Team No.412

Problem B

General Method to Roller Coaster Design and Analysis Based on NURBS Curves

Abstract

This paper offers a general method to design an exciting, yet safe roller coaster with control points inputs and practical theoretical simulation method. According to that method, we designed a roller coaster of height 30 m and 50 seconds decline period. Our method is based on NURBS curves and Newton's Second Law.

1 Introduction

Roller coaster is always famous for its fascinated high speed and breath taking experience, which tightly draws every traveller's mind. In this paper, we are going to design a roller coaster to be as exciting as possible. The following paper will mainly focus on description on path, analysis on its dynamics, and the final simulation. We modeled it to be a general one, which can design and simulate any track according to several straight forward input parameters.
We first start with a simple but general model to analyze the acceleration. A roller coaster loop can be determined by a simple curve, which represents the path of the loop, and a roll angle, which represents the bank inclination of track. Knowing the related parameters, we can apply Newton's Law to derive the acceleration acted on the track, and compare it with the maximum acceleration allowed to act on human body. Additionally, to boost passengers' mood, we design a curve that can provide both the biggest acceleration and various large values of velocity in different directions. After combining them together, we finally derive a solution with Mathematica and plot it out.

2 Model

2.1 Assumptions

By the problem description, we know that gravitational potential is the only energy source for the roller coaster's motion.

We assume that there is no friction everywhere in the roller coaster. So the law of Conservation of Energy always holds. We believe that this assumption is reasonable because in the experiment of Beowulf, his data of acceleration shows that the magnitude of sternumward acceleration (which will be introduced later, see Figure 2) is actually very small[1]. Hence, the friction, which is usually parallel to the sternumward acceleration, can be ignored correctly.

To avoid the influence of rotational reference frame, we consider the heart of the passenger as a mass point, and the passenger coordinate system (which will be explained later) is a translational reference frame. We will first design the heart-line curve. Then we consider the motion of the "passenger", the mass point at the passenger's heart, on this curve, using Newton's law to solve the acceleration of the motion on every point of the curve.

Since modern coasters are mainly heart-lined, actually the track curve and the roll angle are dependent on the heart-line curve and the motion situation[4]. In our model, to simplify the problem, we use a method to eliminate the lateral acceleration (eyes left and eyes right) acting on the passenger. We first analysis the motion of the "car" on every point of the curve. Then we can calculate the direction of the supporting force provided by the track to the car. Then we rotate the passenger around the heart to make the track plane perpendicular with the supporting force. The rotation is shown in figure 1.

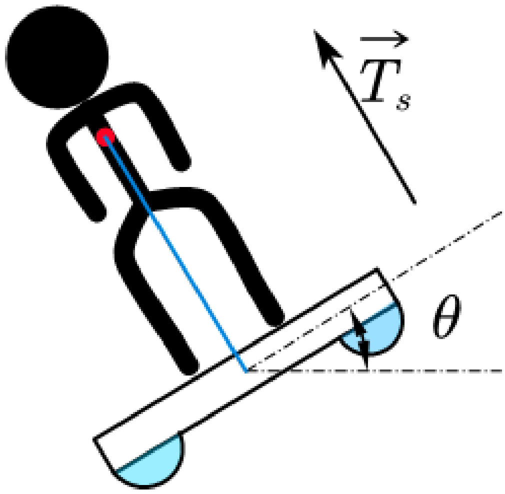

Figure 1: The rotation of the car to avoid lateral acceleration

Then, using this method, we can eliminate the lateral acceleration acted on the car, effectively reduce the possibility of sides lip or rollover of the car. Also, we use this method to design the position of track after we have designed the heart-line curve.

2.2 Acceleration Limits

According to Standard Practice for Design of Amusement Rides and Devices as required by: State of Indiana, the safety limits of the roller coaster are established as below [3].

Terminology Literature shows that whether acceleration is perpendicular or parallel to the spine affects human tolerance greatly [2]. In this model, a reasonable way to describe the acceleration effect on roller coaster passengers is to consider headward, sternumward, spineward and tailward acceleration, as shown in Figure 2.

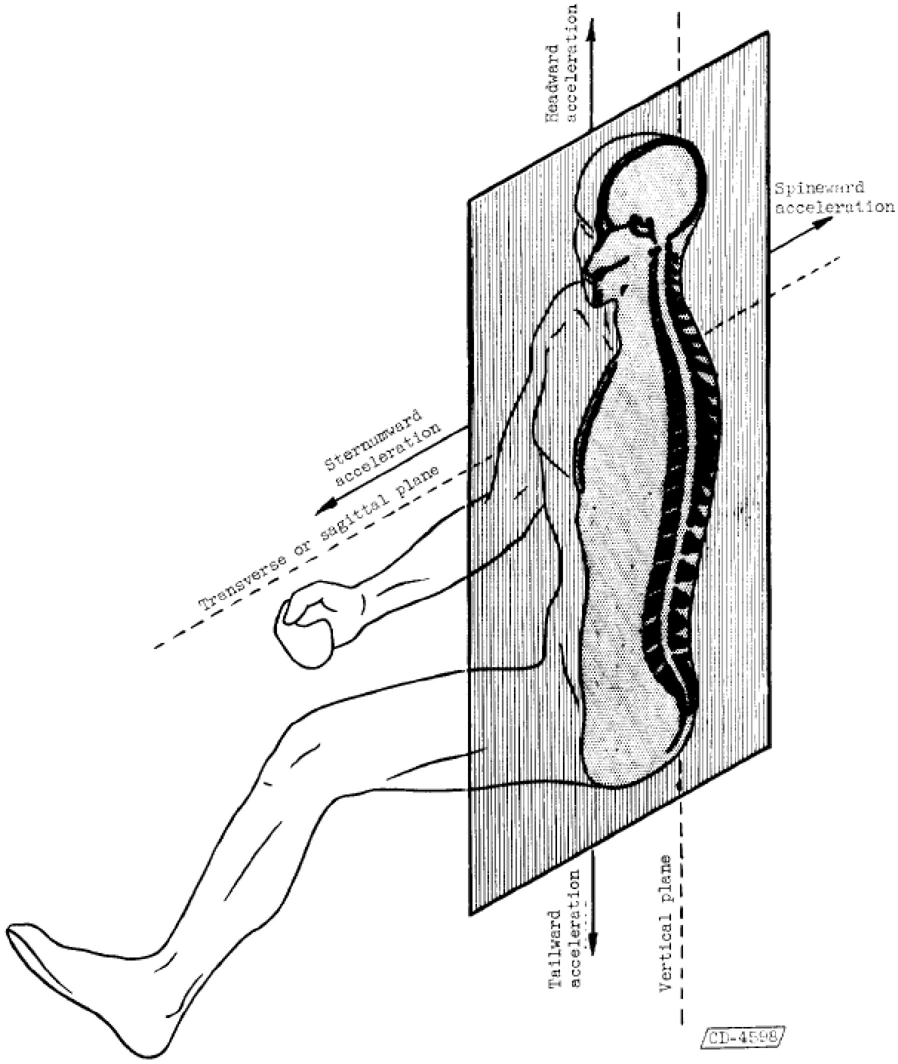

Figure 2: Sectional view for accelerations acted on people[2]

To simplify the notation, headward acceleration is defined as +Z acceleration, while sternumward acceleration is defined as +X acceleration. The corresponding tailward acceleration is thus -Z acceleration and spineward acceleration is -X acceleration. The Y axis points leftward. We note this coordination as passenger coordination. It's shown in Figure 3.

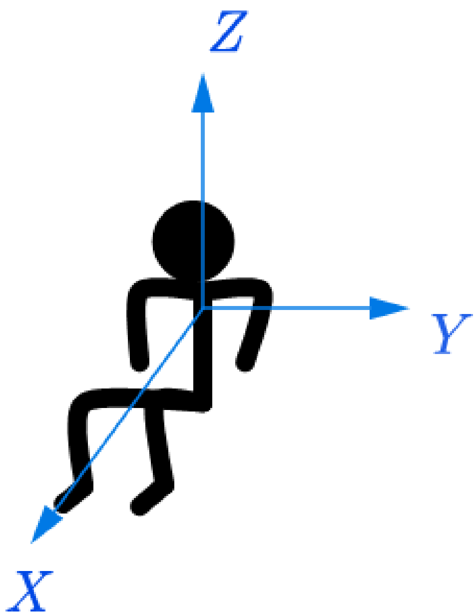

Figure 3: Passenger coordination

Our acceleration evaluation system is established under the gravitational field on Earth. While the passenger is at rest, he experiences only gravity. We say his acceleration is of $a_x, a_y, a_z = 0, 0, -g$ if $g = 9.8m/s^2$ denotes acceleration of gravity.

The origin of the passenger coordinate system is established at the heart of the passenger. If the passenger sits comfortably, his heart is approximately 1.0 metres higher than his feet, with his feet 0.3 m higher than the track level.

Translational Acceleration Limitations Translational acceleration must be limited to a certain range so that the safety of the passengers are ensured. We take the limitations formulated by ASTM International. To simplify their complicated industrial standards which include discussion about exceptions, we design our roller coaster within the basic limits as shown in the figure below.

