

VP160 Midterm Review Class

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UM-SJTU Joint Institute

June 18, 2021

1. Scientific Notations

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2. Unit Prefix and Conversion

- **k** (unit prefix) **m** (unit)
- Some commonly-used unit prefixes:

p	n	μ	m	c	k	M	G
10^{-12}	10^{-9}	10^{-6}	10^{-3}	10^{-2}	10^3	10^6	10^9

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e.g.

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3. Basic Units & Derived Units

- SI system of units:

Quantity	L	m	t	I	T	n	lv
Unit	m	kg	s	A	K	mol	cd

Dimensional Analysis: System of Units

- ① We can first select some physical quantities as the “**basic quantities**” and specify a “**unit for measurement**” for each basic quantity, the other physical quantities’ units can be derived from the relation between them and the fundamental quantities. These physical quantities are called **derived quantities** and their units It’s called **derived unit**.

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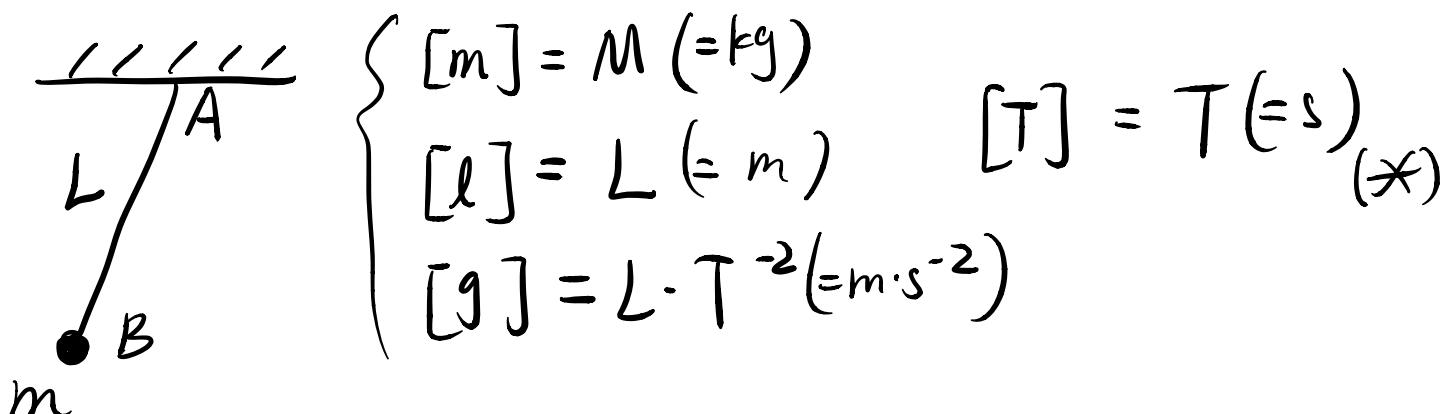
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- ② A set of units derived in this way, is called a **system of units**.
- ③ We often use capital letter to represent a “dimensional quantity”, and use $[x]$ to represent the “dimensional quantity” of specific physical quantity x .
e.g. The dimensional quantity of a particle of mass m is written as:
 $M = [m]$.

Dimensional Analysis: Method of Undetermined Coefficients

Exercise 1

A simple pendulum consists of a light inextensible string AB with length l , with the end A fixed, and a point mass m attached to B. The pendulum oscillates with a small amplitude, and the period of oscillation is T . It is suggested that T is proportional to the product of powers of m , l and g , where g is the acceleration due to gravity. Use dimensional analysis to find this relationship.



$$g = 9.8 \text{ m}\cdot\text{s}^{-2}$$

We assume that $T = k \cdot m^x l^y g^z$, k is a const.
(no unit)

$$\stackrel{(*)}{\Rightarrow} T = M^x L^y (L \cdot T^{-2})^z \quad \leftarrow$$

$$= M^x L^{y+z} \cdot T^{-2z}$$

$$\Rightarrow \begin{cases} x=0 \\ y+z=0 \\ -2z=1 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=\frac{1}{2} \\ z=-\frac{1}{2} \end{cases}$$

$$\text{Thus, } T \propto l^{\frac{1}{2}} g^{-\frac{1}{2}}$$

$$T \propto \sqrt{l} \quad (T = 2\pi\sqrt{\frac{l}{g}})$$

Back-of-the-envelop Problems

Definition

A quick estimation of some physical quantities.

