VP160 Recitation Class V

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Work

Definition

$$\delta W = \vec{F} \circ d\vec{r} \tag{1}$$

$$W_{AB} = \int_{\Gamma_{AB}} \vec{F} \circ d\vec{r} \tag{2}$$

In general, $\vec{F} = \vec{F}(\vec{r})$ (position-dependent force; vector field)

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$$W_{AB} = \int_{\Gamma_{AB}} \vec{F} \circ d\vec{r} = \int_{\Gamma_{AB}} \begin{pmatrix} F_x(x,y,z) \\ F_y(x,y,z) \\ F_z(x,y,z) \end{pmatrix} \circ \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \tag{2}$$

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Methods for Calculation

1. Constant force on a straight line
2. Varying force on a straight line (Integral)
3. Varying force on a curve (Line integral)

Exercise 1

Find work done by the force $\mathbf{F_1}(x, y) = -x\hat{n}_x - y\hat{n}_y$ and by the force $\mathbf{F_2}(x, y) = (2xy + y)\hat{n}_x + (x^2 + 1)\hat{n}_y$ if a particle is being moved from $(-1, 0)$ to $(0, 1)$ along

(a) the straight line connecting these points,

(b) the (shorter) arc of the circle $x^2 + y^2 = 1$,

(c) the axes of the Cartesian coordinate system: first from $(-1, 0)$ to $(0, 0)$ along the x axis, then from $(0, 0)$ to $(0, 1)$ along the y axis.

$\text{rot}\, \vec{F_1} = 0 \Rightarrow F_1$ is a conservative force

so for (a) (b) (c) works are the same

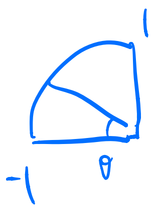
$$\int_T \vec{F_1} \circ d\vec{l} = \int_T \begin{pmatrix} -x \\ -y \end{pmatrix} \circ \begin{pmatrix} dx \\ dy \end{pmatrix} = \int_{-1}^{0} -x\,dx + \int_0^1 -y\,dy = 0$$

$\text{rot}\, \vec{F_2} = 2x+1 - 2x = 1 \neq 0 \Rightarrow F_2$ is non-conservative

(a) $\begin{cases} y = t+1 \\ x = t \end{cases} \Rightarrow \begin{cases} dy = dt \\ dx = dt \end{cases}$

$$\int_{T_a} \vec{F_2} \circ d\vec{l} = \int_T \begin{pmatrix} 2t(t+1) + t+1 \\ t^2+1 \end{pmatrix} \circ \begin{pmatrix} dt \\ dt \end{pmatrix} = \int_{-1}^{0} (3t^2+3t+2)\,dt = 1.5\,J$$

(b) $\begin{cases} y = \sin\theta \\ x = -\cos\theta \end{cases} \Rightarrow \begin{cases} dy = \cos\theta\, d\theta \\ dx = -\sin\theta\, d\theta \end{cases}$



$$\int_{T_b} \vec{F_2} \circ d\vec{l} = \int_T \begin{pmatrix} -2\sin\theta\cos\theta + \sin\theta \\ \cos^2\theta + 1 \end{pmatrix} \begin{pmatrix} -\sin\theta\, d\theta \\ \cos\theta\, d\theta \end{pmatrix} = \int_0^{\frac{\pi}{2}} (2\sin^2\theta\cos\theta - \sin^2\theta + \cos^3\theta + \cos\theta)\,d\theta$$

$$= 1.548\,J$$

(c) $\int_{-1}^{0} F_{2x}\,dx + \int_0^1 F_{2y}\,dy$

$$= \int_{-1}^{0} (2xy+y)\big|_{y=0}\,dx + \int_0^1 (x^2+1)\big|_{x=0}\,dy$$

$$= 1\,J$$

Recall that $\delta W = \vec{F} \circ d\vec{r}$, so the rate of work being done by the net force on a particle

$$\frac{\delta W}{dt} = \vec{F} \circ \frac{d\vec{r}}{dt} = \vec{F} \circ \vec{v} = m\dot{\vec{v}} \circ \vec{v} = \frac{d}{dt}(\frac{1}{2}mv^2)$$

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- Average Power:$P_{av} = \frac{W}{\Delta t}$
- Instantaneous power:$P_{ins} = \frac{\delta W}{dt} = \vec{F} \circ \vec{v}$

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2D: $\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} = 0$

$$\text{rot}\,\vec{F} = \begin{vmatrix} \hat{n}_x & \hat{n}_y & \hat{n}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \hat{n}_x \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \hat{n}_y \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \hat{n}_z \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) = 0$$

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  - Relation with Potential Energy:

$$\vec{F} = -\bigtriangledown U = -\frac{\partial U}{\partial x}\hat{n}_x - \frac{\partial U}{\partial y}\hat{n}_y - \frac{\partial U}{\partial z}\hat{n}_z$$

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- Non-conservative force? Collision, friction force

Exercise 2

Consider a 3D force

$$\vec{F} = \begin{pmatrix} 6xyz^2 + 3y^2 \\ 3x^2z^2 + 6xy - z \\ 6x^2yz - y \end{pmatrix}$$

Is it a conservative force? How to find the corresponding potential energy?

