VP160 Final Exam Review Class

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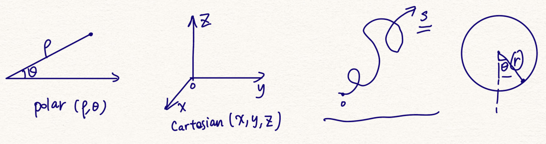
Generalized coordinates

Any coordinates describing motions.

e.g.



Degree of freedom (f)

The minimum number of independent generalized coordinates needed to describe the system's motions.

$$f = 3N - m$$

where N is the number of particles, and m is the number of constraints (equations that relate unknowns).

Hamilton's Principle (math detail don't required)

$$\text{Real path} \iff \delta S = 0$$

(δ: variational differential, S is a functional: a function that maps functions into numbers.)

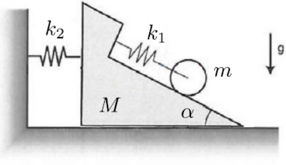
Euler-Lagrange Equation

For $i = 1, 2, ..., f$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

Exercise 1

(*16 points*) A wedge with mass M is placed on a frictionless horizontal surface. The wedge is inclined at an angle α to the horizontal. A uniform cylinder of mass m, and radius R can roll on the wedge without slipping. An axle, that coincides with the symmetry axis of the cylinder, is connected to the top of wedge by a massless spring, obeying Hooke's law with the spring constant k_1. The wedge is connected to a non-moving wall by another massless spring, satisfying Hooke's law with the spring constant k_2. The acceleration due to gravity is g.



(a) How many degrees of freedom does the system have? Define the corresponding generalized coordinates.

(b) Find expressions for the kinetic energy and the potential energy for the system.

(c) Use the Lagrange formalism to find the equations of motion of the system. (Do not attempt to solve them!)

Definition

$$\vec{p} = m\vec{v}$$

Rewrite Newton's second law

$$\vec{F} = \frac{d\vec{p}}{dt}$$

(when m is not varying, $F = m\frac{d\vec{v}}{dt} = m\vec{a}$)

Impulse Theorem

$$\vec{p_2} - \vec{p_1} = \int_{t_1}^{t_2} \vec{F} dt$$

- If $\vec{F_{ext}} = 0$, for a system, $\Delta\vec{p} = 0 \Leftrightarrow \vec{p} = \text{Const}$ (Conservation of momentum)

Collision

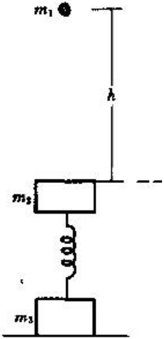
- Non-central Collision (e.g. explosion)

$$\vec{p_{before}} = \vec{p_{after}}$$

- Central Collision
  - Elastic
    - $e = (\vec{v_2}' - \vec{v_1}')/(\vec{v_1} - \vec{v_2}) = 1$ (seperating speed / approaching speed)
    - Conservation of energy
  - Inelastic
    - $e < 1$
    - Energy loss
  - Completely Inelastic
    - $e = 0$
    - stick to each other

Exercise 2

Assume m_1, m_2, m_3, k is known. Release m_1, the collision between m_1 and m_2 is completely inelastic. Find h so that m_3 can just leave the ground.



Answer: $\frac{(m_1+m_2)[(m_1+m_2+m_3)^2-m_2^2]}{2km_1^2}$

Center of Mass

$$\vec{r_C} = \frac{\sum m_i \vec{r_i}}{\sum m_i}$$

$$\vec{r_C} = \frac{\int \vec{r_i} dm}{\int dm}$$

Pappus Law (Another way to derive center of mass)

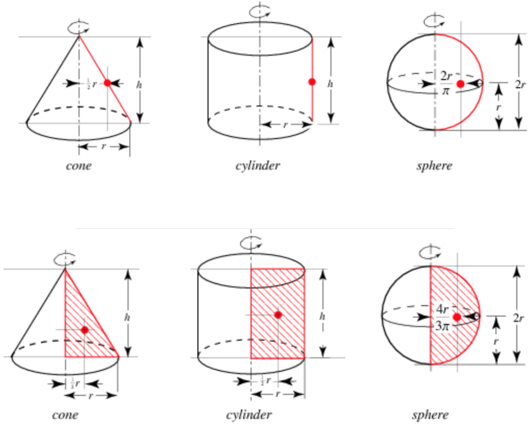
First Theorem (for object with linear mass density):

$$S = 2\pi s x \ (s \text{ the curve length})$$

Second Theorem (for plane):

$$V = 2\pi A x \ (A \text{ the plane area})$$

where x is the distance from the reference axis and the center of mass.


