Angular Momentum

- Derive: $\vec{F} = \frac{d\vec{p}}{dt}$

$$\underbrace{\vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt}}$$

$$\frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d}{dt}\vec{r} \times \vec{p} + \vec{r} \times \frac{d}{dt}\vec{p} = \vec{v} \times m\vec{v} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}$$

So, $\underbrace{\vec{r} \times \vec{F}}_{\text{torque}} = \frac{d}{dt}\underbrace{(\vec{r} \times \vec{p})}_{\text{angular momentum}}$

$\vec{\tau} \stackrel{def}{=} \vec{r} \times \vec{F}$ $\quad$ $[N \cdot m]$

$\vec{L} \stackrel{def}{=} \vec{r} \times \vec{p}$ $\quad$ $[kg \cdot m^2/s]$

$$\boxed{\vec{\tau} = \frac{d\vec{L}}{dt} \Rightarrow \vec{L}(t_2) - \vec{L}(t_1) = \int_{t_1}^{t_2} \vec{\tau}\, dt}$$

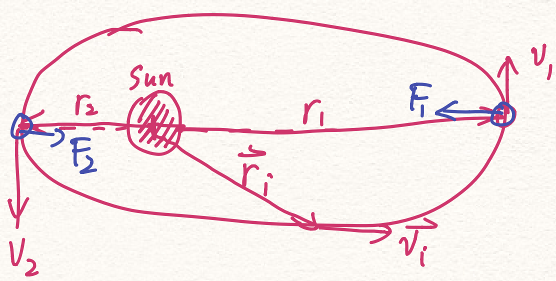
The angular momentum theorem

$$\boxed{\text{If } \vec{\tau} = 0 \Rightarrow \vec{L} = \text{Const}}$$

Law of conservation of angular momentum.

Application: ① central force field.

example. (astronomical body problems)



$\vec{L} = \text{const}$

$$\Rightarrow m r_1 V_1 = m r_2 V_2 = |m\vec{r_i} \times \vec{v_i}|$$

$$= L_0$$

② Aerial Velocity

$$\frac{dA}{dt} = \frac{1}{2}\left|\vec{r} \times \frac{d\vec{r}}{dt}\right| = \frac{1}{2}\left|(\vec{r} \times \vec{v})\right| = \frac{L}{2m}$$

△ Angular Momentum in System of Particles

Similarly, $\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}_{ext}}$

Conservation of the Angular Momentum Law

If the net torque of external forces on a system of particles is equal to zero, then the total angular momentum of that system is conserved.

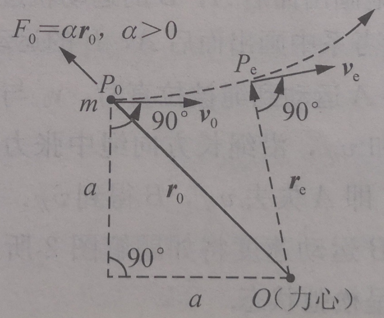
Application: Can be combined with

Conservation of Energy

Conservation of Momentum

【题 1】

质量为 m 的质点相对固定力心 O 的矢径为 $\boldsymbol{r}$ 时受力 $\boldsymbol{F}=\alpha\boldsymbol{r}$，其中 α 是正的常量．质点初始速度 $\boldsymbol{v}_0$ 及其初始位置 P_0 与力心间的相对几何关系如图所示．质点运动到图中 P_e 位置时，速度方向恰好与其相对力心的矢径方向垂直．设 $4\alpha a^2=mv_0^2$，其中 a 为图示几何参量，试求质点位于 P_e 时的速度大小 v_e 与初始速度大小 v_0 的比值 r，答案只能用数字表述．



Force field : $\vec{F}(r) = \alpha \vec{r}$

$$-\frac{\partial U}{\partial r} = \vec{F}(r) \Rightarrow U(r) = -\frac{1}{2}\alpha r^2 \quad \text{potential Energy.}$$

