Excercise 1

1. (a) For $f_{p,\theta} = 0$ to be a pdf, we must show that;

$$\int_{-\infty}^{\infty} f_{p,\theta} = 1$$

$$\int_{-\infty}^{\infty} f_{p,\theta} = \begin{cases} 1 & \text{if x} \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\int_{0}^{\infty} \frac{\theta^p}{\Gamma(p)} e^{-\theta x} x^{p-1} dx$$

We know that $\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx$
Let $t = \theta x;\ x = \frac{t}{\theta};\ \frac{dx}{dt} = \frac{1}{\theta};\ dx = \frac{dt}{\theta}$

$$= \int_{0}^{\infty} \frac{\theta^p}{\Gamma(p)} e^{-t} (\frac{t}{\theta})^{p-1} \frac{dt}{\theta}$$

$$= \int_{0}^{\infty} \frac{\theta^{p-1}}{\Gamma(p)} e^{-t} \frac{t^{p-1}}{\theta^{p-1}} dt$$

$$= \frac{1}{\Gamma(p)} \int_{0}^{\infty} e^{-t} t^{p-1} dt$$

$$= \frac{\Gamma(p)}{\Gamma(p)}$$

$$= 1$$

(b)

$$E(X^n) = \int_{0}^{\infty} x^n \frac{\theta^p}{\Gamma(p)} x^{p-1} e^{-\theta x} dx$$

$$\frac{\theta^p}{\Gamma(p)} \int_{0}^{\infty} x^{n+p-1} e^{-\theta x} dx$$

$$\frac{\theta^p}{\Gamma(p)} \times \frac{\Gamma(n+p)}{\theta^{n+p}} = \frac{\Gamma(n+p)}{\Gamma(p)\theta^n}$$

$$E(X^n) = \frac{\Gamma(n+p)}{\theta^n \Gamma(p)}$$

2. (a) U=X+Y where $X \sim \Gamma(p,\theta)$ and $Y \sim \Gamma(q,\theta)$

$$f_{X+Y}(u) = \int_0^u f_X(x) f_Y(u-x)dx$$

$$f_{X+Y}(u) = \int_0^u \frac{\theta^p}{\Gamma(p)} e^{-\theta x} x^{p-1} . \frac{\theta^q}{\Gamma(q)} e^{-\theta(u-x)} (u-x)^{q-1} dx$$

Recall that

$$\beta(p,q) = \int_0^1 u^{p-1}(1-u)^{q-1} du = \frac{\Gamma(p)\Gamma(p)}{\Gamma(p+q)}$$

so,

$$f_{X+Y}(u) = \int_0^u \frac{\theta^p}{\Gamma(p)} x^{p-1} \frac{\theta^q}{\Gamma(q)} (u-x)^{q-1} e^{-\theta u} dx$$

Let $x = ut;\ dx = udt$

$$\begin{aligned}
f(u) &= \int_0^{\frac{x}{t}} \frac{\theta^p}{\Gamma(p)} (ut)^{p-1} \frac{\theta^q}{\Gamma(q)} (u-ut)^{q-1} e^{-\theta u} udt \\
&= \frac{e^{-\theta u}\theta^p\theta^q}{\Gamma(p)\Gamma(q)} (u)^{q-1}(u)^p \int_0^1 (t)^{p-1}(1-t)^{q-1} dt \\
&= \frac{e^{-\theta u}\theta^p\theta^q}{\Gamma(p)\Gamma(q)} (u)^{q-1}(u)^p . \frac{\Gamma(p)\Gamma(p)}{\Gamma(p+q)} \\
&= \frac{e^{-\theta u}\theta^p\theta^q}{\Gamma(p+q)} (u)^{q-1}(u)^p . \\
&= \frac{e^{-\theta u}\theta^{p+q}}{\Gamma(p+q)} u^{(p+q)-1} \sim \Gamma(p+q,\theta)
\end{aligned}$$

(b) $V = aX;\ X = \frac{v}{a};\ \frac{dx}{dv} = \frac{1}{a}$. Using jacobian, we get

$$\begin{aligned}
f_{ax}(v) &= |\frac{dx}{dv}| f_x \frac{v}{a} = \frac{1}{a} f_x \frac{v}{a} \\
&= \frac{1}{a} \frac{\theta^p}{\Gamma(p)} e^{-\theta\frac{v}{a}} (\frac{v}{a})^{p-1} \\
&= \frac{a^{-p}\theta^p}{\Gamma(p)} e^{-\frac{\theta}{a}v} v^{p-1} \\
&= \frac{(\frac{\theta}{a})^p}{\Gamma(p)} e^{-\frac{\theta}{a}v} v^{p-1} \sim \Gamma(p, \frac{\theta}{a})
\end{aligned}$$

