4 Differentiation

4.1 Derivatives of some common functions

First consider the exponential function:

$$f(x) = a^x,$$

where a is a positive parameter and $a \neq 1$.

By definition,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{a^{x+h}-a^x}{h} \\ &= a^x \lim_{h\to 0} \frac{a^h-1}{h} \\ &= a^x L(a), \end{aligned}$$

where $L(a) = \lim\limits_{h\to 0} \frac{a^h-1}{h}$ is independent of x and h, but only depends upon the value of a. Namely, for a fixed value of a, $L(a)$ is also fixed. Therefore $L(a)$ can be seen as a (monotonic) function of a. The real number e is chosen such that $L(a) = 1$, or you may understand in the way that e is <u>defined</u> as $L^{-1}(1)$.

Therefore, $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^x\right) = \mathrm{e}^x$, and for any other base a,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(a^x\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(\mathrm{e}^{\ln a}\right)^x\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x\ln a}\right) \\ &= \mathrm{e}^{x\ln a}\cdot \ln a && \text{(chain rule)} \\ &= a^x \ln a. \end{aligned}$$

In fact e is an irrational number, and its approximate value is $\mathrm{e} \approx 2.718\,281\,828\,459\,045\cdots$.

The **natural logarithm** of a number is its logarithm to the base e: $\ln x = \log_{\mathrm{e}} x$.

The notation $\log x$ without an explicit base may also refer to the natural logarithm. This usage is common in mathematics and some scientific contexts as well as in many programming languages. In some other contexts, however, $\log x$ can be used to denote the common (base 10) logarithm.

Exercise 22.

1. Differentiate the following functions with respect to x:

   (a) $y = \mathrm{e}^{3x+1}$

   (b) $y = 2^{\sqrt{x}}$

   (c) $y = \dfrac{3}{2+\mathrm{e}^{1-x}}$

   (d) $y = \dfrac{\mathrm{e}^{3x}-2\mathrm{e}^{1-x}}{3\mathrm{e}^{x^2}}$

   (e) (†) $\quad y = \left(\dfrac{1}{2}\right)^{(2^x)}$

2. Find the stationary point of the graph of $y = 6e^x - e^{3x}$, and determine its nature.

3. Find the stationary point of the graph of $y = e^{3x^2+6x}$, and determine its nature.

4. Find the equation of the normal to the curve $y = \sqrt{e^x - x}$ at the point where $x = 0$.

5. (†) Referring to the graph of $y = e^x$, show that for any real numbers a and b, with $a < b$,

$$e^a < \frac{e^a - e^b}{a - b} < e^b.$$

Then we look at the logarithmic function:

$$f(x) = \log_a x,$$

where a is a positive parameter and $a \neq 1$.

From $y = \log_a x$, we can write $x = a^y = \mathrm{e}^{y \ln a}$, then differentiate this equation on both hand with respect to x to obtain:

$$1 = \mathrm{e}^{y \ln a} \cdot \ln a \cdot \frac{\mathrm{d}y}{\mathrm{d}x}.$$

Hence

$$f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{e}^{y \ln a} \cdot \ln a} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}.$$

In particular, when the base is e,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}.$$

Note that $y = f(x) = \mathrm{e}^x$ and $y = f^{-1}(x) = \ln x$ are a pair of inverse functions. A point $A(p, q)$, where $q = \mathrm{e}^p$, on the curve of f correspond to the point $B(q, p)$ on the curve of f^{-1}. You may verify that

$$f'(p) \cdot \left(f^{-1}\right)'(q) = 1.$$

Think about whether the statement above holds for all pairs of inverse functions.

Exercise 23.

1. Differentiate the following functions with respect to x:
   (a) $y = \ln\left(x^2 + 1\right)$
   (b) $y = \log_3(2x - 5)$
   (c) $y = \ln\left(x^3(x^2 - 1)^4\right)$
   (d) $y = \ln \dfrac{4x + 3}{x^3 - 1}$
   (e) $y = \dfrac{1}{\ln(x + 1)}$
   (f) $y = \log_x \mathrm{e}^2$

2. Find the stationary point of the graph of $y = \ln\left(x + \dfrac{1}{x}\right)$.

3. Find the equation of the normal to the curve $y = 4 + \ln(x+1)$ at the point where $x = 0$.

4. Find the equation of the tangent to the curve $y = \dfrac{2\mathrm{e}}{\ln(x+\mathrm{e})}$ at the point where $x = 0$.

