VP160 Recitation Class VI

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UM-SJTU Joint Institute

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Generalized coordinates

Any coordinates describing motions.

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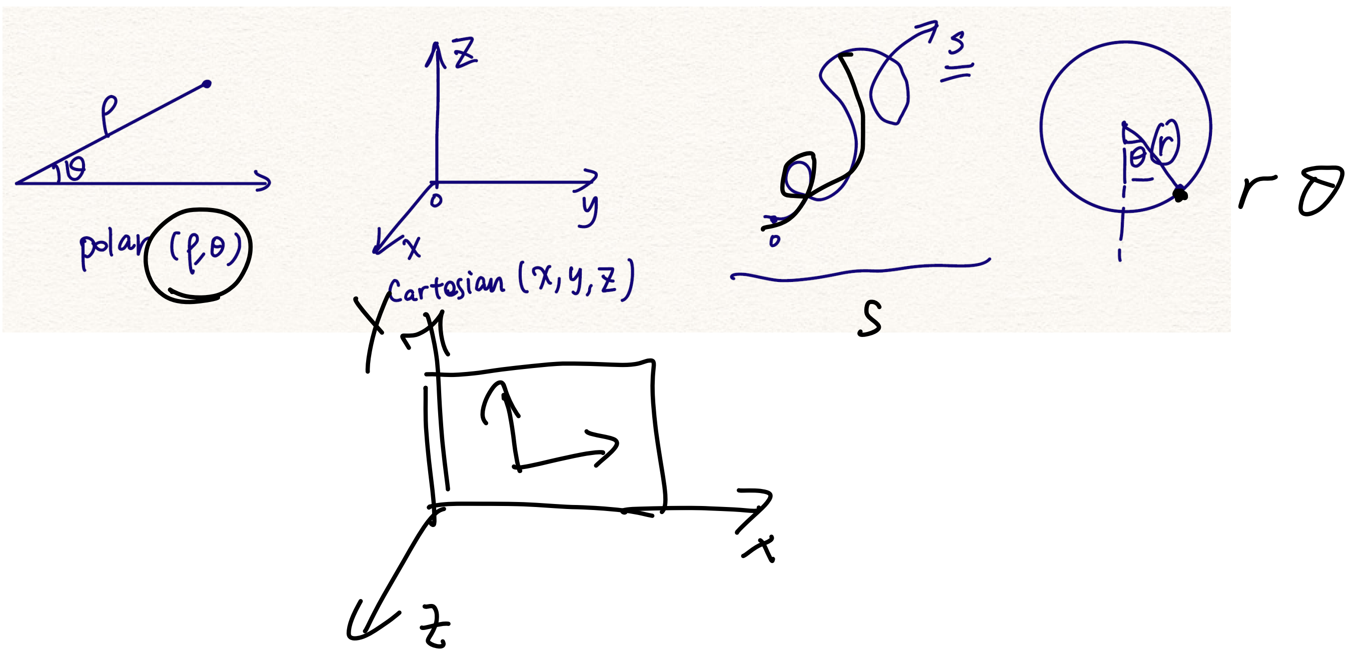
e.g.



Generalized coordinates

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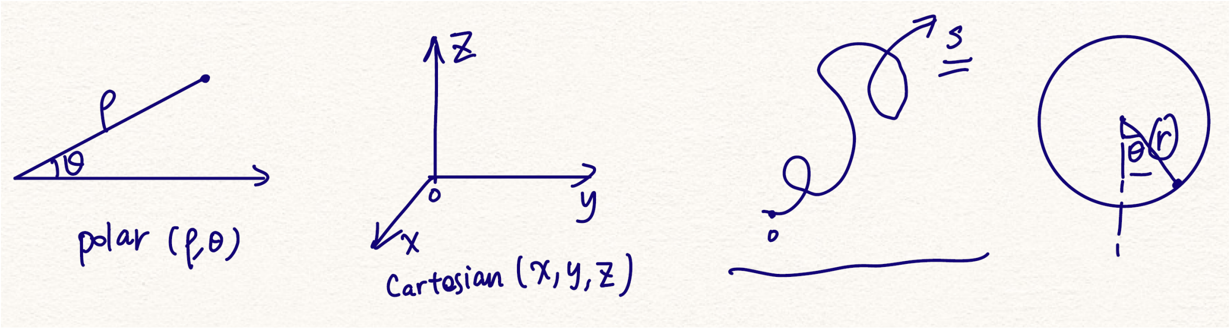

$$\begin{cases} q_1(t) \\ q_2(t) \\ \cdots \\ q_n(t) \end{cases} \Rightarrow \begin{cases} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \cdots \\ \dot{q}_n(t) \end{cases} \text{(generalized velocity)}$$

Degree of freedom (usually denoted by f)

The minimum number of independent generalized coordinates needed to describe the system's motions.

