Chapter 14 – Rigid Body Dynamics (I) Angular Momentum of a Rigid Body. Moment of Inertia Tensor

UM-SJTU Joint Institute
Honors Physics I (Summer 2020)
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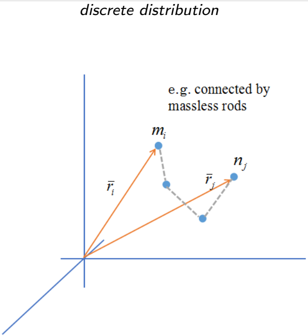
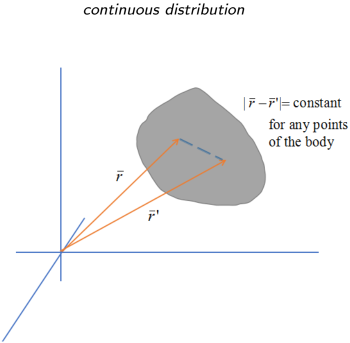
Agenda

Introduction

Rigid Body





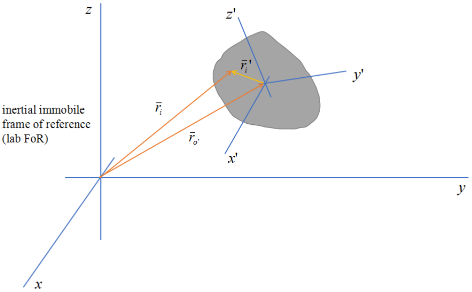
A body is called **rigid** if $|\overline{r}_i - \overline{r}_j| = \text{const}$ for any $i, j = 1, 2, \ldots, N$.

Form the previous discussion of the momentum and angular momentum

$$\overline{F}^{\text{ext}} = 0 \implies \overline{P} = \text{const}$$
$$\overline{\tau}^{\text{ext}} = 0 \implies \overline{L} = \text{const}$$

If, initially, the rigid body is at rest ($\overline{P} = 0$ and $\overline{L} = 0$), it will remain at rest — *STATICS*.

Mathematical Description of a Rotating Rigid Body



FoR associated with the rigid body is, in general, non-inertial —the body can move arbitrarily.

O' — a point of the body

We have (see the derivation of dynamics in non-inertial FoRs)

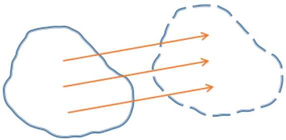
$$
\begin{aligned}
\overline{r}_i &= \overline{r}_{O'} + \overline{r}'_i, \\
\overline{v}_i &= \overline{v}_{O'} + \underbrace{\overline{v}'_i}_{=0} + \overline{\omega} \times \overline{r}'_i,
\end{aligned}
$$

where $\overline{v}'_i = 0$ due to the fact that the body is rigid (no relative motion of the rigid body's points).

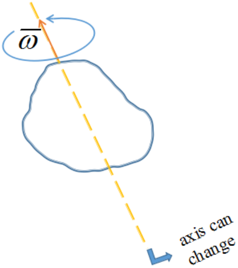
Eventually, the velocity of any point of a rigid body

$$\overline{v}_i = \overline{v}_{O'} + \overline{\omega} \times \overline{r}'_i.$$

The first term on the right hand side corresponds to the **translational motion**, while the second term to the **rotational motion** about an *instantaneous axis of rotation*.



translational motion



rotational motion

Momentum of a Rigid Body

Consequently, the total momentum of an arbitrarily moving rigid body (in lab FoR) is

$$\begin{aligned}\overline{P} &= \sum_{i=1}^{N} m_i \overline{v}_i = \underbrace{\sum_{i=1}^{N} m_i}_{M} \overline{v}_{O'} + \sum_{i=1}^{N} m_i (\overline{\omega} \times \overline{r}_{i'}) = \\ &= M\overline{v}_{O'} + \overline{\omega} \times \underbrace{\sum_{i=1}^{N} m_i \overline{r}_{i'}}_{M\overline{r}'_{\text{cm}}} = \underbrace{M\overline{v}_{O'}}_{\substack{\text{translational} \\ \text{motion}}} + \underbrace{M\overline{\omega} \times \overline{r}'_{\text{cm}}}_{\substack{\text{rotational} \\ \text{motion}}}\end{aligned}$$