① Conservation of angular momentum :

$$r_e m v_e = r_0 m v_0 \cdot \cos 45^\circ = a m v_0$$

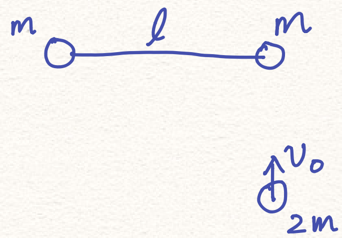
② Conservation of E :

$$\frac{1}{2} m v_e^2 - \frac{1}{2}\alpha r_e^2 = \frac{1}{2} m v_0^2 - \frac{1}{2}\alpha (a^2 + a^2)$$

$$4\alpha a^2 = m v_0^2$$

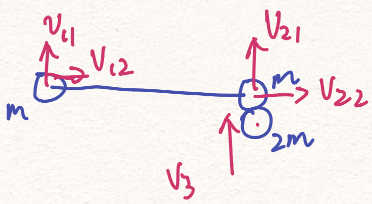
Finally : $v_e = \frac{1}{2}\sqrt{1+\sqrt{5}}\, v_0$.

[Q2]



What will happen after "2m" hit "m"?

①



There must be :

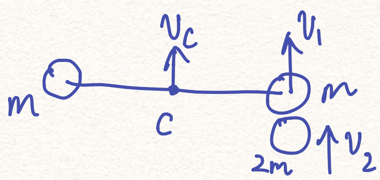
$v_{11} = 0$

$v_{22} = 0$

$v_{12} = 0$

why?

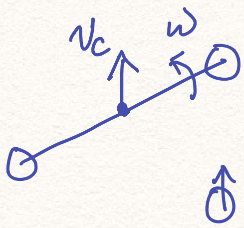
②



$$\begin{cases} 2mv_0 = mv_1 + 2mv_2 \\ v_0 - 0 = v_1 - v_2 \end{cases} \Rightarrow \begin{cases} v_2 = \frac{1}{3}v_0 \\ v_1 = \frac{4}{3}v_0 \end{cases}$$

$$v_c = \frac{1}{2}v_1 = \frac{2}{3}v_0$$

③

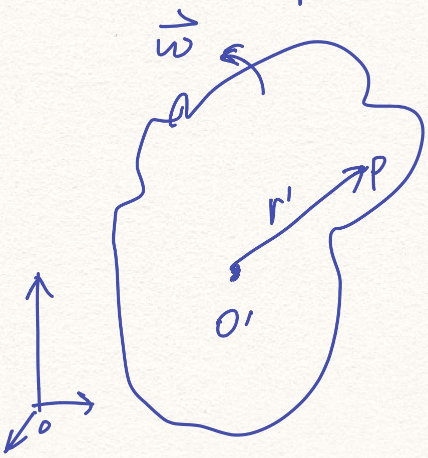


$$\omega = \frac{v_1 - v_c}{\frac{1}{2}l} = \frac{\frac{2}{3}v_0}{\frac{1}{2}l} = \frac{4}{3}\frac{v_0}{l}$$

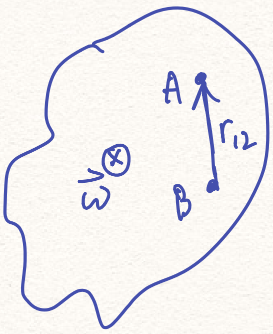
△ Rigid Body Dynamics

∘ Rigid Body : A body is called rigid if $|\vec{r}_i - \vec{r}_j| = \text{Const}$ for any i, j.

