VP160 Recitation Class IV

Non-inertial FoR

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Recall

The non-inertial FoR is a frame of reference that is not in a linear motion with uniform velocity relative to an inertial frame of reference.

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Basic Formula

$$\vec{a}' = \vec{a} - \vec{a}'_O - \frac{d\vec{\omega}}{dt} \times \vec{r}' - 2(\vec{\omega} \times \vec{v}') - \vec{\omega} \times (\vec{\omega} \times \vec{r}')$$

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$$m\vec{a}' = \vec{F} - m\vec{a}'_O - m\frac{d\vec{\omega}}{dt} \times \vec{r}' - 2m(\vec{\omega} \times \vec{v}') - m\vec{\omega} \times (\vec{\omega} \times \vec{r}')$$

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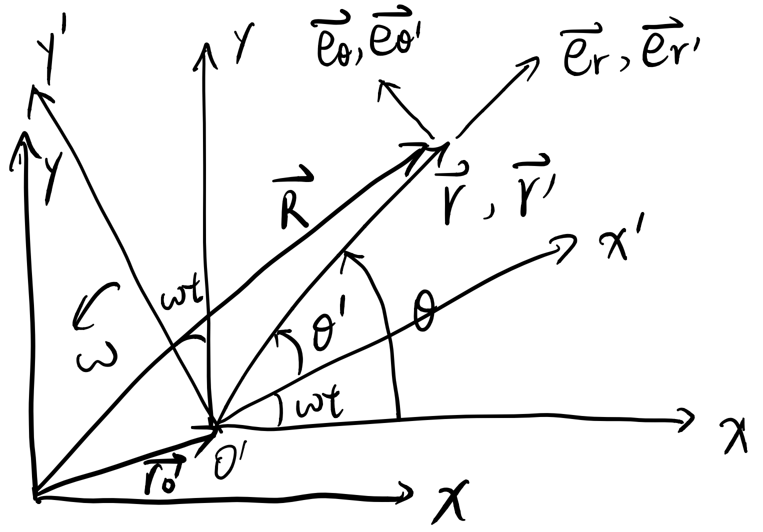
How to derive the formula?

$\vec{R} = \vec{r}' + \vec{r_0}'$, $\theta = \theta' + \omega t$ Let $\vec{r} = \vec{R} - \vec{r_0}'$

$\Rightarrow \quad \frac{d\vec{R}}{dt} = \frac{d\vec{r}'}{dt} + \frac{d\vec{r_0}'}{dt}$, $\frac{d^2\vec{R}}{dt^2} = \frac{d^2\vec{r}'}{dt^2} + \frac{d^2\vec{r_0}'}{dt^2}$

$\frac{d\theta}{dt} = \frac{d\theta'}{dt} + \omega$, $\frac{d^2\theta}{dt^2} = \frac{d^2\theta'}{dt^2}$



$v_r' = \dot{r}'$, $v_\theta' = r'\dot{\theta}'$

$a_{r'} = \ddot{r}' - r'\dot{\theta}'^2$, $a_\theta' = r'\ddot{\theta}' + 2\dot{r}'\dot{\theta}'$

$\vec{a_r} = [\ddot{r} - r\dot{\theta}^2]\vec{e_r}$

$= [\ddot{r}' - r'(\dot{\theta}' + \omega)^2]\vec{e_r}$

$= [\ddot{r}' - r'\dot{\theta}'^2 - 2r'\dot{\theta}\omega - r'\omega^2]\vec{e_{r'}}$

$= \vec{a_r}' - 2\vec{v_\theta}' \times \vec{\omega} - \omega^2\vec{r}'$

$\vec{a_\theta} = (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e_\theta} = [r'\ddot{\theta}' + 2\dot{r}'(\dot{\theta}' + \omega)]\vec{e_{\theta'}}$

$= \vec{a_\theta}' - 2\vec{v_r}' \times \vec{\omega}$

$\Rightarrow \quad \vec{a}' = \vec{a_r}' + \vec{a_\theta}' = (\vec{a_r} + \vec{a_\theta}) + \omega^2\vec{r}' + 2(\vec{v_r}' + \vec{v_\theta}') \times \vec{\omega}$

