VP160 Recitation Class VII

Angular Momentum & Rigid Body Dynamics Part I

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$$\Rightarrow \underbrace{\vec{r} \times \vec{F}}_{\vec{\tau}} = \frac{d}{dt} \underbrace{\vec{r} \times \vec{p}}_{\vec{L}}$$

$\vec{\tau}$: torque

Angular Momentum Theorem

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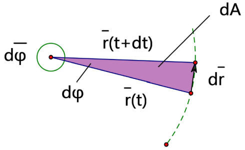
Applications:

- Central force field ($\vec{\tau} = \vec{r} \times \vec{F} = 0$)
- Aerial velocity, e.g. motion of planets, Kepler's Second Laws

For planer motion, the **aerial velocity** may be defined



The surface area swept by $\bar{r}$ over the time dt is $dA = \left|\frac{1}{2}\bar{r} \times d\bar{r}\right|$ and the rate of change of that area

$$\frac{dA}{dt} = \frac{1}{2}\left|\bar{r} \times \frac{d\bar{r}}{dt}\right| = \frac{1}{2}\left|\bar{r} \times \bar{v}\right|.$$

Aerial velocity vector (direction — right-hand rule)

$$\boxed{\bar{\sigma} = \frac{1}{2}(\bar{r} \times \bar{v})} \qquad \text{(direction same as } d\bar{\varphi}\text{)}$$

Recall: $\bar{L} = \bar{r} \times \bar{p} = \underbrace{\bar{r} \times m\bar{v}}_{=\bar{\sigma}\cdot 2m}$. Hence $\bar{L} = const \Leftrightarrow \bar{\sigma} = const$.

Consequently, for motion in a central force field $\bar{\sigma} = \text{const}$.

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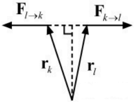
Why $\vec{\tau_{int}} = 0$?

For any two particles k, l in the system,

$$\vec{\tau_{k \to l}} = -\vec{\tau_{l \to k}}$$



Applications:

Use with other conservation laws:

- Conservation of Energy
- Conservation of Momentum

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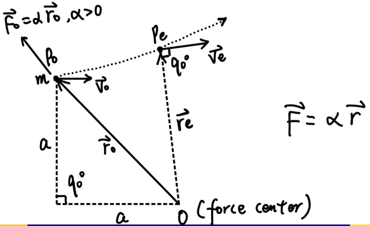
Exercise 1

A particle with mass m is put into a force field $\vec{F} = \alpha\vec{r}$, where α is a positive constant. the Particle's initial velocity is $\vec{v_0}$ and its initial position is P_0, when it moves to the position P_e, the instantaneous velocity $\vec{v_e}$ is orthogonal to its radius vector $\vec{r_e}$. Take $\alpha = \frac{mv_0^2}{4a^2}$ and calculate the value of $\frac{|\vec{v_e}|}{|\vec{v_0}|}$.



Rigid Body

A body is called rigid if $|\vec{r}_i - \vec{r}_j| = \text{const}$ for any point i, j in the body.

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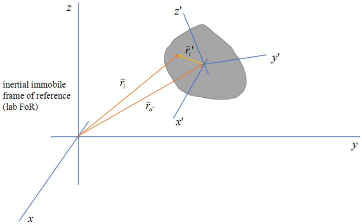


FoR associated with the rigid body is, in general, non-inertial —the body can move arbitrarily.

O' — a point of the body

We have (see the derivation of dynamics in non-inertial FoRs)

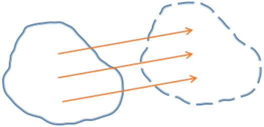
$$\begin{aligned} \bar{r}_i &= \bar{r}_{O'} + \bar{r}'_i, \\ \bar{v}_i &= \bar{v}_{O'} + \underbrace{\bar{v}'_i}_{=0} + \bar{\omega} \times \bar{r}'_i, \end{aligned}$$

where $\bar{v}'_i = 0$ due to the fact that the body is rigid (no relative motion of the rigid body's points).

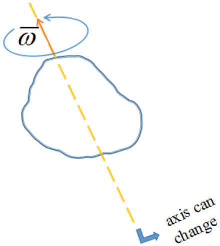
Eventually, the velocity of any point of a rigid body

$$\overline{v}_i = \overline{v}_{O'} + \overline{\omega} \times \overline{r}'_i.$$

The first term on the right hand side corresponds to the **translational motion**, while the second term to the **rotational motion** about an *instantaneous axis of rotation*.



translational motion



rotational motion

Consequently, the total momentum of an arbitrarily moving rigid body (in lab FoR) is

$$\begin{aligned}\overline{P} &= \sum_{i=1}^{N} m_i \overline{v}_i = \underbrace{\sum_{i=1}^{N} m_i}_{M} \overline{v}_{O'} + \sum_{i=1}^{N} m_i (\overline{\omega} \times \overline{r}_{i'}) = \\ &= M\overline{v}_{O'} + \overline{\omega} \times \underbrace{\sum_{i=1}^{N} m_i \overline{r}_{i'}}_{M\overline{r}'_{\mathrm{cm}}} = \underbrace{M\overline{v}_{O'}}_{\substack{\text{translational} \\ \text{motion}}} + \underbrace{M\overline{\omega} \times \overline{r}'_{\mathrm{cm}}}_{\substack{\text{rotational} \\ \text{motion}}}\end{aligned}$$

