1

Part **(a)** was done reasonably well by the majority of candidates. They knew that it was necessary to expand, $(\cos\theta + \mathrm{i}\sin\theta)^8$, using the binomial theorem and pick the imaginary part to find $\sin 8\theta$. They also attempted to substitute $(1-\sin^2\theta)$ for $\cos\theta$, which earned them most of the marks. There were, however, quite a few unnecessary slips with signs and algebra.

Many could not get started on part **(b)**, but those who realised that they needed to find the imaginary part of the sum of an appropriate geometric progression made good progress. The most efficient solutions retained the complex numbers in exponential form for as long as possible. Some candidates showed impressive mastery of this area of the syllabus.

Answer: **(a)** -128.

2	**EITHER**				
	Uses de Moivre's theorem and binomial theorem.	$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ and $\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$	B1 M1A1		
	(If line 1 missing, or no reference made to $\cos 3\theta$ and $\sin 3\theta$ being real and imaginary parts.)	Award B0M1A1M1M1A0 i.e.4/6			
	Equates real and imaginary parts.	$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$ $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$	M1		
	Uses $\tan A = \dfrac{\sin A}{\cos A}$	$\tan 3\theta = \dfrac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}$ $\therefore \tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ (*) (AG)	M1 A1	6	
		$\dfrac{\pi}{12}$, $\dfrac{5\pi}{12}$, $\dfrac{9\pi}{12} = \dfrac{3\pi}{4}$	B1 (one) B1 (all)	2	
		Put $\tan 3\theta = 1$ in (*) $\Rightarrow t^3 - 3t^2 - 3t + 1 = 0$	M1		
	States roots of $\tan 3\theta = 1$ between 0 and π.	Roots are $\tan\dfrac{\pi}{12}, \tan\dfrac{5\pi}{12}, \tan\dfrac{3\pi}{4}$	A1 (one) A1 (all)	3	
	Obtains cubic equation and solves.	$(t+1)(t^2 - 4t + 1) = 0$ $\Rightarrow t = -1$, $2 \pm \sqrt{3}$	M1		
	Evaluates each root.	$\tan\dfrac{3\pi}{4} = -1$ $\quad\tan\dfrac{\pi}{12} = 2 - \sqrt{3}$ $\tan\dfrac{5\pi}{12} = 2 + \sqrt{3}$	A1 (one) A1 (all)	3	**[14]**

3	Obtains all fifth roots.	$z=\cos\frac{2k\pi}{5}+\mathrm{i}\sin\frac{2k\pi}{5}, k=0, \pm1, \pm2.$	B1B1	2	
	Simplifies expression.	$x^2-2\cos\frac{2\pi}{5}x+1$	M1A1	2	
	Obtains factors.	$\left(x^2-2\cos\frac{2\pi}{5}+1\right)\left(x^2-2\cos\frac{4\pi}{5}+1\right)(x-1)$	M1A1	2	
	Solves quadratic in x^3.	$x^3=\frac{1}{2}\pm\mathrm{i}\frac{\sqrt{3}}{2}=\cos\frac{\pi}{3}\pm\mathrm{i}\sin\frac{\pi}{3}$	M1A1 A1		
	Expresses them in polar form.	or $\cos\frac{7\pi}{3}\pm\mathrm{i}\sin\frac{7\pi}{3}$ or $\cos\frac{13\pi}{3}\pm\mathrm{i}\sin\frac{13\pi}{3}$	A1		
		$x=\cos\frac{\pi}{9}\pm\mathrm{i}\sin\frac{\pi}{9}, \cos\frac{7\pi}{9}\pm\mathrm{i}\sin\frac{7\pi}{9}, \cos\frac{13\pi}{9}\pm\mathrm{i}\sin\frac{13\pi}{9}$	M1A1	6	
	Finds factors.	$\left(x^2-2\cos\frac{\pi}{9}x+1\right)\left(x^2-2\cos\frac{7\pi}{9}+1\right)$ $\left(x^2-2\cos\frac{13\pi}{9}+1\right)$ (ACF)	M1A1	2	**[14]**