Comment. Separation of the translational and the rotational motion is again visible in the formula. Note that $\overline{r}'_{\text{cm}}$ is the position of center of mass in the primed (i.e. rigid body's) FoR. Therefore, the term corresponding to the rotational motion will vanish if we choose O' to be the rigid body's center of mass (then $\overline{r}'_{\text{cm}} = 0$).

Angular Momentum of a Rigid Body Moving Arbitrarily

Angular Momentum of Rigid Body Moving Arbitrarily

In the lab FoR

$$
\begin{aligned}
\overline{L} &= \sum_{i=1}^{N} \overline{L}_i = \sum_{i=1}^{N} m_i \overline{r}_i \times \overline{v}_i = \sum_{i=1}^{N} [m_i(\overline{r}_{O'} + \overline{r}'_i) \times (\overline{v}_{O'} + \overline{\omega} \times \overline{r}'_i)] \\
&= \sum_{i=1}^{N} m_i (\overline{r}_{O'} \times \overline{v}_{O'}) + \sum_{i=1}^{N} m_i \overline{r}_{O'} \times (\overline{\omega} \times \overline{r}'_i) + \\
&\quad + \sum_{i=1}^{N} m_i \overline{r}'_i \times \overline{v}_{O'} + \sum_{i=1}^{N} m_i \overline{r}'_i \times (\overline{\omega} \times \overline{r}'_i) \\
&= M\overline{r}_{O'} \times \overline{v}_{O'} + M\overline{r}_{O'} \times (\overline{\omega} \times \overline{r}'_{\text{cm}}) + M\overline{r}'_{\text{cm}} \times \overline{v}_{O'} + \\
&\quad + \sum_{i=1}^{N} m_i \overline{r}'_i \times (\overline{\omega} \times \overline{r}'_i)
\end{aligned}
$$

1. For the time being, we want to discuss only effects due to rotational motion. Therefore, we assume $\overline{v}_{O'} = 0$ (no translational motion).
2. We will discuss rotations about the center of mass, so choose O' such that $\overline{r}'_{\text{cm}} = 0$, that is O' at the center of mass of the body.

Note. If $O = O'$, the two conditions imply: $\overline{r}'_i = \overline{r}_i$, $\overline{r}'_{\text{cm}} = \overline{r}_{\text{cm}}$.

Eventually, taking these assumptions into account

$$\overline{L} = \sum_{i=1}^{N} m_i \overline{r}'_i \times (\overline{\omega} \times \overline{r}'_i).$$

Using the identity $\overline{a} \times (\overline{b} \times \overline{c}) = \overline{b}(\overline{a} \circ \overline{c}) - \overline{c}(\overline{a} \circ \overline{b})$, we have

$$\overline{L} = \sum_{i=1}^{N} m_i \overline{r}'_i \times (\overline{\omega} \times \overline{r}'_i) = \sum_{i=1}^{N} m_i \left[\overline{\omega}(\overline{r}'_i)^2 - \overline{r}'_i(\overline{\omega} \circ \overline{r}'_i)\right].$$

In terms of the Cartesian components in the FoR associated with the rigid body)

$$L_{\alpha'} = \sum_{i=1}^{N} m_i \big[\omega_{\alpha'}(r'_i)^2 - r'_{i\alpha}\big(\sum_{\beta'} \omega_{\beta'} r_{i\beta'}\big)\big],$$

where $\alpha', \beta' \in \{x', y', z'\}$ and $\sum_{\beta'} \cdots = \sum_{\beta' \in \{x', y', z'\}} \cdots$

