VP160 Recitation Class II

Kinematics

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May 23, 2021

Kinematics in 3D: Cartesian Coordinates

By convention, we write

$$\frac{d\alpha}{dt} = \dot{\alpha}, \quad \frac{d^2\alpha}{dt^2} = \ddot{\alpha}$$

for convenience.

Kinematics in 3D: Cartesian Coordinates

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for convenience.

$\vec{r}(t) = (x(t), y(t), z(t))$ already describe all information of the motion.

Basic formulas in Cartesian Coordinates

$$\vec{r} = x\hat{n}_x + y\hat{n}_y + z\hat{n}_z$$

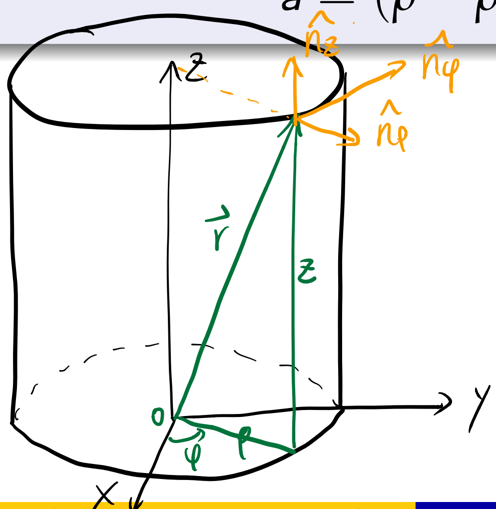
$$\vec{v} = \dot{x}\hat{n}_x + \dot{y}\hat{n}_y + \dot{z}\hat{n}_z$$

$$\vec{a} = \ddot{x}\hat{n}_x + \ddot{y}\hat{n}_y + \ddot{z}\hat{n}_z$$

Kinematics in 3D: Cylindrical Coordinates

Basic Formulas in Cylindrical Coordinates

$$\vec{r} = \rho \hat{n}_\rho + z \hat{n}_z$$
$$\vec{v} = \dot{\rho} \hat{n}_\rho + \rho \dot{\phi} \hat{n}_\phi + \dot{z} \hat{n}_z$$
$$\vec{a} = (\ddot{\rho} - \rho \dot{\phi}^2) \hat{n}_\rho + (\rho \ddot{\phi} + 2\dot{\rho}\dot{\phi}) \hat{n}_\phi + \ddot{z} \hat{n}_z$$



Jump to derivation on slide 8

Kinematics in 3D: Cylindrical Coordinates

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Tips:

- if $z = 0$, they become kinematics formulas in polar coordinates(see next slide).

Kinematics in 3D: Cylindrical Coordinates

Basic Formulas in Cylindrical Coordinates

$$
\begin{aligned}
\vec{r} &= \rho \hat{n}_\rho + z \hat{n}_z \\
\vec{v} &= \dot{\rho} \hat{n}_\rho + \rho \dot{\phi} \hat{n}_\phi + \dot{z} \hat{n}_z \\
\vec{a} &= (\ddot{\rho} - \rho \dot{\phi}^2) \hat{n}_\rho + (\rho \ddot{\phi} + 2\dot{\rho}\dot{\phi}) \hat{n}_\phi + \ddot{z} \hat{n}_z
\end{aligned}
$$

Tips:

- if $z = 0$, they become kinematics formulas in polar coordinates(see next slide).
- Very useful, do remember this set of formulas, otherwise you may have to derive them by yourselves during the exam!

Kinematics in 2D: Polar Coordinates

Basic Formulas in Polar Coordinates

$$\vec{r} = r\hat{n}_r \tag{1}$$

$$\vec{v} = \dot{r}\hat{n}_r + r\dot{\theta}\hat{n}_\theta \tag{2}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{n}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{n}_\theta \tag{3}$$

Can be seen as a special case of cylindrical coordinates when $z = 0$.

Kinematics in 2D: Polar Coordinates

Basic Formulas in Polar Coordinates

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Can be seen as a special case of cylindrical coordinates when $z = 0$.

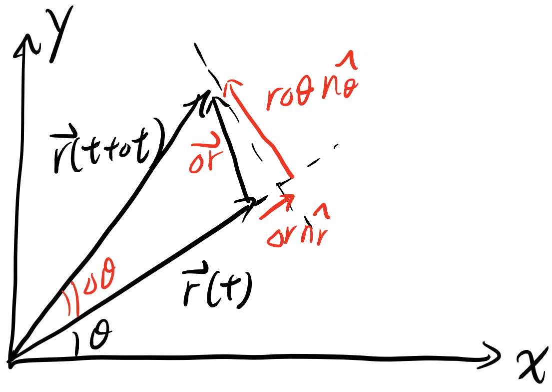
Relations with cartesian coordinates

$$x = rcos\theta, \ y = rsin\theta$$

$$d\vec{r} = dr\hat{n}_r + rd\theta\hat{n}_\theta, \ |d\vec{r}| = \sqrt{(dr)^2 + (rd\theta)^2}$$

$$dx = d(r\cos\theta) = \cos\theta\, dr + r(-\sin\theta)\, d\theta = \cos\theta\, dr - r\sin\theta\, d\theta.$$

$$dy = d(r\sin\theta) = \sin\theta\, dr + r\cos\theta\, d\theta.$$



$$d\vec{x} = \cos\theta\, dr\, \hat{n}_x - r\sin\theta\, d\theta\, \hat{n}_x$$

$$d\vec{y} = \sin\theta\, dr\, \hat{n}_y + r\cos\theta\, d\theta\, \hat{n}_y$$

x: $dr\hat{n}_r\cos\theta - rd\theta\sin\theta = dx = \cos\theta\, dr - r\sin\theta\, d\theta.$

y: ...

