1 The Frenet frame

Given a curve $C(t)$ parameterized by the variable t, let $s(t)$ denote the arc length along the curve from $C(0)$ to $C(t)$.

Definitions:

The *tangent vector*:

$$T = \frac{\mathrm{d}C}{\mathrm{d}s}. \tag{1}$$

The *velocity*:

$$v = ds/dt. \tag{2}$$

The *curvature*:

$$\kappa = |\frac{\mathrm{d}T}{\mathrm{d}s}|. \tag{3}$$

The *normal vector*:

$$N = \kappa^{-1}\frac{\mathrm{d}T}{\mathrm{d}s}. \tag{4}$$

The *binormal vector*:

$$B = T \times N. \tag{5}$$

The *torsion*:

$$\tau = \frac{\mathrm{d}N}{\mathrm{d}s} \cdot B. \tag{6}$$

Note that if we reparametrise a curve then only the velocity changes. For example, if $0 \leq t \leq 1$, then the same set of points

$$\{C(t) \quad | \quad 0 \leq t \leq 1\}$$

can be parametrized by letting $u = 3t + 5$, and looking at the set of points

$$\{C(u) \quad | \quad 5 \leq u \leq 8\}.$$

However the velocity is three times as slow: $\mathrm{d}s/\,\mathrm{d}u = (1/3)\,\mathrm{d}s/\,\mathrm{d}t$. Often, calculations are easier without using arc-length parametrization s (but theorems are usually easier with s).

These quantities satisfy a number of simple properties:

Theorem

1. $|T| = 1$, i.e. T is a unit vector.
2. $T \cdot N = 0$, i.e. tangent and normal are perpendicular.
3. $$T \cdot \frac{\mathrm{d}N}{\mathrm{d}s} = -\kappa. \tag{7}$$
4. $$\frac{\mathrm{d}N}{\mathrm{d}s} = -\kappa T + \tau B. \tag{8}$$
5. $$\frac{\mathrm{d}B}{\mathrm{d}s} = -\tau N. \tag{9}$$

Proof

1. Let $C(t) = (x(t), y(t), z(t))$. Then
$$\frac{\mathrm{d}C}{\mathrm{d}s} = \frac{\mathrm{d}t}{\mathrm{d}s}\left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right)$$
and so
$$\begin{aligned} \left|\frac{\mathrm{d}C}{\mathrm{d}s}\right|^2 &= \left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 \left(\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right) && (10) \\ &= \left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 \left(\frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}{\mathrm{d}t^2}\right) && (11) \\ &= \left(\frac{\mathrm{d}t^2}{\mathrm{d}s^2}\right)\left(\frac{\mathrm{d}s^2}{\mathrm{d}t^2}\right) = 1. && (12) \end{aligned}$$
2. As $|T| = 1$, $T \cdot T = 1$. Take the derivative of this:
$$0 = 2T \cdot \frac{\mathrm{d}T}{\mathrm{d}s} = 2T \cdot \kappa N. \tag{13}$$
3. From the previous part, $T \cdot N = 0$. Take the derivative of this:
$$0 = \frac{\mathrm{d}T}{\mathrm{d}s} \cdot N + T \cdot \frac{\mathrm{d}N}{\mathrm{d}s} = \kappa N \cdot N + T \cdot \frac{\mathrm{d}N}{\mathrm{d}s}.$$
As N is a unit vector, $N \cdot N = 1$, and $0 = \kappa + T \cdot \mathrm{d}N/\mathrm{d}s$.

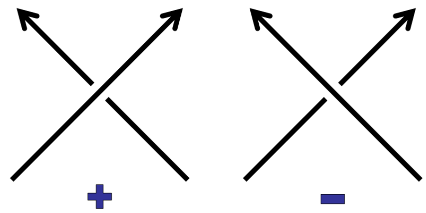

Figure 1: crossings of two curves in a plane can have two signs, depending on which curve is 'on top'.

