VP160 Recitation Class IV

Angular Momentum & Rigid Body Dynamics Part I

Zeyi Ren

UM-SJTU Joint Institute

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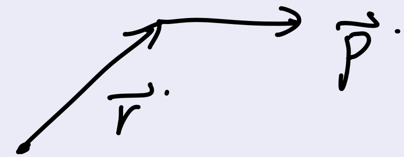
Angular Momentum

$$\vec{L} = \vec{r} \times \vec{p}$$

$$[kg \cdot m^2 / s]$$

r



Angular Momentum

$$\vec{L} = \vec{r} \times \vec{p}$$

$$[kg \cdot m^2 / s]$$

How to derive?

$$\vec{F} = \frac{d\vec{p}}{dt} = m \cdot a + \frac{dm}{dt} \cdot v$$

Angular Momentum

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$$[kg \cdot m^2 / s]$$

How to derive?

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$\Rightarrow \vec{r} \times \vec{F} = \boxed{\vec{r} \times \frac{d\vec{p}}{dt}} = \boxed{\frac{d}{dt}(\vec{r} \times \vec{p}) - \frac{d\vec{r}}{dt} \times \vec{p}}$$

Angular Momentum

$$\vec{L} = \vec{r} \times \vec{p}$$

$$[kg \cdot m^2 / s]$$

How to derive?

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$\Rightarrow \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) - \underbrace{\frac{d\vec{r}}{dt} \times \vec{p}}_{\shortparallel\ 0.}$$

Notice $\boxed{\frac{d\vec{r}}{dt} \times \vec{p}} = \vec{v} \times (m\vec{v}) = 0$

$\underline{\vec{v} \times m(\vec{v})}$

Angular Momentum

$$\vec{L} = \vec{r} \times \vec{p}$$

$$[kg \cdot m^2 / s]$$

How to derive?

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$\Rightarrow \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) - \frac{d\vec{r}}{dt} \times \vec{p}$$

$$\text{Notice } \frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times (m\vec{v}) = 0$$

$M = \vec{r} \times \vec{F}$

$M = \frac{d\vec{L}}{dt}$

$$\Rightarrow \underbrace{\vec{r} \times \vec{F}}_{\vec{\tau}} = \frac{d}{dt}(\underbrace{\vec{r} \times \vec{p}}_{\vec{L}})$$

$\vec{\tau}$: torque

Angular Momentum Theorem

$$\vec{\tau} = \frac{d\vec{L}}{dt} \Rightarrow \vec{L}(t_2) - \vec{L}(t_1) = \int_{t_1}^{t_2} \vec{\tau}\, dt$$

Angular Momentum Theorem

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$L(t) = L(0)$

Law of Conservation of Angular Momentum

If $\vec{\tau} = 0 \Rightarrow \vec{L} = \text{const}$

$\vec{F}$ $\hat{r}$

$\underline{\vec{r} \times \vec{F}}$

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Applications:

- Central force field ($\vec{\tau} = \vec{r} \times \vec{F} = 0$)

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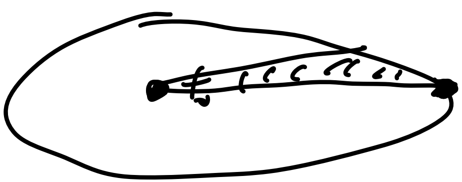
Applications:

- Central force field ($\vec{\tau} = \vec{r} \times \vec{F} = 0$)
- Aerial velocity, e.g. motion of planets, Kepler's Second Laws



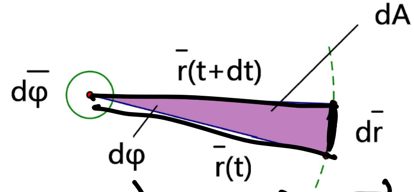
$\boxed{\delta t}$

For planer motion, the **aerial velocity** may be defined

dt

$\frac{dA}{dt} = \frac{1}{2} \left| \vec{r} \times \frac{d\vec{r}}{dt} \right|$

$= \frac{1}{2} \left| \vec{r} \times \vec{v} \right|$



$\vec{L} = \vec{r} \times \vec{p}$

$\vec{L} = \text{const}$

$\vec{r} \times \vec{v}$

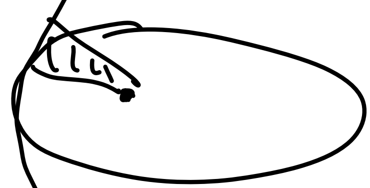
The surface area swept by $\bar{r}$ over the time dt is $dA = \left|\frac{1}{2}\bar{r} \times d\bar{r}\right|$ and the rate of change of that area

$$\sigma = \frac{dA}{dt} = \frac{1}{2}\left|\bar{r} \times \frac{d\bar{r}}{dt}\right| = \frac{1}{2}\left|\bar{r} \times \bar{v}\right|.$$