Examples

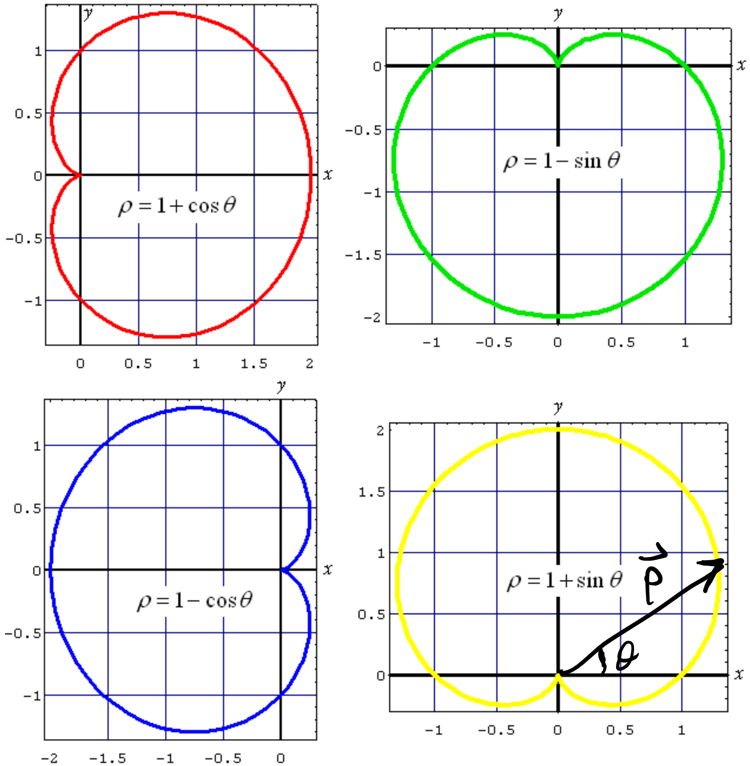


Figure: Cardiod.

Examples Lemniscate:

$$r^2 = 2A^2 cos 2\theta.$$

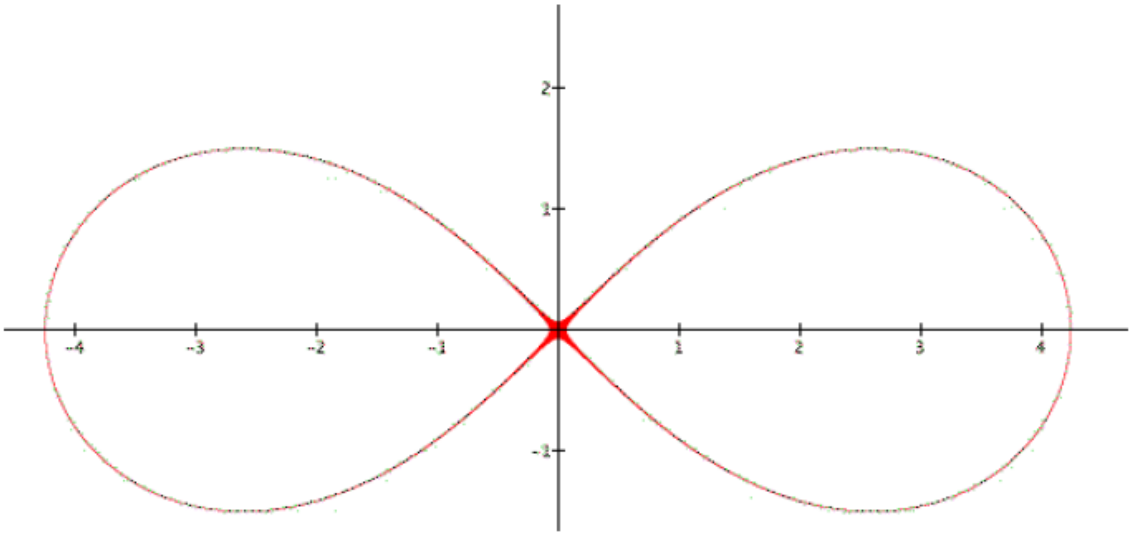


Figure: Lemniscate.

Exercise 1

How to derive this set of formula(1)(2)(3)?

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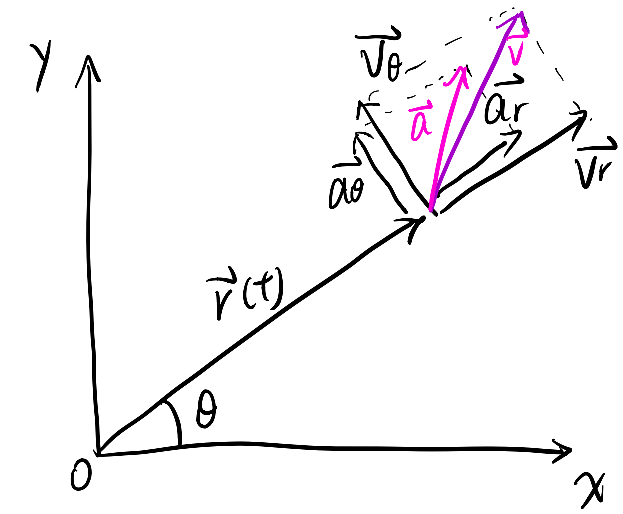
How to derive this set of formula$(1)(2)(3)$?

Tips:

If you can derive it by yourself, you can be very confident about kinematics problems in exams.



$$\begin{cases} \dot{\hat{n}}_r = \dot{\theta}\,\hat{n}_\theta \\ \dot{\hat{n}}_\theta = -\dot{\theta}\,\hat{n}_r \end{cases} \qquad (*)$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d(r\cdot\hat{n}_r)}{dt} = \frac{dr}{dt}\hat{n}_r + r\cdot\frac{d\hat{n}_r}{dt}$$

$$r\frac{d\hat{n}_r}{dt} = r\dot{\hat{n}}_r = r\dot{\theta}\hat{n}_\theta$$

$$\Rightarrow \vec{v} = \dot{r}\hat{n}_r + r\dot{\theta}\hat{n}_\theta = \vec{V_r} + \vec{V_\theta}$$

