Level 4: Hyperbolic PDEs: Wave Equation

Pavni: We've studied the heat equation as an example of a parabolic PDE. What about hyperbolic ones? Where do we begin?

Acharya: A natural starting point is the **wave equation**:

$$u_{tt} = c^2 u_{xx}.$$

Pavni: Oh! That's the equation for vibrations of a string, right?

Acharya: Exactly. Imagine a taut string. Newton's law applied to a small element gives:

$$\rho u_{tt} = T u_{xx},$$

where T is tension and ρ is density. Dividing through, we get

$$u_{tt} = \left(\tfrac{T}{\rho}\right) u_{xx},$$

so the wave speed is $c = \sqrt{T/\rho}$.

Pavni: So physically, it describes oscillations moving along the string. But why do we call this *hyperbolic*?

Acharya: Let's look at the general second-order PDE in two variables:

$$Au_{xx} + 2Bu_{xt} + Cu_{tt} = 0.$$

Its type depends on the discriminant $D = B^2 - AC$.
- If $D < 0$, it's elliptic.
- If $D = 0$, parabolic.
- If $D > 0$, hyperbolic.

Pavni: For the wave equation, we have $A = -c^2, B = 0, C = 1$. Then

$$D = B^2 - AC = c^2 > 0.$$

Acharya: Exactly — that's why it's **hyperbolic**.

Pavni: Does that mean it has some special geometry?

Acharya: Yes. Notice how the operator factors:

$$u_{tt} - c^2 u_{xx} = (\partial_t - c\partial_x)(\partial_t + c\partial_x)u.$$

This reveals the **characteristics**: the lines

$$x - ct = \text{constant}, \quad x + ct = \text{constant}.$$

Pavni: So along those lines, the solution behaves in a simple way?

Acharya: Precisely. The general solution is

$$u(x,t) = f(x - ct) + g(x + ct).$$

It's just the sum of two waves, one traveling right, one traveling left.

Pavni: That's beautiful! It really captures the idea of finite-speed propagation.

Acharya: Indeed. If you disturb the string at one point, the influence spreads only within the cone $|x - x_0| \leq c(t - t_0)$.
That's the hallmark of hyperbolic PDEs: signals travel with finite speed, unlike diffusion where influence is instant.

Pavni: So the wave equation is the prototype for hyperbolic PDEs?

Acharya: Exactly. From here, we can explore more complex hyperbolic equations — nonlinear ones, conservation laws, and shock waves — but the wave equation is where the story begins.

```
\begin{tikzpicture}[scale=1.2]
  % Axes
  \draw[->] (-2,0) -- (2,0) node[right] {$x$};
  \draw[->] (0,0) -- (0,3) node[above] {$t$};

  % Point of disturbance
  \fill (0,0) circle (2pt) node[below left] {$(x_0,t_0)$};

  % Characteristics
  \draw[thick,red] (0,0) -- (1.5,3) node[right] {$x = x_0 + c(t-t_0)$};
  \draw[thick,red] (0,0) -- (-1.5,3) node[left] {$x = x_0 - c(t-t_0)$};

  % Shaded influence region
  \fill[blue!20,opacity=0.5] (-1.5,3) -- (0,0) -- (1.5,3) -- cycle;

  % Labels
  \node at (0,1.8) {Region of influence};
\end{tikzpicture}
```