$\vec{F} = -\frac{GMm}{r^3}\vec{r}$

Aerial velocity vector (direction — right-hand rule)

$$\boxed{\bar{\sigma} = \frac{1}{2}(\bar{r} \times \bar{v})} \qquad \text{(direction same as } d\bar{\varphi}\text{)}$$



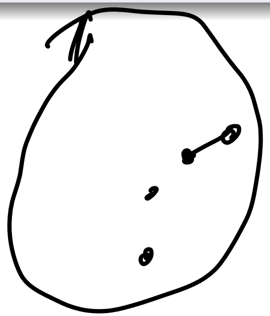
Recall: $\bar{L} = \bar{r} \times \bar{p} = \underbrace{\bar{r} \times m\bar{v}}_{=\bar{\sigma}\cdot 2m}$. Hence $\bar{L} = const \Leftrightarrow \bar{\sigma} = const$.

Consequently, for motion in a central force field $\bar{\sigma} = \text{const}$.

Angular Momentum in System of Particles

Conservation of the Angular Momentum Law

If the net torque of external forces on a system of particles is equal to zero, then the total angular momentum of that system is conserved.



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$$\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}} = \vec{\tau}_{ext} + \underbrace{\vec{\tau}_{int}}_{=0} = 0$$

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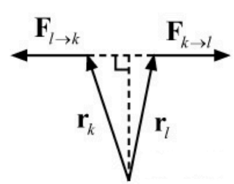
$$\frac{d\vec{L}}{dt} = \vec{\tau} = \vec{\tau_{ext}} + \underbrace{\vec{\tau_{int}}}_{=0}$$

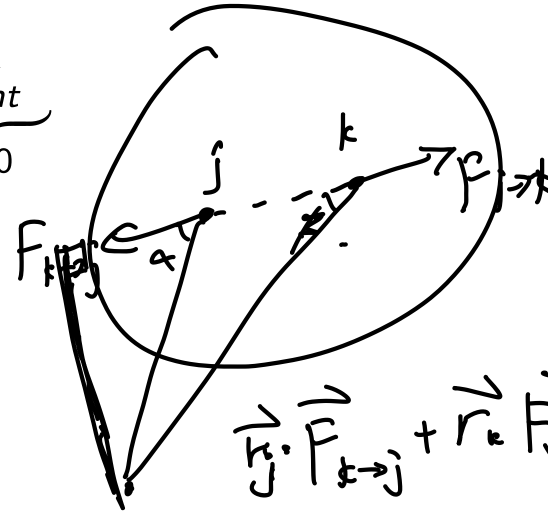
$\tau = 0$

Why $\vec{\tau_{int}} = 0$?

For any two particles k, l in the system,

$$\vec{\tau_{k\to l}} = -\vec{\tau_{l\to k}}$$





$\vec{r_j} \cdot \vec{F}_{k\to j} + \vec{r_k} \vec{F}_{j\to k}$
$= 0$

Applications:

Use with other conservation laws:

- Conservation of Energy
- Conservation of Momentum

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$-\frac{\partial U}{\partial r} = F(r) = \alpha r$.

$\Rightarrow U(r) = C - \frac{1}{2}\alpha r^2 = U(0) - \frac{1}{2}\alpha r^2$

$U(r)$

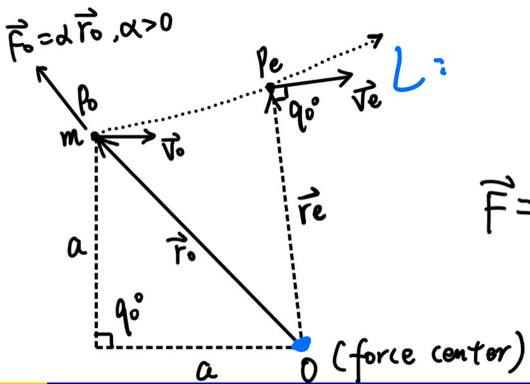
Exercise 1

A particle with mass m is put into a force field $\vec{F} = \alpha \vec{r}$, where α is a positive constant. The particle's initial velocity is $\vec{v_0}$ and its initial position is P_0, when it moves to the position P_e, the instantaneous velocity $\vec{v_e}$ is orthogonal to its radius vector $\vec{r_e}$. Take $\alpha = \frac{mv_0^2}{4a^2}$ and calculate the value of $\frac{|\vec{v_e}|}{|\vec{v_0}|}$.