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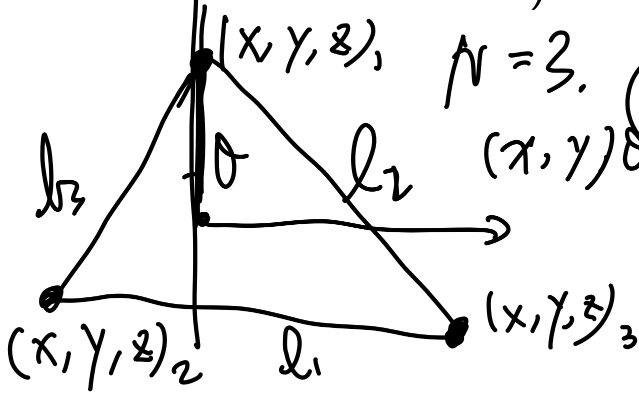
In general,

$$f = 3N - m$$

where N is the number of particles, and m is the number of constraints (number of equations that relate unknowns).

$f = 3 \times 3 - 3 - 3$

$= 3$



$N = 3.$

$(x, y)\ \theta$

$z_1 = z_2 = z_3 = 0.$

$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = l_2^2$

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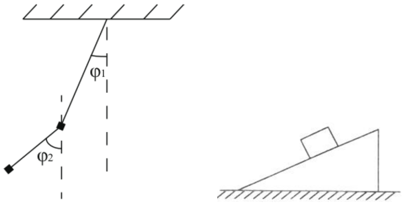
Exercise 1

Find the degree of freedom:

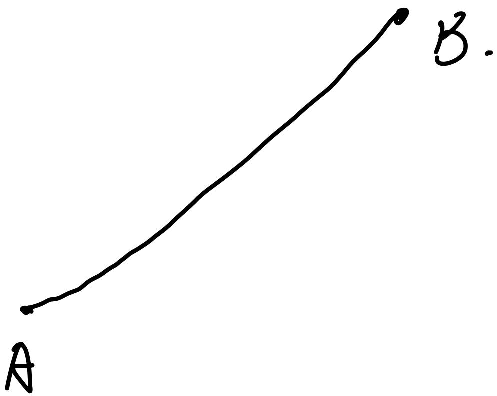


(e) (f)

Hamilton's Principle

$$\text{Real path} \iff \delta S = 0$$

(δ: variational differential, S is a functional: a function that maps functions into numbers.)



$\delta \bar{q}$, $\delta \dot{\bar{q}}$, δt.

Hamilton's Principle

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(δ: variational differential, S is a functional: a function that maps functions into numbers.)

$\Downarrow$ How?

Euler-Lagrange Equation

For $i = 1, 2, ..., f$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

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Learn more about variational and how to derive Hamilton's Principle, visit
`https://zhuanlan.zhihu.com/p/126115834`
`https://zhuanlan.zhihu.com/p/139018146`

Exercise 2

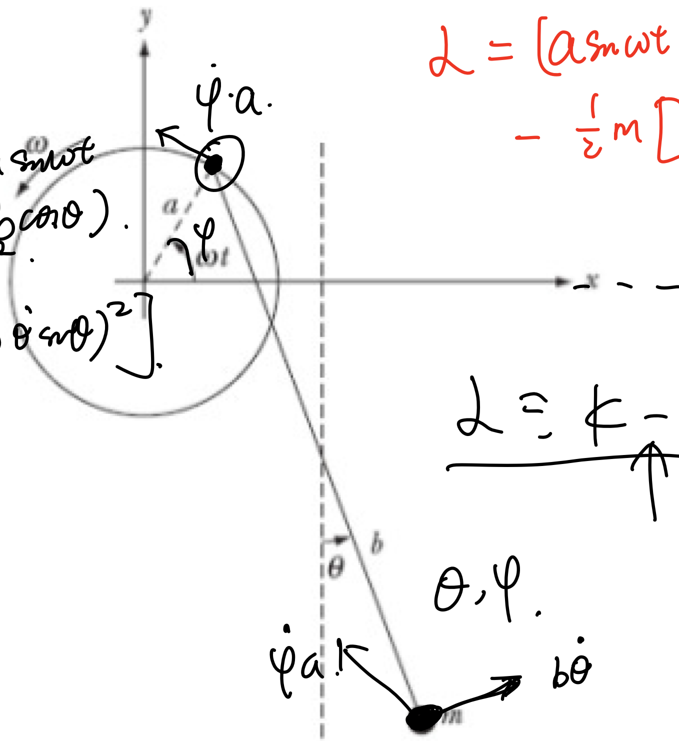
A simple pendulum of length b and mass m moves attached to a massless rim of radius a rotating with constant angular velocity ω. How many degrees of freedom do we have here? Find the Lagrangian.