① $\nabla \times \vec{F} = \begin{vmatrix} \hat{n}_x & \hat{n}_y & \hat{n}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{pmatrix} 6x^2z-1-(6x^2z-1) \\ -(12yzx - 12xyz) \\ 6xz^2+6y-(6xz^2+6y) \end{pmatrix} = \vec{0}$

So it's conservative.

② Then what's the U?

$\frac{\partial U}{\partial x} = -F_x = -(6xyz^2+3y^2) \Rightarrow U = -3x^2yz^2 - 3xy^2 + C_1(y,z)$

$\frac{\partial U}{\partial y} = -F_y = z - 3x^2z^2 - 6xy \Rightarrow U = yz - 3x^2yz^2 - 3xy^2 + C_2(z)$

$\underbrace{C_1(y,z) = yz + C(z)}$

$\frac{\partial U}{\partial z} = -F_z = y - 6x^2yz \Rightarrow U = yz - 3x^2yz^2 + C(x,y)$

So $U = yz - 3x^2yz^2 - 3xy^2 + C$

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The work done by the net force (by **Work-Kinetic Energy Theorem**):

$$\delta W = \vec{F} \circ d\vec{r} = \vec{F}_{con} \circ d\vec{r} + \vec{F}_{\text{n-cons}} \circ d\vec{r} = dK$$

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and the total work done from A to B

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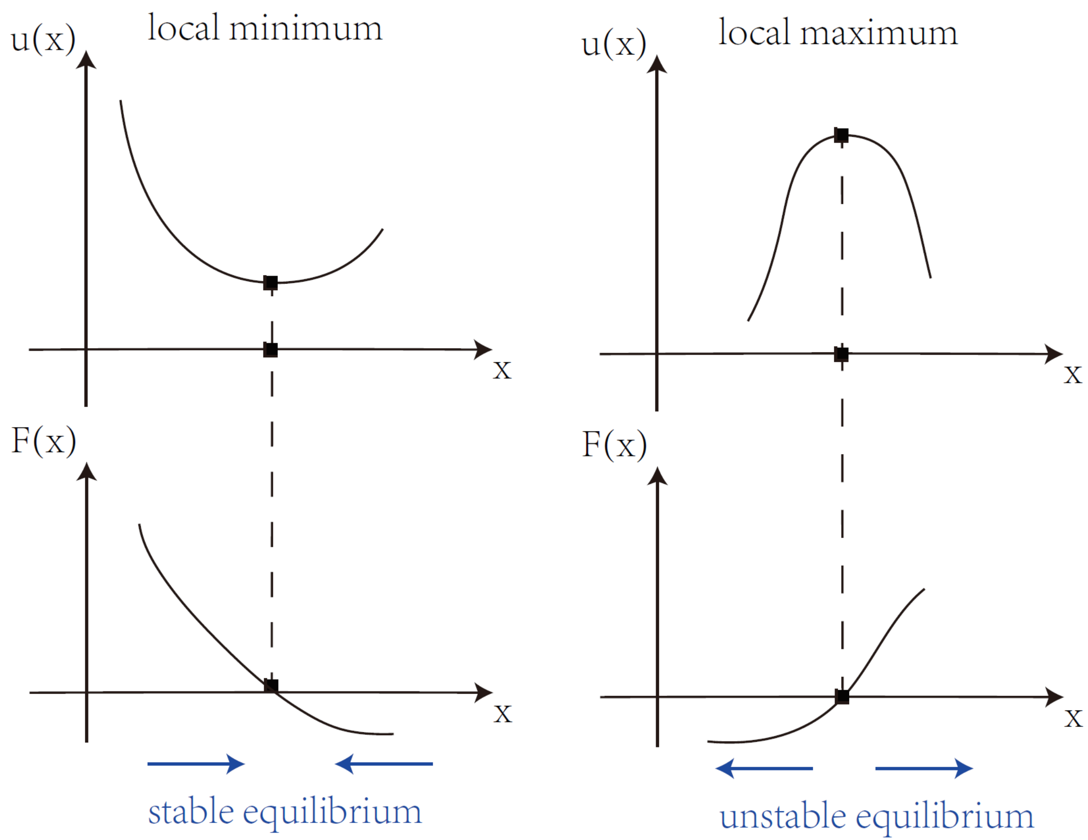
and we finally get the **Law of Conservation of the Total Energy**:

$$\mathbf{\Delta K + \Delta U + \Delta U_{int}} = 0$$

Energy Diagram



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