$\vec{r} \times \vec{F} = 0.$

E: $\frac{1}{2} m v_0^2 - \frac{1}{2}\alpha r_0^2 = \frac{1}{2} m V_e^2 - \frac{1}{2}\alpha r_e^2$

L: $m V_0 r_0 \frac{\sqrt{2}}{2} = m V_e \cdot r_e.$

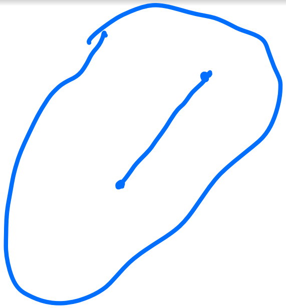


$\vec{F} = \alpha \vec{r}$

$\frac{|\vec{V_e}|}{|\vec{V_0}|} = \frac{1}{2}\sqrt{1+\sqrt{5}}$

Rigid Body

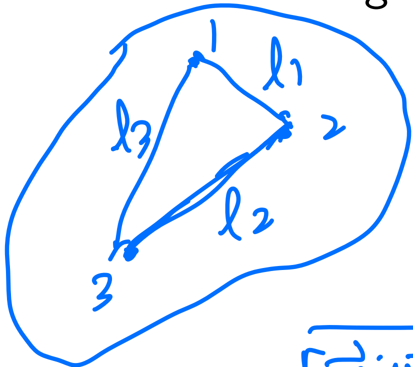
A body is called rigid if $|\vec{r}_i - \vec{r}_j| = \text{const}$ for any point i, j in the body.



Rigid Body

A body is called rigid if $|\vec{r}_i - \vec{r}_j| =$ const for any point i, j in the body.

Degree of freedom of a rigid body?



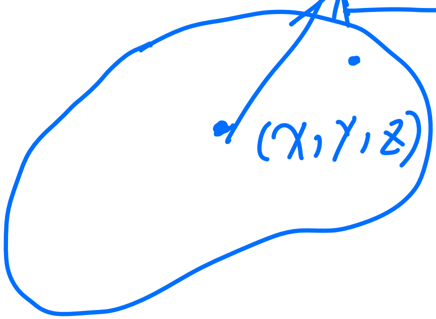
$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = l_1^2$$
$$= l_2^2$$
$$= l_3^2$$

$3 \times 3 - 3 = 6.$

$\boxed{\vec{\omega} \times \vec{r_i}} + \vec{v_i}$

3.

$3 + 3 = 6$



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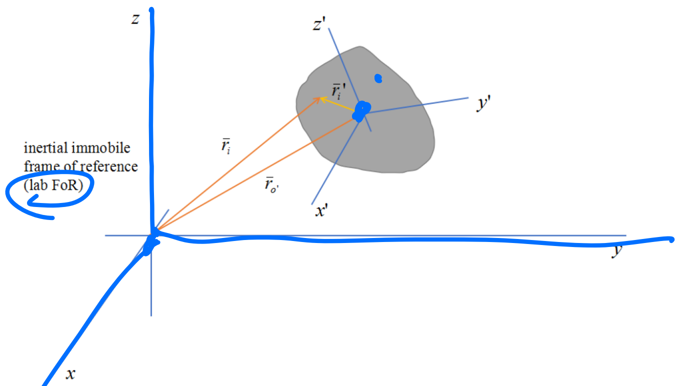


FoR associated with the rigid body is, in general, non-inertial —the body can move arbitrarily.