$\vec{V_r} = \dot{r}\hat{n}_r$, $\vec{V_\theta} = r\dot{\theta}\hat{n}_\theta$

radius component   transversial component

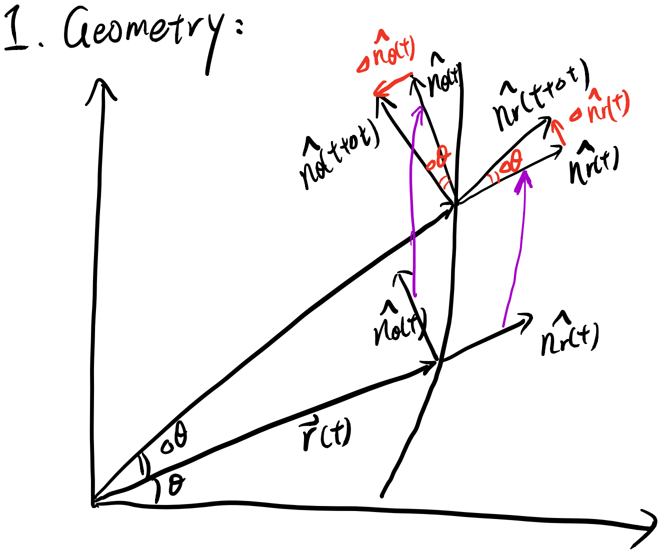
$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{r}\hat{n}_r + r\dot{\theta}\hat{n}_\theta) = \frac{d\dot{r}}{dt}\hat{n}_r + \dot{r}\frac{d\hat{n}_r}{dt} + \frac{dr\dot{\theta}}{dt}\hat{n}_\theta + r\dot{\theta}\frac{d\hat{n}_\theta}{dt}$$

$$= \ddot{r}\hat{n}_r + \dot{r}\dot{\theta}\hat{n}_\theta + \dot{r}\dot{\theta}\hat{n}_\theta + r\cdot\ddot{\theta}\hat{n}_\theta - r\dot{\theta}^2\hat{n}_r$$

$$\Rightarrow \vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{n}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{n}_\theta$$

How to derive (*):

1. Geometry:



① $\Delta\hat{n}_\theta(t) = -|\hat{n}_\theta(t)|\Delta\theta\,\hat{n}_r$ $|\hat{n}| = 1$

$$\frac{\Delta\hat{n}_\theta(t)}{\Delta t} = -\frac{\Delta\theta}{\Delta t}\cdot\hat{n}_r$$

$$\Rightarrow \frac{d\hat{n}_\theta(t)}{dt} = -\frac{d\theta}{dt}\hat{n}_r$$

$$\dot{\hat{n}}_\theta = -\dot{\theta}\hat{n}_r$$

② $\Delta\hat{n}_r(t) = |\hat{n}_r(t)|\Delta\theta\cdot\hat{n}_\theta$

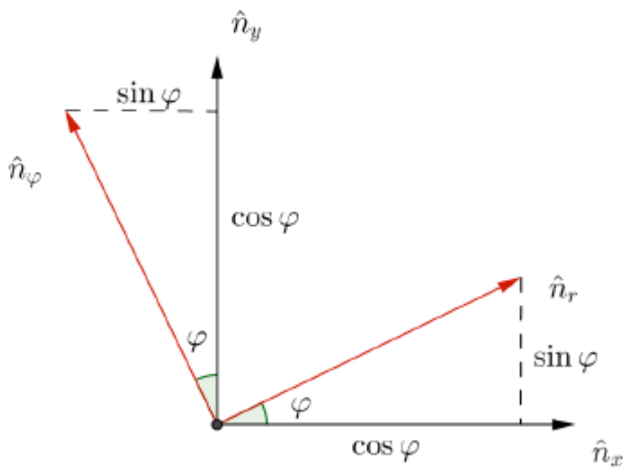
$$\frac{\Delta\hat{n}_r(t)}{\Delta t} = \frac{\Delta\theta}{\Delta t}\hat{n}_\theta$$

$$\Rightarrow \frac{d\hat{n}_r(t)}{dt} = \frac{d\theta}{dt}\hat{n}_\theta$$

$$\dot{\hat{n}}_r = \dot{\theta}\hat{n}_\theta$$

2. derivative

How to find the derivative $\hat{n}_r$ (and $\hat{n}_\varphi$) ?



$$\hat{n}_r = \cos\varphi\hat{n}_x + \sin\varphi\hat{n}_y$$
$$\hat{n}_\varphi = -\sin\varphi\hat{n}_x + \cos\varphi\hat{n}_y$$

$$\begin{aligned}\boxed{\dot{\hat{n}}_r} &= -\dot{\varphi}\sin\varphi\hat{n}_x + \dot{\varphi}\cos\varphi\hat{n}_y \\ &= \dot{\varphi}(-\sin\varphi\hat{n}_x + \cos\varphi\hat{n}_y) = \\ &= \boxed{\dot{\varphi}\hat{n}_\varphi}\end{aligned}$$

Similarly,

$$\begin{aligned}\boxed{\dot{\hat{n}}_\varphi} &= -\dot{\varphi}\cos\varphi\hat{n}_x - \dot{\varphi}\sin\varphi\hat{n}_y \\ &= -\dot{\varphi}(\cos\varphi\hat{n}_x + \sin\varphi\hat{n}_y) = \\ &= \boxed{-\dot{\varphi}\hat{n}_r}\end{aligned}$$

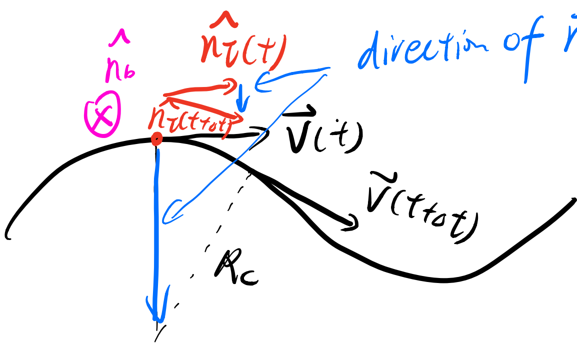
Kinematics in 3D: Natural Coordinates

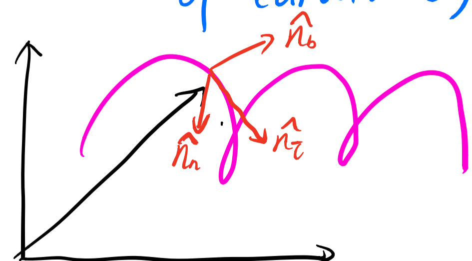
Basic Vectors

1. $\hat{n}_\tau$: along the direction of $\vec{v}$
2. $\hat{n}_n$ and $\hat{n}_b$: perpendicular to the direction of $\vec{v}$

$$\underbrace{\hat{n}_\tau}_{\text{tangential}} = \frac{\vec{v}}{|\vec{v}|}, \quad \underbrace{\hat{n}_n}_{\text{normal}} = \frac{\dot{\hat{n}}_\tau}{|\dot{\hat{n}}_\tau|}, \quad \underbrace{\hat{n}_b}_{\text{binormal}} = \hat{n}_\tau \times \hat{n}_n$$

direction of $\dot{\hat{n}}_\tau$ (point to the center of radius of curvature)





$\hat{n}_\tau$ and $\hat{n}_n$ is enough to describe the motion in 3D coordinates.