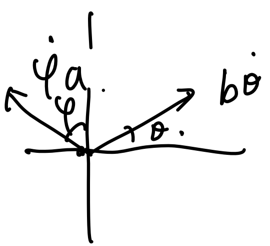
$f = 2$

$y = 0$ $U = 0$.

$U(\varphi, \theta) = mga\sin\omega t + mg(a\sin\omega t - b\cos\theta)$.

$K = \frac{1}{2}m\left[(b\dot\theta\cos\theta - \dot\varphi a\sin\varphi)^2 + (\dot\varphi a\cos\varphi + b\dot\theta\sin\theta)^2\right] + \frac{1}{2}m(\dot\varphi a)^2$.



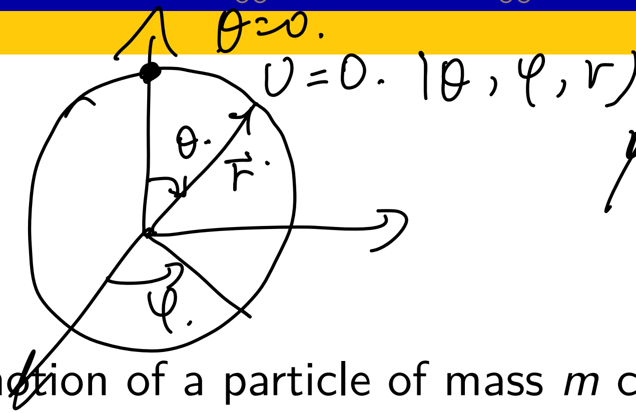


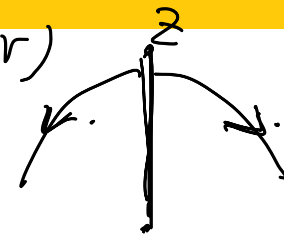
$\mathcal{L}$

$\mathcal{L} = (a\sin\omega t - b\cos\theta)mg - \frac{1}{2}m\left[(a\omega\cos\omega t + b\dot\theta\sin\theta)^2 + (a\omega\sin\omega t - b\dot\theta\cos\theta)^2\right]$

$\mathcal{L} = K - U$

θ, φ.





Exercise 3

Find the equations of motion of a particle of mass m constrained to move on the surface of a sphere, acted upon a conservative force $\boxed{\mathbf{F} = F_0 \hat{n}_\theta}$ with F_0 a constant.

U

Hint. To find the potential energy find the scalar product $\mathbf{F} \cdot d\mathbf{r}$ for the infinitesimal displacement on the sphere and use the fact that it is equal to $-dU$ (the force is conservative).

$$-dU = \vec{F} \cdot d\vec{r}$$

$$U = \int_0^{U_0} dU = -\int_0^{\theta_0} F_0 \cdot r \, d\theta = -F_0 r \theta \quad \checkmark$$

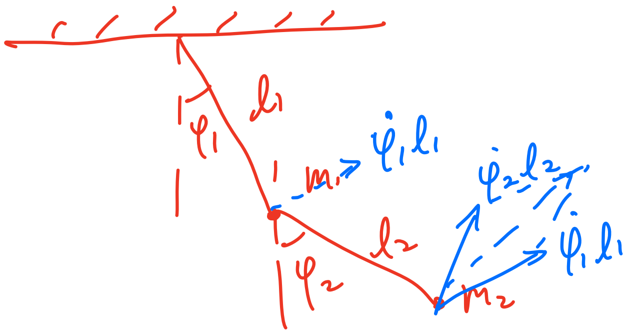
Exercise 4

Double pendulum:

