Assignment 3: Complex numbers

Name: Group: Due:

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1. (a) Use de Moivre's theorem to show that $\sin 8\theta$ can be expressed in the form [6]

$$\sin\theta\cos\theta(a\sin^6\theta + b\sin^4\theta + c\sin^2\theta + d),$$

where the value of the constant a is to be found and b, c, d are constants whose values need not be found.

(b) Use de Moivre's theorem to show that [7]

$$\sum_{n=1}^{N}\frac{\sin n\theta}{2^n} = \frac{2^{N+1}\sin\theta + \sin N\theta - 2\sin(N+1)\theta}{2^N(5-4\cos\theta)}.$$

2. Use de Moivre's theorem to prove that [6]

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

State the exact values of θ, between 0 and π, that satisfy $\tan 3\theta = 1$. [2]

Express each root of the equation $t^3 - 3t^2 - 3t + 1 = 0$ in the form $\tan(k\pi)$, where k is a positive rational number. [3]

For each of these values of k, find the exact value of $\tan(k\pi)$. [3]

3. State the fifth roots of unity in the form $\cos\theta + \mathrm{i}\sin\theta$, where $-\pi < \theta \leqslant \pi$. [2]

Simplify [2]

$$\left(x - \left[\cos\left(\frac{2}{5}\pi\right) + \mathrm{i}\sin\left(\frac{2}{5}\pi\right)\right]\right)\left(x - \left[\cos\left(\frac{2}{5}\pi\right) - \mathrm{i}\sin\left(\frac{2}{5}\pi\right)\right]\right).$$

Hence find the real factors of [2]

$$x^5 - 1.$$

Express the six roots of the equation

$$x^6 - x^3 + 1 = 0$$

as three conjugate pairs, in the form $\cos\theta \pm \mathrm{i}\sin\theta$. [6]

Hence find the real factors of [2]

$$x^6 - x^3 + 1.$$

4. (a) Write down the five fifth roots of unity. [2]

(b) Hence find all the roots of the equation

$$z^5 + 16 + (16\sqrt{3})\mathrm{i} = 0,$$

giving answers in the form $r\mathrm{e}^{\mathrm{i}q\pi}$, where $r > 0$ and q is a rational number. Show these roots on an Argand diagram. [4]

Let Let ω be a root of the equation in part (b).

(c) Show that [3]

$$\sum_{k=0}^{4} \left(\frac{\omega}{2}\right)^k = \frac{3 + \mathrm{i}\sqrt{3}}{2 - \omega}.$$

(d) Identify the root for which $|2 - \omega|$ is least. [2]

5. Prove de Moivre's theorem for a positive integral exponent:

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$$

for all positive integers n. [5]

Use de Moivre's theorem to show that [4]

$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

Hence obtain the roots of the equation

$$128x^7 - 224x^5 + 112x^3 - 14x + 1 = 0$$

in the form $\cos q\pi$, where q is a rational number. [4]

6. Use de Moivre's theorem to express $\sin^6\theta$ in the form

$$a + b\cos 2\theta + c\cos 4\theta + d\cos 6\theta,$$

where a, b, c, d are constants to be found. [5]

Hence evaluate

$$\int_0^{\frac{1}{4}\pi} \sin^6 2x \, \mathrm{d}x\,,$$

leaving your answer in terms of π. [4]

7. Let $\omega = \cos \frac{1}{5}\pi + \mathrm{i} \sin \frac{1}{5}\pi$. Show that $\omega^5 + 1 = 0$ and deduce that [2]

$$\omega^4 - \omega^3 + \omega^2 - \omega = -1.$$

Show further that [3]

$$\omega - \omega^4 = 2\cos\frac{1}{5}\pi, \qquad \text{and} \qquad \omega^3 - \omega^2 = 2\cos\frac{3}{5}\pi.$$

Hence find the values of [4]

$$\cos\frac{1}{5}\pi + \cos\frac{3}{5}\pi \qquad \text{and} \qquad \cos\frac{1}{5}\pi\cos\frac{3}{5}\pi.$$

Find a quadratic equation having roots $\cos \frac{1}{5}\pi$ and $\cos \frac{3}{5}\pi$ and deduce the exact value of $\cos \frac{1}{5}\pi$. [4]

8. By considering $\sum_{k=0}^{n-1}(1+\mathrm{i}\tan\theta)^k$, show that

$$\sum_{k=0}^{n-1}\cos k\theta\sec^k\theta = \cot\theta\sin n\theta\sec^n\theta,$$

provided θ is not an integer multiple of $\frac{1}{2}\pi$. [7]

Hence or otherwise show that [2]

$$\sum_{k=0}^{n-1}2^k\cos\left(\frac{1}{3}k\pi\right) = \frac{2^n}{\sqrt{3}}\sin\left(\frac{1}{3}n\pi\right).$$

Given that $0 < x < 1$, show that [4]

$$\sum_{k=0}^{n-1}\frac{\cos\left(k\cos^{-1}x\right)}{x^k} = \frac{\sin\left(n\cos^{-1}x\right)}{x^{n-1}\sqrt{1-x^2}}.$$