h
$\frac{1}{2}r$
r
cone
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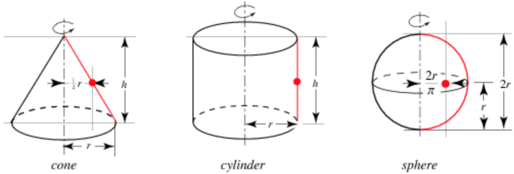
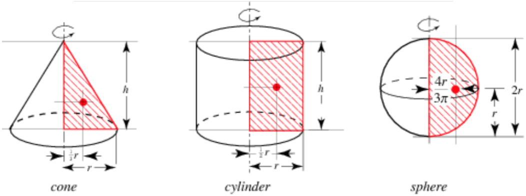
2r
r
sphere





- An important fact (for any system (total mass = constant), e.g. rigid body):

$$\vec{F_{ext}} = \dot{\vec{p}} = M\dot{\vec{v_c}}$$

Mass Variation Problem

Rocket Propulsion

$$mv + Fdt = (m + dm)(v + dv) - udm$$

$$m\frac{dv}{dt} = (u - v)\frac{dm}{dt} + F$$

Reminder: What FoR are we looking at?

Some string problems?

Tips:

- infinitesimal methods
- write out $F = \frac{dp}{dt}$, find out what is p, e.g. $p = \lambda vx$.

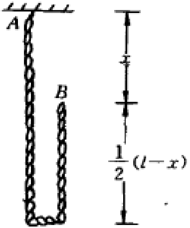
Exercise 3

An unpowered aircraft of mass m and initial velocity v_0 is moving in space dust. During the movement, the aircraft will absorb the dust. The mass absorbed onto the craft is proportional to the distance s it traveled, the coefficient is α, $m_{absorb} = \alpha s$.
i) Determine the total distance traveled by the aircraft before stopping.
ii) Determine the relationship between aircraft movement speed and time.

Answer: i) $s \to \infty$ ii) $v = \frac{v_0}{\sqrt{1+\frac{2\alpha v_0}{m_0}t}}$

Exercise 4

A uniform soft rope of length l and mass linear density λ is hung on the ceiling. At the beginning, both ends A and B are hung on the fixed point together, then B begins to fall freely from the suspension point. When the drop height of the B end is $x < l$, try to find the magnitude of the tensile force at the suspension point.



Answer: $T = \frac{1}{2}(l + 3x)\lambda g$

Equilibrium

$$\vec{F_{ext}} = 0$$

$$\vec{\tau_{ext}} = 0$$

- The sum of all the external forces is equal to zero
- The sum of all torques of external forces about any point is equal to zero

$\Rightarrow$ If the object is initially at rest, then it will remain at rest.

Center of Gravity

A point from which the weight of a body or system may be considered to act. In uniform gravity it is the same as the center of mass.

- For translational motion:

$$\vec{G} = M\vec{a_c}$$

- For rotation:

$$\vec{\tau_{tot}} = \sum \vec{r_i} \times \vec{G_i}$$

If in a uniform gravitaional field (mostly):

$$\vec{\tau_{tot}} = \sum m_i \vec{r_i} \times \vec{g} = M\frac{\sum m_i \vec{r_i}}{\sum m_i} \times \vec{g} = M\vec{r_c} \times \vec{g} = \vec{r_c} \times \vec{G}$$

Methods for Solving Statics

Review RC_wk10_withnotes for detail!

1. Equilibrium equations.
    - Translational motion: Force
    - Rotation: Torque
2. Infinitesimal methods
3. Derivation of energy
    - $\frac{\partial U}{\partial q_i} = 0$, the system potential energy reaches an local minimum.
    - useful for low degree of freedom system.

Stress

Stress is the force per unit area.

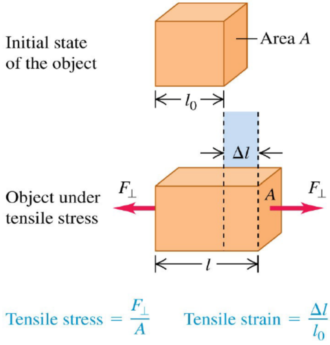
Strain

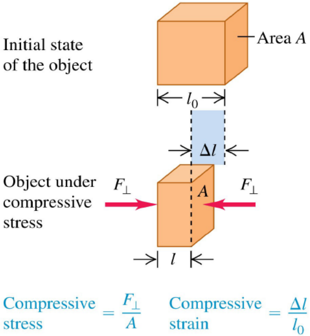
Strain is the fractional deformation due to the stress.

