

Level 4: Hyperbolic PDEs: Wave Equation



**Pavni:** We've studied the heat equation as an example of a parabolic PDE. What about hyperbolic ones? Where do we begin?

**Acharya:** A natural starting point is the **wave equation**:

$$u_{tt} = c^2 u_{xx}.$$

**Pavni:** Oh! That's the equation for vibrations of a string, right?

**Acharya:** Exactly. Imagine a taut string. Newton's law applied to a small element gives:

$$\rho u_{tt} = T u_{xx},$$

where  $T$  is tension and  $\rho$  is density. Dividing through, we get

$$u_{tt} = \left(\frac{T}{\rho}\right) u_{xx},$$

so the wave speed is  $c = \sqrt{T/\rho}$ .

**Pavni:** So physically, it describes oscillations moving along the string. But why do we call this *hyperbolic*?

**Acharya:** Let's look at the general second-order PDE in two variables:

$$A u_{xx} + 2B u_{xt} + C u_{tt} = 0.$$

Its type depends on the discriminant  $D = B^2 - AC$ .

- If  $D < 0$ , it's elliptic.
- If  $D = 0$ , parabolic.
- If  $D > 0$ , hyperbolic.

**Pavni:** For the wave equation, we have  $A = -c^2, B = 0, C = 1$ . Then

$$D = B^2 - AC = c^2 > 0.$$

**Acharya:** Exactly — that's why it's **hyperbolic**.

**Pavni:** Does that mean it has some special geometry?

**Acharya:** Yes. Notice how the operator factors:

$$u_{tt} - c^2 u_{xx} = (\partial_t - c \partial_x)(\partial_t + c \partial_x)u.$$

This reveals the **characteristics**: the lines

$$x - ct = \text{constant}, \quad x + ct = \text{constant}.$$

**Pavni:** So along those lines, the solution behaves in a simple way?

**Acharya:** Precisely. The general solution is

$$u(x, t) = f(x - ct) + g(x + ct).$$

It's just the sum of two waves, one traveling right, one traveling left.

**Pavni:** That's beautiful! It really captures the idea of finite-speed propagation.

**Acharya:** Indeed. If you disturb the string at one point, the influence spreads only within the cone  $|x - x_0| \leq c(t - t_0)$ .

That's the hallmark of hyperbolic PDEs: signals travel with finite speed, unlike diffusion where influence is instant.

**Pavni:** So the wave equation is the prototype for hyperbolic PDEs?

**Acharya:** Exactly. From here, we can explore more complex hyperbolic equations — nonlinear ones, conservation laws, and shock waves — but the wave equation is where the story begins.

```
\begin{tikzpicture}[scale=1.2]
% Axes
\draw[->] (-2,0) -- (2,0) node[right] {$x$};
\draw[->] (0,0) -- (0,3) node[above] {$t$};

% Point of disturbance
\fill (0,0) circle (2pt) node[below left] {$(x_0, t_0)$};

% Characteristics
\draw[thick, red] (0,0) -- (1.5,3) node[right] {$x = x_0 + c(t-t_0)$};
\draw[thick, red] (0,0) -- (-1.5,3) node[left] {$x = x_0 - c(t-t_0)$};

% Shaded influence region
\fill[blue!20, opacity=0.5] (-1.5,3) -- (0,0) -- (1.5,3) -- cycle;

% Labels
\node at (0,1.8) {Region of influence};
\end{tikzpicture}
```