VP160 Recitation Class V

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Work

Definition

$$\delta W = \vec{F} \circ d\vec{r} \tag{1}$$

$$W_{AB} = \int_{\Gamma_{AB}} \vec{F} \circ d\vec{r} \tag{2}$$

In general, $\vec{F} = \vec{F}(\vec{r})$ (position-dependent force; vector field)

Methods for Calculation

1. Constant force on a straight line
2. Varying force on a straight line (Integral)
3. Varying force on a curve (Line integral)

Exercise 1

Find work done by the force $\mathbf{F_1}(x, y) = -x\hat{n_x} - y\hat{n_y}$ and by the force $\mathbf{F_2}(x, y) = (2xy + y)\hat{n_x} + (x^2 + 1)\hat{n_y}$ if a particle is being moved from $(-1, 0)$ to $(0, 1)$ along
(a) the straight line connecting these points,
(b) the (shorter) arc of the circle $x^2 + y^2 = 1$,
(c) the axes of the Cartesian coordinate system: first from $(-1, 0)$ to $(0, 0)$ along the x axis, then from $(0, 0)$ to $(0, 1)$ along the y axis.

Recall that $\delta W = \vec{F} \circ d\vec{r}$, so the rate of work being done by the net force on a particle

$$\frac{\delta W}{dt} = \vec{F} \circ \frac{d\vec{r}}{dt} = \vec{F} \circ \vec{v} = m\dot{\vec{v}} \circ \vec{v} = \frac{d}{dt}(\frac{1}{2}mv^2)$$

- Kinetic Energy:$E_k = \frac{1}{2}mv^2$
- Work-Kinetic Energy Theorem: $\delta W = dE_k$
- Average Power:$P_{av} = \frac{W}{\Delta t}$
- Instantaneous power:$P_{ins} = \frac{\delta W}{dt} = \vec{F} \circ \vec{v}$

Conservative Force

- Path-Independent.
  - If $\vec{F}$ is conservative $\Rightarrow \Delta W_{AB} = Constant$
  - e.g. Gravitational Force, Elastic Force
- How to determine?
  - In a simple connected region, $rot\vec{F} = 0$

$$\text{rot } \vec{F} = \begin{vmatrix} \hat{n}_x & \hat{n}_y & \hat{n}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \hat{n}_x \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \hat{n}_y \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \hat{n}_z \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) = 0$$

  - Relation with Potential Energy:

$$\vec{F} = - \bigtriangledown U = -\frac{\partial U}{\partial x}\hat{n_x} - \frac{\partial U}{\partial y}\hat{n_y} - \frac{\partial U}{\partial z}\hat{n_z}$$

- Non-conservative force?

Exercise 2

Consider a 3D force

$$\vec{F} = \begin{pmatrix} 6xyz^2 + 3y^2 \\ 3x^2z^2 + 6xy - z \\ 6x^2yz - y \end{pmatrix}$$

Is it a conservative force? How to find the corresponding potential energy?

The net force may be the sum of a conservative force and a non-conservative force

$$\vec{F} = \vec{F}_{con} + \vec{F}_{\text{n-cons}}$$

The work done by the net force (by **Work-Kinetic Energy Theorem**):

$$\delta W = \vec{F} \circ d\vec{r} = \vec{F}_{con} \circ d\vec{r} + \vec{F}_{\text{n-cons}} \circ d\vec{r} = dK$$

However, only for conservative force $\vec{F}_{con} \circ d\vec{r} = -dU$. Hence, the elementary work done by the non-conservative force is

$$\delta W_{\text{n-cons}} = \vec{F}_{\text{n-cons}} \circ d\vec{r} = d(K + U)$$

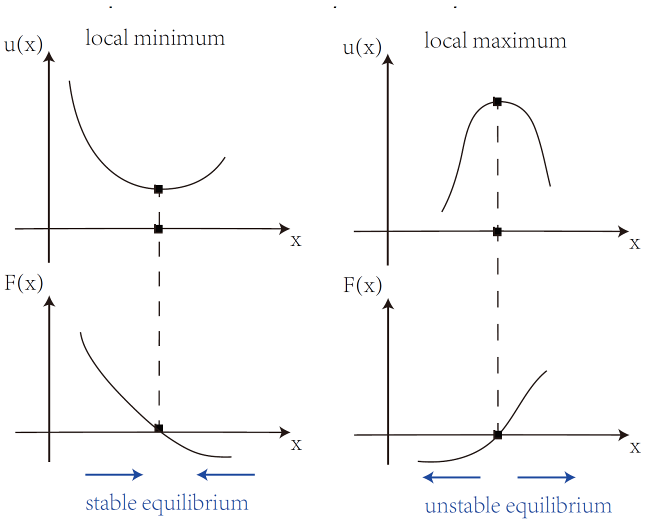
and the total work done from A to B

$$W_{\text{n-cons}} = \int_{\Gamma_{AB}} \vec{F}_{\text{n-cons}} \circ d\vec{r} = E(B) - E(A) = \Delta E = -\Delta U_{int}$$

and we finally get the **Law of Conservation of the Total Energy**:

$$\Delta \mathbf{K} + \Delta \mathbf{U} + \Delta \mathbf{U}_{\mathbf{int}} = 0$$

Energy Diagram



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