

4.1 The Hilbert-style proof System

- ①
1. $\{\neg A \rightarrow A, \neg A\} \vdash \neg A$ Assumption
 2. $\{\neg A \rightarrow A, \neg A\} \vdash \neg A \rightarrow A$ Assumption
 3. $\{\neg A \rightarrow A, \neg A\} \vdash A$ MP 1, 2
 4. $\{\neg A \rightarrow A, \neg A\} \vdash A \rightarrow (\neg A \rightarrow \text{false})$ Theorem
 5. $\{\neg A \rightarrow A, \neg A\} \vdash \neg A \rightarrow \text{false}$ MP 3, 4
 6. $\{\neg A \rightarrow A, \neg A\} \vdash \text{false}$ MP 1, 5
 7. $\{\neg A \rightarrow A\} \vdash \neg A \rightarrow \text{false}$ Deduction
 8. $\{\neg A \rightarrow A\} \vdash A$ Theorem
 9. $\vdash (\neg A \rightarrow A) \rightarrow A$ Deduction.
- ②
1. $\{\neg A, \neg B \rightarrow A\} \vdash (\neg B \rightarrow A) \rightarrow (\neg A \rightarrow B)$ Axiom 3.
 2. $\{\neg A, \neg B \rightarrow A\} \vdash \neg B \rightarrow A$ Assumption
 3. $\{\neg A, \neg B \rightarrow A\} \vdash \neg A \rightarrow B$ MP 1, 2
 4. $\{\neg A, \neg B \rightarrow A\} \vdash \neg A$ Assumption
 5. $\{\neg A, \neg B \rightarrow A\} \vdash B$ MP 3, 4
 6. $\{\neg A\} \vdash (\neg B \rightarrow A) \rightarrow B$ Deduction

4.2 ND proof System

- ①
- | | | |
|----|-----------------------|-----------------|
| 1. | $\neg(\neg p \vee q)$ | Premise |
| 2. | $\neg p$ | Assumption |
| 3. | $\neg p \vee q$ | $\vee i: 2$ |
| 4. | \perp | $\perp i: 1, 3$ |
5. $\neg(\neg p)$ $\neg i: 2-4$
 6. p $\neg e: 5$

- ②
- | | | |
|----|----------------------------|------------------|
| 1. | $p \wedge q \rightarrow r$ | Premise |
| 2. | p | Assumption |
| 3. | q | Assumption |
| 4. | p | Reflexivity: 2 |
| 5. | $p \wedge q$ | $\wedge i: 3, 4$ |

| | | |
|----|-------------------|----------------------|
| b. | r | $\rightarrow e: 1-5$ |
| 7 | $q \rightarrow r$ | $\rightarrow i: 3-6$ |

8. $p \rightarrow (q \rightarrow r)$ $\rightarrow i: 2-7$

③

1. $(p \vee q) \vee r$ Premise

2. r Assumption

3. $q \vee r$ $\vee i: 2$

4. $p \vee (q \vee r)$ $\vee i: 3$

5. $p \vee q$ Assumption

6. q Assumption

7. $q \vee r$ $\vee i: 6$

8. $p \vee (q \vee r)$ $\vee i: 7$

9. p Assumption

10. $p \vee (q \vee r)$ $\vee i: 9$

11. $p \vee (q \vee r)$ $\vee e: 5-8, 9-10$

12. $p \vee (q \vee r)$ $\vee e: 1-2, 4, 5-11$

④

1. $p \wedge (q \vee r)$ Premise

2. $q \vee r$ $\wedge e: 1$

3. p $\wedge e: 1$

4. r Assumption

5. $p \wedge r$ $\wedge i: 3, 4$

6. $(p \wedge q) \vee (p \wedge r)$ $\vee i: 5$

7. q Assumption

8. $p \wedge q$ $\wedge i: 3, 7$

9. $(p \wedge q) \vee (p \wedge r)$ $\vee i: 8$

10. $(p \wedge q) \vee (p \wedge r)$ $\vee e: 2, 4, 6, 7, 9$

⑤

1. $\neg(p \vee q)$ Premise

2. p Assumption

3. $p \vee q$ $\vee i: 2$

| | | |
|-----|------------------------|-----------------|
| 4. | \perp | $\perp i: 1.3$ |
| 5. | $\neg p$ | $\neg i: 2-4$ |
| 6. | q | Assumption |
| 7 | $p \vee q$ | $\vee i: 6$ |
| 8 | \perp | $\perp i: 1.7$ |
| 9. | $\neg q$ | $\neg i: 6-8$ |
| 10. | $\neg p \wedge \neg q$ | $\wedge i: 5.9$ |

