3 Trigonometric Functions

3.1 Definitions, graphs & properties of trigonometric functions (revisited)

Sketch the graphs of the following functions:

$y = \sin x$ $y = \cos x$

$y = \tan x$ $y = \cot x$

$y = \sec x$ $y = \csc x$

Exercise 16.

1. State the domain and range for each of the above functions.

2. Which of the above functions are bounded? For these bounded functions, find their extrema (if any).

3. Which of the above functions are continuous?

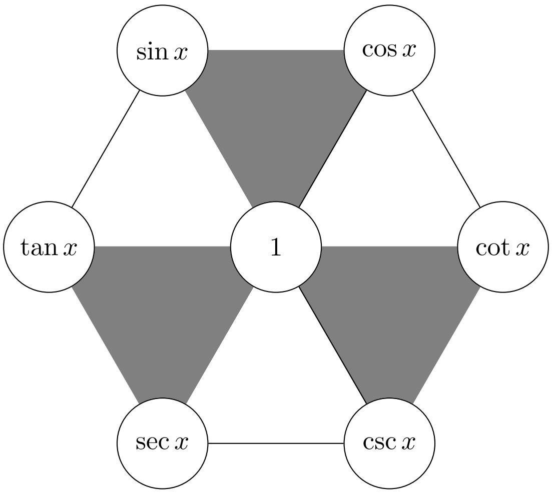
4. Investigate the symmetry property for each of the above functions.

5. Which of the above functions are periodical? For these periodic functions, determine their (least positive) period.

6. State the complementary relations among these function.

3.2 Trigonometric identities

You may summarize the relations among the trigonometric functions using the following diagram:



- Products:

- Reciprocals:

- Sum of squares:

Exercise 17.

1. Evaluate each of the following expressions.

   (a) $\tan \phi + \sec \phi$, given that $\sin \phi = -0.6$, and that ϕ is a fourth-quadrant angle

   (b) $\sin \xi + \cot \xi$, given that $\cos \xi = -\frac{12}{13}$, and that ξ is a third-quadrant angle

(c) $\sin^2\alpha - 3\sin\alpha\cos\alpha - 2\cos^2\alpha$, given that $\cot\alpha = -\frac{8}{15}$

(d) $\dfrac{\sec\mu + \csc\mu}{\sec\mu - \csc\mu}$, given that $\tan\mu = -4$

2. Prove the following identities.

(a) $\dfrac{1 - 2\sin x\cos x}{\cos^2 x - \sin^2 x} = \dfrac{1 - \tan x}{1 + \tan x}$

(b) $\tan^2\alpha\sec^2\beta + \sec^2\alpha\tan^2\beta - \tan^2\alpha\tan^2\beta = \sec^2\alpha\sec^2\beta - 1$

3. Solve the following equations, giving all roots between 0° and 360° inclusive, correct to 1 decimal place.

(a) $3\tan x = 10\csc x$

(b) $\sec 2x + \cos 2x = 4$

(c) $2\tan^2 2x = 3\sec 2x$

(d) $(\tan x - 2\cot x)(\tan x + 3\cot x) = 2$

3.3 Sum and difference formulae

- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
- $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

Derive the formula of $\sin(A + B)$ for both A and B being acute angles, and think about why it is true in general.

Hence deduce the rest of the formulae by using this one.

Exercise 18.

1. Evaluate the exact values of $\sin \frac{\pi}{12}$.

2. Express $\tan\left(\frac{5}{6}\pi - x\right)$ in terms of $\tan x$.

3. Given that $\sin A = \frac{4}{5}$ and $\cos B = -\frac{5}{13}$, where A and B are both obtuse angles, find the exact values of:

$$\sin(A + B), \qquad \cos(A - B), \qquad \tan(B - A)$$

4. Solve the equation $\sin\left(x+\frac{1}{6}\pi\right)\sin\left(x-\frac{1}{6}\pi\right)=\frac{1}{4}$, in the interval $-2\pi \leq x \leq 2\pi$.

5. Solve the equation $\sin\left(x+\frac{1}{4}\pi\right) = 2\cos\left(x+\frac{1}{3}\pi\right)$, in the interval $-2\pi \leq x \leq 2\pi$, correct your answers to 2 decimal places.

6. Given that $\cos\alpha+\cos\beta=\frac{1}{2}$, and $\sin\alpha+\sin\beta=\frac{1}{3}$, find the value of $\cos(\alpha-\beta)$.