Kinematics in 3D: Natural Coordinates

Basic Vectors

1. $\hat{n}_\tau$: along the direction of $\vec{v}$
2. $\hat{n}_n$ and $\hat{n}_b$: perpendicular to the direction of $\vec{v}$

$$\hat{n}_\tau = \frac{\vec{v}}{|\vec{v}|}, \quad \hat{n}_n = \frac{\dot{\hat{n}}_\tau}{|\dot{\hat{n}}_\tau|}, \quad \hat{n}_b = \hat{n}_\tau \times \hat{n}_n$$

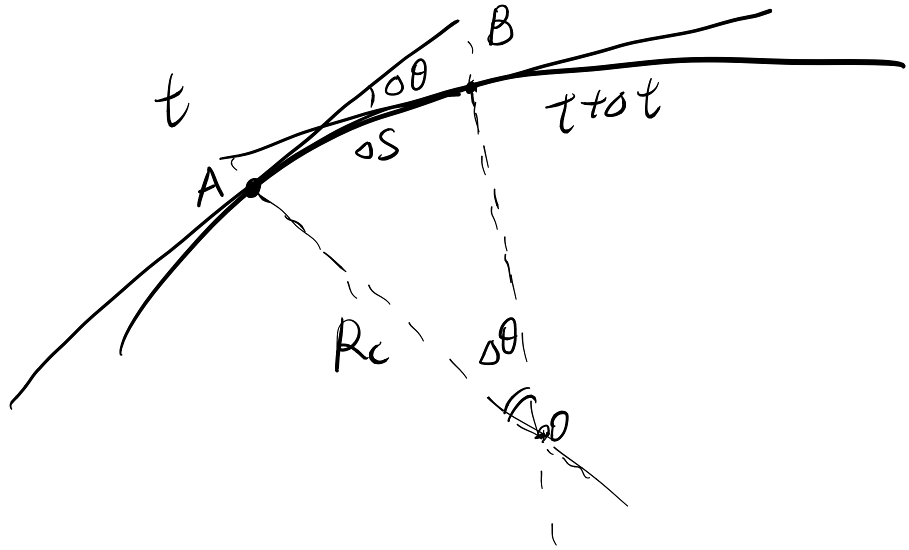
Basic Formulas

$$\vec{v} = v\hat{n}_\tau$$

$$\vec{a} = \dot{v}\hat{n}_\tau + \frac{v^2}{R_c}\hat{n}_n$$

centralpetal acceleration of an instantaneous circular motion

R_c means radius of curvature, what is radius of curvature?



$$R_c = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta \theta}$$

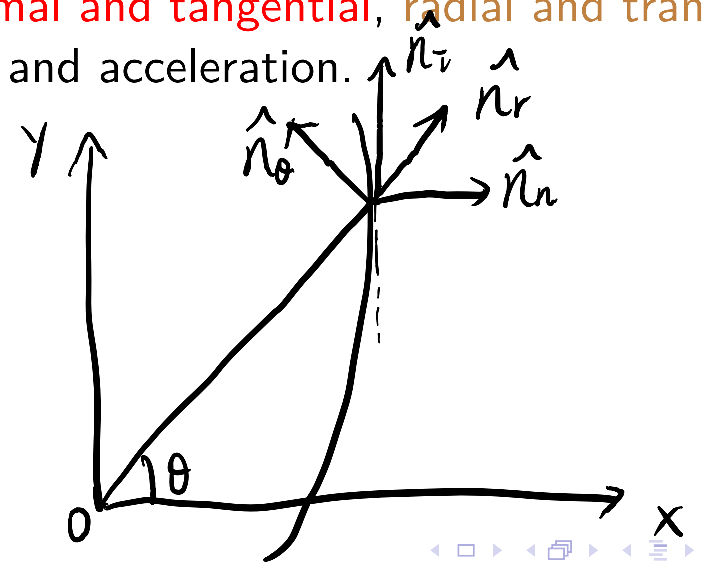
General formula: $$R_c = \left| \frac{(1+f'^2)^{\frac{3}{2}}}{f''} \right| \quad \left(\begin{array}{l} f' = \dfrac{df(x)}{dx} \\ f'' = \dfrac{d^2 f(x)}{dx^2} \end{array} \right)$$

Important Tips

1. Difference between $\dot{\vec{v}}$ and $\dot{v}$.

Important Tips

1. Difference between $\dot{\vec{v}}$ and $\dot{v}$.
2. Difference between normal and tangential, radial and transversal components of velocity and acceleration.



Kinematics in 3D: Spherical Coordinates (Optional)

Basic Formulas in Spherical Coordinates

$$
\begin{aligned}
\vec{r} &= r\hat{n}_r \\
\vec{v} &= \dot{r}\hat{n}_r + rcos\phi\dot{\theta}\hat{n}_\theta + r\dot{\phi}\hat{n}_\phi \\
\vec{a} &= (\ddot{r} - r\dot{\theta}^2cos^2\phi - r\dot{\phi}^2)\hat{n}_r \\
&\quad + (2\dot{r}\dot{\theta}cos\phi - 2r\dot{\theta}\dot{\phi}sin\phi + r\ddot{\theta}cos\phi)\hat{n}_\theta \\
&\quad + (2\dot{r}\dot{\phi} + r\dot{\theta}^2cos\phi sin\phi + r\ddot{\phi})\hat{n}_\phi
\end{aligned}
$$

Kinematics in 3D: Spherical Coordinates (Optional)

Basic Formulas in Spherical Coordinates

$$
\begin{aligned}
\vec{r} &= r\hat{n}_r \\
\vec{v} &= \dot{r}\hat{n}_r + rcos\phi\dot{\theta}\hat{n}_\theta + r\dot{\phi}\hat{n}_\phi \\
\vec{a} &= (\ddot{r} - r\dot{\theta}^2cos^2\phi - r\dot{\phi}^2)\hat{n}_r \\
&\quad + (2\dot{r}\dot{\theta}cos\phi - 2r\dot{\theta}\dot{\phi}sin\phi + r\ddot{\theta}cos\phi)\hat{n}_\theta \\
&\quad + (2\dot{r}\dot{\phi} + r\dot{\theta}^2cos\phi sin\phi + r\ddot{\phi})\hat{n}_\phi
\end{aligned}
$$