4	**(i)**	$\exp(2\pi ki/5)$, $k = 0, 1, 2, 3, 4$ (AEF)	M1A1 [2]
		M1 for 1 correct fifth root of unity A1 for exactly 5 distinct, correct roots	
	(ii)	$z^5 = 32\exp(-2\pi i/3)$	M1
		$z_k = 2\exp(-2\pi i/15 + 2\pi ki/5)$	A1
		Roots equally spaced on circle $\lvert z\rvert = 2$; correctly placed	M1A1 [4]
	(iii)	$\sum_{k=0}^{4}(w/2)^k = \dfrac{[1-(w/2)^5]}{[1-w/2]}$	M1
		$= \dfrac{1-(1/32)(-16-16\sqrt{3}i)}{[1-w/2]}$	A1
		$= \ldots = (3+\sqrt{3}i)/(2-w)$ (AG)	A1 [3]
	(iv)	Deduces from diagram in **(ii)** that minimum of $\lvert 2-w\rvert$ occurs when $w = 2e^{-2\pi i/15}$ or $2e^{\frac{28\pi}{15}i}$	M1A1 [2]
		OR Evaluates 5 possible values of $\lvert 2-w\rvert$	M1
		Identifies minimum of $\lvert 2-w\rvert$ correctly	A1

5

Most candidates managed to make good progress with this question. Errors occurred mainly in the first and final parts of the question.

The majority of candidates proved, or attempted to prove de Moivre's theorem for a positive integral index by induction. The comments with regard to the inductive hypothesis made in this report for **Question 4** apply here. Moreover, the working for the central part of the proof where it is necessary to *prove* that $(\cos\theta + \mathrm{i}\sin\theta)(\cos k\theta + \mathrm{i}\sin k\theta) = \cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta$ was deficient in a number of scripts.

Almost all candidates produced a complete and correct response to the second part of the question. The large amounts of detailed working on display were impressive and they provided evidence of a candidature well prepared for this type of problem.

For the concluding part of this question, there were many incomplete responses. Almost everyone got as far as showing that $\cos 7\theta = -\frac{1}{2}$ and hence that all the roots of the equation could be expressed in the required form by appropriate choice of θ. Nevertheless only about half of all candidates obtained exactly 7 distinct roots.

Answer: $\cos\left(\frac{2\pi}{21} + \frac{2k\pi}{7}\right)$, $k = 0, 1, \ldots, 6$.

6

$64\sin^6\theta = -(z - 1/z)^6 \left(z = \mathrm{e}^{i\theta}\right)$	M1
$= -\left(z^6 + 1/z^6\right) + 6\left(z^4 + 1/z^4\right) - 15\left(z^2 + 1/z^2\right) + 20$	M1A1
$\sin^6\theta = 5/16 - (15/32)\cos 2\theta + (3/16)\cos 4\theta - (1/32)\cos 6\theta$	M1A1
$\sin^6 2x = 5/16 - (15/32)\cos 4x + (3/16)\cos 8x - (1/32)\cos 12x$	M1
Any one of $\int_0^{\pi/4} \cos kx\, dx = 0$ for $k = 1, 2, 3$	B1
Further **B1** if all 3 are written down or implied	B1
$\int_0^{\pi/4} \sin^6 2x\, dx = 5\pi/64$ or $\pi a/4$ (CWO)	A1
OR for last 4 marks	
$I = (1/2)\int_0^{\pi/2} \sin^6 u\, du = (1/64)\int_0^{\pi/2} (10 - 15\cos 2u + 6\cos 4u - \cos 6u)\, du$	M1
$= (1/64)\left[10u - 15\sin 2u/2 + 3\sin 4u/2 - \sin 6u/6\right]_0^{\pi/2}$	M1A1
$= 5\pi/64$ (CWO)	A1