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$$\Rightarrow \vec{p} = M\vec{v}_c$$

In the lab FoR

$$\begin{aligned}
\overline{L} &= \sum_{i=1}^{N} \overline{L}_i = \sum_{i=1}^{N} m_i \overline{r}_i \times \overline{v}_i = \sum_{i=1}^{N} [m_i(\overline{r}_{O'} + \overline{r}'_i) \times (\overline{v}_{O'} + \overline{\omega} \times \overline{r}'_i)] \\
&= \sum_{i=1}^{N} m_i(\overline{r}_{O'} \times \overline{v}_{O'}) + \sum_{i=1}^{N} m_i \overline{r}_{O'} \times (\overline{\omega} \times \overline{r}'_i) + \\
&\quad + \sum_{i=1}^{N} m_i \overline{r}'_i \times \overline{v}_{O'} + \sum_{i=1}^{N} m_i \overline{r}'_i \times (\overline{\omega} \times \overline{r}'_i) \\
&= M \overline{r}_{O'} \times \overline{v}_{O'} + M \overline{r}_{O'} \times (\overline{\omega} \times \overline{r}'_{\mathrm{cm}}) + M \overline{r}'_{\mathrm{cm}} \times \overline{v}_{O'} + \\
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$\vec{L}' = \sum_{i=1}^{N} m_i \bar{r}'_i \times (\bar{\omega} \times \bar{r}'_i)$: Rigid body's angular momentum w.r.t its center of mass

Tensor of Inertia

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$$I = \begin{bmatrix} I_{x'x'} & I_{x'y'} & I_{x'z'} \\ I_{y'x'} & I_{y'y'} & I_{y'z'} \\ I_{z'x'} & I_{z'y'} & I_{z'z'} \end{bmatrix} =$$

$$\begin{bmatrix} \sum_{i=1}^{N} m_i \left(y_i'^2 + z_i'^2\right) & \sum_{i=1}^{N} -m_i x_i' y_i' & \sum_{i=1}^{N} -m_i x_i' z_i' \\ \sum_{i=1}^{N} -m_i y_i' x_i' & \sum_{i=1}^{N} m_i \left(x_i'^2 + z_i'^2\right) & \sum_{i=1}^{N} -m_i y_i' z_i' \\ \sum_{i=1}^{N} -m_i z_i' x_i' & \sum_{i=1}^{N} -m_i z_i' y_i' & \sum_{i=1}^{N} m_i \left(x_i'^2 + y_i'^2\right) \end{bmatrix}$$

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$\vec{L}$ can not be always parallel to $\vec{\omega}$, when will it be?
By a random choice of x', y', z', e.g. $\omega_x', \omega_y', \omega_z'$ **all contributes to $\vec{\mathbf{L}}_{\mathbf{x}}'$**, it's hard to see, but if we ...

By choosing a better set of $\tilde{x}', \tilde{y}', \tilde{z}'$, we can obtain a diagonal form of I.

$$I = \begin{bmatrix} I_{\tilde{x}'\tilde{x}'} & 0 & 0 \\ 0 & I_{\tilde{y}'\tilde{y}'} & 0 \\ 0 & 0 & I_{\tilde{z}'\tilde{z}'} \end{bmatrix}$$

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Then, ω_x' **only contributes to $\vec{\mathbf{L}}_{\mathbf{x}}'$, so do** ω_y' **and** ω_z'

$$\Rightarrow L_{x'} = I_{\tilde{x}'\tilde{x}'} \cdot \omega_{x'}, \quad L_{y'} = I_{\tilde{y}'\tilde{y}'} \cdot \omega_{y'}, \quad L_{z'} = I_{\tilde{z}'\tilde{z}'} \cdot \omega_{z'}$$

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The axis in this special sets of axes is called the **Pricipal axis**, which is our main focus.

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4. Plug back λ_i into the equation $(I - \lambda_i\mathbb{1})\vec{u_i} = 0$, find the solution $\vec{u_1}, \vec{u_2}, \vec{u_3}$

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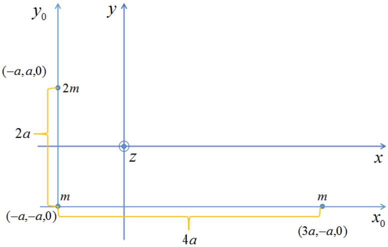
- Use symmetry to "guess" the principle axes.
- Why does this methods always works?
- Recall the form of I, it's a self-adjoint matrix.
  To learn the mathematical details, have a look at:
  zxj_Eigenvalue & Diagonalization.pdf (under canvas RC folder).

Exercise 2

Use the example in slide(s-21hp14) to practice.



(answer: in slide)

Reference

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