Rewriting the first term as $\omega_{\alpha'}(r_i')^2 = \sum_{\beta'} \omega_{\beta'}(r_i')^2\delta_{\alpha'\beta'}$, where $\delta_{\alpha'\beta'}$ is the *Krönecker symbol* defined as $\delta_{\alpha'\beta'} = \begin{cases} 1 & \text{if } \alpha' = \beta' \\ 0 & \text{if } \alpha' \neq \beta' \end{cases}$, we have

$$
\begin{aligned}
\boxed{L_{\alpha'}} &= \sum_{i=1}^{N} \sum_{\beta'} \left[\omega_{\beta'}(r_i')^2\delta_{\alpha'\beta'} - r_{i\alpha'}r_{i\beta'}\omega_{\beta'}\right] \\
&= \sum_{\beta'} \underbrace{\left\{\sum_{i=1}^{N} m_i\left[(r_i')^2\delta_{\alpha'\beta'} - r_{i\alpha'}r_{i\beta'}\right]\right\}}_{I_{\alpha'\beta'}} \omega_{\beta'} = \boxed{\sum_{\beta'} I_{\alpha'\beta'}\omega_{\beta'}}
\end{aligned}
$$

The physical quantity

$$
\boxed{I_{\alpha'\beta'} = \sum_{i=1}^{N} m_i\left[(r_i')^2\delta_{\alpha'\beta'} - r_{i\alpha'}r_{i\beta'}\right]}
$$

is called the **tensor of the moment of inertia** (or in short, the **tensor of inertia**) for a rigid body with respect to the axes x', y', z', with the origin at the body's center of mass.

Tensor of Inertia

Mathematically, the tensor of inertia is a 3×3 matrix. Since $I_{\alpha'\beta'} = I_{\beta'\alpha'}$, the matrix is *symmetric*.

$$[I_{\alpha'\beta'}]_{\alpha',\beta'=x',y',z'} = \begin{bmatrix} I_{x'x'} & I_{x'y'} & I_{x'z'} \\ I_{y'x'} & I_{y'y'} & I_{y'z'} \\ I_{z'x'} & I_{z'y'} & I_{z'z'} \end{bmatrix} =$$

$$= \begin{bmatrix} \sum\limits_{i=1}^{N} m_i(y_i'^2 + z_i'^2) & -\sum\limits_{i=1}^{N} m_i x_i' y_i' & -\sum\limits_{i=1}^{N} m_i x_i' z_i' \\ -\sum\limits_{i=1}^{N} m_i y_i' x_i' & \sum\limits_{i=1}^{N} m_i(x_i'^2 + z_i'^2) & -\sum\limits_{i=1}^{N} m_i y_i' z_i' \\ -\sum\limits_{i=1}^{N} m_i z_i' x_i' & -\sum\limits_{i=1}^{N} m_i z_i' y_i' & \sum\limits_{i=1}^{N} m_i(x_i'^2 + y_i'^2) \end{bmatrix}$$

The off-diagonal terms are called *deviation moments* and the diagonal terms — moments of inertia with respect to axes the x', y', z' through the center of mass.

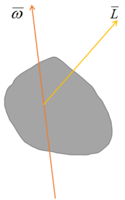
Analogously, the tensor of inertia can be defined for a continuous distribution of mass ($\sum_i m_i \cdots \longrightarrow \int \ldots \, \mathrm{d}m$).

Tensor of Inertia. Physical Significance

What is the physical consequence of the relationship

$$L_{\alpha'} = \sum_{\beta'} I_{\alpha'\beta'}\omega_{\beta'} \quad ?$$

Observation. In general $\overline{L}' \nparallel \overline{\omega}'$. That is, in general, the angular momentum of a rigid body *is not* parallel to the axis of rotation. Note that



$$\begin{aligned} L_{x'} &= I_{x'x'}\omega_{x'} + I_{x'y'}\omega_{y'} + I_{x'z'}\omega_{z'} \\ L_{y'} &= I_{y'x'}\omega_{x'} + I_{y'y'}\omega_{y'} + I_{y'z'}\omega_{z'} \\ L_{z'} &= I_{z'x'}\omega_{x'} + I_{z'y'}\omega_{y'} + I_{z'z'}\omega_{z'} \end{aligned}$$