7	**EITHER**				
	Verifies that ω is a root.	$\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)^5+1=\cos\pi+i\sin\pi+1=0$	B1		
	Factorises.	$(\omega^5+1)=(\omega+1)(\omega^4-\omega^3+\omega^2-\omega+1)=0$			
		$\omega\neq-1\Rightarrow\omega^4-\omega^3+\omega^2-\omega+1=0$			
		$\Rightarrow\omega^4-\omega^3+\omega^2-\omega=-1$	B1	2	
		$\omega=\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}$			
	Finds ω^4	$\Rightarrow\omega^4=\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}=-\cos\frac{\pi}{5}+\sin\frac{\pi}{5}$	M1		
	and subtracts.	$\Rightarrow\omega-\omega^4=2\cos\frac{\pi}{5}$	A1		
	Finds ω^3	$\omega^3=\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}$			
	and ω^2	$\omega^2=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}=\cos\frac{3\pi}{5}-i\sin\frac{3\pi}{5}$	M1		
	and subtracts	$\omega^3-\omega^2=2\cos\frac{3\pi}{5}$	A1	4	
		$-2\cos\frac{\pi}{5}-2\cos\frac{3\pi}{5}=-1\Rightarrow\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$	M1A1		
		$\cos\frac{\pi}{5}\cos\frac{3\pi}{5}=\frac{1}{4}(\omega-\omega^4)(\omega^3-\omega^2)$	M1		
		$=\frac{1}{4}(\omega^4-\omega^3-\omega^7+\omega^6)$			
		$=\frac{1}{4}(\omega^4-\omega^3+\omega^2-\omega)=-\frac{1}{4}$	A1	4	
		Equation with roots $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$ is:			
	Finds required quadratic equation.	$x^2-\frac{1}{2}x-\frac{1}{4}=0$ or $4x^2-2x-1=0$	M1		
	Solves for x.	$\Rightarrow x=\frac{2\pm2\sqrt{5}}{8}$	M1A1		
	States required value.	$\Rightarrow\cos\frac{\pi}{5}=\frac{1+\sqrt{5}}{4}$ (since $0<\cos\frac{\pi}{5}<1$)	A1	4	**[14]**

$(1+i\tan\theta)^k = \sec^k\theta(\cos k\theta + i\sin k\theta)$ M1A1

$\sum_{k=0}^{n-1}(1+i\tan\theta)^k = [(1+i\tan\theta)^n - 1]/i\tan\theta$ M1A1

$= \sec^n\theta\sin n\theta\cot\theta + i(\cot\theta - \sec^n\theta\cos n\theta\cot\theta)$ A1

$\Rightarrow \sum_{k=0}^{n-1}\sec^k\theta\cos k\theta = \sec^n\theta\sin n\theta\cot\theta$ (AG) M1A1

Put $\theta = \pi/3$ to obtain second result (AG) M1A1

$x = \cos\theta \Rightarrow \sec\theta = 1/x$, $\cot\theta = x/\sqrt{1-x^2}$ M1A1

Uses these results to obtain final result (AG) M1A1

First 7 marks

$$\sum_{0}^{n-1}(1+i\tan\theta)^k = \underbrace{\sum_{0}^{n-1}(\cos k\theta + i\sin k\theta)\sec^k\theta}_{\text{M1A1}} = \underbrace{\frac{\{1-(1+i\tan\theta)^n\}}{-i\tan\theta}}_{\text{M1A1}}$$

$$\therefore \sum_{0}^{n-1}\cos k\theta\sec^k\theta = \mathrm{Re}\left\{\frac{1-(\cos n\theta + i\sin n\theta)\sec^n\theta}{i\tan\theta}\right\}$$ M1A1

$$= \sin n\theta\sec^n\theta\cot\theta$$ (AG) A1