- Math Description



$$\begin{cases} \vec{r}_p = \vec{r}_{o'} + \vec{r}' \\ \underline{\vec{v}_p = \vec{v}_{o'} + \vec{\omega} \times \vec{r}'} \end{cases}$$



At any time, there is an instantaneous axis of rotation, which satisfy: $\underline{\vec{v}_1 - \vec{v}_2 = \vec{\omega} \times \vec{r}_{12}}$

☆ Very important :

$$\begin{cases} \underline{\vec{P} = \vec{P}_c = m\vec{v}_c} \\ \vec{L} = \vec{L}_c + \vec{L}_{in\text{-}c} \\ \quad = \underline{m \cdot \vec{r}_c \times \vec{v}_c + \vec{L}_{in\text{-}c}} \end{cases}$$

△ Tensor of Inertia

Definition : $\boxed{\vec{L} = I \cdot \vec{\omega}}$

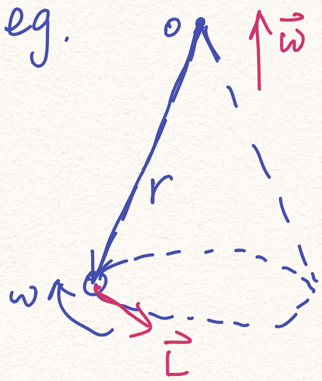
I : A Tensor, a linear map

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \quad \text{with} \quad I_{\alpha\beta} = I_{\beta\alpha}$$

$$= \begin{bmatrix} \sum m_i (y_i^2 + z_i^2) & -\sum m_i x_i y_i & -\sum m_i x_i z_i \\ -\sum m_i y_i x_i & \sum m_i (x_i^2 + z_i^2) & -\sum m_i y_i z_i \\ -\sum m_i z_i x_i & -\sum m_i z_i y_i & \sum m_i (y_i^2 + x_i^2) \end{bmatrix}$$

① $\vec{L}$ can be not parallel to $\vec{\omega}$

eg.



$\vec{L} \not\parallel \vec{\omega}$

② If we want $\vec{L} \parallel \vec{\omega}$

"I" should be a diagonal form. $I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$

And this axis is called "$\boxed{\text{principal axis}}$"

△ Why we want to find "principal axis"?

① Easy to represent $\vec{L}$ and K_{rot}.

$$\vec{L} = \begin{pmatrix} I_{xx} \cdot \omega_x \\ I_{yy} \cdot \omega_y \\ I_{zz} \cdot \omega_z \end{pmatrix}$$

$$K_{rol} = \frac{1}{2}\left(I_{xx} \cdot \omega_x^2 + I_{yy} \cdot \omega_y^2 + I_{zz} \cdot \omega_z^2 \right)$$

② We will be interested in $\vec{\tau} = \frac{d\vec{L}}{dt} = \underbrace{\frac{d(I \cdot \vec{\omega})}{dt}}$

In principle axis :

$$\frac{d\vec{L}}{dt} = \begin{pmatrix} I_{xx} \cdot \beta_x \\ I_{yy} \cdot \beta_y \\ I_{zz} \cdot \beta_z \end{pmatrix}$$

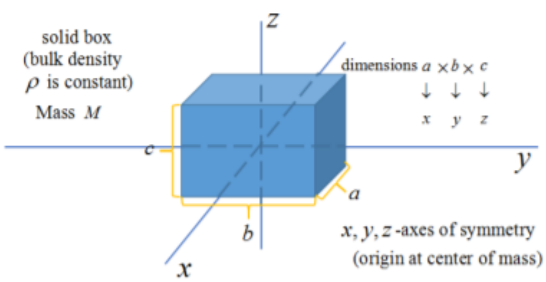
In other axis : very complex.

△ example of calculation :

① Calculation of I when "o-xyz" fixed.

Example (b). Uniform solid box

We will find the tensor of inertia and show that x, y, z are the principal axes.



