Lecture 14. Tail Bounds, Randomized Incremental Construction (Also Some Computational Geometry)

CpSc 8400: Algorithms and Data Structures
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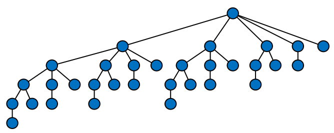

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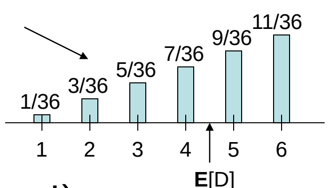
Spring, 2016

Expected Value

- The expected value of a discrete random variable X, denoted **E**[X], is defined as
  $$\mathbf{E}[X] = \Sigma_{\text{values } v}\, v\, \mathbf{Pr}[X = v].$$
- Think of **E**[X] informally as the “center of mass” of X’s probability distribution.
- Example: Let D be the max of two dice rolls. Recall that D has this probability distribution.
  – Thus, **E**[D] = 1(1/36) + 2(3/36) + 3(5/36) + 4(7/36) + 5(9/36) + 6(11/36) = 161/36 = $4\ {}^{17}/_{36}$



- Careful: Don’t write **E**[A] if A is an event (another syntax error!)

Computing Expected Values

There are generally 4 different ways we will compute expected values in this class:

1. Directly using the definition **E**[X] = Σ_v v **Pr**[X = v].
2. The special case of an **indicator** random variable.
3. The special case of a **geometric** random variable.
4. Expressing a complicated random variable in terms of a sum of simpler r.v.'s and applying **linearity of expectation**.

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Randomized Quicksort Revisited

- Let's think again about randomized quicksort.
- We've already shown an O(n log n) running time both w.h.p and also in expectation.
- There is another alternate expected running time proof that corresponds exactly to the w.h.p. proof:
- **W.h.p. proof:**

  Randomized reduction lemma → O(log n) / element w.h.p.

  Union bound → O(n log n) time for all elements w.h.p.
- **Expected running time proof:**

  Linearity of expectation → O(log n) expected time / element

  Linearity of expectation → O(n log n) total expected time.

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Wald's Theorem

- Linearity of expectation tells us that for N identically distributed random variables $X_1 \ldots X_N$,

  $$E[X_1 + \ldots + X_N] = N\,E[X_1].$$

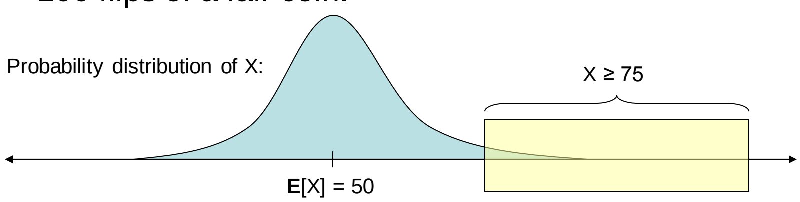
- What if N itself is a random variable though? (i.e., we have a random number of trials, each of which have identically-distributed random running times)
- **Wald's theorem** says that, as we might expect:

  $$E[X_1 + \ldots X_N] = E[N]\,E[X_1].$$

  (this requires a few technical conditions to hold; for example, N must be independent of the X_i's)

Tail Bounds

- We are often interested in the probability that a random variable X deviates significantly from its mean, **E**[X], when we randomly sample X from its distribution.
- Examples:
  - Show that **Pr**[$T = O(\log n)$] $\geq 1 - 1/n^c$ for arbitrary $c > 0$.
  - What is **Pr**[$X \geq 75$] if X denotes the number of heads in 100 flips of a fair coin.