$= \vec{a} - \vec{\omega} \times (\vec{\omega} \times \vec{r}') - 2\vec{v}' \times \vec{\omega}$

If $\omega \neq$ const

$\theta = \theta' + \int_0^t \omega(\tau)d\tau$ $\quad \frac{d\theta}{dt} = \frac{d\theta'}{dt} + \omega(\tau)$ $\quad \frac{d^2\theta}{dt^2} = \frac{d^2\theta'}{dt^2} + \beta(t)$

$\vec{a_\theta} = (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e_\theta} = [r'(\ddot{\theta}' + \beta) + 2\dot{r}'(\dot{\theta}' + \omega)]\vec{e_\theta}$

$= [r'\ddot{\theta}' + 2\dot{r}'(\dot{\theta}' + \omega) + r'\beta]\vec{e_\theta}$

$\vec{a_r}$ remains the same $\quad = \vec{a_\theta}' - 2\vec{v_r}' \times \vec{\omega} - \vec{r}' \times \vec{\beta}$

$\Rightarrow$ In all, $\vec{a}' = \vec{a} - \vec{a_0}' - \vec{\omega} \times (\vec{\omega} \times \vec{r}') - 2\vec{\omega} \times \vec{v}' - \vec{\beta} \times \vec{r}'$

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Newton's Second Law doesn't hold in non-inertial FoR. To describe the motion in non-inertial FoR, we need to add the forces of inertia (pseudo-forces) into "Newton's Second Law" in non-inertial FoR.

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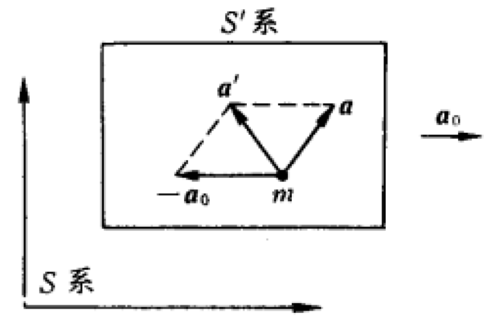
$$\mathbf{F}_{\mathbf{Unreal}} = -m\vec{a'}_O - m\frac{d\vec{\omega}}{dt} \times \vec{r'} - 2m(\vec{\omega} \times \vec{v'}) - m\vec{\omega} \times (\vec{\omega} \times \vec{r'})$$

Drift "Force" $-m\vec{a}_{O'}$



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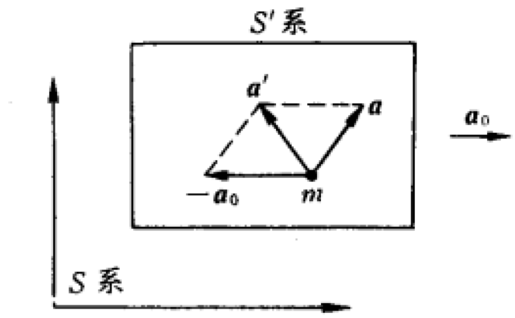

Newton's Second Law in inertia FoR S:

$$\mathbf{F} = m\mathbf{a}$$

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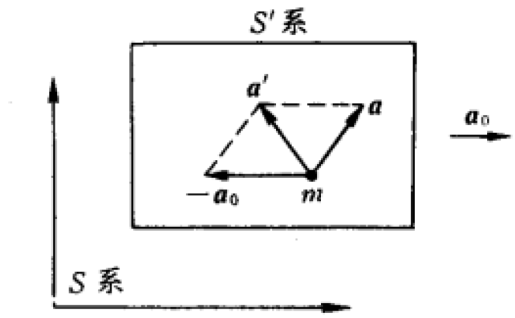