4. Follows from the previous item, plus the definition of τ.

5. By the product rule and $B = T \times N$,

$$\frac{\mathrm{d}B}{\mathrm{d}s} = \frac{\mathrm{d}T}{\mathrm{d}s} \times N + T \times \frac{\mathrm{d}N}{\mathrm{d}s}. \tag{14}$$

But $\mathrm{d}T/\,\mathrm{d}s = \kappa N$ so the first term vanishes. Thus

$$\frac{\mathrm{d}B}{\mathrm{d}s} = T \times \frac{\mathrm{d}N}{\mathrm{d}s} = T \times (-\kappa T + \tau B) = \tau T \times B. \tag{15}$$

2 Linking number, Twist, and Writhe

Recall that the Gauss integral for the linking number between a curve $X(\tau)$ and a curve $Y(\sigma)$ is

$$\alpha = \frac{1}{4\pi} \oint_X \oint_Y \frac{\mathrm{d}X}{\mathrm{d}\sigma} \cdot \frac{\mathbf{r}}{r^3} \times \frac{\mathrm{d}Y}{\mathrm{d}\tau}\,\mathrm{d}\tau\,\mathrm{d}\sigma. \qquad (\mathbf{r} = Y - X) \tag{16}$$

We can also describe the linking number in terms of the crossings seen when the curve is projected onto a plane. A crossing can be either positive or negative (see figure 1). Then if we count the number N^+ of positive crossings and subtract the number N^- of negative crossings we can obtain the linking number:

$$\alpha = \frac{1}{2}(N^+ - N^-). \tag{17}$$

This greatly assists us in quickly determining the linking number, without having to do a rather complicated integral!

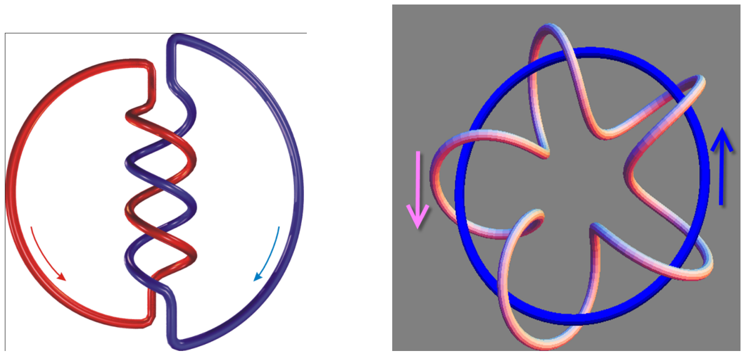

Figure 2: Left: $\alpha = -2$. Right: $\alpha = +5$.

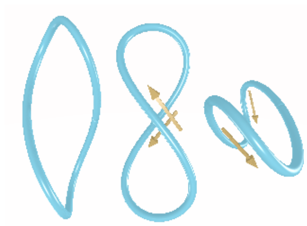

Figure 3: The same curve seen from three different viewing angles,

But what if we do the same for a single closed curve (a knot)? A single curve can exhibit a different number of crossings from different viewing angles (see figure 2). So the number $1/2(N^+ - N^-)$ is now a function of viewing angle. We can, however, obtain a quantity which does not depend on viewing angle, simply by averaging over *all* viewing angles! This quantity will be called *the writhe*. Note that the sign of a crossing stays the same if we reverse the direction of the curve, because both segments above and below a crossing change direction.

But first, let us examine what happens to linking number when the two curves are almost parallel.

2.1 Ribbons

What happens when the two curves are almost parallel?