(c) $Z = \frac{X}{Y};\ U = Y;\ X = \frac{Z}{U}$

$$J \begin{bmatrix} z & u \\ 1 & 0 \end{bmatrix} = u$$

$$f_{U,Z}(u,z) = \int_0^\infty \frac{\theta^p}{\Gamma(p)} e^{-\theta z u}(zu)^{p-1} \cdot \frac{\theta^q}{\Gamma(q)} e^{-\theta(u)}(u)^{q-1} u du$$

$$= \frac{\theta^p}{\Gamma(p)} \frac{\theta^q}{\Gamma(q)} (z)^{p-1} \int_0^\infty e^{-\theta u(z+1)}(u)^{(p+q)-1} du$$

$$v = \theta u(z+1);\ u = \frac{v}{\theta u(z+1)};\ dv = \theta u(z+1)du;\ du = \frac{dv}{\theta(z+1)};$$

$$= \frac{\theta^p}{\Gamma(p)} \frac{\theta^q}{\Gamma(q)} (z)^{p-1} \int_0^\infty e^{-v} \frac{(v)^{(p+q)-1}}{(\theta(z+1))^{q+p}} dv$$

$$= \frac{\theta^p}{\Gamma(p)} \frac{\theta^q}{\Gamma(q)} (z)^{p-1} \frac{1}{\theta(z+1)^{q+p}} \int_0^\infty e^{-v}(v)^{(p+q)-1} dv$$

$$= \frac{\theta^p}{\Gamma(p)} \frac{\theta^q}{\Gamma(q)} (z)^{p-1} \frac{1}{\theta(z+1)^{q+p}} \Gamma(p+q)$$

$$= \frac{\theta^{p+q}}{\beta} \times \frac{z^{p-1}}{\theta(z+1)^{q+p}} \sim \beta(p,q)\ type\ II$$

3.

$$S_n = \sum_{i=1}^n X_i$$

$$X_i \leq i \leq n \sim \Gamma(1,0)$$

Using its characteristic function,

$$\Psi_{1,0}(t) = \Pi_{i=1}^n (\frac{\theta}{\theta - it}) = (\frac{\theta}{\theta - it})^n$$

This resulting characteristic function follows the probability law $\sim \Gamma(n,\theta)$

4. If X follows the standard normal distribution law, then $X^2 \sim \chi^2$ with one degree of freedom

5. (a) $Z_n = \sum_{i=1}^{n} X_i^2$ which follows $X^2 \sim (v)$,and the probability density function of Z_n is ;

$$f(x) = \frac{e^{\frac{-x}{2}} X^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} \quad x \geq 0$$

$$E(Z_n) = E(X) = \int_0^{\infty} X f(x) dx$$

$$= \frac{e^{\frac{-x}{2}} X^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} dx$$

$$= \frac{1}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} \int_0^{\infty} X^{\frac{v}{2}} e^{\frac{-x}{2}} dx = \frac{1}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} \int_0^{\infty} X^{\frac{v}{2}+1-1} e^{\frac{-x}{2}} dx$$

$$= \frac{2^{\frac{v}{2}}.2.\frac{v}{2}\Gamma(\frac{v}{2})}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} = 2.\frac{v}{2}$$

$$E(X) = v = \mathbf{E}(Z_n)$$

$$\mathbf{E}(X^2) = \int_0^{\infty} \frac{X^2 X^{\frac{v}{2}-1} e^{\frac{-x}{2}}}{2^{\frac{v}{2}\Gamma(\frac{v}{2})}} dx = \frac{1}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} \int_0^{\infty} X^{\frac{v}{2}+2-1} e^{\frac{-x}{2}} dx$$

$$\frac{1}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} .2^{\frac{v}{2}+2}\Gamma(\frac{v}{2}+2)$$

$$\frac{2^{\frac{v}{2}}.2^2.(\frac{v}{2}+1)(\frac{v}{2})\Gamma(\frac{v}{2})}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})}$$

$$= 2^3.(\frac{v}{2}+1)(\frac{v}{2}) = 2^2(\frac{v^2}{4}+\frac{v}{2}) = 4(\frac{v^2}{4}+\frac{v}{2})$$

$E(x^2) = v^2{+}2v var(Z_n) = var(X) = E(x^2){-}[E(x)^2]v^2{+}2v{-}v^2 = 2v$

Therefore, $var(Z_n) = 2v$ and $\mathbf{E}(Z_n) = v$

(b)

$$F = \frac{\frac{1}{n}\sum_{i}^{n} X_i^2}{\frac{1}{m}\sum_{i}^{n} Y_i^2} \sim f - distribution(n, m)$$

with the *pdf* given by (probability law).

$$F_{n,m}(X) = \frac{\Gamma(\dfrac{n+m}{2})m^{\frac{m}{2}}n^{\frac{n}{2}}}{\Gamma(\dfrac{n}{2})\Gamma(\dfrac{m}{2})}\frac{X^{\frac{m}{2}}-1}{(m+nx)^{\frac{n+m}{2}}}$$

let $U' = \sum X_i^2 \sim X_{(n)^2}$
$V' = \sum Y_i^2 \sim X_{(m)^2}$

F $= \dfrac{U'}{V'}$

$U \sim X_{(n)^2}$ and $V \sim X_{(m)^2}$ and U and V are independent.

