AIME Problems 2018

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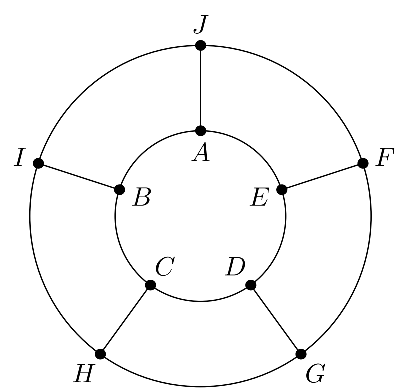
by AlcumusGuy, DaniyalQazi2, lucasxia01, brainiac1, franchester, AOPS12142015, nosysnow, GeronimoStilton, Vfire, djmathman, rrusczyk

–	I
–	March 6th, 2018
1	Let S be the number of ordered pairs of integers (a,b) with $1 \leq a \leq 100$ and $b \geq 0$ such that the polynomial x^2+ax+b can be factored into the product of two (not necessarily distinct) linear factors with integer coefficients. Find the remainder when S is divided by 1000.
2	The number n can be written in base 14 as $\underline{a}\ \underline{b}\ \underline{c}$, can be written in base 15 as $\underline{a}\ \underline{c}\ \underline{b}$, and can be written in base 6 as $\underline{a}\ \underline{c}\ \underline{a}\ \underline{c}$, where $a>0$. Find the base-10 representation of n.
4	In $\triangle ABC, AB=AC=10$ and $BC=12$. Point D lies strictly between A and B on $\overline{AB}$ and point E lies strictly between A and C on $\overline{AC}$ so that $AD=DE=EC$. Then AD can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.
5	For each ordered pair of real numbers (x,y) satisfying $$\log_2(2x+y)=\log_4(x^2+xy+7y^2)$$ there is a real number K such that $$\log_3(3x+y)=\log_9(3x^2+4xy+Ky^2).$$ Find the product of all possible values of K.
6	Let N be the number of complex numbers z with the properties that $\|z\|=1$ and $z^{6!}-z^{5!}$ is a real number. Find the remainder when N is divided by 1000.
7	A right hexagonal prism has height 2. The bases are regular hexagons with side length 1. Any 3 of the 12 vertices determine a triangle. Find the number of these triangles that are isosceles (including equilateral triangles).
8	Let $ABCDEF$ be an equiangular hexagon such that $AB=6, BC=8, CD=10$, and $DE=12$. Denote d the diameter of the largest circle that fits inside the hexagon. Find d^2.
9	Find the number of four-element subsets of $\{1,2,3,4,\dots,20\}$ with the property that two distinct elements of a subset have a sum of 16, and two distinct elements of a subset have a sum of 24. For example, $\{3,5,13,19\}$ and $\{6,10,20,18\}$ are two such subsets.

10 The wheel shown below consists of two circles and five spokes, with a label at each point where a spoke meets a circle. A bug walks along the wheel, starting at point A. At every step of the process, the bug walks from one labeled point to an adjacent labeled point. Along the inner circle the bug only walks in a counterclockwise direction, and along the outer circle the bug only walks in a clockwise direction. For example, the bug could travel along the path $AJABCHCHIJA$, which has 10 steps. Let n be the number of paths with 15 steps that begin and end at point A. Find the remainder when n is divided by 1000.



11 Find the least positive integer n such that when 3^n is written in base 143, its two right-most digits in base 143 are 01.

12 For every subset T of $U = \{1, 2, 3, \ldots, 18\}$, let $s(T)$ be the sum of the elements of T, with $s(\emptyset)$ defined to be 0. If T is chosen at random among all subsets of U, the probability that $s(T)$ is divisible by 3 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

13 Let $\triangle ABC$ have side lengths $AB = 30$, $BC = 32$, and $AC = 34$. Point X lies in the interior of $\overline{BC}$, and points I_1 and I_2 are the incenters of $\triangle ABX$ and $\triangle ACX$, respectively. Find the minimum possible area of $\triangle AI_1I_2$ as X varies along $\overline{BC}$.

14 Let $SP_1P_2P_3EP_4P_5$ be a heptagon. A frog starts jumping at vertex S. From any vertex of the heptagon except E, the frog may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Find the number of distinct sequences of jumps of no more than 12 jumps that end at E.

15 David found four sticks of different lengths that can be used to form three non-congruent convex cyclic quadrilaterals, A, B, C, which can each be inscribed in a circle with radius 1. Let φ_A denote the measure of the acute angle made by the diagonals of quadrilateral A, and define φ_B and φ_C similarly. Suppose that $\sin\varphi_A = \frac{2}{3}$, $\sin\varphi_B = \frac{3}{5}$, and $\sin\varphi_C = \frac{6}{7}$. All three quadrilaterals have the same area K, which can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

– II

– March 21st - 23rd, 2018

1 Points A, B, and C lie in that order along a straight path where the distance from A to C is 1800 meters. Ina runs twice as fast as Eve, and Paul runs twice as fast as Ina. The three runners start running at the same time with Ina starting at A and running toward C, Paul starting at B and running toward C, and Eve starting at C and running toward A. When Paul meets Eve, he turns around and runs toward A. Paul and Ina both arrive at B at the same time. Find the number of meters from A to B.

2 Let $a_0 = 2$, $a_1 = 5$, and $a_2 = 8$, and for $n > 2$ define a_n recursively to be the remainder when $4(a_{n-1} + a_{n-2} + a_{n-3})$ is divided by 11. Find $a_{2018} \cdot a_{2020} \cdot a_{2022}$.