O' — a point of the body

We have (see the derivation of dynamics in non-inertial FoRs)

$$\boxed{\overline{r}_i = \overline{r}_{O'} + \overline{r}_i',}$$
$$\overline{v}_i = \boxed{\overline{v}_{O'}} + \underbrace{\overline{v}_i'}_{=0} + \boxed{\overline{\omega} \times \overline{r}_i',}$$

$\omega \times \vec{r}_i'$

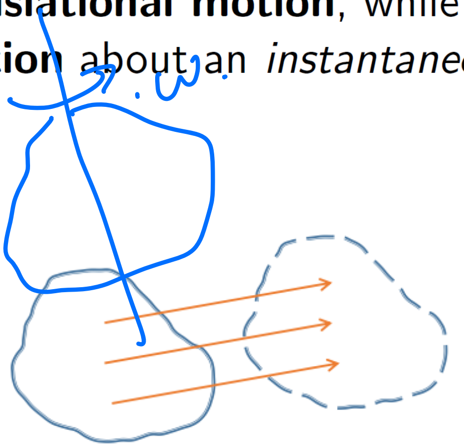
where $\overline{v}_i' = 0$ due to the fact that the body is rigid (no relative motion of the rigid body's points).

Eventually, the velocity of any point of a rigid body

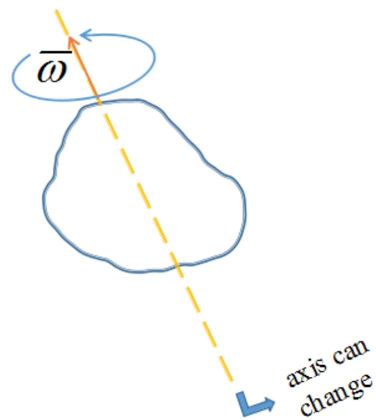
$$\overline{v}_i = \overline{v}_{O'} + \overline{\omega} \times \overline{r}'_i.$$

The first term on the right hand side corresponds to the **translational motion**, while the second term to the **rotational motion** about an *instantaneous axis of rotation*.

t.



translational motion



rotational motion

Consequently, the total momentum of an arbitrarily moving rigid body (in lab FoR) is

$(\vec{v}_{0}' + \vec{\omega} \times \vec{r}_{i}')$

$$\overline{P} = \sum_{i=1}^{N} m_i \boxed{\overline{v}_i} = \underbrace{\sum_{i=1}^{N} m_i}_{M} \overline{v}_{O'} + \sum_{i=1}^{N} m_i(\overline{\omega} \times \overline{r}_{i'}) =$$

$$= M\overline{v}_{O'} + \overline{\omega} \times \underbrace{\boxed{\sum_{i=1}^{N} m_i \overline{r}_{i'}}}_{M\overline{r}'_{\mathrm{cm}}} = \underbrace{M\boxed{\overline{v}_{O'}}}_{\text{translational motion}} + \underbrace{\boxed{M\overline{\omega} \times \overline{r}'_{\mathrm{cm}}}}_{\text{rotational motion}} = 0$$

$M \boxed{\dot{\vec{r}}_{cm}}$

$M \cdot \boxed{\dfrac{\sum_{i=1} m_i \vec{r}_i'}{\boxed{M}}}$, $= \sum m_i$

$=0$

$\vec{P} = \boxed{M \cdot \vec{v}_c}$

Consequently, the total momentum of an arbitrarily moving rigid body (in lab FoR) is

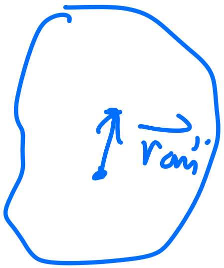
$$\overline{P} = \sum_{i=1}^{N} m_i \overline{v}_i = \underbrace{\sum_{i=1}^{N} m_i}_{M} \overline{v}_{O'} + \sum_{i=1}^{N} m_i(\overline{\omega} \times \overline{r}_{i'}) =$$

$$= M\overline{v}_{O'} + \overline{\omega} \times \underbrace{\sum_{i=1}^{N} m_i \overline{r}_{i'}}_{M\overline{r}'_{\text{cm}}} = \underbrace{M\overline{v}_{O'}}_{\text{translational motion}} + \underbrace{M\overline{\omega} \times \overline{r}'_{\text{cm}}}_{\text{rotational motion}}$$