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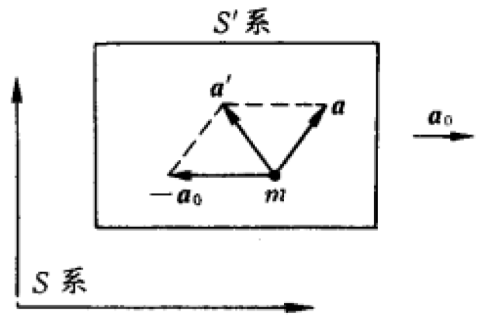

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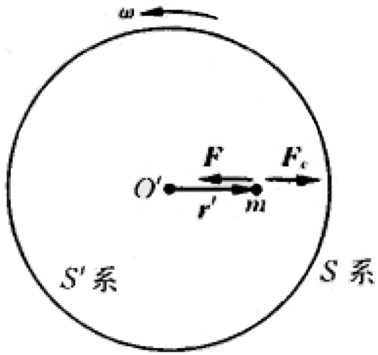
$$\Rightarrow \mathbf{F}' = m\mathbf{a}' = m\mathbf{a} + m(-\mathbf{a_0}) = \mathbf{F} + m(-\mathbf{a_0})$$

Centrifugal "Force" $-m\vec{\omega} \times (\vec{\omega} \times \vec{r'})$



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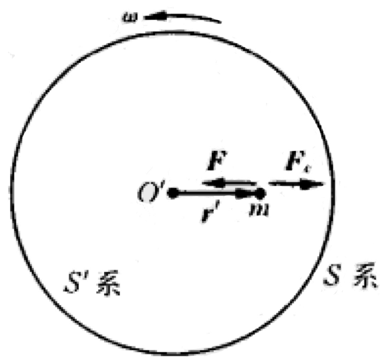
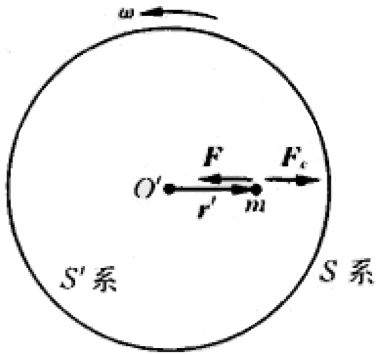


Uniform circular motion in S :

$$\mathbf{a} = -\omega^2 \mathbf{r}', \quad \mathbf{F} = m\mathbf{a} = -m\omega^2 \mathbf{r}'$$

Centrifugal "Force" $-m\vec{\omega} \times (\vec{\omega} \times \vec{r'})$



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In S':

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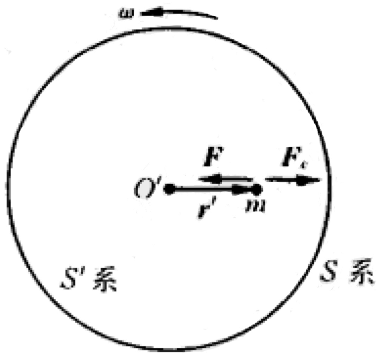

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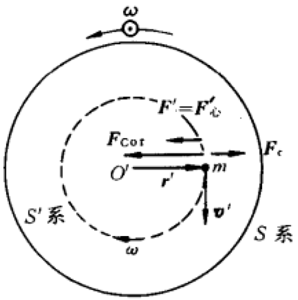
Thus, we need

$$\mathbf{F_c} = m\omega^2 \mathbf{r}' \quad \text{s.t.} \quad \mathbf{F}' = \mathbf{F} + \mathbf{F_c} = 0$$

Coriolis "Force" $-2m(\vec{\omega} \times \vec{v'})$



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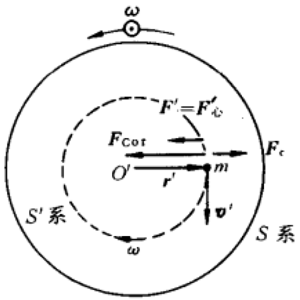

m stays still in S:

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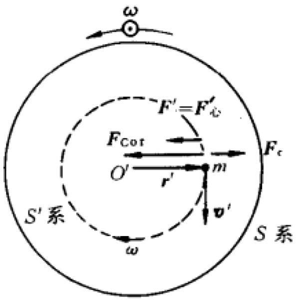

