Assignment 10.

1. Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (\cos x \cos y)^2,$$

obtaining an expression for $\tan y$ in terms of x. [6]

2. Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \ln(x^y),$$

obtaining an expression for y in terms of x. [5]
Given further that $y = 1$ when $x = \mathrm{e}$, find the value of y when $x = 1$. [2]

3. Given that the curve, whose equation satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x\sqrt{(x^2+1)(y+1)},$$

passes through the point $(1, 1)$, find an expression of y in terms of x. [7]

4. In ecology, a common model of population growth was proposed by *Pierre-François Verhulst*, where the rate of reproduction is proportional to both the existing population and the amount of available resources, *ceteris paribus* (all else being equal). The model is formalized by the differential equation:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP \cdot \left(1 - \frac{P}{K}\right),$$

where P represents population size, t represents time, and r, K are two positive constants.

(a) Given the initial condition: $P = P_0$, when $t = 0$, solve the differential equation and express P in terms of t, r, K and P_0. [7]

(b) According to *Verhulst*'s model, what is the limiting population size in the long run? [1]

5. A tank is being filled with water. At time t minutes after filling begins, the volume of water is V liters. Water is poured in at a constant rate of 9 liters per minute, but owing to leakage, it is lost at a rate proportional to V. Initially the tank is empty. When $V = 4$, $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 7$.

(a) Show that V satisfies the differential equation: $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 9 - \frac{1}{2}V$. [2]

(b) Solve the above differential equation, expressing V in terms of t. [4]

(c) Calculate the time taken to fill the tank with 9 liters of water. [2]

6. (†) Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 3y^2,$$

such that $y = 2$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4$ when $x = 1$. [7]

Hint: Prove that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = z\dfrac{\mathrm{d}z}{\mathrm{d}y}$, where $z = \dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Total mark of this assignment: $36 + 7$.

The symbol (†) indicates a bonus question. Finish other questions before working on this one.