7. Given that $\sin\phi+\cos\phi=-\frac{1}{5}$, find all possible values of $\tan\phi$.

8. Simplify the expression $\sin x+\sin\left(x+\frac{2}{3}\pi\right)+\cos\left(x+\frac{5}{6}\pi\right)$.

9. Prove the identity $\dfrac{\sin(2\xi+\eta)}{\sin\xi}-2\cos(\xi+\eta)=\dfrac{\sin\eta}{\sin\xi}$.

3.4 Double angle formulae

Double angle formulae follow immediately from the respective sum formulae, by setting $A = B$:

- $\sin 2A = 2\sin A \cos A$

- $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

- $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

Exercise 19.

1. If $\cos 2\theta = -\frac{7}{18}$, find the possible values of $\cos\theta$ and $\sin\theta$.

2. If $\tan 2\alpha = 1$, find the possible values of $\tan\alpha$. Hence state the exact value of $\tan \frac{1}{8}\pi$.

3. Solve the equation $2\cos 2x + 1 + \sin x = 0$ in the interval $0 \leq x \leq 2\pi$.

4. Solve the equation $\tan 2x + 5\tan x = 0$.

5. Deduce that $\sin^2 \frac{1}{2}A = \frac{1}{2}(1 - \cos A)$, hence express $\sin \frac{1}{2}A$ in terms of $\cos A$. Derive a similar result for $\cos \frac{1}{2}A$.

6. Derive the tripe angle formulae of $\sin 3A$, $\cos 3A$ and $\tan 3A$.

7. Prove the following identities.

(a) $\tan \frac{1}{2}A = \dfrac{\sin A}{1 + \cos A} = \dfrac{1 - \cos A}{\sin A} = \dfrac{\tan A}{1 + \sec A}$

(b) $\tan \dfrac{x + y}{2} = \dfrac{\sin x + \sin y}{\cos x + \cos y}$

(c) $\tan \left(\frac{1}{2}\theta + \frac{1}{4}\pi\right) = \sec \theta + \tan \theta$

(d) $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \left|\dfrac{1-\tan\frac{1}{2}\theta}{1+\tan\frac{1}{2}\theta}\right|$

(e) $\cos 4\alpha + 4\cos 2\alpha + 3 = 8\cos^4\alpha$

(f) $\dfrac{1+\sin 2x}{2\cos^2 x + \sin 2x} = \frac{1}{2}(\tan x + 1)$

(g) (†) $\cos^2\phi + \cos^2\left(\frac{2}{3}\pi - \phi\right) + \cos^2\left(\frac{2}{3}\pi + \phi\right) = \frac{3}{2}$

8. Given two functions $f(x) = \sin^{-1}(\sin x)$, and $g(x) = \cos(2\cos^{-1} x)$, sketch their graphs. Hence show that $f(x)$ is an odd function, while $g(x)$ is an even function.

9. Simplify the following expressions.

   (a) $\dfrac{1+\sin 2\theta-\cos 2\theta}{1+\sin 2\theta+\cos 2\theta}$

   (b) $\dfrac{1+\sin 2\alpha}{1+\sin 2\alpha+\cos 2\alpha}-\frac{1}{2}\tan\alpha$

   (c) (†) $\tan 2\alpha\cdot\tan\left(\frac{1}{6}\pi-\alpha\right)+\tan 2\alpha\cdot\tan\left(\frac{1}{3}\pi-\alpha\right)+\tan\left(\frac{1}{6}\pi-\alpha\right)\cdot\tan\left(\frac{1}{3}\pi-\alpha\right)$

10. α and β are two acute angles such that $\tan\alpha$ and $\tan\beta$ are the two roots of the equation $6x^2-5x+1=0$, find the exact values of $\tan(\alpha+\beta)$ and $\tan\frac{\alpha+\beta}{2}$.

11. Given that α is obtuse and β is acute, and that $\sin\alpha=\frac{4}{5}$, $\sin\beta=\frac{12}{13}$, find the value of $\cos\frac{\alpha-\beta}{2}$.

3.5 The sinusoidal form

If a and b are positive,

- $a \sin x \pm b \cos x$ can be written in the form ____________________,

- $a \cos x \pm b \sin x$ can be written in the form ____________________,

where $R =$ _______________, and $\alpha =$ _____________.