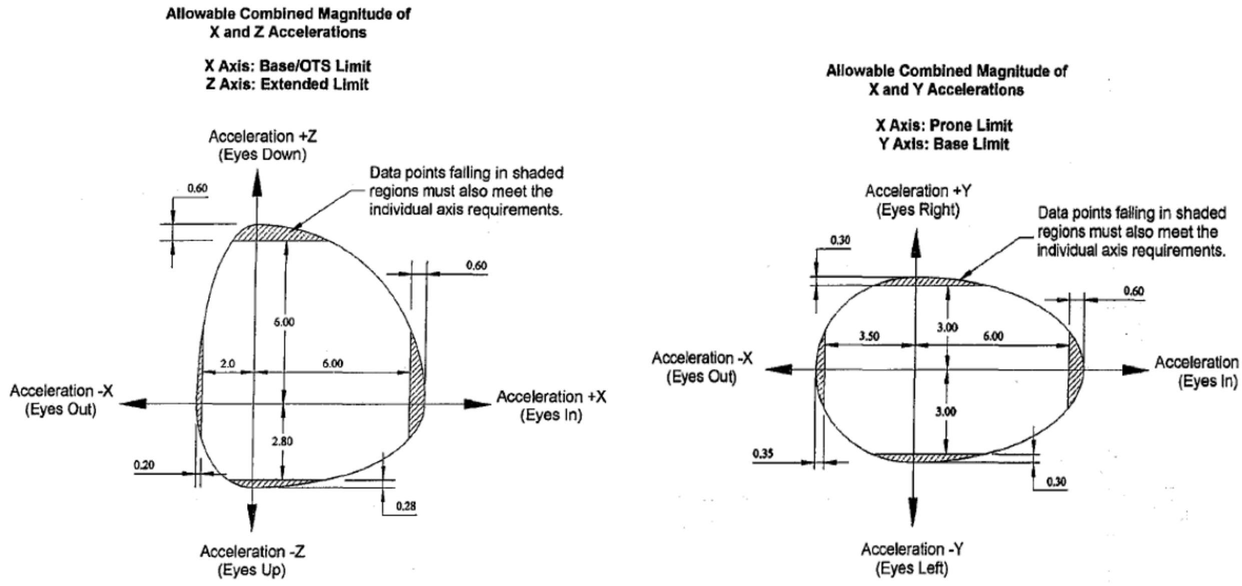

Figure 4: Allowable magnitude of accelerations [2]

A maximum of 6.60g of spineward(-X) acceleration will provide an exciting yet pleasant experience, while sternumard(X) acceleration must be limited to 2.2g to prevent spine or brain injury. A tailward acceleration (-Z) of 6.60g is irritative yet won't let the passenger faint out because all his blood flows out from his brain.

2.3 NURBS Curves

We want our roller coaster follow a smooth curve, which mean differentiable for at least three times so that there is no sudden change of acceleration, therefore The n-th degree NUBRS curve is given by

$$\gamma(p) = \sum_{i=0}^{k} R_{i,n}(p) P_i \tag{1}$$

where the P_i are control points and the rational basis function $R_{i,n}$ is

$$R_{i,n}(p) = \sum_{i=0}^{k} \frac{N_{i,n}(p) w_i P_i}{N_{i,n}(p) w_i} \tag{2}$$

where $N_{i,n}$ is the basis function determined by the degree of the curve (see Figure 5).

This curve can be (p-1)th differentiable, and the curve will start at the first point and end at the last point.

Therefore, the track can be generated be adding the control points. For example, choose the control points $P = \{(0,0,0), (-3,30,0), (-6,20,40), (-9,10,0), (-12,40,0)\}$, then the result is given in Figure 6.

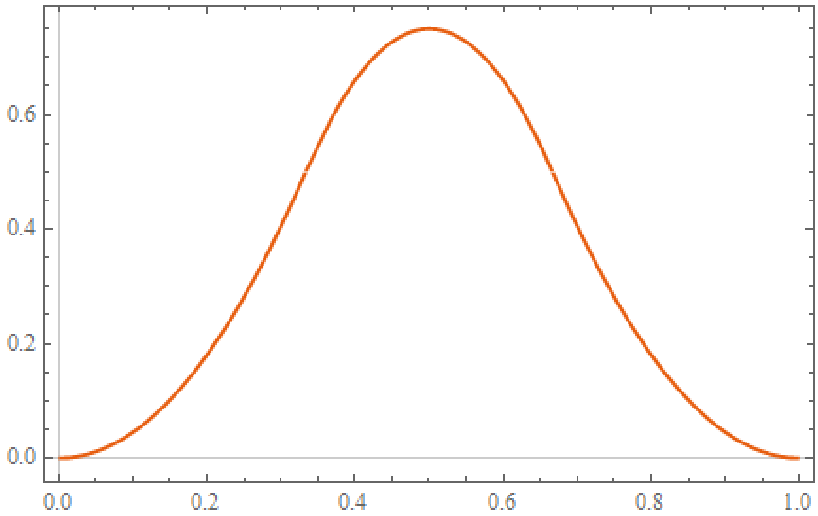

Figure 5: The first basis function of degree 3

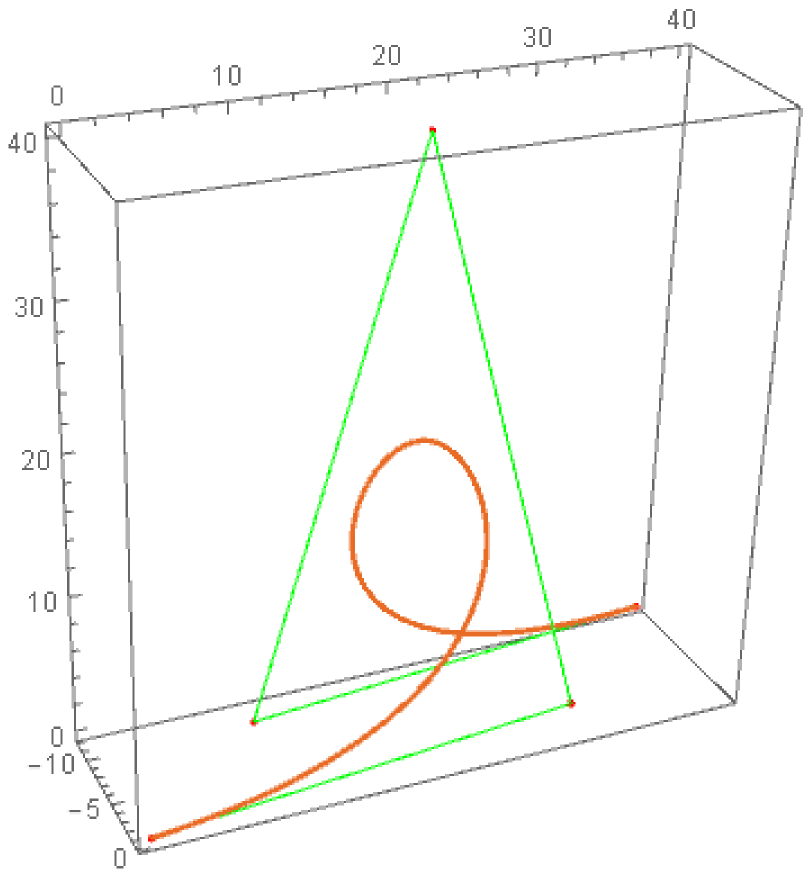

Figure 6: The clothoid curve (control line is green and generated curve is orange)

2.4 Model overview

In order to design an exciting and safe roller coaster, we should make the velocity of the car changing rapidly and can reach a big value. Also, we should try our best to make the acceleration bigger, but still in a safe range.

We will first design a parametrized NURBS curve for the heart-line. While designing, we will set a series of parameters determining the curve. Then we will calculate the velocity and acceleration of the passenger(heart) on the curve. By analysing the free body diagram of passenger, we can find the direction of supporting force, and further find the direction of passenger's headward. Then we project the acceleration into the passenger coordinate system, and get a curve in the acceleration phase diagram. Finally we adjust the parameter to let the curve in phase diagram reach the limitation value as far as possible. Finally we can finish our designing by adding other curves to finish a loop.

2.5 Basic model

2.5.1 Initial and terminal state

The initial and terminal part is trivial and do not need difficult analysis. We assume the lift onto the initial hill is completed in 30 seconds. As for the terminal part, the vehicle will slow down with a constant acceleration in 20m. Therefore, the acceleration of slowing down is,

$$a_{terminal} = \frac{v_0^2}{2s} = 14.82\ m/s^{-2} \tag{3}$$

which is within the ASTM safety range.

2.5.2 Basic motion equation of a 3D curve

We begin by constructing a curve parametrization $\gamma(p)$ in $\mathbb{R}^3$:

$$\gamma : [0,1] \to \mathbb{R}^3 \tag{4}$$

$$\gamma(p) = \begin{pmatrix} \gamma_1(p) \\ \gamma_2(p) \\ \gamma_3(p) \end{pmatrix} = \begin{pmatrix} \gamma_1 \circ p(t) \\ \gamma_2 \circ p(t) \\ \gamma_3 \circ p(t) \end{pmatrix} \tag{5}$$

with a undetermined function p(t):

$$p : [0, t_{\max}] \to [0,1] \tag{6}$$

Now, for the motion of a car on the curve, we write down the motion equations:

$$\vec{r}(t) = \gamma(p(t)) \tag{7}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d\gamma}{dp} \cdot \frac{dp}{dt} = \gamma'(p(t)) \cdot p'(t) \tag{8}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} \tag{9}$$

Since the law of energy conservation always exists, with given initial velocity v_0, we have:

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + mgx_3 \tag{10}$$

Then we can derive a differential equation to solve for p(t):

$$v_0^2 - 2g\gamma_3(p(t)) = \left\|\gamma'(p(t))\right\|^2 \cdot (p'(t))^2 \tag{11}$$

Once p(t) is solved, we can know all the velocity and acceleration everywhere on the curve.

