1 Velocity and acceleration

1.1 Terminologies

Note the following terms:

- velocity, displacement
- SI units
- displacement-time graph
- velocity-time graph

Exercise 1

1. Light travels at a speed of 3.00 $\times 10^8$ m s^{-1}. Light from the star Sirius takes 8.65 years to reach the earth. What is the distance of Sirius from the earth in kilometres?
2. The speed limit on a motorway is 120 km per hour. What is this in SI units?
3. An aircraft flies due east at 800 km per hour from Kingston to Antigua, a displacement of about 1600 km. Model the flight by drawing
   (a) a displacement-time graph,
   (b) a velocity-time graph.

   Label your graphs to show the numbers 800 and 1600 and to indicate the units used.

1.2 Equations for constant acceleration

The rate at which the velocity increases is called the ________________.

The velocity-time graph:

Equations for constant acceleration:

Exercise 2

1. A racing car enters the final straight travelling at 35 $\mathrm{m\,s^{-1}}$, and covers the 600 m to the finishing line in 12 s. Assuming constant acceleration, find its speed as it crosses the finishing line.
2. A cyclist reaches the top of a slope with a speed of 1.5 $\mathrm{m\,s^{-1}}$, and accelerates at 2 $\mathrm{m\,s^{-2}}$. The slope is 22 m long. How long does she take to reache the bottom of the slope, and how fast is she moving then?
3. A train is travelling at 80 $\mathrm{m\,s^{-1}}$ when the driver applies the brakes, producing a deceleration of 2 $\mathrm{m\,s^{-2}}$ for 30 second. How fast is the train then travelling, and how far does it travel while the brakes are on?
4. A ballon at a height of 300 m is descending at 10 $\mathrm{m\,s^{-1}}$ and decelrating at a rate of 0.4 $\mathrm{m\,s^{-2}}$. How long will it takes for the ballon to stop descending, and what will its height be then?

Exercise 3

1. The barrel of a shotgun is 0.9 m long, and the shot emerges from the muzzle with a speed of 240 $\mathrm{m\,s^{-1}}$. Find the acceleration of the shot in the barrel, and the length of time the shot is in the barrel after firing.

2. The driver of a car travelling at 96 k.p.h. in mist suddenly sees a stationary bus 100 metres ahead. Within the brakes full on, the car can decelerate at 4 $\mathrm{m\,s^{-2}}$ in the prevailing road conditions. Can the driver stop in time?

   In practise, there would be some react time as the driver sees the bus. What is the maximum react time if the driver can avoid the accident?

3. A car travelling at 10 $\mathrm{m\,s^{-1}}$ is 25 metres from a pededtrain crossing when the traffic light changes from green to amber. The light remains at amber for 2 seconds before it changes to red. The driver has two choices: to accelerate so as to reach the crossing before the light changes to red, or to try to stop at the light. What is the least acceleration which would be necessary in the first case, and the least deceleration which would be necessary in the second?

4. A freight train $\frac{1}{4}$ km long takes 20 seconds to pass a signal. The train is descelerating at a constant rate, and by the time the rear truck has passed the signal it is moving 10 kilometers per hour slower than it was when the front of the train passed the signal. Find the deceleration in kilometer-hour units, and the speed at which the train is moving when the rear truck has just passed the signal.

5. A cheetah is pursuing an impala. The impala is running in a straight line at a constant speed of 16 $\mathrm{m\,s^{-1}}$. The cheetah is 10 m behind the impala, running at 20 $\mathrm{m\,s^{-1}}$ but tiring, so that it is decelerating at 1 $\mathrm{m\,s^{-2}}$. Find an expression for the gap between the cheetah and the impala t seconds later. Will the impala get away?

1.3 Multi-stage problems

A journey can often be broken down into several stages, in each of which there is constant velocity, or constant acceleration or deceleration.

For example, a car might accelerate at different rates in different gears, or it might slow down to go through a villiage and then speed up again.

You can analyse situations like these by applying the formulae to each stage separately, or you can use the velocity-time graph.

Graphs with discontinuities: bouncing or being struck

Exercise 4

1. A sprinter in a 100-meter race pushes off the starting block with a speed of 6 $\mathrm{m\,s^{-1}}$, and accelerates at a constant rate. He attains his maximum speed of 10 $\mathrm{m\,s^{-1}}$ after 40 meters, and then continus at that speed for the rest of the race. What is his time for the whole race?

2. Two stops on a tramline are 960 meters apart. A tram starts from one stop, accelerates at a constant rate to its maximum speed of 15 $\mathrm{m\,s^{-1}}$, maintains this speed for some time and then decelerates at a constant rate to come to rest at the other stop. The total time between the stops is 84 seconds.

   (a) For how many seconds does the tram travel at its maximum speed?

   (b) If the tram accelerates at 0.5 $\mathrm{m\,s^{-2}}$, at what rate does it decelerate?

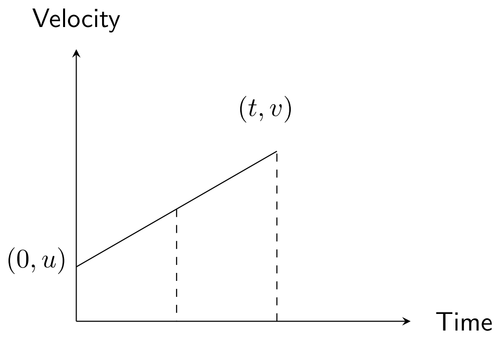
3. A truck is travelling at a constant rate speed of 96 k.p.h. The driver of a car, also going at 96 k.p.h., decides to overtake it. The car accelerates up to 120 k.p.h., then immediately starts to decelerates until its speed has again dropped to 96 k.p.h. The whole manoeuvre takes half a minute. If the gap between the car and the truck was originally 35 metres, the truck is 10 metres long and the car is 4 metres long, what will be the gap between the truck and the car afterwards?

1.4 Average velocity

The constant acceleration formula $s = \frac{1}{2}(u+v)t$ can be rearranged as

$$\frac{s}{t} =$$

The fraction on the left, the displacement divided by the time, is called the ______________________.



Notice that the area under the graph for the first half of the period is less than the area of the second half.

Exercise 5

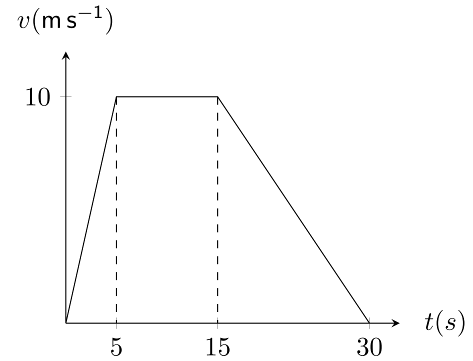
1. A passenger notices that a train covers 4 km in 3 minutes, and 2 km in the next minute. Assuming that the acceleration is constant, find how fast the train is travelling at the end of the fourth minute.
2. A cyclist travels from A to B, a distance of 240 metres. He passes A at 12 m s^{-1}, maintains this speed for as long as he can, and then brakes so that he comes to a stop at B. If the maximum deceleration he can achieve when braking is 3 m s^{-2}, what is the least time in which he can get from A to B?
3. A car comes to a stop from a speed of 30 m s^{-1} in a distance of 804 m. The driver brakes so as to produce a deceleration of $\frac{1}{2}$ m s^{-2} to begin with, and then brakes harder to produce a deceleration of $\frac{3}{2}$ m s^{-2}. Find the speed of the car at the instant when the deceleration is increased, and the total time the car takes to stop.
4. A cyclist is free-wheeling down a long straight hill. The times between passing successive kilometer post are 100 seconds and 80 seconds. Assuming his acceleration is constant, find this accelertion.
5. A particle is moving along a straight line with constant acceleration. In an interval of T seconds it moves D meters, in the next interval of $3T$ seconds it moves $9D$ metres. How far does it move in a further interval of T seconds.

Miscellaneous exercise 1

1. As a car passes the point A on a straight road, its speed is 10 m s^{-1}. The car moves with constant acceleration a m s^{-2} along the road for T seconds until it reaches the point B, where its speed is V m s^{-1}. The car travels at this speed for a further 10 seconds, when it reaches the point C. From C it travels for a further T seconds with constant acceleration 3a m s^{-2} until it reaches a speed of 20 m s^{-1} at the point D. Sketch the (t, v) graph for the motion, and show that $V = 12.5$.

   Given that the distance between A and D is 675 m, find the value of a and T.

2. The figure shows the (t, v) graph for the motion of a cyclist; the graph consists of three straight line segments. Use the information given on the graph to find the acceleration of the cyclist when $t = 2$ and the total distance travelled by the cyclist for $0 \leq t \leq 30$. Without making any detailed calculations, sketch the displacement-time graph for this motion.



3. A woman stands on the bank of a frozen lake with a dog by her side. She skims a bone across the ice at a speed of 3 m s^{-1}. The bone slows down with deceleration 0.4 m s^{-2}, and the dog chases it with acceleration 0.6 m s^{-2}. How far out from the bank does the dog catch up with the bone?

4. A ball is released from rest 20 m above the ground and accelerates under gravity at 10 m s^{-2}. When it bounces its speed halves. If bounce n occurs at time t_n the speed after the bounce is v_n. Show that $v_n = 15 - 2.5t_n$ and deduce that, despite infinitely many bounces, the ball stops bouncing after 6 s.

5. Two trains are travelling towards each other, one heading north at a constant speed of u m s^{-1} and the other heading south at a constant speed of v m s^{-1}. When the trains are a distance d m aprat, a fly leaves the northbound train at a constant speed of w m s^{-1}. As soon as it reaches the other train, in instantly turns back travelling at w m s^{-1} in the other direction. Show that the fly meets the southbound train having travelled a distance of $\frac{wd}{w+v}$ and returns to the northbound train when the train has travelled a distance of $\frac{2uwd}{(w+v)(w+u)}$.

6. Two cars are on the same straight road, the first one s m ahead of the second and travelling in the same direction. The first car is moving at initial spped v m s^{-1} away from the second car. The second car is moving at initial speed u m s^{-1}, where $u > v$. Both cars decelerates at a constant rate of a m s^{-2}.
   (a) Show that the second car overtakes at time $t = \frac{s}{u-v}$ irrespective of the deceleration, provided the cars do not come to rest before the second one passes.
   (b) Show also that the distance from the starting point of the second car to the point where it overtakes depends on a and find a formula for that distance.

Exam-style Questions 1

1. A car starts from rest and moves in a straight line from point A with constant acceleration 3 m s^{-2} for 10 s. The car then travels at constant speed for 30 s before decelerating uniformly, coming to rest at point B. The distance AB is 1.5 km.

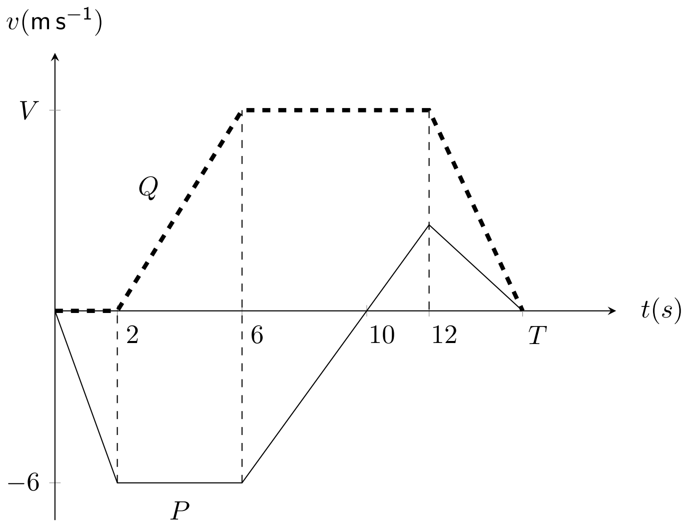
   (i) Find the total distance travelled in the first 40 s of motion. [3]

   When the car has been moving for 20 s, a motorcycle starts from rest and accelerates uniformly in a straight line from point A to a speed V m s^{-1}. It then maintains this speed for 30 s before decelerating uniformly to rest at point B. The motorcycle comes to rest at the same time as the car.

   (ii) Given that the magnitude of the acceleration a m s^{-2} of the motorcycle is three times the magnitude of its deceleration, find the value of a. [6]

   (iii) Sketch the displacement-time graph for the motion of the car. [3]

2. The diagram shows the velocity-time graphs for two particles, P and Q, which are moving in the same straight line. The graph for P consists of four straight line segments. The graph for Q consists of three straight line segments. Both particles start from the same initial position O on the line. Q starts 2 seconds after P and both particles come to rest at time $t = T$. The greatest velocity of Q is V m s^{-1}.



   (i) Find the displacement of P from O at $t = 10$. [1]

   (ii) Find the velocity of P at $t = 12$. [2]

   (iii) Given that the total distance covered by P during the T seconds of its motion is 49.5 m, find the value of T. [3]

   (iv) Given also that the acceleration of Q from $t = 2$ to $t = 6$ is 1.75 m s^{-2}, find the value of V and hence find the distance between the two particles when they both come to rest at $t = T$. [3]

3. The top of a cliff is 40 metres above the level of the sea. A man in a boat, close to the bottom of the cliff, is in difficulty and fires a distress signal vertically upwards from sea level. Find

   (i) the speed of projection of the signal given that it reaches a height of 5 m above the top of the cliff, [2]

   (ii) the length of time for which the signal is above the level of the top of the cliff. [2]

   The man fires another distress signal vertically upwards from sea level. This signal is above the level of the top of the cliff for $\sqrt{17}$ s.

   (iii) Find the speed of projection of the second signal. [3]

4. A particle starts from rest at a point X and moves in a straight line until, 60 seconds later, it reaches a point Y. At time t s after leaving X, the acceleration of the particle is

0.75m s^{-2}	for	$0 < t < 4$
0m s^{-2}	for	$4 < t < 54$
-0.5m s^{-2}	for	$54 < t < 60$

(i) Find the velocity of the particle when $t = 4$ and $t = 60$, and sketch the velocity-time graph. [5]

(ii) Find the distance XY. [2]

5. Particles P and Q have a total mass of 1 kg. The particles are attached to opposite ends of a light inextensible string which passes over a smooth fixed pulley. P is held at rest and Q hangs freely, with both straight parts of the string vertical. Both particles are at a height of h m above the floor (see Figure.1). P is released from rest and the particles start to move with the string taut. Figure.2 shows the velocity-time graphs for P's motion and for Q's motion, where the positive direction for velocity is vertically upwards. Find

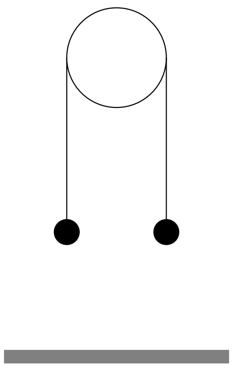


Figure 1

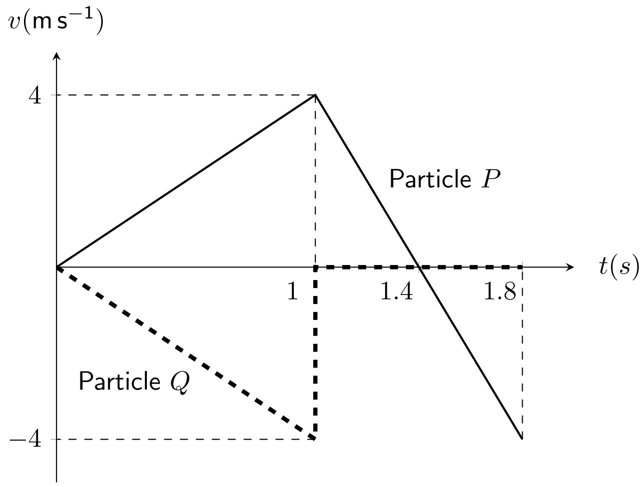


Figure 2

(i) the magnitude of the acceleration with which the particles start to move and the mass of each of the particles, [5]

(ii) the value of h, [1]

(iii) the greatest height above the floor reached by particle P. [2]

2 Force and motion in one dimension

2.1 Terminologies

Note the following terms:

- Newton's first law
- Newton's second law, SI units
- Tension, compression, thrust, driving force
- Friction, air resistance
- Gravity
- The particle model

Exercise 6

1. In the sports of curling, a stone of mass 18 kg is placed on ice and given a push. If this produces a speed of 2 $\mathrm{m\,s^{-1}}$, and the stone goes 30 metres before coming to rest, calculate the deceleration, and find the frictional force between the stone and the ice.
2. The World's Strongest Man has a cable attached to a harness round his shouleders. The cable is horizontal, and the other end is attached to a 20 tonne truck. The man starts to pull so that the tension in the cable is 800 N. How long will it take for the truck to move 1 metre from rest?
3. A hocky player hits a stationary ball, of mass 0.2 kg. The contact time between the stick and the ball is 0.15 s and the force exerted on the ball by the stick is 60 N. Find the speed with which the ball leaves the stick.
4. A car and driver, of total mass 1350 kg, are moving at 30 $\mathrm{m\,s^{-1}}$ on a horizontal road when the driver sees roadworks 400 m ahead. She brakes, decelerating with a constant force of 600 N until arriving at the roadworks. Find the time elapsed before arriving at the roadwoks.
5. A block is being dragged along a horizontal surface by a constant horizontal force of size 45 N. It covers 8 m in the first 2 s and 8.5 m in the next 1 s. Find the mass of the block.