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m moves in a uniform circular motion in S':

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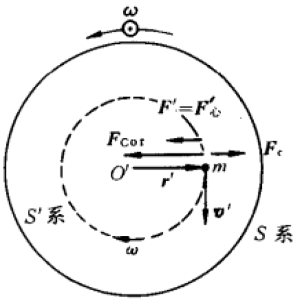

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Thus, we need (Don't forget the centrifugal force we added)

$$\mathbf{F_{Cor}} = -2m\omega^2\mathbf{r}' \quad \text{s.t.} \quad \mathbf{F}' = \mathbf{F} + \mathbf{F_c} + \mathbf{F_{Cor}} = 0 + m\omega^2\mathbf{r}' + (-2m\omega^2\mathbf{r}') = m\mathbf{a}'$$

Euler "force" $-m\frac{d\vec{\omega}}{dt} \times \vec{r}'$

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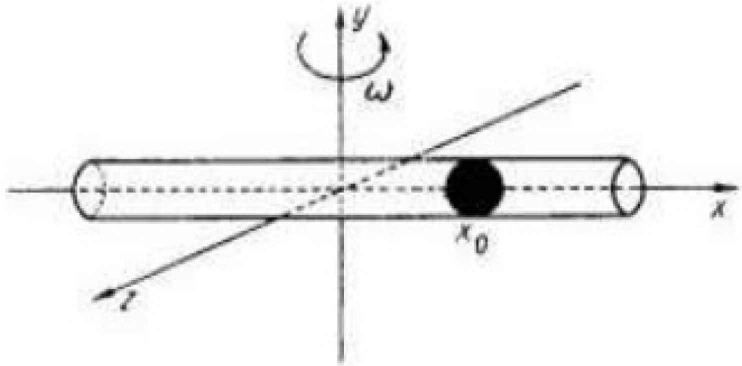
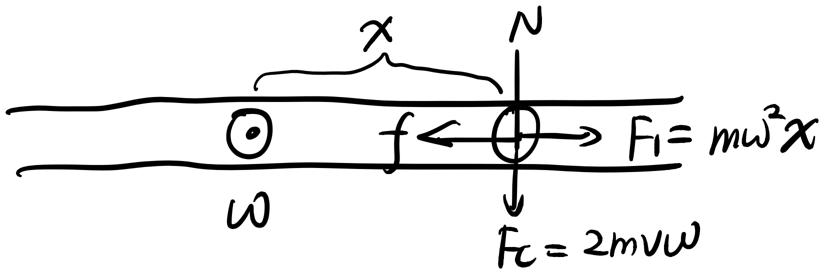
- Conventionally, we use $\vec{\beta}$ to denote the angular acceleration $\frac{d\vec{\omega}}{dt}$.

Exercise 1

A particle with mass m is inside a pipe that rotates with constant angular velocity ω about an axis perpendicular to the pipe.The kinetic coefficient of friction is equal to μ_k. Write down (do not solve!) the equation of motion for this particle in the non-inertial frame of reference of the rotating pipe. There is no gravitational force in this problem.





$$N = 2mv\omega$$

$$f = \mu_k N = 2m\mu_k v\omega$$

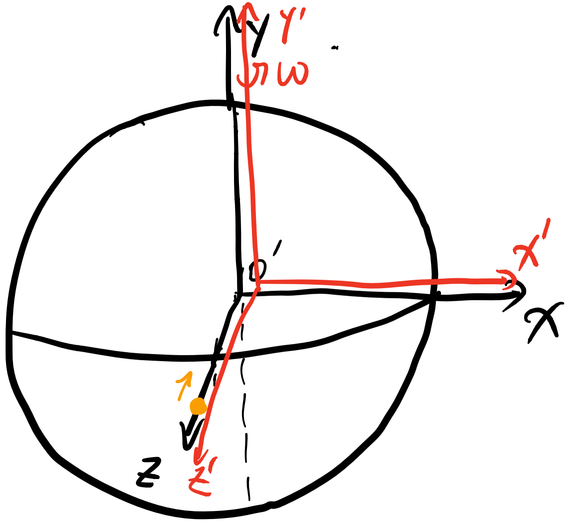
$$F_1 - f = m\omega^2 x - 2m\mu_k \omega \dot{x} = ma$$