$$\Rightarrow \vec{p} = M\vec{v}_c$$

In the lab FoR

$$
\begin{aligned}
\bar{L} &= \sum_{i=1}^{N} \bar{L}_i = \sum_{i=1}^{N} m_i \bar{r}_i \times \bar{v}_i = \sum_{i=1}^{N} [m_i \boxed{(\bar{r}_{O'} + \bar{r}'_i) \times (\bar{v}_{O'} + \varpi \times \bar{r}'_i)]} \\
&= \sum_{i=1}^{N} m_i (\bar{r}_{O'} \times \bar{v}_{O'}) + \sum_{i=1}^{N} m_i \bar{r}_{O'} \times (\varpi \times \bar{r}'_i) + \\
&\quad + \sum_{i=1}^{N} m_i \bar{r}'_i \times \bar{v}_{O'} + \sum_{i=1}^{N} m_i \bar{r}'_i \times (\varpi \times \bar{r}'_i) \\
&= M \bar{r}_{O'} \times \bar{v}_{O'} + \boxed{M \bar{r}_{O'} \times (\varpi \times \bar{r}'_{\mathrm{cm}}) + M \bar{r}'_{\mathrm{cm}} \times \bar{v}_{O'}} + \\
&\quad + \sum_{i=1}^{N} m_i \bar{r}'_i \times (\varpi \times \bar{r}'_i)
\end{aligned}
$$



In the lab FoR

$$
\begin{aligned}
\overline{L} &= \sum_{i=1}^{N}\overline{L}_i = \sum_{i=1}^{N} m_i\overline{r}_i \times \overline{v}_i = \sum_{i=1}^{N}[m_i(\overline{r}_{O'} + \overline{r}'_i) \times (\overline{v}_{O'} + \overline{\omega} \times \overline{r}'_i)] \\
&= \sum_{i=1}^{N} m_i(\overline{r}_{O'} \times \overline{v}_{O'}) + \sum_{i=1}^{N} m_i\overline{r}_{O'} \times (\overline{\omega} \times \overline{r}'_i) + \\
&\quad + \sum_{i=1}^{N} m_i\overline{r}'_i \times \overline{v}_{O'} + \sum_{i=1}^{N} m_i\overline{r}'_i \times (\overline{\omega} \times \overline{r}'_i) \\
&= M\overline{r}_{O'} \times \overline{v}_{O'} + M\overline{r}_{O'} \times (\overline{\omega} \times \overline{r}'_{cm}) + M\overline{r}'_{cm} \times \overline{v}_{O'} + \\
&\quad + \sum_{i=1}^{N} m_i\overline{r}'_i \times (\overline{\omega} \times \overline{r}'_i)
\end{aligned}
$$

$$
\Rightarrow \vec{L} = \boxed{\underbrace{\vec{L}_c}_{=M\vec{r}_c \times \vec{v}_c}} + \vec{L}'
$$

$M\vec{r}_c \times \vec{v}_c$

In the lab FoR

$$
\begin{aligned}
\overline{L} &= \sum_{i=1}^{N} \overline{L}_i = \sum_{i=1}^{N} m_i \overline{r}_i \times \overline{v}_i = \sum_{i=1}^{N} [m_i(\overline{r}_{O'} + \overline{r}'_i) \times (\overline{v}_{O'} + \overline{\omega} \times \overline{r}'_i)] \\
&= \sum_{i=1}^{N} m_i(\overline{r}_{O'} \times \overline{v}_{O'}) + \sum_{i=1}^{N} m_i \overline{r}_{O'} \times (\overline{\omega} \times \overline{r}'_i) + \\
&\quad + \sum_{i=1}^{N} m_i \overline{r}'_i \times \overline{v}_{O'} + \sum_{i=1}^{N} m_i \overline{r}'_i \times (\overline{\omega} \times \overline{r}'_i) \\
&= M\overline{r}_{O'} \times \overline{v}_{O'} + M\overline{r}_{O'} \times (\overline{\omega} \times \overline{r}'_{cm}) + M\overline{r}'_{cm} \times \overline{v}_{O'} + \\
&\quad + \sum_{i=1}^{N} m_i \overline{r}'_i \times (\overline{\omega} \times \overline{r}'_i)
\end{aligned}
$$

$$\Rightarrow \vec{L} = \underbrace{\vec{L}_c}_{= M\vec{r}_c \times \vec{v}_c} + \vec{L}'$$

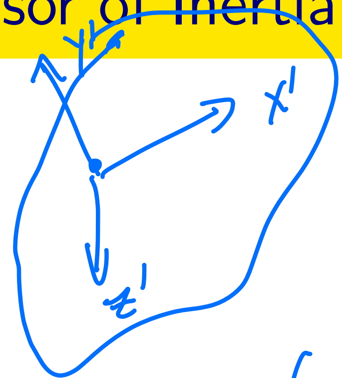
$\vec{L}' = \sum_{i=1}^{N} m_i \overline{r}'_i \times (\bar{\omega} \times \overline{r}'_i)$: Rigid body's angular momentum w.r.t its center of mass

