Chapter 13 – Angular Momentum and Torque

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Honors Physics I (Summer 2020)
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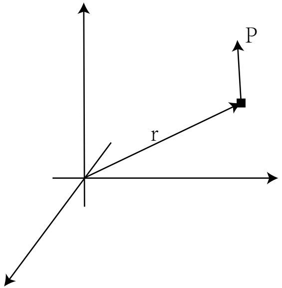
Agenda

1. Single Particle
   - Angular Momentum and Torque. 2^{nd} Law of Dynamics for Orbital Motion
   - Aerial Velocity
   - Planar Motion. Example
2. System of Particles
   - Torque: Internal vs External Forces
   - Conservation of Angular Momentum Law

Single Particle

Angular Momentum and Torque. Single Particle



By 2nd law of dynamics, for the net force $\overline{F}$ acting the particle,

$$\overline{F} = \frac{\mathrm{d}\overline{p}}{\mathrm{d}t}.$$

Multiplying, from the left, by $\overline{r}$

$$\overline{r} \times \overline{F} = \overline{r} \times \frac{\mathrm{d}\overline{p}}{\mathrm{d}t}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\overline{r} \times \overline{p}) = \frac{\mathrm{d}\overline{r}}{\mathrm{d}t} \times \overline{p} + \overline{r} \times \frac{\mathrm{d}\overline{p}}{\mathrm{d}t} = \underbrace{\overline{v} \times m\overline{v}}_{=0} + \overline{r} \times \frac{\mathrm{d}\overline{p}}{\mathrm{d}t} = \overline{r} \times \frac{\mathrm{d}\overline{p}}{\mathrm{d}t}.$$

Hence,

$$\underbrace{\overline{r} \times \overline{F}}_{\text{torque}} = \frac{\mathrm{d}}{\mathrm{d}t} \underbrace{(\overline{r} \times \overline{p})}_{\text{angular momentum}}$$

For a single particle, the **angular momentum** and the **torque**, with respect to the origin, are defined as

$$\boxed{\overline{L} \overset{\text{def}}{=} \overline{r} \times \overline{p}} \qquad \boxed{\overline{\tau} \overset{\text{def}}{=} \overline{r} \times \overline{F}}$$

SI Units. Torque [N·m]; angular momentum [kg· m^2/s]

Magnitude. $|\overline{L}| = |\overline{r}|\,|\overline{p}| \sin\angle(\overline{r}, \overline{p})$ and $|\overline{\tau}| = |\overline{r}|\,|\overline{F}| \sin\angle(\overline{r}, \overline{F})$

Hence, the relationship between the two quantities

$$\boxed{\overline{\tau} = \frac{\mathrm{d}\overline{L}}{\mathrm{d}t}}$$

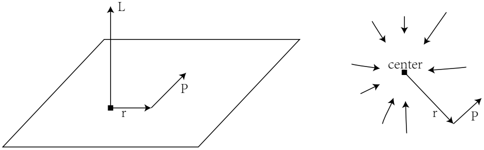
The rate of change of the angular momentum *of a particle equals the* torque *of the net force acting on that particle.*

Comment. This can be treated as the rotational (orbital motion) analogue of the 2nd law of dynamics for translational motion.Consequently, its integral form (the angular momentum–torque impulse theorem) can be found as

$$\overline{\tau} = \frac{\mathrm{d}\overline{L}}{\mathrm{d}t} \implies \mathrm{d}\overline{L} = \overline{\tau}\,\mathrm{d}t \implies \boxed{\overline{L}(t_2) - \overline{L}(t_1) = \int_{t_1}^{t_2} \overline{\tau}\,\mathrm{d}t}$$

Discussion and Interpretation

1. If $\overline{\tau} = 0$, then $\overline{L} = \overline{r} \times m\overline{v} = \text{const}$. Constant direction of $\overline{L}$ means that the motion is *planar*.The particle moves always in one plane (spanned by $\overline{r}$ and $\overline{v}$, i.e. perpendicular to $\overline{L}$).



(a) (b)

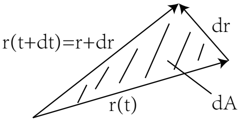
2. In particular, for a particle acted upon a central force $\overline{F}(\overline{r}) = f(r)\,\overline{r}$, where $r = |\overline{r}|$, the torque

$$\tau = \overline{r} \times \overline{F} = \overline{r} \times f(r)\overline{r} = f(r)\,(\underbrace{\overline{r} \times \overline{r}}_{=0}) = 0.$$

Therefore, for any central force, the torque of the force w.r.t. the center is equal to zero, hence the angular momentum is constant and the motion is planar.

