3 Complex number

3.1 De Movire's theorem

The modulus-argument form of the complex number $z = x + \mathrm{i}y$ is

$$z = r(\cos\theta + \mathrm{i}\sin\theta)$$

where

- r is called the ____________, $r \geqslant 0$.
- θ is called the principal _______________, $-\pi < \theta \leqslant \pi$.

Multiplication and Division:

For two complex numbers $z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$

- $z_1 z_2 =$
- $\dfrac{z_1}{z_2} =$

In the case $r = 1$, thus $z = \cos\theta + \mathrm{i}\sin\theta$, it follows that

$$\begin{aligned} z^2 &= (\cos\theta + \mathrm{i}\sin\theta)^2 = \qquad\qquad\qquad\qquad = \\ z^3 &= \\ z^4 &= \end{aligned}$$

A conjecture can be made that, for any positive integer, n is

$$z^n = (\cos\theta + \mathrm{i}\sin\theta)^n = \tag{3.1}$$

This is **de Movire's theorem**. Furthermore, It will be shown that it is true for any interger n.

Proof:

- For positive integer n:

- For negative integer n:

Notice: if stated in exponential form, de Movire's theorem can be written as:

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \qquad\qquad = \qquad\qquad = \qquad\qquad\qquad .$$

However, above deduction can not be used to prove the theorem directly.

1. Find the value of each of the following complex numbers.

   (a) $\left(\cos\dfrac{\pi}{4} + \mathrm{i}\sin\dfrac{\pi}{4}\right)^8$

   (b) $\left(\sqrt{3} - \mathrm{i}\right)^{18}$

   (c) $\left(\dfrac{1}{1+\mathrm{i}}\right)^6$

   (d) $\dfrac{\left(\cos\dfrac{9}{17}\pi + \mathrm{i}\sin\dfrac{9}{17}\pi\right)^5}{\left(\cos\dfrac{2}{17}\pi - \mathrm{i}\sin\dfrac{2}{17}\pi\right)^3}$

Trigonometric functions of multiple angles

Applying binomial expansion in de Moivre's Theorem, we can find multiple-angle formulae.

2. Find an expression for $\cos 5\theta$ in terms of $\cos\theta$.

$$\begin{aligned}\cos 5\theta + \mathrm{i}\sin 5\theta &= (\cos\theta + \mathrm{i}\sin\theta)^5\\ &= (C + \mathrm{i}S)^5\\ &=\\ &=\\ &=\end{aligned}$$

Hence,

$$\begin{aligned}\cos 5\theta &= \mathrm{Re}\{\cos 5\theta + \mathrm{i}\sin 5\theta\}\\ &=\\ &=\\ &=\\ &=\end{aligned}$$

3. (a) Find $\sin 3\theta$ in terms of $\sin\theta$.

   (b) Find $\dfrac{\sin 6\theta}{\sin\theta}$, $\theta \neq n\pi$, $n \in \mathbf{Z}$, in terms of powers of $\cos\theta$.

4. By considering the form of $\tan 3\theta$, solve the cubic equation $3t^3 + 6t^2 - 9t - 2 = 0$.

5. Show that $\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$, find all solutions for the polynomial

$$2x^5 - 5x^4 - 20x^3 + 10x^2 + 10x = 1.$$

Powers of trigonometric functions:

Given that $z = \cos\theta + \mathrm{i}\sin\theta$, then $\dfrac{1}{z} =$ $\qquad\qquad\qquad$ $=$ $\qquad\qquad\qquad$, hence

$$z + \frac{1}{z} = \tag{3.2}$$

$$z - \frac{1}{z} = \tag{3.3}$$

According to de Movire's theorem, $z^n =$ $\qquad\qquad\qquad$, and $\dfrac{1}{z^n} =$ $\qquad\qquad\qquad$, it follows that

$$z^n + \frac{1}{z^n} = \tag{3.4}$$

$$z^n - \frac{1}{z^n} = \tag{3.5}$$

Use binomial expansion and above results, we can find the expression of powers of trigonometric functions.