Let $\ell =$ length of the X curve. Suppose that $Y(t) = X(t) + \epsilon \hat{V}(t)$ where

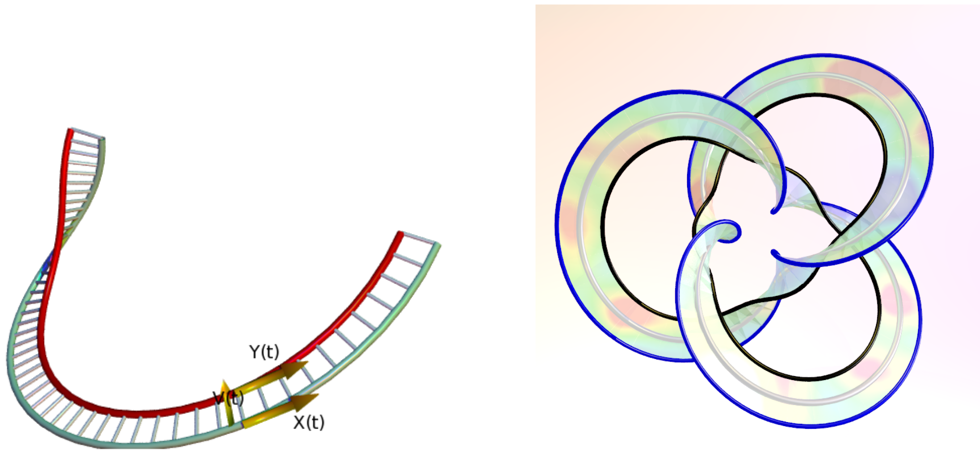

Figure 4: What is the linking number of a ribboned trefoil knot?

$|\hat{V}| = 1$ and $\epsilon << \ell$.

For this ribbon, $\alpha = 6$.

How much of that $\alpha = 6$ is due to the *twist* inside the ribbon, and how much is due to the *writhe* of the trefoil?

The answer, using the formulae below, is: Twist: $T = 2.29$. Writhe: 3.71.

2.2 Călugăreanu Theorem

Suppose we write the Gauss linking integral for a ribbon:

$$\alpha = \frac{1}{4\pi} \oint_X \oint_{X+\epsilon\hat{V}} \frac{\mathrm{d}X}{\mathrm{d}\sigma} \cdot \frac{\mathbf{r}}{r^3} \times \frac{\mathrm{d}(X+\epsilon\hat{V})}{\mathrm{d}\tau} \,\mathrm{d}\tau\,\mathrm{d}\sigma. \qquad \big(\mathbf{r} = X(\tau) - (X(\sigma) + \epsilon\hat{V}(\sigma))\big) \tag{18}$$

What happens when we take $\epsilon \to 0$? Do we get

$$\alpha \to \frac{1}{4\pi} \oint_X \oint_X \frac{\mathrm{d}X}{\mathrm{d}\sigma} \cdot \frac{\mathbf{r}}{r^3} \times \frac{\mathrm{d}X}{\mathrm{d}\tau} \,\mathrm{d}\tau\,\mathrm{d}\sigma? \tag{19}$$

No!!!!

In 1961 Călugăreanu found that in the limit $\epsilon \to 0$,

$$\alpha = \mathcal{T} + \mathcal{W} \tag{20}$$

where

$$\mathcal{W} = \frac{1}{4\pi} \oint_X \oint_X \frac{\mathrm{d}X}{\mathrm{d}\sigma} \cdot \frac{\mathbf{r}}{r^3} \times \frac{\mathrm{d}X}{\mathrm{d}\tau} \,\mathrm{d}\tau \,\mathrm{d}\sigma \tag{21}$$

and $\mathcal{T}$ measures the internal twist of the ribbon,

$$\mathcal{T} = \frac{1}{2\pi} \oint \frac{\mathrm{d}X}{\mathrm{d}\tau} \cdot \hat{V}(\tau) \times \frac{\mathrm{d}\hat{V}(\tau)}{\mathrm{d}\tau} \,\mathrm{d}\tau. \tag{22}$$

2.3 Basic Properties of Linking number, Twist, and Writhe

1. α is invariant to deformations of the two curves, as long as the two curves are not allowed to cross through each other.

2. α is an integer.

