Roots of Polynomial mark scheme

1 (i)	$\alpha+\beta+\gamma+\delta=-4$	B1 (1)
(ii)	$\alpha^2+\beta^2+\gamma^2+\delta^2=(-4)^2-2\times 2=12$	M1A1 (2)
(iii)	$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{-(-4)}{1}=4$	M1A1 (2)
(iv)	$\frac{\alpha}{\beta\gamma\delta}+\frac{\beta}{\alpha\gamma\delta}+\frac{\gamma}{\alpha\beta\delta}+\frac{\delta}{\alpha\beta\gamma}=\frac{\alpha^2+\beta^2+\gamma^2+\delta^2}{\alpha\beta\gamma\delta}=\frac{12}{1}=12$	M1A1 (2)
	$y=x+1\Rightarrow x=y-1$	
	$(y-1)^4+4(y-1)^3=y^4-6y^2+8y-3$	M1A1
	$2(y-1)^2-4(y-1)+1=2y^2-8y+7$	A1
	$\Rightarrow x^4+4x^3+2x^2-4x+1=y^4-4y^2+4=0$	A1
	$(y^2-2)^2=0\Rightarrow y=\pm\sqrt{2}$ (twice).	A1
	$\Rightarrow x=\pm\sqrt{2}-1$ (twice). (Some indication of four roots for final mark.)	M1A1 (7) **[14]**

2	$\alpha+\beta+\gamma=7$, $\alpha\beta+\beta\gamma+\gamma\alpha=2$, $\alpha\beta\gamma=3$ (Stated or implied by working.)	B1
(i)	$\frac{1}{(\alpha\beta)(\beta\gamma)(\gamma\alpha)}=\frac{1}{(\alpha\beta\gamma)^2}=\frac{1}{9}$	B1
(ii)	$\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}=\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}=\frac{7}{3}$	M1A1
(iii)	$\frac{1}{\alpha^2\beta\gamma}+\frac{1}{\alpha\beta^2\gamma}+\frac{1}{\alpha\beta\gamma^2}=\frac{1}{\alpha\beta\gamma}\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)=\frac{1}{\alpha\beta\gamma}\left(\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}\right)=\frac{2}{9}$	M1A1 (6)
	$\Rightarrow x^3-\frac{7}{3}x^2+\frac{2}{9}x-\frac{1}{9}=0\Rightarrow 9x^3-21x^2+2x-1=0$	M1A1√ (2) **Total: 8**

3 Good responses to the first part of this question were common, but in contrast, there were many failures, both tactical and strategic, in the working for the remainder.

Very few candidates failed to obtain the correct value of the sum of the squares of the roots of the given equation. On the other hand, arguments to show that the given equation has exactly one real root were frequently hazy, to say the least. In this context, it is necessary to highlight the essential aspects of the current mathematical situation, which are:

- since the degree of the given polynomial equation is 3, then it has 3 complex roots
- sum of squares of roots < 0 implies that not all roots are real
- all coefficients of the given polynomial equation are real implies that complex roots occur in conjugate pairs.

The conclusion is then immediate. However, few candidates were able to produce arguments of this clarity and/or completeness.

For the rest of the question, most candidates argued that $f(x) \equiv x^3 + x + 12 \Rightarrow f(-3) = -18$, $f(-2) = 2$ and hence that as $f(-3) < 0$ and $f(-2) > 0$, then $-3 < \alpha < -2$.

They also comprehended that the result $\alpha\beta\beta^* = -12$ (A), which may be obtained directly from the equation, was the key to the obtaining of the final inequality. Generally, this was combined well with (A) so as to obtain the final result. However, a significant minority went wrong at this stage and there were even those who did not recognise that there were 2 inequalities to prove in this question. This suggests that failure to read the question properly induced some reduction in the overall quality of responses.

Answer: Sum of squares of roots of equation = –2.

4	$(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha) = \alpha^2+\beta^2+\gamma^2 \Rightarrow \sum\alpha\beta = 1$	M1A1	2	
	Either			
	Required equation is $x^3 - 4x^2 + x + c = 0$	M1		
	$\Rightarrow \sum\alpha^3 - 4\sum a^2 + 4 + 3c = 0$	M1		
	$\Rightarrow 3c = 56 - 34 - 4 = 18 \Rightarrow c = 6$ (AG)	A1		
	Or			
	$\alpha^3+\beta^3+\gamma^3 - 3\alpha\beta\gamma = (\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha)$	(M1)		
	(or some other appropriate identity, e.g.			
	$(\alpha+\beta+\gamma)^3 = \alpha^3+\beta^3+\gamma^3+3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha) - 3\alpha\beta\gamma)$	(M1A1)		
	$\Rightarrow \ldots \Rightarrow \alpha\beta\gamma = -6$			
	$\Rightarrow x^3 - 4x^2 + x + 6 = 0$ (AG)			
	$\Rightarrow (x+1)(x-2)(x-3) = 0 \Rightarrow x = -1, 2, 3.$	A1		
		M1A1	6	**[8]**