How to derive them?

http://output.to/sideway/default.aspx?qno=140700003

Exercise 2

Cartesian coordinates and natural coordinates

A particle moves in the x-y plane so that

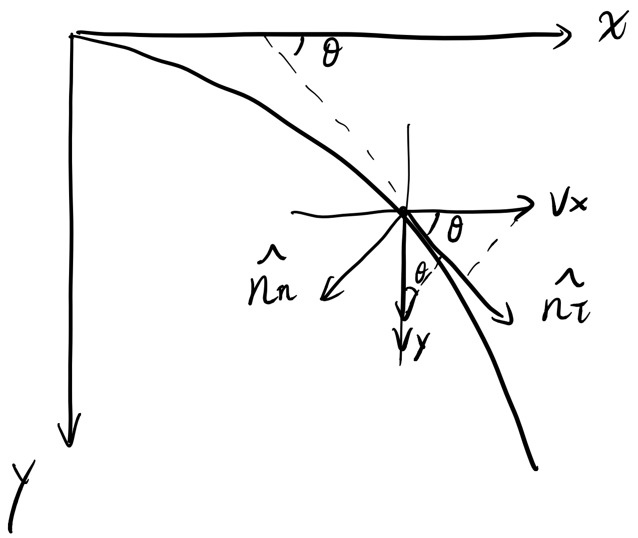
$$x(t) = at,\ \ y(t) = bt^2,$$

where a, b are positive constants. Find its trajectory, velocity, and acceleration (its tangential and normal components).

Trajectory: $t = \frac{x}{a}$, $y = b \cdot \frac{x^2}{a^2} = \frac{b}{a^2} x^2$

$v_x = a$, $v_y = 2bt$

$a_x = 0$, $a_y = 2b$



Two ways:

① Geometry relationships

$$\tan\theta = \frac{dy}{dx} = \frac{2bx}{a^2} = \frac{2b}{a} t$$

$$v_\tau = v_x \cos\theta + v_y \sin\theta = \ldots$$

$$v_n = -v_x \sin\theta + v_y \cos\theta = \ldots$$

a_τ, a_n similarly

② Calculate $\hat{n}_\tau$, $\hat{n}_n$ directly

$$\hat{n}_\tau = \frac{\vec{v}}{|\vec{v}|} = \begin{pmatrix} a/\sqrt{a^2+4b^2t^2} \\ 2bt/\sqrt{a^2+4b^2t^2} \end{pmatrix}$$

$$\hat{n}_n = \begin{pmatrix} -2bt/\sqrt{a^2+4b^2t^2} \\ a/\sqrt{a^2+4b^2t^2} \end{pmatrix}$$

$\vec{v_\tau} = \vec{v} \cdot \hat{n}_\tau$, $\vec{v_n} = \vec{v} \cdot \hat{n}_n$

$\vec{a_\tau} = \vec{a} \cdot \hat{n}_\tau$, $\vec{a_n} = \vec{a} \cdot \hat{n}_n$

Exercise 3

Cartesian coordinates and natural coordinates

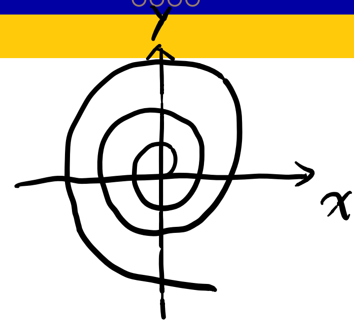
The velocities of two particles observed from a fixed frame of reference are given in the Cartesian coordinates by vectors $\vec{v}_1(t) = (0,2,0) + (3,1,2)t^2$ and $\vec{v}_2(t) = (1,0,1)$. At the initial instant of time $t = 0$, the positions of these particles are $\vec{r}_1(0) = (1,0,0)$ and $\vec{r}_2(0) = (0,1,1)$.

Find: the positions of both particles and the acceleration of particle 1 (and its tangential and normal components), relative position, and relative acceleration of particle 1 with respect to particle 2 at any instant of time t.

$\vec{a}_1 - \vec{a}_2$

$\vec{r}_1 - \vec{r}_2$

Tip: use method ② in Ex. 2



Exercise 4

Polar coordinates (Archimedes' spiral)

A disc of radius R rotates about its axis of symmetry (perpendicular to the disk surface) with constant angular velocity $\dot{\varphi} = \omega = \text{const}$. At the instant of time $t = 0$ a beetle starts to walk with constant speed v_0 along a radius of the disk, from its center to the edge.

Find

(a) the position of the beetle and its trajectory in the Cartesian and polar coordinate systems,

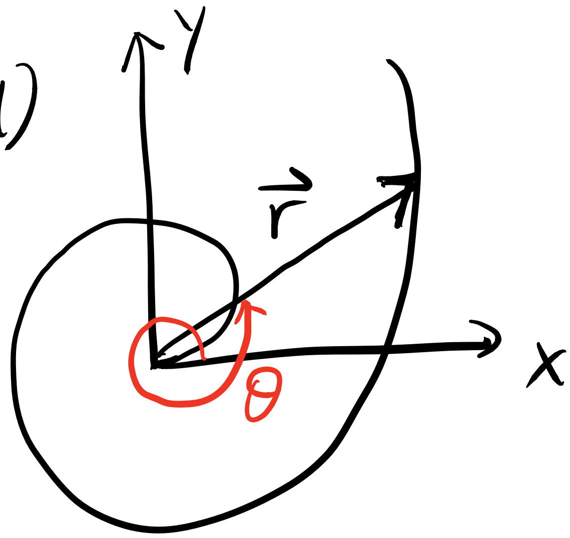
(b) its velocity both systems,

(c) its acceleration in both systems (Cartesian components, polar components, as ~~well as tangential~~ and normal components).

$$\vec{v} = \dot{r}\hat{n}_r + r\dot{\theta}\hat{n}_\theta$$

$$\boxed{\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{n}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{n}_\theta}$$

(a)



