

2 Exponential & Logarithmic Functions

2.1 Exponential functions

Rules of exponents:

- $a^{m+n} = a^m \cdot a^n$
- $a^{m-n} = \frac{a^m}{a^n}$
- $(a^m)^n = a^{mn}$
- $(a \cdot b)^m = a^m \cdot b^m$
- $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

For a fixed number a , where $a > 0$ and $a \neq 1$, define the value of a^x properly, as the range of x extends from integers to rational numbers, and then to real numbers. This gives the idea of the **exponential function**.

Sketch the graphs of $y = a^x$ for both $0 < a < 1$ and $a > 1$, stating the domain and range for each case, and noting the difference in monotonicity.

Exercise 12.

1. State the natural domain and range of the function $y = (0.14)^{\frac{1}{x}}$.
2. Sketch the graphs of $f(x) = -3^{x-1} + 1$, $g(x) = 2^{1-2x}$ and $h(x) = \left|\left(\frac{1}{2}\right)^x - 2\right|$.

3. Compare the following pairs of values:

(a) $3^{2.8}$ and $3^{2.7}$

(b) $0.7^{-0.1}$ and $0.7^{-0.2}$

(c) $1.23^{-1.1}$ and $1.23^{-0.9}$

(d) 0.6^{-3} and 0.6^{-4}

4. Given that $0.2^m > 0.2^n$, compare the values of m and n .

5. For different possible values of a , solve the inequality $a^{3x-5} > a^{4x+1}$.

6. Determine whether the equation $(a^b)^c = a^{(b^c)}$ holds in general.

7. Given that the maximum value of the function $f(x) = a^{2x} + 2a^x - 1$, where $a > 0$ and $a \neq 1$, on the interval $-1 \leq x \leq 1$ is 14, find the value of a .

8. (†) If a non-zero continuous function $f(x)$ defined on \mathbb{R} has the property $f(x+y) = f(x)f(y)$ for all real numbers x and y , show that $f(x) = a^x$ for some $a > 0$.

2.2 Logarithms

For $a > 0$ and $a \neq 1$, if $y = a^x$, then we define the **logarithm to base a** as $x = \log_a y$.

Rules of logarithms include:

- $\log_a 1 = 0$; $\log_a a = 1$; $\log_a a^x = a^{\log_a x} = x$
- $\log_a(xy) = \log_a x + \log_a y$; $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
- $\log_a x^n = n \log_a x$; $\log_{a^n} x = \frac{1}{n} \log_a x$; $\log_{a^m} x^n = \frac{n}{m} \log_a x$
- $\log_a b = \frac{1}{\log_b a}$; $\log_a b = \frac{\log_c b}{\log_c a}$

You may wish to write respective rules of exponents for the above equations.

Special bases: $\lg A = \log_{10} A$, and $\ln B = \log_e B$, where $e = 2.718\,281\,828\,459\,045 \dots$ is a special irrational number.

Exercise 13.

1. Simplify the following expressions:

$$\log_5 125; \quad \log_6 \left(\frac{1}{216}\right); \quad \log_{\frac{1}{4}} 8\sqrt{2}; \quad \log_2 \left(4^7 \times \sqrt{8^3}\right); \quad (\log_4 3 + \log_8 3)(\log_3 2 + \log_9 2)$$

2. Find the value of xy given that $\log_{11} x = 2$ and $\log_y 27 = -3$.

3. Find the root of the equation $8^{2+x} = 3 \times 9^x$, giving your answer exactly in terms of logarithms.

4. Solve the equation $3^x = 2^{1-2x}$, giving your answer to three decimal places.

5. Solve the simultaneous equations:

$$\begin{cases} 3^x = 5^y \\ x + y = 1 \end{cases},$$

giving your answers exactly in terms of logarithms.

6. Express $\log_{10} (5\sqrt{10}) - \frac{1}{3} \log_{10} 2.7 - \log_{10} \left(\frac{10}{9}\right)$ in the form of $c + \log_{10} d$, where c and d are rational numbers.

7. Prove the following identities:

(a) $a^{\log_c b} = b^{\log_c a}$

(b) $\log_a \left(\frac{u}{v}\right) + \log_a \left(\frac{v}{w}\right) + \log_a \left(\frac{w}{u}\right) = 0$.

(c) $\log_a b \log_c d = \log_a d \log_c b$.