5. Find the coordinates of the stationary point of the curve $y = \ln\left(x^2 + 6x + 10\right)$ and show that it is a minimum.

6. It is given that $f(x) = \dfrac{x^2 - 1}{x}$, for $x > 0$.

   (a) Find its range.

   (b) Find its inverse function $f^{-1}(x)$.

   (c) Find the equation of the tangent to the curve of f^{-1} at the point $(0, 1)$.

Now we consider trigonometric functions.

The derivative of $y = \sin x$ is, by definition,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \sin x \cdot \left(\lim_{h\to 0} \frac{\cos h - 1}{h}\right) + \cos x \cdot \left(\lim_{h\to 0} \frac{\sin h}{h}\right).
\end{aligned}$$

When h is positive but sufficiently close to 0,

$$\sin h < h < \tan h = \frac{\sin h}{\cos h}.$$

Therefore,

$$\cos h < \frac{\sin h}{h} < 1,$$

for small values of h.

As $h \to 0$, $\cos h \to 1$, thus $\lim_{h\to 0} \frac{\sin h}{h} = 1$.

Hence

$$\begin{aligned}
\lim_{h\to 0} \frac{\cos h - 1}{h} &= \lim_{h\to 0} \frac{\cos h - 1}{h} \cdot \frac{\cos h + 1}{\cos h + 1} \\
&= \lim_{h\to 0} \frac{-\sin^2 h}{h(\cos h + 1)} \\
&= \left(\lim_{h\to 0} \frac{\sin h}{h}\right) \cdot \left(\lim_{h\to 0} \frac{-\sin h}{\cos h + 1}\right) \\
&= 1 \cdot 0 \\
&= 0.
\end{aligned}$$

Therefore, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x.$$

Similarly,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \cos x \cdot \left(\lim_{h\to 0} \frac{\cos h - 1}{h}\right) - \sin x \cdot \left(\lim_{h\to 0} \frac{\sin h}{h}\right) \\
&= -\sin x.
\end{aligned}$$

An alternative is to consider $\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$, then apply chain rule. Try to work out full details by yourself:

$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) =$

Find the following derivatives:

- $\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) =$

- $\frac{\mathrm{d}}{\mathrm{d}x}(\csc x) =$

Exercise 24.

1. Differentiate the following functions.

   (a) $y = \sin(x^2)$

   (b) $y = \sqrt{\cos x + 1}$

   (c) $y = \mathrm{e}^{3\sin 2x}$

   (d) $y = \ln \sec x$

   (e) $y = \sin\left(\mathrm{e}^{2x}\right)$

   (f) $y = \cos^2 x$

   (g) $y = \sin^2 x$

   Explain the relation between the derivatives of the last two functions.

2. Find the equation of the tangent where $x = \frac{1}{4}\pi$ on the curve $y = \ln \sec x$.

3. Find the equation of the normal where $x = \frac{1}{6}\pi$ on the curve $y = 2\sin 3x - 3\cos^2 x$.

4. The equation of a curve is $y = x + 2\cos x$, where $0 \le x \le 2\pi$. Find the x-coordinates of the stationary points of the curve, and determine the nature of each of these stationary points.

5. Differentiate the function $y = \cos^4\left(x + \frac{1}{4}\pi\right)$, simplifying your answer.

6. Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{\cos x}\right) = \dfrac{1}{\csc x - \sin x}$.

   Hence find an expression for $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}(\sec x)$, simplifying your answer.

7. Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\ln \tan x) = 2 \csc 2x$.

8. Let $f(x) = \sin \frac{1}{2}x - \cos \frac{1}{3}x$.

   (a) Find $f'(x)$.

   (b) Find the values of $'(0)$ and $f'(0)$.

   (c) State a value of x, other than 0, such that $f(x) = f(0)$ and $f'(x) = f'(0)$.

   (d) Hence suggest the period of $f(x)$.

9. Differentiate the inverse trigonometric functions: $y = \sin^{-1} x$ and $y = \cos^{-1} x$.
   Explain why these two derivatives only differ by a negative sign.

10. Differentiate the following functions, and (†) sketch their graphs.

    (a) $y = \cos\sqrt{x}$, for $x > 0$

    (b) $y = \sqrt{\cos x}$, for $0 < x < \frac{1}{2}\pi$

    (c) $y = \sin x^2$

    (d) $y = \sin \dfrac{1}{x}$

4.2 Product and quotient rules

Consider a function $y = f(x)$ which can be written as a product of two, usually simpler, functions: $f(x) = u(x)v(x)$.