If $\overline{L}' \parallel \overline{\omega}'$ — that is the angular momentum *was* parallel to $\overline{\omega}$ (and hence the rotation axis), we would need to have $\overline{L}' = \text{const} \cdot \overline{\omega}'$, i.e.

$$L_{x'} = \text{const} \cdot \omega_{x'}, \qquad L_{y'} = \text{const} \cdot \omega_{y'}, \qquad L_{z'} = \text{const} \cdot \omega_{z'}.$$

When $\overline{L} \parallel \overline{\omega}$?

Recall that $I_{\alpha'\beta'}$ corresponds to a particular choice of the direction of the axes x', y', z'.

Fact. For any tensor of inertia we can find a set of three axes $\tilde{x}', \tilde{y}', \tilde{z}'$, passing through the center of mass, such that $I_{\tilde{\alpha}'\tilde{\beta}'}$ is a diagonal tensor (all off-diagonal elements are zero).[1]

$$[I_{\tilde{\alpha}'\tilde{\beta}'}] = \begin{bmatrix} I_{\tilde{x}'\tilde{x}'} & 0 & 0 \\ 0 & I_{\tilde{y}'\tilde{y}'} & 0 \\ 0 & 0 & I_{\tilde{z}'\tilde{z}'} \end{bmatrix}$$

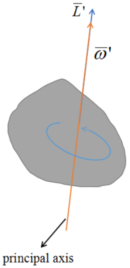
then we have $L_{\tilde{\alpha}'} = I_{\tilde{\alpha}'\tilde{\alpha}'}\omega_{\tilde{\alpha}'}$.

Hence, if the body rotates about one of these axes, *e.g.*, $\overline{\omega} = (0, \omega_{\tilde{y}'}, 0)^T$ for rotation about $\tilde{y}'$, then the angular momentum has only one non-zero component corresponding to this axis and $\overline{L} \parallel \overline{\omega}$.

[1] This fact follows from a general theorem proved in linear algebra. .

Principal Axes

An axis of rotation with the property that, for rotation about this axis, $\overline{L} \parallel \overline{\omega}$ is called a **principal axis** of the tensor of inertia and the corresponding value of $I_{\tilde{\alpha}'\tilde{\alpha}'}$ is called the **principal moment of inertia** about that given axis.



Principal axes often correspond to the axes of symmetry of the rigid body.

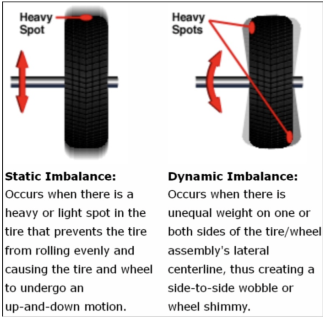
If a rigid body has at least two equal principal moments of inertia, it is called a **symmetrical top**. If all three principal moments are equal, it is called a spherical top. Hence, for a spherical top

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Practical Application: Balancing a Tyre

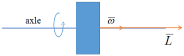
If the axle of a car/motorcycle does not coincide with a principal axis of a rotating wheel, the angular momentum vector is not parallel to the angular velocity $\overline{L} \nparallel \overline{\omega}$ and the wheel wobbles. This wobbling, transmitted to the steering system can be felt directly when you put hands on the steering wheel of a moving car. We then say that the car's tyres are *not balanced*.

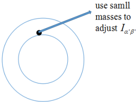
Source: `zx14ninjaforum.com`



To fix this, small weights are attached to the rim of the wheel. In that way, the distribution of the mass over a tyre (and hence the tensor of inertia) is adjusted so that the car's axle becomes one of the tyre's principal axes, so that $\overline{L} \parallel \overline{\omega}$. Then the tyre is said to be *balanced* and the wobbling disappears.

Source: `www.lesschwab.com`





Principal Axes and Moments of Inertia about Principal Axes

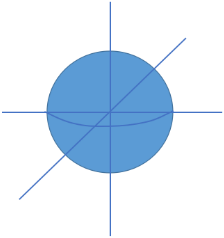
How to Find Principal Axes and Principal Moments?