$\theta = \omega t \qquad t = \frac{\theta}{\omega}$

$r = v_0 t$

$x = r\cos\theta = v_0 t \cos\omega t$

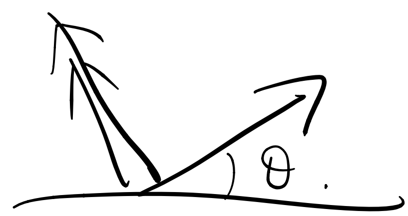
$y = r\sin\theta = v_0 t \sin\omega t.$

$r(\theta) = v_0 \frac{\theta}{\omega}.$

(b) $v_r = \dot{r} = v_0$

$v_\theta = r\dot{\theta} = v_0 t \cdot \omega = v_0 \omega t.$

$\vec{V} = v_0 \hat{n}_r + v_0 \omega t \cdot \hat{n}_\theta$



$\hat{n}_r = \hat{n}_x \cos\theta + \hat{n}_y \sin\theta$

$\hat{n}_\theta = -\hat{n}_x \sin\theta + \hat{n}_y \cos\theta$

(c) $\ddot{r} = \frac{dv_0}{dt} = 0 \qquad r\ddot{\theta} = 0 \qquad \boxed{\dot{\omega}} = 0$

$-r\dot{\theta}^2 = -v_0 t \cdot \omega^2 \qquad 2\dot{r}\dot{\theta} = 2v_0 \omega$

(a) $\begin{cases} \varphi(t) = \omega t \\ r(t) = v t \end{cases} \Rightarrow r = \frac{v_0}{\omega}\varphi$ $\quad\begin{cases} x = v_0 t \cos\omega t \\ y = v_0 t \sin\omega t \end{cases} \to \underline{x^2+y^2 = v_0^2 t^2}$

(b) $\vec{v} = v_0 \cdot \hat{n}_r + v_0 \omega t \cdot \hat{n}_\varphi$

$= (v_0 \cos\omega t - v_0 t\,\omega \sin\omega t)\,\hat{n}_x + (v_0 \sin\omega t + v_0 t\omega \cos\omega t)\,\hat{n}_y$

(c) $\vec{a} = (\ddot{r} - r\dot{\varphi}^2)\hat{n}_r + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{n}_\varphi$

$= -(v_0 t \cdot \omega^2)\cdot \hat{n}_r + 2v_0\omega\,\hat{n}_\varphi$

(d) $(ds)^2 = (dr)^2 + (r\,d\varphi)^2$

$= (v_0 dt)^2 + (v_0\omega t\,dt)^2$

$= (v_0^2 + v_0^2\omega^2 t^2)(dt)^2$

$ds = v_0\sqrt{1+\omega^2 t^2}\;dt$

$s = \int_0^t ds = \int_0^t v_0\sqrt{1+\omega^2 t^2}\,dt$

(e) In polar system: $(\hat{r}, \hat{\varphi})$

$\hat{n}_z = \begin{pmatrix} 1/\sqrt{1+\omega^2 t^2} \\ \omega t/\sqrt{1+\omega^2 t^2} \end{pmatrix} \qquad \hat{n}_n = \begin{pmatrix} -\omega t/\sqrt{1+\omega^2 t^2} \\ 1/\sqrt{1+\omega^2 t^2} \end{pmatrix}$

$a_n = \vec{a} \cdot \hat{n}_n$

$= \frac{v_0 t\omega^2 \cdot \omega t}{\sqrt{1+\omega^2 t^2}} + \frac{2v_0\omega}{\sqrt{1+\omega^2 t^2}} = \frac{v_0\omega(2+\omega^2 t^2)}{\sqrt{1+\omega^2 t^2}}$

$R_c = \frac{v^2}{a_n} = \frac{v_0^2(1+\omega^2 t^2)}{a_n}.$

Exercise 5

Polar coordinates

A particle moves along a hyperbolic spiral (i.e. a curve $r = c/\varphi$, where c is a positive constant), so that $\varphi(t) = \varphi_0 + \omega t$, where φ_0 and ω are positive constants. **Find** its velocity and acceleration (all components and magnitudes of both vectors).

$$\varphi(t) = \varphi_0 + \omega t \qquad \dot{\varphi} = \omega. \qquad \ddot{\varphi} = 0$$

$$r(t) = \frac{c}{\varphi} = \frac{c}{\varphi_0 + \omega t} \qquad \dot{r} = \frac{-\omega c}{(\varphi_0+\omega t)^2} \qquad \ddot{r} = \frac{+2(\varphi_0+\omega t)\cdot\omega^2 c}{(\varphi_0+\omega t)^4}$$

$$= \frac{2\omega^2 c}{(\varphi_0+\omega t)^3}$$

$$\vec{v}(t) = \dot{r}\,\hat{n}_r + r\dot{\varphi}\cdot\hat{n}_\varphi$$

$$= \frac{-\omega c}{(\varphi_0+\omega t)^2}\cdot\hat{n}_r + \frac{c\omega}{\varphi_0+\omega t}\cdot\hat{n}_\varphi$$

$$\vec{a}(t) = (\ddot{r} - r\dot{\varphi}^2)\hat{n}_r + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{n}_\varphi$$

$$= \left(\frac{2c\omega^2}{(\varphi_0+\omega t)} - \frac{c\omega^2}{\varphi_0+\omega t}\right)\hat{n}_r + \left(0 + \frac{-2\omega^2 c}{(\varphi_0+\omega t)^2}\right)\hat{n}_\varphi$$

Exercise 4

Polar coordinates (Archimedes' spiral)

A disc of radius R rotates about its axis of symmetry (perpendicular to the disk surface) with constant angular velocity $\dot{\varphi} = \omega =$ const. At the instant of time $t = 0$ a beetle starts to walk with constant speed v_0 along a radius of the disk, from its center to the edge.

Find

(a) the position of the beetle and its trajectory in the Cartesian and polar coordinate systems,

(b) its velocity both systems,

(c) its acceleration in both systems (Cartesian components, polar components, as well as tangential and normal components).

Exercise 6

For the situation discussed in Exercise 3, answer the following questions.
(a) What is the distance covered by the beetle (write down the integral only, do not evaluate it)?
(b) What is the radius of curvature of the trajectory?

Exercise 7

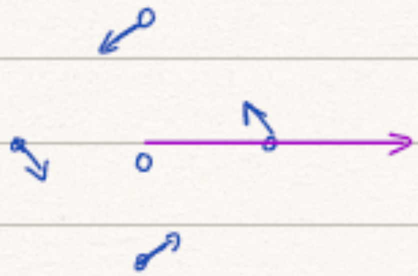
Four spiders are initially placed at the four corners of a square with side length a. The spiders crawl counter-clockwise at the same speed and each spider crawls directly toward the next spider at all times. They approach the center of the square along spiral paths.