By definition,

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \underline{\qquad\qquad\qquad\qquad\qquad\qquad} \\
&= \lim_{h \to 0} \frac{u(x+h)v(x+h) + [-u(x+h)v(x) + u(x+h)v(x)] - u(x)v(x)}{h} \\
&= \lim_{h \to 0} \left[u(x+h) \cdot \frac{v(x+h) - v(x)}{h} \right] + \lim_{h \to 0} \left[\frac{u(x+h)v(x) - u(x)}{h} \cdot v(x) \right] \\
&= \underline{\qquad\qquad\qquad} + \underline{\qquad\qquad\qquad}.
\end{aligned}
$$

Therefore we conclude the **product rule** of differentiation, in short notation, as

$$(uv)' = u'v + uv'.$$

If instead we are given a quotient: $y = f(x) = \dfrac{u}{v}$, then we may apply both the chain rule and the product rule to obtain the following **quotient rule**

$$
\begin{aligned}
f'(x) &= \frac{\mathrm{d}}{\mathrm{d}t} \left(u(x) \cdot \frac{1}{v(x)} \right) \\
&= u'(x) \cdot \frac{1}{v(x)} + u(x) \cdot \left(-\frac{1}{v^2(x)} \right) v'(x) \\
&= \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)},
\end{aligned}
$$

or written in the simple form as

$$\left(\frac{u}{v} \right)' = \frac{u'v - uv'}{v^2}.$$

Exercise 25.

1. Differentiate the following functions with respect to x.
   (a) $y = \tan x$
   (b) $y = \cot x$
   (c) $y = x^2 \ln x$
   (d) $y = \tan\left(x^2 + 1\right)$
   (e) $y = \dfrac{\mathrm{e}^{2x} + 1}{x}$
   (f) $y = \left(x^2 - 2\right) \sin x^2$

2. Find the maximum and minimum point(s) of the function $y = \left(x^2 - 3\right) \mathrm{e}^x$.

3. Given that $f(x) = \mathrm{e}^x \sin x$, prove that its fourth derivative satisfies: $f''''(x) = -4f(x)$.

4. Find the coordinates of the maximum point of the curve $y = \dfrac{\ln x}{x}$, where $x > 1$.

5. Find the coordinates of the maximum point of the curve $y = \dfrac{x}{x^2 + 1}$, where $x > 0$.

6. The curve with equation $y = \mathrm{e}^{-x}\sin x$ has one stationary point for which $0 \leq x \leq \pi$.

   (a) Find the x-coordinate of this point.

   (b) Determine whether this point is a maximum or a minimum point.

7. Given that $f(x) = u_1(x)u_2(x)\cdots u_k(x)$, derive a formula for the derivative $f'(x)$.

8. Given that $f(x) = u(x)v(x)$, derive a formula for the second derivative $f''(x)$, the third derivative $f'''(x)$, and more generally, the nth derivative $\frac{\mathrm{d}^n}{\mathrm{d}x^n}f(x)$, in terms of derivatives of $u(x)$ and $v(x)$.

Exercise 26.

(†) Enjoy the following challenges.

1. (a) Given the function $y = x \sin x$, sketch its graph.
   Find all points on its graph where the tangent to the curve passes through the origin.
   (b) Sketch the graph of $y = x \sin \dfrac{1}{x}$, noticing its behavior as $|x| \to \infty$.

2. By first taking logarithms, find the derivative of the function $y = x^4 \left(x^2 - 2\right)^3 e^{2x+1} \cos \sqrt{x}$.

3. Prove that the rectangle of greatest perimeter which can be inscribed in a given circle is a square.
   The result changes if, instead of maximizing the sum of lengths of sides of the rectangle, we seek to maximize the sum of nth powers of the lengths of those sides for $n \geq 2$. What happens if $n = 2$? What happens if $n = 3$? Justify your answers.

4. Let
$$f(x) = ax - \frac{x^3}{1 + x^2},$$
   where a is a constant. Show that, if $a \geq \frac{9}{8}$, then $f'(x) \geq 0$ for all x.

5. Let $f(x) = x^m (x-1)^n$, where m and n are both integers greater than 1. Show that the curve $y = f(x)$ has a stationary point with $0 < x < 1$. By considering $f''(x)$, show that this stationary point is a maximum if n is even and a minimum if n is odd. Sketch the graphs of $f(x)$ in the four cases that arise according to the values of m and n.