Aerial Velocity (more details in the chapter on gravitation)

Let $\mathrm{d}A$ denote the area swept by $\overline{r}$ over an infinitesimal time interval $\mathrm{d}t$. Then



$$\mathrm{d}A = \frac{1}{2}|\overline{r} \times \mathrm{d}\overline{r}| \qquad \Longrightarrow \qquad \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}\left|\overline{r} \times \frac{\mathrm{d}\overline{r}}{\mathrm{d}t}\right| = \frac{1}{2}|\overline{r} \times \overline{v}|.$$

Define the **aerial velocity** as a vector $\overline{\sigma}$ whose magnitude is equal to $\mathrm{d}A/\mathrm{d}t$ and its direction is determined by right hand rule for $\overline{r} \times \overline{v}$. Then,

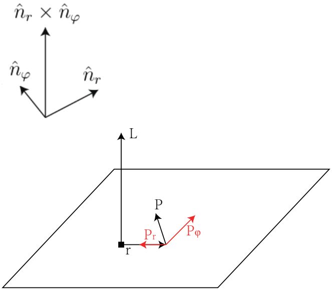
$$\boxed{\overline{\sigma} = \frac{1}{2}(\overline{r} \times \overline{v})}$$

and

$$\boxed{\overline{\sigma}} = \frac{1}{2}(\overline{r} \times \overline{v}) = \frac{1}{2m}(\overline{r} \times m\overline{v}) = \boxed{\frac{\overline{L}}{2m}}.$$

Hence $\overline{L} = \text{const} \Leftrightarrow \overline{\sigma} = \text{const}$. This will be discussed in the context of Kepler's laws.

Single Particle in Planar Motion



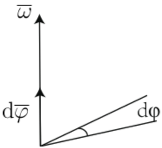
In polar coordinates $\overline{p} = \overline{p}_r + \overline{p}_\varphi = m\dot{r}\hat{n}_r + mr\dot{\varphi}\hat{n}_\varphi$. Hence the angular momentum (w.r.t. origin)

$$\begin{aligned} \overline{L} &= \overline{r} \times \overline{p} = r\hat{n}_r \times (m\dot{r}\hat{n}_r + mr\dot{\varphi}\hat{n}_\varphi) \\ &= mr^2\dot{\varphi}(\hat{n}_r \times \hat{n}_\varphi) = I\dot{\varphi}(\hat{n}_r \times \hat{n}_\varphi) \end{aligned}$$

where $I = mr^2$ is the *moment of inertia of a particle* about the origin.

Recall that angular velocity is defined as a vector $\overline{\omega} = d\overline{\varphi}/dt$. Hence, for a single particle in planar motion

$$\overline{L} = I\overline{\omega}.$$



Using the 2nd law of dynamics for orbital motion, we have for a particle in planar motion

$$\overline{\tau} = \frac{\mathrm{d}\overline{L}}{\mathrm{d}t} = \frac{\mathrm{d}(I\overline{\omega})}{\mathrm{d}t}$$

Discussion and Example

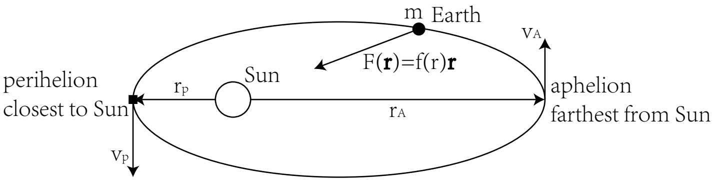
Note. If $\overline{\tau} = 0$ then $\mathrm{d}(I\overline{\omega})/\mathrm{d}t = 0$ and, consequently, $I\overline{\omega} = \text{const}$.

Note. If $I = \text{const}$ (e.g. particle in circular motion), then

$$\overline{\tau} = I\frac{\mathrm{d}\overline{\omega}}{\mathrm{d}t} = I\overline{\varepsilon},$$

where $\overline{\varepsilon}$ denotes angular acceleration.

Example. The Earth orbiting around the Sun.



The gravitational force acting on the Earth is central, (centered at the Sun), and hence produces no torque about the Sun

$$\overline{\tau} = 0 \qquad \Longrightarrow \qquad I\overline{\omega} = \text{const} \qquad \Longrightarrow \qquad I_P \omega_P = I_A \omega_A.$$

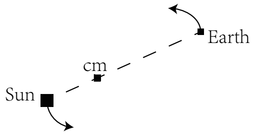
For the moment of inertia,

$$I_P = mr_P^2 < mr_A^2 = I_A \qquad \Longrightarrow \qquad \omega_P > \omega_A,$$

hence the relationship between the magnitudes of linear velocities $\overline{v} = \overline{\omega} \times \overline{r}$ is

$$\boxed{\frac{v_P}{v_A}} \approx \frac{\omega_P r_P}{\omega_A r_A} = \frac{I_A r_P}{I_p r_A} = \frac{m r_A^2 r_P}{m r_P^2 r_A} = \frac{r_A}{r_P} \boxed{> 1}.$$

Comment. The reason why $\approx$ is used in the equation above is that in fact, both the Sun and the Earth move about the center of mass of the Sun-Earth system.