Markov's Inequality

- X is any *nonnegative* random variable, then

  $\mathbf{Pr}[X \geq k\mathbf{E}[X]] \leq 1/k.$
- Sometimes written as $\mathbf{Pr}[X \geq a] \leq \mathbf{E}[X] / a$.
- Example: let X be the number of heads we see when flipping 100 coins.
  - $E[X] = 50$.
  - $Pr[X \geq 75] = Pr[X \geq (3/2)E[X]] \leq 2/3$.
- Due to its generality, Markov's inequality is a rather weak bound, although it's still quite useful.
- If expected running time = T, then probability our algorithm takes ≥ kT time is at most 1/k.

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Chernoff Bounds

- Suppose $X = X_1 + X_2 + \ldots + X_n$ is a sum of independent indicator (0/1) random variables (a common scenario).
- Then the tails of X's distribution drop off very quickly:
  - $\mathbf{Pr}[X \geq \mathbf{E}[X] + t] \leq e^{-2t^2/n}$
  - $\mathbf{Pr}[X \leq \mathbf{E}[X] - t] \leq e^{-2t^2/n}$
  - $\mathbf{Pr}[X \leq (1 - \varepsilon)\mathbf{E}[X]] \leq e^{-\varepsilon^2 E[x]/2}$
  - $\mathbf{Pr}[X \geq (1 + \varepsilon)\mathbf{E}[X]] \leq e^{-\varepsilon^2 E[x]/4}$ (if $\varepsilon \leq 2e - 1$)
  - $\mathbf{Pr}[X \geq (1 + \varepsilon)\mathbf{E}[X]] \leq 2^{-(\varepsilon+1)E[x]}$ (if $\varepsilon > 2e - 1$)
- Example: If X denotes the number of heads we see when flipping 100 fair coins, then
  - $\mathbf{Pr}[X \geq 75] = \mathbf{Pr}[X \geq \mathbf{E}[X] + 25] \leq e^{-25^2/50} = e^{-12.5} \leq 0.000004$
- Much stronger than Markov's inequality, but only applies when X is a sum of independent indicator random variables.

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Proving the Randomized Reduction Lemma

- **Theorem:** Suppose every iteration of our algorithm has probability $\geq p$ of reducing our problem size to at most $\leq q$ times its original size. Then given any constant $c > 0$, we can find another constant k such that **Pr**$[T \geq k \log n] \leq 1 / n^c$, where T is the total number of iterations of our algorithm.
- **Proof:**
  - Suppose we run our algorithm for $k \log n$ iterations.
  - Let X denote the # of "good" iterations among these (a good iteration is one where we reduce the problem size to $\leq q$ times original).
  - Now note that **Pr**$[T \geq k \log n] \leq$ **Pr**$[X \leq \log_{(1/q)} n]$, since the event that "$T \geq k \log n$" is a subset of the event that "$X \leq \log_{(1/q)} n$".
  - So now let's find a constant k such that **Pr**$[X \leq \log_{(1/q)} n] \leq 1/n^c$.
  - Continued…

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Proving the Randomized Reduction Lemma

- **Proof Continued:**
  - We run our algorithm for only $L = k \log_2 n$ iterations.
  - X = # of good iterations among these.
  - We want to show that **Pr**$[X \leq \log_{(1/q)} n] \leq 1/n^c$.
  - Write $X = X_1 + X_2 + \ldots + X_L$, where X_j is an indicator r.v. whose value is 1 if the jth iteration is good. Note that **E**$[X_j] \geq p$.
  - **E**$[X]$ = **E**$[X_1] + \ldots +$ **E**$[X_L] \geq Lp = kp \log_2 n$.
  - So **Pr**$[X \leq \log_{(1/q)} n]$

    $\leq$ **Pr**$[X \leq (1 / kp \log_2(1/q))$ **E**[**X**]]

    $=$ **Pr**$[X \leq (r / k)$ **E**[**X**]], where $r = 1 / [p \log_2 (1/q)]$ is a constant.
  - Now we use the Chernoff bound with $(1 - \varepsilon) = r / k$:
    Pr$[X \leq (1 - \varepsilon)$**E**$[X]] \leq e^{-\varepsilon^2 E[x]/2} = e^{-(1-r/k)^2 E[x]/2} \leq e^{-E[x]/8}$, if we choose k sufficiently large so that $1 - r / k \geq ½$.
  - Thus, **Pr**$[X \leq \log_{(1/q)} n] \leq e^{-E[x]/8} \leq e^{-(kp/8)\log n} = e^{-(kp/8 \ln 2)\ln n} = 1 / n^{kp/8\ln 2}$.
  - So **Pr**$[X \leq \log_{(1/q)} n] \leq 1/n^c$ if we choose $k \geq (8 \ln 2)/p$.