6. A function $f(x)$ is said to be **convex** in the interval $a < x < b$ if $f''(x) \geq 0$ for all x in this interval.
   (a) Sketch on the same axes the graphs of $y = \frac{2}{3} \cos^2 x$ and $y = \sin x$ in the interval $0 \leq x \leq 2\pi$.
   The function $f(x)$ is defined for $0 < x < 2\pi$ by $f(x) = e^{\frac{2}{3} \sin x}$. Determine the intervals in which $f(x)$ is convex.
   (b) The function $g(x)$ is defined for $0 < x < \frac{1}{2}\pi$ by $g(x) = e^{-k \tan x}$. If $k = \sin 2\alpha$ and $0 < alpha < \frac{1}{4}\pi$, show that $g(x)$ is convex in the interval $0 < x < \alpha$, and give one other interval in which $g(x)$ is convex.

7. The curve $y = \left(\dfrac{x-a}{x-b}\right) e^x$, where a and b are constants, has two stationary points. Show that
$$a - b < 0 \quad \text{or} \quad a - b > 4.$$
   (a) Show that, in the case $a = 0$ and $b = \frac{1}{2}$, there is one stationary point on either side of the curve's vertical asymptote, and sketch the curve.
   (b) Sketch the curve in the case $a = \frac{9}{2}$ and $b = 0$.

8. Prove the identity
$$4 \sin\theta \sin\left(\tfrac{\pi}{3} - \theta\right) \sin\left(\tfrac{\pi}{3} + \theta\right) = \sin 3\theta. \qquad (*)$$
   (a) By differentiating $(*)$, show that
$$\cot \tfrac{1}{9}\pi - \cot \tfrac{2}{9}\pi + \cot \tfrac{4}{9}\pi = \sqrt{3}.$$
   (b) By setting $\theta = \frac{1}{6}\pi - \phi$ in $(*)$, obtain a similar identity for $\cos 3\theta$ and deduce that
$$\cot\theta \cot\left(\tfrac{\pi}{3} - \theta\right) \cot\left(\tfrac{\pi}{3} + \theta\right) = \cot 3\theta.$$
   Show that
$$\csc \tfrac{1}{9}\pi - \csc \tfrac{5}{9}\pi + \csc \tfrac{7}{9}\pi = 2\sqrt{3}.$$

9. An accurate clock has an hour hand of length a and a minute hand of length b (where $b > a$), both measured from the pivot at the center of the clock face. Let x be the distance between the ends of the hands when the angle between the hands is θ, where $0 \leq \theta < \pi$.
   Show that the rate of increase of x is greatest when $x = (b^2 - a^2)^{\frac{1}{2}}$.
   In the case when $b = 2a$ and the clock starts at mid-day (with both hands pointing vertically upwards), show that this occurs for the first time a little less than 11 minutes later.

4.3 Parametric equations

If a parametric equation is given as

$$x = \phi(t); \qquad y = \psi(t),$$

then,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} \qquad \Rightarrow \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Note that a parametric equation may not qualify as a function.

Find the equation of the tangent to the curve defined as

$$x = 3\cos t; \qquad y = 4\sin t,$$

at the point where $t = \frac{1}{6}\pi$.

We first apply the formula to find the derivative,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4\cos t}{-3\sin t} = -\frac{4}{3\tan t}.$$

When $t = \frac{\pi}{6}$, $x = \frac{3\sqrt{3}}{2}$, $y = 2$, and $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=\frac{\pi}{4}} = -\frac{4}{\sqrt{3}}$.

Thus the equation of the desired tangent is

$$y - 2 = -\frac{4}{\sqrt{3}}\left(x - \tfrac{3\sqrt{3}}{2}\right),$$

or

$$y = -\tfrac{4}{\sqrt{3}}x + 8.$$

Exercise 27.

1. Find the equation of the normal at $(-8, 4)$ to the curve which is given parametrically by $x = t^3$, $y = t^2$. Sketch the curve, showing the normal.

2. A curve is given parametrically by $x = a\cos^3 t$, $y = a\sin^3 t$, where a is a positive constant, for $0 \le t < 2\pi$. The tangent at any point P meets the x-axis at A and the y-axis at B. Prove that the length of AB is constant.