2.5.3 Finding the supporting force T_s

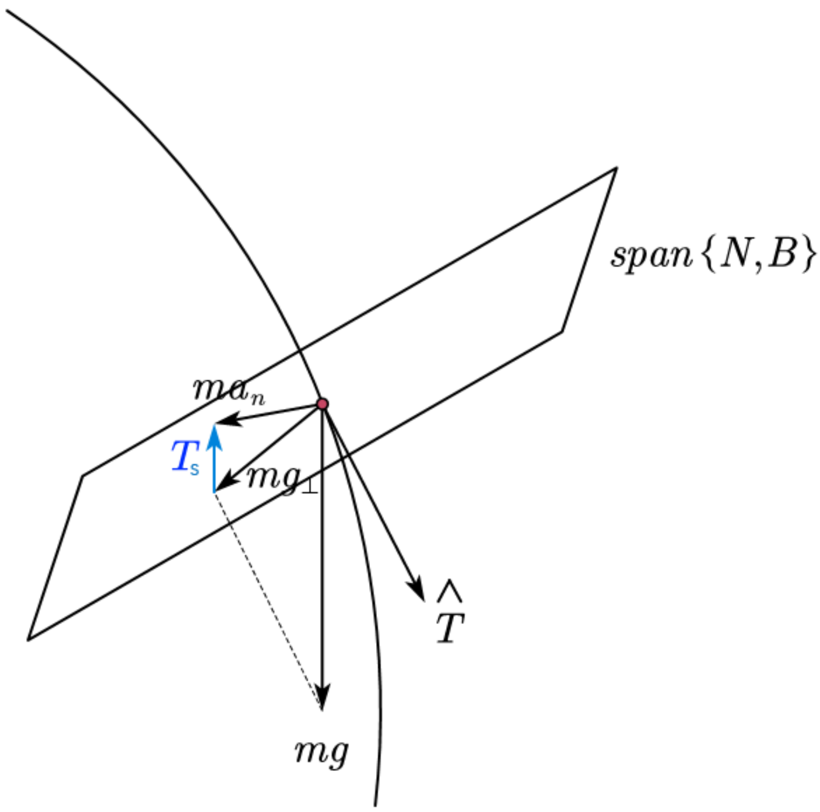

Figure 7: The calculation for the supporting force T_s

As shown in the figure, we know that the track can only provide the passenger a supporting force which is perpendicular to the tangent vector $\hat{T}$(the direction of velocity), i.e, the supporting force T_s must be in the plate spanned by vectors $\hat{N}$ and $\hat{B}$. So we must analysis the free body diagram in this plane. We should find the projection of gravity force in this plane. Also, we should find the centripetal acceleration a_n. By using the math knowledge of curves, we know:
The tangent vector:

$$\hat{T} \circ \gamma(p(t)) = \frac{\gamma'(p(t))}{\|\gamma'(p(t))\|} \tag{12}$$

The normal vector:

$$\hat{N} \circ \gamma(p(t)) = \frac{\left(\hat{T} \circ \gamma\right)' (p(t))}{\left\| \left(\hat{T} \circ \gamma\right)' (p(t)) \right\|} \tag{13}$$

The binormal vector:

$$\hat{B} = \hat{T} \times \hat{N} \tag{14}$$

And the curvature:

$$\kappa \circ \gamma(p(t)) = \frac{\|\gamma'(p(t)) \times \gamma''(p(t))\|}{\|\gamma'(p(t))\|^3} \tag{15}$$

So the radius of curvature:

$$\rho = \frac{1}{\kappa} \tag{16}$$

So, now we can calculate :

$$a_n = \frac{v^2}{\rho} \tag{17}$$

$$\overrightarrow{mg_\perp} = \langle \overrightarrow{mg}, \hat{B} \rangle \cdot \hat{B} + \langle \overrightarrow{mg} \cdot \hat{N} \rangle \cdot \hat{N} \tag{18}$$

$$\overrightarrow{ma_n} = m\frac{v^2}{\rho} \cdot \hat{N} = m\kappa \cdot \gamma(p(t)) \cdot v^2 \cdot \hat{N} \tag{19}$$

And finally, by Newton's law we know that:

$$\overrightarrow{T_s} = \overrightarrow{ma_n} - \overrightarrow{ma_g} \tag{20}$$

2.5.4 Get the acceleration in passenger coordinate system

After we get the direction of supporting force T_s, now we know the direction of the passenger by using the method we explained in the assumption part. We know that the track plane should be perpendicular to the supporting force. Introducing the passenger coordinate system, now we have:

$$\hat{X} = \hat{T} \tag{21}$$

$$\hat{Z} = \frac{\vec{T_s}}{\|\vec{T_s}\|} \tag{22}$$

$$\hat{Y} = \hat{Z} \times \hat{X} \tag{23}$$

The following figure shows the geometrical relationship between these vectors. This figure is from the perspective along the moving direction of the roller coaster.

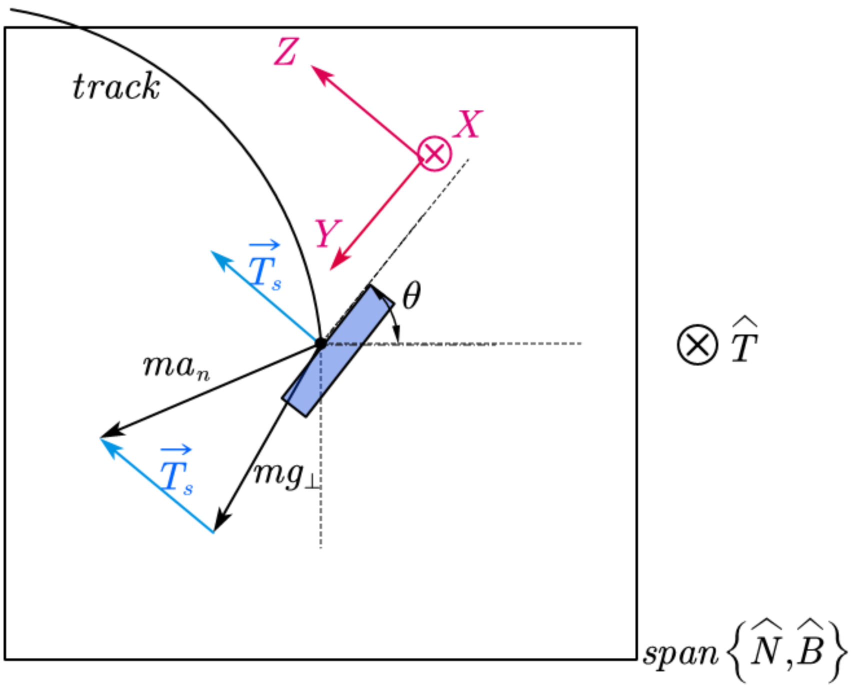

Figure 8: The relationship between the two coordinate systems

Finally, we project the acceleration vector to the passenger coordinate system.

$$a_x = \langle \vec{a}, \hat{X} \rangle \cdot \hat{X} \tag{24}$$

$$a_y = \langle \vec{a}, \hat{Y} \rangle \cdot \hat{Y} \tag{25}$$

$$a_z = \langle \vec{a}, \hat{Z} \rangle \cdot \hat{Z} \tag{26}$$

Then we get the acceleration that the passengers feel in the passenger coordinate system.

2.6 Evaluation

2.6.1 Excitement

Roller coaster physical experience mainly depends on its highest speed, its acceleration, how long overweight and weightlessness last, and how rapid these two states switches. Our model will give velocity, acceleration with respect to time. We aim to vary the physical experience, so that we want the acceleration to vary in direction, magnitude and duration.

2.6.2 Gidiness

Though high acceleration, long term overview and weightlessness and rapid acceleration change can be irritative, it may also cause gidiness and unpleasant experiences. Rapid rotation is the main cause of it. Our model will give rotation with respect to passenger coordinate, so that we can evaluate that.

3 Results

To design an exciting and safe roller coaster, we start with identifying several typical tracks that will provide various acceleration experiences. We then connect them together with elegant smooth curve with the NURBS curve method provided above.

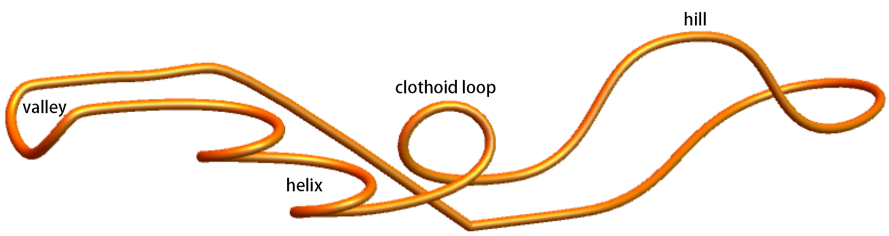

Figure 9: The roller coaster

3.1 Design Parts

Clothoid Loop Clothoid Loop provides a 360 degree overturn with overweight $\rightarrow$ weightlessness $\rightarrow$ overweight experiences. The clothoid loop is often the sailing point of massive colthoid loops. We define the clothoid loop with the following control points an then calculate NURBS curve of 5 degrees.

x_1	x_2	x_3
0	0	0
-2.5	36	0
-5	18	25
-7.5	0	0
-10	36	0

Table 1: The control points for clothoid loop

Then, we use this parametrization, using our model, we can plot out he track of clothoid loop:

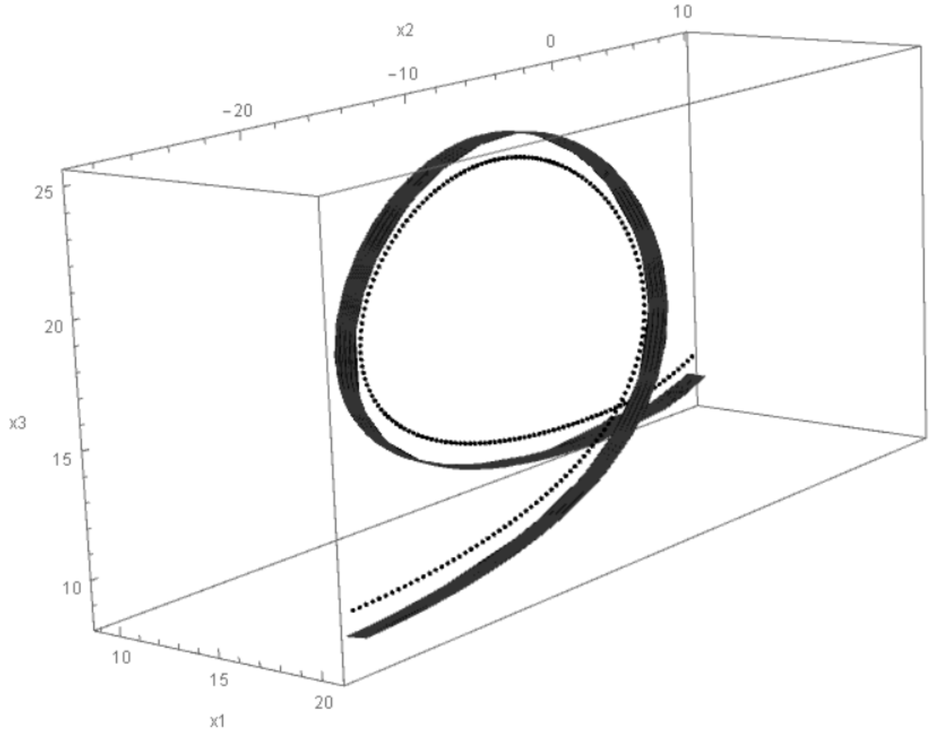

Figure 10: The track of clothoid loop(1)

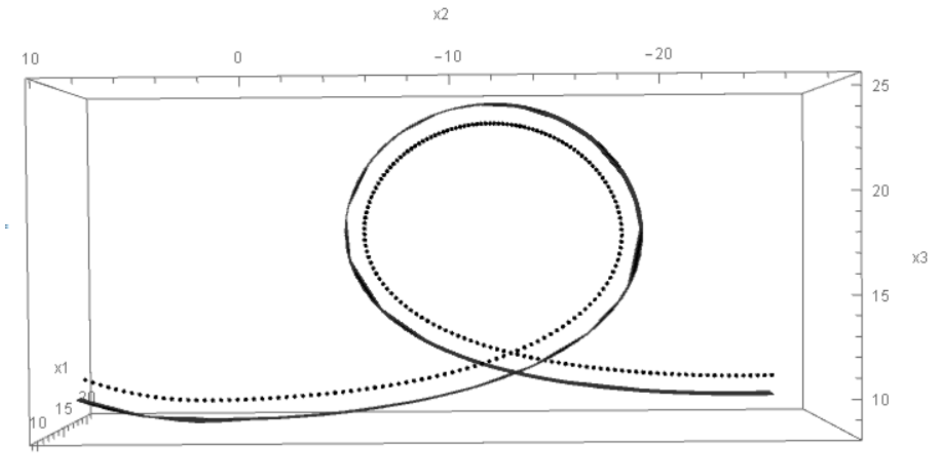

Figure 11: The track of clothoid loop(2)

This is a colthoid of height 13.8 meter. In this figure, the long black belt represent the track, and the black imaginary line represent the heart-line.

The passenger enter the curve at a speed of $19.8702m/s$, and leave the curve at a speed of $19.3913m/s$. The length of the time interval when the passenger go through this colthoid curve is $4.5s$. The following are the figures of the passenger's speed and acceleration when going through the track.

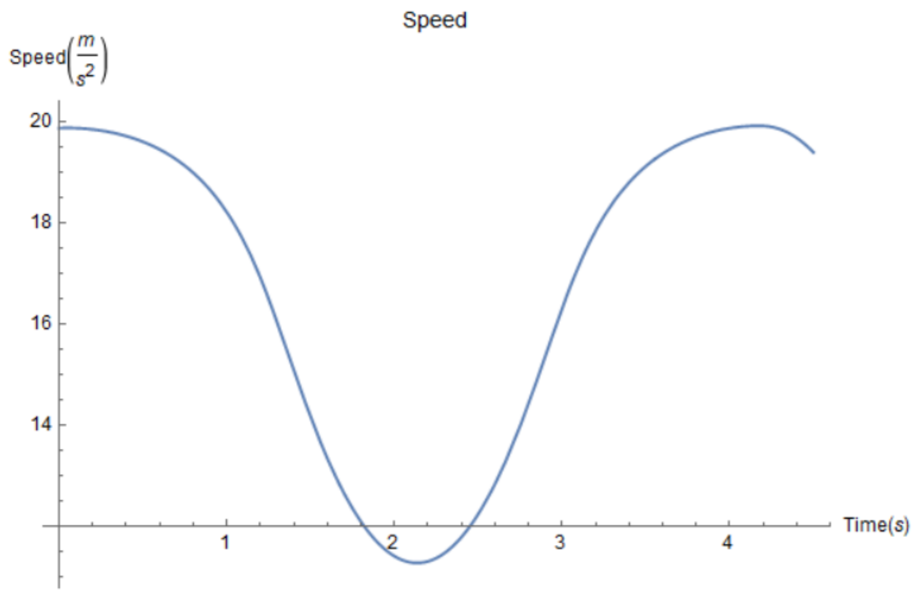

Figure 12: The speed of the passenger

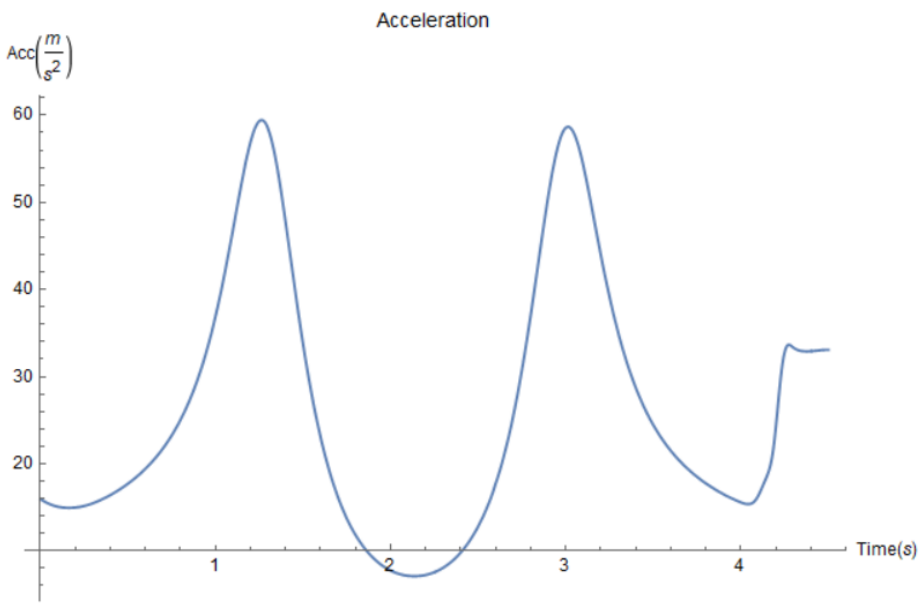

Figure 13: The acceleration of the passenger

In the clothoid, the passenger experiences maximum speed of $59m/s$ and acceleration from $12.6m/s^2 - 19.8m/s^2$. The weightlessness lasts $0s$ while overweight lasts $4.5s$. Also, the following figure shows the acceleration phase diagram in X-Z plane of the passenger coordinate.

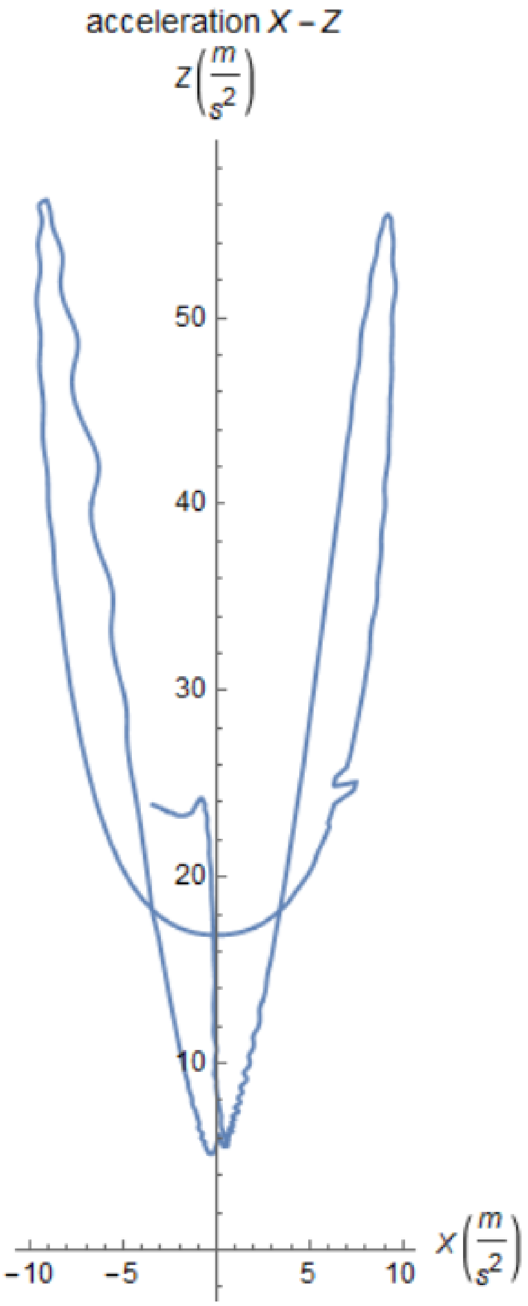

Figure 14: The acceleration phase diagram in X-Z plane

Such experience is quite exciting, while within the acceleration limitation provided by ASTM. We can also have passenger head direction with respect to time so that we know about passenger rotation. The rotation is quite smooth so the passenger won't feel dizzy.