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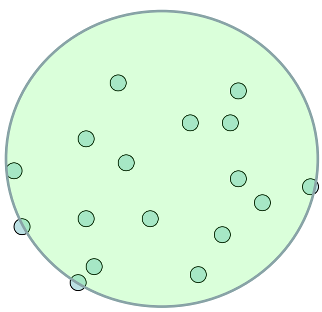
Randomized Incremental Construction

- When building a BST or a binary heap, inserting elements in certain orderings may lead to inefficient running times.
- However, inserting in a random order is (with high probability) quite efficient.
- These are examples of a general algorithm design technique called **randomized incremental construction**, where we build a solution by incorporating one element at a time, processing elements in random order.
- R.I.C. algorithms are especially common in computational geometry:
  - Examples: Convex hulls (2D and 3D), half-space intersections, Voronoi diagrams, Delaunay triangulations, smallest enclosing circle, low-dimensional linear programming, binary space partition trees, trapezoidal decompositions, closest pair, etc.

Example: Smallest Enclosing Circle

- Given n points in the plane, find the smallest circle enclosing them all.



Smallest Enclosing Circle

- Given n points in the plane, find the smallest circle enclosing them all.
- The optimal circle will be determined by at most 3 points, leading to an $O(n^4)$ "brute force" solution.
- With randomized incremental construction, however, we can solve this problem in only O(n) expected time!

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3 Simple Steps…

- Suppose by magic that we already know 2 points on the boundary of an optimal circle…
  - Now it's easy to compute the answer in O(n) time:

    Let p_1 and p_2 be the points we know.

    Start with a circle C having p_1 and p_2 as endpoints of its diameter.

    Let the remaining points $p_3 \ldots p_n$ be arbitrarily ordered.

    Process $p_3 \ldots p_n$ in sequence, enlarging C when necessary so it remains an optimal circle for the set of points considered thus far.

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3 Simple Steps…

- Suppose by magic that we already know 2 points on the boundary of an optimal circle…
  - Now it's easy to compute the answer in O(n) time.
- Now suppose (also by magic) that we already know only 1 point on the boundary of an optimal circle…
  - We can still compute the optimal circle in O(n) expected time…
  - Let p_1 be the point we know, and let $p_2 \ldots p_n$ be randomly ordered. Start with a zero-area circle C centered at p_1.
  - Process $p_2 \ldots p_n$ in sequence, updating C as we go.

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3 Simple Steps…

- Suppose by magic that we already know 2 points on the boundary of an optimal circle…
  - Now it's easy to compute the answer in O(n) time.
- Now suppose (also by magic) that we already know only 1 point on the boundary of an optimal circle…
  - We can still compute the optimal circle in O(n) expected time…
- Finally, suppose we know none of the points on the boundary of an optimal circle…

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3 Simple Steps…

- Suppose by magic that we already know 2 points on the boundary of an optimal circle…
  - Now it's easy to compute the answer in O(n) time.
- Now suppose (also by magic) that we already know only 1 point on the boundary of an optimal circle…
  - We can still compute the optimal circle in O(n) expected time…
- Finally, suppose we know none of the points on the boundary of an optimal circle…
- Final running time: O(d! n), where d is the dimensionality of our space.