2.2 Forces acting together

If several forces act on an object **parallel** to a given direction, then the ________________ is the sum of the forces in that direction minus the sum of the forces in the opposite direction.

If the ________________ is zero, the forces on the object are said to be in ________________. The object them remains ________________, or moves with ________________. (Newton's first law)

The ________________ is equal to the product of the mass of the object and its acceleration in the given direction. (Newton's second law)

Always draw a **force diagram**. In mechanics your normally draw a circle or rectangles and show forces as arrows going out form the object. Accelerations are shown beside the diagram using a double arrow.

Exercise 7

1. A heavy box of mass 32 kg has a handle on one side. Two children try to move it across the floor. One pulls horizontally on the handle with a force of 20 N, the other pushes from the other side of the box with a force of 25 N, but the box does not move. Find the frictional force resisting the motion.
2. A wagon of mass 250 kg is pulled by a horizontal cable along a straight level track against a resisting force of 150 N. The wagon starts from rest. After 10 seconds it has covered a distance of 60 m. Find the tension in the cable.
3. A small boat of mass 90 kg is moved across a horizontal beach at a steady speed of 2 $\mathrm{m\,s^{-1}}$. One of the crew pulls with a force of P newtons, the other pushes with a force of $(P + 15)$ newtons. The frictional force resisting the motion is 105 newtons. Find P.
4. One horse pulls, with a force of X N, a cart of mass 800 kg along a horizontal road at constant speed. Three horses, eaching pulling a force of X N, give the cart an acceleration of 0.8 $\mathrm{m\,s^{-2}}$. Find the time it would take two horses to increase the speed of the cart from 2 $\mathrm{m\,s^{-1}}$ to 5 $\mathrm{m\,s^{-1}}$, given that each horse pulls with a force of X N, and that the resistance to motion has the same constant value at all times.
5. A car of mass 1400 kg slows down from 30 $\mathrm{m\,s^{-1}}$ to 20 $\mathrm{m\,s^{-1}}$ when the driver sees a sign for reduced speed limit 400 m ahead. There is air resistance of 1000 N. Determine whether the driver needs to provide a braking force or just reduce the amount of driving force exerted, and find the size of the force.

2.3 Weight and motion due to gravity

All objects, when dropped, fall towards the earth in a vertical line with the same constant acceleration, provided that there is no air resistance. This acceleration is called the ________________.

In this book, we use the simpler value ___________ for g.

The weight of an object on or near the surface of the earth is the force of gravity with which the earth attracts it, so is measured in newtons. Further more, the weight of an object of mass m is given by :

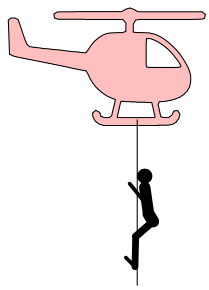
$$W =$$

Exercise 8

1. An injured sailor is being winched up to rescue helicopter. The mass of the sailor is 55 kg. Find the tension in the cable when the sailor is being raised

   (a) at a steady speed of 4 $\mathrm{m\,s^{-1}}$,

   (b) with an acceleration of 0.8 $\mathrm{m\,s^{-2}}$.



2. Machinery of tatal mass 280 kg is being lowered to the bottom of a mine by means of two ropes attached to a cage of mass 20 kg. For the first 3 seconds of the descent, the tension in each rope is 900 N. Then for a further 16 seconds, the tension in each rope is 1500 N. For the final 8 seconds, the tension in each rope is 1725 N. Find the depth of the mine.

3. A load of weight 7 kN is being raised from rest with constant acceleration by a cable. After the load has been raised 20 metres, the cable suddenly becomes slack. The load continues upwards for a distance of 4 metres before coming to instantaneous rest. Assuming no air resistance, find the tension in the cable before it became slack.

Exercise 9

1. (a) A ball of mass 0.2 kg is thrown vertically upwards out of a window 4 m above the ground. The ball is released with speed 8 m s^{-1}. Assuming there is no air resistance, find how long it takes to hit the ground.

   (b) If instead there is a constant air resistance of 0.1 N against the direction of motion, find how long the ball takes to hit the ground.

2. (a) A ball is dropped from a height of 30 m above the ground. Two seconds later, another ball is thrown upwards from the ground with a speed of 5 m s^{-1}. They collide at a time t s after the first ball was dropped. Find t.

   (b) The ball collide at a height h m above the ground. Find h.

3. A parachutist of mass 70 kg falls out of an aeroplane from a height of 2000 m and falls under gravity until 600 m from the ground when he opens his parachute. The parachute provides a resistance of 2330 N. Find the speed at which the parachuist is travelling when he reaches the ground.

4. A pebble is dropped from rest into a deep well. At time t s later it splashes into the water at the bottom of the well. Sound travels at 340 m s^{-1} and is heard at the top of the well 5 s after the pebble was released. Find the depth of the well.

5. A ball of mass 2 kg is projected up in the air from ground level with speed 20 m s^{-1}. It experiences constant air resistance R. It returns to ground level with speed 15 m s^{-1}. Find R.

2.4 Normal contact force

When an object is in contact with a surface, there is a force on the object at right angles to the region of contact. This is called the ________________, or the ________________.

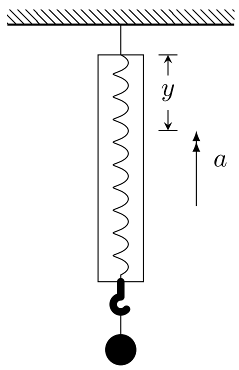
For a rounded object like a pebble or a ball there may be only one point of contact with the surface, rather than a region. The surface is then a tangent plane to the object, and the contact force is at right angles to this plane.

Difference between weight and mass:

Exercise 10

1. A mechanics students lives on the tenth floor of a tall building. She has just bought new bathroom scales, and decides to try them out by standing on them as she goes up in the lift. Initially the scales read 50 kg. After the doors have closed the reading briefly goes up to 60 kg, but them returns to 50 kg. As the lift nears the tenth floor, the reading drops to 35 kg. Explain.

2. A heavy mass m kg is suspended from the roof of a lift by a wire. The wire is cut, and a spring balance is inserted between the two free ends. When the lift is accelerating upwards at a m s^{-2}, the reading on the balance is y kg. Find the equation connecting a and y.



3. A man of mass M kg and his son of mass m kg are standing in a lift. When the lift is accelerating upwards with magnitude 1 m s^{-2} the magnitude of the normal contact force exerted on the man by the lift floor is 880 N. When the lift is moving with constant speed the combined magnitude of the normal contact forces exerted on the man and the boy by the lift floor is 1000 N. Find the values of M and m.

4. (a) A crane is lifting a pallet on that rests a stone block of mass 5 kg. The motion is vertically upwards. The crane lifts the pallet from rest to a speed of 3 m s^{-1} in 6 m. Find the normal contact force on the stone block during the acceleration.

   (b) If the normal contact force exceeds 650 N, the pallet may break and so this situation is considered unsafe. Assuming the same acceleration as in part (4a), find how many stone blocks the crane can lift safely.

Miscellaneous exercise 2

1. A ballon of total mass 680 kg is descending with a constant acceleration of 0.4 $\mathrm{m\,s^{-2}}$. Find the upthrust acting on the ballon. When the ballon is moving at 1.5 $\mathrm{m\,s^{-1}}$, enough ballst is released for the ballon to fall with a deceleration of 0.2 $\mathrm{m\,s^{-2}}$. Calculate

   (a) how much ballst was released.

   (b) the time for which the ballon continues to fall before it begins to rise.

2. A box of weight W rests on a platform. When the platform is moving upwards with acceleration a, the normal contact force from the platform on the box has magnitude kW. When the platform is moving downwards with acceleration $2a$, the box remains in contact with it. Find the normal contact force in terms of k and W, and deduce that $k < \frac{3}{2}$.

3. An acrobat of mass m slides down a vertical rope of height h. For the first three-quarters of her descent she grips the rope with her hands and legs so as to produce a frictional force equal to five-ninths of her weight. She then tightens her grip so that she comes to rest at the bottom of the rope. Sketch a (t, v) graph to illustrate her descent, and find the frictional force she must produce in the last quarter. If the rope is 60 m high, calculate

   (a) her greatest speed,

   (b) the time she takes to descend.

4. A car of mass 350 kg is travelling 30 $\mathrm{m\,s^{-1}}$ when it starts to slow down, 100 m from a junction. At first, it slows just using the air resistance of 200 N. Then, at a distance of s m from the junction, it slows using brakes, providing a force of 2000 N as well as the air resistance. Find the distance from the junction at which the brakes must be applied if the car is to stop at the junction.

5. A boy drags a cart of mass 5 kg with force 10 N along a horizontal road. There is air resistance of 2 N. At some point the boy let go of the cart and the cart slows down due to air resistance until coming to rest. In total, the cart has travelled 36 m. Find the length of time the boy was dragging the cart.

6. An air hockey table is 2 m long. A puck of mass 50 g is on the table at the middle point. A player hits the puck with initial speed 4 $\mathrm{m\,s^{-1}}$ directly towards one side. Once it is moving there is air resistance of R N. Every time the puck hits a side, the speed is reduced by 20%.

   (a) Show that if $R < \frac{32}{205}$, the puck returns past the middle point of the table.

   (b) Given that the puck does not return to the middle point a second time, find a lower bound for R.

Exam-style Questions 2

1. A particle P is projected vertically upwards, from a point O, with a velocity of 8 m s^{-1}. The point A is the highest point reached by P. Find

   (i) the speed of P when it is at the mid-point of OA, [4]

   (ii) the time taken for P to reach the mid-point of OA while moving upwards. [2]

2. A ball A is released from rest at the top of a tall tower. One second later, another ball B is projected vertically upwards from ground level near the bottom of the tower with a speed of 20 m s^{-1}. The two balls are at the same height 1.5 s after ball B is projected.

   (i) Show that the height of the tower is 50 m. [3]

   (ii) Find the length of time for which ball B has been in motion when ball A reaches the ground. Hence find the total distance travelled by ball B up to the instant when ball A reaches the ground. [5]

3. A particle P is projected vertically upwards from horizontal ground with speed 12 m s^{-1}

   (i) Find the time taken for P to return to the ground. [2]

   The time in seconds after P is projected is denoted by t. When $t = 1$, a second particle Q is projected vertically upwards with speed 10 m s^{-1} from a point which is 5 m above the ground. Particles P and Q move in different vertical lines.

   (ii) Find the set of values of t for which the two particles are moving in the same direction. [4]

4. A particle of mass 3 kg falls from rest at a point 5 m above the surface of a liquid which is in a container. There is no instantaneous change in speed of the particle as it enters the liquid. The depth of the liquid in the container is 4 m. The downward acceleration of the particle while it is moving in the liquid is 5.5 m s^{-2}.

   (i) Find the resistance to motion of the particle while it is moving in the liquid. [2]

   (ii) Sketch the velocity-time graph for the motion of the particle, from the time it starts to move until the time it reaches the bottom of the container. Show on your sketch the velocity and the time when the particle enters the liquid, and when the particle reaches the bottom of the container. [7]

5. A particle P_1 is projected vertically upwards, from horizontal ground, with a speed of 30 m s^{-1}. At the same instant another particle P_2 is projected vertically upwards from the top of a tower of height 25 m, with a speed of 10 m s^{-1}. Find

   (i) the time for which P_1 is higher than the top of the tower, [3]

   (ii) the velocities of the particles at the instant when the particles are at the same height, [5]

   (iii) the time for which P_1 is higher than P_2 and is moving upwards. [3]

3 Resolving forces in two dimensions

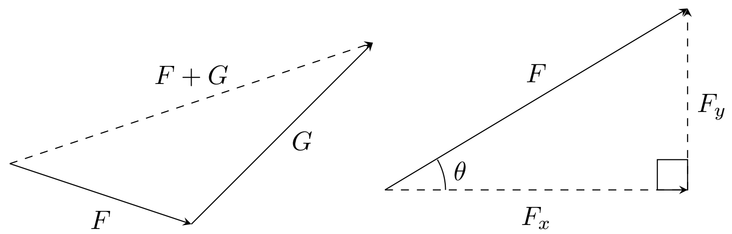
3.1 Resolving forces horizontally and vertically in equilibrium problems

A force is a vector quantity.

- When vectors are added it is equivalent of joining one vector on to the ___________ of the other.

- In the reverse we can split a vector into the sum of two others called ___________.

- For convenience, we choose the two vectors to be in perpendicular directions, in particular, horizontal and vertical directions.



Component are not extra forces.

Equations can be formed by find the net compoent horizontally and vertically. This is called ___________ the force, a convenient shorthand for this is write $\mathcal{R}(\rightarrow)$ and $\mathcal{R}(\uparrow)$.

Exercise 11

1. A particle of mass 4 kg is held in place by a force of magnitude 100 N acting at an angle θ above the horizontal and a horizontal force of F N. Find the value of θ and F.

2. A boat is held in place by a force of 5 N due east, a force of 10 N due south and a force F N, on a bearing of θ. Find the value of F and θ.

3. A box of mass 15 kg is dragged along the floor at a constant speed of 1.2 m s^{-1} by means of a rope at 30° to the horizontal. Then tension from the rope is 50 N. Calculate the frictional force resisting the motion and the normal contact force from the floor.

4. A small child is strapped into the seat of a swing which is supported by two ropes. To start her off, her father pulls the swing back with a horizontal force, so that the ropes make an angle of 20° with the vertical. The child and swing together have mass 18 kg. Calculate the tension in each rope, and the force exerted by her father before he lets go.

Exercise 12

1. A lamp is supported in equilibrium by two chains fixed to two points A and B at the same level. The lengths of the chains are 0.3 m and 0.4 m and the distance between A and B is 0.5 m. Given that the tension in the longer chain is 36 N, show by resolving horizontally that the tension in the shorter chain is 48 N. By resolving vertically, find the mass of the lamp.

2. A concrete slab of mass m kg is being raised vertically, at a constant speed, by two cables. One of the cables is inclined at $10°$ to the vertical and has a tension of 2800 N, the other cable has a tension of 2400 N. Calculate the angle at which the cable is inclined to the vertical, and also find the value of m, assuming there is no air resistance.

3. A man is pulling a chest of mass 40 kg along a horizontal floor with a force of 140 N inclined at $30°$ to the horizontal. His daughter is pushing with a force of 50 N directed downwards at $10°$ to the horizontal. The chest is moving with constant speed. Calculate the magnitude, F N, of the frictional force, and the magnitude, R N, of the normal contact force from the ground on the chest. Show that the ratio $F : R$ lies between 0.50 and 0.51.

4. A ship is being blown by a breeze with a force of 100 N on a bearing of $280°$. It is pulled by a rope attached to the shore with force 50 N on a bearing of $170°$. A tugboat holds it in place. Find the size and bearing of the force F applied by the tugboat.

5. A wooden block is held in position by three horizontal forces, as shown in Figure 3a . One acts to the left with force 56 N. One acts with force F at an angle θ, where $\sin\theta = \frac{3}{5}$, above the rightwards horizontal. One acts with force G at an angle ϕ, where $\sin\phi = \frac{15}{17}$, below the rightwards horizontal. Find F and G.

6. A particle has three forces actiong on it, as shown in Figure 3b, where $\sin\theta = \frac{3}{5}$. Show that $F + G = 150\sqrt{3}$ by resolving horizontally, and write down another equation by resolving vertically. Hence show that $G = 75\sqrt{3} + 100$ and find F.

7. A particle has three horizontal forces acting on it, as shown in Figure 3c . Show that $\cos\alpha = \frac{14-13\cos\beta}{15}$ and find an expression for $\sin\alpha$. Use $\cos^2\alpha + \sin^2\alpha = 1$ to get an equation in β. Hence, find α and β.

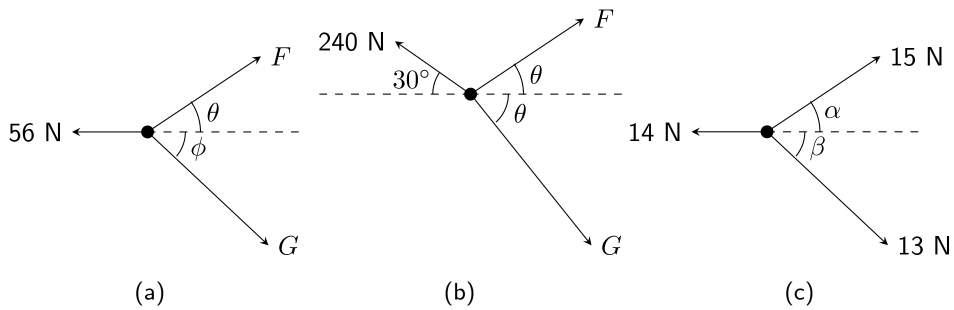


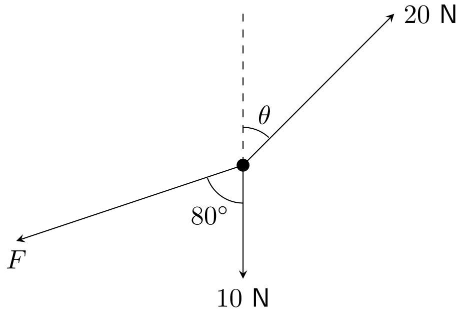
Figure 3

3.2 Resolving forces at other angles in equilibrium problems

You can write equations of resolving in any direction you like, not just horizontally and vertically.