6. Express $\cos^5\theta$ in the form $a\cos 5\theta + b\cos 3\theta + c\cos\theta$, where a, b and c are constants.

7. Prove that

$$\sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta.$$

8. (a) Express $\sin^4\theta$ in the form of $d\cos 4\theta + e\cos 2\theta + f$, where d, e and f are constants.

(b) Hence find the exact value of $\displaystyle\int_0^{\frac{\pi}{2}} \sin^4\theta \,\mathrm{d}\theta$.

9. By considering $\left(z - \frac{1}{z}\right)^2 \left(z + \frac{1}{z}\right)^4$, find the constants p, q, r and s such that

$$\sin^2\theta\cos^4\theta = p + q\cos 2\theta + r\cos 4\theta + s\cos 6\theta.$$

Using the substitution $x = 2\cos\theta$, show that

$$\int_1^2 x^4\sqrt{4-x^2}\,\mathrm{d}x = \frac{4}{3}\pi + \sqrt{3}$$

3.2 The nth roots of unity

Every polynomial equation of degree n has exactly n roots, including repeated roots. Therefore equation

$$z^n = 1$$

has n roots. $z = 1$ is one of these roots, if n is even, then $z = -1$ is another. All of the other roots are **complex** roots.

1. (a) Write down the two roots of the equation $z^2 = 1$ and show them on an Argand diagram.

   (b) Use $z^3 - 1 = (z-1)(z^2 + z + 1)$ to find the three roots of $z^3 = 1$. Show them on the Argand diagram.

   (c) Find the four roots of $z^4 = 1$ and show them on Argand diagram.

From the above example, you may have noticed that:

- all the roots lie on a ______________________
- one root at ___________.

In fact, every root of the equation $z^n = 1$ lies on the unit circle, and is equally distributed around the circle.

Proof:

Suppose $z = r(\cos\theta + \mathrm{i}\sin\theta)$, $r > 0$ and $-\pi < \theta \leqslant \pi$, it follows that

$$z^n = r^n(\cos\theta + \mathrm{i}\sin\theta)^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta).$$

Since $|z^n| = 1$, it is true that

$$r^n = 1,$$

it can be deduced that

$$r = \qquad\qquad, \qquad \cos n\theta = \qquad\qquad, \qquad \sin n\theta = \qquad\qquad.$$

Therefore,

$$r = \qquad\qquad, \qquad \theta = \frac{2k\pi}{n}.$$

Generally, as k takes values 0, 1, 2, $\cdots$, $n-1$ the corresponding values of θ are:

$$0, \frac{2\pi}{n}, \frac{4\pi}{n}, \frac{6\pi}{n}, \cdots, \frac{2(n-1)\pi}{n}$$

giving n distinct values of z.

When $k = n$ then $\theta = \qquad\qquad$, which gives the same z as $\theta = 0$. Similarly, any integer values of k larger than n will differ from 0, 1, 2, $\cdots$, $(n-1)$ by a multiple of n, and so gives a value of θ differing by a multiple of 2π from one already listed; the same applies when k is any negative integer.

Therefore, the equation $z^n = 1$ has exactly n roots. These are

$$z_k = \cos\frac{2k\pi}{n} + \mathrm{i}\sin\frac{2k\pi}{n}, \qquad k = 0, 1, 2, 3, \cdots, (n-1) \tag{3.6}$$

These n complex numbers are called the n**th roots of unity**.

It is clear that these roots are on a unit circle and equally distributed around it.