Find

(a) polar coordinates of a spider at any instant of time, assuming the origin is at the center of the square,

(b) the time after which all spiders meet,

(c) the trajectory of a spider in polar coordinates.



(a) $\begin{cases} \dot{r} = v_r = -\frac{\sqrt{2}}{2} v_0 & ① \\ r\dot{\varphi} = v_\theta = \frac{\sqrt{2}}{2} v_0 & ② \end{cases}$

①: $r = \int \dot{r}\, dt = \int_0^t \left(-\frac{\sqrt{2}}{2} v_0\right) dt = r_0 - \frac{\sqrt{2}}{2} v_0 t$

$$r(t) = \frac{\sqrt{2}}{2}(a - v_0 t) \quad ③$$

③ → ② $\quad \dot{\varphi} = \dfrac{\frac{\sqrt{2}}{2} v_0}{\frac{\sqrt{2}}{2}(a - v_0 t)} = \dfrac{v_0}{a - v_0 t}$

$\dfrac{d\varphi}{dt} = \dfrac{v_0}{a - v_0 t} \qquad \underline{d\varphi = \dfrac{v_0}{a - v_0 t}\, dt} \quad ④$

$$\varphi - 0 = \int_a^{a - v_0 t} \frac{\cancel{v_0}}{a - v_0 t}\, d(a - v_0 t) \cdot \frac{-1}{\cancel{v_0}}$$

$$= -\ln \frac{a - v_0 t}{a} = \ln \frac{v_0 t - a}{a}$$

$$\varphi(t) = \ln\left(\frac{v_0 t - a}{a}\right)$$

(c) $\varphi = -\ln \dfrac{\sqrt{2} r}{a} \qquad \ln \dfrac{\sqrt{2} r}{a} = -\varphi$

$$\underbrace{r = \frac{a}{\sqrt{2}} e^{-\varphi}}$$

Or by one of our students:

$\dfrac{①}{②} = \dfrac{dr}{r} = -d\varphi.$

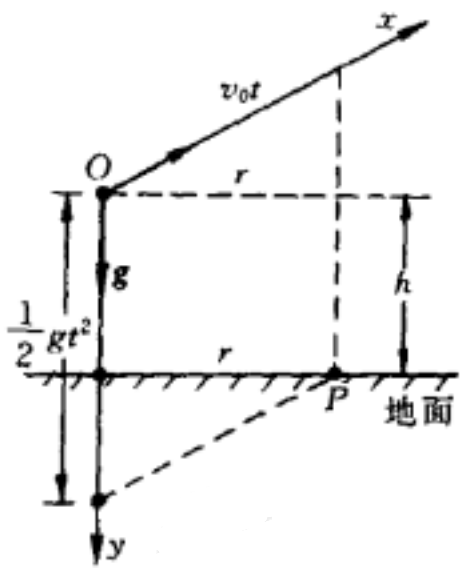
$\int \dfrac{dr}{r} = \int -d\varphi$

$\ln r = -\varphi + C$

when $\varphi = 0$, $\ln r(0) = C = \ln \frac{\sqrt{2}}{2} a$. $\Rightarrow \ln \dfrac{r}{\frac{\sqrt{2}}{2} a} = -\varphi \Rightarrow r = \frac{\sqrt{2}}{2} a e^{-\varphi}$

Exercise 8 (supplement)

The hall chandelier with a height of h above the ground exploded into fragments, shooting in all directions, the initial velocity was the same as v_0, and the fragments would not bounce when them reach the ground. Try to find the radius R of the debris distribution area on the ground.



解 某碎片于 $t=0$ 时刻于 O 点以初速 $\boldsymbol{v}_0$ 抛出，设置沿 $\boldsymbol{v}_0$ 方向的 x 轴和竖直向下的 y 轴，t 时刻碎片落于地面 P 处，如图 1-4 所示. P 到灯柱距离记为 r，则应有

$$r^2=(v_0t)^2-\left(\frac{1}{2}gt^2-h\right)^2$$

$$=-\frac{1}{4}g^2t^4+(v_0^2+gh)t^2-h^2.$$

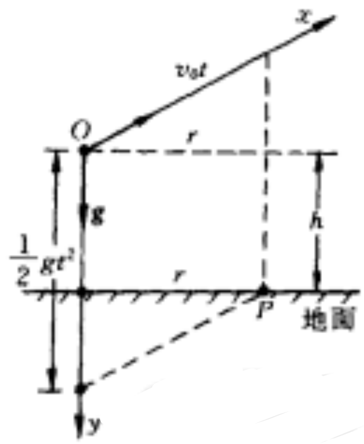


图 1-4

不同的碎片有不同的 $\boldsymbol{v}_0$，对应不同的落地时间 t，落地点 P 有不同的 r 值，所求 R 即为这些 r 中的极大值. 因 r^2 是 t^2 的二次函数，当

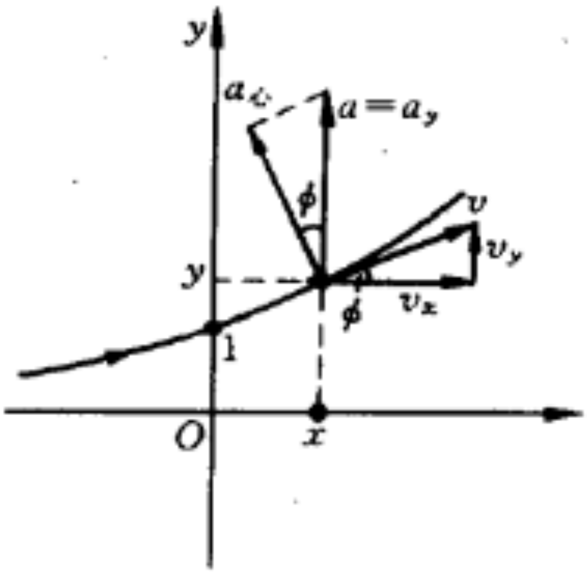
$$t^2=\frac{2}{g^2}(v_0^2+gh)$$

时，r^2 取得极大，对应 r 取得极大值，即有

$$R=\frac{v_0}{g}\sqrt{v_0^2+2gh}.$$

Exercise 9 (supplement)

Using kinematics method to solve the distribution function $\rho(x)$ of the radius of curvature of the curve $y = e^x$.