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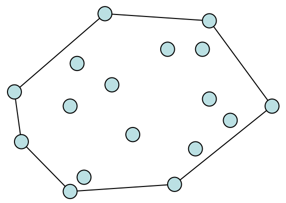
The Convex Hull Problem

- The **convex hull** of a set of n points is the smallest convex polygon containing all n points:



- The convex hull problem:
  - **Input:** A list of n points $(x_1, y_1) \ldots (x_n, y_n)$.
  - **Output:** An array or linked list specifying the points around the boundary of the hull in clockwise (or counterclockwise) order.

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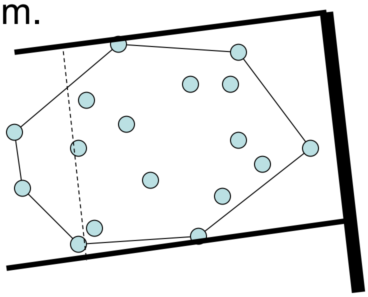
Farthest Pair of Points (2D)

- **Claim:** The farthest pair of points in a point set lies on the convex hull.
  - The two points will be on “opposite sides” of the hull.
- Once we’ve found the convex hull, we can therefore find the farthest pair of points in O(n) time using the “rotating calipers” algorithm.

  (we could call this a “sort and scan” approach…)



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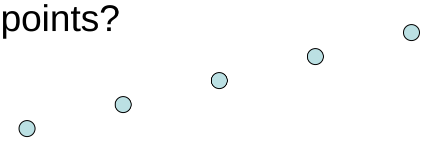
Collinear Points and Other Special Cases…

- Do all of these points belong to the convex hull, or just the endpoints?



- In general, computational geometry problems tend to be plagued by special cases like this.
  - This plus the danger of round-off errors can make it somewhat tricky to implement computational geometry algorithms correctly in practice!
- Typical assumption: no 3 of our points are collinear.

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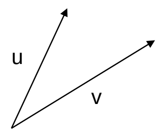
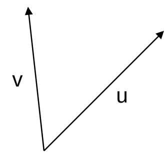
Primitive Operations

- Consider the following simple geometric questions:
  - Point P on line L?
  - Points P and Q on same side of line L?
  - Line segments S_1 and S_2 intersect?
  - Point P in angle formed by two rays R_1 and R_2?
  - Point P in convex polygon Q?
- A single “trick” makes all of these easy: look at the sign of the z component of the cross product!



$(u \times v)_z < 0$



$(u \times v)_z > 0$

$$(u \times v)_z = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

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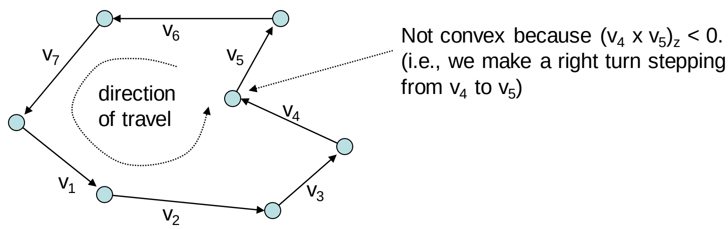
Checking Convexity

- Using the cross product test on consecutive pairs of vectors v, we can test a polygon for convexity:



- As we walk around the polygon in a counter-clockwise direction, we should make only “left turns” (z component of cross product positive).

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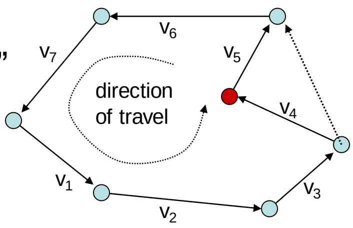
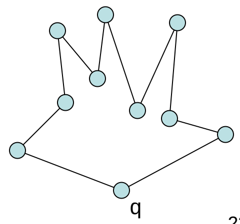
Convexifying a Polygon

- By splicing out points at which we turn right, we can "convexify" any polygon in O(n) time.



- This gives a simple O(n log n) algorithm (known as the "package wrapping" algorithm) for convex hulls:
  - Pick a "reference" point q known to be on the hull (e.g, with minimum y coordinate).
  - Sort remaining points by angle form q.
  - Use this sorted ordering to build a polygon and then convexify it.