2. Solve the equation $z^6 = 1$. Show the roots on a Argand diagram.

Exponential form of roots of unity

Since
$$e^{i\theta} = \cos\theta + i\sin\theta,$$
substitute $\theta = 0$, we can have $e^{i0} = \cos 0 + \sin 0 = 1$, it follows that
$$\begin{aligned} z^n &= 1 \\ z^n &= e^{i0} \\ z^n &= e^{i(0+2k\pi)} \\ (z^n)^{\frac{1}{n}} &= \left(e^{i(2k\pi)}\right)^{\frac{1}{n}} \\ z &= e^{i\left(\frac{2k\pi}{n}\right)}. \end{aligned}$$
Therefore we can also have exactly n distinct roots of unity:
$$z_k = e^{i\left(\frac{2k\pi}{n}\right)}, \qquad k = 0, 1, 2, 3, \cdots, (n-1). \tag{3.7}$$
These are the exponential form of the n**th roots of unity**.

3. Find in exponential form, the 7th roots of unity, and show the roots on a Argand diagram.

Properties of the nth root of unity

It is common to use Greek letter ω for the root with the smallest positive argument:
$$\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = e^{i\left(\frac{2\pi}{n}\right)}.$$
Then by the de Movire's theorem:
$$\omega^k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$$
so the nth roots of unity can be written as:
$$1, \omega, \omega^2, \omega^3, \cdots, \omega^{n-1}.$$

The roots of unity have some important properties:

- Complex roots of unity occur in conjugate pairs. In particular,

$$(z_k)^* = z_{n-k}. \tag{3.8}$$

- The sum of the nth roots of unity is always 0 because

$$z_0 + z_1 + z_2 + \cdots + z_{n-k} = 1 + \omega^1 + \omega^2 + \cdots + \omega^{n-1} = \qquad\qquad = \qquad . \tag{3.9}$$

$z^n - 1$ can be factorised into linear and quadratic factors with real coefficients:

$$\begin{aligned} z^n - 1 &= (z - z_0)(z - z_1)(z - z_2)\cdots(z - z_{n-2})(z - z_{n-1}) \\ &= (z - z_0)(z - z_1)(z - z_{n-1})(z - z_2)(z - z_{n-2})\cdots \\ &= (z - 1)(z^2 - \qquad - \qquad)(\qquad\qquad)\cdots \\ &= \\ &= \\ &= \end{aligned}$$

Notice, for even and odd number of n, the factorization can be different:

- for odd n,

$$z^n - 1 = \underline{(\qquad)(\qquad)\qquad\qquad\qquad\qquad} \tag{3.10}$$

- for even n,

$$z^n - 1 = \underline{(\qquad)\qquad\qquad\qquad\qquad\qquad} \tag{3.11}$$

4. Factorise the following in linear or quadratic factors.

   (a) $z^5 - 1$

   (b) $z^6 - 1$

5. If ω is a complex cube root of unity,

   (a) Simplify $(1+\omega)(1+\omega^2)$ and $(1+6\omega)(1+6\omega^2)$.

   (b) Prove that $(a+b)(a+\omega b)(a+\omega^2 b) = a^3+b^3$.

6. Find the fifth root of 1 and hence prove that

$$\cos\frac{2}{5}\pi + \cos\frac{4}{5}\pi = -\frac{1}{2}, \quad \text{and} \quad \cos\frac{1}{5}\pi + \cos\frac{3}{5}\pi = \frac{1}{2}.$$

7. (a) Draw an Argand diagram showing the point 1, ω, ω^2, ω^3, ω^4, where $\omega = \cos\frac{2}{5}\pi + \mathrm{i}\sin\frac{2}{5}\pi$.

   (b) If $\alpha = \omega^2$, show that the point 1, α, α^2, α^3, α^4 are the same as the points in the first part, but in a different order. Indicate this order by labelling the points clearly.

   (c) Repeat the second part with α replaced by β, where $\beta = \omega^3$.

8. Prove that all the roots of

$$z^n = (z-1)^n$$

have real part $\frac{1}{2}$.

9. (a) By considering the solutions of the equation $z^n - 1 = 0$ prove that

$$(z-\omega)(z-\omega^2)(z-\omega^3)\cdots(z-\omega^{n-1}) = z^{n-1} + z^{n-2} + \cdots + z + 1$$

where $\omega = \cos\frac{2\pi}{n} + \mathrm{i}\sin\frac{2\pi}{n}$.