$$\text{elastic modulus} = \frac{\text{stress}}{\text{strain}}$$

Young's modulus: tensile stress divided by tensile strain

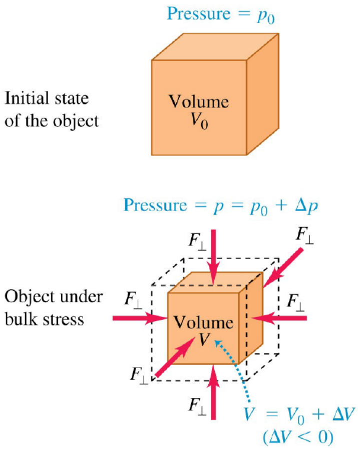
$$Y = \frac{\frac{F\perp}{A}}{\frac{\Delta l}{L}}$$





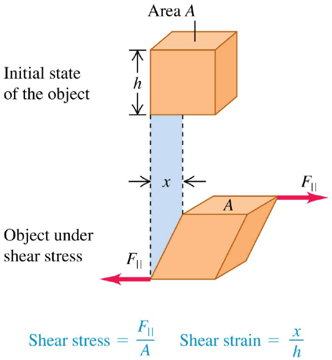
Bulk's modulus: bulk stress divided by bulk strain

$$B = -\frac{\Delta p}{\frac{\Delta v}{V}}$$



Shear modulus: shear stress divided by shear strain

$$S = \frac{\frac{F_{\parallel}}{A}}{\frac{X}{h}}$$



Pressure at a Depth

$$dp = -\rho g dy$$

$$p = p_0 + \rho g h$$

Pascal's Law

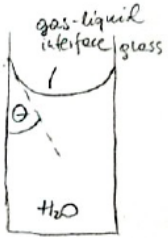
Pressure applied to an enclosed incompressible fluid is transmitted undiminished to every portion of the liquid and the walls of the container.

Guage Pressure vs. Absolute Pressure

Absolute: $p = p_{atm} + p_{gauge}$
Gauge: $p_{gauge} = p - p_{atm}$

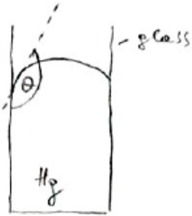
$1atm = 101325Pa = 760mmHg$
Every $10m$ depth of water adds to a pressure of $1atm$.

Surface Tension:



wetting liquid $0 < \theta < \frac{\pi}{2}$

adhesion forces dominate



non-wetting liquid $\frac{\pi}{2} < \theta < \pi$

cohesion forces dominate

Continuity Equation

$$A_1 v_1 = A_2 v_2$$

Bernoulli's Equation

$$\delta W = dK + dU$$
$$(p_1 - p_2)dV = \frac{1}{2}\rho(v_2^2 - v_1^2)dV + \rho g(y_2 - y_1)dV$$

$$\Rightarrow p + \frac{1}{2}\rho v^2 + \rho g y = \text{const}$$

Basic formulas

Force:

$$\vec{F} = -\frac{GMm}{r^3}\vec{r}$$

Gravitational Potential Energy:

$$U = -\frac{GMm}{r} + C$$

(C depends on the choice of zero potential. Treat $r = \infty$ as the zero potential, then $C = 0$)

Kepler's Laws

1. Each planet moves in an elliptical orbit with the Sun at one of the focal points.
2. The line from the Sun to a given planet sweeps out equal areas in equal times.
3. T^2/a^3 is a constant $(= 4\pi^2/GM$ if $M >> m)$

Comment:

1. Law 1: hyperbolic curve: $U + K < 0$ ellipse; $U + K = 0$ parabola; $U + K > 0$ hyperbola.
2. Law 2: The result of the conservation of angular momentum, or to say, aerial velocity = constant.
3. a is the semi-major axis of the ellipse(半长轴).

Tips on solving satellite motion problems:

Use Conservation of Energy and Conservation of Angular Momentum!

E = const:

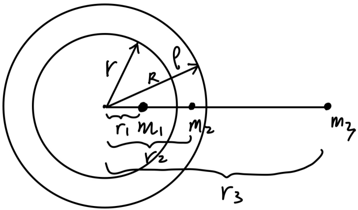
$$\frac{1}{2}mv_1^2 + (-\frac{GMm}{r_1^2}) = \frac{1}{2}mv_2^2 + (-\frac{GMm}{r_2^2})$$

L = const:

$$mv_1 r_1 = mv_2 r_2$$

Exercise 5

Consider a hollow thick spherical shell with a mass density of ρ as showed in the figure, calculate the gravitational force exerted on the three particles m_1, m_2, m_3 (only consider the force between the spherical shell and the particle)



How to calculate the gravitaional filed caused by a spherically symmetric mass distribution (what is spherically symmetric distribution?)

recall hw12 p6 Gauss Law for gravitational force: $\iint_{\sum} \mathbf{E_G} \hat{n} dS = -4\pi G M_{\sum}$

GOOD LUCK!