$$E(f) = E(\frac{U'}{V'}) = EU'EU'$$

$$= \frac{1}{n}E(U).mE\frac{1}{v} = \frac{m}{n}E(U)E(\frac{1}{v})$$

We know that $\mathbf{E}(U) = n$

$$E(f) = \frac{m}{n}nE(\frac{1}{v}) = mE(\frac{1}{v})$$

$$E(\frac{1}{v}) = \int_0^\infty \frac{1}{v}f(v)dv$$

where $f(v)$ is the X_m^2 density

$$f(v) = \frac{1}{2^{\frac{m}{2}}\Gamma(\frac{m}{2})}v^{\frac{m}{2}-1}e^{\frac{-v}{2}}dv$$

$$= \frac{1}{\Gamma(\frac{m}{2})}\int_0^\infty \frac{1}{v}(\frac{v}{2}^{\frac{m}{2}-1})e^{\frac{-v}{2}}\frac{dv}{2}$$

let w $= \frac{v}{2}$; $dw = \frac{dv}{2}$. $v = 2w$ we have

$$= \frac{1}{\Gamma(\frac{m}{2})} \int_0^\infty \frac{1}{2w} w^{\frac{m}{2}-1} e^{-w} dw$$

$$= \frac{1}{2\Gamma(\frac{m}{2})} \int_0^\infty w^{\frac{m}{2}-2} e^{-w} dw$$

$$= \frac{1}{2\Gamma(\frac{m}{2})} \int_0^\infty w^{(\frac{m}{2}-1)-1} e^{-w} dw$$

$$= \frac{\Gamma(\frac{m}{2}-1)}{2\Gamma(\frac{m}{2})}$$

$=E(\frac{1}{v}) = \dfrac{\Gamma(\frac{m}{2}-1)}{2\Gamma(\frac{m}{2}-1)\Gamma(\frac{m}{2}-1)} = \dfrac{1}{2(\frac{m}{2}-1)}$

$$E(\frac{1}{v}) = \frac{1}{m-2}$$

$\therefore \mathbf{E}(F) = m\mathbf{E}(\dfrac{1}{v}) \Rightarrow m(\dfrac{1}{m-2})$

So, $E(F) = \dfrac{m}{m-2}$

Excercise 2

1. The likelihood of the Lebesgue measure it follows is;

$$L(\theta; x_1, ..., x_n) = \prod_{i=1}^{n} \frac{1}{\theta} 1_{x_i \leq \theta} = 1_{max\{X_i\} \leq \theta} \theta^{-n}$$

   From the above equation, there exists a maximum likehood because the function $\theta \to \theta^{-n}$ decreases. Therefore, we conlude that $\hat{\theta} = max\{X_i\}$

2. Let $t \in \mathbb{R}^+$. For find the law, we have;

$$\mathbb{P}_\theta(\hat{\theta} \leq t) = \mathbb{P}_\theta(\forall i \epsilon [1,n], X_i \leq t) = \mathbb{P}_\theta(X_1 \leq t)^n = (\frac{t}{\theta})^n 1_{t \leq \theta} + 1_{t \leq \theta}$$

From this, we deduced that the density is θ given by;

$$f_n(t) = \frac{n}{\theta}(\frac{t}{\theta})^{n-1} 1_{[0,\theta]}(t)$$

So the bias is;

$$E_\theta[\hat{\theta}] = \int_{\mathbb{R}} t f_n(t) dt = \frac{n}{\theta^n} \int_0^\theta t^n dt = \frac{n}{n+1}\theta$$

3. We fix $t \in \mathbb{R}^+$ and we let $n \geq \frac{t}{\theta}$. We get;

$$\mathbb{P}_\theta(n(\theta - \hat{\theta}) \geq t) = \mathbb{P}_\theta(\hat{\theta} \leq \theta - \frac{t}{n}) = (\frac{\theta - \frac{t}{n}}{\theta})^n$$

$$= (1 - \frac{t}{n\theta})^n \to e^{\frac{-t}{\theta}} as\ n \to \infty$$

Therefore,

$$\mathbb{P}_\theta(n|\theta - \hat{\theta}|) \to 1 - e^{\frac{-t}{\theta}} = -Y$$

.The speed of convergence follows an exponential law $\frac{1}{\theta}$

4. The quadratic risk is as thus;

$$E_\theta[(\hat{\theta} - \theta)^2] = E_\theta[(\hat{\theta})^2] - 2\theta E_\theta[\hat{\theta}] + \theta^2$$

$$= \frac{n}{\theta^n} \int_0^\theta t^{n+1} dt - \frac{2n\theta^2}{n+1} + \theta^2$$

$$= (\frac{n}{n+2} - \frac{2n}{n+1} + 1)\theta^2$$

$$= \frac{2\theta^2}{(n+1)(n+2)}$$