(b) There are n points equally spaced around the circumference of a unit circle. Prove that the product of the distances from one of these points to each of the others is n.

(c) By finding expressions for the distances in the previous part, deduce that

$$\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)\cdots\sin\left(\frac{(n-1)\pi}{n}\right) = \frac{n}{2^{n-1}}.$$

3.3 The nth roots of a complex number

To solve the equation

$$z^n = u$$

where u is a non-zero complex number, we first rewrite the complex number u in modulus argument form:

$$u = s(\cos\phi + \mathrm{i}\sin\phi), \quad s > 0, \text{ and } 0 \leqslant \phi < 2\pi$$

and then set $z = r(\cos\theta + \mathrm{i}\sin\theta)$, it follows that

$$\begin{aligned}(r(\cos\theta + \mathrm{i}\sin\theta))^n &= s(\cos\phi + \mathrm{i}\sin\phi) \\ r^n(\cos n\theta + \mathrm{i}\sin n\theta) &= s(\cos(\phi + 2k\pi) + \mathrm{i}\sin(\phi + 2k\pi))\end{aligned}$$

hence

$$r^n = s, \quad \text{and} \quad n\theta = \phi + 2k\pi.$$

It can then be deduced that

$$r = s^{\frac{1}{n}} \quad \text{and} \quad \theta = \frac{\phi + 2k\pi}{n} \quad (k = 0, 1, 2, \cdots, n-1)$$

The first expression indicates that all the roots have the same ____________, and the second one implies that there are ______ distinct nth roots of a complex number.

Therefore the non-zero complex number u has precisely n different nth roots, which are

$$s^{\frac{1}{n}}\left(\cos\left(\frac{\phi + 2k\pi}{n}\right) + \mathrm{i}\sin\left(\frac{\phi + 2k\pi}{n}\right)\right) = s^{\frac{1}{n}}\mathrm{e}^{\mathrm{i}\left(\frac{\phi+2k\pi}{n}\right)}, \tag{3.12}$$

where $k = 0, 1, 2, 3, \cdots, n-1$.

These roots can also be expressed in terms of the nth roots of unity as

$$\alpha, \alpha\omega, \alpha\omega^2, \cdots \alpha\omega^{n-1} \tag{3.13}$$

where

$$\alpha = s^{\frac{1}{n}}\left(\cos\left(\frac{\phi}{n}\right) + \mathrm{i}\sin\left(\frac{\phi}{n}\right)\right) = s^{\frac{1}{n}}\mathrm{e}^{\mathrm{i}\left(\frac{\phi}{n}\right)} \quad \text{and} \quad \omega = \cos\left(\frac{2\pi}{n}\right) + \mathrm{i}\sin\left(\frac{2\pi}{n}\right) = \mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{n}\right)}. \tag{3.14}$$

1. Solve

$$z^{12} = -1,$$

giving all answers in the form $z = r\mathrm{e}^{\mathrm{i}\theta}$.

2. Solve

$$z^6 = 4 + 4\sqrt{3}\mathrm{i},$$

giving all answers in the form $z = r\mathrm{e}^{\mathrm{i}\theta}$. Show all your solutions on an Argand diagram.

3. Find the roots of the equation

$$z^3 = -4\sqrt{3} + 4\mathrm{i},$$

giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \leqslant \theta < 2\pi$.

Denoting these roots by z_1, z_2, z_3, show that, for every positive integer k,

$$z_1^{3k} + z_2^{3k} + z_3^{3k} = 3\left(2^{3k}\mathrm{e}^{\frac{5}{6}k\pi\mathrm{i}}\right).$$

4. The polynomial P is given by

$$(z+2-3\mathrm{i})^3 = 2+2\mathrm{i}.$$

(a) Given that $\omega = z+2-3\mathrm{i}$, find the roots ω, ω^2, ω^3. State, in terms of the z-plane, the value of $\omega+\omega^2+\omega^3$.