Continuous distribution
$(\Sigma \longrightarrow \int)$

$dm = \rho \cdot dx\,dy\,dz$

$$\underset{\text{triple integral}}{\int_{\Omega} (\ldots)\, dm} = \varrho \underset{\text{iterated integral}}{\int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} (\ldots)\, dz}$$

$$[I]_{\alpha\beta} = \begin{bmatrix} \int_{\Omega} (y^2 + z^2)\, dm & -\int_{\Omega} xy\, dm & -\int_{\Omega} xz\, dm \\ -\int_{\Omega} yx\, dm & \int_{\Omega} (x^2 + z^2)\, dm & -\int_{\Omega} yz\, dm \\ -\int_{\Omega} zx\, dm & -\int_{\Omega} yz\, dm & \int_{\Omega} (x^2 + y^2)\, dm \end{bmatrix}$$

Explicitly, for the box

$$I_{xx} = \varrho \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} (y^2+z^2)\,dz = \varrho \underbrace{\int_{-a/2}^{a/2} dx}_{=a} \int_{-b/2}^{b/2} \left[y^2 z + \frac{1}{3}z^3\right]_{z=-\frac{c}{2}}^{z=\frac{c}{2}} d$$

$$= \varrho a \int_{-b/2}^{b/2} \left(y^2 c + \frac{1}{3}\frac{c^3}{4}\right) dy = \varrho a \left[\frac{1}{3}y^3 c + \frac{1}{12}c^3 y\right]_{y=-\frac{b}{2}}^{y=\frac{b}{2}}$$

$$= \varrho a \left(\frac{1}{3}\frac{b^3}{4}c + \frac{1}{12}c^3 b\right) = \frac{\varrho abc}{12}\left(b^2+c^2\right) = \boxed{\frac{M}{12}\left(b^2+c^2\right)}.$$

Analogously,

$$\boxed{I_{yy} = \frac{M}{12}\left(a^2+c^2\right)}, \qquad \boxed{I_{zz} = \frac{M}{12}\left(a^2+b^2\right)}$$

and $\boxed{I_{xy} = I_{xz} = I_{yz} = 0}$ (integrals of odd functions over symmetric intervals).

Eventually,

$$[I] = \begin{bmatrix} \frac{M}{12}\left(b^2+c^2\right) & 0 & 0 \\ 0 & \frac{M}{12}\left(a^2+c^2\right) & 0 \\ 0 & 0 & \frac{M}{12}\left(a^2+b^2\right) \end{bmatrix}.$$

Hence, since the tensor is diagonal, the axes x, y, z are the principal axes indeed.

② How to find principle axis

i) Find the mass center C, Let C be the new origin.

ii) Use "C-xyz" to calculate out the I.

iii) Let $\det(I - \lambda \cdot \mathbb{1}) = 0$, find $\lambda_1, \lambda_2, \lambda_3$

iv) Plug back λ_i, find $\vec{u}_1, \vec{u}_2, \vec{u}_3$

v) use the direction of $\vec{u}_1, \vec{u}_2, \vec{u}_3$ as axis,

write out the final new I_p.

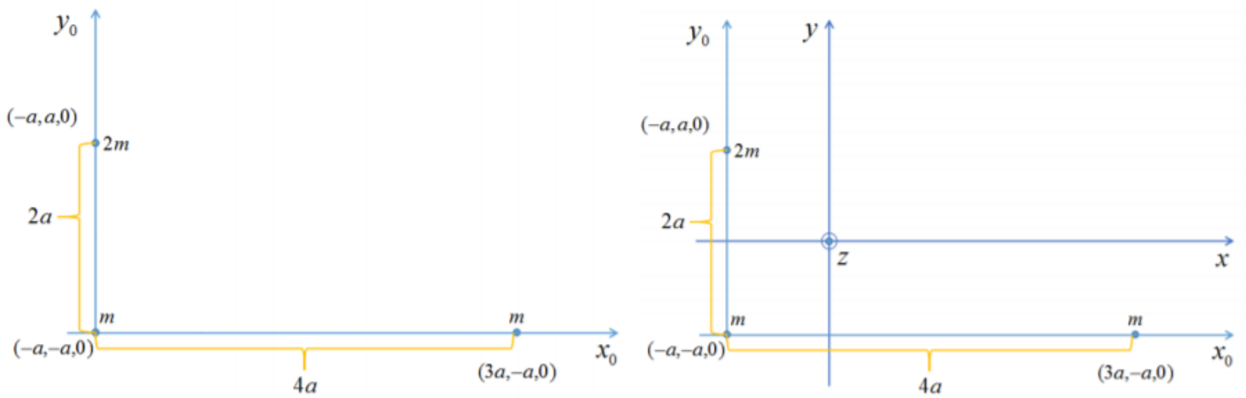
Why this always works?

