Eigenvalue & Diagonalization

Definition

Eigenvalue, Eigenvector, Eigenspace

Let A be a $n \times n$ matrix. Then a number λ such that

$$Ax = \lambda x$$

is called an **eigenvalue** of A. Any x such that the above equation holds is called an **eigenvector** for the eigenvalue λ. The subspace

$$V_\lambda = \{x \in V : Ax = \lambda x\}$$

is called the **eigenspace** for the eigenvalue λ.

(Notice: In our case, the tensor of inertia has three eigenvector, whose directions coincide with the principal axes.)

Characteristic Polynomial

$$Ax = \lambda x \Leftrightarrow (A - \lambda E) = 0$$

where E is the unit matrix. This is a homogeneous system of linear equations which has a solution $x \in \mathbb{R}^n$ iff

$$p(\lambda) = \det(A - \lambda E) = 0$$

(Recall *Fredholm Alternative*. If you forget that, I recommend you restudy VV285!)

here $p(\lambda)$ is a polynomial of degree n, called the **characteristic polynomial**.

Geometric & Algebraic Multiplicity

1. $\dim V_\lambda$ is called the **geometric multiplicity** of λ. (indicates the number of independent eigenvectors associated to each eigenvalue)
2. the multiplicity of the zero of $p(\lambda) = 0$ is called the **algebraic multiplicity** of λ. (indicates how often the eigenvalue repeats in the characteristic polynomials)

A important relation: for any eigenvalue λ of A, geometric multiplicity $\leq$ algebraic multiplicity. (Result: We may fail to use eigenvectors to span the whole space.)

(Why do we hope they are equal: In this lucky case we can use the set of all eigenvectors as a basis and represent x by them. So the matrix operation becomes much simpler!)

Diagonalizable, Diagonal Matrix, Diagonal Form

A matrix $A \in \mathtt{mat}(n; \mathbb{R})$ whose eigenvectors form a basis of $\mathbb{R}^n$ is called **diagonalizable**.

Suppose that $A \in \mathrm{Mat}(n \times n, \mathbb{R})$ is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding independent eigenvectors $v_1, \ldots, v_n$. Then

$$U = (v_1, \ldots, v_n)$$

is invertible. The matrix U represents the transformation of the standard basis $(e_1, \ldots, e_n)$ into the basis of eigenvectors. It follows that

$$D := U^{-1}AU = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_n \end{pmatrix}$$

is a **diagonal matrix** whose elements are the eigenvalues of A. We say that D is the **diagonal form** of A.

proof:

The k^{th} column vector of D is given by

$$De_k = U^{-1}AUe_k = U^{-1}Av_k = U^{-1}\lambda_k v_k = \lambda_k U^{-1}v_k = \lambda_k e_k$$

(Equivalently, $A = UDU^{-1}$. This is what we essentially want when looking for principal axes: to find some suitable U and perform transformation of coordinates. $I\omega = UI_DU^{-1}\omega = UI_D\omega_D \Leftarrow$ essentially a transformation of basis.)

Self-Adjoint Matrices

[Why we care self-adjoint matrices: **Not every matrix is diagonalizable** (algebraic & geometric multiplicities may differ). However, this never happens for a self-adjoint matrix.]

If $A = A^*$, the map A is said to be **self-adjoint**. In the case of matrices, if $A \in \mathrm{Mat}(n \times n; \mathbb{C})$,

$$A = A^* \quad \Leftrightarrow \quad A = \bar{A}^T$$

where A^T denotes the transpose of A and $\bar{A}$ is the complex conjugate of A taken in each component. (Notice that the tensor of inertia I is symmetric, and it is real obvious. So it is self-adjoint.)

Theorem

Distinct Eigenvalues $\Rightarrow$ Independent Set

If all the eigenvalues are distinct with each other, the set of associated eigenvectors is an independent set.

(Corollary) A linear map $A \in \mathrm{mat}(n; \mathbb{F})$ can have at most n distinct eigenvectors.

(Corollary) If $A \in \mathtt{mat}(n; \mathbb{F})$ has n distinct eigenvalues, then it has precisely n independent eigenvectors and they constitute a basis of $\mathbb{R}^n$.

All eigenvalues of a self-adjoint matrix A are real.

proof:

$$\lambda\|x\|^2 = \lambda\langle x, x\rangle = \langle x, Ax\rangle = \langle Ax, x\rangle = \langle \lambda x, x\rangle = \bar{\lambda}\langle x, x\rangle = \bar{\lambda}\|x\|^2$$

(Corollary) A linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $A = A^T$ has at least one eigenvalue (real).

*** Spectral Theorem**

Let $A = A^* \in \mathrm{Mat}(n \times n; \mathbb{R})$ be a self-adjoint matrix. Then there exists an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

(Corollary) Every self-adjoint matrix A is diagonalizable. Furthermore, if $(v_1, \ldots, v_n)$ is an orthonormal basis of eigenvectors and $U = (v_1, \ldots, v_n)$, then $U^{-1} = U^*$. Hence, if A is self-adjoint, there exists an orthogonal matrix U such that

$$D = U^* AU$$

is the diagonalization of A.