Tips

- ① Try to remember the order of magnitude of some important constant.
- ② This type of questions may occur in exams.

Basic Vector operations

- Addition & Subtraction

$$\vec{u} \pm \vec{v} = (u_1 \pm v_1, u_2 \pm v_2, u_3 \pm v_3)$$

- Scalar Multiplication

$$\lambda \vec{u} = (\lambda u_1, \lambda u_2, \lambda u_3)$$

- Dot Product

$$\vec{u} \cdot \vec{v} = |u||v| \cos\theta = u_1 v_1 + u_2 v_2 + u_3 v_3$$

- Orthogonal Projection Vector of the vector \vec{u} onto the vector \vec{v}

$$\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \cdot \frac{\vec{v}}{|\vec{v}|}$$

Basic Vector operations

• Cross Product

- Magnitude: $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$
- Direction: determined by **Right Hand Rule**
- Matrix expression(Using determinant):

$$\begin{aligned}\vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= (u_2 v_3 - u_3 v_2) \hat{i} + (u_3 v_1 - u_1 v_3) \hat{j} + (u_1 v_2 - u_2 v_1) \hat{k}\end{aligned}$$

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• anti-commutative

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

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- Scalar Triple Product:

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{v} \cdot (\vec{w} \times \vec{u}) = \vec{w} \cdot (\vec{u} \times \vec{v})$$

1D Kinematics

- Average vs. Instantaneous Quantities

$$v_{x,A} = \frac{x(t + \Delta t) - x(t)}{\Delta t} \quad a_{x,A} = \frac{v(t + \Delta t) - v(t)}{\Delta t}$$

- Relationships

$$x = x(t)$$

$$v = v(t) = \frac{d}{dt}x(t)$$

$$a = a(t) = \frac{d}{dt}v(t) = \frac{d^2}{dt^2}x(t)$$

- Relative motion

$$x = x_r + x'; \quad v = v_r + v'; \quad a = a_r + a'$$

Kinematics in 3D: Cylindrical Coordinates

Basic Formulas in Cylindrical Coordinates

$$\vec{r} = \rho \hat{n}_\rho + z \hat{n}_z$$

$$\vec{v} = \dot{\rho} \hat{n}_\rho + \rho \dot{\phi} \hat{n}_\phi + \dot{z} \hat{n}_z$$

$$\vec{a} = (\ddot{\rho} - \rho \dot{\phi}^2) \hat{n}_\rho + (\rho \ddot{\phi} + 2\dot{\rho}\dot{\phi}) \hat{n}_\phi + \ddot{z} \hat{n}_z$$

Tips:

if $z = 0$, they become kinematics formulas in polar coordinates (see next slide).

Kinematics in 2D: Polar Coordinates

Basic Formulas in Polar Coordinates

$$\vec{r} = r \hat{n}_r \quad (1)$$

$$\vec{v} = \dot{r} \hat{n}_r + r \dot{\theta} \hat{n}_\theta \quad (2)$$

$$\vec{a} = (\ddot{r} - r \dot{\theta}^2) \hat{n}_r + (r \ddot{\theta} + 2\dot{r}\dot{\theta}) \hat{n}_\theta \quad (3)$$

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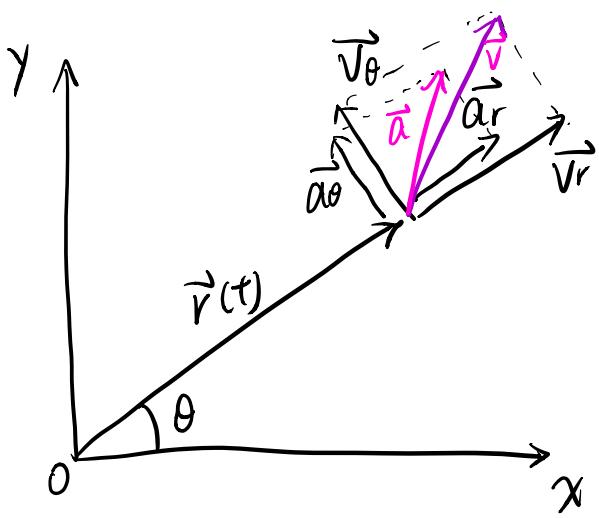
$$\vec{v} = \dot{r} \hat{n}_r + r \dot{\theta} \hat{n}_\theta \quad (2)$$