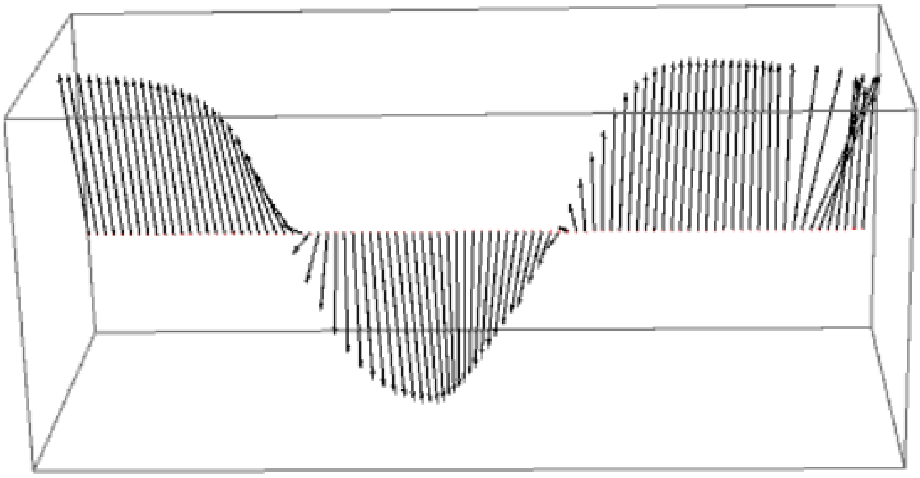

Figure 15: The rotation of the passenger with respect to time

Hill A hill provides a raise followed by a long, exciting drop. The hill we consider is 2D. We define the hill with the following control points an then calculate NURBS curve of 5 degrees.

x_1	x_2	x_3
0	-35	0
0	-17.5	0
0	-15.2174	20
0	0	20
0	15.2174	20
0	17.5	0
0	35	0

Table 2: The control points for hill

Then, we use this parametrization, using our model, we can plot out he track of hill:

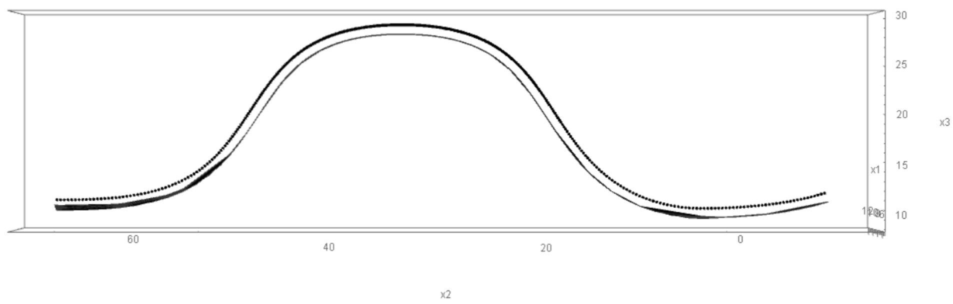

Figure 16: The track of hill

This is a hill of height 17.8 meter. In this figure, the long black belt represent the track, and the black imaginary line represent the heart-line.

The passenger enter the curve at a speed of $19.1189m/s$, and leave the curve at a speed of $19.3433m/s$. The length of the time interval when the passenger go through this hill curve is $9.5s$. The following are the figures of the passenger's speed and acceleration when going through the track.

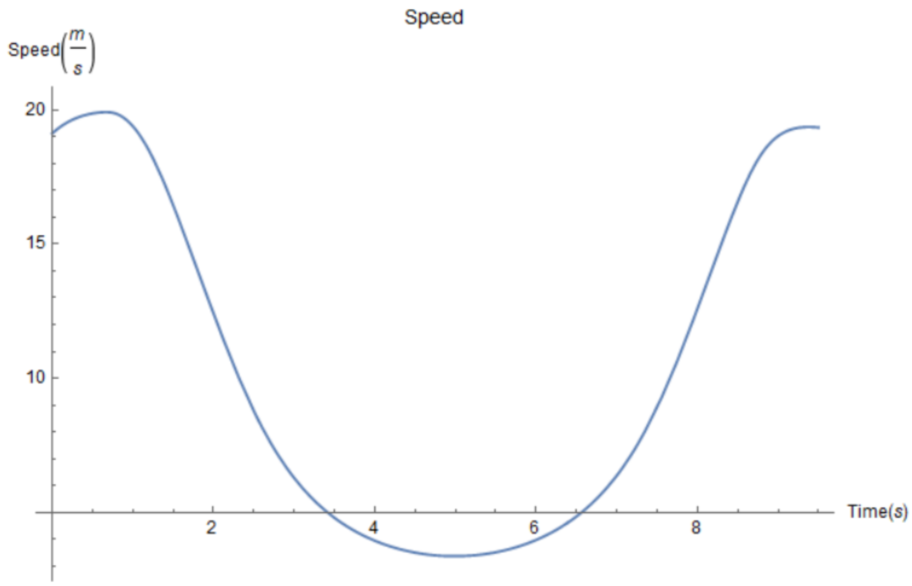

Figure 17: The speed of the passenger

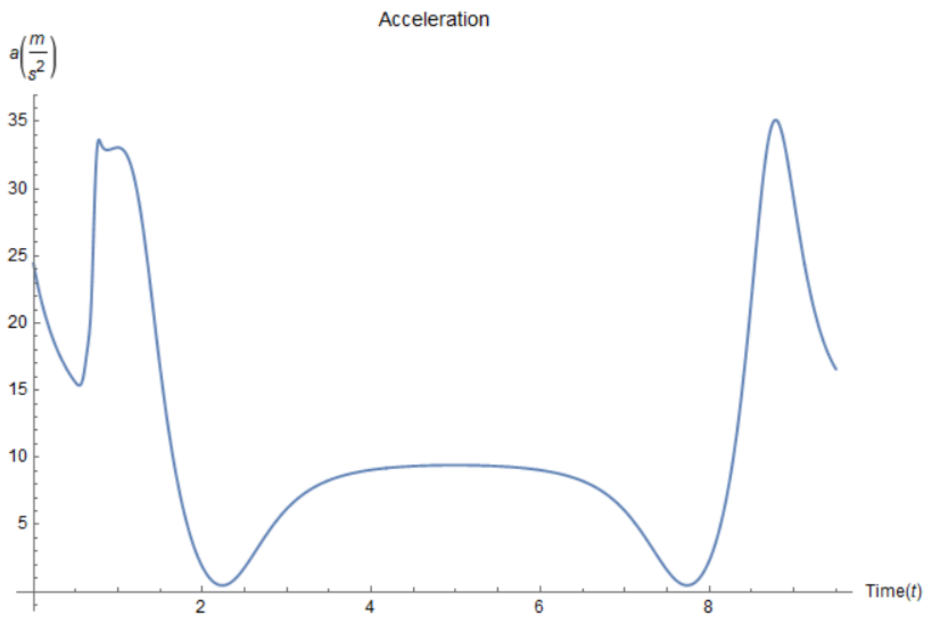

Figure 18: The acceleration of the passenger

In the hill, the passenger experiences maximum speed of $20m/s$ and acceleration from $0m/s^2 - 34m/s^2$. The weightlessness lasts $0.5s$ for two times while overweight lasts $0.5s$ for two times. Also, the following figure shows the acceleration phase diagram in X-Z plane of the passenger coordinate.

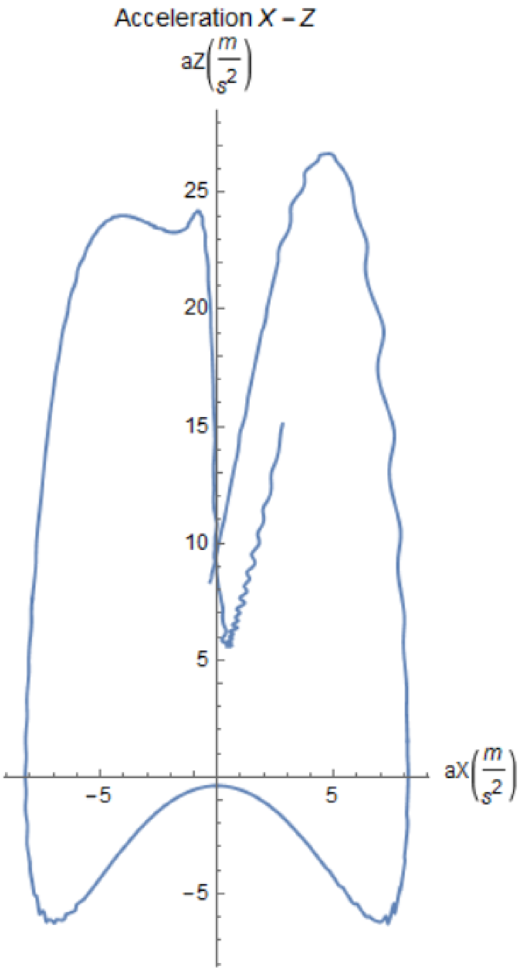

Figure 19: The acceleration phase diagram in X-Z plane

We can also have passenger head direction with respect to time so that we know about passenger rotation.