[4.3] Formalization

"Train Arrives Late": p

"No taxi at the station": q

"Late for the meeting": r

$\{ p, \neg r, (p \wedge q) \rightarrow r \} \vdash \neg q$

1. p Premise

2. $\neg r$ Premise

3. $p \wedge q \rightarrow r$ Premise

4. $\neg(p \wedge q)$ MT. 2.3

5. q Assumption

6. $p \wedge q$ $\wedge i: 1.5$

7. \perp $\perp i: 4.6$

8. $\neg q$ $\neg i: 5-7$

[4.4] CNF Reduction

$$\begin{aligned}
 (p \rightarrow q) \rightarrow (q \rightarrow r) &\equiv (p \vee q) \rightarrow (\neg q \vee r) \equiv \neg(p \vee q) \vee (\neg q \vee r) \\
 &\equiv (\neg p \wedge \neg q) \vee (\neg q \vee r) \equiv (\neg p \vee \neg q \vee r) \wedge (\neg q \vee \neg q \vee r) \\
 &\equiv (\neg p \vee \neg q \vee r) \wedge (\neg q \vee r) \equiv (\neg q \vee r)
 \end{aligned}$$

[4.5] Resolution.

① $p \equiv p$

② $p \rightarrow ((q \vee r) \wedge \neg(q \wedge r)) \equiv \neg p \vee ((q \vee r) \wedge \neg(q \wedge r)) \equiv (\neg p \vee (q \vee r)) \wedge (\neg p \vee \neg(q \wedge r))$
 $\equiv (\neg p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$

$$\textcircled{3} P \rightarrow ((S \vee t) \wedge \neg(S \wedge t)) \equiv \neg P \vee ((S \vee t) \wedge \neg(S \wedge t)) \equiv (\neg P \vee (S \vee t)) \wedge (\neg P \vee \neg(S \wedge t)) \\ \equiv (\neg P \vee S \vee t) \wedge (\neg P \vee \neg S \vee \neg t)$$

$$\textcircled{4} S \rightarrow Q \equiv (\neg S \vee Q)$$

$$\textcircled{5} R \rightarrow T \equiv (R \vee T)$$

$$\textcircled{6} T \rightarrow S \equiv (\neg T \vee S)$$

$$C = \{P, \neg P \vee Q \vee R, \neg P \vee \neg Q \vee \neg R, \neg P \vee S \vee t, \neg P \vee \neg S \vee \neg t, \neg S \vee Q, R \vee T, \neg T \vee S\}$$

$$\{P\}, \{\neg P, Q, R\}, \{\neg P, \neg Q, \neg R\}, \{\neg P, S, t\}, \{\neg P, \neg S, \neg t\}, \{\neg S, Q\}, \{R, T\}, \{\neg T, S\}$$

$$\textcircled{1} \frac{\neg P \vee \neg Q \vee R \quad \neg S \vee Q}{\neg P \vee \neg R \vee \neg S}$$

$$\textcircled{2} \frac{\neg P \vee \neg S \vee \neg t \quad R \vee T}{\neg P \vee \neg S \vee R}$$

$$\textcircled{3} \frac{\neg P \vee \neg R \vee \neg S \quad \neg P \vee \neg S \vee R}{\neg P \vee \neg S}$$

$$\textcircled{4} \frac{R \vee T \quad \neg T \vee S}{R \vee S}$$

$$\textcircled{5} \frac{R \vee S \quad \neg P \vee \neg S \vee R}{\neg P}$$

$$\textcircled{6} \frac{\neg P \quad P}{\perp}$$