Because it's a self-Adjoint Matrix.

Example (c). Non-Symmetric Discrete Mass Distribution

Consider a system of three masses arranged as in the figure. First we need to find the center of mass of this system (in the $x_0 O y_0$ plane).

$\overline{r}_{0,cm} = \frac{1}{4m}\left[m \cdot (0,0,0) + 2m \cdot (0,2a,0) + m \cdot (4a,0,0)\right] = (a,a,0)$



Then we can find the tensor of inertia with respect to x, y, z (passing through the center of mass). For the diagonal terms we have

$$I_{xx} = \sum_i m_i\left(y_i^2 + z_i^2\right) = ma^2 + ma^2 + 2ma^2 = 4ma^2,$$

$$I_{yy} = \sum_i m_i\left(x_i^2 + z_i^2\right) = ma^2 + 9ma^2 + 2ma^2 = 12ma^2,$$

$$I_{zz} = \sum_i m_i\left(x_i^2 + y_i^2\right) = m(a^2 + a^2) + m(9a^2 + a^2) + 2m(a^2 + a^2) = 16ma^2$$

For the off-diagonal terms

$$I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0$$

and

$$I_{xy} = I_{yx} = -\sum_i m_i x_i y_i = -\left(ma^2 - 3ma^2 - 2ma^2\right) = 4ma^2.$$

Eventually, the tensor of inertia with respect to the axes x, y, z has the form

$$I = \begin{bmatrix} 4ma^2 & 4ma^2 & 0 \\ 4ma^2 & 12ma^2 & 0 \\ 0 & 0 & 16ma^2 \end{bmatrix}$$

Step 1. Eigenvalues

Solve the characteristic equation $\det(I - \lambda E) = 0$ to get

$$\det \begin{bmatrix} 4ma^2 - \lambda & 4ma^2 & 0 \\ 4ma^2 & 12ma^2 - \lambda & 0 \\ 0 & 0 & 16ma^2 - \lambda \end{bmatrix} = 0.$$

Hence

$$\Rightarrow (16ma^2 - \lambda)\left[(4ma^2 - \lambda)(12ma^2 - \lambda) - 16\left(ma^2\right)^2\right] = 0$$

$$\Rightarrow (16ma^2 - \lambda)\left[\lambda^2 - 16ma^2\lambda + 32\left(ma^2\right)^2\right] = 0.$$

And eventually, after solving the quadratic equation,

$\boxed{\lambda_1 = 16ma^2,}$ $\boxed{\lambda_2 = 4(2 + 2\sqrt{2})ma^2,}$ $\boxed{\lambda_3 = 4(2 - \sqrt{2})ma^2.}$

Step 2. Eigenvectors

Find eigenvectors by plugging back the found eigenvalues one by one to the equation $(I - \lambda E)\overline{u} = 0$.

$\boxed{\lambda_1}$

$$\begin{bmatrix} -12ma^2 & 4ma^2 & 0 \\ 4ma^2 & -4ma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{1,x} \\ u_{1,y} \\ u_{1,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{cases} u_{1,x} & = 0 \\ u_{1,y} & = 0 \\ u_{1,z} & = \alpha, \quad \alpha \in \mathbb{R} \end{cases}$$

Hence, the eigenvector corresponding to λ_1 is of the form $\overline{u}_1 = (0, 0, \alpha)$. We may require this vector to be a unit vector. Then it defines the direction $\boxed{(0, 0, 1)}$, that is the original z-axis.