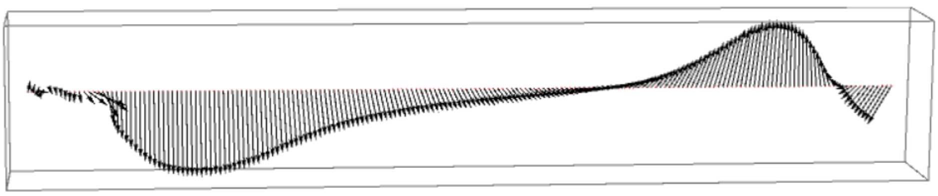

Figure 20: The rotation of the passenger with respect to time

valley A valley starts with a small drop followed by a small raise.In this particular part, we also combine a left turn to enhance excitement. We define the valley with the following control points an then calculate NURBS curve of 5 degrees. The rotation is quite smooth so the passenger won't feel dizzy.

x_1	x_2	x_3
0	10	0
0	5	0
0	3.333333	-0.33333
0	0	-0.33333
0	-3.33333	-0.33333
0	-5	0
0	-10	0

Table 3: The control points for valley

Then, we use this parametrization, using our model, we can plot out he track of valley:

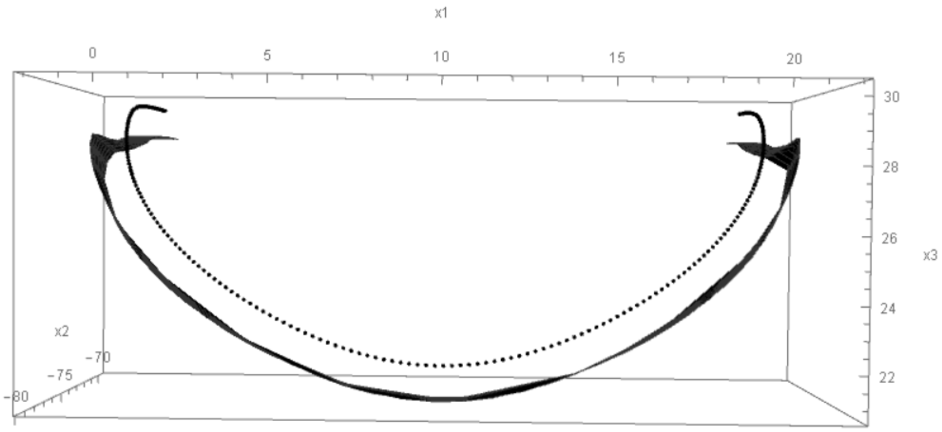

Figure 21: The track of valley(1)

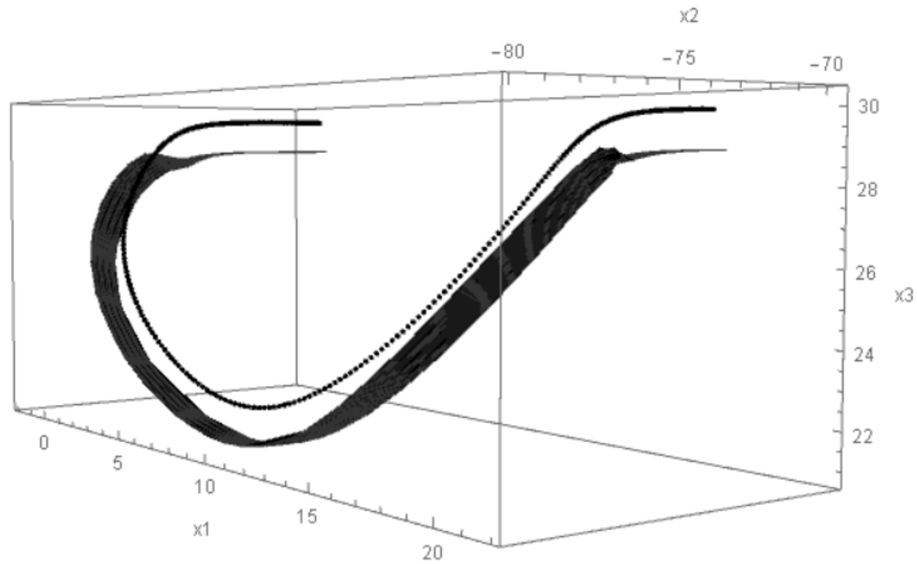

Figure 22: The track of valley(2)

This is a valley of depth 7.8 meter. In this figure, the long black belt represent the track, and the black imaginary line represent the heart-line.

The passenger enter the curve at a speed of $2.20974 m/s$, and leave the curve at a speed of $2.41137 m/s$. The length of the time interval when the passenger go through this valley curve is $7s$. The following are the figures of the passenger's speed and acceleration when going through the track.

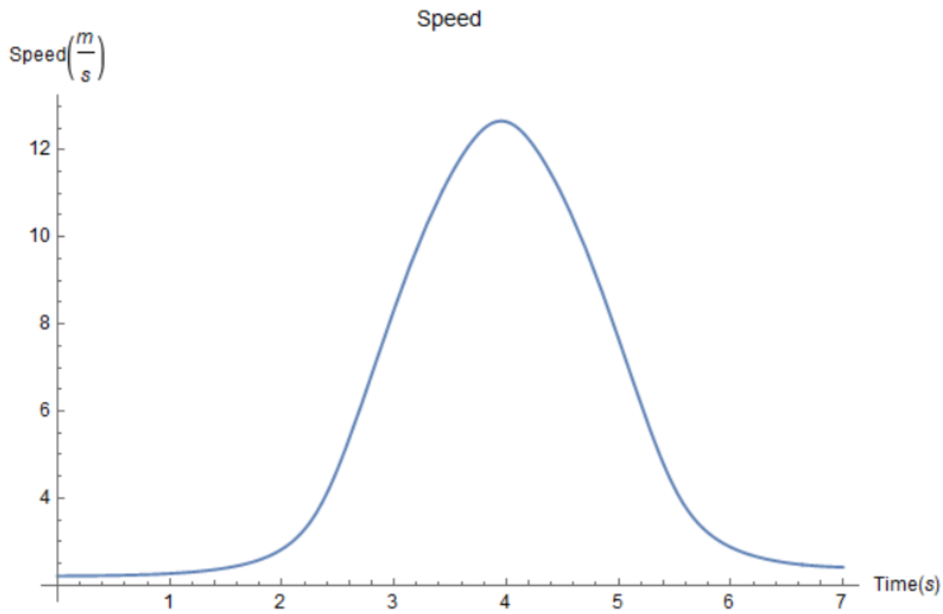

Figure 23: The speed of the passenger

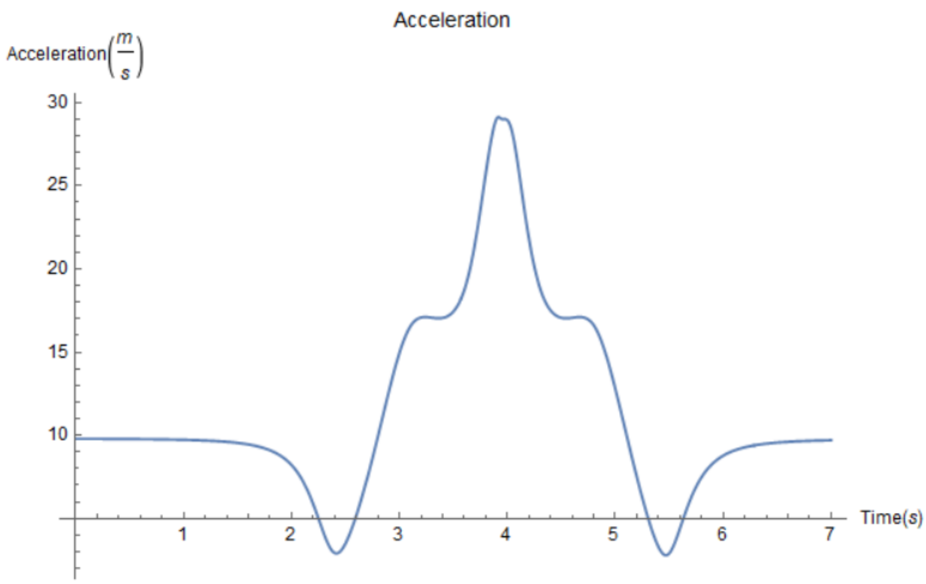

Figure 24: The acceleration of the passenger

In the valley, the passenger experiences maximum speed of $11.7m/s$ and acceleration from $2.5m/s^2-$

$29m/s^2$. The weightlessness lasts $0.5s$ for two times while overweight lasts $2.5s$. Also, the following figure shows the acceleration phase diagram in X-Z plane of the passenger coordinate.

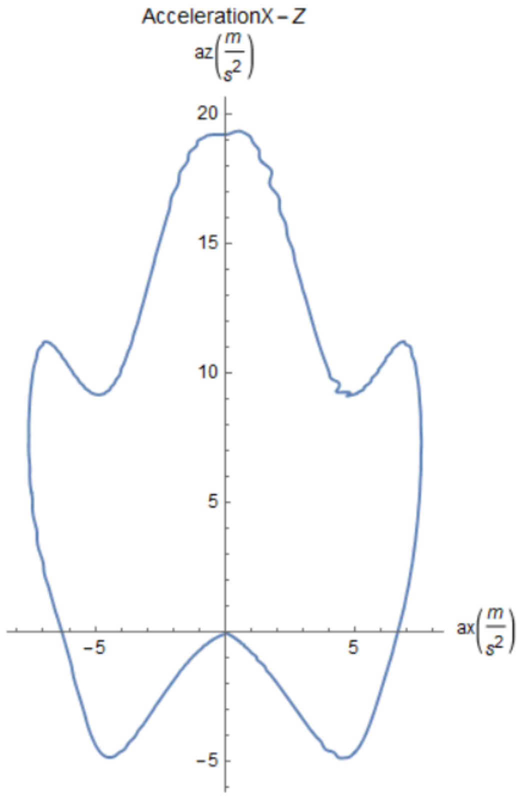

Figure 25: The acceleration phase diagram in X-Z plane

We can also have passenger head direction with respect to time so that we know about passenger rotation. The rotation is quite smooth so the passenger won't feel dizzy.