解 设质点沿 $y=e^x$ 轨道运动过程中的 x 方向分运动为匀速直线运动，即

$$x = v_0 t,$$

则 y 方向分运动为

$$y = e^{v_0 t}.$$

继而可得

$$v_x = v_0, \quad v_y = v_0 e^{v_0 t}.$$

$$a_x = 0, \quad a_y = v_0^2 e^{v_0 t}.$$

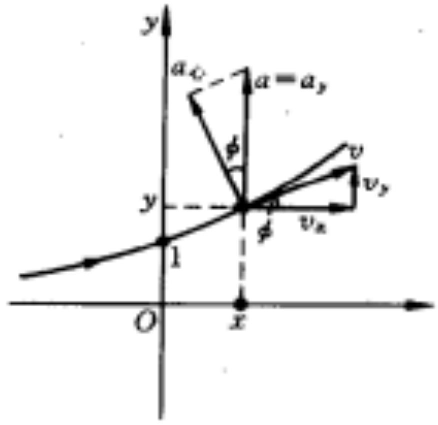


图 1-7

参考图 1-7，有

$$a_{心} = a\cos\phi = a_y \frac{v_x}{v} = v_0^3 \frac{e^{v_0 t}}{v},$$

$$\rho = \frac{v^2}{a_{心}} = \frac{v^3}{v_0^3 e^{v_0 t}}.$$

将

$$v = \sqrt{v_x^2 + v_y^2} = v_0\sqrt{1+e^{2v_0 t}}$$

代入，有

$$\rho = (1+e^{2v_0 t})^{\frac{3}{2}}/e^{v_0 t}.$$

$v_0 t$ 用 x 替换，即得

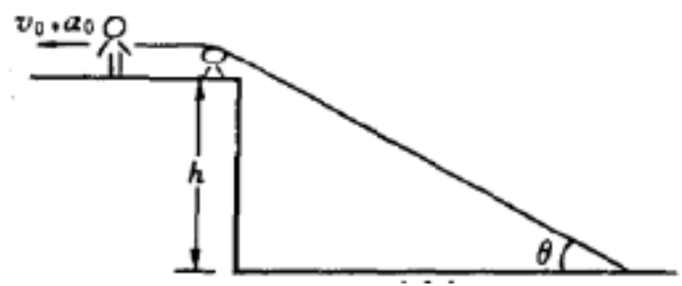
$$\rho = (1+e^{2x})^{\frac{3}{2}}/e^x,$$

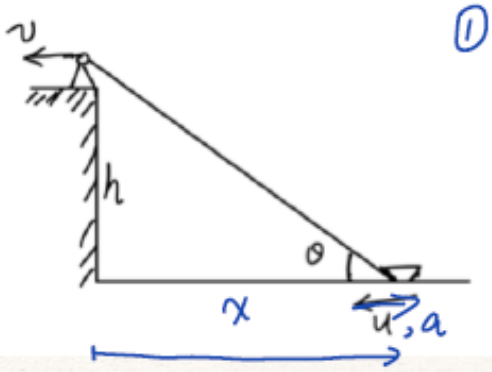
或

$$\rho = (1+y^2)^{\frac{3}{2}}/y.$$

Exercise 10 (supplement)

The height of the river bank is h, and people use ropes to pull the boat to shore. If the angle between the rope and the water surface is θ, the speed of the human left is v_0 and the acceleration is a_0. Try to find the speed v and acceleration a of the ship at this time.





① Method 1: use x. (Chain rule)

$$x = h/\tan\theta = \frac{h}{\tan\theta}$$

$$u = \dot{x} = \frac{dx}{dt} = \frac{dx}{d\theta}\cdot\frac{d\theta}{dt}$$

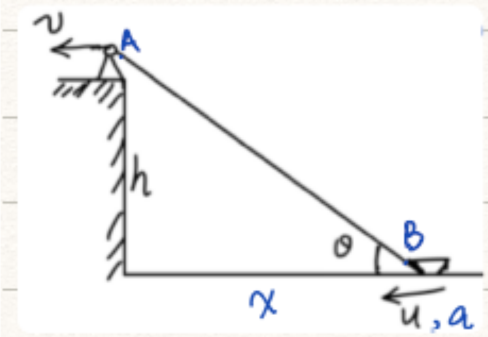
$$\frac{dx}{d\theta} = \left(\frac{h}{\tan\theta}\right)' = \frac{-\frac{h}{\cos^2\theta}}{\tan^2\theta} = \frac{-h}{\sin^2\theta}$$

$$l = \frac{h}{\sin\theta}, \quad -\frac{dl}{dt} = v \iff -\frac{dl}{d\theta}\cdot\frac{d\theta}{dt} = v \iff \frac{+\cos\theta h}{\sin^2\theta}\cdot\frac{d\theta}{dt} = v$$

$$\frac{d\theta}{dt} = \frac{+\sin^2\theta v}{\cos\theta h} = \frac{+\sin\theta\tan\theta v}{h}.$$

So finally $u = \frac{dx}{d\theta}\cdot\frac{d\theta}{dt} = \frac{-h}{\sin^2\theta}\cdot\frac{+\sin\theta\tan\theta v}{h} = \frac{-v}{\cos\theta}$

$$a = \dot{u} = \frac{du}{dt} = \frac{du}{d\theta}\cdot\frac{d\theta}{dt} = \frac{-\sin\theta v}{\cos^2\theta}\cdot\frac{+\sin\theta\tan\theta v}{h} = \frac{-\tan^3\theta v^2}{h}.$$



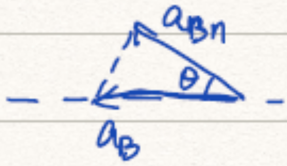
Method 2: $v_{An} = v_{Bn}$

$$v = u\cos\theta \Rightarrow u = \frac{v}{\cos\theta}$$

$$a_{An} = 0 \qquad a_{(B-A)_n} = w^2\cdot l$$

$$w = \frac{u\sin\theta}{l} = \frac{v\tan\theta}{h/\sin\theta} = \frac{v\tan\theta\sin\theta}{h}$$

$$a_{(B-A)n} = w^2\cdot l = \frac{v^2\tan^2\theta\sin^2\theta}{h^2}\cdot\frac{h}{\sin\theta} = \frac{v^2\tan^2\theta\sin\theta}{h} = a_{Bn}$$



$$a_B = \frac{a_{Bn}}{\cos\theta} = \frac{v^2\tan^3\theta}{h}.$$

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