$$\vec{a} = (\ddot{r} - r \dot{\theta}^2) \hat{n}_r + (r \ddot{\theta} + 2\dot{r}\dot{\theta}) \hat{n}_\theta \quad (3)$$

Relations with cartesian coordinates

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$d\vec{r} = dr \hat{n}_r + rd\theta \hat{n}_\theta, \quad |d\vec{r}| = \sqrt{(dr)^2 + (rd\theta)^2}$$



$$\begin{cases} \dot{\hat{n}_r} = \dot{\theta} \hat{n}_\theta \\ \dot{\hat{n}_\theta} = -\dot{\theta} \hat{n}_r \end{cases} \quad (*)$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d(r \cdot \hat{n}_r)}{dt} = \frac{dr}{dt} \hat{n}_r + r \cdot \frac{d\hat{n}_r}{dt}$$

$$r \frac{d\hat{n}_r}{dt} = r \dot{\hat{n}}_r = r \dot{\theta} \hat{n}_\theta$$

$$\Rightarrow \vec{v} = \dot{r} \hat{n}_r + r \dot{\theta} \hat{n}_\theta = \vec{v}_r + \vec{v}_\theta$$

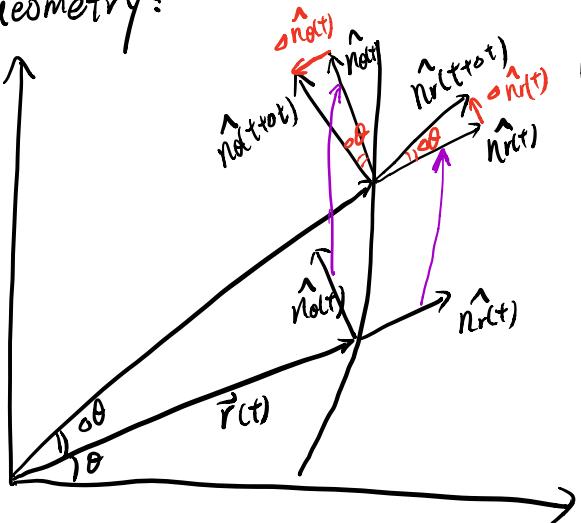
$$\vec{v}_r = \dot{r} \hat{n}_r, \vec{v}_\theta = r \dot{\theta} \hat{n}_\theta.$$

radius component transversial component

$$\begin{aligned} \vec{a} &= \frac{d\vec{v}}{dt} = \frac{d}{dt} (\dot{r} \hat{n}_r + r \dot{\theta} \hat{n}_\theta) = \frac{d\dot{r}}{dt} \hat{n}_r + \dot{r} \frac{d\hat{n}_r}{dt} + \frac{d\dot{\theta}}{dt} \hat{n}_\theta + r \dot{\theta} \frac{d\hat{n}_\theta}{dt} \\ &= \ddot{r} \hat{n}_r + \dot{r} \dot{\theta} \hat{n}_\theta + \dot{r} \dot{\theta} \hat{n}_\theta + r \ddot{\theta} \hat{n}_\theta - r \dot{\theta}^2 \hat{n}_r \\ \Rightarrow \vec{a} &= (\ddot{r} - r \dot{\theta}^2) \hat{n}_r + (2\dot{r}\dot{\theta} + r \ddot{\theta}) \hat{n}_\theta \end{aligned}$$

How to derive (*):

1. Geometry:



$$\textcircled{1} \quad \omega \hat{n}_\theta(t) = -|\hat{n}_\theta(t)| \omega \theta \hat{n}_r \quad |\hat{n}| = 1$$