One important question arises: How to find principal axes, *i.e.*, directions with respect to which the tensor of inertia is diagonal?

- Use the symmetry,
- or use a theorem stating that *any symmetric matrix can be diagonalized*, i.e. use the mathematical procedure of matrix diagonalization.
  [*See a tutorial on Canvas for some background information.*]

Example (a). Uniform ball

Any three axes that are mutually perpendicular and pass through the ball's center of mass are the principal axes.

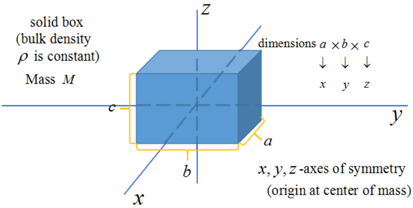


$$\begin{bmatrix} I_0 & 0 & 0 \\ 0 & I_0 & 0 \\ 0 & 0 & I_0 \end{bmatrix}$$

Hence a uniform ball is a spherical top.

Example (b). Uniform solid box

We will find the tensor of inertia and show that x, y, z are the principal axes.



Continuous distribution
$(\sum \longrightarrow \int)$

$$\int\limits_{\Omega} (\ldots)\, \mathrm{d}m = \varrho \int\limits_{-a/2}^{a/2} \mathrm{d}x \int\limits_{-b/2}^{b/2} \mathrm{d}y \int\limits_{-c/2}^{c/2} (\ldots)\, \mathrm{d}z$$

triple integral

iterated integral

$$[I]_{\alpha\beta} = \begin{bmatrix} \int\limits_{\Omega} (y^2+z^2)\, \mathrm{d}m & -\int\limits_{\Omega} xy\, \mathrm{d}m & -\int\limits_{\Omega} xz\, \mathrm{d}m \\ -\int\limits_{\Omega} yx\, \mathrm{d}m & \int\limits_{\Omega} (x^2+z^2)\, \mathrm{d}m & -\int\limits_{\Omega} yz\, \mathrm{d}m \\ -\int\limits_{\Omega} zx\, \mathrm{d}m & -\int\limits_{\Omega} yz\, \mathrm{d}m & \int\limits_{\Omega} (x^2+y^2)\, \mathrm{d}m \end{bmatrix}$$

Note. From now on, to simplify the notation, we will omit primes in the formulas for $I_{\alpha'\beta'}$.

Explicitly, for the box

$$
\begin{aligned}
I_{xx} &= \varrho \int\limits_{-a/2}^{a/2} \mathrm{d}x \int\limits_{-b/2}^{b/2} \mathrm{d}y \int\limits_{-c/2}^{c/2} (y^2 + z^2)\,\mathrm{d}z = \varrho \underbrace{\int\limits_{-a/2}^{a/2} \mathrm{d}x}_{=a} \int\limits_{-b/2}^{b/2} \left[y^2 z + \frac{1}{3} z^3 \right]_{z=-\frac{c}{2}}^{z=\frac{c}{2}} \\
&= \varrho a \int\limits_{-b/2}^{b/2} \left(y^2 c + \frac{1}{3}\frac{c^3}{4} \right) \mathrm{d}y \;=\; \varrho a \left[\frac{1}{3} y^3 c + \frac{1}{12} c^3 y \right]_{y=-\frac{b}{2}}^{y=\frac{b}{2}} \\
&= \varrho a \left(\frac{1}{3}\frac{b^3}{4} c + \frac{1}{12} c^3 b \right) \;=\; \frac{\varrho abc}{12}\left(b^2 + c^2\right) \;=\; \boxed{\frac{M}{12}\left(b^2 + c^2\right)}.
\end{aligned}
$$

Analogously,

$$
\boxed{I_{yy} = \frac{M}{12}\left(a^2 + c^2\right)}, \qquad \boxed{I_{zz} = \frac{M}{12}\left(a^2 + b^2\right)}
$$

and $\boxed{I_{xy} = I_{xz} = I_{yz} = 0}$ (integrals of odd functions over symmetric intervals).