Step 2. Eigenvectors (contd)

$\boxed{\lambda_2}$

$$\begin{bmatrix} -4ma^2(1+\sqrt{2}) & 4ma^2 & 0 \\ 4ma^2 & 4ma^2(1-\sqrt{2}) & 0 \\ 0 & 0 & 4ma^2(2-\sqrt{2}) \end{bmatrix} \begin{bmatrix} u_{1,x} \\ u_{1,y} \\ u_{1,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{cases} u_{1,x} & = \alpha, \quad \alpha \in \mathbb{R} \\ u_{1,y} & = (1+\sqrt{2})\alpha \\ u_{1,z} & = 0 \end{cases}$$

The eigenvector corresponding to λ_2 is $\overline{u}_2 = (\alpha, (1+\sqrt{2})\alpha, 0)$, and hence defines the direction $\boxed{\left(1, (1+\sqrt{2}), 0\right).}$

$\boxed{\lambda_3}$

$$\begin{bmatrix} -4ma^2(1-\sqrt{2}) & 4ma^2 & 0 \\ 4ma^2 & 4ma^2(1+\sqrt{2}) & 0 \\ 0 & 0 & 4ma^2(2+\sqrt{2}) \end{bmatrix} \begin{bmatrix} u_{1,x} \\ u_{1,y} \\ u_{1,z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} u_{1,x} & = \alpha, \quad \alpha \in \mathbb{R} \\ u_{1,y} & = (1-\sqrt{2})\alpha \\ u_{1,z} & = 0 \end{cases}$$

Hence $\overline{u}_3 = (\alpha, (1-\sqrt{2})\alpha, 0)$, defining the direction $\boxed{\left(1, (1-\sqrt{2}), 0\right).}$

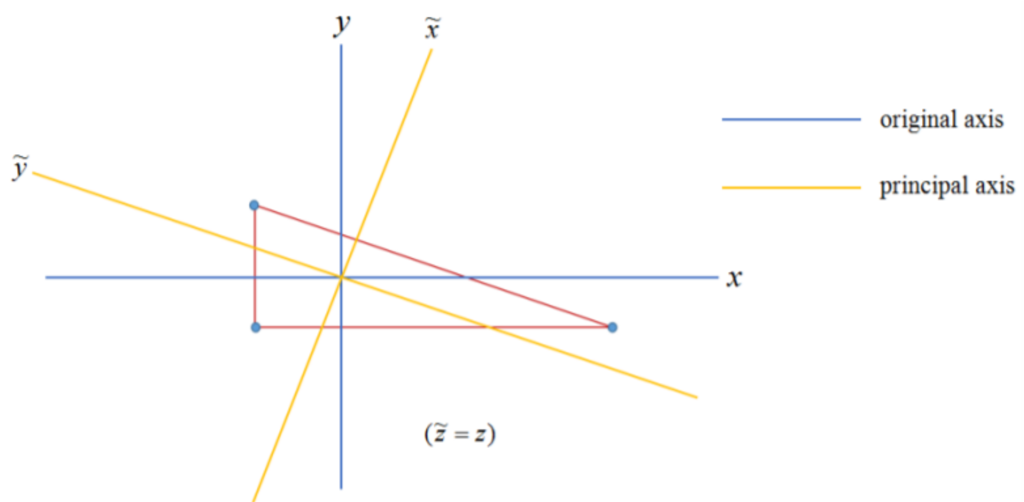
Summary of Example (c)

Principal axes are given by the directions

$$\boxed{(0,0,1), \qquad (1, 1+\sqrt{2}, 0), \qquad (1, 1-\sqrt{2}, 0),}$$

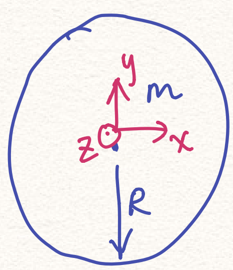
and the corresponding moments of inertia about these principal axes are

$$\boxed{I_{\tilde{z}\tilde{z}} = 16ma^2, \quad I_{\tilde{x}\tilde{x}} = 4(2+\sqrt{2})ma^2, \quad I_{\tilde{y}\tilde{y}} = 4(2-\sqrt{2})ma^2}$$