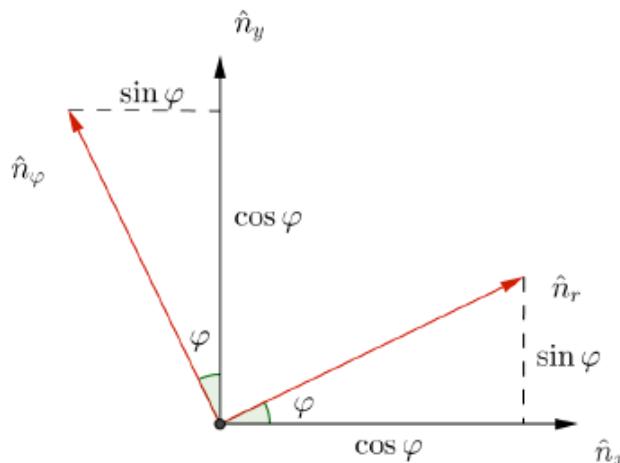
$$\begin{aligned} \frac{\omega \hat{n}_\theta(t)}{\omega t} &= -\frac{\omega \theta}{\omega t} \cdot \hat{n}_r \\ \Rightarrow \frac{d\hat{n}_\theta(t)}{dt} &= -\frac{d\theta}{dt} \hat{n}_r \\ \dot{\hat{n}}_\theta &= -\dot{\theta} \hat{n}_r \end{aligned}$$

$$\textcircled{2} \quad \omega \hat{n}_r(t) = |\hat{n}_r(t)| \omega \theta \cdot \hat{n}_\theta$$

$$\begin{aligned} \frac{\omega \hat{n}_r(t)}{\omega t} &= \frac{\omega \theta}{\omega t} \hat{n}_\theta \\ \Rightarrow \frac{d\hat{n}_r(t)}{dt} &= \frac{d\theta}{dt} \hat{n}_\theta \\ \dot{\hat{n}}_r &= \dot{\theta} \hat{n}_\theta \end{aligned}$$

2. derivative

How to find the derivative \hat{n}_r (and \hat{n}_φ) ?



$$\hat{n}_r = \cos \varphi \hat{n}_x + \sin \varphi \hat{n}_y$$

$$\hat{n}_\varphi = -\sin \varphi \hat{n}_x + \cos \varphi \hat{n}_y$$

$$\begin{aligned}\dot{\hat{n}}_r &= -\dot{\varphi} \sin \varphi \hat{n}_x + \dot{\varphi} \cos \varphi \hat{n}_y \\ &= \dot{\varphi}(-\sin \varphi \hat{n}_x + \cos \varphi \hat{n}_y) = \\ &= \boxed{\dot{\varphi} \hat{n}_\varphi}\end{aligned}$$

Similarly,

$$\begin{aligned}\dot{\hat{n}}_\varphi &= -\dot{\varphi} \cos \varphi \hat{n}_x - \dot{\varphi} \sin \varphi \hat{n}_y \\ &= -\dot{\varphi}(\cos \varphi \hat{n}_x + \sin \varphi \hat{n}_y) = \\ &= \boxed{-\dot{\varphi} \hat{n}_r}\end{aligned}$$

Kinematics in 3D: Natural Coordinates

Basic Vectors

- ① \hat{n}_τ : along the direction of \vec{v}
- ② \hat{n}_n and \hat{n}_b : perpendicular to the direction of \vec{v}

$$\hat{n}_\tau = \frac{\vec{v}}{|\vec{v}|}, \quad \hat{n}_n = \frac{\dot{\hat{n}_\tau}}{|\dot{\hat{n}_\tau}|}, \quad \hat{n}_b = \hat{n}_\tau \times \hat{n}_n$$