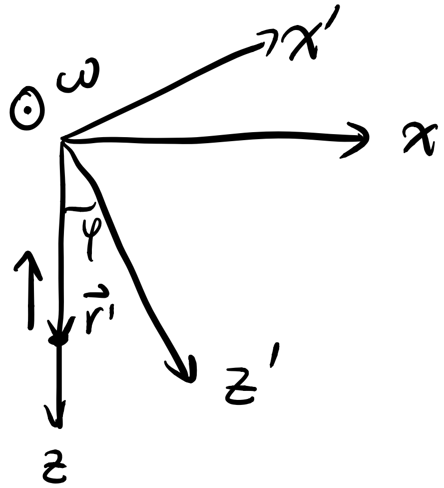
$$\Rightarrow \quad m\ddot{x} - m\omega^2 x + 2m\mu_k \omega \dot{x} = 0$$

equation of motion must contain a.

Exercise 2

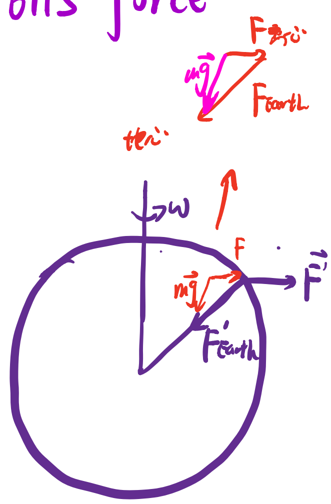
If we let go an object $100m$ above the equator. Ignore the air drag, calculate the deviation caused by Coriolis Force.





$$m\vec{a}' = \vec{F}_{Earth} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}') - \underbrace{2m(\vec{\omega}\times\vec{v}')}_{\text{coriolis "force"}}$$

$$\vec{v}' = \frac{d\vec{r}'}{dt} = \frac{dz}{dt}\hat{k} = gt\hat{k}$$



$$\vec{a}' = \frac{\vec{F}_{Earth}}{m} - \omega\hat{j}' \times\left[\omega\hat{j}'\times(x'\hat{i}'+z'\hat{k}')\right] - 2\left(\omega\hat{j}\times\frac{dz}{dt}\hat{k}\right)$$

$$\vec{a}' = \underbrace{\frac{\vec{F}_{Earth}}{m} + \omega^2\vec{r}'}_{\vec{g}} - 2(\vec{\omega}\times\vec{v}')$$

$$\vec{a}' = -g\hat{k}' - 2\omega\frac{dz}{dt}\hat{i} = -g\hat{k}' - 2\omega g t\hat{i}$$

$$\hat{i} = \hat{k}'\sin\varphi + \hat{i}'\cos\varphi \Rightarrow \vec{a}' = -(g + 2\omega g t\sin\omega t)\hat{k}' - 2\omega g t\cos\omega t\,\hat{i}'$$

$$V_{z'} = \int_0^t a_{z'}\,dt = -\frac{2g}{\omega}\sin\omega t + 2gt\cos\omega t - gt \approx -gt$$

$$V_{x'} = \int_0^t a_{x'}\,dt = 2g\frac{1-\cos\omega t}{\omega} - 2gt\sin\omega t \approx -\omega g t^2$$

t

$$T = \sqrt{\frac{2h}{g}} \approx 4.52s$$

$$\Delta x' = \int_0^T -\omega g t^2\,dt = -\frac{1}{3}\omega g T^3 = -\frac{1}{3}\times\frac{2\pi}{24\times3600}\times 9.8\times 4.52^3 = -0.022m$$

$$\Rightarrow 2.2cm$$

Reference

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