Exercise 20.

1. Express each of the following expressions in the given form(s), where R is positive, and α an acute angle.
   (a) $7 \sin x + 2 \cos x$; $\quad R \sin(x + \alpha)$; $\quad R \cos(x - \beta)$
   (b) $4 \cos x - 5 \sin x$; $\quad R \cos(x + \alpha)$
   (c) $3 \sin x - 5 \cos x$; $\quad R \sin(x - \alpha)$

2. Solve the equation $4 \cos \frac{1}{2}x + 5 \sin \frac{1}{2}x + 2 = 0$ for values of x between 0 and 2π.

3. Simplify $\sqrt{3} \sin\left(\frac{1}{6}\pi - \alpha\right) - \cos\left(\frac{1}{6}\pi - \alpha\right)$.

4. State the range of the function $y = \dfrac{2}{8\cos 3x - 15\sin 3x + 20}$.

5. Find the maximum and minimum values of the function $y = 12\sin 2x + 5\cos 2x - 4$, stating the value(s) of x with least modulus where the maximum or minimum value is attained.

6. State the period of the function $y = 3\sin 3x - 5\cos 3x + 4$, and sketch its graph for at least two periods.

7. Determine the period of the function $f(x) = \cos^2 x + 2\sqrt{3}\sin x\cos x - \sin^2 x$.

3.6 (†) Other trigonometric formulae [EXTRA]

- The **sum-product** formulae:

$$\begin{aligned}
\sin x \pm \sin y &\equiv 2\sin\frac{x\pm y}{2}\cos\frac{x\mp y}{2}\\
\cos x + \cos y &\equiv 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}\\
\cos x - \cos y &\equiv -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}
\end{aligned}$$

- The **product-sum** formulae:

$$\begin{aligned}
\sin x\sin y &\equiv -\frac{\cos(x+y)-\cos(x-y)}{2}\\
\sin x\cos y &\equiv \frac{\sin(x+y)+\sin(x-y)}{2}\\
\cos x\cos y &\equiv \frac{\cos(x+y)+\cos(x-y)}{2}
\end{aligned}$$

- The **tangent half-angle** formulae:

$$\begin{aligned}
\sin x &\equiv \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\\
\cos x &\equiv \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\\
\tan x &\equiv \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}
\end{aligned}$$

Exercise 21.

(†) Enjoy the following challenges.

1. The two roots of the quadratic equation $ax^2 + (2a-3)x + (a-2) = 0$ are $\tan\alpha$ and $\tan\beta$, respectively.
   (a) Find the value of $\tan(\alpha+\beta)$ if $a = \frac{1}{2}$.
   (b) Find the set of values of $\tan(\alpha+\beta)$ as a varies ($a \neq 0$).

2. For a certain value of k, $f(\theta,\alpha) = \dfrac{\sin^2(\theta-\alpha)+\sin^2\theta+\sin^2(\theta+\alpha)+k}{\cos^2(\theta-\alpha)+\cos^2\theta+\cos^2(\theta+\alpha)+k} \equiv m$, for all values of θ and α.
   Find the values of k and m.

3. Let $a_1 = \cos x$ with x acute, and $b_1 = 1$. Given that
   $$a_{n+1} = \tfrac{1}{2}(a_n + b_n), \qquad b_{n+1} = \sqrt{a_{n+1}b_n},$$
   find a_2 and b_2 and show that
   $$a_3 = \cos\tfrac{x}{2}\cos^2\tfrac{x}{4}, \text{ and } b_3 = \cos\tfrac{x}{2}\cos\tfrac{x}{4}.$$
   Determine general expressions for a_n and b_n for $n \geq 2$.

4. Prove the identity
   $$\cos x\cos y\sin^2\tfrac{x+y}{2} - \sin x\sin y\cos^2\tfrac{x+y}{2} = \tfrac{1}{2}\cos(x+y)(1-\cos(x-y)).$$

5. The points P, Q, R and S have coordinates $(a\cos p, b\sin p)$, $(a\cos q, b\sin q)$, $(a\cos r, b\sin r)$, and $(a\cos s, b\sin s)$ respectively, where $0 \leq p < q < r < s < 2\pi$, and a and b are positive.
   Given that neither of the lines PQ and RS is vertical, show that these lines are parallel if and only if
   $$r + s - p - q = 2\pi.$$