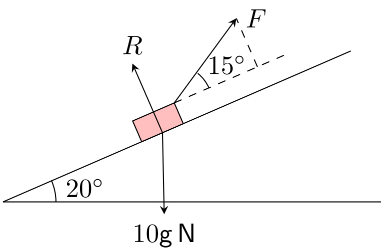
A force has no component in the direction ___________ to its line of action.

Choose the directions carefully so there are as few unknowns as possible in each direction, to make solving the equations easier. Here is one example:



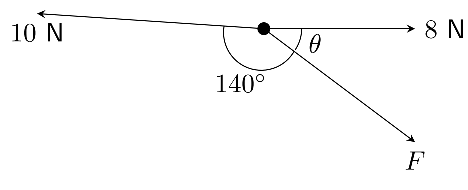
In problems that involve a slope, you should resolve forces ___________ and ___________ to the slope.

Here is one example: A block of mass 10 kg is held in equilibrium on a slope at an angle of $20°$ to the horizontal by a force, F, acting at $15°$ above the slope. Find F and the normal contact force.



Exercise 13

1. A particle has three forces actiong on it, as shown in the diagram. By resolving perpendicular to and parallel to F, find F and θ.



2. A boat is held in equilibrium by two tugboats. One pulls with a force of 100 N on a bearing of $190°$. One pulls on a bearing of $340°$ with tension T. The wind blows with a force on the boat of F on a bearing of $50°$. By resolving perpendicular to T, find T. Find also F.
3. A wooden block of mass 4 kg is held at rest on a slope at angle θ to the horizontal by a force of 12 N acting up the slope and parallel to it. Find the slope's angle and the normal contact force.
4. A girl is dragging a sled of mass 20 kg up a slope at angle $14°$ to the horizontal. She pulls at an angle of θ above the slope with a force of 70 N. She maintains a constant speed despite friction of 10 N parallel to the slope. Find θ and the normal contact force.

3.3 The triangle of forces and Lami's theorem ⋇

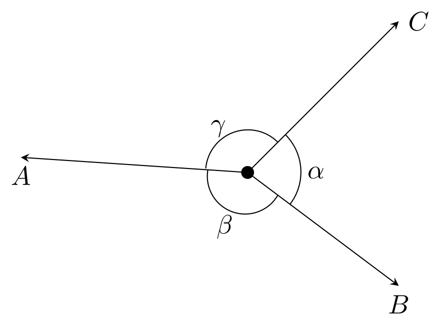
If three forces act on an object to keep it in equilibrium, they will have no resultant.

Draw them end to end will form a triangle.

Sine rule:

Cosine rule:

Lami's theorem: for a particle in equilibrium with three forces on it, the ratio of the magnitude of the force with the sine of the angle between the other two forces is the same for each force.



Exercise 14

1. An object is in equilibrium by the action of forces of 10 N, 8 N and 9 N, as shown in the Figure 4. Find the value of θ and ϕ.
2. Figure 5 shows that a ship is held in equilibrium by ropes on bearings of $120°$ and $220°$. The wind is blowing due north and exerting a force of 90 N on the ship. Find the tensions in the two ropes.
3. A particle is held in equilibrium by three forcces, as shown in the Figure 6. Find the size of F and α.
4. A box has two ropes holding it in place. It is pushed by a force of 10 N. The angles between the force and the ropes are $120°$ and $150°$. Find the tension in the ropes.
5. Four forces on an object, A, B, C and D, result in no net force. If the angle between forces A and B is α and the angle between forces C and D is γ, show that $A^2+B^2+2AB\cos\alpha = C^2+D^2+2CD\cos\gamma$.

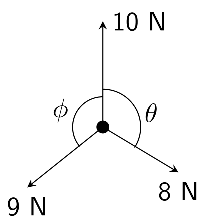


Figure 4

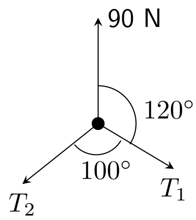


Figure 5

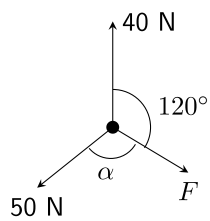
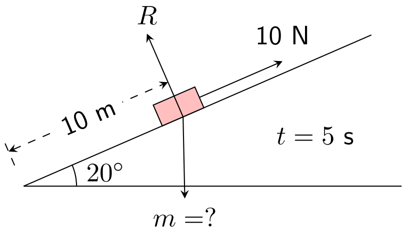


Figure 6

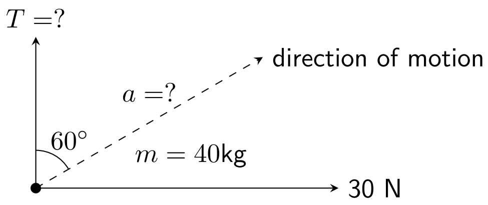
3.4 Non-equilibrium problems for objects on slopes and known directions of acceleration

The net force in the direction ___________ to the acceleration is zero.

For objects on a slope, any acceleration will be___________ to the slope, either up or down it, we should resolve in directions perpendicular and parallel to the slope.

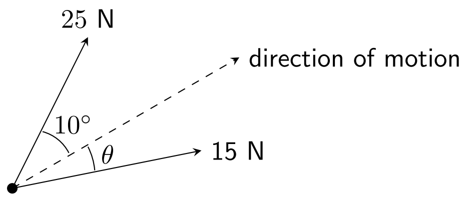


Furthermore, if the direction of motion is known, then there will be no acceleration ___________ to the direction of motion, so we should resolve in directions perpendicular and parallel to the motion.



Exercise 15

1. A ship of mass 10000 kg is being towed due north by two tugboats with acceleration 0.1 m s^{-2}. One pulls with a tension of 2000 N on a bearing of $330°$. The other pulls with a tension, T, on a bearing of θ. There is resistance against the motion of 1000 N. Find T and θ.
2. A log of mass 200 kg is dragged up a slope at an angle of $13°$ to the horizontal by a rope attached to a truck. The rope is at an angle of $20°$ above the slope. The log accelerates at 0.3 m s^{-2}. Find the tension in the rope.
3. A buoy of mass 12 kg is on the surface of a lake. The tide pushes it with a force of 25 N and the wind pushes it with a force of 15 N, as shown in the diagram. The buoy moves in the direction shown. Find the value of θ and the acceleration.



4. A cyclist of mass 70 kg(including her bicycle) arrives at an uphill stretch of road of length 30 m with an angle $9°$ to the horizontal, travelling at 10 m s^{-1}. She exerts a force of 15 N parallel to the slope and there is wind resistance of 5 N against her. Find the time taken to reach the top of the slope.
5. A ball of mass m kg is rolled up a slope at an angle θ to the horizontal, where $\sin\theta = \frac{2}{5}$. The ball passes a point A with speed 7 m s^{-1}. A point B is 5 m further up the slope than point A. Find the time between passing B on the way up and returning to B on the way down.

3.5 Non-equilibrium problems and finding resultant forces and directions acceleration

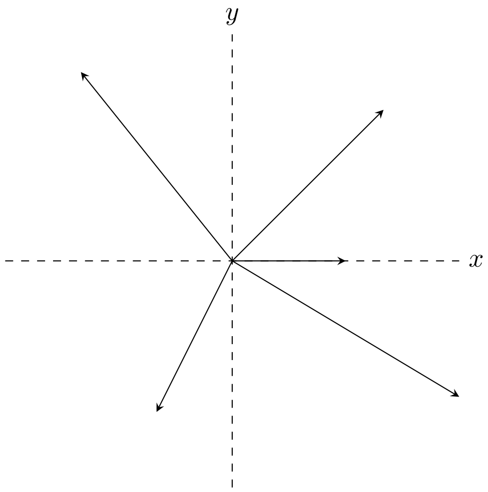
A common way of dealing thess problems is

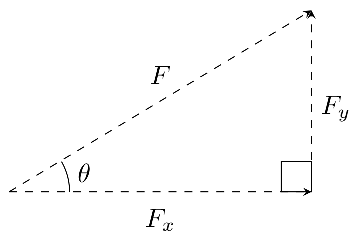
Step 1: Resolve forces horizontally and vertically. (Similarly, along the slope and perpendicular to the slope)

Stpe 2: Adding these components separately, we get ____________ and ____________.

Stpe 3: The magnitude of the resultant force, F, is given by $F =$ ____________

Step 4: The direction of the resultant force, is given by $\theta =$ __________________.





Exercise 16

1. A boat of mass 100 kg experiences a force of 30 N eastward from the wind and a force of 40 N from the tide on a bearing of $35°$, as shown in Figure 7. Find the direction of the subsequent motion and the acceleration.
2. A particle of mass 3 kg is attached to three ropes in the horizontal plane with forces of 2 N, 4 N and 3 N, as shown in Figure 8. Find the acceleration of the subsequent motion and the acceleration.

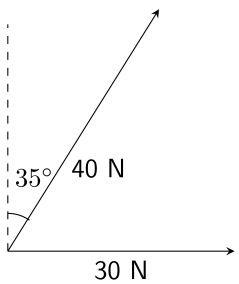


Figure 7

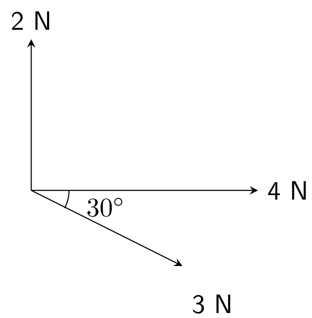
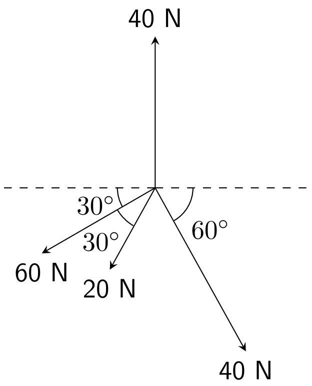


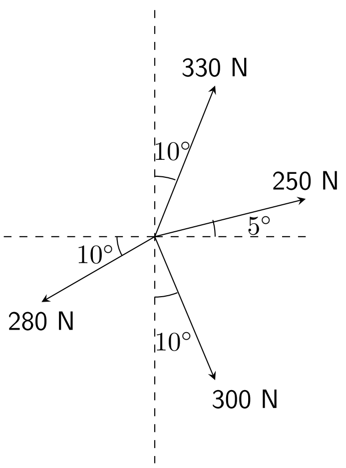
Figure 8

Exercise 17

1. Three coplanar forces act on a particle, as shown in the following diagram. Show that the x component and y component of the resultant are equal. Hence, determine the direction of the resultant.



2. In a competition of strength, four people pull a mass with ropes at different angles. The direction in which the mass moves determines the winner. Arjun wants the mass to go north, Bob wants it to go east, Chen wants it to go south and David wants it to go west. The men pull will the forces in the directions shown in the diagram. Find the direction of the resultant motion and determine who wins.



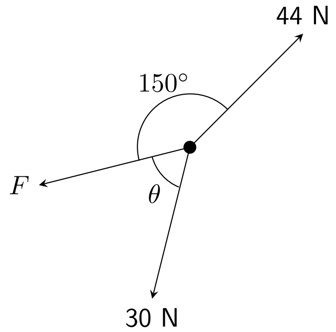
3. A rowing boat of mass 120 kg is being pulled from the rest by three boats. One pulls north with a force of 100 N, the second pulls on a bearing of $020°$ with a force of 80 N and the third pulls on a bearing of $045°$ with a force of 90 N. There is resistance from the water of 200 N directly against the motion. Find the bearing and acceleration of the resultant motion.

4. The wind is blowing a boat with force F. The motor of the boat can exert a driving force of D N, where $D < F$. Show with a diagram that, whatever direction the wind is taking the boat with the motor switched off, the motor is capable of deflecting the direction by a maximum of $\sin^{-1} \frac{D}{F}$.

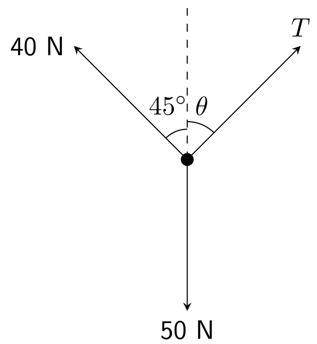
5. A building is unstable after a natural disaster. A car is stuck under the building and needs to be dragged out as quickly as possible, although the exact direction is less important. Three people can pull ropes, one due north, one at a bearing of $010°$ and one at a bearing of $030°$. Akhil can pull with a force of 300 N, Ben can pull with a force of 240 N and Khadijah can pull with a force of 210 N. Find who should pull each rope to maximise the acceleration and what the net force will be.

Miscellaneous exercise 3

1. Three forces act on a particle in equilibrium in the horizontal plane, as shown in Figure . By resolving in a direction perpendicular to F, show that $\theta = 47.2°$ and find F.



2. A boat is in equilibrium held by a rope to the shore. The rope exerts a force T at an angle θ from north. The wind blows the boat with force 40 N in a northwest direction. The current pushes it south with a force of 50 N. Show that $T \sin\theta = 20\sqrt{2}$ and find an expression for $T \cos\theta$. Hence, show that $\tan\theta = \frac{8+10\sqrt{2}}{17}$ and find θ and T.



3. Three boys are having a strength competiton. They hold ropes attached to the same object of mass 10 kg. One pulls due north with force 32 N and another pulls on a bearing of $200°$ with force 45 N. The third wants to make the object accelerate due east and pulls with a force of 24 N.
   (a) Find the bearing at which the third boy should pull.
   (b) Find the resultant acceleration.

4. A girl can drag a stone block of mass 18 kg up a slope at an angle of $13°$ to the horizontal with an acceleration of 0.7 m s^{-2}. Assuming this is the maximum force she can exert to drag the block, find the mass of the heaviest stone block she would be able to drag up the slope.

5. The four atheletes in a bobsleigh team start the race by running along the ice. They push for 40 m on a horizontal track, providing an average horizontal force of 180 N each. The total mass of the bobsleight and the four athletes is 600 kg.
   (a) Find the speed a the end of the horizontal stretch of track.

   The athletes then get into the bobsleigh. The track continues with a downhill stretch of length 1300 m on a slope at an angle of $5°$ to the horizontal. There is air resistance of 175 N.

   (b) Find the total time to complete the entire track.

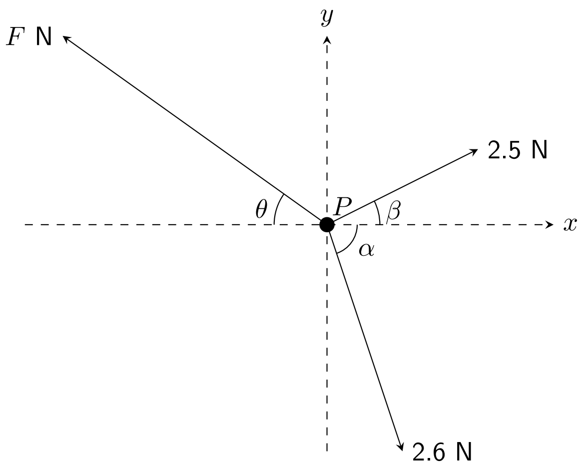
6. A ball of mass m kg slides down a slope, which is at an angle of θ to the horizontal. It passes two light gates x m apart. At the first gate, the speed of the ball is measured as u m s^{-1}, and at the second its speed is measured as v m s^{-1}. Assuming the resistance is constant, show the resistance force has a total size of $\frac{m}{2x}(2xg\sin\theta + u^2 - v^2)$.

Exam-style Questions 3

1. A particle P of mass 0.5 kg lies on a smooth horizontal plane. Horizontal forces of magnitudes F N, 2.5 Nand 2.6 N act on P. The directions of the forces are as shown in the diagram, where $\tan\alpha = \frac{12}{5}$ and $\tan\beta = \frac{7}{24}$.

   (i) Given that P is in equilibrium, find the values of F and $\tan\theta$. [6]

   (ii) The force of magnitude F N is removed. Find the magnitude and direction of the acceleration with which P starts to move. [3]



2. Forces of magnitudes P N and 25 N act at right angles to each other. The resultant of the two forces has magnitude R N and makes an angle of $\theta°$ with the x-axis (see diagram). The force of magnitude P N has components -2.8 N and 9.6 N in the x- direction and the y-direction respectively, and makes an angle of $\alpha°$ with the negative x-axis.

