We presuppose a universe Λ of *labels*. For subsets $A \subseteq \Lambda$, we use A^ϵ to denote $A \cup \{\epsilon\}$, where $\epsilon \notin \Lambda$ is a special symbol signifying the empty string.

Definition 1 (transducer). *Let $A, C \subseteq \Lambda$ be sets of actions. An A to C-*transducer *is a tuple $S = \langle Q, Q^\mathsf{S}, Q^\mathsf{F}, \iota, T\rangle$ where*
- *Q is a set of states;*
- *$Q^\mathsf{S} \subseteq Q$ is the subset of* steady *states;*
- *$Q^\mathsf{F} \subseteq Q^\mathsf{S}$ is the subset of* final *states;*
- *$\iota \in Q^\mathsf{S}$ is the initial state;*
- *$T \subseteq (Q \backslash Q^\mathsf{F}) \times A \times C^* \times Q$ is a transition relation, such that $(q, a, \sigma, q'), (q', b, \rho, q'') \in T$ with $q' \notin Q^\mathsf{S}$ implies $a = b$.*

A transducer is

- large-step *if $Q^\mathsf{S} = Q$,*
- small-step *if $(q, a, \sigma, q') \in T$ implies $\sigma \in C^\epsilon$,*
- pure *if $(q, a, \sigma, q') \in T$ implies $\sigma \in C$, and*
- *a* transition system *if $Q^\mathsf{S} = Q$ and $C = \emptyset$ (hence $(q, a, \sigma, q') \in T$ implies $\sigma = \epsilon$).*

A state that is not steady is called *transient*; we use $Q^\mathsf{T} = Q \setminus Q^\mathsf{S}$ to denote the set of transient states. If S is large-step then we use $q \overset{a}{\underset{\sigma}{\Longrightarrow}} q'$ to denote $(q, a, \sigma, q') \in T$, whereas if S is small-step we use $q \xrightarrow[\sigma]{a} q'$; in both cases, if $\sigma = \epsilon$ we typically leave it out altogether. Finally, we write $q\bullet$ to denote $q \in Q^\mathsf{S}$, $q\circ$ to denote $q \in Q^\mathsf{T}$ and $q\!\uparrow$ to denote $q \in Q^\mathsf{F}$.

An A-to-C transducer S establishes a relation between abstract actions (in A) and (sequences) of concrete actions (in C). The general intuition behind a transition (q, a, σ, q') with $q, q' \in Q^\mathsf{S}$ is that this causes a sequence of actions σ to occur, while posing to the environment that a has occurred. If either q or q' is transient, however, then σ is only *part* of the execution of a; the entire sequence of concrete actions that constitutes a can be found by concatenating sequences of transitions starting and ending at steady states. That is, if S is a small-step transducer, then there is a corresponding large-step transducer $\bar{S}$ with the state set $\bar{Q} = Q^\mathsf{S}$, final states $\bar{Q}^\mathsf{F} = Q^\mathsf{F}$, initial state $\bar{\iota} = \iota$, and $\bar{T}$ generated by

$$\frac{q_0 \xrightarrow[c_1]{a} q_1 \xrightarrow[c_2]{a} \cdots \xrightarrow[c_n]{a} q_n \qquad q_0, q_n\bullet \qquad q_1, \ldots, q_{n-1}\circ}{q_1 \overset{a}{\underset{c_1\cdots c_n}{\Longrightarrow}} q_n}$$

The effect of transducers is essentially established by their *composition*, $S_1 :: S_2$, where S_i for $i = 1, 2$ are A_i-to-C_i transducers with $C_1 = A_2$. The result is an A_1-to-C_2 transducer defined by the following rules. If the S_i are large-step, then $S_1 :: S_2$ is also a large-step transducer, defined by state set $Q = \{q_1 :: q_2 \mid q_1 \in Q_1, q_2 \in Q_2\}$, steady states, $Q^\mathsf{S} = \emptyset$, initial state $\iota = \iota_1 :: \iota_2$, and Q^F and T generated by

$$\frac{q_1 \overset{a}{\underset{c_1\cdots c_n}{\Longrightarrow}} q_1' \qquad q_2 \overset{c_1}{\underset{\sigma_1}{\Longrightarrow}} \cdots \overset{c_n}{\underset{\sigma_n}{\Longrightarrow}} q_2'}{q_1 :: q_2 \overset{a}{\underset{\sigma_1\cdots\sigma_n}{\Longrightarrow}} q_1' :: q_2'} \qquad \frac{q_1\!\uparrow \qquad q_2\!\uparrow}{q_1 :: q_2 \uparrow}$$

For small-step transducers, the composition is similar, but also needs a rule for stability:

$$\frac{q_1 \xrightarrow[c]{a} q_1' \qquad q_2 \xrightarrow[e]{c} q_2'}{q_1 :: q_2 \xRightarrow[e]{a} q_1' :: q_2'} \qquad \frac{q_1 \xrightarrow[\epsilon]{a} q_1'}{q_1 :: q_2 \xRightarrow[\epsilon]{a} q_1' :: q_2} \qquad \frac{q_1\bullet \qquad q_2\bullet}{q_1 :: q_2 \bullet} \qquad \frac{q_1\uparrow \qquad q_2\uparrow}{q_1 :: q_2 \uparrow}$$

This brings us to our first result:

Proposition 1. *If S_1, S_2 are small-step A_i-to-C_i-transducers with $C_2 = A_1$, then $\bar{S}_1 :: \bar{S}_2 = \overline{S_1 :: S_2}$.*