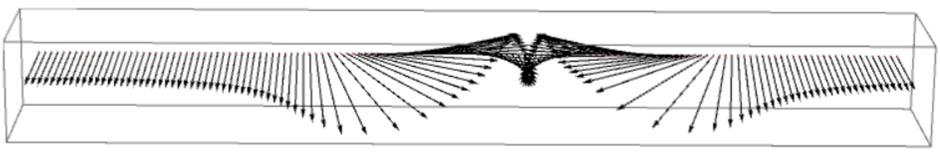

Figure 26: The rotation of the passenger with respect to time

Helix A helix is circular motion combined with vertical and horizontal motion. It will provide large accelerate due to the circular motion and high speed due to decrease of altitude. We define the helix with the following control points an then calculate NURBS curve of 5 degrees. The rotation is quite smooth so the passenger won't feel dizzy.

x_1	x_2	x_3
0	0	0
0	15	-2
15	15	-4
15	0	-6
0	0	-8
0	15	-10

Table 4: The control points for helix

Then, we use this parametrization, using our model, we can plot out he track of helix:

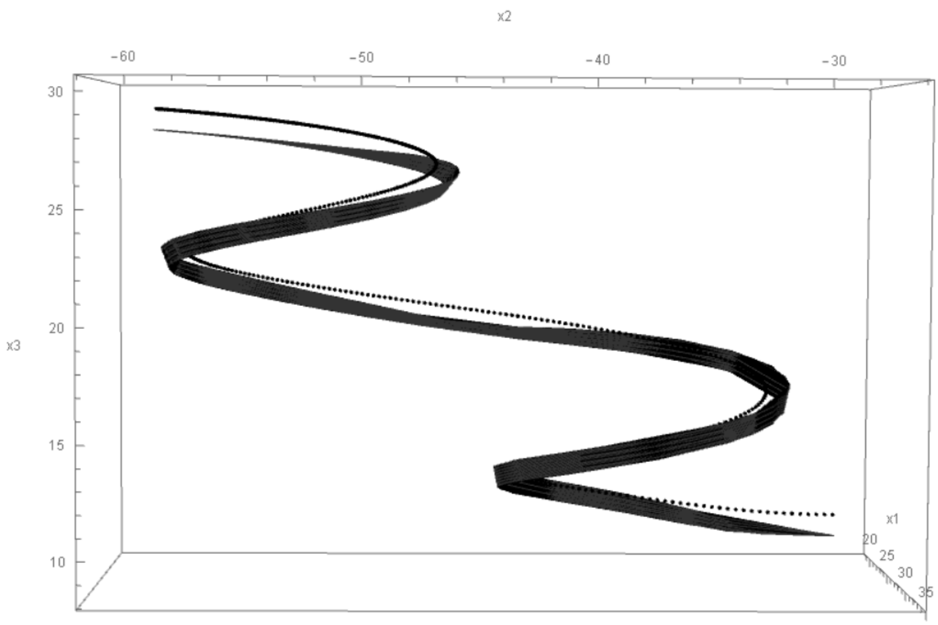

Figure 27: The track of helix

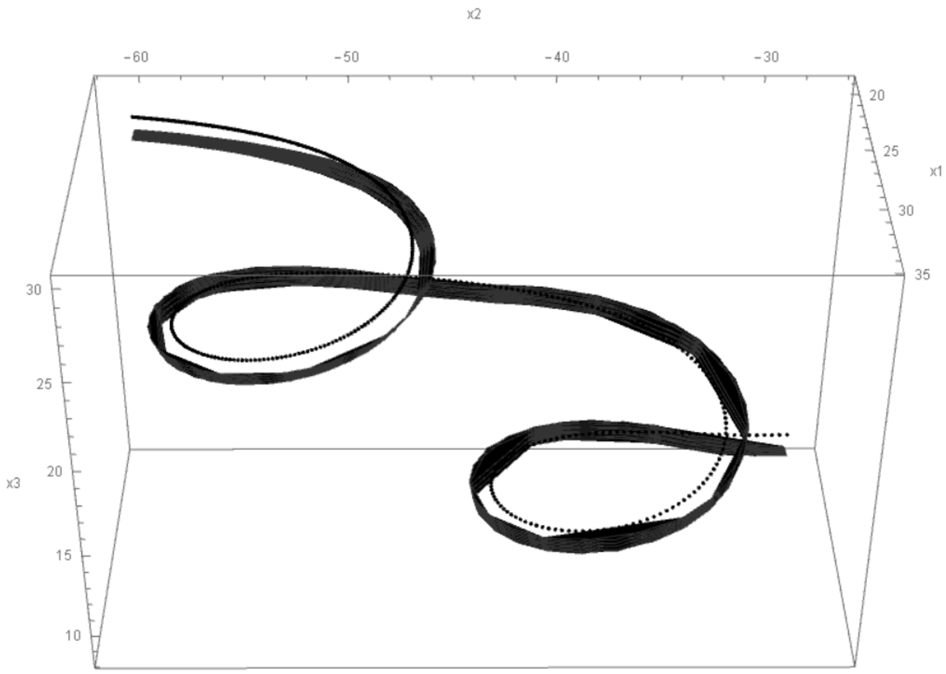

Figure 28: The track of helix(2)

This is a helix of depth 19.5 meter. In this figure, the long black belt represent the track, and the black imaginary line represent the heart-line.

The passenger enter the curve at a speed of $3.70204 m/s$, and leave the curve at a speed of $19.8702 m/s$. The length of the time interval when the passenger go through this helix curve is $9.5s$. The following are the figures of the passenger's speed and acceleration when going through the track.

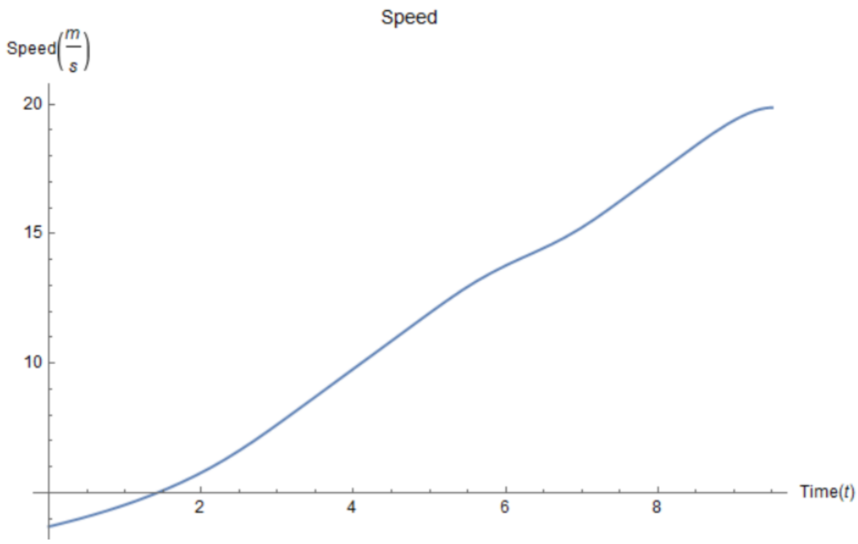

Figure 29: The speed of the passenger

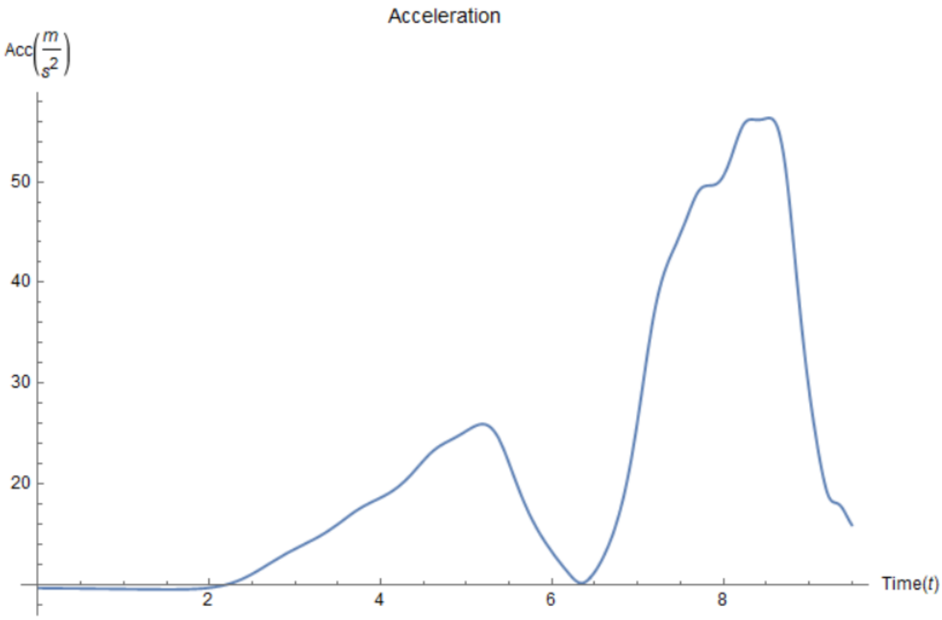

Figure 30: The acceleration of the passenger

In the helix, the passenger experiences maximum speed of $19.8m/s$ and acceleration from $0m/s^2 - 57m/s^2$. The weightlessness lasts $2.2s$ for two times while overweight lasts $3.6s$. Also, the following figure shows the acceleration phase diagram in X-Z plane of the passenger coordinate.

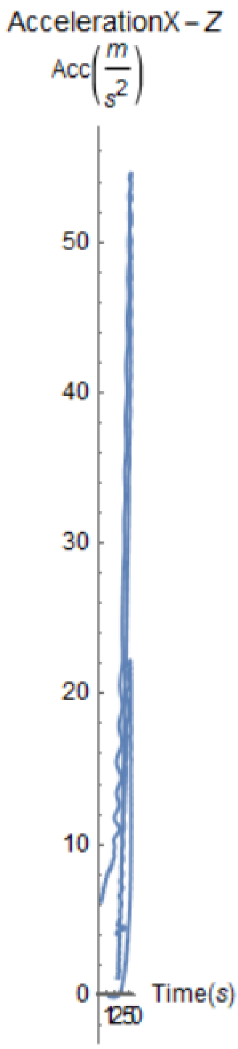

Figure 31: The acceleration phase diagram in X-Z plane

We can also have passenger head direction with respect to time so that we know about passenger rotation. The rotation is quite smooth so the passenger won't feel dizzy.