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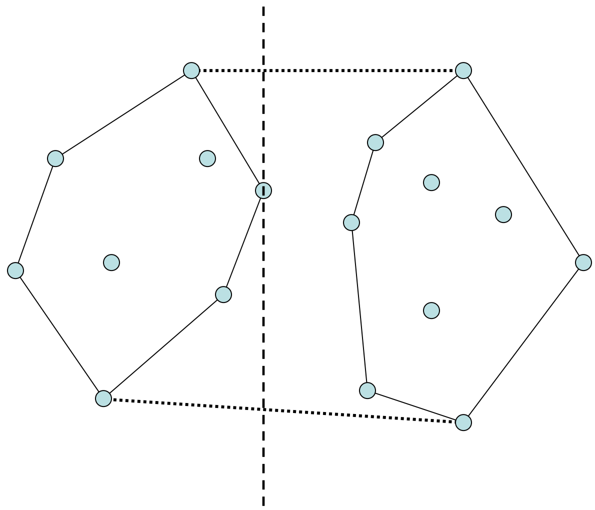
"QuickSort"



- How do we link two disjoint convex polygons in O(n) time?

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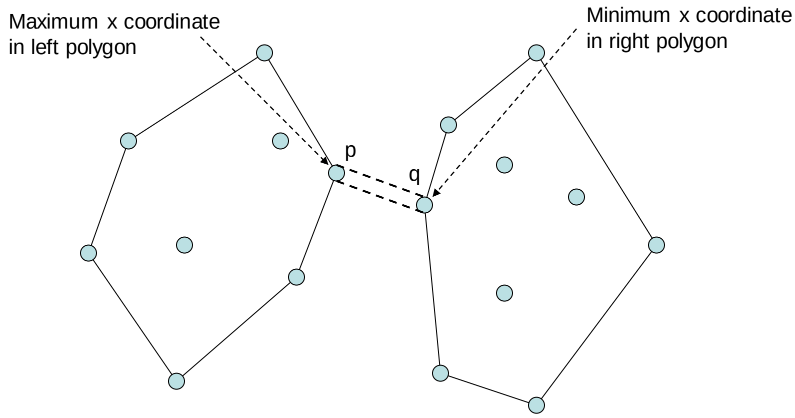
Linking Two Disjoint Polygons



- Splice polygons together into a non-convex polygon with a "doubled edge", then convexify!

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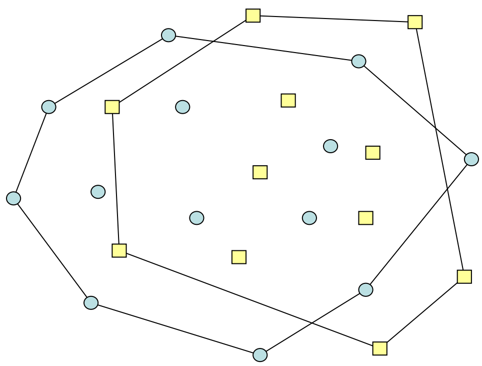
"Merge Sort"



- We can merge two (possibly overlapping) convex polygons in O(n) time by sweeping monotonically around each one in order of angle.
  - Just like merging two sorted sequences, only slightly more geometric details to deal with along the way (further details omitted…)

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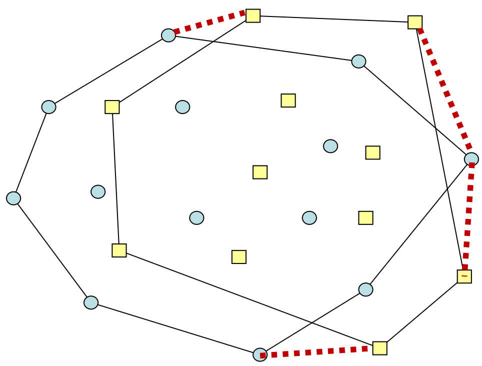
"Merge Sort"