Eventually,

$$[I] = \begin{bmatrix} \frac{M}{12}\left(b^2+c^2\right) & 0 & 0 \\ 0 & \frac{M}{12}\left(a^2+c^2\right) & 0 \\ 0 & 0 & \frac{M}{12}\left(a^2+b^2\right) \end{bmatrix}.$$

Hence, since the tensor is diagonal, the axes x, y, z are the principal axes indeed.

In particular, for a cube $(a = b = c)$

$$[I] = \begin{bmatrix} \frac{1}{6}Ma^2 & 0 & 0 \\ 0 & \frac{1}{6}Ma^2 & 0 \\ 0 & 0 & \frac{1}{6}Ma^2 \end{bmatrix},$$

so a uniform cube is a spherical top.

Diagonalization of the Tensor of Inertia

Eigenvalues and Eigenvectors

If a rigid body is not regular, then we cannot use symmetry. We need a *general* procedure to find the values of the principal moments and the directions of principal axes.

This procedure is known as diagonalization of the tensor. It requires us to find **eigenvalues** and **eigenvectors** of a matrix representing the tensor.

What are *eigenvalues and eigenvectors of a matrix*? An eigenvalue of a matrix I is a number λ, such that

$$\boxed{I\,\overline{u} = \lambda\,\overline{u}.}$$

The corresponding vector $\overline{u}$ is called called the **eigenvector** corresponding to the eigenvalue λ

Comment. (fact from linear algebra) A symmetric $n \times n$ real matrix (such as the tensor of inertia) has n real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_3$, each coming with an eigenvector. Not always all eigenvalues are different.

Physical Interpretation of Eigenvalues and Eigenvectors of the Tensor of Inertia

For the tensor of inertia (a 3×3 matrix) its three eigenvectors vectors $\overline{u}_i$ (eigenvectors) define three mutually perpendicular directions of **principal axes**. The tensor of inertia evaluated in the coordinate system of principal axes, is always diagonal, with the eigenvalues λ_i on the diagonal

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

λ_i is called the **principal moment of inertia** about the axis defined by $\overline{u}_i$ for $i = 1, 2, 3$.

For any shape of a rigid body, even very irregular (imagine a potato), such three directions always exist!

How to Find Eigenvalues and Eigenvectors

Note that $I\,\overline{u} = \lambda\,\overline{u}$ implies

$$(I - \lambda E)\overline{u} = 0,$$

where E denotes the unit 3×3 matrix, and the zero on the right hand side represents the zero vector.

The above equation has non-zero solutions if and only if

$$\boxed{\det(I - \lambda E) = 0}.$$

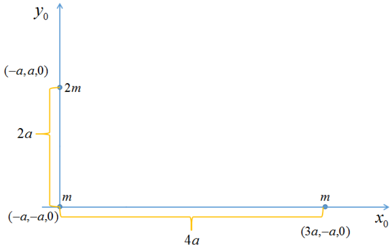
This algebraic equation for λ is known as the **characteristic equation** or the **secular equation**.

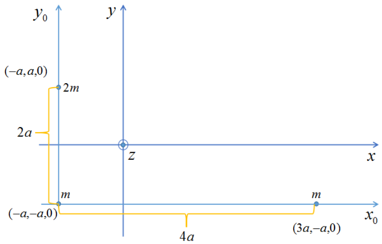
1. Solve the characteristic equation to get the values of λ_i, with $i = 1, 2, 3$, that is the eigenvalues.
2. For each eigenvalue λ_i, plug it back into the equation $(I - \lambda_i E)\overline{u}_i = 0$ and solve for $\overline{u}_i$. This yields the eigenvector $\overline{u}_i$ corresponding to the eigenvalue λ_i.