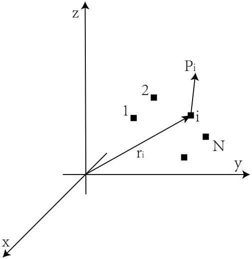


System of Particles

Angular Momentum

Recall that for the total momentum of a system of particles

$$\frac{d\overline{P}}{dt} = \frac{d}{dt}\left(\sum_{i=1}^{N} \overline{p}_i\right) = \sum_{i=1}^{N} \overline{F}_i^{ext} = \overline{F}^{ext}.$$



Hence, the angular momentum and its rate of change

$$\begin{aligned}
\frac{d\overline{L}}{dt} &= \frac{d}{dt}\left(\sum_{i=1}^{N} \overline{L}_i\right) = \frac{d}{dt}\left(\sum_{i=1}^{N} \overline{r}_i \times \overline{p}_i\right) = \sum_{i=1}^{N}\left(\frac{d\overline{r}_i}{dt} \times \overline{p}_i + \overline{r}_i \times \frac{d\overline{p}_i}{dt}\right) \\
&= \sum_{i=1}^{N}\left(\underbrace{\overline{v}_i \times m_i \overline{v}_i}_{=0} + \overline{r}_i \times \frac{d\overline{p}_i}{dt}\right) = \sum_{i=1}^{N}\left(\overline{r}_i \times \frac{d\overline{p}_i}{dt}\right) =
\end{aligned}$$

$$\overset{\text{2nd law}}{=\!=} \quad \sum_{i=1}^{N} \overline{r}_i \times \overline{F}_i = \sum_{i=1}^{N} \overline{r}_i \times (\overline{F}_i^{\text{int}} + \overline{F}_i^{\text{ext}})$$

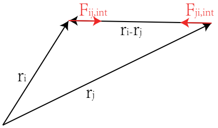
$$= \quad \sum_{i=1}^{N} \overline{r}_i \times \overline{F}_i^{\text{int}} + \sum_{i=1}^{N} \overline{\tau}_i^{\text{ext}} = \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \overline{r}_i \times \overline{F}_{ij}^{\text{int}} + \overline{\tau}^{\text{ext}}$$

$$= \quad \sum_{i=1}^{N} \sum_{j=1, j<i}^{N} (\overline{r}_i \times \overline{F}_{ij}^{\text{int}} + \overline{r}_j \times \overline{F}_{ji}^{\text{int}}) + \overline{\tau}^{\text{ext}}$$

$$\underset{\overline{F}_{ij}^{\text{int}} = -\overline{F}_{ji}^{\text{int}}}{\overset{\text{3rd law}}{=\!=}} \quad \sum_{i=1}^{N} \sum_{j=1, j<i}^{N} (\overline{r}_i \times \overline{F}_{ij}^{\text{int}} - \overline{r}_j \times \overline{F}_{ij}^{\text{int}}) + \overline{\tau}^{\text{ext}}$$

$$= \quad \sum_{i=1}^{N} \sum_{j=1, j<i}^{N} (\overline{r}_i - \overline{r}_j) \times \overline{F}_{ij}^{\text{int}} + \overline{\tau}^{\text{ext}} = \overline{\tau}^{\text{ext}}$$

But $(\overline{r}_i - \overline{r}_j) \times \overline{F}_{ij}^{\text{int}} = 0$, since $(\overline{r}_i - \overline{r}_j) \parallel \overline{F}_{ij}^{\text{int}}$.



Eventually,

$$\boxed{\frac{\mathrm{d}\overline{L}}{\mathrm{d}t} = \overline{\tau}^{\text{ext}}.}$$

Conclusion The total angular momentum of a system of particles can only be changed by a non-zero net torque of external forces.

Conservation of the Angular Momentum Law

If the net torque of external forces on a system of particles is equal to zero, then the total angular momentum of that system is conserved.

Comment.* The law of conservation of the angular momentum is another conservation principle that we come across. Together with the laws of conservation of the (linear) momentum and the energy, the three conservation principles are very closely related to symmetries of space–time [*Noether's Theorem*; see Canvas for details].