- We can merge two (possibly overlapping) convex polygons in O(n) time by sweeping monotonically around each one in order of angle.
  - Just like merging two sorted sequences, only slightly more geometric details to deal with along the way (further details omitted…)

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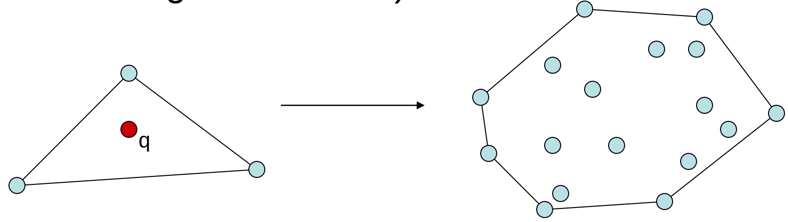
A Randomized Incremental Construction Convex Hull Algorithm

- We'll build up a convex hull by starting with 3 randomly-chosen points (a triangle) and adding the remaining points in random order, updating the hull when necessary.
- Maintain a point q inside the hull (the centroid of our initial triangle works fine).



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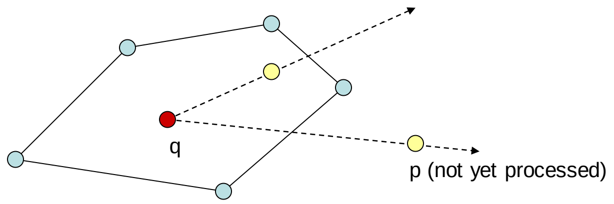
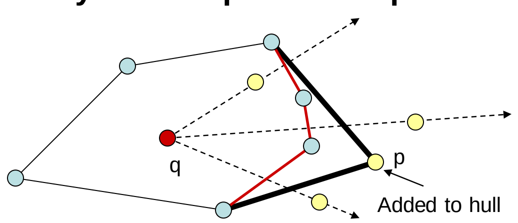
Randomized Incremental Construction of a Convex Hull

- Maintain an ordered linked list of all points (+edges) in clockwise order around the current hull.
- For each point p not yet processed, maintain bi-directional pointers between p and the edge in our hull crossed by the ray qp (with this extra information, we can quickly test if p is inside or outside the current hull).



Randomized Incremental Construction of a Convex Hull

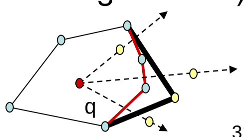
- When we add a new point p,
  - Check in O(1) time if p is inside or outside current hull.
  - If inside, nothing happens.
  - If outside, update the hull by removing necessary edges (shown in red below) and adding two new edges (the thick edges below); also **update the pointers of any affected yet-to-be-processed points**.



Running Time Analysis

- Add up running time per point p:
  - O(1) total work during initialization.
  - O(1) total work checking if p is inside or outside the current hull when p is finally selected.
  - O(1) total work inserting p into the hull (if at all).
  - O(1) total work later deleting p from the hull (if at all).
  - O(log n) expected total work for changing p's "current hull edge" pointer during the course of the algorithm

    (this is the only "expensive" part of the whole algorithm…)



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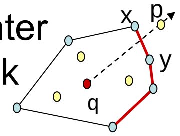
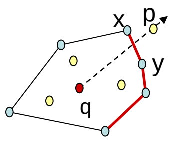
Probability of Pointer Change: Thinking Backwards Again…

- Consider the point in time at which we have added k points to our instance (some on the current hull, some inside the hull).
- p: not-yet-added point with pointer to hull edge xy.
- Only deletion of x or y causes p's pointer to change!
- We pick a point to delete at random from the set of all k outstanding points
- So w/ probability exactly 2/k, p's pointer changes when shrinking from a size-k point set to a size-(k-1) point set.



