6.1 Free and Bound Variables

1. there is no free variable
2. Yes.

   $\forall x(\exists y P(x, f(y)) \to P(x, \underline{y}))$: y is free
3. Yes.

   $\forall z(P(z) \to \exists y(\exists x Q(x, y, z) \vee Q(z, y, \underline{x})))$: x is free
4. there is no free variable.
5. u is free.

6.2 Semantics

1. $4 \times (2+1) = 12$
2. $P(f(2,4), g(2,1))^{(I,E)} = P(6,2) = \text{false} = 0$
3. $g(0, f(1,2)) = g(0,3) = 0$ $\Rightarrow$ therefore, the result should be 1
4. $P(6, g(2,1)) = 0$

6.3 Semantics II

1. $S(aa + abaa) = aaabaaa$
2. $S(aba + S(aba + aa)) = S(aba + abaaaa) = abaabaaaaa$

6.4. Semantic entailment

1. No. when $E(x) = b$ $E(y) = a$, we don't have $I \models \phi$
2. $E(x) = a$ $E(y) = a$ then $E(z)$ can be anything

   $E(x) = a$ $E(y) = b$ then $E(z) = c$

   $E(x) = a$ $E(y) = c$ then $E(z)$ can be anything

   $E(x) = b$ $E(y) = a$ then $E(z)$ can be anything

   $E(x) = b$ $E(y) = b$ then $E(z)$ can be anything

   $E(x) = b$ $E(y) = c$ then $E(z) = b$

   $E(x) = c$ $E(y) = a$ then $E(z)$ can be anything

   $E(x) = c$ $E(y) = b$ then $E(z) = c$

   $E(x) = c$ $E(y) = c$ then $E(z)$ can be anything

   $\Rightarrow$ therefore, the answer is yes. it's valid.

6.5. Formal Proof.

①

1. $\forall x P(x) \vee \forall x Q(x)$ — Premise
2. $\forall x P(x)$ — Assumption
3. y fresh.
4. $P(y)$ — $\forall e: 2$
5. $P(y) \vee Q(y)$ — $\vee i: 4$
6. $\forall x (P(x) \vee Q(x))$ — $\forall i: 3\text{-}5$
7. $\forall x Q(x)$ — Assumption
8. y fresh
9. $Q(y)$ — $\forall e: 7$
10. $P(y) \vee Q(y)$ — $\vee i: 9$
11. $\forall x (P(x) \vee Q(x))$ — $\forall i: 8\text{-}9$
12. $\forall x (P(x) \vee Q(x))$ — $\forall i: 2\text{-}6.\ 8\text{-}9$

②

1. $\neg \exists x P(x)$ — Premise
2. $\neg \forall x (\neg P(x))$ — Assumption.
3. u fresh.
4. $\neg(\neg P(u))$ — $\forall e: 2$
5. $P(u)$ — $\neg\neg e: 3$
6. $\exists x P(x)$ — $\exists i: 5$
7. $\perp$ — Reflexivity: 1.6
8. $\neg(\neg \forall x (\neg P(x)))$ — $\neg i: 2\text{-}7$
9. $\forall x \neg P(x)$ — $\neg\neg e: 8$

③

1. $\forall x (Q(x) \rightarrow R(x))$ — Premise
2. $\exists x (P(x) \wedge Q(x))$ — Premise
3. $P(u) \wedge Q(u)$ u fresh — Assumption
4. $Q(u) \rightarrow R(u)$ — $\forall e: 1$
5. $Q(u)$ — $\wedge e: 3$
6. $R(u)$ — $\rightarrow e: 4,5$

7. $P(u)$ $\wedge e: 3$

8. $P(u) \wedge R(u)$ $\wedge i: 6, 7$

9. $\exists x(P(x) \wedge R(x))$ $\exists i: 8$

10. $\exists x(P(x) \wedge R(x))$ $\exists e: 2, 3\text{-}10$

④

1. $\forall x(P(x) \to Q(x))$ Premise

2. $\forall x P(x)$ Assumption

3. u fresh

4. $P(u)$ $\forall e: 2$

5. $P(u) \to Q(u)$ $\forall e: 1$

6. $Q(u)$ $\to e: 4, 5$

7. $\forall y Q(y)$ $\forall e: 3\text{-}6$

8. $\forall x P(x) \to \forall y Q(y)$ $\to i: 2\text{-}7$