(b) Sketch the three solutions on an Argand diagram.

5. Solve the equation

$$(z+\mathrm{i})^n + (z-\mathrm{i})^n = 0.$$

6. (a) If $u = \cos\theta + \mathrm{i}\sin\theta$, show that

$$\frac{1+u}{1-u} = \mathrm{i}\cot\left(\frac{\theta}{2}\right).$$

(b) Write down the roots of

$$u^n = -1,$$

where n is a positive integer.

(c) Hence, by writing u as $\dfrac{x-1}{x+1}$, prove that the roots of $(x-1)^n = -(x+1)^n$ are

$$\mathrm{i}\cot\left(\frac{(2r+1)\pi}{2n}\right)$$

for $r = 0, 1, 2, \cdots, n-1$.

3.4 Complex summations

Geometric sequence

The sum, S_n, of the first n terms of a geometric sequence is

$$S_n = \frac{a(1-r^n)}{1-r}$$

where a is the first term and r is the common ratio, and r is not equal to one.

- 'Coefficient' of common ratio is "1"

Suppose $z = \cos\theta + \mathrm{i}\sin\theta = \mathrm{e}^{\mathrm{i}\theta}$, then summation of a geometric sequence such as

$$\sum_{n=1}^{N} z^n = z + z^2 + z^3 + \cdots + z^N$$

has common ratio z with 'coefficient' 1, this means $1z$, not $2z$, $3z$ or otherwise.

There are usually two methods carried out to convert the **complex denominator** into **real** one.

Method 1: (conjugate multiplication)

Since

$$(1-z)(1-z^{-1}) = 1 - (z + z^{-1}) + 1 = 2 - 2\cos\theta,$$

it follows that

$$\sum_{n=1}^{N} z^n = \frac{z(1-z^N)}{1-z} = \frac{z(1-z^N)(1-z^{-1})}{(1-z)(1-z^{-1})} = \frac{(1-z^N)(z-1)}{2-2\cos\theta} = \frac{z-1-z^{N+1}+z^N}{2-2\cos\theta} \tag{3.15}$$

Method 2: (factorization)

De Movire's theorem is also valid for rational number, hence

$$z^{\frac{1}{2}} + z^{-\frac{1}{2}} = 2\cos\frac{1}{2}\theta$$
$$z^{\frac{1}{2}} - z^{-\frac{1}{2}} = 2\mathrm{i}\sin\frac{1}{2}\theta$$

it follows that

$$\sum_{n=1}^{N} z^n = \frac{z(1-z^N)}{1-z} = \frac{z(1-z^N)}{z^{\frac{1}{2}}(z^{-\frac{1}{2}} - z^{\frac{1}{2}})} = \frac{z^{\frac{1}{2}}(1-z^N)}{-2\mathrm{i}\sin\frac{1}{2}\theta} = \frac{\mathrm{i}z^{\frac{1}{2}}(1-z^N)}{2\sin\frac{1}{2}\theta} = \frac{\mathrm{i}\left(z^{\frac{1}{2}} - z^{N+\frac{1}{2}}\right)}{2\sin\frac{1}{2}\theta} \tag{3.16}$$

1. By considering the expansion of $\sum_{n=0}^{N-1} z^n$, show that

$$\sum_{n=0}^{N-1} \cos n\theta = \frac{1}{2}\left[\sin\left(N - \frac{1}{2}\right)\theta \operatorname{cosec}\left(\frac{1}{2}\theta\right) + 1\right].$$

Proof:

Let $C = 1 + \cos\theta + \cos 2\theta + \cdots + \cos(N-1)\theta$,
and $S = \sin\theta + \sin 2\theta + \cdots + \sin(N-1)\theta$.

Thus $C + iS = 1 + z + z^2 + \cdots + z^{N-1} = \sum_{n=0}^{N-1} z^n$, it follows that $C = \mathrm{Re}\left\{\sum_{n=0}^{N-1} z^n\right\}$.