Qu No	Commentary	Solution	Marks	Part Mark	Total
5	**EITHER** Substitute α into equation. Multiply by α^n. Obtain result.	α is a root $\Rightarrow \alpha^4 - 3\alpha^2 + 5\alpha - 2 = 0$ $\Rightarrow \alpha^{n+4} - 3\alpha^{n+2} + 5\alpha^{n+1} - 2\alpha^n = 0$ Repeat for β, γ, δ and sum $\Rightarrow S_{n+4} - 3S_{n+2} + 5S_{n+1} - 2S_n = 0$ (AG)	M1 A1	2	
(i)	Uses $\sum\alpha^2 = \left(\sum\alpha\right)^2 - 2\sum\alpha\beta$ Finds S_4 from formula.	$S_2 = 0 - 2\times(-3) = 6$ $S_4 = 3\times 6 - 5\times 0 + 2\times 4 = 26$	B1 M1A1	3	
(ii)	$S_{-1} = \dfrac{\sum\alpha\beta\gamma}{\alpha\beta\gamma\delta}$ Finds S_3 from formula. Finds S_5 from formula.	$S_{-1} = \dfrac{-5}{-2} = \dfrac{5}{2}$ $S_3 = 3\times 0 - 5\times 4 + 2\times\dfrac{5}{2} = -15$ $S_5 = 3\times(-15) - 5\times 6 + 2\times 0 = -75$	M1A1 M1A1 M1A1	6	
		$\sum\alpha^2\beta^3 = S_2S_3 - S_5$ $= 6\times(-15) - (-75) = -15$	M1 M1A1	3	**[14]**

6	$y = \dfrac{1}{x+1} \quad \therefore x = \dfrac{1-y}{y}$	M1	use in given cubic equation	
	Gives $6y^3 - 7y^2 + 3y - 1 = 0$ **AG**	A1		[2]
	$n = 1$: given expression = sum of roots = 7/6	B1		
	$n = 2$: $\sum\dfrac{1}{(\alpha+1)^2} = \left(\sum\dfrac{1}{(\alpha+1)}\right)^2 - 2\sum\text{"}\alpha\beta\text{"} = \dfrac{13}{36}$	B1		[2]
	From cubic in y,			
	$6\sum\left(\dfrac{1}{\alpha+1}\right)^3 - 7.\dfrac{13}{36} + 3\left(\dfrac{7}{6}\right) - 3 = 0$	M1		
	$\sum\left(\dfrac{1}{\alpha+1}\right)^3 = 73/216$	A1		[2]
	LHS = $\sum\left(\dfrac{(\beta+1)(\gamma+1)(\alpha+1)}{(\alpha+1)^3}\right)$	M1		
	$= \left(\dfrac{1}{6}\right)^{-1} \times \dfrac{73}{216}$	M1	recognise product of roots	
	= 73/36 **AG**	A1		[3]

7	Obtains an equation in y not involving radicals, e.g., $y(y+1)^2 = 1$ $\Rightarrow ... \Rightarrow y^3 + 2y^2 + y - 1 = 0$ (AG)	M1 A1 [2]
	(i) $S_2 = -2$ $S_4 = 4 - 2 = 2$	B1 M1A1 [3]
	(ii) $S_6 = -2S_4 - S_2 + 3 = 1$	M1A1
	OR $\Sigma\alpha^2 = -2$, $\Sigma\alpha^2\beta^2 = 1$, $\alpha^2\beta\gamma^2 = 1$ $S_6 = (\Sigma\alpha^2)^3 - 3\Sigma\alpha^2\Sigma\alpha^2\beta^2 + 3\alpha^2\beta^2\gamma^2$ $= (-2)^3 - 3 \times (-2) \times 1 + 3$ $= -8 + 6 + 3$ $= 1$	 M1 A1
	$S_8 = -2S_6 - S_4 + S_2 = -6$	M1A1 [4]

8	**(i)** $x = 1/y \Rightarrow 2y^4 - 4y^3 - cy^2 - y - 1 = 0$	M1A1 [2]
	(ii) $\sum\alpha^2 = 1 - 2c$	M1A1
	$\sum\alpha^{-2} = 4 + c$	A1
	(M1 is for use of $\sum\alpha^2 = \left(\sum\alpha\right)^2 - 2\sum\alpha\beta$ in either part.)	[3]
	(iii) $S = \sum\left(\alpha - \alpha^{-1}\right)^2 = \sum\alpha^2 + \sum\alpha^{-2} - 8 = -c - 3$	M1A1√
	A1ft is for adding answers to **(ii)** correctly and subtracting 8.	[2]
	(iv) $c = -3 \Rightarrow S = 0$ so that if all roots are real then $\alpha = \pm 1$	
	and similarly for β, γ, δ	M1A1 CWO
	This is impossible since e.g., $\alpha\beta\gamma\delta = -2$, or any other contradiction	A1 CWO [3]