(1) identify the generalized coordinates;

(2) find the Lagrangian;

(3) write down the Euler-Lagrange equations of motion;



$$U = -l_1\cos\varphi_1 m_1 g - (l_1\cos\varphi_1 + l_2\cos\varphi_2)m_2 g$$

$$K = \frac{1}{2}m_1(l_1\dot\varphi_1)^2 + \frac{1}{2}m_2\left[(l_1\dot\varphi_1\cos\varphi_1 + l_2\dot\varphi_2\cos\varphi_2)^2 + (l_1\dot\varphi_1\sin\varphi_1 + l_2\dot\varphi_2\sin\varphi_2)^2\right]$$

$$\mathcal{L} = K - U$$

Definition

$$\vec{p} = m\vec{v}$$

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Rewrite Newton's second law

$$\vec{F} = \frac{d\vec{p}}{dt}$$

(when m is not varying, $F = m\frac{d\vec{v}}{dt} = m\vec{a}$)

$\vec{F} = m\vec{a}$ $m = \text{const}$

$F = ma \xrightarrow{t} \int F \cdot dt = \int m a\, dt = m \Delta v \Rightarrow$ Impulse Theorem

$F = ma \xrightarrow{\vec{r}} \int \vec{F} \cdot dr = \int m \cdot a\, dr = \int m \cdot \frac{dv}{dr} \frac{dr}{dt} \cdot dr$

$= \underline{\frac{1}{2} m \Delta (v^2)}.$

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Impulse Theorem

$$\vec{p}_2 - \vec{p}_1 = \int_{t_1}^{t_2} \vec{F}\,dt$$

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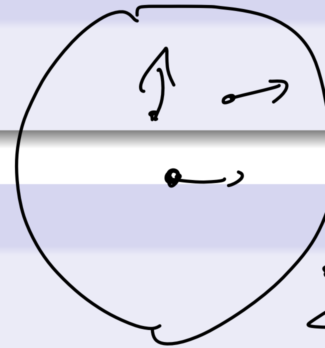


Rewrite Newton's second law

$$\vec{F} = \frac{d\vec{p}}{dt}$$

(when m is not varying, $F = m\frac{d\vec{v}}{dt} = m\vec{a}$)

$\sum_i m_i \vec{V_i} = \text{Cons}$

$\vec{r_c} = \frac{\sum m_i \vec{r_i}}{\sum m_i}$

$\frac{d}{dt}\sum m_i \vec{r_c} = \sum m_i \frac{d\vec{r_i}}{dt} = \vec{P}$.

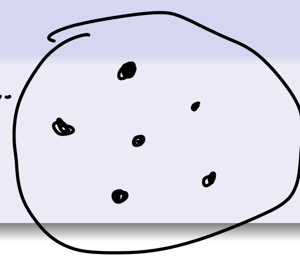
$\sum m_i \cdot \frac{d\vec{r_c}}{dt} = \vec{P}$

Impulse Theorem

$M \cdot \vec{V_c} = \vec{P}$.

$$\vec{p}_2 - \vec{p}_1 = \int_{t_1}^{t_2} \vec{F}dt$$

$\vec{F_{ext}}$.



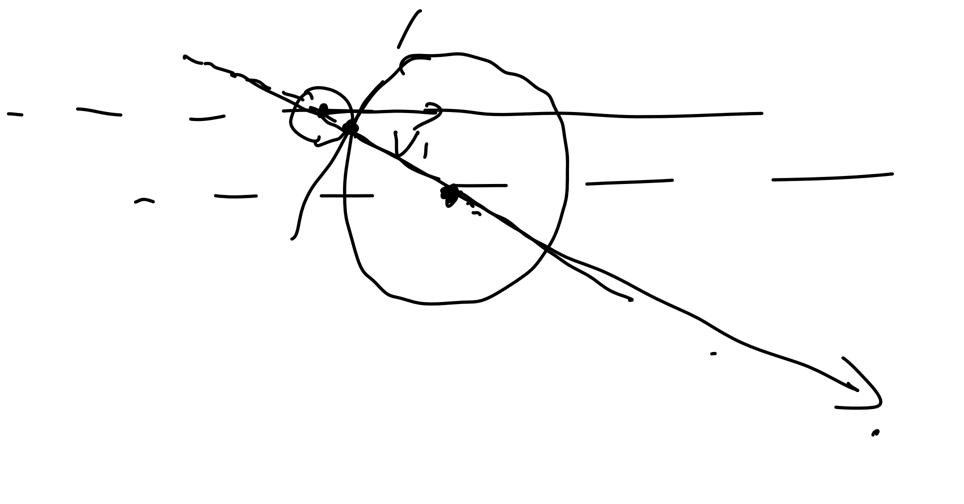
- If $\vec{F_{ext}} = 0$, for a system, $\Delta\vec{p} = 0 \Leftrightarrow p = \text{Const}$ (Conservation of momentum)

