*$. Moreover, $\Gamma\backslash D^*$ and thus $\Gamma\backslash D^{**}$ admit complex structures, see [35].

Theorem 4.7. (1) *The arithmetic quotients $\Gamma\backslash D^{**}$ and $\Gamma\backslash D^*$ are locally compact and Hausdorff.*
(2) *$\Gamma\backslash D$ is open and dense in $\Gamma\backslash D^{**}$.*
(3) *Let $\phi : \Delta^* \to \Gamma\backslash D$ be a period map such that the nilpotent logarithm N has order 2 and satisfies the condition $(\star)$. Then ϕ can be extended holomorphically to a map $\overline{\phi} : \Delta \to \Gamma\backslash D^{**}$.*

The statements in the theorem are results in Usui's papers [34, §A.3] and [35, Thm. 5.1]. We focus our attention on the limit point of the extension map $\overline{\phi}$. It is described by Schmid's SL_2-orbit theory for which basic notions can be found in Appendix A. An SL_2-orbit is a pair $(\rho, \mathbf{r})$ with a Lie group homomorphism

$$\rho : SL(2, \mathbb{C}) \to G_{\mathbb{C}}$$

and a reference point $\mathbf{r} \in D$ such that $\rho(SL(2, \mathbb{R})) \subset G_{\mathbb{R}}$, and the differential map on the Lie algebra

$$\rho_* : \mathfrak{sl}_2\mathbb{C} \to \mathfrak{g}_{\mathbb{C}}$$

is *horizontal* at $\mathbf{r}$. The group $SL(2, \mathbb{C})$ (resp. $G_{\mathbb{C}}$) acts transitively on $\mathbb{P}^1$ (resp. $\check{D}$). Then ρ induces a holomorphic, horizontal and equivariant embedding

$$\tilde{\rho} : \mathbb{P}^1 \to \check{D}$$

given by

$$\tilde{\rho}(g \cdot i) = \rho(g) \cdot \mathbf{r},\ \ i \in \mathbb{P}^1, \forall g \in SL(2, \mathbb{C}).$$

Note that the upper half plane $\mathfrak{h}$ is an $SL(2, \mathbb{R})$-orbit of i. We have $\tilde{\rho}(\mathfrak{h}) \subset D$.

Let $\tilde{\phi} : \mathfrak{h} \to D$ be the lifting of the period map ϕ to the upper half plane. Let F be the limiting Hodge filtration associated to ϕ, and

$$\theta : \mathfrak{h} \to \check{D}, z \to \exp(zN) \cdot F$$

be the nilpotent orbit. There exists a rational SL_2-orbit $(\rho, \mathbf{r})$ (by rational we mean ρ is defined over $\mathbb{Q}$) such that $\rho_*(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) = N$. The SL_2-orbit theorem asserts that the nilpotent orbit θ, as well as the lifting $\tilde{\phi}$, are asymptotically approximated by the induced embedding $\tilde{\rho}$. Let $F_{\mathbf{r}}$ be the corresponding Hodge filtration of $\mathbf{r} \in D$. It follows from Theorem A.5 that $(W(N), e^{-iN}F_{\mathbf{r}})$ is an S-polarized mixed Hodge structure. Since $N(W_\lambda) \subset W_{\lambda-2}$ the restriction of the action of N on Gr_λ^W is zero. Hence the Hodge structures on Gr_λ^W induced by the filtrations $e^{-iN}F_{\mathbf{r}}$ and $F_{\mathbf{r}}$ are the same. To the rational $SL(2)$-orbit $(\rho, \mathbf{r})$ one assigns the boundary point

$$b(\rho, \mathbf{r}) := (\psi, F_{\mathbf{r}}(Gr_k^W)) \in B(W, p, N)$$

where $\psi = S(N^{-1}\cdot, \cdot)$. It can be proved that

$$\lim_{\operatorname{Im} z \to \infty} \tilde{\phi}(z) = r_\infty \cdot b(\rho, \mathbf{r})$$

in the Satake topology on D^{**} for some r_∞ in the centralizer of the boundary bundle $\mathcal{B}(W_{k-1}, p)$, see [34, A.4.1], [7, 6.5]. As a consequence, we have

$$\overline{\phi}(0) = b(\rho, \mathbf{r}) \in \Gamma \backslash D^{**}.$$

Corollary 4.8. *Let $\phi : \Delta^* \to \Gamma \backslash D$ be the period map in Theorem 4.7, and let $\overline{\phi}$: $\Delta \to \overline{\Gamma \backslash D}$ be the extension map. Let $(W(N), F)$ be the limit mixed Hodge structure associated to ϕ. Then the induced Hodge structure on the graded subquotient Gr_k^W is exactly the limit point $\overline{\phi}(0)$ in the boundary component of $\overline{\Gamma \backslash D}$.*

Proof. For the monodromy weight filtration $W(N)$, the isotropic subspace W_{k-1} has dimension 1. Then the equivalence class of polarizing forms on W_{k-1} is unique. By definition the boundary component $B(W, p, N)$ equals to the classifying space of $\tilde{S}$-polarized Hodge structrues of type $\{p_k^{a,b}\}$ on Gr_k^W.