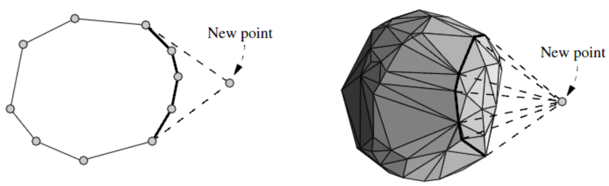
Linearity of Expectation

- With probability exactly 2/k, p's pointer changes when shrinking from a size-k point set to a size-(k-1) point set.
- X: Total number of changes to p's pointer
- X_k: indicator r.v. telling us whether p's pointer changes when adding the kth point.
- $E[X] = \sum_k E[X_k] = 2 \sum_k 1/k = O(\log n)$.
- And using randomized reduction, we can also argue that E[X] = O(log n).



Convex Hulls in Higher Dimensions

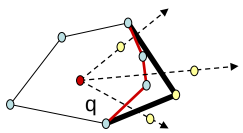
- Our randomized incremental construction algorithm extends readily to 3D, and keeps its O(n log n) expected running time!

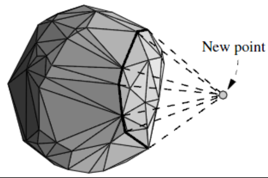


- In D>3 dimensions, the complexity of the hull can be as bad as $O(N^{\lfloor D/2 \rfloor})$. But we can compute convex hulls this quickly…

Running Time Analysis: 3D Convex Hull

- Add up running time per point p:
  - O(1) total work during initialization.
  - O(1) total work checking if p is inside or outside the current hull when p is finally selected.
  - ~~O(1) total work inserting p into the hull (if at all).~~
  - ~~O(1) total work later deleting p from the hull (if at all).~~
  - O(log n) expected total work for changing p's "current hull edge" pointer during the course of the algorithm

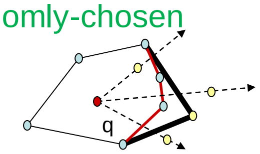


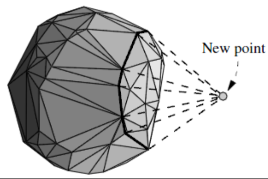


Running Time Analysis: 3D Convex Hull

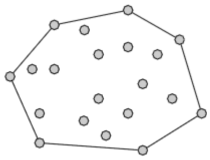
- Add up running time per point p:
  - O(1) total work during initialization.
  - O(1) total work checking if p is inside or outside the current hull when p is finally selected.
  - O(1) expected work inserting p into the hull (if at all).
  - O(1) expected work later deleting p from the hull (if at all).

(Euler's formula tells us that the average # of neighbors of a point in a polyhedron is O(1), so the expected # of neighbors of a randomly-chosen point is also O(1)).

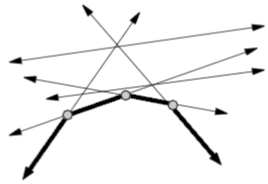




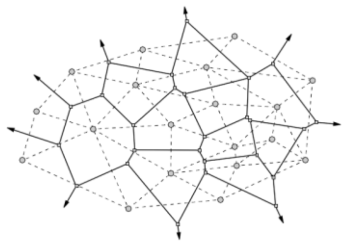
Some Prominent Problems in Computational Geometry…



Convex hull
Upper / lower hull



Lower / upper envelope



Voronoi diagram (solid)
Delaunay triangulation (dashed)

Geometric Duality

- In computational geometry, we often exploit geometric duality – exchanging the roles of points and lines.
  - Slope-intercept duality: point (a,b) ↔ line y = ax – b.
  - Polar duality: point (a,b) ↔ line ax + by = 1.
- The roles of points and lines reverse when we dualize:
  - Recall that two points determine a unique line, and likewise that two lines determine a unique point.
  - If L1 and L2 are two lines intersecting at point P, then dual(L1) and dual(L2) are two points that lie on the common line dual(P).
  - If P1 and P2 are two points determining line L, then dual(L) is the point of intersection between the lines dual(P1) and dual(P2).