Application of Conservative of Momentum

- Non-central Collision (e.g. explosion)

$$p_{\vec{before}} = p_{\vec{after}}$$

$$\vec{p_0} = \vec{p_1} + \vec{p_2}$$



Application of Conservative of Momentum

- Non-central Collision (e.g. explosion)

$$\vec{p_{before}} = \vec{p_{after}}$$

- Central Collision
  - Elastic
    - $e = (\vec{v_2}' - \vec{v_1}')/(\vec{v_1} - \vec{v_2}) = 1$
    - Conservation of energy

Application of Conservative of Momentum

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    - $e = (\vec{v_2}' - \vec{v_1}')/(\vec{v_1} - \vec{v_2}) = 1$
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  - Inelastic
    - $e < 1$
    - Energy loss

Application of Conservative of Momentum

- Non-central Collision (e.g. explosion)

$$\vec{p_{before}} = \vec{p_{after}}$$

m_1 m_2 m_1 m_2

V_1 V_2

- Central Collision
  - Elastic
    - $\boxed{e = (\vec{v_2}' - \vec{v_1}')/(\vec{v_1} - \vec{v_2}) = 1}$
    - $\boxed{\text{Conservation of energy}}$
  - Inelastic
    - $e < 1$
    - Energy loss
  - Completely Inelastic
    - $e = 0$
    - stick to each other

$\rightarrow m_1V_1 + m_2V_2 = m_1 v_1' + m_2V_2'$

$\rightarrow \boxed{\frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 = \frac{1}{2}m_1V_1'^2 + \frac{1}{2}m_2V_2'^2}$

$\boxed{\frac{V_2' - V_1'}{V_1 - V_2} = 1}$

$\frac{0}{\square - \square} = 0.$

7

Exercise 5

Assume m_1, m_2, m_3, k is known. Release m_1, the collision between m_1 and m_2 is completely inelastic. Find h so that m_3 can just leave the ground.

$v_1 = \sqrt{2hg}$

$m_1 v_1 = (m_1 + m_2) v_1'$

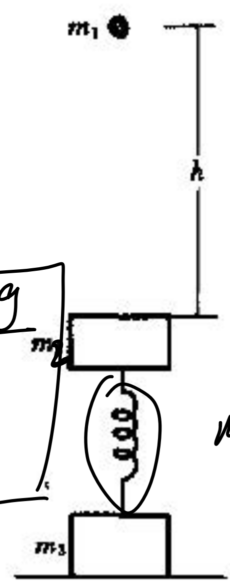
$v_1' = \frac{m_1}{m_1+m_2}\sqrt{2hg}$.

m_1+m_2

k.

$\Delta x_0 = \frac{(m_1+m_2)g}{k}$.

$\Delta x_0' = \frac{m_2 g}{k}$.



$m_3 g$.

$A = \sqrt{\frac{2m_1^2 g}{m_1+m_2} + \frac{m_1^2 g^2}{k^2}}$.

$A = \Delta x_0 + \frac{m_3 g}{k}$

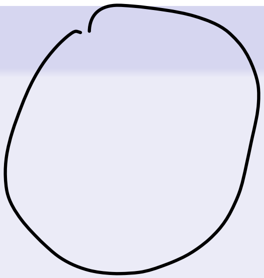
$h = \frac{(m_1+m_2)\left[(m_1+m_2+m_3)^2 - m_2^2\right]}{2k m_1^2}$

$\boxed{\frac{1}{2}kA^2} = \frac{1}{2}(m_1+m_2)v_1'^2 + \frac{1}{2}k \cdot \left(\frac{m_1 g}{k}\right)^2$.