3 Find the sum of all positive integers $b < 1000$ such that the base-b integer 36_b is a perfect square and the base-b integer 27_b is a perfect cube.

4 In equiangular octagon $CAROLINE$, $CA = RO = LI = NE = \sqrt{2}$ and $AR = OL = IN = EC = 1$. The self-intersecting octagon $CORNELIA$ encloses six non-overlapping triangular regions. Let K be the area enclosed by $CORNELIA$, that is, that total area of the six triangular regions. Then $K = \frac{a}{b}$ where a and b are relatively prime positive integers. Find $a + b$.

5 Suppose that x, y, and z are complex numbers such that $xy = -80 - 320i$, $yz = 60$, and $zx = -96+24i$, where $i = \sqrt{-1}$. Then there are real numbers a and b such that $x+y+z = a+bi$. Find $a^2 + b^2$.

6 A real number a is chosen randomly and uniformly from the interval $[-20, 18]$. The probability that the roots of the polynomial

$$x^4 + 2ax^3 + (2a-2)x^2 + (-4a+3)x - 2$$

are all real can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

7 Triangle ABC has sides $AB = 9$, $BC = 5\sqrt{3}$, and $AC = 12$. Points $A = P_0, P_1, P_2, \dots, P_{2450} = B$ are on segment $\overline{AB}$ with P_k between P_{k-1} and P_{k+1} for $k = 1, 2, \dots, 2449$, and points $A = Q_0, Q_1, Q_2, \dots, Q_{2450} = C$ for $k = 1, 2, \dots, 2449$. Furthermore, each segment $\overline{P_kQ_k}$, $k = 1, 2, \dots, 2449$, is parallel to $\overline{BC}$. The segments cut the triangle into 2450 regions, consisting of 2449 trapezoids and 1 triangle. Each of the 2450 regions have the same area. Find the number of segments $\overline{P_kQ_k}$, $k = 1, 2, \dots, 2450$, that have rational length.

8 A frog is positioned at the origin in the coordinate plane. From the point (x, y), the frog can jump to any of the points $(x+1, y)$, $(x+2, y)$, $(x, y+1)$, or $(x, y+2)$. Find the number of distinct sequences of jumps in which the frog begins at $(0, 0)$ and ends at $(4, 4)$.

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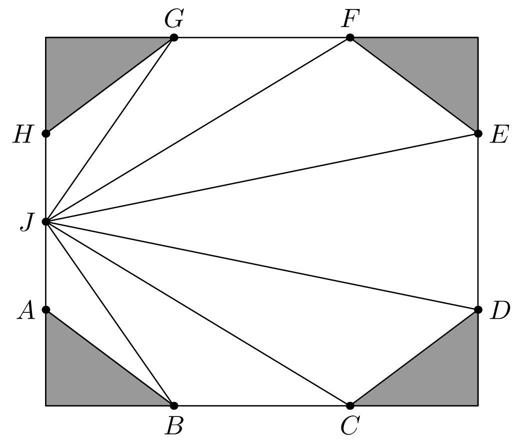
9 Octagon $ABCDEFGH$ with side lengths $AB = CD = EF = GH = 10$ and $BC = DE = FG = HA = 11$ is formed by removing four $6-8-10$ triangles from the corners of a 23×27 rectangle with side $\overline{AH}$ on a short side of the rectangle, as shown. Let J be the midpoint of $\overline{HA}$, and partition the octagon into 7 triangles by drawing segments $\overline{JB}$, $\overline{JC}$, $\overline{JD}$, $\overline{JE}$, $\overline{JF}$, and $\overline{JG}$. Find the area of the convex polygon whose vertices are the centroids of these 7 triangles.



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10 Find the number of functions $f(x)$ from $\{1, 2, 3, 4, 5\}$ to $\{1, 2, 3, 4, 5\}$ that satisfy $f(f(x)) = f(f(f(x)))$ for all x in $\{1, 2, 3, 4, 5\}$.

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11 Find the number of permutations of $1, 2, 3, 4, 5, 6$ such that for each k with $1 \leq k \leq 5$, at least one of the first k terms of the permutation is greater than k.

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12 Let $ABCD$ be a convex quadrilateral with $AB = CD = 10$, $BC = 14$, and $AD = 2\sqrt{65}$. Assume that the diagonals of $ABCD$ intersect at point P, and that the sum of the areas of $\triangle APB$ and $\triangle CPD$ equals the sum of the areas of $\triangle BPC$ and $\triangle APD$. Find the area of quadrilateral $ABCD$.

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13 Misha rolls a standard, fair six-sided die until she rolls 1-2-3 in that order on three consecutive rolls. The probability that she will roll the die an odd number of times is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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14 The incircle of ω of $\triangle ABC$ is tangent to $\overline{BC}$ at X. Let $Y \neq X$ be the other intersection of $\overline{AX}$ with ω. Points P and Q lie on $\overline{AB}$ and $\overline{AC}$, respectively, so that $\overline{PQ}$ is tangent to ω at Y. Assume that $AP = 3, PB = 4, AC = 8$, and $AQ = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

15 Find the number of functions f from $\{0,1,2,3,4,5,6\}$ to the integers such that $f(0)=0$, $f(6)=12$, and

$$|x-y| \leq |f(x)-f(y)| \leq 3|x-y|$$

for all x and y in $\{0,1,2,3,4,5,6\}$.

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