VP160 Recitation Class VI

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UM-SJTU Joint Institute

July 5, 2021

Generalized coordinates

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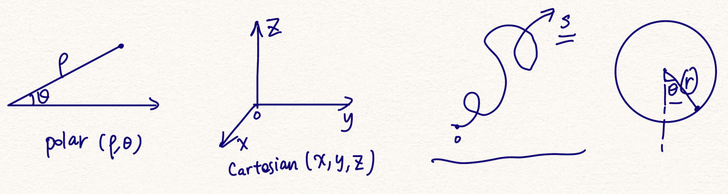
e.g.



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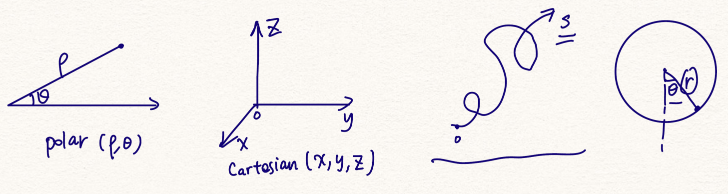

$$\begin{cases} q_1(t) \\ q_2(t) \\ \dots \\ q_n(t) \end{cases} \Rightarrow \begin{cases} \dot{q_1}(t) \\ \dot{q_2}(t) \\ \dots \\ \dot{q_n}(t) \end{cases} \text{(generalized velocity)}$$

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The minimum number of independent generalized coordinates needed to describe the system's motions.

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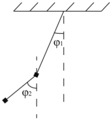
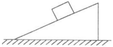
Exercise 1

Find the degree of freedom:



(e)



(f)

Hamilton's Principle

$$\text{Real path} \Longleftrightarrow \delta S = 0$$

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Euler-Lagrange Equation

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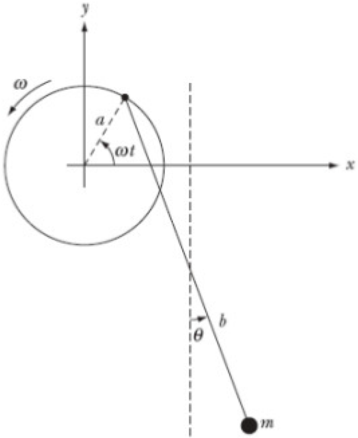
Learn more about variational and how to derive Hamilton's Principle, visit

`https://zhuanlan.zhihu.com/p/126115834`

`https://zhuanlan.zhihu.com/p/139018146`

Exercise 2

A simple pendulum of length b and mass m moves attached to a massless rim of radius a rotating with constant angular velocity ω. How many degrees of freedom do we have here? Find the Lagrangian.



Exercise 3

Find the equations of motion of a particle of mass m constrained to move on the surface of a sphere, acted upon a conservative force $\mathbf{F} = F_0 \hat{n}_\theta$ with F_0 a constant.

Hint. To find the potential energy find the scalar product $\mathbf{F} \cdot d\mathbf{r}$ for the infinitesimal displacement on the sphere and use the fact that it is equal to $-dU$ (the force is conservative).

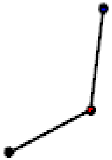
Exercise 4

Double pendulum:

(1) identify the generalized coordinates;

(2) find the Lagrangian;

(3) write down the Euler-Lagrange equations of motion;



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- If $\vec{F_{ext}} = 0$, for a system, $\Delta\vec{p} = 0 \Leftrightarrow p = \text{Const}$ (Conservation of momentum)

Application of Conservative of Momentum

- Non-central Collision (e.g. explosion)

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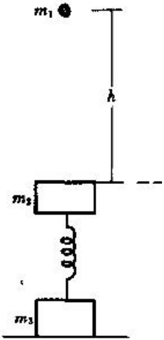
  - Inelastic
    - $e < 1$
    - Energy loss
  - Completely Inelastic
    - $e = 0$
    - stick to each other

Exercise 5

Assume m_1, m_2, m_3, k is known. Release m_1, the collision between m_1 and m_2 is completely inelastic. Find h so that m_3 can just leave the ground.



Center of Mass

$$r_C = \frac{\sum m_i r_i}{\sum m_i}$$

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Pappus Law

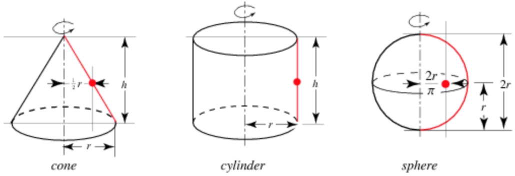
First Theorem:

$$S = 2\pi s x$$

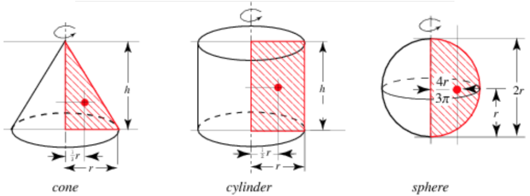
Second Theorem:

$$V = 2\pi A x$$

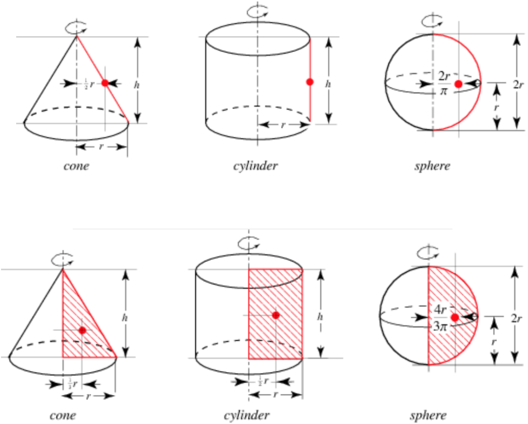
where x is the distance from the reference axis and the center of mass.



$\frac{1}{2}r$
h
r
cone
h
r
cylinder
$\frac{2r}{\pi}$
$2r$
r
sphere



h
$\frac{1}{3}r$
r
cone
h
$\frac{1}{2}r$
r
cylinder
$\frac{4r}{3\pi}$
$2r$
r
sphere



- An important fact:

$$\vec{F_{ext}} = 0 \Leftrightarrow \vec{p} = Const \Leftrightarrow \vec{v_c} = Const$$

Rocket Propulsion

$$mv + Fdt = (m + dm)(v + dv) - udm$$

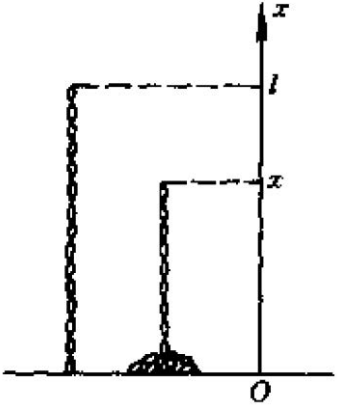
$$m\frac{dv}{dt} = (u - v)\frac{dm}{dt} + F$$

Reminder

What FoR are we looking at?

Exercise 6

A rope with length l and mass m is placed vertically. At the beginning, the lower end of the rope just touches the ground. Release the rope, find the support force of the ground with respect to x.



Reference

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