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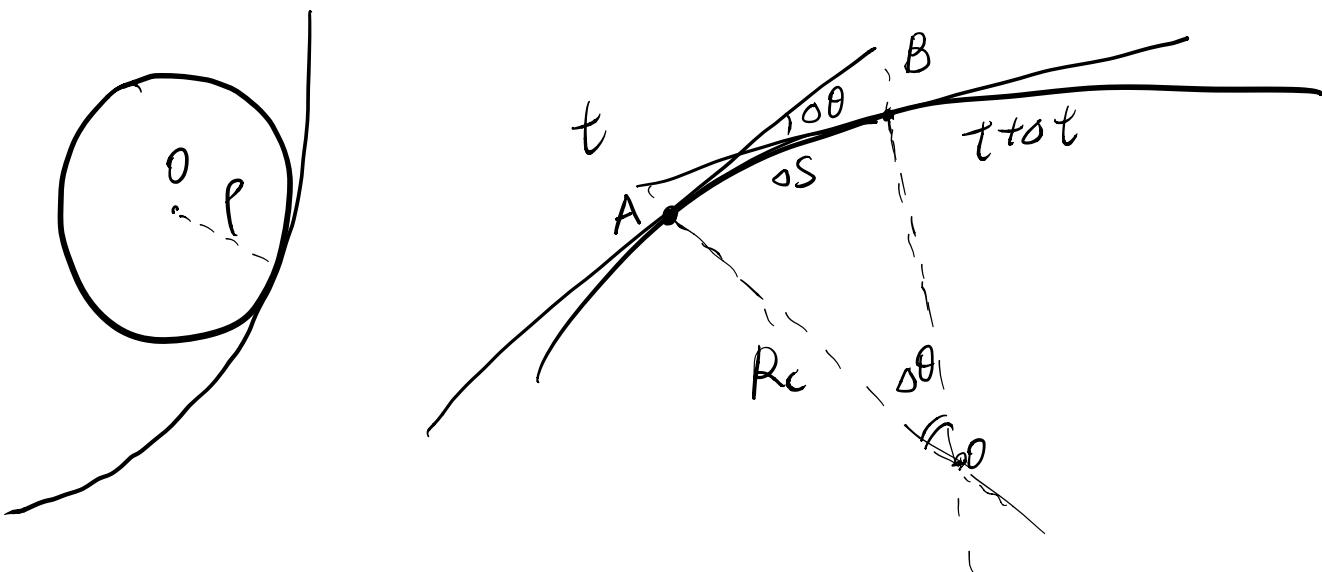
$$\hat{n}_\tau = \frac{\vec{v}}{|\vec{v}|}, \quad \hat{n}_n = \frac{\dot{\hat{n}}_\tau}{|\dot{\hat{n}}_\tau|}, \quad \hat{n}_b = \hat{n}_\tau \times \hat{n}_n$$

Basic Formulas

$$\vec{v} = v \hat{n}_\tau$$

$$\vec{a} = \dot{v} \hat{n}_\tau + \frac{v^2}{R_c} \hat{n}_n$$

R_c means radius of curvature, what is radius of curvature?



$$R_c = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta \theta}$$

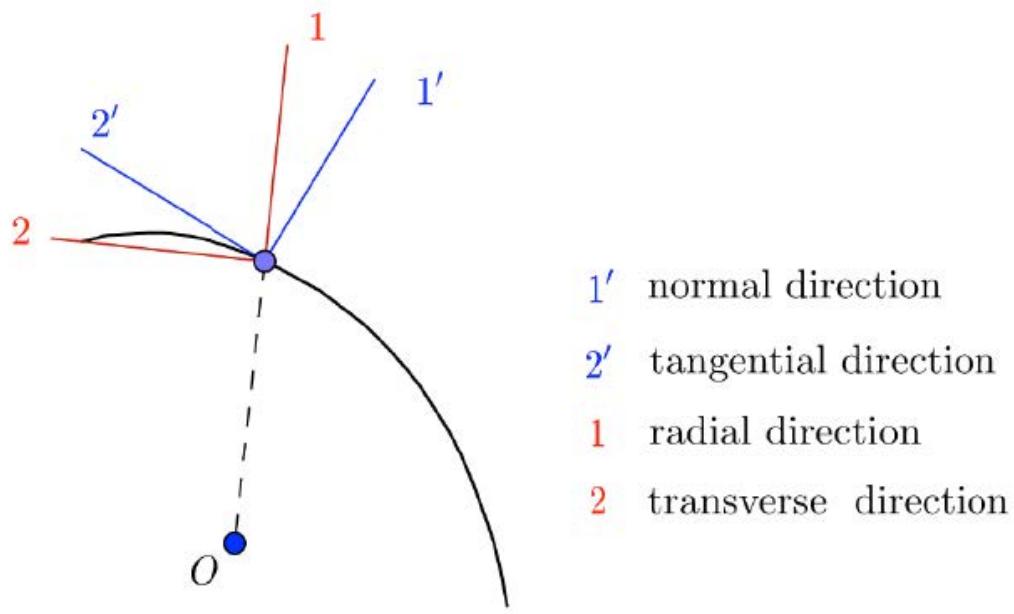
General formula:

$$R_c = \left| \frac{(1+f'^2)^{\frac{3}{2}}}{f''} \right| \quad \begin{pmatrix} f' = \frac{df(x)}{dx} \\ f'' = \frac{d^2f(x)}{dx^2} \end{pmatrix}$$

Normal and Tangential, Radial and Transversal

CAUTION!

In general, radial \neq normal, nor transverse \neq tangential!



Useful Methods

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$$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v$$

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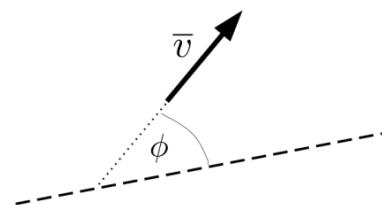
- ③ Integral by parts:

$$\int u'v = uv - \int v'u$$

Exercise 2

(12 points) A particle moves in a plane so that the angle between the particle's instantaneous velocity \bar{v} and its instantaneous acceleration \bar{a} is constant and equal to α . Let ϕ be the angle that the vector \bar{v} forms with a fixed direction on that plane (see the figure). Initially, the speed of the particle $|\bar{v}(0)| = v_0$ and $\phi(0) = \phi_0$.

Find the instantaneous speed of the particle as a function of the angle ϕ .



Force and Inertial Frame of Reference

Force

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Inertial frame of reference

In an inertial FoR, a physical object with zero net force acting on it moves with a constant velocity (which might be zero), or, equivalently, it is a frame of reference in which Newton's first law of motion holds.

Using Newton's Law in Complex System

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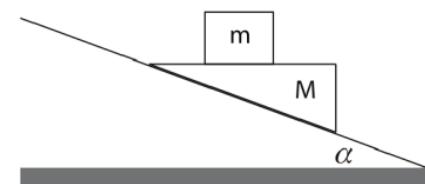
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- ① Objects at relatively rest → see as an whole.
- ② Analyze forces between objects (relatively at rest or in motion) → isolate each of them.
- ③ To maintain a relatively static condition: $|f| \leq \mu_s N$
- ④ Use inertia force in Non-inertia FoR (usually objects with constant acceleration \vec{a}), $\vec{F}' = m(-\vec{a})$.

Exercise 3

(12 points) A wedge with mass M is placed on a frictionless fixed plane inclined at an angle α . The upper surface of the wedge is horizontal. A block with mass m is placed on top of the wedge. The system is released from rest. Assuming that there is no friction between M and m , find the acceleration of the block m with respect to the wedge M just after the system is released from rest. The acceleration due to gravity g is known.



Motion with Air/Fluid Drag

Consider a particle with linear drag $\mathbf{F} = -k\mathbf{v}$ and initial velocity $\mathbf{v}_0 = v_0 \cos(\alpha) \hat{n}_x + v_0 \sin(\alpha) \hat{n}_y$, what's its trajectory?

Two recommended ways:

- ① decompose the drag force
- ② decompose the velocity

What if quadratic drag force?