Tensor of Inertia

$$\vec{L}' = \sum_{i=1}^{N} m_i \bar{r}_i' \times (\bar{\omega} \times \bar{r}_i') = \boxed{I\vec{\omega}}$$

I

Tensor of Inertia



$\vec{L}$ $\vec{\omega}$

$$\vec{L}' = \sum_{i=1}^{N} m_i \bar{r}_i' \times \left(\bar{\omega} \times \bar{r}_i'\right) = I\vec{\omega}$$

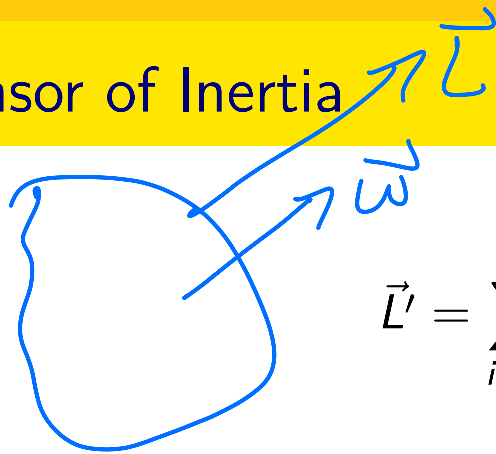
$$\begin{pmatrix} L_{x'} \\ L_{y'} \\ L_{z'} \end{pmatrix}$$

$$I = \begin{bmatrix} I_{x'x'} & I_{x'y'} & I_{x'z'} \\ I_{y'x'} & I_{y'y'} & I_{y'z'} \\ I_{z'x'} & I_{z'y'} & I_{z'z'} \end{bmatrix} \begin{pmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{pmatrix} =$$

$$\begin{bmatrix} \sum_{i=1}^{N} m_i \left(y_i'^2 + z_i'^2\right) & \sum_{i=1}^{N} -m_i x_i' y_i' & \sum_{i=1}^{N} -m_i x_i' z_i' \\ \sum_{i=1}^{N} -m_i y_i' x_i' & \sum_{i=1}^{N} m_i \left(x_i'^2 + z_i'^2\right) & \sum_{i=1}^{N} -m_i y_i' z_i' \\ \sum_{i=1}^{N} -m_i z_i' x_i' & \sum_{i=1}^{N} -m_i z_i' y_i' & \sum_{i=1}^{N} m_i \left(x_i'^2 + y_i'^2\right) \end{bmatrix}$$

$$L_{x'} = I_{xx'} \omega_{x'} + I_{x'y'} \omega_{y'} + I_{x'z'} \omega_{z'}$$

Tensor of Inertia



$$\vec{L}' = \sum_{i=1}^{N} m_i \bar{r}_i' \times (\bar{\omega} \times \bar{r}_i') = I\vec{\omega}$$

$$\begin{pmatrix} L_{x'} \\ L_{y'} \\ L_{z'} \end{pmatrix} = \lambda \begin{pmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z} \end{pmatrix},$$

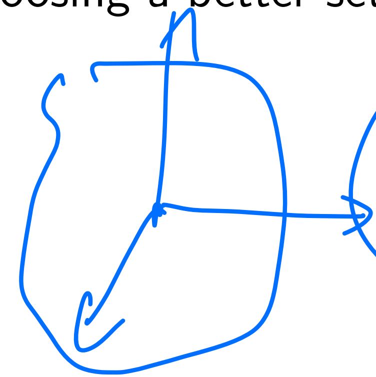
$$I = \begin{bmatrix} I_{x'x'} & I_{x'y'} & I_{x'z'} \\ I_{y'x'} & I_{y'y'} & I_{y'z'} \\ I_{z'x'} & I_{z'y'} & I_{z'z'} \end{bmatrix} =$$

$$\begin{bmatrix} \sum_{i=1}^{N} m_i \left(y_i'^2 + z_i'^2\right) & \sum_{i=1}^{N} -m_i x_i' y_i' & \sum_{i=1}^{N} -m_i x_i' z_i' \\ \sum_{i=1}^{N} -m_i y_i' x_i' & \sum_{i=1}^{N} m_i \left(x_i'^2 + z_i'^2\right) & \sum_{i=1}^{N} -m_i y_i' z_i' \\ \sum_{i=1}^{N} -m_i z_i' x_i' & \sum_{i=1}^{N} -m_i z_i' y_i' & \sum_{i=1}^{N} m_i \left(x_i'^2 + y_i'^2\right) \end{bmatrix}$$

$\vec{L}$ can not be always parallel to $\vec{\omega}$, when will it be?