Center of Mass

$$\vec{r_C} \doteq \frac{\sum m_i \vec{r_i}}{\sum m_i}$$

$$\vec{r_C} \doteq \frac{\int \vec{r_i} dm}{\int dm}$$



$$\frac{\iiint \rho(x,y,z) \cdot dV}{\int dm}$$

Center of Mass

$$r_C = \frac{\sum m_i r_i}{\sum m_i}$$

$$r_C = \frac{\int r_i dm}{\int dm}$$

Pappus Law

First Theorem:

$$4\pi r^2 = 2\pi \cdot \pi r \cdot X$$

$$S = 2\pi s x$$

$$X = \frac{2r}{\pi}.$$

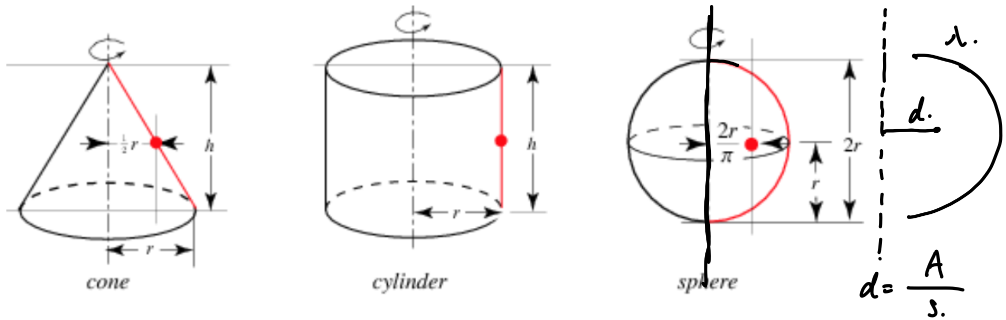
Second Theorem:

$$V = 2\pi A x$$

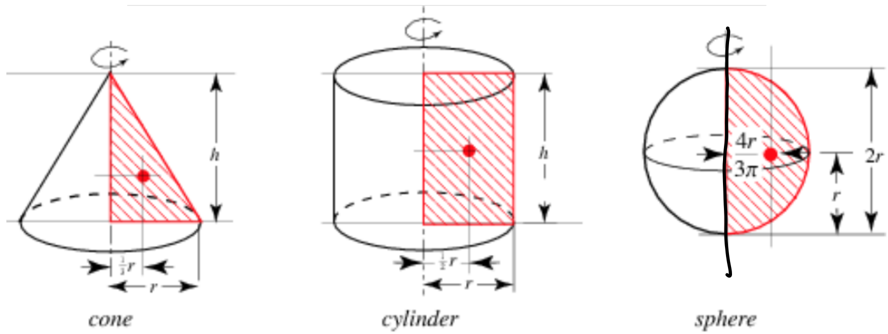
$$\frac{4}{3}\pi r^3 = 2\pi \cdot \pi r^2 \cdot X$$

$$X = \frac{4r}{3\pi}$$

where x is the distance from the reference axis and the center of mass.



h
r
$\frac{1}{2}r$
cone
h
r
cylinder
$\frac{2r}{\pi}$
2r
r
sphere
λ.
d.
$d = \frac{A}{s.}$