A consequence [32, Cor. 6.21] of the SL_2-orbit theorem shows that the limit Hodge filtration F and the reference filtration $F_{\mathbf{r}}$ induces the same graded polarized Hodge structures on Gr_λ^W. Hence

$$\overline{\phi}(0) = F_{\mathbf{r}}(Gr_k^W) = F(Gr_k^W)$$

in $B(W, p, N)$. □

Appendix A. SL(2)-orbits and mixed Hodge structures

Let $(V_{\mathbb{Q}}, V^{p,q}, S)$ denote a Hodge structure of weight k with the polarization form S. Denote by $\mathfrak{g}_R$ the Lie algebra of the orthgonal group $G_R = \mathrm{Aut}(V, S)$. The Hodge decomposition $\{V^{p,q}\}$ induces a Hodge structure of weight zero on $\mathfrak{g}_{\mathbb{R}}$ in the following manner

$$\mathfrak{g}^{r,-r} := \{X \in \mathfrak{g}_{\mathbb{C}} \mid X(V^{p,q}) \subset V^{p+r,q-r}, \forall p, q\}.$$

Example A.1. *Consider weight one Hodge structures on $\mathbb{C}^2$ polarized by the standard symplectic form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The orthogonal group $G_{\mathbb{R}}$ is $SL(2,\mathbb{R})$ and the classifying space D is the upper half plane $\mathfrak{h}$. Let $i \in \mathfrak{h}$ correspond to the Hodge decomposition*

$$H^{1,0} = \mathbb{C} \cdot ie_1 + e_2, H^{0,1} = \mathbb{C} \cdot e_1 + ie_2.$$

Then the induced weight zero Hodge structure on $\mathfrak{sl}_2\mathbb{R}$ is

$$(\mathfrak{sl}_2)^{1,-1} = \mathbb{C} \cdot \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix},\ (\mathfrak{sl}_2)^{0,0} = \mathbb{C} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ (\mathfrak{sl}_2)^{-1,1} = \mathbb{C} \cdot \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}.$$

Denote $Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $X_+ = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$, $X_- = \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}$.

Definition A.2. *Fix a polarized Hodge structure $\{V^{p,q}\}$ on V. A Lie algebra homomorphism $\rho_* : \mathfrak{sl}_2\mathbb{C} \to \mathfrak{g}_{\mathbb{C}}$ is called horizontal at $\{V^{p,q}\}$ if it is a morphism of real Hodge structures of type $(0,0)$, i.e. $\rho_*(X_-) \in \mathfrak{g}^{-1,1}$.*

If a Lie algebra homomorphism $\rho_* : \mathfrak{sl}_2\mathbb{C} \to \mathfrak{g}_{\mathbb{C}}$ is horizontal at a reference point $\mathbf{r} := \{V^{p,q}\} \in D$, then it is lifted to a group homomorphism

$$\rho : SL(2,\mathbb{C}) \to G_{\mathbb{C}}$$

of complex Lie groups such that $\rho(SL(2,\mathbb{R})) \subset G_{\mathbb{R}}$. The group $SL(2,\mathbb{C})$ (resp. $G_{\mathbb{C}}$) acts transitively on $\mathbb{P}^1$ (resp. $\check{D}$). Then ρ induces a holomorphic, horizontal, and equivariant embedding

$$\tilde{\rho} : \mathbb{P}^1 \to \check{D}$$

given by

$$\tilde{\rho}(g \cdot i) = \rho(g) \cdot \mathbf{r}, \forall g \in SL(2,\mathbb{C})$$

Note that the upper half plane $\mathfrak{h}$ is the $G_{\mathbb{R}}$-orbit of the point i in $\mathbb{P}^1$. Then $\psi(\mathfrak{h}) \subset D$. We call such a pair $(\rho, \mathbf{r})$ an $SL(2)$-orbit.

To a nilpotent element $N \in \mathfrak{gl}(V_{\mathbb{C}})$ with $N^{k+1} = 0$, one can associate the monodromy weight filtration $W(N)$, cf. (1.2). Let $W_l := W(N)_{l+k}$ denote the weight filtration shift by k. Then the weight filtration $W_\bullet$ satisfies

(1) $N(W_l) \subset W_{l-2}$;
(2) $N^l : Gr_l^W \xrightarrow{\sim} Gr_{-l}^W$.

A grading of the weight filtration $W_\bullet$ that is compatible with N is a decomposition $V_{\mathbb{C}} = \bigoplus_l H_l$ into complex subspaces H_l such that

$$W_l = \bigoplus_{\lambda \le l} H_\lambda, N(H_l) \subset H_{l-2}.$$

Such a grading corresponds to a semisimple element $Y \in \mathfrak{gl}(V_{\mathbb{C}})$ such that H_λ is the λ-eigenspace of Y. It implies $[Y, N] = -2N$. Let

$$\mathbf{n}_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{n}_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be the standard basis of $\mathfrak{sl}_2(\mathbb{C})$. The pair (N, Y) induces a Lie algebra homomorphism

$$\rho_* : \mathfrak{sl}_2\mathbb{C} \to \mathfrak{gl}(V_{\mathbb{C}})$$

with $\rho_*(\mathbf{n}_-) = N, \rho_*(\mathbf{y}) = Y$. Conversely, for any $\mathfrak{sl}_2$-representation ρ_* with $\rho_*(\mathbf{n}_-) = N$, the semisimple element $Y = \rho_*(\mathbf{y})$ has integral eigenvalues. Then the eigenspaces $H_\lambda(Y)$ form a grading of the weight filtration $W_\bullet$ that is compatible with N.