By a random choice of x', y', z', e.g. $\omega_x', \omega_y', \omega_z'$ **all contributes to $\vec{L}_x'$**, it's hard to see, but if we ...

By choosing a better set of $\tilde{x}', \tilde{y}', \tilde{z}'$, we can obtain a diagonal form of I.



$$I = \begin{bmatrix} I_{\tilde{x}'\tilde{x}'} & 0 & 0 \\ 0 & I_{\tilde{y}'\tilde{y}'} & 0 \\ 0 & 0 & I_{\tilde{z}'\tilde{z}'} \end{bmatrix}$$

By choosing a better set of $\tilde{x}', \tilde{y}', \tilde{z}'$, we can obtain a diagonal form of I.

$$I = \begin{bmatrix} I_{\tilde{x}'\tilde{x}'} & 0 & 0 \\ 0 & I_{\tilde{y}'\tilde{y}'} & 0 \\ 0 & 0 & I_{\tilde{z}'\tilde{z}'} \end{bmatrix}$$

Then, ω'_x **only contributes to $\vec{\mathbf{L}}'_{\mathbf{x}}$, so do** ω'_y **and** ω'_z

$$\Rightarrow L_{x'} = \boxed{I_{\tilde{x}'\tilde{x}'}} \cdot \omega_{x'}, \quad L_{y'} = I_{\tilde{y}'\tilde{y}'} \cdot \omega_{y'}, \quad L_{z'} = I_{\tilde{z}'\tilde{z}'} \cdot \omega_{z'}$$

L_x

x' ω

$\vec{L} = \boxed{I}\,\vec{\omega}$

By choosing a better set of $\tilde{x}', \tilde{y}', \tilde{z}'$, we can obtain a diagonal form of I.

$$I = \begin{bmatrix} I_{\tilde{x}'\tilde{x}'} & 0 & 0 \\ 0 & I_{\tilde{y}'\tilde{y}'} & 0 \\ 0 & 0 & I_{\tilde{z}'\tilde{z}'} \end{bmatrix}$$

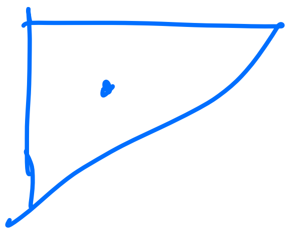
Then, ω'_x **only contributes to** $\vec{\mathbf{L}}'_{\mathbf{x}}$**, so do** ω'_y **and** ω'_z

$$\Rightarrow L_{x'} = I_{\tilde{x}'\tilde{x}'} \cdot \omega_{x'}, \quad L_{y'} = I_{\tilde{y}'\tilde{y}'} \cdot \omega_{y'}, \quad L_{z'} = I_{\tilde{z}'\tilde{z}'} \cdot \omega_{z'}$$

The axis in this special sets of axes is called the **Pricipal axis**, which is our main focus.

How to find principle axis

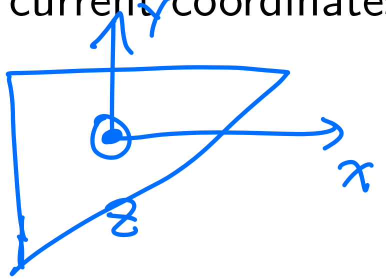
1. Find the mass center C of the rigid body, let C be the origin



O'

How to find principle axis

1. Find the mass center C of the rigid body, let C be the origin
2. Use the current coordinates x, y, z to derive the tensor of inertia I



How to find principle axis

1. Find the mass center C of the rigid body, let C be the origin
2. Use the current coordinates x, y, z to derive the tensor of inertia I
3. Let $\boxed{det(I - \lambda \mathbb{1})} = 0$ to find $\lambda_1, \lambda_2, \lambda_3$.

$$\begin{pmatrix} 1 & & \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

$$I \quad \begin{pmatrix} I_{xx}-\lambda & & \\ & I_{yy}-\lambda & \\ & & \ddots \end{pmatrix}$$