h
$\frac{1}{3}r$
r
cone
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r
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$\frac{4r}{3\pi}$
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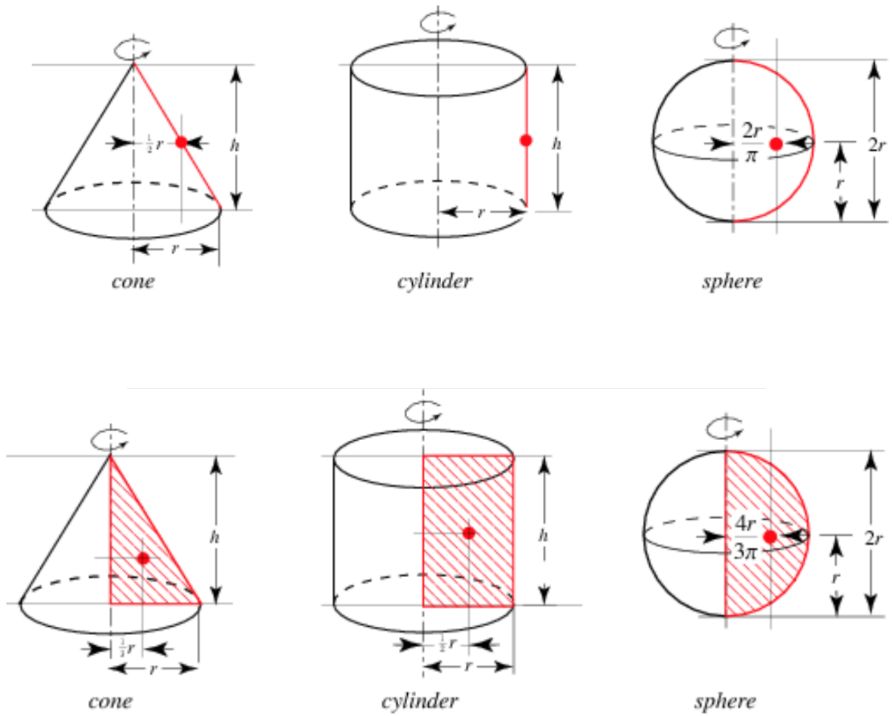
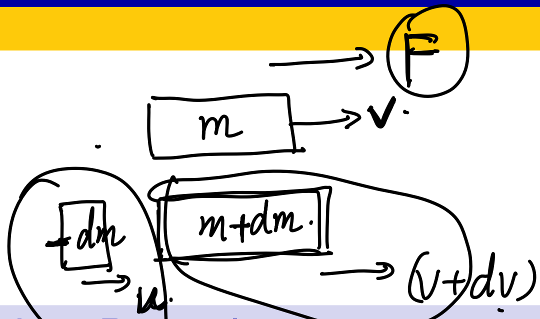
sphere



- An important fact:

$$\vec{F_{ext}} = 0 \Leftrightarrow \vec{p} = Const \Leftrightarrow \vec{v_c} = Const$$



$$\boxed{Fdt = p_2 - p_1}$$

$\boxed{dm}$

$$Fdt = (m+dm)(v+dv)+(-dm)\cdot u - mv$$

Rocket Propulsion

$$mv + Fdt = (m + dm)(v + dv) - udm \quad \Leftarrow \quad ①$$

$$\boxed{m\frac{dv}{dt} = (u - v)\frac{dm}{dt} + F}$$

Reminder

What FoR are we looking at?

In the ground FoR

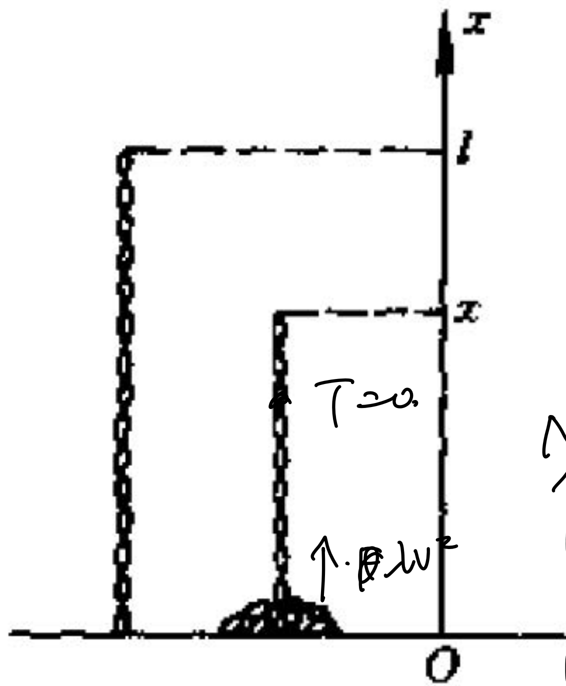
Exercise 6

A rope with length l and mass m is placed vertically. At the beginning, the lower end of the rope just touches the ground. Release the rope, find the support force of the ground with respect to x.

Δt.

$\boxed{v \to 0}$ $\lambda \cdot v\Delta t$.

$\lambda(l-x)g$.



$$\boxed{F\Delta t = (\lambda \cdot v\Delta t)\cdot v}$$

$$F = \lambda v^2$$

$$= \lambda\left(\sqrt{2g(l-x)}\right)^2$$

$\downarrow g$.

$\uparrow F$ $F = 2\lambda(l-x)g$

N.

$\downarrow F$ $\downarrow \lambda(l-x)g$.

$$N = F + \lambda(l-x)g = 3\lambda(l-x)g$$

Reference

- Yigao Fang.
  VP160 Recitation Slides.
  2020
- Haoyang Zhang.
  VP160 Recitation Slides.
  2020