Example (c). Non-Symmetric Discrete Mass Distribution

Consider a system of three masses arranged as in the figure. First we need to find the center of mass of this system (in the $x_0 O y_0$ plane).

$$\bar{r}_{0,cm} = \frac{1}{4m}\left[m \cdot (0,0,0) + 2m \cdot (0,2a,0) + m \cdot (4a,0,0)\right] = (a,a,0)$$





Then we can find the tensor of inertia with respect to x, y, z (passing through the center of mass). For the diagonal terms we have

$$\begin{aligned}
I_{xx} &= \sum_i m_i \left(y_i^2 + z_i^2\right) = ma^2 + ma^2 + 2ma^2 = 4ma^2, \\
I_{yy} &= \sum_i m_i \left(x_i^2 + z_i^2\right) = ma^2 + 9ma^2 + 2ma^2 = 12ma^2, \\
I_{zz} &= \sum_i m_i \left(x_i^2 + y_i^2\right) = m(a^2 + a^2) + m(9a^2 + a^2) + 2m(a^2 + a^2) = 16ma^2
\end{aligned}$$

For the off-diagonal terms

$$I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0$$

and

$$I_{xy} = I_{yx} = -\sum_i m_i x_i y_i = -\left(ma^2 - 3ma^2 - 2ma^2\right) = 4ma^2.$$

Eventually, the tensor of inertia with respect to the axes x, y, z has the form

$$I = \begin{bmatrix} 4ma^2 & 4ma^2 & 0 \\ 4ma^2 & 12ma^2 & 0 \\ 0 & 0 & 16ma^2 \end{bmatrix}$$

Step 1. Eigenvalues

Solve the characteristic equation $\det(I - \lambda E) = 0$ to get

$$\det \begin{bmatrix} 4ma^2 - \lambda & 4ma^2 & 0 \\ 4ma^2 & 12ma^2 - \lambda & 0 \\ 0 & 0 & 16ma^2 - \lambda \end{bmatrix} = 0.$$

Hence

$$\Rightarrow (16ma^2 - \lambda)\left[(4ma^2 - \lambda)(12ma^2 - \lambda) - 16\left(ma^2\right)^2\right] = 0$$

$$\Rightarrow (16ma^2 - \lambda)\left[\lambda^2 - 16ma^2\lambda + 32\left(ma^2\right)^2\right] = 0.$$

And eventually, after solving the quadratic equation,

$$\boxed{\lambda_1 = 16ma^2,} \qquad \boxed{\lambda_2 = 4(2 + 2\sqrt{2})ma^2,} \qquad \boxed{\lambda_3 = 4(2 - \sqrt{2})ma^2.}$$

Step 2. Eigenvectors

Find eigenvectors by plugging back the found eigenvalues one by one to the equation $(I - \lambda E)\overline{u} = 0$.

$\boxed{\lambda_1}$

$$\begin{bmatrix} -12ma^2 & 4ma^2 & 0 \\ 4ma^2 & -4ma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{1,x} \\ u_{1,y} \\ u_{1,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \begin{cases} u_{1,x} & = 0 \\ u_{1,y} & = 0 \\ u_{1,z} & = \alpha, \qquad \alpha \in \mathbb{R} \end{cases}$$

Hence, the eigenvector corresponding to λ_1 is of the form $\overline{u}_1 = (0, 0, \alpha)$. We may require this vector to be a unit vector. Then it defines the direction $\boxed{(0, 0, 1)}$, that is the original z-axis.

Step 2. Eigenvectors (contd)

$$\boxed{\lambda_2} \qquad \begin{bmatrix} -4ma^2(1+\sqrt{2}) & 4ma^2 & 0 \\ 4ma^2 & 4ma^2(1-\sqrt{2}) & 0 \\ 0 & 0 & 4ma^2(2-\sqrt{2}) \end{bmatrix} \begin{bmatrix} u_{1,x} \\ u_{1,y} \\ u_{1,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{cases} u_{2,x} & = \alpha, \quad \alpha \in \mathbb{R} \\ u_{2,y} & = (1+\sqrt{2})\alpha \\ u_{2,z} & = 0 \end{cases}$$