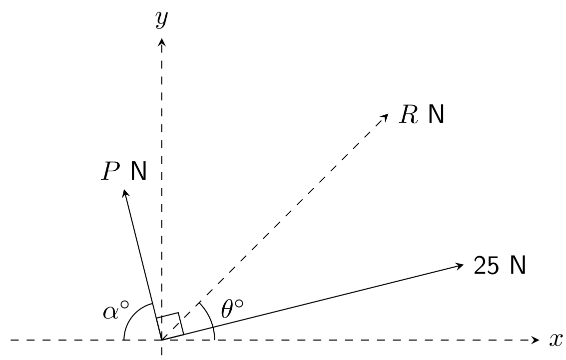
   (i) Find the values of P and R. [3]

   (ii) Find the value of α, and hence find the components of the force of magnitude 25 N in

   (a) the x-direction,

   (b) the y-direction. [4]

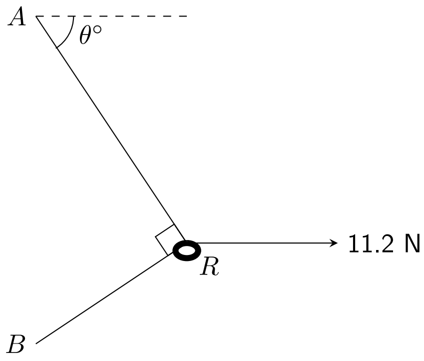
   (iii) Find the value of θ. [3]



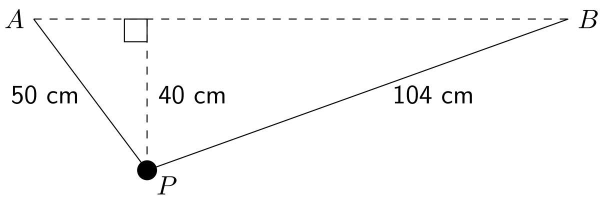
3. A smooth ring R of mass 0.16 kg is threaded on a light inextensible string. The ends of the string are attached to fixed points A and B. A horizontal force of magnitude 11.2 N acts on R, in the same vertical plane as A and B. The ring is in equilibrium. The string is taut with angle $ARB = 90°$, and the part AR of the string makes an angle of $\theta°$ with the horizontal (see diagram). The tension in the string is T N.

(i) Find two simultaneous equations involving $T \sin \theta$ and $T \cos \theta$. [3]

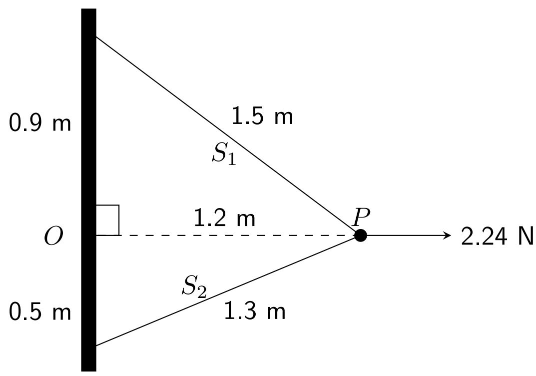
(ii) Hence find T and θ. [3]



4. A particle P of mass 2.1 kg is attached to one end of each of two light inextensible strings. The other ends of the strings are attached to points A and B which are at the same horizontal level. P hangs in equilibrium at a point 40 cm below the level of A and B, and the strings PA and PB have lengths 50 cm and 104 cm respectively (see diagram). Show that the tension in the string PA is 20 N, and find the tension in the string PB. [5]



5. A particle P of weight 1.4 N is attached to one end of a light inextensible string S_1 of length 1.5 m, and to one end of another light inextensible string S_2 of length 1.3 m. The other end of S_1 is attached to a wall at the point 0.9 m vertically above a point O of the wall. The other end of S_2 is attached to the wall at the point 0.5 m vertically below O. The particle is held in equilibrium, at the same horizontal level as O, by a horizontal force of magnitude 2.24 m acting away from the wall and perpendicular to it (see diagram). Find the tensions in the strings.



4 Friction

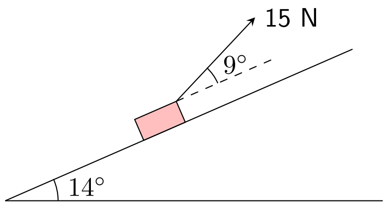
4.1 Basic properties

Note the folloing terms:

- rough, smooth, coefficient of friction
- direction of motion or possible motion
- limiting friction, limiting equilibrium
- total contact force, direction of the total contact force
- line of greatest slope, up or down the slope

Exercise 18

1. A sledge of mass 200 kg is being pulled by a woman along rough horizontal ground. She exerts a force of 500 N at $18°$ above the horizontal and the sledge is on the point of slipping. Find the coefficient of friction.
2. A waster container of mass 400 kg is in equilibrium on a rough slope at an angle of $18°$ to the horizontal. The coefficient of friction between the slope and the skip is 0.3. It is held in equilibrium by a winch with tension T N. Find the range of possible values for T.
3. A book of mass 4 kg is at rest on a rough slope at angle $14°$ to the horizontal. Find the magnitude of the total contact force in each of these cases.
   (a) No other force acts on the book.
   (b) The book is pulled down the slope by a force of 5 N parallel to the line of greatest slope
   (c) The book is pulled up the slope by a force of 15 N at $9°$ above the line of greatest slope, as shown in the diagram.



4. Two men are trying to drag a bin of mass 100 kg up a rough slope at an angle $20°$ to the horizontal. The coefficient of friction is 0.25. One man pulls up the slope with a force of 400 N. The other tries to lift the bin perpendicular to the slope, providing a force such that the bin is on the point of slipping up the slope. Find the force exerted by the second man.

Exercise 19

1. A box of mass 30 kg is at rest on a rough slope at an angle of 20° to the horizontal. When a girl pushes up the slope along the line of greatest slope with a force of 25 N, the box does not slip down. Find the range of values for the coefficient of friction between the box and the slope.

2. A box of mass 50 kg is at rest on a slope, which is at an angle of 26° to the horizontal. The coefficient of friction is 0.4. The box is held in place by a rope attached to a winch pulling up the slope and parallel to it. Find the minimum and maximum possible values for the tension, T, which the winch could provide for the box to remain in equilibrium.

3. A car of mass 1350 kg is at rest on a rough slope at an angle of 7° to the horizontal. A man tries to push it down the slope, exerting a force of 500 N, but cannot get it to move.

   (a) Find the angle that the total contact force makes with the slope.

   (b) When the man stops pushing, the car remains in equilibrium. Find the angle that the total contact force makes with the slope.

4. A ring of mass 3 kg is at rest on a rough horizontal wire. It is attached to a string that is at an angle of 60° above the horizontal. The coefficient of friction between the ring and the wire is 0.7. Find the set of values for the tension, T, which will allow the ring to remain in equilibrium.

4.2 Limiting friction

Case 1: if motion is taking place (whether ________________ or with ___________), friction is **limiting** and in a direction opposite to the direction of motion.

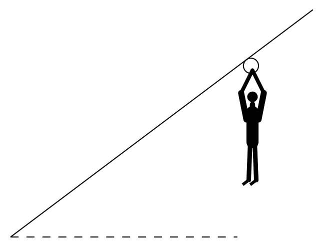
Case 2: if the object is on the point of moving, friction is **limiting** and in a direction opposite to that in which the object is about to move.

If friction is limiting, then $F = \mu R$.

Whether friction is limiting or not, $F \leqslant \mu R$.

Exercise 20

1. A bag of sand of mass 200 kg is being winched up a slope of length 10 m, which is at angle of $6°$ to the horizontal. The slope is rough and the coefficient of friction is 0.4. The winch provides a force of 1000 N parallel to the slope. At the bottom of the slope the bag is moving at 2 m s^{-1}. Find the distance it has moved when its speed has reduce to 1.5 m s^{-1}.
2. A snooker ball of mass 0.4 kg is struck towards a cushion from 0.8 m away with speed 3 m s^{-1}. The surface of the snooker table has a coefficient of friction of 0.3. When the ball bounces from the cushion its speed is reduced by 20%. Find how far from the cushion it stops.
3. A wooden block of mass 10 kg is on rough horizontal ground with coefficient of friction 0.6. It is dragged by force of 80 N acting at $15°$ to the horizontal.
   (a) Find the acceleration if the force is above the horizontal.
   (b) Find the acceleration if the force is below the horizontal.
4. Part of an army assult course consists of a taut cable 25 metres long fixed at $35°$ to the horizontal. A light rope ring is placed round the cable at its upper end. A solider of mass 80 kg grabs hold of the ring and slides down the cable. If the coefficient of friction between the ring and the cable is 0.4, find how fast the solider is moving when he reaches the bottom.



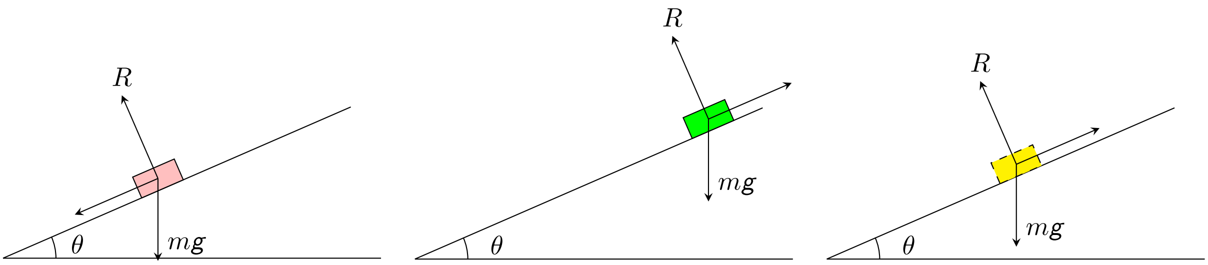
5. A block of weight 200 N is placed on a slope at $\beta°$ to the horizontal, where $\sin\beta° = 0.6$ and $\cos\beta° = 0.8$. It is kept from moving by a horizontal force of P newtons. For different values of the coefficient of friction μ, find the range of possible values of P.

4.3 Change of direction of friction

When an object is moving upwards a slope, the friction will be limiting to start with and act ___________ the slope to stop moving up the slope.

Once the object comes to rest, friction will act ___________ the slope to try to prevent the objects moving back down the slope.

If the force due to gravity is large enough, the object will start moving back down the slope and friction will again become limiting, but will now act ___________ the slope.



Exercise 21

1. A ball of mass 3 kg rolls up a slope with initial speed 10 m s^{-1}. The slope is at an angle of $20°$ to the horizontal and the coefficient of friction is 0.3. By modelling the ball as a particle, find the distance up the slope when the ball comes to rest.

   Show that after coming to rest the ball starts to roll down the slope, and find the speed of the ball when it returns to its starting point.

2. A car of mass 1250 kg is at rest on a rough slope at an angle of $35°$ to the horizontal. It takes a force of 13000 N to move it up the slope. Show that without any force the car would slide dwon the slope, and find the minimum force to prevent it moving down.

3. A pinball game involves hiting a ball up a slope whenever it reaches the bottom the the slope. The pinball has mass 0.2 kg and rolls down a rough slope of length 1.2 m at angle $12°$ to the horizontal and with coefficient of friction 0.1. The ball starts at the top of the slope at rest. When it reaches the bottom of the slope it is hit back up and its speed is increased by 50$\%$.

   (a) Find the maximum height up the slope the pinball reaches after it has been hit back up the slope.

   (b) What assumptions have been made to answer to question.

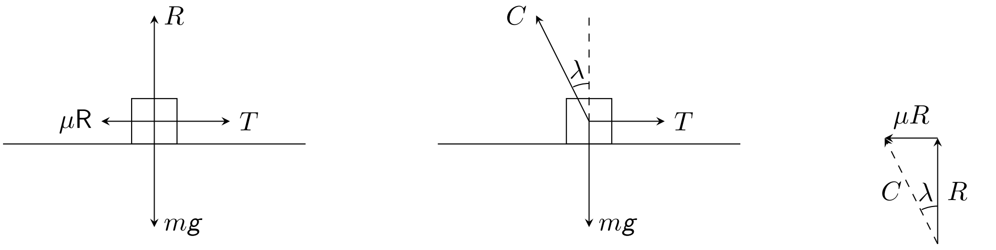
4. A boy drags a sledge of mass 4 kg from rest down a rough slope at an angle of $18°$ to the horizontal. He pulls it with a force of 8 N for 3 s by a rope that is angled at $10°$ above the parallel down the slope. After 3 s the rope becomes detached from the sledge. The coefficient of friction between the slope and the sledge is 0.4. Find the total distance the sledge has moved down the slope from when the boy started dragging it until it comes to rest.

5. A particle slides up a slope at angle $34°$ to the horizontal with coefficient of friction 0.4. It passes a point P on the way up the slope with speed 3 m s^{-1} and passes it on the way down the slope with speed 2 m s^{-1}. Find the coefficient of friction between the particle and the slope.

6. A toy car of mass 80 g rolls from rest 80 cm down a rough slope at an angle of $16°$ to the horizontal. When it hits a rubber barrier at the bottom of the slope it bounces back up the slope with its speed havled, and reaches a height of 10 cm. Find the coefficient of friction between the car and the slope.

4.4 Angle of friction ⋇

When friction is limiting, the angle of friction λ is the angle between the ________ and the ___________ .



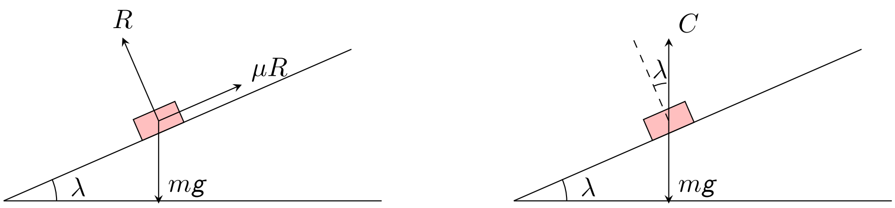
Notice that

$$\tan\lambda = \frac{\mu R}{R} = \mu.$$

Therefore, the angle of friction is related to the coefficient of friction by

$$\lambda = \tan^{-1}\mu.$$

By ways of considering total contact force would reduce forces in some problems, thus easier to deal with.



The angle of friction is the steepest slope which an object can remain at rest without slipping under gravity.

Exercise 22

1. A man tries to drag a suitcase of mass 18 kg along a rough horizontal surface. He drags it with a rope at an angle of $20°$ above the horizontal. The coefficient of friction between the ground and the suitcase is 0.4. The suitcase is in limiting equilibrium. Find the tension in the rope.

2. (a) A 2 kg brick is at rest on a plank. The plank is lifted at one end to make an angle of $20°$ with the horizontal and the brick remains stationary on the plank. Find the total contact force between the brick and the plank.
   (b) The plank is lifted further to an angle of $25°$ and the brick is on the point of slipping down the slope. Find the coefficient of friction between the plank and the brick.
   (c) The plank is lifted further to an angle of $35°$ and the brick is held in place by a force at an angle of $15°$ above the angle of the upwards slope. Find the size of the force.

3. A metal block of mass 20 kg is o n a rough slope at an angle of $12°$ to the horizontal. The coefficient of friction between the book and the slope is 0.4. A boy is trying to move the block up the slope by pushing parallel to the slope. He increases the force until equilibrium breaks. Find the maximum size of the force the boy pushes with before the block slips.

4. A box of mass 12 kg is at rest on a rough horizontal surface with coefficient of friction 0.6. A force is exerted on it at an angle of θ above the horizontal so that the force required to break equilibrium is minimised. Show that θ is the angle of friction and find the size of the force required to break equilibrium.

5. A box has mass 40 kg and is on a rough slope with coefficient of friction 0.3. It is pulled up the slope by a force of 300 N at $10°$ above the slope and is in limiting equilibrium. Find the angle that the slope makes with the horizontal.

Miscellaneous exercise 4

1. A mass of 6 kg is on a slope at an angle of $14°$ to the horizontal. The coefficient of friction between the slope and the mass is 0.4. There is force of 5 N acting down the slope and parallel to it.

   (a) Show that the force is not great enough to overcome friction, and find the magnitude of the total contact force between the mass and the slope.

   (b) When the force of 5 N is removed, find the total contact force and the angle it makes with the slope.

2. A particle of mass 6 kg is on a slope at an angle of $20°$ to the horizontal. The coefficient of friction between the particle and the slope is 0.1. The particle is 5 m from the bottom of the slope. It is projected up the slope with speed 4 $\mathrm{m\,s^{-1}}$.

   (a) Find the distance travelled up the slope from the starting point until the particle comes to rest.

   (b) Find the time the particle reaches the bottom of the slope.

3. A particle of mass 8 kg is at rest on a slope at angle $15°$ to the horizontal. The coefficient of friction between the particle and the slope is 0.05. The particle is pulled up the slope by a rope with tension 30 N at an angle of $20°$ above the line of the slope.

   (a) Find the acceleration of the particle.

   After travelling 10 m the string is cut and there is no tension.

   (b) Find the speed of the particle when the string is cut.

   The particle slows down until coming to rest.

   (c) Find how far the particle has travelled in total when it reaches its highest point on the slope.

   (d) Find the total time until it reaches that point.

4. A valley is formed between two hills, which are at angles of $\alpha°$ and $\beta°$ to the horizontal. A skier starts from rest on the slope of the first hill, at a height h above the valley floor. Find his acceleration down the first hill and his deceleration up the opposite hill, supposing that there is no friction. Show that, if he exerts no force with his ski sticks, he ends up at the same height h as he started.

   Suppose now that $\alpha = 15$, $\beta = 10$ and that the coefficient of friction is 0.1. Show that he ends up at a height of about 0.4h above the valley floor.

5. A mass of m is at rest on a plank of wood on level ground with coefficient of friction μ_1. One end of the plank is lifted until the mass starts to slip. The angle at which this happens is α.

   (a) Show that $\mu_1 = \tan\alpha$.

   The angle of the plank is then raised to an angle β and the mass is held in place. The mass is then released and travels a distance x down the slope. At the end of the slope the particle slides along the level ground, slowing down under friction where the coefficient of friction is μ_2, until coming to rest at a distance y from the bottom of the slope. You may assume the mass starts sliding along the floor at the same speed as it has when it reaches the end of the slope.

   (b) Show that $\mu_2 = \dfrac{x(\sin\beta - \tan\alpha\cos\beta)}{y}$.