8. Solve the following equations

(a) $\log_2(x+2) + \log_2 x = 3$

(b) $\sqrt{3\log_5 x - 2} - 3\log_5 x + 4 = 0$

(c) $3 \cdot 4^x - 10 \cdot 2^x + 3 = 0$

(d) $7^{-x} + 7^{|x|} = 4$

(e) $\ln(x-3) + 2\ln(x+1) = 3\ln(x-1)$

(f) $\log_{\frac{1}{2}}(9^x + 1) + \log_{\sqrt{2}}(3^x - 2) + 1 = 0$

(g) $\frac{\log_3 x + 2}{\log_9 x - 2} = \log_{\sqrt{3}} x - 4$

(h) (†) $\log_x 2 \cdot \log_x 3 + 6 = \log_x 72$

2.3 Logarithmic functions

Sketch the graphs of $y = \log_a x$ in two cases: $0 < a < 1$ and $a > 1$, stating the domain and range for each case, and noting the difference in monotonicity.

Exercise 14.

1. Compare the following pairs of values:

- (a) $\log_7 0.6$ and $\log_7 0.8$
- (b) $\log_{0.4} 4$ and $\log_{0.4} 5$
- (c) $\log_{\frac{3}{5}} 1.4$ and $\log_{\frac{3}{5}} 1.5$
- (d) $\log_{2.1} 0.9$ and $\log_{2.2} 0.9$
- (e) $\log_{\frac{4}{5}} \pi$ and $\log_{\frac{5}{6}} \pi$

2. Sketch the graphs of:

- (a) $f_1(x) = \log_{\frac{1}{4}}(2 - x)$
- (b) $f_2(x) = 2 - \log_{\frac{1}{4}} x$
- (c) $g_1(x) = |\log_3(4x + 1)|$
- (d) $g_2(x) = \log_3 |4x + 1|$
- (e) $g_3(x) = \log_3(4|x| + 1)$

3. Given that $\log_{0.2} m < \log_{0.2} n$, compare the values of m and n .
4. Given two functions $f(x) = \log_a(1+x)$, and $g(x) = \log_a(1-x)$, where $a > 0$ and $a \neq 1$.
- (a) State the natural domain of $h(x) = f(x) + g(x)$.
- (b) Determine whether h is an even function or an odd, or neither.
5. The function f is given by $f: x \mapsto \log_a(1+x)$, where $x > -1$. Define its inverse, f^{-1} , in a similar form.
6. Solve the inequality $\log_2(a^{2x} + 2a^x - 2) < 0$, where $a > 0$ and $a \neq 1$.
7. (†) Given that the function $f(x) = \log_a(a - ka^x)$, where $a > 0$ and $a \neq 1$, is a **self-inverse** function, find the value of k . When k has this value, solve the inequality $f(x) \geq 1$.
8. (†) Given that the domain of the function $f(x) = \log_3 \frac{mx^2 + 8x + n}{x^2 + 1}$ is \mathbb{R} , and that its range is $0 \leq x \leq 2$, find the values of m and n .

Exercise 15.

1. A radioactive source has a half-life of 20 years. Initially its mass is 4.5 kg, find
 - (a) the remaining mass after 76 years,
 - (b) the time taken such that the remaining mass is only 0.05 kg.

2. An experiment was conducted to discover how fast a particular type of tree grows over time. The results are summarized in the table below, where t is measured in years, and h is the height of the tree being measured, in meters.

t	10	20	30	40
h	2.50	3.10	3.45	3.70

A model is given by $h = a \log_{10} t + b$, where a and b are constants. Find the values of a and b , and estimate by how many years this tree will grow up to the height of 5 meters.

3. The time taken, T seconds, to compile a file with a source code of x lines by a particular programme is modeled as an exponential function: $T = Ac^x$. An experiment was conducted, and the results are summarized in the table below.

x	50	100	150	200
T	0.658	1.082	1.779	2.926

Find the values of A and c , and estimate the shortest source code such that its compiling time exceeds 1 minute.

4. (†) To nine decimal places, $\log_{10} 2 = 0.301029996$, and $\log_{10} 3 = 0.477121255$.
 - (a) Calculate $\log_{10} 5$ and $\log_{10} 6$ to three decimal places.
 - (b) Show that $5 \times 10^{47} < 3^{100} < 6 \times 10^{47}$.
 - (c) Hence write down the first digit of 3^{100} .
 - (d) Find the first digit of each of the following numbers: 2^{1000} ; 2^{10000} ; and 2^{100000} .
5. (‡) If a continuous function $f(x)$ defined on \mathbb{R}^+ has the property $f(xy) = f(x) + f(y)$ for all positive numbers x and y , show that $f(x) = \log_a x$ for some $a > 0$ and $a \neq 1$.