$$F_x = -k v_x, \quad F_y = -k v_y.$$

$$x: -kv_x = m\ddot{x}$$

$$y: -kv_y - mg = m\ddot{y}.$$

$$x: m \frac{dv_x}{dt} = -kv_x.$$

$$\int_{v_{0\cos\alpha}}^{v_x} \frac{dv_x}{v_x} = \int_0^t -\frac{k}{m} dt.$$

$$\ln \frac{v_x}{v_{0\cos\alpha}} = -\frac{k}{m} t$$

$$\Rightarrow v_x = v_{0\cos\alpha} e^{-\frac{k}{m} t}.$$

$$y: m \frac{dv_y}{dt} = -kv_y - mg.$$

$$v_y' = v_y + \frac{mg}{k}.$$

$$\frac{dv_y}{dt} = -\frac{k}{m} v_y - g.$$

$$\frac{dv_y'}{dt} = -\frac{k}{m} v_y'$$

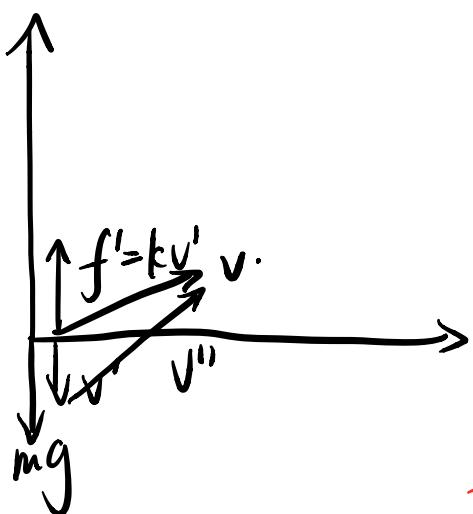
$$\int_{v_{0\sin\alpha} + \frac{mg}{k}}^{v_y'} \frac{dv_y'}{v_y'} = \int_0^t -\frac{k}{m} dt$$

$$\ln \frac{v_y'}{v_{0\sin\alpha} + \frac{mg}{k}} = -\frac{k}{m} t$$

$$\Rightarrow v_y = \left(v_{0\sin\alpha} + \frac{mg}{k} \right) e^{-\frac{k}{m} t} - \frac{mg}{k}$$

$$dx = v_x dt \Rightarrow \int_0^x dx = \int_0^t v_x dt \Rightarrow \left\{ x = v_{0\cos\alpha} \frac{m}{k} (1 - e^{-\frac{k}{m} t}) \right.$$

$$dy = v_y dt \Rightarrow \int_0^y dy = \int_0^t v_y dt \quad \left\{ y = \left(v_{0\sin\alpha} + \frac{mg}{k} \right) \frac{m}{k} (1 - e^{-\frac{k}{m} t}) - \frac{mg}{k} t \right.$$



$$\vec{V} = \vec{V}' + \vec{V}''$$

$f' = kV' = mg$, $V' = \frac{mg}{k}$ equivalent with G.

$\frac{\vec{V}''}{|\vec{V}''|}$: So \vec{V}' remain the same, in this direction, rectilinear motion.

$\frac{\vec{V}''}{|\vec{V}''|}$: \vec{V}'' , only the 'drag force', with acceleration

$$a'' = \frac{kV''}{m}.$$

$$\frac{dV''}{dt} = -\frac{kV''}{m}. \quad \int_{V''(0)}^{V''} \frac{dV''}{V''} = \int_0^t -\frac{k}{m} dt$$

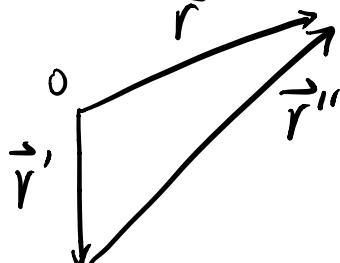
$$V''(0)^2 = (V \sin \theta + V')^2 + (V \cos \theta)^2$$

$$V''(0) = \sqrt{(V \sin \theta + \frac{mg}{k})^2 + (V \cos \theta)^2}$$

$$V'(t) = \sqrt{(V \sin \theta + \frac{mg}{k})^2 + (V \cos \theta)^2} e^{-\frac{k}{m}t}.$$

$$\Rightarrow \vec{V} = \vec{V}''(t) + \vec{V}'$$

$$\vec{r} = \vec{r}'' + \vec{r}' \quad \vec{r}(t) \text{ can be solved.}$$



this method can sometimes be more useful than the upper one when dealing linear force.

☰ Question

A projectile is launched at an angle to the horizontal, close to the earth surface. Assuming quadratic air drag, when is the magnitude of the projectile's total acceleration maximum?