Exam-style Questions 4

1. A particle is projected from a point P with initial speed u m s^{-1} up a line of greatest slope PQR of a rough inclined plane. The distances PQ and QR are both equal to 0.8 m. The particle takes 0.6 s to travel from P to Q and 1 s to travel from Q to R.

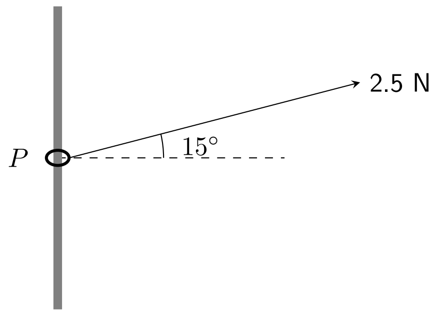
   (i) Show that the deceleration of the particle is $\frac{2}{3}$ m s^{-2} and hence find u, giving your answer as an exact fraction. [6]

   (ii) Given that the plane is inclined at $3°$ to the horizontal, find the value of the coefficient of friction between the particle and the plane. [4]

2. A particle of mass 20 kg is on a rough plane inclined at an angle of $60°$ to the horizontal. Equilibrium is maintained by a force of magnitude P N acting on the particle, in a direction parallel to a line of greatest slope of the plane. The greatest possible value of P is twice the least possible value of P.

   Find the value of the coefficient of friction between the particle and the plane. [7]

3. A small ring P of mass 0.03 kg is threaded on a rough vertical rod. A light inextensible string is attached to the ring and is pulled upwards at an angle of $15°$ to the horizontal. The tension in the string is 2.5 N (see diagram). The ring is in limiting equilibrium and on the point of sliding up the rod. Find the coefficient of friction between the ring and the rod. [4]



4. A particle of mass 18 kg is on a plane inclined at an angle of $30°$ to the horizontal. The particle is projected up a line of greatest slope of the plane with a speed of 20 m s^{-1}.

   (i) Given that the plane is smooth, find the distance the particle moves up the plane before coming to instantaneous rest. [4]

   (ii) Given instead that the plane is rough and the coefficient of friction between the particle and the plane is 0.25, find the speed of the particle as it returns to its starting point. [8]

5. A block of mass 3 kg is initially at rest on a rough horizontal plane. A force of magnitude 6 N is applied to the block at an angle of θ above the horizontal, where $\cos\theta = \frac{24}{25}$. The force is applied for a period of 5 s, during which time the block moves a distance of 4.5 m.

   (i) Find the magnitude of the frictional force on the block. [4]

   (ii) Show that the coefficient of friction between the block and the plane is 0.165, correct to 3 significant figures. [3]

   (iii) When the block has moved a distance of 4.5 m, the force of magnitude 6 N is removed and the block then decelerates to rest. Find the total time for which the block is in motion. [4]

5 Connected particles

5.1 Forces in pairs

Newton's third law :

Normal contact force

Frictional forces

Gravity forces

Exercise 23

1. A pick-up truck of mass 1200 kg tows a trailer of mass 400 kg. There is air resistance of 140 N on the truck, but the resisitance to the motion of the trailer is negligible. A coupling connects the trailer to the truck. Find the force from the coupling, and the driving force on the truck, when the truck and trailer accelerate at 0.5 $\mathrm{m\,s^{-2}}$.
2. A man of weight 750 N tries to push a bookcase of weight 1200 N across the floor. The coefficient of friction between the bookcase and the floor is 0.4. How rough must the contact between his shoes and the floor be for this to be possible?
3. A bar magnet of mass 0.2 kg hangs from a string. A meatal sphere, of mass 0.5 kg, is held underneath the magnet by a magnetic force of 20 N. The string is then pulled upwards with a force of T N. Find the least possible value of T if the sphere is not to separate from the magnet.
4. A student has two books lying flat on the table, one on top of the other. She wants to consult the lower book. To extract it, she pushes it to the left with a force of Q N. To prevent the upper book moving as well, she exerts a force of P N on the upper book to the right. The lower book then slides out, and the upper book remains stationary. The weights of the upper and the lower books are 8 N and 7 N respectively. Between the twoo books the coefficient of friction is 0.25, and between the lower book and the table it is 0.4. Calculate P and Q.

5.2 Strings, rods, pegs and pulleys

Equilibrium of the string

Newton's third law at both ends

Tension:

Thrust (difference between rod and string):

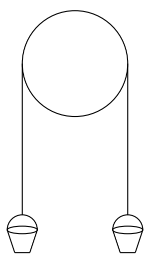
Property of conneted particles:

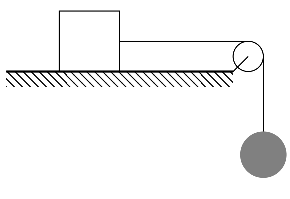
Note: when a string passes round a **smooth** peg, or a **light** pulley with **smooth** bearings, the tension in the string is the same on either side.

Exercise 24

1. Repairs are being carried out in a tall building. A wheel is attached at the top of the scafflolding with its axis horizontal. A rope runs over the rim of the wheel and has buckets of mass 2 kg tied to it at both ends. One bucket is filled with 8.5 kg of rubble and then released, so that it descends to ground leve. With what acceleration does it move.

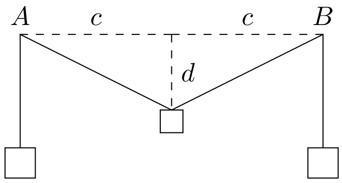
2. A box of mass 2 kg is placed on a table. A string attached to the box passes over a smooth peg at the edge of the table, and a ball of mass 1 kg is tied to the other end. The two straight sections of the string are horizontal and vertical. If the coefficient of friction between the box and the table is 0.2, find the acceleration of the box and the ball.





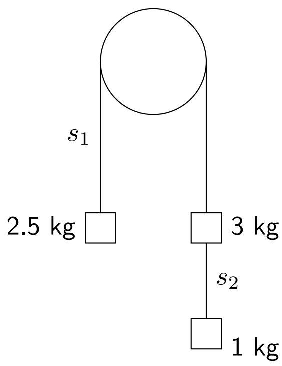
Exercise 25

1. In an apparatus illustrated in Figure a mass m is attached to two strings of equal length, each of which carries a mass M at its other end. The strings are placed symmetrically over smooth nails A and B, which are at the same level. The whole system is in equilibrium. The distance between the nails is $2c$, and the mass m is at a depth d below the mid-point of AB. Find an equation connecting M, m, c and d.



2. On a construction site a truck of mass 400 kg is pulled up a $10°$ slope by a chain. The chain runs parallel to the slope up to the top, where it passes over a cog wheel of negligible mass. It then runs horizontally and is attached to the rear of a locomotive of mass 2000 kg. Neglecting any resistances, calculate the driving force needed to accelerate the truck up the slope at 0.1 m s^{-2}.

3. A particle of mass 3 kg is attached to one end of each of two strings, s_1 and s_2. A particle of mass 2.5 kg is attached to the other end of s_1, and a particle of mass 1 kg is attached to the other end of s_2. The particles are held in the positions shown, with the strings taut and s_1 passing over a smooth fixed peg. The system is released from rest. Find the acceleration of the particles, and the tensions in s_1 and s_2.



4. Two particles are connected by a light inextensible string which passes over a smooth fixed peg. The heavier particle is held so that the string is taut, and the parts of the string not in contact with the pulley are vertical. When the system is released from rest the particles have an acceleration of $\frac{1}{2}g$. Find the ratio of the masses of the particles.

5.3 Internal and external forces

When an object is made up of two parts, each of which has the same velocity and acceleration, you can apply Newton's second law either to the object as a whole or to parts separately.

For the object as a whole, forcees of interaction between the two parts are _________________, and are **NOT** included in the equation.

For the separate parts, the forces of interaction of each on the other are _________________, and are included in the equations.

Exercise 26

1. A dynamo of mass 1500 kg is placed in a cage of mass 500 kg, which is raised vertically by a cable from a crane. The tension in the cable is 20400 N. Find the acceleration of the cage, and the contact force between the cage and the dynamo.
2. Three barges travel down a river in line. Only the rear barge has an engine, which produces a forward force of 400 kN. The masses of the front, middle and rear barges are 1600 tonnes, 1400 tonnes and 2000 tonnes, and the water exerts on them resistances o f 100 kN, 20 kN and 30 kN respectively. Find the forces in the couplings joining the barges.
3. A car of mass M pulls a trailer of mass m down a straight hill which is inclined at angle $\alpha°$ to the horizontal. Resistive forces of magnitudes P and Q act on the car and the trailer respectively, and the driving force on the car is F. Find an expression for the acceleration of the car and trailer, in terms of F, P, Q, M, m and α.

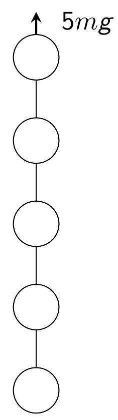
   Show that the tension in the towbar is independent of α.

   If $F = P + Q$, show that the acceleration is $g \sin \alpha°$ and that the tension in the towbar is Q.
4. Five spheres, each of mass m, are joined together by four inextensible strings. The spheres hang in a vertical lines as shown in the diagram, and are held at rest by a force applied to the uppermost sphere, of magnitude 5mg, acting vertically upwards. Find the tension in each of the strings.

   The force on the uppper most sphere is now removed. If the total air resistance acting vertically upwards on the spheres is $\frac{1}{2}mg$, find the acceleration of the system in the subsequent motion. Find also the tension in each of the four strings if

   (a) the air resistance on each individual sphere is $\frac{1}{10}mg$,

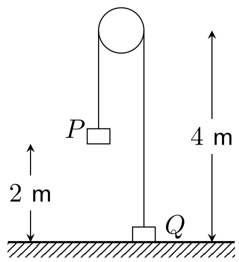
   (b) the air resistance on the uppermost sphere is $\frac{3}{10}mg$ and the air resistance on each of the other four spheres is $\frac{1}{20}mg$.



Miscellaneous exercise 5

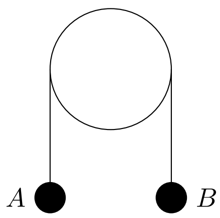
1. Two small bodies P and Q, of masses 6 kg and 2 kg respectively, are attached to the ends of a light inextensible string. The string passes over a pulley fixed at a height of 4 m above the ground. Initially Q is held on the ground and P hangs in equilibrium at a height of 2 m above the ground. Both hanging parts of the string are vertical. Q is released. The modelling assumptions are that there is no air resistance and that the pulley is smooth. Find the speed of Q when P hits the ground, and find also the greatest height above the ground, reached by Q in the subsequent motion. When Q reaches its highest point the string is cut. Find the speed of Q just before it hits the ground.

   Without further calculation, sketch the (t, v) graph of the motion of Q from the start until it hits the ground. Show clearly, by shading, a region on your sketch whose area is equal to the greatest height, above the ground, reached by Q.



2. The diagram shows a light inextensible string passing over a fixed smooth pulley. Particles A and B, of masses 0.03 kg and 0.05 kg respectively, are attached to the ends of the string. The system is held at rest with A and B at the same horizontal level and the string taut. The two parts of the string not in contact with the pulley are vertical. The system is released at time $t = 0$, where t is measured in seconds. The particle B moves downwards for 2 s before being brought to rest as it hits the floor. The string then becomes slack and B remains at rest. Neglecting air resistance, show that the string becomes taut again when $t = 3$.

   Draw, on separate diagrams, the (t, v) graphs for A and B, for $0 \le t \le 3$, clearly indicating the velocity of A when $t = 2$ and when $t = 3$.

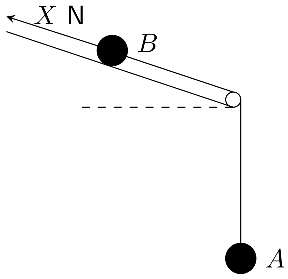


3. Particles A and B, of masses 0.5 kg and 0.8 kg respectively, are joined by a light inextensible string. A is held at rest on a smooth horizontal platform. The string passes over a small pulley at the edge of the platform, and B hangs vertically below the pulley. A is 1.3 m from the pulley. A is released, with the string taut, and the particles start to move. Find the tension in the string, and the speed of A immediately before it reaches the pulley, stating any assumption you make.

   Immediately before A raches the puly it becomes detached from the string. Given that B reaches the floor 1.21 s after the release of A, calculate the initial height of B above the floor.

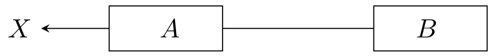
4. Particles A and B, of masses 0.2 kg and 0.1 kg respectively, are joined by a light inextensible string. Particle A is placed on a fixed smooth plane inclined at $10°$ to the horizontal, and is held at rest by a force of magnitude X newtons which acts in a direction parallel to a line of greatest slope of the plane. The string passes over a smooth pulley P fixed at the bottom of the plane, and the part PB of the string hangs vertically, as shown in the diagram. Find X.

   The force of magnitude X newtons is now removed. Ignoring air resistance, find the tension in the string in the subsequent motion.



5. Two trucks A and B, of masses 6000 kg and 4000 kg respectively, are connected by a horizontal coupling. An engine pulls the trucks along a straight horizontal track, exerting a constant horizontal force of magnitude X newtons on truck A. The resistance to motion for truck A may be modelled by a constant horizontal force of magnitude 360 N; for truck B the resistance may be modelled by a constant horizontal force of magnitude 240 N. Given that the tension in the coupling is T newtons and that the acceleratation of the trucks is a m s^{-2}, show that $T = \frac{2}{5}X$, and express a in terms of X. Given that the trucks are slowing down, obtain an inequality satisfied by X.

   The model is changed so that the resistance for truck B is modelled by a constant force of magnitude 200 N. The resistance for truck A remains unchanged. For this changed model find the range of possible values of X for which the force in the coupling is compressive (i.e. the force in the coupling acting on B is directed from A to B).

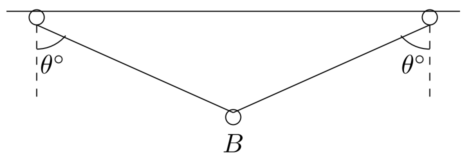


6. A smooth bead B of mass 0.6 kg is threaded on a light inextensible string whose ends are attached to two identical rings, each of mass 0.4 kg. The rings can move on a fixed straight horizontal wire. The system rests in equilibrium with each section of the string making an angle θ with the vertical, as shown in the figure.

   (a) Find the magnitude of the normal contact force exerted on each ring by the wire.

   (b) Find, in terms of θ, the magnitude of the frictional force on each ring.

   Given that the coefficient of friction between each ring and the wire is 0.3, find the greatest possible value of θ for the syetem to be in equilibrium.

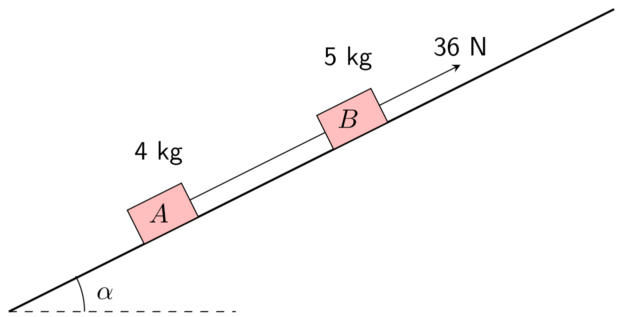


Exam-style Questions 5

1. Two blocks A and B of masses 4 kg and 5 kg respectively are joined by a light inextensible string. The blocks rest on a smooth plane inclined at an angle α to the horizontal, where $\tan\alpha = \frac{7}{24}$. The string is parallel to a line of greatest slope of the plane with B above A. A force of magnitude 36 N acts on B, parallel to a line of greatest slope of the plane (see diagram).

   (i) Find the acceleration of the blocks and the tension in the string. [5]

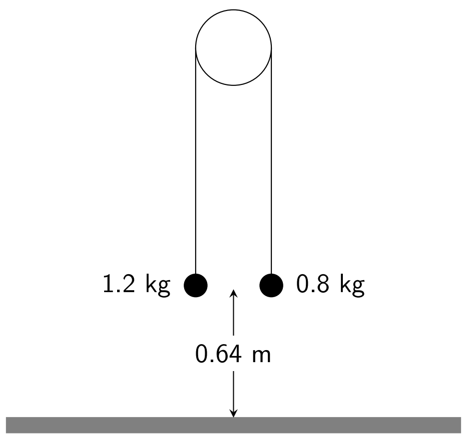
   (ii) At a particular instant, the speed of the blocks is $1 \mathrm{m\,s^{-1}}$. Find the time, after this instant, that it takes for the blocks to travel 0.65 m. [2]



2. Two particles of masses 1.2 kg and 0.8 kg are connected by a light inextensible string that passes over a fixed smooth pulley. The particles hang vertically. The system is released from rest with both particles 0.64 m above the floor (see diagram). In the subsequent motion the 0.8 kg particle does not reach the pulley.