△ Some useful Moment of Inertia

① Disk (Uniform)



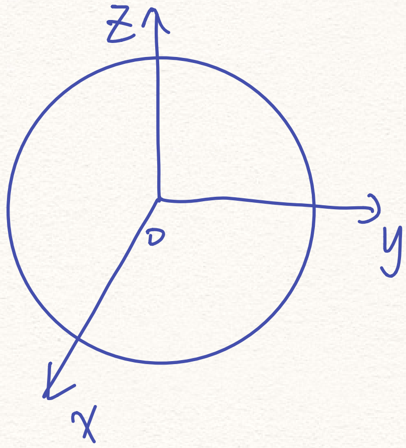
$$I = \begin{pmatrix} \frac{1}{4}mR^2 & & \\ & \frac{1}{4}mR^2 & \\ & & \frac{1}{2}mR^2 \end{pmatrix}$$

$$\boxed{I_{zz} = I_{xx} + I_{yy}}$$ For planar objects

② Uniform Cylinder

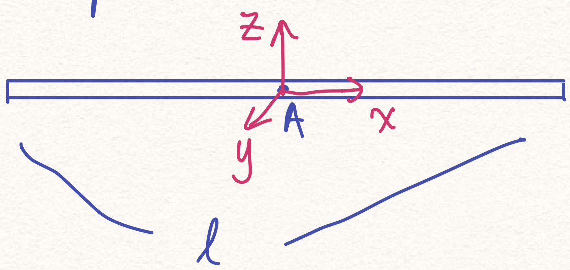
$$I_{zz} = \frac{1}{2}mR^2$$

③ Uniform ball

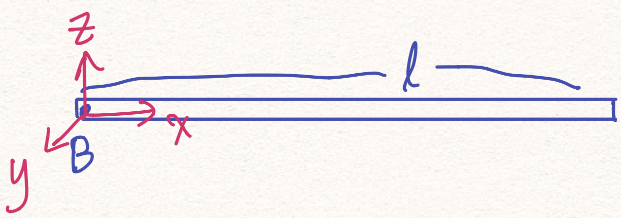


$$I = \begin{pmatrix} \frac{2}{5}mR^2 & & \\ & \frac{2}{5}mR^2 & \\ & & \frac{2}{5}mR^2 \end{pmatrix}$$

④ Uniform Rod



$$A: \begin{cases} I_{xx} = 0 \\ I_{yy} = \frac{1}{12}ml^2 \\ I_{zz} = \frac{1}{12}ml^2 \end{cases}$$



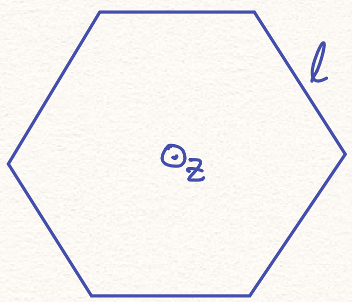
$$B: \begin{cases} I_{xx} = 0 \\ I_{yy} = \frac{1}{3}ml^2 \\ I_{zz} = \frac{1}{3}ml^2 \end{cases}$$

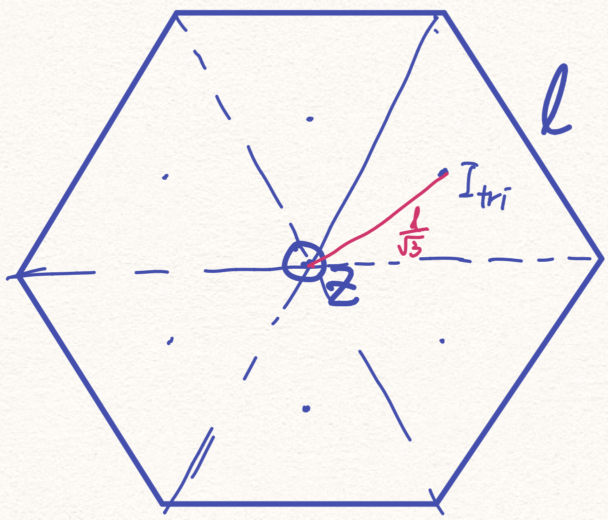
Parallel Axis Theorem: Assume "A" is the center of mass.

$$I_B = I_A + m\, l_{AB}^2$$

There, $I_B = I_A + m l_{AB}^2 = \frac{1}{12}ml^2 + m \cdot \left(\frac{l}{2}\right)^2 = \frac{1}{3}ml^2$.