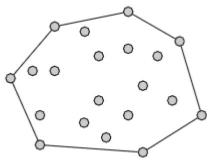
Duality – Example

- Given N points in the 2D plane, find the two that determine a line with maximum slope.

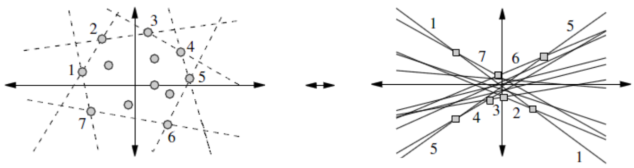
Duality – Example

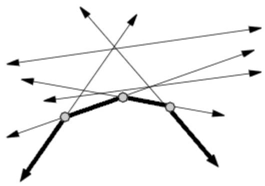
- Given N points in the 2D plane, find the two that determine a line with maximum slope.
- Apply slope-intercept duality: $(a,b) \leftrightarrow y = ax - b$
- Now we have a set of N lines, and we are looking for the two lines that determine a point with maximum coordinate!
- This point will occur between two lines with adjacent slopes (after sorting by slope); hence, in our original problem, the optimal line will occur between two points with adjacent x coordinates (after sorting by x coordinate).

Geometric Duality Between Upper/Lower Hulls and Lower/Upper Envelopes



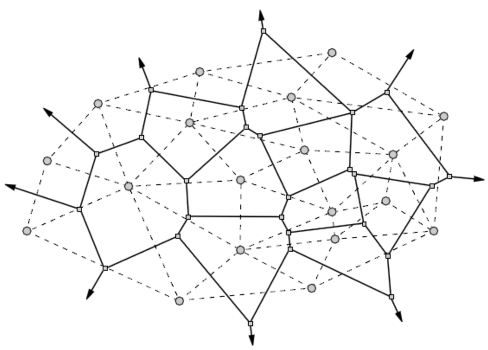
Convex hull
Upper / lower hull





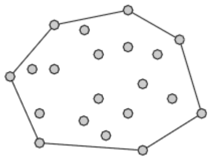
Lower / upper envelope

Planar Graph Duality Between Voronoi Diagrams and Delaunay Triangulations

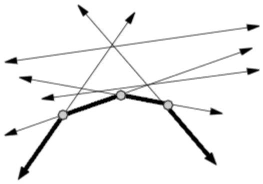


- There are many ways (most rather complicated!) to compute both objects in 2D in O(N log N) time (and there is a matching lower bound)
- In D>2 dimensions, the complexity of these objects can be as bad as $O(N^{\lfloor (D+1)/2 \rfloor})$.

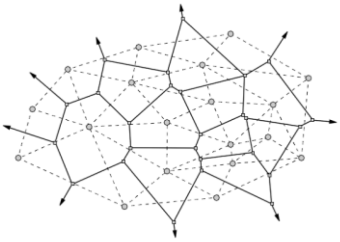
Back to Our High-Level List of Problems…



Convex hull
Upper / lower hull



Lower / upper envelope

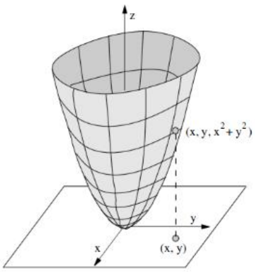


Voronoi diagram
Delaunay triangulation

We'll now discuss the *parabolic lifting* transformation, which allows us to transform between Voronoi/Delaunay problems in D dimensions and hull/envelope problems in D+1 dimensions!...

Parabolic Lifting

- Map each of our D-dimensional input points up to a (D+1)-dimensional point by lifting it onto a paraboloid ($z = x^2 + y^2$ in 2D):



- Then draw tangent planes to the paraboloid at each of the N lifted points…

Parabolic Lifting

- By projecting the upper envelope of our tangent planes back down onto D dimensions, we get the Voronoi diagram of our original point set!
- By projecting the lower hull of our lifted points back down to D dimensions, we get the Delaunay triangulation of the original point set!