   (i) Show that the acceleration of the particles is 2 $\mathrm{m\,s^{-2}}$ and find the tension in the string. [4]

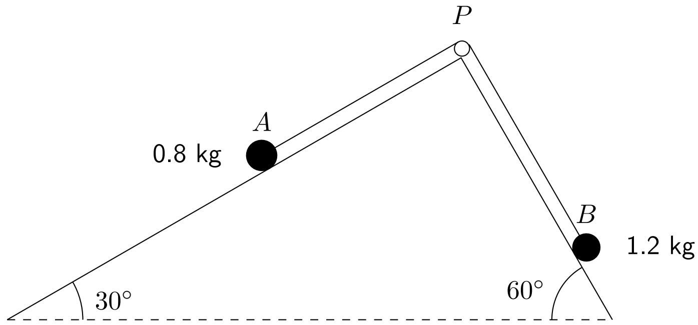
   (ii) Find the total distance travelled by the 0.8 kg particle during the first second after the particles are released. [8]



3. As shown in the diagram, a particle A of mass 0.8 kg lies on a plane inclined at an angle of $30°$ to the horizontal and a particle B of mass 1.2 kg lies on a plane inclined at an angle of $60°$ to the horizontal. The particles are connected by a light inextensible string which passes over a small smooth pulley P fixed at the top of the planes. The parts AP and BP of the string are parallel to lines of greatest slope of the respective planes. The particles are released from rest with both parts of the string taut.

   (i) Given that both planes are smooth, find the acceleration of A and the tension in the string. [6]

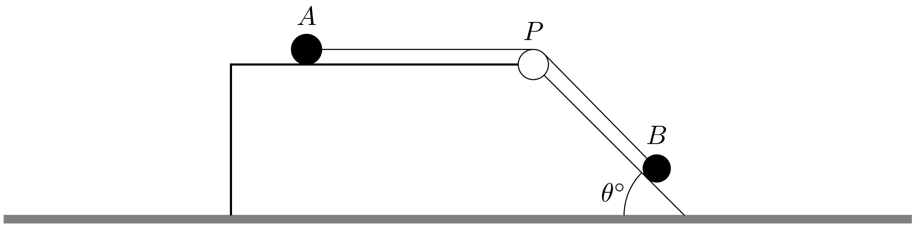
   (ii) It is given instead that both planes are rough, with the same coefficient of friction, μ, for both particles. Find the value of μ for which the system is in limiting equilibrium. [6]



4. The diagram shows a fixed block with a horizontal top surface and a surface which is inclined at an angle of θ to the horizontal, where $\sin\theta = \frac{3}{5}$ A particle A of mass 0.3 kg rests on the horizontal surface and is attached to one end of a light inextensible string. The string passes over a small smooth pulley P fixed at the edge of the block. The other end of the string is attached to a particle B of mass 1.5 kg which rests on the sloping surface of the block. The system is released from rest with the string taut.

   (i) Suppose the block is smooth, find the acceleration of particle A and the tension in the string. [5]

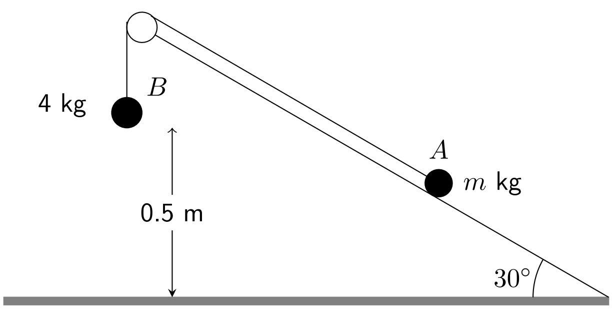
   (ii) It is given instead that the block is rough. The coefficient of friction between A and the block is μ and the coefficient of friction between B and the block is also μ. In the first 3 seconds of the motion, A does not reach P and B does not reach the bottom of the sloping surface. The speed of the particles after 3 s is 5 m s^{-1}. Find the acceleration of particle A and the value of μ. [9]



5. Two particles A and B of masses m kg and 4 kg respectively are connected by a light inextensible string that passes over a fixed smooth pulley. Particle A is on a rough fixed slope which is at an angle of $30°$ to the horizontal ground. Particle B hangs vertically below the pulley and is 0.5 m above the ground (see diagram). The coefficient of friction between the slope and particle A is 0.2.

   (i) In the case where the system is in equilibrium with particle A on the point of moving directly up the slope, show that $m = 5.94$, correct to 3 significant figures. [6]

   (ii) In the case where $m = 3$, the system is released from rest with the string taut. Find the total distance travelled by A before coming to instantaneous rest. You may assume that A does not reach the pulley. [8]



6 General motion in a straight line

6.1 Displacement, velocity and acceleration

Velocity is the rate of change of ___________. i.e. the derivative of displacement with respect to time:

$$v = \frac{\mathrm{d}s}{\mathrm{d}t}$$

Acceleration is the rate of change of ___________. i.e. the derivative of velocity with respect to time.

Acceleration is also the second derivative of displacement with respect to time.

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$$

Exercise 27

1. A ball moves in a straight line so that its displacement, s m, at time t s is given by $s = 2t^3 - 10t^2$. Find its speed when $t = 2$.
2. A particle moves forwards and backwards along a straight line so that its displacement, s meters from the initial position, at time t seconds is given by $s = 2t^3 - 12t^2 + 18t$. Find the distance that it travels in the first 5 s.
3. A car moves in a straight line so that its velocity, v m s^{-1}, at time t s is given by $v = 5t^2 - t^3$, for $0 < t < 4$. Find its acceleration when $t = 2$.
4. A particle moves in a straight line so that its displacement, s m, at time t s $(0 \le t \le 10)$ is given by $s = \frac{1}{3}t^3 - 6t^2 + 15t$.
   (a) Sketch the shape of the velocity-time graph for the particle.
   (b) Hence, find the maximum speed of the particle.
5. A robot moves along a straight line for 3 s. The displacement of the robot, s m, from its initial position is given by $s = At^3 + Bt^2 + Ct$ for constants A, B and C, where t is measured in seconds and $0 < t < 3$. The robot starts with velocity 2 m s^{-1}. It travels 6 m before coming to rest at time $t = 3$. Work out the values of A, B and C.
6. A train is travelling in a straight line. Alice is sitting on the train and is using her mobile phone, which is being tracked. The position of the phone, measured from when the tracking began, is given by $s = t^2 - 2t^3 + 75t$, where s is measured in km and t is measured in hours. The phone is tracked for 2 hours, so $0 < t < 2$. Initially the train speeds up but then it slows down again.
   (a) After how long is the train travelling at its fastest speed?
   (b) Find the maximum velocity.
   (c) Find how far the train travels before it starts to slow down.

6.2 The reverse problem

We know how to find the acceleration, and therefore the force, if we have the displacement-time graph.

But more often, you want to work the other way around, that is, if the force is known, then we can find how fast an object is moving, and how far it has travelled, after a given time.

$$v = \int a \, \mathrm{d}t \qquad \text{and} \qquad s = \int v \, \mathrm{d}t.$$

Note: The integration involves an arbitrary constant. You can find this by knowing the initial velocity, the initial displacement; these are the values of v and s when $t = 0$.

For the displacement s, you can always use definite integral! However, to find the velocity at same particular time t, you should first find the expression for v in terms of t, then substitue the value of t to get the velocity.

Exercise 28

1. A particle moves in a staright line so that its velocity, v m s^{-1}, at time t is given by $v = t^3 - 5t^2$.
   (a) Find the displacement of the particle after 5 s.
   (b) Sketch the velocity-time graph for $0 \leq t \leq 10$.
   (c) Work out the distance that the particle travels in the first 10 s.
   (d) Find the time when the particle passes through the start position.
2. A circus performer rides a unicycyle along a stretched tightrope. The tightrope is modelled as a straight horizontal line and the velocity, v m s^{-1}, is modelled as:
   $$\begin{aligned} v &= 1 && \text{for} \quad 0 \leq t \leq 2 \\ v &= 2t - 3 && \text{for} \quad 2 \leq t \leq 4 \\ v &= 7 - \sqrt{t} && \text{for} \quad 4 \leq t \leq T \end{aligned}$$
   where t is the time, in seconds, from when the performer starts to cycle along the tightrope.
   The performer stops at time T s, having just reached the other end of the tightrope.
   (a) Find the value of T.
   (b) Calculate how far the performer cycles.
3. A particle moves in a straight line so that its acceleration, a m s^{-2}, at time t is given by $a = 8 - 4t$. It starts with velocity 2 m s^{-1}. Find an expression for its velocity at time t.
4. A particle moves in a straight line so that its acceleration, a m s^{-2}, at time t is given by $a = 12 - 4t - 0.6t^2$. The particle comes to rest after 5 s.
   (a) Find the initial velocity of the particle.
   (b) Find the displacement after 2 s.
5. A train of mass 500 tonnes is travelling on a straight track at 48 m s^{-1} when the driver sees an amber light ahead. He applies the brakes for a period of 30 seconds with a force given by the formula $4t(30 - t)$ kN, where t is the time in seconds after the brakes are applied. Find how fast the train is moving after 30 seconds, and how far it has travelled in that time.

Miscellaneous exercise 6

1. A particle, P, starts from rest at a point O and travels in a horizontal straight line. For $0 < t < 20$, where t is the time in s, the velocity of P in m s^{-1} is given by $v = 1.2t - 0.03t^2$. When $t = 20$, P collides with another particle. After the collision the direction of travel of P is reversed. For $20 < t < 30$, the velocity of P in m s^{-1} is given by $v = 0.3t - 9$. The particle comes to rest and stops when $t = 30$.

   (a) Find the speed of P immediately before the collision and immediately after the collision.

   (b) Find the total distance travelled by the particle.

2. A particle moves in a straight line, starting at time $t = 0$ and continuing until it comes to rest. While it is moving the particle has acceleration a m s^{-2}, in the positive direction along the line. This acceleration is given by $a = 0.1 - 0.01t$, where t is the time from the start, in seconds. The particle starts with speed 4 m s^{-1} and finishes with speed 0 m s^{-1}.

   (a) Find the maximum speed of the particle.

   (b) Find the time when the particle comes to rest.

3. A car is moving in a staright line. The acceleration, a m s^{-2}, at time t s after the car starts to move is modelled as $a = A(1 + 4t)$ for $0 \leq t < 1$ and $a = B(30 - \frac{10}{t^3})$ for $1 \leq t \leq 5$, where A and B are constants.

   (a) Show that $A = 4B$.

   At $t = 5$ the velocity of the car is 31.8 m s^{-1}.

   (b) Show that $A = 1$.

   (c) Work out the distance travelled in the time interval $0 \leq t \leq 5$.

   (d) By considering the accelration-time graph at $t = 1$, criticise the model.

4. A hocky ball is hit so that it moves in a horizontal straight line with acceleration a m s^{-2}, along the direction of travel; $a = -0.6t$, where t is the time from when the ball was hit, in seconds. The initial speed of the ball is 14 m s^{-1}.

   (a) Find the speed of the ball when it has travelled 57.5 m.

   (b) Find the distance that the ball has travelled when the ball is first momentarily stationary.

   (c) Find the value of t when the ball has travelled 40 m.

5. Two walkers, P and Q, travels along a straight track ABC. Both walkers start from point A at time $t = 0$ s and pass through point B at time $t = 10$ s. They both finish at point C. P starts from point A with speed 2 m s^{-1} and accelerates with constant acceleration 0.1 m s^{-2} until reaching point B.

   (a) Show that the distance from A to B is 25 m.

   (b) Find the speed of P on reaching point B.

   Q starts form point A and moves with speed v_1 m s^{-1}, given by $v_1 = 0.003t^2 + 0.06t + k$. When Q passes through point B both walkers have the same speed.

   (c) Find the value of the constant k.

   P moves from point B to point C with speed v_2 m s^{-1}, given by $v_2 = 4 - 0.1t$, and comes to rest as C is reached.

   (d) Show that the distance from A to C is 70 m.

   Q moves from point B to point C with speed v_3 m s^{-1}, given by $v_3 = 0.4t - 0.01t^2$.

   (e) Show that Q reaches point C first.

Exam-style Questions 6

1. A particle P moves in a straight line starting from a point O and comes to rest 35 s later. At time t s after leaving O, the velocity v m s^{-1} of P is given by

$$\begin{aligned} v &= \frac{4}{5}t^2 && 0 \le t \le 5, \\ v &= 2t + 10 && 5 \le t \le 15, \\ v &= a + bt^2 && 15 \le t \le 35, \end{aligned}$$

where a and b are constants such that $a > 0$ and $b < 0$.

(i) Show that the values of a and b are 49 and -0.04 respectively. [3]

(ii) Sketch the velocity-time graph. [4]

(iii) Find the total distance travelled by P during the 35 s. [5]

2. A particle moves in a straight line. It starts from rest at a fixed point O on the line. Its acceleration at time t s after leaving O is a m s^{-2}, where $a = 0.4t^3 - 4.8t^{\frac{1}{2}}$.

(i) Show that, in the subsequent motion, the acceleration of the particle when it comes to instantaneous rest is 16 m s^{-2}. [6]

(ii) Find the displacement of the particle from O at $t = 5$. [3]

3. A particle moves in a straight line starting from rest from a point O. The acceleration of the particle at time t s after leaving O is a m s^{-2}, where

$$a = 5.4 - 1.62t.$$

(i) Find the positive value of t at which the velocity of the particle is zero, giving your answer as an exact fraction. [4]

(ii) Find the velocity of the particle at $t = 10$ and sketch the velocity-time graph for the first ten seconds of the motion. [3]

(iii) Find the total distance travelled during the first ten seconds of the motion. [5]

4. A particle moves in a straight line. The particle is initially at rest at a point O on the line. At time t s after leaving O, the acceleration a m s^{-2} of the particle is given by $a = 25 - t^2$ for $0 \le t \le 9$.

(i) Find the maximum velocity of the particle in this time period. [4]

(ii) Find the total distance travelled until the maximum velocity is reached. [2]

The acceleration of the particle for $t > 9$ is given by $a = -3t^{-\frac{1}{2}}$.

(iii) Find the velocity of the particle when $t = 25$. [4]

5. A particle moves in a straight line, starting from rest at a point O, and comes to instantaneous rest at a point P. The velocity of the particle at time t s after leaving O is v m s^{-1}, where

$$v = 0.6t^2 - 0.12t^3.$$

(i) Show that the distance OP is 6.25 m. [5]

On another occasion, the particle also moves in the same straight line. On this occasion, the displacement of the particle at time t s after leaving O is s m, where

$$s = kt^3 + ct^5.$$

It is given that the particle passes point P with velocity 1.25 m s^{-1} at time $t = 5$.

(ii) Find the values of the constants k and c. [5]

7 Momentum

7.1 Introduction

In Newtonian mechanics, momentum is the product of the mass and velocity of an object.

If m is an object's mass and v is its velocity , then the object's momentum is

$$\mathbf{p} =$$

Momentum is a vector quantity, having the same direction as the velocity.

The units of momentum are ___________.

Note: in SI units, momentum is measured in kilogram meters per second (kg m s^{-1}).

Change of momentum: an object with mass m has final velocity v_2, initial velocity v_1, then the change of momentum is

$$\mathbf{\Delta p} = mv_2 - mv_1.$$

Exercise 29

1. A ball of mass 50 g hits the ground with speed 10 m s^{-1} and rebounds with speed 6 m s^{-1}. Find the change in momentum that occurs in the bounce.
2. A ball of mass 0.2 kg falls 1.25 m vertically downwards to the ground, starting from rest. It hits the ground and rebounds. The downwards momentum of the ball changes by 1.6 N s in the bounce.
   (a) What height does the ball reach after this bounce?
   (b) By considering the modelling assumptions, explain why the height might be less than this.
3. A ball bearing of mass 25 g is thrown vertically upwards, and is caught on the way back down. The ball bearing has an initial speed of 3 m s^{-1} upwards and is travelling at 2 m s^{-1} when it is caught. Find the change in its momentum.
4. A hocky ball of mass 0.2 kg is hit so that it has an initial speed of 8 m s^{-1}. The ball travels in a horizontal straight line with acceleration a m s^{-2} given by $a = -0.5 - kt$ where t is the time in seconds, measured from when the ball was hit. After 2 s the ball has travelled $\frac{41}{3}$ m. It is then intercepted by a player from the other team. This player hits the ball so that its direction of travel is reversed and its speed is now 5 m s^{-1}. Show that when the ball is hit by the second player its momentum changes in magnitude by 2 N s.
5. A man strikes a snooker ball so that it travels horizontally across a snoker table and makes a direct hit against the end cushion of the table. The ball rebounds from the cushion and travels to the other end of the table where it rebounds from the cushion at that end. The ball finishes at exactly the same point at which it started. The distance between the two cushions is 3.5 m and the initial speed of the ball is 10 m s^{-1}. The ball is slowed by friction, resulting in a constant deceleration of 1 m s^{-2}. At each rebound the direction of the ball is reversed and the magnitude of the momentum after the rebound is 50% of the magnitude of the momentum before. Work out the distance that the ball travels before it reaches the first cushion.

7.2 Conservation of momentum

General form of Impulse-momentum theorem (Further Mechancis):

$$J = \Delta \mathbf{p}$$

Impulse-momentum theorem for constant force and mass:

$$F\Delta t = mv_2 - mv_1$$

During an impact when two bodies collide, the contact forces between the two objects involved in the impact are equal and ___________, that is

$$F\Delta t = m_1 v_1 - m_1 u_1 \quad \text{and} \quad -F\Delta t = m_2 v_2 - m_2 u_2$$

It follows that

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

Momentum is conserved in impacts. The total momentum is constant.

Coalesce and ___________ can be considered as special patterns of impact.

Exercise 30

1. Two ball bearings are moving directly towards one another. The first ball bearing has mass 20 g and is moving at 3 $\mathrm{m\,s^{-1}}$. The second ball bearing has mass 25 g and is moving at 1 $\mathrm{m\,s^{-1}}$. After the collision the first ball bearing is stationary. What is the speed of the second ball bearing after the collision.