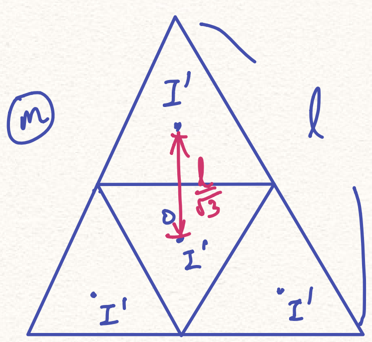
[Q3] Caculate the Moment of Inertia I_{zz} of a Uniform regular hexagon.





$$m = \frac{1}{6}M$$

$$I_H = 6\cdot\left(I_{tri} + \frac{1}{6}M\cdot\left(\frac{l}{\sqrt{3}}\right)^2\right)$$



Assume $I_{tri} = x m l^2$

by dimensional analysis : $I' = x\cdot\frac{1}{4}m\cdot\left(\frac{l}{2}\right)^2$

$$= \frac{1}{16}x m l^2$$

$$I_{tri} = I' + 3\cdot\left(I' + \frac{1}{4}m\cdot\left(\frac{l}{\sqrt{3}}\right)^2\right)$$

$$= \frac{1}{16}x m l^2 + 3\left(\frac{1}{16}x m l^2 + \frac{1}{12}m l^2\right)$$

$$x m l^2 = \frac{1}{4}x m l^2 + \frac{1}{4}m l^2 \Rightarrow x = \frac{1}{3}$$

$$\Rightarrow I_{tri} = \frac{1}{3}m l^2$$

$$\Rightarrow I_H = 6\left(\frac{1}{3}\cdot\frac{1}{6}M\cdot l^2 + \frac{1}{6}M\cdot\frac{l^2}{3}\right) = \frac{2}{3}M l^2$$

△ Now, having $\vec{L} = I\vec{\omega}$, we can use new form of angular momentum.

Also,

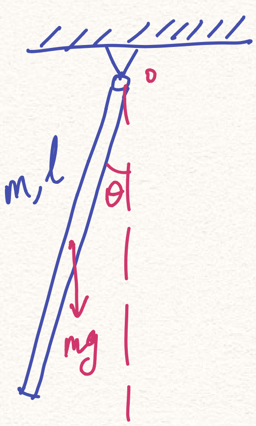
$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

$$= \frac{d(I\cdot\vec{\omega})}{dt} \underset{\text{axis}}{\overset{\text{principle}}{=}} I\cdot\frac{d\vec{\omega}}{dt}$$

$$\boxed{\vec{\tau} = I\cdot\frac{d\vec{\omega}}{dt}}$$ "Law of Rotation"

(you can compare it with $\underbrace{\vec{F} = m\cdot\frac{d\vec{v}}{dt}}$)

[Q4]



What's the period of this oscillation?
(small amplitute)

$$\tau = mg\cdot\frac{l}{2}\sin\theta$$

$$\approx \frac{1}{2}mgl\theta$$

$$I = \frac{1}{3}ml^2$$

$$\tau = I\cdot\frac{d\omega}{dt} = I\ddot{\theta}$$

$$\frac{1}{2}mgl\theta = \frac{1}{3}ml^2\ddot{\theta} \Rightarrow \ddot{\theta} - \frac{3g}{2l}\theta = 0$$

$$\omega = \sqrt{\frac{3g}{2l}}, \quad T = 2\pi\sqrt{\frac{2l}{3g}}$$

[Q5]