Level 2: Classification of PDEs

Pavni: Acharya, last time you told me how PDEs were born from strings, heat, and flows. But there seem to be so many kinds of PDEs. How do we organize them?

Acharya: A good question, Pavni. Mathematicians classify PDEs in several ways, much like a botanist classifies plants. Let us begin with the simplest.

Pavni: I am listening.

Acharya: First, by **order**. A PDE is first-order if the highest derivative is first order, second-order if the highest is second, and so on. For example, the transport equation $u_t + cu_x = 0$ is first-order, while the heat equation $u_t = \alpha u_{xx}$ is second-order.

Pavni: That part seems simple enough. What else?

Acharya: Next comes **linearity**. A PDE is linear if the unknown function and its derivatives appear only linearly — not multiplied together, not inside a sine or square. The heat equation is linear. But if you had a term like uu_x, that would make it nonlinear.

Pavni: So, nonlinear PDEs are trickier?

Acharya: Very much so! Nonlinearity makes life both harder and richer.

Pavni: Are there other distinctions?

Acharya: Yes. Equations can be **homogeneous** or **inhomogeneous** depending on whether the right-hand side is zero. Laplace's equation is homogeneous, Poisson's equation is inhomogeneous.

Acharya: More generally, a non-homogeneous PDE can be written as:

$$F(t, x_1, x_2, \dots, x_n, u, u_t, u_{x_1}, \dots, u_{x_n}, u_{tt}, u_{x_1x_1}, \dots) = g(x_1, x_2, \dots, x_n, t),$$

where g is a nonzero source or forcing term. If $g \equiv 0$, the PDE is homogeneous.

Pavni: So the right-hand side introduces an external influence, like heat sources or forces?

Acharya: Exactly. For example, $u_{xx} + u_{yy} = f(x, y)$ is the Poisson equation with a source term $f(x, y)$. .

Pavni: I think I follow. But I have also heard words like elliptic and hyperbolic. What do they mean?

Acharya: Ah, those arise for **second-order PDEs in two variables**. Suppose we have:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + \cdots = 0.$$

We look at the discriminant: $B^2 - 4AC$.
- If it is less than 0, the PDE is **elliptic** (like Laplace's equation).
- If it equals 0, the PDE is **parabolic** (like the heat equation).
- If it is greater than 0, the PDE is **hyperbolic** (like the wave equation).

Pavni: This reminds me of conic sections! Circles, parabolas, and hyperbolas.

Acharya: Exactly. The analogy is deliberate — both come from the same quadratic form.

Pavni: And physically?

Acharya: Elliptic equations describe steady states — like equilibrium temperature distributions. Parabolic equations describe diffusion, the smoothing of irregularities over time. Hyperbolic equations describe wave-like motion, signals traveling with finite speed.

Pavni: That helps me imagine them. So classification is not just a game, but it tells us the nature of the solutions.

Acharya: Well said, Pavni. And it also guides how we design numerical methods. An elliptic problem requires different strategies than a hyperbolic one.

Pavni: Then I am eager to learn those strategies!

Acharya: Patience, Pavni. First we must prepare the ground. Classification is the map; numerical methods are the journey.

Mini Quizzes

Quiz 1: Identify the order
Which of the following is a second-order PDE?
1. $ u_t + cu_x = 0 $
2. $ u_t = u_{xx} $
3. $ u u_x = 0 $

Answer 1

Equation (2) $ u_t = u_{xx} $ is second-order because of the u_{xx} term.

Quiz 2: Linearity check
Which PDE is nonlinear?
1. $ u_t = u_{xx} $
2. $ u_t + u u_x = 0 $

Answer 2

Equation (2) is nonlinear because of the product term uu_x.

Quiz 3: Homogeneous or inhomogeneous
Classify: $ u_{xx} + u_{yy} = f(x,y) $.

Answer 3

It is **inhomogeneous**, since the right-hand side is not zero.

Quiz 4: Type of second-order PDE
For $ u_{xx} + 2u_{xy} + u_{yy} = 0 $, compute $ B^2 - 4AC $. What type is it?

Answer 4

Here, $A = 1, B = 2, C = 1$.
$ B^2 - 4AC = 2^2 - 4(1)(1) = 0 $.
So it is **parabolic**.

Quiz 5: Physical meaning
Match each equation with its physical interpretation:
- Heat equation
- Wave equation
- Laplace's equation

(a) Steady state

(b) Wave-like motion

(c) Diffusion in time

Answer 5

- Heat equation → (c) Diffusion in time
- Wave equation → (b) Wave-like motion
- Laplace's equation → (a) Steady state