2. A girl is sitting on a sledge. The girl and the sledge have a combined mass of 50 kg. When the girl and the sledge are moving at 2 $\mathrm{m\,s^{-1}}$ her sister standing in front of the sledge throws a snowball at the sledge. The snowball has mass 0.2 kg and is travelling at 10.55 $\mathrm{m\,s^{-1}}$ when it hits the sledge, head on. The snowball, the girl and the sledge continue together. Assuming that the total momentum is unchanged, find the new speed of the sledge.

3. Two snooker balls are travelling towards one another in a straight line when they make a direct impact. Before the impact the first ball had speed 12 $\mathrm{m\,s^{-1}}$ and the second ball had speed 8 $\mathrm{m\,s^{-1}}$. After the impact both balls have reversed their dirction and each has speed 10 $\mathrm{m\,s^{-1}}$. It is claimed that the balls are not both real snooker balls because they have different masses. Find the ratio of the masses of the balls.

4. Particle A, B and C, of masses 0.01 kg, 0.06 kg and 0.12 kg respectively, are at rest in a straight line on a smooth horizontal sureface, with B between A and C. A is given an initial velocity of 4 $\mathrm{m\,s^{-1}}$ towards B. After this impact A rebounds with velocity 2 $\mathrm{m\,s^{-1}}$ and B goes on to hit C. After the second impact B comes to rest. Find the speed of C after the second impact.

5. Jayne is performing in a show on ice. She is pushed onto the ice while sitting on a chair. The chair slides across the ice and Jayne then stands up and moves away from the chair. Jayne has speed 4 $\mathrm{m\,s^{-1}}$ when she is sitting on the chair and speed 5 $\mathrm{m\,s^{-1}}$ when she moves away from the chair. Jayne has mass 60 kg and the chair has mass 6 kg.

   (a) Find the velocity of the chair as Jayne moves away from it.

   (b) What assumption have you made on this model.

Miscellaneous exercise 7

1. Particle A has mass 4 kg. It moves with speed 3 m s^{-1} in a straight line on a smooth horizontal surface. Particle B has mass 6 kg and is at rest on the surface. Particle A collides with particle B. After the collision, A and B move away from each other with speed v m s^{-1} and $4v$ m s^{-1}. Find the value of v.

2. Two balls are travelling towards one another along the x -axis. The first ball hass mass 2 kg and is travelling at 3 m s^{-1} in the positive x -direction. The second ball has mass 5 kg is travelling at 1 m s^{-1} in the negative x -direction. The balls collide and after the collision the balls are travelling at the same speed but in opposite directions. Work out the speed of the balls after the collision.

3. Ball X has mass 0.03 kg. It falls vertically rest from a window that is 30 m above the ground. Ball Y has mass 0.01 kg. At the same time that the ball X starts to fall, ball Y is projected vertically upwards from ground level directly towards ball X. The initial speed of ball Y is 20 m s^{-1} vertically upwards.

   (a) Find the downward momentum of each ball just before they meet.

   The ball coalesce and the combined object falls to the ground.

   (b) Show that the combined object reaches the ground 2.68 seconds after ball X started to fall.

4. Three balls, A, B and C, are in that order in a straight line on a smooth horizontal surface. A has mass 0.4 kg and is moving at 4 m s^{-1} towards B. B hass mass m kg and is stationary. C has mass 0.25 kg and is moving at 0.8 m s^{-1} away from B. A hits B and then B hits C. There are no further impacts. A and C now each have a speed of 1 m s^{-1} and are both moving in directions away from B. Find the range of possible values of m.

5. Balls X, Y and Z lie at rest on a smooth horizontal surface, with Y between X and Z. Balls X and Z each have mass 2 kg and ball Y has mass 1 kg. Ball X is given a velocity of 1 m s^{-1} towards ball Y. Ball X and Y collide. After the collision the speed of ball Y is three times the speed of ball X. Ball Y goes on to collide with ball Z. After this collision the speed of ball Y is the same as the speed of ball X, and the speed of ball Z is twice the speed of ball Y. Finally ball Y collides with ball X again. After this collision the speed of ball Y is twice the speed of ball X, and the speed of ball Z is four times the speed of ball Y. Show that the balls are now all travelling in the same direction and that no further collision occur.

Exam-style Questions 7

1. Particles P of mass 0.4 kg and Q of mass 0.5 kg are free to move on a smooth horizontal plane. P and Q are moving directly towards each other with speeds 2.5 m s^{-1} and 1.5 m s^{-1} respectively. After P and Q collide, the speed of Q is twice the speed of P. Find the two possible values of the speed of P after the collision. [4]

2. Small smooth spheres A and B, of equal radii and of masses 4 kg and 2 kg respectively, lie on a smooth horizontal plane. Initially B is at rest and A is moving towards B with speed 10 m s^{-1}. After the spheres collide A continues to move in the same direction but with half the speed of B.

   (i) Find the speed of B after the collision. [2]

   A third small smooth sphere C, of mass 1 kg and with the same radius as A and B, is at rest on the plane. B now collides directly with C. After this collision B continues to move in the same direction but with one third the speed of C.

   (ii) Show that there is another collision between A and B. [3]

3. Three particles P, Q and R, of masses 0.1 kg, 0.2 kg and 0.5 kg respectively, are at rest in a straight line on a smooth horizontal plane. Particle P is projected towards Q at a speed of 5 m s^{-1}. After P and Q collide, P rebounds with speed 1 m s^{-1}.

   (i) Find the speed of Q immediately after the collision with P. [3]

   Q now collides with R. Immediately after the collision with Q, R begins to move with speed V m s^{-1}.

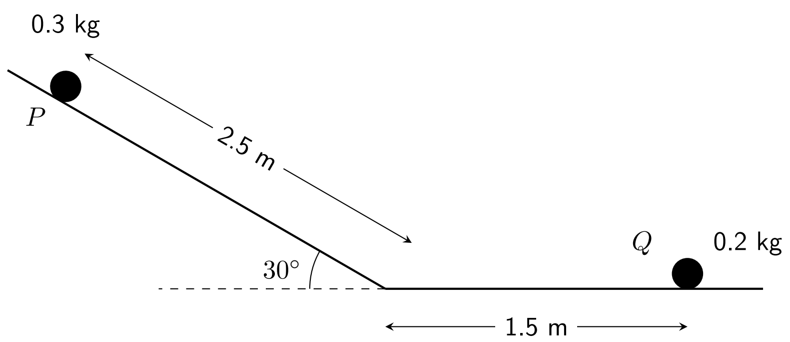
   (ii) Suppose no subsequent collision occurs between P and Q, find the maximum value of V. [3]

4. A particle A is projected vertically upwards from level ground with an initial speed of 30 m s^{-1}. At the same instant a particle B is released from rest 15 m vertically above A. The mass of one of the particles is twice the mass of the other particle. During the subsequent motion A and B collide and coalesce to form particle C. Find the difference between the two possible times at which C hits the ground. [8]

5. A particle P of mass 0.3 kg, lying on a smooth plane inclined at $30°$ to the horizontal, is released from rest. P slides down the plane for a distance of 2.5 m and then reaches a horizontal plane. There is no change in speed when P reaches the horizontal plane. A particle Q of mass 0.2 kg lies at rest on the horizontal plane 1.5 m from the end of the inclined plane (see diagram). P collides directly with Q.

   (i) It is given that the horizontal plane is smooth and that, after the collision, P continues moving in the same direction, with speed 2 m s^{-1}. Find the speed of Q after the collision. [5]

   (ii) Given that the horizontal plane is rough and that when P and Q collide, they coalesce and move with speed 1.2 m s^{-1}. Find the coefficient of friction between P and the horizontal plane. [5]



8 Work, energy and power

8.1 The work-energy equation

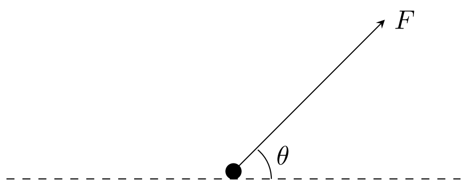
Basic form: if a constant force acts on an object over a certain distance, the ____________ by the force is equal to the gain in the ____________ of the object.

$$\mathbf{F} \cdot \mathbf{s} = \frac{1}{2}mv^2 - \frac{1}{2}mu^2.$$

The unit for the work done is joule, or J. We have 1 J = 1 N · m.

General form: if an object moves through a distance s along a line under the action of a force of magnitude F at an angle θ to the line, the work done by the force is $Fs\cos\theta$, thus

$$\mathbf{F} \cdot \mathbf{s} \cdot \cos\theta = \frac{1}{2}mv^2 - \frac{1}{2}mu^2.$$



Work done by the resistance & work done against resistance:

Work-Energy Principal: the net work on an object is equal to the change in that object's kinetic energy,

$$W_{\text{net}} = \Delta KE$$

Note: for the left side, to calculate the total work done W_{net}, you can either add up each individual work done by each individual force,

$$W_{\text{net}} = W_1 + W_2 + W_3 + \cdots,$$

or find the total net force and calculate the work done by it,

$$W_{\text{net}} = \mathbf{F}_{\text{net}} \cdot \mathbf{s} \cdot \cos\theta.$$

Exercise 31

1. A car and driver have a total mass of 1000 kg. The car gains speed from 7 m s^{-1} to 13 m s^{-1} with constant acceleration over a distance of 200 metres. Calculate the driving force.
2. A cyclist and her machine together have a mass of 100 kg. She free-wheels down a hill inclined at α to the horizontal, where $\sin\alpha = 0.05$, for a distance of 500 metres. If her speed at the top was 5 m s^{-1}, and there is air resistance of 40 newtons, how fast will she be going at the bottom of the hill.
3. A nail is being hammered into a plank. The mass of the hammer is 200 grams, and at each stroke the hammer is raised 15 cm above the nail. If the average force used to bring the hammer down is 10 times the average force used to raise the hammer, find the speed, to 2 significant figures, with which the hammer hits the nail.

8.2 Power

Definition 1: the rate at which a person or an engine works is called **power**.

$$P =$$

The unit of power is the joule per second, or the **watt**, abbreviated to W, thus 1 W = 1 J s^{-1}.

Note: reversely, we can find the work done by an object, if it exerts ___________ power over a given time.

Definition 2: if an engine drives an object at velocity v by means of a force F in the direction of motion, the power developed by the engine is

$$P =$$

In many cases, the greatest power of an engine is a constant over a range of speeds. This means that, when v gets larger, F ___________, and by Newton's second law, it follows that the acceleration ___________.

The $v - t$ graph:

Exercise 32

1. A car of mass 1500 kg arrives at the foot of a straight hill travelling at 30 m s^{-1}. It reaches the top of the hill 40 seconds later travelling at 10 m s^{-1}. The length of the hill is 1000 metres, and the gain in height is 120 metres. The average resistance to motion is 500 N. Find the average power developped by the engine.
2. A swimmer of mass 50 kg pushes off from the side of a pool with a speed of 0.8 m s^{-1}. She can develop power of 200 W, and the resistance of the water is 200 N.
   (a) At what rate can she accelerate away from the side of the pool?
   (b) Assuming the resistance remains the same, what is her greatest possible speed?
3. A car of mass 900 kg descends a straight hill which is inclined at 2° to the horizontal. The car passes through the points A and B with speed 14 m s^{-1} and 28 m s^{-1} respectively. The distance AB is 500 metres. Assuming there are no resisting forces, and that the driving force produced by the car's engine is constant, calculate the power of the car's engine at A and at B.
4. A car of mass 960 kg moves along a straight horizontal road with its engine working at a constant rate of 20 kW. Its speed at the point A on the road is 10 m s^{-1}. Assuming that there is no resistance to motion, calculate the time taken for the car to travel from A until its speed reaches 20 m s^{-1}.

   Assume now that there is a constant resistance to motion and that the car's engine continues to work at 20 kW. It takes 12 seconds for the car's speed to increase from 10 m s^{-1} to 20 m s^{-1}. During this time the car travels 190 metres. Calculate the work done against the resistance and hence find the magnitude of the resistance.
5. A racing car of mass 1830 kg is being tested out at high speed. Running at full power, it is found that the greatest speed the car can achieve is 80 m s^{-1}. With the same power output, at a speed of 64 m s^{-1}, the car accelerates at 0.5 m s^{-2}. Assuming that the resistance to motion is proportional to the square of the speed, find the accelration of the car at full power when its speed is 75 m s^{-1}.

Miscellaneous exercise 8

1. A car of mass 1000 kg travels on a horizontal straight road. The resistance to motion is modelled as a constant force of magnitude 380 N. Find the power of the car's engine at an instant when the car has a speed of 12 m s^{-1} and an acceleration of 0.7 m s^{-2}.

2. A car of mass 1220 kg travels up a straight road which is inclined at an angle α to the horizontal, where $\sin\alpha = 0.05$. The resistance to motion are modelled as a constant force of magnitude 1400 N. The car travels a distance of 25.8 m whilst increasing its speed from 8 m s^{-1}, at the point X, to 12 m s^{-1} at the point Y. Calculate the work done by the car's engine in travelling from X to Y.

   The car's engine works at a constant rate of 40 kW. Calculate the time taken to travel from X to Y.

3. A car of mass 700 kg descends a straight hill which is inclined at an angle of $3°$ to the horizontal. The car passes through the points P and Q with speeds of 12 m s^{-1} and 30 m s^{-1} respectively. The distance PQ is 500 metres. Assuming there are no resistances to motion, calculate the work done by the car's engine for the journey from P to Q.

   Assuming further that the driving force produced by the car's engine is constant, calculate the power of the car's engine at P, at Q, and the mid-point of PQ.

4. A car starts from rest and travels on a horizontal straight road. A resisting force acts on the car. By modelling the resisting force as a constant force of magnitude 750 N acting in the direction opposite to the motion of the car, calculate the maximum speed which the car can reach with its engine working at a constant rate of 30 kW.

   The car, with its engine switched off, can easily be pushed by one person along the horizontal road. State, giving a reason, whether or not the model for the resisting force is realistic at low speeds.

   The maximum power of the car is 40 kW and the mass of the car is 1250 kg. Calculate the maximum speed the car can attain after starting from rest and while travelling up a staight hill inclined at $3°$ to the horizontal, assuming that the resistance of 750 N continues to act.

5. If the air resistance to the motion of an airliner at speed v m s^{-1} is given by kv^2 newtons at ground level, then at 6000 metres the corresponding formula is $0.55kv^2$, and at 12000 metres it is $0.3kv^2$. If an airliner can cruise at 220 m s^{-1} at 12000 metres, at what speed will it travel at 6000 metres with the same power output from the engines?

   Suppose that $k = 2.5$ and that the mass of the airliner is 250 tonnes. As the airliner takes off its speed is 80 m s^{-1} and it is immediately starts to climb with the engines developing three times the cruising power. At what angle to the horizontal does it climb?

Exam-style Questions 8

1. A load is pulled along a horizontal straight track, from A to B, by a force of magnitude P N which acts at an angle of $30°$ upwards from the horizontal. The distance AB is 80 m. The speed of the load is constant and equal to 1.2 m s^{-1} as it moves from A to the mid-point M of AB.

   (i) For the motion from A to M the value of P is 25. Calculate the work done by the force as the load moves from A to M. [2]

   The speed of the load increases from 1.2 m s^{-1} as it moves from M towards B. For the motion from M to B the value of P is 50 and the work done against resistance is the same as that for the motion from A to M. The mass of the load is 35 kg.

   (ii) Find the gain in kinetic energy of the load as it moves from M to B and hence find the speed with which it reaches B. [5]

2. A particle of mass 0.8 kg slides down a rough inclined plane along a line of greatest slope AB. The distance AB is 8 m. The particle starts at A with speed 3 m s^{-1} and moves with constant acceleration 2.5 m s^{-2}.

   (i) Find the speed of the particle at the instant it reaches B. [2]

   (ii) Given that the work done against the frictional force as the particle moves from A to B is 7 J, find the angle of inclination of the plane. [4]

   When the particle is at the point X its speed is the same as the average speed for the motion from A to B.

   (iii) Find the work done by the frictional force for the particle's motion from A to X. [3]

3. A van of mass 3200 kg travels along a horizontal road. The power of the van's engine is constant and equal to 36 kW, and there is a constant resistance to motion acting on the van.

   (i) When the speed of the van is 20 m s^{-1}, its acceleration is 0.2 m s^{-2}. Find the resistance force. [3]

   When the van is travelling at 30 m s^{-1}, it begins to ascend a hill inclined at $1.5°$ to the horizontal. The power is increased and the resistance force is still equal to the value found in part (i).

   (ii) Find the power required to maintain this speed of 30 m s^{-1}. [3]

   (iii) The engine is now stopped, with the van still travelling at 30 m s^{-1}, and the van decelerates to rest. Find the distance the van moves up the hill from the point at which the engine is stopped until it comes to rest. [4]

4. A car of mass 1400 kg travelling at a speed of v m s^{-1} experiences a resistive force of magnitude $40v$ N. The greatest possible constant speed of the car along a straight level road is 56 m s^{-1}.