The eigenvector corresponding to λ_2 is $\overline{u}_2 = (\alpha, (1+\sqrt{2})\alpha, 0)$, and hence defines the direction $\boxed{\left(1, (1+\sqrt{2}), 0\right).}$

$$\boxed{\lambda_3} \qquad \begin{bmatrix} -4ma^2(1-\sqrt{2}) & 4ma^2 & 0 \\ 4ma^2 & 4ma^2(1+\sqrt{2}) & 0 \\ 0 & 0 & 4ma^2(2+\sqrt{2}) \end{bmatrix} \begin{bmatrix} u_{3,x} \\ u_{3,y} \\ u_{3,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} u_{1,x} & = \alpha, \quad \alpha \in \mathbb{R} \\ u_{1,y} & = (1-\sqrt{2})\alpha \\ u_{1,z} & = 0 \end{cases}$$

Hence $\overline{u}_3 = (\alpha, (1-\sqrt{2})\alpha, 0)$, defining the direction $\boxed{\left(1, (1-\sqrt{2}), 0\right).}$

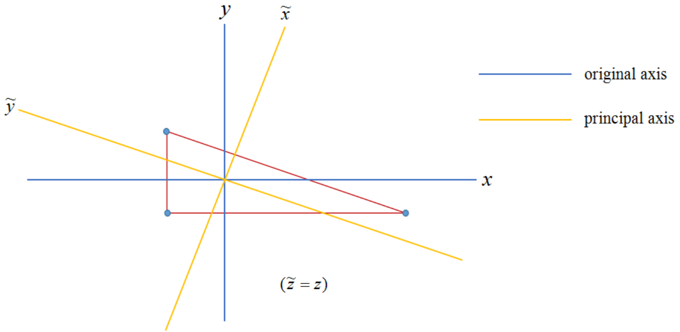
Summary of Example (c)

Principal axes are given by the directions

$$\boxed{(0,0,1), \qquad (1, 1+\sqrt{2}, 0), \qquad (1, 1-\sqrt{2}, 0),}$$

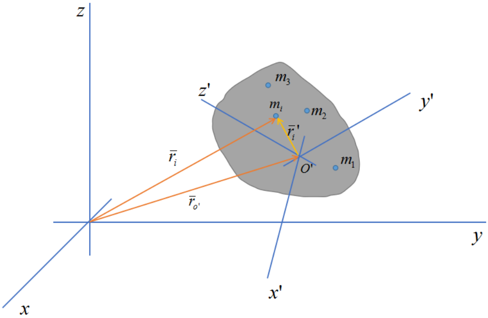
and the corresponding moments of inertia about these principal axes are

$$\boxed{I_{\tilde{z}\tilde{z}} = 16ma^2, \quad I_{\tilde{x}\tilde{x}} = 4(2+\sqrt{2})ma^2, \quad I_{\tilde{y}\tilde{y}} = 4(2-\sqrt{2})ma^2}$$



Kinetic Energy of a Rotating Rigid Body

Kinetic Energy of a Rotating Rigid Body



$$K = \sum_{i=1}^{N} K_i = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2$$

$\downarrow$

in the lab frame of reference

As before, for any point of the rigid body $\overline{v}_i = \overline{v}_{O'} + \overline{\omega} \times \overline{r}'_i$, and assuming $\overline{v}_{O'} = 0$ (no translational motion), $\overline{r}'_{\text{cm}} = 0$ ($O^{'}$ is chosen to be the center of mass of the rigid body), we can find (see Problem Set)

$$\boxed{K = \frac{1}{2} \sum_{\alpha'\beta'} I_{\alpha'\beta'} \omega_{\alpha'} \omega_{\beta'} .}$$

or in the matrix notation

$$K = \frac{1}{2} \underbrace{\overline{\omega}^T I \overline{\omega}}_{\substack{\downarrow \\ \text{quadratic} \\ \text{form}}} .$$

Of course, since K is the kinetic energy, $K \geq 0$ (the quadratic form non-negative definite).

In the next lectures we will discuss a much simpler case: rotation about a principal axis.