3. The parametric equations of a curve are $x = t + \mathrm{e}^{-t}$, $y = 1 - \mathrm{e}^{-t}$, where $t \in \mathbb{R}$.
Express $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t, and hence find the value of t for which the gradient of the curve is 2, giving your answer in the logarithmic form.

4. A curve has parametric form: $x = 1 + \frac{1}{t}$, $y = t^3\mathrm{e}^{-t}$, where $t \neq 0$. Find the coordinates of the points on the curve where the tangent is parallel to the axes.

5. The parametric equations of the **cycloid** are: $x = a(t - \sin t)$, $y = a(1 - \cos t)$, where a is a positive constant and t the parameter. Prove that the cycloid satisfies the equation:

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{2a}{y} - 1.$$

6. A curve is defined by the parametric equations: $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$.

   (a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at the point where $t = 2$.

   (b) (†) Find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at the point where $t = 2$.

7. (†) The distinct points P and Q, with coordinates $(ap^2, 2ap)$ and $(aq^2, 2aq)$ respectively, lie on the curve $y^2 = 4ax$. The tangents to the curve at P and Q meet at the point T. Show that T has coordinates $(apq, a(p+q))$. The point F has coordinates $(a, 0)$ and ϕ is the angle TFP. Show that

$$\cos\phi = \frac{pq + 1}{\sqrt{(p^2 + 1)(q^2 + 1)}}$$

and deduce that the line FT bisects the angle PFQ.

4.4 Differentiating implicit functions

The function $y = \arctan x$ is equivalent to $x = \tan y$, where x and y are well defined. Taking derivatives with respect to x gives:

$$1 = \sec^2 y \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\therefore \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sec^2 y} = \frac{1}{\tan^2 y + 1} = \frac{1}{x^2 + 1}.$$

Therefore we may conclude:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1} x\right) = \frac{\mathrm{d}}{\mathrm{d}x}(\arctan x) = \frac{1}{x^2 + 1}.$$

The curve is defined implicitly as

$$x \sin y = \frac{1}{2}.$$

Find its gradient at the point $\left(1, \frac{1}{6}\pi\right)$.

Taking derivatives with respect to x to obtain:

$$\sin y + x \cos y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Hence, $\displaystyle\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\tan y}{x}$, and $\displaystyle\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\left(1, \frac{1}{6}\pi\right)} = -\tfrac{1}{\sqrt{3}}$.

Exercise 28.

1. Find the gradient of the normal to the curve $4x \arctan x + y \cos y = 0$ at the point $(1, \pi)$.

2. Find the equation of the tangent to the curve $x^2 - 2xy + 2y^2 = 5$ at the point $(1, 2)$.

3. Find the points of the curve $3x^2 - 2xy + y^2 = 18$ at which the tangent is either horizontal or vertical.

4. Find the stationary points of the curve defined implicitly by the equation $y - 2x^2 = xy^2$.
Then determine the nature of the stationary points.

5. Find the coordinates of the point where the normal to the curve $xy + y^2 = 2x$ at the point $(1, 1)$ meets the curve again.

6. The curve C, whose equation is $x^2 + y^2 = \mathrm{e}^{x+y} - 1$, passes through the origin O. Find the values of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at O.

7. Given that $\mathrm{e}^y = \mathrm{e}^x + \mathrm{e}^{-x}$, show that $\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 4\mathrm{e}^{-2y} = 1$.

8. A curve has equation $x \ln y = y^2$. Find the coordinates of the point on the curve where the tangent is vertical. Also show that the tangent to the curve is never horizontal.

9. A curve has equation $x^4 + xy + y^2 = 13$. Find the coordinates of the stationary points, and determine their nature.

10. (a) (†) Show that the gradient of the curve $\dfrac{a}{x} + \dfrac{b}{y} = 1$, where $b \neq 0$, is $-\dfrac{ay^2}{bx^2}$.

    The point (p, q) lies on both the straight line $ax + by = 1$ and the curve $\dfrac{a}{x} + \dfrac{b}{y} = 1$, where $ab \neq 0$. Given that, at this point, the line and the curve have the same gradient, show that $p = \pm q$.
    Show further that either $(a - b)^2 = 1$ or $(a + b)^2 = 1$.

    (b) (†) Show that if the straight line $ax + by = 1$, where $ab \neq 0$, is a normal to the curve $\dfrac{a}{x} - \dfrac{b}{y} = 1$, then $a^2 - b^2 = \frac{1}{2}$.