3. α equals half the signed number of crossings of the two curves as seen in any plane projection:

$$\alpha = \frac{1}{2}(N^+ - N^-). \tag{23}$$

4. α is independent of the direction of the axis curve, i.e. α does not change if $s \to -s$. (For two arbitrary curves, α changes sign if one of the two curves reverses its direction. However for ribbons both curves must change direction together.)

2.3.1 Writhe

$$\mathcal{W} \equiv \frac{1}{4\pi} \oint_{\mathbf{x}} \oint_{\mathbf{x}} \hat{T}(s) \times \hat{T}(s') \cdot \frac{\mathbf{x}(s) - \mathbf{x}(s')}{|\mathbf{x}(s) - \mathbf{x}(s')|^3} \,\mathrm{d}s \;\, \mathrm{d}s'. \tag{24}$$

1. $\mathcal{W}$ depends only on the axis curve $\mathbf{x}$.

2. $\mathcal{W}$ equals the signed number of crossings of the axis curve with itself, averaged over all possible projection angles (i.e. over all directions on the sphere S^2).

3. $\mathcal{W}$ is independent of the direction of the axis curve.

2.3.2 Twist

$$\mathcal{T} \equiv \frac{1}{2\pi} \oint_{\mathbf{x}} \hat{T}(s) \cdot \hat{V}(s) \times \frac{\mathrm{d}\hat{V}(s)}{\mathrm{d}s}\, \mathrm{d}s \tag{25}$$

$$= \frac{1}{2\pi} \oint_{\mathbf{x}} \frac{1}{|\mathbf{v}|^2(s)} \hat{T}(s) \cdot \mathbf{v}(s) \times \frac{\mathrm{d}\mathbf{v}(s)}{\mathrm{d}s}\, \mathrm{d}s \tag{26}$$

where $\mathbf{v} = \epsilon \hat{V}$.

1. $\mathcal{T}$ has a local density along the curve, i.e. it is meaningful to write

$$\mathcal{T} = \frac{1}{2\pi} \oint_{\mathbf{x}} \frac{\mathrm{d}\mathcal{T}}{\mathrm{d}s}\, \mathrm{d}s. \tag{27}$$

2. $\mathrm{d}\mathcal{T}/\mathrm{d}s$ measures the rotation rate of the secondary curve about the axis curve. At each point on the axis curve $\mathbf{x}$, define the plane perpendicular to $\hat{T}(s)$. The offset vector $\hat{V}(s)$ lives in this plane, and rotates at a rate

$$\hat{T}(s) \cdot \hat{V}(s) \times \frac{\mathrm{d}\hat{V}(s)}{\mathrm{d}s} = 2\pi \frac{\mathrm{d}\mathcal{T}}{\mathrm{d}s}. \tag{28}$$

3. Consider a ribbon whose edges are two neighboring magnetic field lines. In this case, the twist is related to the parallel electric current. Let $\mathbf{J} = \nabla \times \mathbf{B}/\mu_0$ be the electric current associated with the magnetic field $\mathbf{B}$, and $J_\parallel = \mathbf{J} \cdot \mathbf{B}/|\mathbf{B}|$. Then

$$\frac{\mathrm{d}\mathcal{T}}{\mathrm{d}s} = \frac{\mu_0 J_\parallel}{4\pi|\mathbf{B}|}. \tag{29}$$

   Similarly, if we measure how much two neighboring flow lines in a fluid twist about each other, then

$$\frac{\mathrm{d}\mathcal{T}}{\mathrm{d}s} = \frac{\omega_\parallel}{4\pi|\mathbf{V}|}, \tag{30}$$

   where $\mathbf{V}$ is the fluid velocity and ω is the vorticity.

4. $\mathcal{T}$ is independent of the direction of the axis curve. For example, suppose the axis is a vertical straight line, and the secondary is a right helix (positive twist). Turning the two upside down will still give a right helix of the same pitch.