6 Assignment 6: FOL Semantics and Proof (100 points)

6.1 Free and Bound Variables (20 points)

Is there any free variable in the following formulas? If so, please also point out which variable(s) or which occurrence(s) of a variable is free.

1. $\forall x(P(x) \to \exists y \neg Q(f(x), y, f(y)))$
2. $\forall x(\exists y R(x, f(y)) \to R(x, y))$
3. $\forall z(P(z) \to \exists y(\exists x Q(x, y, z) \vee Q(z, y, x)))$
4. $\forall z \exists u \exists y(Q(z, y, g(u, y)) \vee R(u, g(z, u)))$
5. $\forall z \exists x \exists y(Q(z, u, g(u, y)) \vee R(u, g(z, u)))$

6.2 Semantics (20 points)

For a language $\mathscr{L} = \{\mathbb{N}, 0, 1, P^{(2)}, f^{(2)}, g^{(2)}\}$, define an interpretation $\mathcal{I}$:

- Domain: $\mathbb{N}$
- Constants: $0, 1$
- Predicates: $P^{(2)}$: less than or equal to.
- Functions: $f^{(2)}$ for sum and $g^{(2)}$ for multiplication.

Define an environment $E(x_n) = 2n$. For the following terms or formulas, write down their values.

1. $g(x_2, f(x_1, 1))^{(\mathcal{I},E)}$
2. $P(f(x_1, x_2), g(x_1, f(0, 1)))^{(\mathcal{I},E)}$
3. $g(x_0, f(1, x_1)) = 0^{(\mathcal{I},E)}$
4. $\forall x_1 P(f(x_1, x_2), g(x_1, f(0, 1)))^{(\mathcal{I},E)}$

6.3 Semantics II (20 points)

For a language $\mathscr{L} = \{D, 0, s^{(1)}, +^{(2)}\}$, define an interpretation $\mathcal{I}$:

- Domain D: the collection of all words over the alphabet $\{a, b\}$.
- Constant 0: a
- Function $s^{(1)}$: appends a to the end of a string

- Function $+^{(2)}$: concatenation

Define an environment $E(x) = aba$. Write down the values for the following terms or formulas.

1. $s(s(0) + s(x))^{(\mathcal{I},E)}$
2. $s(x + s(x + s(0)))^{(\mathcal{I},E)}$

6.4 Semantic entailment (20 points)

Let ϕ be the sentence $\forall x \forall y \exists z (R(x, y) \to R(y, z))$, where R is a binary predicate.

1. Let $dom(\mathcal{I}) \stackrel{\text{def}}{=} \{a, b, c, d\}$, and $R^{\mathcal{I}} \stackrel{\text{def}}{=} \{(b, c), (b, b), (b, a)\}$. Do we have $\mathcal{I} \vDash \phi$? Answer "yes" or "no" and also justify your answer.
2. Let $dom(\mathcal{I}) \stackrel{\text{def}}{=} \{a, b, c\}$, and $R^{\mathcal{I}} \stackrel{\text{def}}{=} \{(b, c), (a, b), (c, b)\}$. Do we have $\mathcal{I} \vDash \phi$? Answer "yes" or "no" and also justify your answer.

6.5 Formal proof (20 points)

Use the ND proof system of FOL to prove the following:

1. $\forall x P(x) \vee \forall x Q(x) \vdash \forall x (P(x) \vee Q(x))$
2. $\neg \exists x P(x) \vdash \forall x \neg P(x)$
3. $\{\forall x (Q(x) \to R(x)), \exists x (P(x) \wedge Q(x))\} \vdash \exists x (P(x) \wedge R(x))$
4. $\forall x (P(x) \to Q(x)) \vdash \forall x P(x) \to \forall y Q(y)$