2. By considering the expansion of $\sum_{n=0}^{N} z^{2n-1}$, show that

$$\sum_{n=0}^{N-1} \cos(2n-1)\theta = \frac{\sin 2N\theta}{2\sin\theta}.$$

- Coefficient of common ratio is NOT "1"

The above Method 1 (conjugate multiplication)is still valid, since

$$(1 - az)(1 - az^{-1}) = 1 - a(z + z^{-1}) + a^2 = 1 + a^2 - 2a\cos\theta \tag{3.17}$$

3. Prove

$$\sum_{n=1}^{N} 2^n \sin n\theta = \frac{2^{N+2}\sin N\theta - 2^{N+1}\sin(N+1)\theta + 2\sin\theta}{5 - 4\cos\theta}.$$

4. By first expanding $\sum_{n=0}^{N-1} \left(\frac{z}{3}\right)^n$, show that

$$\sum_{n=0}^{N} 3^{-n} \cos n\theta = \frac{3^{-N+1}\cos(N-1)\theta - 3^{-N+2}\cos N\theta - 3\cos\theta + 9}{10 - 6\cos\theta}.$$

5. Determine the value of

$$\sum_{n=0}^{\infty} 2^{-n} \sin\left(\frac{n\pi}{2}\right).$$

6. It is given that $u = 1 - \mathrm{e}^{\mathrm{i}\theta}\cos\theta$, where $0 < \theta < \dfrac{\pi}{2}$.

   (a) Express $\mathrm{e}^{\mathrm{i}k\theta}$ and $\mathrm{e}^{-\mathrm{i}k\theta}$ in the form $a + \mathrm{i}b$, show that $u = -\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\sin\theta$.

   (b) Find $|u|$ and $\arg u$. Hence write down the modulus and argument of each of the two roots of u.

   Series C and S are defined by

$$C = \cos\theta\cos\theta + \cos 2\theta\cos^2\theta + \cos 3\theta\cos^3\theta + \cdots + \cos n\theta\cos^n\theta$$
$$S = \sin\theta\cos\theta + \sin 2\theta\cos^2\theta + \sin 3\theta\cos^3\theta + \cdots + \sin n\theta\cos^n\theta$$

   (c) Show that $C + \mathrm{i}S$ is a geometric series, and write down the sum of this series.

   (d) Using the results of the first part, or otherwise, show that $C = \dfrac{\sin n\theta\cos^{n+1}\theta}{\sin\theta}$, and nd a similar expression for S.

Binomial expansion

7. Let $z = \cos\theta + \mathrm{i}\sin\theta$. Use the binomial expansion of $(1+z)^n$, where n is a positive integer, to show that
$$\binom{n}{1}\cos\theta + \binom{n}{2}\cos 2\theta + \cdots \binom{n}{n}\cos n\theta = 2^n \cos^n\left(\frac{1}{2}\theta\right)\cos\left(\frac{1}{2}n\theta\right) - 1.$$
Find
$$\binom{n}{1}\sin\theta + \binom{n}{2}\sin 2\theta + \cdots \binom{n}{n}\sin n\theta$$

8. (a) Show the points 2 and $2 + \cos\frac{2}{3}\pi + \mathrm{i}\sin\frac{2}{3}\pi$ on an Argand diagram, and hence show that

$$2 + \cos\frac{2}{3}\pi + \mathrm{i}\sin\frac{2}{3}\pi = \sqrt{3}\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}.$$

(b) Deduce that

$$\sum_{r=0}^{n} \binom{n}{r} 2^{n-r} \cos\frac{2r\pi}{3} = 3^{\frac{n}{2}} \cos\frac{\pi}{6}.$$

(c) State the corresponding result for

$$\sum_{r=0}^{n} \binom{n}{r} 2^{n-r} \sin\frac{2r\pi}{3}.$$