Lemma A.3. [6, Prop. 2.8] *Let $\mathscr{L}(W, N)$ denote the set of gradings of W that is compatible with N. Set $\mathfrak{c} := \mathrm{Ker}(ad_{\mathfrak{g}} N) \cap \mathrm{Im}(ad_{\mathfrak{g}} N)$.*

(1) *There is a one-to-one correspondence*

$$\{\rho_* : \mathfrak{sl}_2\mathbb{C} \to \mathfrak{gl}(V_{\mathbb{C}}) \mid \rho_*(\mathbf{n}_-) = N\} \to \mathscr{L}(W, N)$$

given by $\rho \mapsto \rho_*(\mathbf{y})$

(2) *The group $\exp(\mathfrak{c})$ acts simply transitive on $\mathscr{L}(W, N)$.*

If S is the polarization form on V and $N \in \mathfrak{g}_{\mathbb{C}}$, it is easy to check $Y \in \mathfrak{g}_{\mathbb{C}}$. Then the homomorphism ρ_* factors through $\mathfrak{g}_{\mathbb{C}}$.

Let $(W(N), F)$ be a mixed Hodge structure on V. Consider Deligne's generalized Hodge decomposition [9]

$$V = \bigoplus_{a,b} I^{a,b}$$

such that

$$W_l = \bigoplus_{a+b \leq l} I^{a,b}, F^p = \bigoplus_{a \geq p} I^{a,b}$$

where the subspaces $I^{p,q}$ are defined to be

$$I^{a,b} := F^p \cap W_{a+b} \cap (\overline{F^b} \cap W_{a+b} + \sum_{j \geq 1} \overline{F^{b-j}} \cap W_{a+b-j-1}).$$

The complex conjugate on $I^{p,q}$ satisfies

$$I^{p,q} = \overline{I^{q,p}} \ (\mathrm{mod} \bigoplus_{a<p, b<q} I^{a,b}).$$

We say the mixed Hodge structure (W, F) splits over $\mathbb{R}$ if $I^{p,q} = \overline{I^{q,p}}$, in which case $I^{p,q} = F^p \cap \overline{F^q} \cap W_{p+q}$.

Proposition A.4. [6, Prop. 2.20] *For any mixed Hodge structure (W, F) on V, there exists a unique operator*

$$\delta \in L_{\mathbb{R}}^{-1,-1}(W, F) := \{T \in End_{\mathbb{R}}(V) \mid T(I^{p,q}) \subset \bigoplus_{a<p, b<q} I^{a,b}\}$$

such that $(W, e^{-i\delta} F)$ splits over $\mathbb{R}$.

Suppose that $(W(N),F)$ is an $\mathbb{R}$-split polarized mixed Hodge structure. The generalized Hodge decomposition gives a canonical grading of $W := W(N)[k]$ with

$$H_l = \bigoplus_{p+q=k+l} I^{p,q}.$$

The corresponding semisimple element Y acts on V by

$$Y(u) = (p+q-k)\cdot u \text{ for } u \in I^{p,q}.$$

Note that the eigenspace H_l is defined over $\mathbb{R}$ if $(W(N),F)$ is $\mathbb{R}$-split . Then $Y \in \mathfrak{g}_{\mathbb{R}}$ and the corresponding homomorphism $\rho_* : \mathfrak{sl}_2\mathbb{C} \to \mathfrak{g}_{\mathbb{C}}$ is defined over $\mathbb{R}$. The following is a remarkable connection between $\mathbb{R}$-split polarized mixed Hodge structures and SL_2-orbits.

Theorem A.5. *Let $(W(N),F)$ be a polarized mixed Hodge structure which split over $\mathbb{R}$. Then the canonical homomorphism $\rho_* : \mathfrak{sl}_2\mathbb{C} \to \mathfrak{g}_{\mathbb{C}}$ is horizontal at $e^{iN}F \in D$. Conversely, if a Lie algebra homomorphism $\rho_* : \mathfrak{sl}_2\mathbb{C} \to \mathfrak{g}_{\mathbb{C}}$ is horizontal at a reference point $\mathbf{r} \in D$. Denote by $N = \rho_*(\mathbf{n}_-)$, and $F_{\mathbf{r}}$ the corresponding Hodge filtration of $\mathbf{r}$. Then $(W(N), e^{-iN}F_{\mathbf{r}})$ is a polarized mixed Hodge structure which split over $\mathbb{R}$.*

Proof. See [5, Prop. 3.9] or [4, Prop. 2.18] □

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