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

$$x_1 \begin{vmatrix} y_2 & y_3 \\ z_2 & z_3 \end{vmatrix} - x_2 \begin{vmatrix} y_1 & y_3 \\ z_1 & z_3 \end{vmatrix} + x_3 \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix}$$

$$z_2 \cdot y_1 - y_2 \cdot z_1$$

How to find principle axis

1. Find the mass center C of the rigid body, let C be the origin
2. Use the current coordinates x, y, z to derive the tensor of inertia I
3. Let $det(I - \lambda\mathbb{1}) = 0$ to find $\boxed{\lambda_1, \lambda_2, \lambda_3.}$
4. Plug back λ_i into the equation $(I - \lambda_i\mathbb{1})\vec{u}_i = 0$, find the solution $\vec{u}_1, \vec{u}_2, \vec{u}_3$

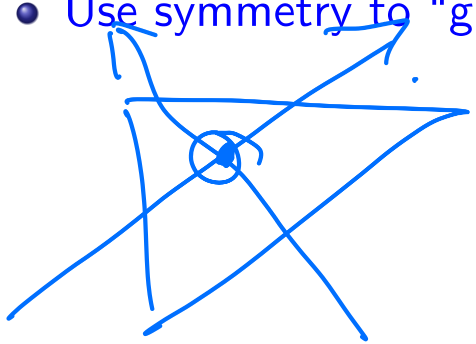
$$(I - \lambda\mathbb{1})\boxed{\vec{u}_i} = 0.$$

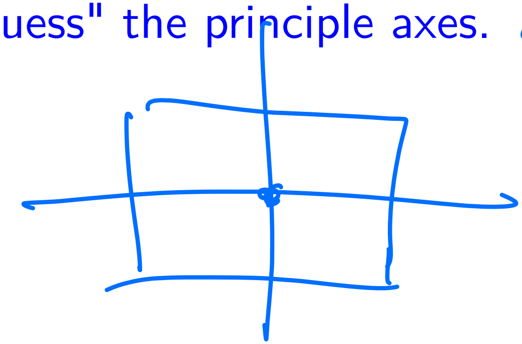
How to find principle axis

1. Find the mass center C of the rigid body, let C be the origin
2. Use the current coordinates x, y, z to derive the tensor of inertia I
3. Let $det(I - \lambda \mathbb{1}) = 0$ to find $\lambda_1, \lambda_2, \lambda_3$.
4. Plug back λ_i into the equation $(I - \lambda_i \mathbb{1})\vec{u}_i = 0$, find the solution $\vec{u}_1, \vec{u}_2, \vec{u}_3$
5. Use the direction of $\vec{u}_1, \vec{u}_2, \vec{u}_3$ as axes, calculate the new I_p.

- Use symmetry to "guess" the principle axes.

$$\begin{pmatrix} I_{xx} & & \\ & I_{yy} & \\ 0 & & I_{zz} \end{pmatrix}$$





How to find principle axis

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- Use symmetry to "guess" the principle axes.
- Why does this methods always works?

How to find principle axis

1. Find the mass center C of the rigid body, let C be the origin
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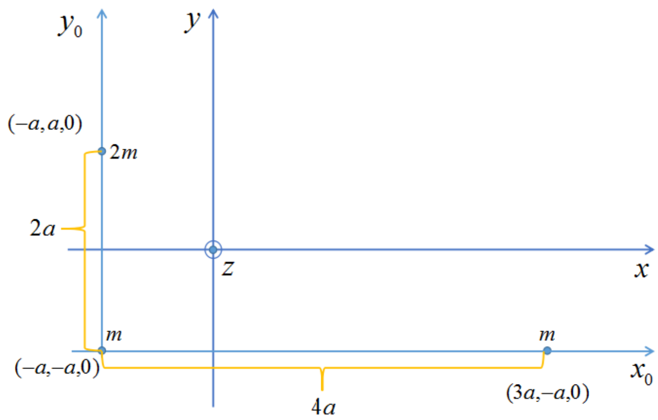
5. Use the direction of $\vec{u_1}, \vec{u_2}, \vec{u_3}$ as axes, calculate the new I_p.

- Use symmetry to "guess" the principle axes.
- Why does this methods always works?
- Recall the form of I, it's a self-adjoint matrix.
  To learn the mathematical details, have a look at:
  zxj_Eigenvalue & Diagonalization.pdf (under canvas RC folder).

Exercise 2

Use the example in slide(s-21hp14) to practice.



(answer: in slide)

Reference

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- Haoyang Zhang.
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