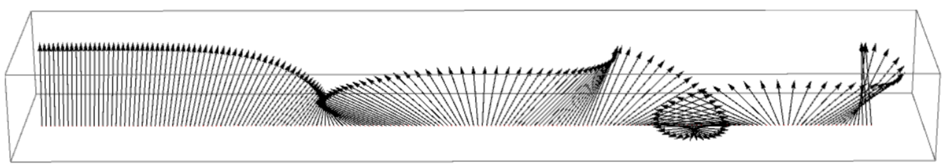

Figure 32: The rotation of the passenger with respect to time

3.2 Roller Coaster Design

To build a roller coaster that provides a continuous experience, we want to connect these parts together smoothly. We combine the control points mentioned above together with some additional points to make sure the loop connects without unreasonable acceleration. Fig. 33 is what we got for the decline part of roller coaster track.

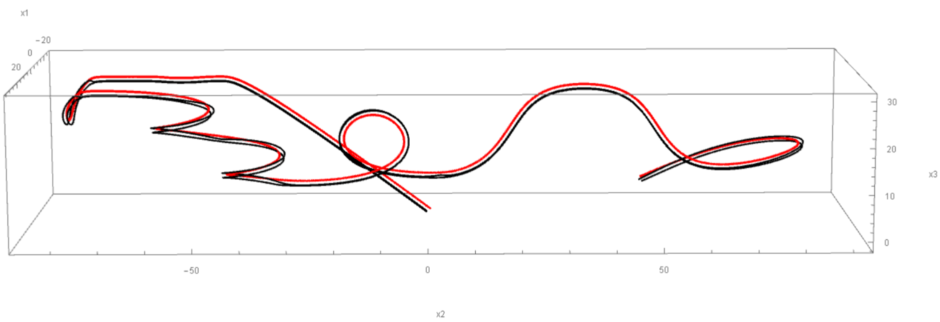

Figure 33: The track of the roller coaster

We plug that directly into our model and get the corresponding acceleration and velocity that passenger experience.

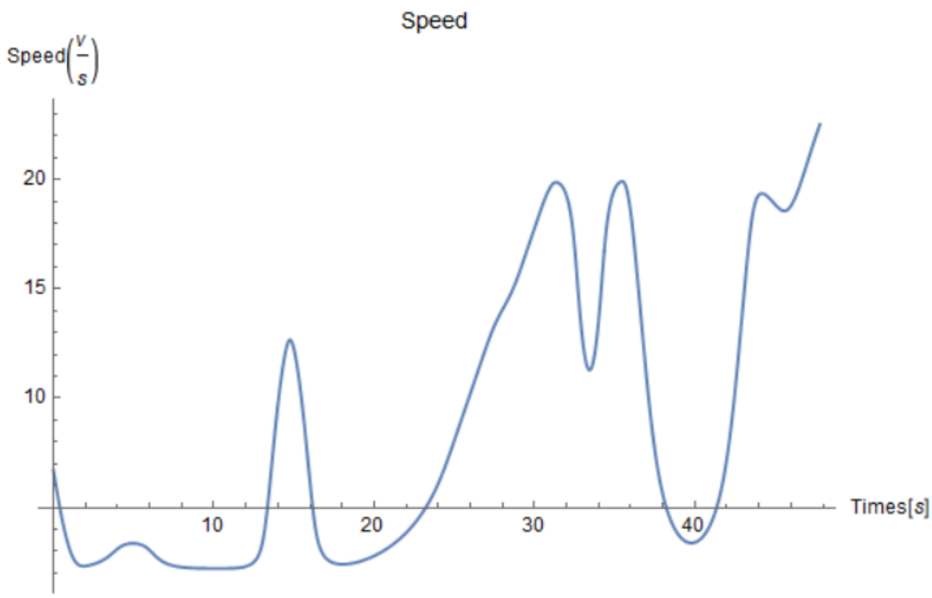

Figure 34: The speed of the passenger

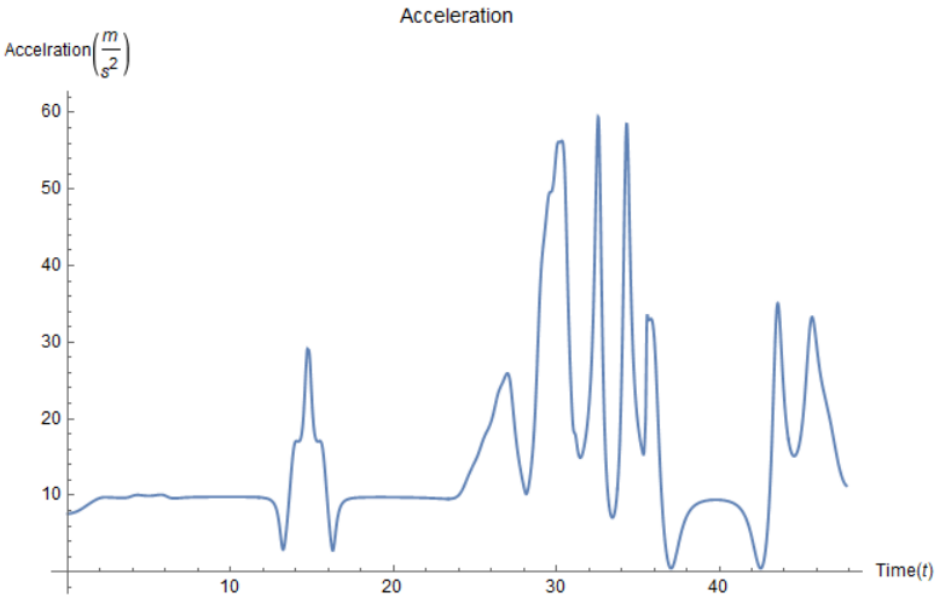

Figure 35: The acceleration of the passenger

If we combine these two figures with the heart-line of the passenger, we have see the acceleration and speed he experience visually.

Figure 36: The acceleration of the passenger (green means low and pink means high)

Figure 37: The velocity of the passenger (blue means low and yellow means high)

The lift will take 30 seconds to the cart to the top. Starting with a high speed by ejecting, the roller coaster trip after the initial lift starts with a relatively gentle valley, combined with a small turn. Then the trolley enters the helix curve, which declines the altitude and provides small, followed by a the first acceleration peak at the second loop. The overweight experience last more than 2 seconds. The train soon enters the clothoid loop at a high initial speed. The passenger will experience overweight → weightlessness → overweight rapid acceleration change. The high hill follow with a long term weightlessness that serves as the highlight of this roller coaster experience. The final round turn provides centrifuge experience, bring the loop back down to the decelerate section and give the passengers some time to slow down their heart rate. In the end, with a high acceleration, when the trip ends at the initial point, the speed reduces to zero. The whole ride will take only 79.4s, with 49.4 s of it on exciting decline and breaking.

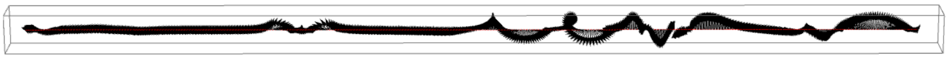

Figure 38: Rotation of the passenger

During the whole ride, the acceleration is controlled below 6g, within the safety range suggested by ASTM but has some peaks to boost up the excitement. The rotation plot (time-eyesup direction) diagram Fig. 38 is still smooth to reduce the possibility of terrible dizziness.

4 Discussion

Our model assumed that on the decline curve, the track only provides supportive force parallel to the eyes up direction of the passenger. We used NURBS curves to generate smooth trail of passenger's heart with several control point. We then calculate the track curve and roll angel. The overall roller coaster curve we generated at last is perfectly smooth, providing high enough but safe acceleration on the passenger with the rotation under control.

4.1 Strength

The highlight of this model is that it is very general. The designer only need to input control points and the model will automatically generate the track curve and return simulation results. The designer can then use these data to evaluate the design and adjust the initial control point, until they get a satisfying result.
Moreover, though we didn't consider friction in this case to simplify the calculation, it can be easily added to the model by replacing the energy conservation equation with Newton's 2rd Law.
Our method to generate the curve ensured that there will be no eye left or eye right acceleration (along Y axis, minimizing the wear of the track. Such characteristic allows our method to be used in special roller coaster designs, for example ice tracks which can merely provide any friction.

5 Limitations

However, our model doesn't allow super weightlessness since we assumed that the track only provide supportive force, which means such curves will cause the cart to fly off the curve. However, improved roller coaster carts can grab on the track, making super weightlessness possible. In that case, our model will fail.

Moreover, due to the huge amount of calculation needed to solve the rotation, though we've got simulation result for roll angle, our limited computational power restricted us from solving for rotation.

6 Conclusion

In this paper, we provides a general method for roller coaster design and simulation. According to that, we designe Our method can provide a great design of the roller coaster track and analysis it thoroughly. Also, we can use the indicators like excitement and gidiness to evaluate the track just by its equation. The track shown in this paper is an example that can provide the high excitement and low gidiness. However, generally, this method can be applied to judge any track.

References

[1] A.M. Pendrill and H. Rödjegard, "A rollercoaster viewed through motion tracker data," *Phys. Educ.*, vol. 40, no. 6, pp. 522–526, Nov. 2005.

[2] A. M. Eiband, "Human Tolerance to Rapidly Applied Accelerations: A Summary of the Literature," Jun. 1959.

[3] ASTM F2291-2006: Standard Practice for Design of Amusement Rides and Devices as required by: State of Indiana, 685 IAC 1-2-9.

[4] G. C. Duran and M. S. G. Tsuzuki, "Procedural Method for Finding Roller Coaster Rails Centerlines Based on Heart-Line Acceleration Criteria," *2018 13th IEEE International Conference on Industry Applications (INDUSCON)*, São Paulo, Brazil, 2018, pp. 1341-1348. doi: 10.1109/INDUSCON.2018.8627334

[5] J. Pombo and J. Ambrosio, "Modelling tracks for roller coaster dynamics," *International Journal of Vehicle Design*, vol. 45, no. 4, pp. 470–500, Jan. 2007.

[6] Wolfram Research, Inc., Mathematica, Version 12.0, Champaign, IL (2019).

A App