   (i) Find, in kW, the greatest possible power of the car's engine. [2]

   (ii) Find the greatest possible acceleration of the car at an instant when its speed on a straight level road is 32 m s^{-1}. [3]

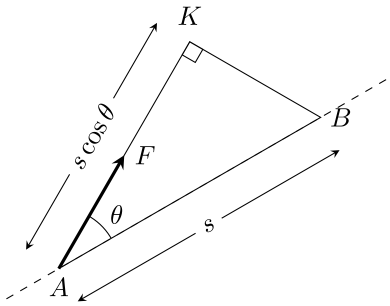
   (iii) The car travels down a hill inclined at an angle of $\theta°$ to the horizontal at a constant speed of 50 m s^{-1}. The power of the car's engine is 60 kW. Find the value of θ. [4]

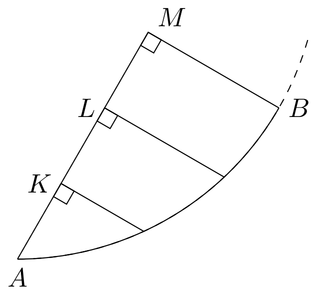
5. A car has mass 1000 kg. When the car is travelling at a steady speed of v m s^{-1}, where $v > 2$, the resistance to motion of the car is $(Av + B)$ N, where A and B are constants. The car can travel along a horizontal road at a steady speed of 18 m s^{-1} when its engine is working at 36 kW. The car can travel up a hill inclined at an angle of θ to the horizontal, where $\sin\theta = 0.05$, at a steady speed of 12 m s^{-1} when its engine is working at 21 kW. Find A and B. [7]

9 Potential energy

9.1 Work done along a curve

Work done: (resolving displacement)





$$W = F(S\cos\theta)$$

$$\begin{aligned} W &= F \times AK + F \times KL + F \times LM + \ldots \\ &= F \times (AK + KL + LM + \ldots) \end{aligned}$$

If an object moves along a curve under the action of a constant force of magnitude F, the work done by the force is equal to the product of F and the distance that the object moves in the ___________ of the force.

Consider three problems with one answer:

1. A stone is thrown vertically upwards with initial speed u. Find its speed v when it has risen to a height h.
2. A block is hit and start to move up a smooth path at an angle α to the horizontal. If its initial speed is u, find its speed v when it it at a height h above its starting point.
3. A ball is thrown at an angle to the horizontal with speed u. Neglecting the effect of air resistance, find how fast it will be moving when it is at a height h above the ground.

Fact: For gravity, the work done by it is determined by ________________ and ______________________.

$$W_G =$$

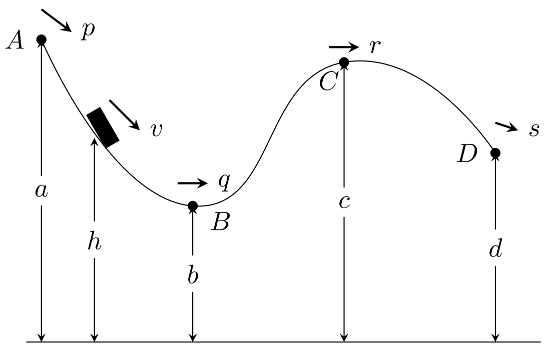
Exercise 33

1. A small sphere of mass m is suspended from a hook by a thread of length l. The sphere is pulled sideways, so that the thread makes an angle of 60° with the downward vertical, and then released from rest. How fast is the sphere moving when the thread becomes vertical?
2. A stone of mass 0.5 kg is attached to one end of a light inextensible string of length 0.4 metres. The other end of the string is attached to a fixed point O. The stone is released from rest with the string taut and inclined at an angle of 40° below the horizontal through O. Calculate the speed of the particle as it passes beneath O. Calculate also the speed of the stone when the string makes an angle of 20° with vertical through O.
3. A particle of mass 0.2 kg is attached to one end of a light rod of length 0.6 metres. The other end of the rod is freely pivoted at a fixed point O. The particle is released from rest with the rod making an angle of 60° with the upward vertical through O. Calculate the speed of the particle when the rod is
   (a) horizontal
   (b) vertical.

9.2 Conservation of energy

Conservative force:

Non-conservative force: **4th** problem: A brick is set in motion with speed u across a rough floor. The frictional force is F. Find the speed at which it is moving when it has gone a distance h horizontally. Suppose that F has the value mg.



Potential energy:

Conservation of energy prinple For an object moving a long a path, if there is no work done by external forces other than the force of gravity, the sum of the potential energy and the kinetic energy is constant.

$$\text{initial KE} + \text{initial GPE} = \text{KE at any point} + \text{GPE at any point} = \text{final KE} + \text{final GPE}$$

Exercise 34

1. A box of mass m kg is initially at rest. It slides down a smooth slope that is inclined at 30° to the horizontal, find the speed of the box after sliding a distance of 3 m.
2. A balll of mass 0.05 kg is thrown vertically upwards from a height of 1.5 m above the ground. The ball rises through a height of 2 m to reach its maximum height at 3.5 m above the ground. Use the conservation of mechanical energy to find the initial speed of the ball.
3. A ball is launched up a smooth slope that makes an angle 30° to the horizontal. The ball travels a distance 2.5 m up the slope before coming to instantaneous rest. Find the lauch speed of the ball.
4. A football is kicked from ground level with speed 15 $\mathrm{m\,s^{-1}}$ and rises to a height of 1.45 m. Assume that air resistance is negligible.

   (a) Find the speed of the ball when it is 1 m above the ground

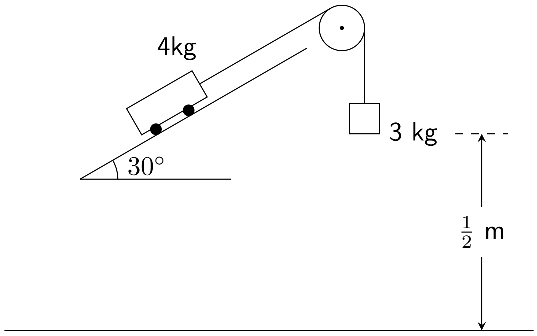
   At the top of its flight the ball is travelling horizontally

   (b) Explain why the horizontal component of the velocity is constant throughout the motion

   (c) Show that the ball was kicked at an angle of 21.0° with the horizontal.

9.3 Systems of connected objects

Conservation of energy can also be used to solve problems about a pair of objects connected by a string.



As a system, there are no external forces other than gravity, so the total mechanical energy is ___________.

The speed of the objects can be found by :

$$KE_1 + GPE_1 + KE_2 + GPE_2 = KE_1' + GPE_1' + KE_2' + GPE_2'$$

The tension can be found by writing a work-energy equation for the truck by itself:

Exercise 35

1. A crate of mass M kg sits at the bottom of a smooth slope that is inclined at an angle θ to the horizontal. A light inextensible rope is attached to the crate and passes over a smooth pulley at the top of the slope. The part of the rope between the crate and the pulley is parallel to the slope. The other end of the rope hangs vertically and at the other end there is a ball of mass m kg. The system is released from rest and the ball reaches the ground with speed v m s^{-1} after descending a distance of h m.

   (a) Find an expression for

      i. the decrease in potential energy for the ball

      ii. the increase in kinetic energy for the ball

      iii. the increase in mechanical energy for the crate.

   (b) Use the work-energy principle to show that $v = \sqrt{20h\left(\dfrac{m - M\sin\theta}{m+M}\right)}$.

9.4 Non-conservative forces

A non-conservative force: any force for which the work done by that force in moving a particle between two points is ________ for different path taken, examples are ___________, ___________, ___________, etc.

Driving force:

Friction:

For an object moving along a path, the total energy (potential and kinetic) is changed by the work done by the non-conservative external forces.

Exercise 36

1. Alan and Bob have mass 30 kg and 40 kg respectively. Alan is standing on the ground, holding one end of a rope. Bob is 5 metres up a tree. Alan tosses the other end of his rope over a high branch. Bob graps it, pull it tight and then uses it to descend to the ground. As he does so, Alan keeps hold of the rope and goes up. As Bob reaches the ground level, both boys are moving at 3 $\mathrm{m\,s^{-1}}$. How much work is done against the frictional force between the rope and the branch?
2. A parcel of mass 3 kg slides 3.5 m down a rough slope inclined at 20° to the horizontal. The coefficient of friction between the parcel and the slope is 0.5. When it reaches the bottom of the slope the parcel has speed 8 $\mathrm{m\,s^{-1}}$. Use energy method or otherwise to find the speed of the parcel at the top of the slope.
3. A car of mass 1500 kg (including the driver) is travelling at 64 $\mathrm{km\,h^{-1}}$ along a level road when the driver sees a ball roll out onto the road and in front of the car. The driver takes 2 s to react and then applies the brakes, using the maximum braking force. The car comes to rest, just missing the ball, after travelling total distance of 80 m from when the driver first saw the ball. Assuming that the wheels lock as soon as the brakes are applied (so the car slides) and that air resistance can be ignored, find the coefficient of friction between the tyres and the road.
4. An airliner of mass 300 tonnes is powered by four engines, each developing 15000 kW. Its speed at take-off is 75 $\mathrm{m\,s^{-1}}$, and it takes 11 minutes to reach its cruising speed of 210 $\mathrm{m\,s^{-1}}$ at a height of 10000 metres. Calculate the work done against air resistance during the climb.
5. A golf ball of mass 45.9 g is hit from a tee with speed 50 $\mathrm{m\,s^{-1}}$. The ball lands in a pond that is 5 m lower than the tee. When the ball lands in the pond it has travelled along a curved path of length 160 m. The resistance acting on the ball has magnitude 0.3 N.

   (a) Find the speed of the ball just before it hits the water.

   The water immediately absorbs 8 J of energy from the ball. The ball then sinks vertically downwards to reach the bottom of the pond. The resistance acting on the ball has magnitude 3 N and the ball just comes to rest as it reaches the bottom of the pond.

   (b) Find the depth of the pond.

Miscellaneous exercise 9

1. Car A, of mass 1250 kg, is travelling along a straight horizontal road at speed 10 $\mathrm{m\,s^{-1}}$. The engine works at a constant rate of 25 kW and the resistance is a constant 500 N. After 5 s the speed of the car has increased to v $\mathrm{m\,s^{-1}}$.

   (a) Use the work-energy principle to find the amount of energy that is dissipated and, hence, find the distance travelled in the 5 s.

   (b) Find an expression for the acceleration at time 5 s as a function of v and show that the acceleration is not constant.

   Car B travels along the same road, starting with speed 10 $\mathrm{m\,s^{-1}}$ and accelerating at a constant rate for 5 s. After 5 s the two cars have the same speed and also have the same accelration as one another.

   (c) Show that v must satisfy the equation $v^2 - 8v = 100$ and, hence, find the speed of the cars at the end of the 5 s.

2. Particles, of mass 1.5 kg and 2 kg, are attached to the ends of a light inextensible string. The string passes over a light pulley on smooth bearings, fixed at the top of the smooth, sloping face of a fixed wedge. The 2 kg mass is at rest on the sloping face, which is inclined at $30°$ to the horizontal. The 1.5 kg mass hangs freely and the string is taut. The particles are released. Find the speed of the particles when they have travelled 0.7 metres, and state the direction of motion of the 1.5 kg mass.

3. A car of mass 650 kg is travelling on a straight road inclined to the horizontal at $5°$. At a certain point P on the road the car's speed is 15 $\mathrm{m\,s^{-1}}$. The point Q is 400 metres down the hill from P, and at Q the car's speed is 35 $\mathrm{m\,s^{-1}}$. For the motion from P to Q, find

   (a) the increase in kinetic energy of the car

   (b) the decrease in gravitational potential energy of the car.

   Neglecting any resistance to the car's motion, and assuming that the car's engine produces a constant tractive force on the car as it moves down the hill from P to Q, calculate the magnitude of the tractive force and the power of the car's engine when the car is at Q.

   Assume instead that resistance to the car's motion between P and Q may be represented by a constant force of magnitdue 900 N. Given that the acceleration of the car at Q is zero, show that the power of the car's engine at this instant is approximately 12 kW.

   Assuming that the power of the car's engine is the same when the car is at P as it is when the car is at Q, calculate the car's acceleration at P.

4. A smooth wire is bent into the shape of the graph of $y = \frac{1}{6}x^3 - \frac{3}{2}x^2 + 4x$ for $0 < x < 6$, the units being metres. Points A, B and C on the wire have coordinates $(0,0)$, $(3,3)$ and $(6,6)$. A bead of mass m kg is projected along the wire from A with speed u $\mathrm{m\,s^{-1}}$ so that it has enough energy to reach B but not C. Prove that u is between 8.16 and 10.95, to 2 decimal places, and that the speed at B is at least 2.58 $\mathrm{m\,s^{-1}}$.

   If $u = 10$, the bead comes to rest at a point D between B and C. Find

   (a) the greatest speed of the bead between B and D,

   (b) the coordinates of D, to 1 decimal place.

   What happens after the bead reaches D?

5. A block of mass M is placed on a rough horizontal table. A string attached to the block runs horizontally to the edge of the table, passes round a smooth peg, and supports a sphere of mass m attached to its other end. The motion of the block on the table is resisted by a frictional force of magnitude F, where $F < mg$. The system is initially at rest.

   (a) Show that, when the block and the sphere have each moved a distance h, their common speed v is given by $v^2 = \frac{2(mg-F)h}{m+M}$.

   (b) Show that the total energy lost by the sphere as it falls through the distance h is $\frac{m(Mg+F)h}{M+m}$. Hence find an expression for the tension in the string.

   (c) Write down an expression for the energy gained by the block as it moves through the distance h. Use your answer to check the experession for the tension which you found in part (b).

Exam-style Questions 9

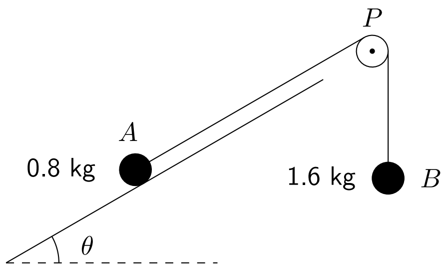
1. A car has mass 1250 kg.

   (i) The car is moving along a straight level road at a constant speed of 36 m s^{-1} and is subject to a constant resistance of magnitude 850 N. Find, in kW, the rate at which the engine of the car is working. [2]

   (ii) The car travels at a constant speed up a hill and is subject to the same resistance as in part (i). The hill is inclined at an angle of $\theta°$ to the horizontal, where $\sin\theta° = 0.1$, and the engine is working at 63 kW. Find the speed of the car. [3]

   (iii) The car descends the same hill with the engine of the car working at a constant rate of 20 kW. The resistance is not constant. The initial speed of the car is 20 m s^{-1}. Eight seconds later the car has speed 24 m s^{-1} and has moved 176 m down the hill. Use an energy method to find the total work done against the resistance during the eight seconds. [5]

2. Two particles A and B, of masses 0.8 kg and 1.6 kg respectively, are connected by a light inextensible string. Particle A is placed on a smooth plane inclined at an angle θ to the horizontal, where $\sin\theta = \frac{3}{5}$. The string passes over a small smooth pulley P fixed at the top of the plane, and B hangs freely (see diagram). The section AP of the string is parallel to a line of greatest slope of the plane. The particles are released from rest with both sections of the string taut. Use an energy method to find the speed of the particles after each particle has moved a distance of 0.5 m, assuming that A has not yet reached the pulley. [6]



3. A particle of mass 18 kg is on a plane inclined at an angle of $30°$ to the horizontal. The particle is projected up a line of greatest slope of the plane with a speed of 20 m s^{-1}.

   (i) Given that the plane is smooth, use an energy method to find the distance the particle moves up the plane before coming to instantaneous rest. [4]

   (ii) Given instead that the plane is rough and the coefficient of friction between the particle and the plane is 0.25, find the speed of the particle as it returns to its starting point. [8]

4. A particle of mass 0.3 kg is released from rest above a tank containing water. The particle falls vertically, taking 0.8 s to reach the water surface. There is no instantaneous change of speed when the particle enters the water. The depth of water in the tank is 1.25 m. The water exerts a force on the particle resisting its motion. The work done against this resistance force from the instant that the particle enters the water until it reaches the bottom of the tank is 1.2 J.

   (i) Use an energy method to find the speed of the particle when it reaches the bottom of the tank. [4]

   When the particle reaches the bottom of the tank, it bounces back vertically upwards with initial speed 7 m s^{-1}. As the particle rises through the water, it experiences a constant resistance force of 1.8 N. The particle comes to instantaneous rest t seconds after it bounces on the bottom of the tank.

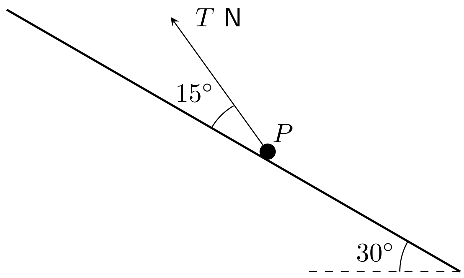
   (ii) Find the value of t. [7]

5. A particle P of mass 0.2 kg rests on a rough plane inclined at 30° to the horizontal. The coefficient of friction between the particle and the plane is 0.3. A force of magnitude T N acts upwards on P at 15° above a line of greatest slope of the plane (see diagram).

   (i) Find the least value of T for which the particle remains at rest. [6]

   The force of magnitude T N is now removed. A new force of magnitude 0.25 N acts on P up the plane, parallel to a line of greatest slope of the plane. Starting from rest, P slides down the plane. After moving a distance of 3 m, P passes through the point A.

   (ii) Use an energy method to find the speed of P at A. [5]



The end of note by Hunter, September 30, 2021