RECALL: V v.s. over $\mathbb{F}$, $\dim(V) = n$
(char($\mathbb{F}$) > 2)

$\Lambda^k V$: k^{TH} exterior algebra

k-vectors: $u_1 \wedge u_2 \wedge \dots \wedge u_k$, $u_i \in V$

k-linear (can break up linearly in any single coordinate),

alternating: $u_1 \wedge \dots \wedge v \wedge \dots \wedge v \wedge \dots \wedge u_k = 0$

same

$u_1 \wedge \dots \wedge u_j \wedge \dots \wedge u_i \wedge \dots \wedge u_k = -(u_1 \wedge \dots \wedge u_i \wedge \dots \wedge u_j \wedge \dots \wedge u_k)$

switched

$$\dim(\Lambda^k V) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

If $\beta = (v_1, \dots, v_n)$ is a basis for V, then

$$\beta_k = \{ v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{i_k} : 1 \le i_1 < i_2 < \dots < i_k \le n \}$$

is a basis for $\Lambda^k V$

We saw that when $k = n$, $\dim(\Lambda^n V) = 1$ and, if $\beta = (v_1, \ldots, v_n)$ is a basis for V, then $\{v_1 \wedge \cdots \wedge v_n\}$ is a basis for $\Lambda^k V$.

We also showed that $u_1 \wedge \cdots \wedge u_k = 0$ iff $\{u_1, \ldots, u_k\}$ is linearly dependent.

<u>RMK</u> Let $F : V^k \longrightarrow W$ be alternating and k-linear function.

We may define $\hat{F} : \Lambda^k V \longrightarrow W$ by $\hat{F}(u_1 \wedge \cdots \wedge u_k) := F(u_1, u_2, \ldots, u_k)$.

The map $\hat{F}$ can be shown to be <u>linear</u>.

<u>HW 7, # 6</u>: linear when $k = n$.

<u>THM</u> Given $T : V \longrightarrow W$ linear, the <u>induced map</u> $T_* : \Lambda^k V \longrightarrow \Lambda^k W$ given by

$$T_* (v_1 \wedge \dots \wedge v_k) := T(v_1) \wedge \dots \wedge T(v_k)$$

on k-vectors is linear.

<u>PF</u> By the above remark, this is equivalent to showing $T_0 : V^k \longrightarrow \Lambda^k W$ defined by $T_0(v_1, \dots, v_k) := T(v_1) \wedge \dots \wedge T(v_k)$ is alternating and k-linear (and then noting $T_* = \hat{T}_0$).

So: $T_0(v_1, \dots, au_i + bv_i, \dots, v_k)$

$$= T(v_1) \wedge \dots \wedge T(au_i + bv_i) \wedge \dots \wedge T(v_k)$$

$$= T(v_1) \wedge \dots \wedge aT(u_i) + bT(v_i) \wedge \dots \wedge T(v_k)$$

$$= a\big(T(v_1) \wedge \dots \wedge T(u_i) \wedge \dots \wedge T(v_k)\big)$$
$$+ \; b\big(T(v_1) \wedge \dots \wedge T(v_i) \wedge \dots \wedge T(v_k)\big)$$

$$= a\,T_0(v_1, \dots, u_i, \dots, v_k) + b\,T_0(v_1, \dots, v_i, \dots, v_k),$$

and so T_0 is k-linear.

Moreover,

$$T_0(\cdots \wedge v \wedge \cdots \wedge v \wedge \cdots) = \cdots \wedge T(v) \wedge \cdots \wedge T(v) \wedge \cdots$$

linear dependence

$$= 0,$$

and T_0 is alternating. $\square$

<u>**LEMMA**</u> If $S : U \to V$ and $T : V \to W$ are linear, then $(TS)_* = T_* S_*$, where

$$S_* : \Lambda^k U \longrightarrow \Lambda^k V, \qquad T_* : \Lambda^k V \to \Lambda^k W.$$

<u>Pf</u> Let $u_1 \wedge \cdots \wedge u_k \in \Lambda^k U$.

Since $(TS)_*$, T^*, S^* are linear by previous THM, we only need to check on k-vectors.

$$(TS)_*(u_1 \wedge \cdots \wedge u_k)$$

$$= TS(u_1) \wedge \cdots \wedge TS(u_k)$$

$$= T_*(S(u_1) \wedge \cdots \wedge S(u_k))$$

$$= T_* S_* (u_1 \wedge \cdots \wedge u_k). \quad \square$$

Now, we apply this theory to define the <u>determinant</u>:

Let $\dim(V) = n$, $T : V \to V$ linear.

Then $T_* : \Lambda^n V \longrightarrow \Lambda^n V$ is <u>linear</u>,

where $T_*(u_1 \wedge \cdots \wedge u_n) := T(u_1) \wedge \cdots \wedge T(u_n)$.

As we have seen, $\dim(\Lambda^n V) = 1$.

If $\beta = (v_1, \ldots, v_n)$ is a basis for V,

then $(v_1 \wedge v_2 \wedge \cdots \wedge v_n)$ is a basis for $\Lambda^n V$.

Thus, if $y \in \Lambda^n V$, then

$$y = c\,(v_1 \wedge v_2 \wedge \cdots \wedge v_n), \quad c \in \mathbb{F}.$$

Since $T_*(v_1 \wedge \cdots \wedge v_n) \in \Lambda^n V$,

$$T_*(v_1 \wedge \cdots \wedge v_n) = d\,(v_1 \wedge \cdots \wedge v_n)$$

Hence
$$T_*(y) = T_*(c\,(v_1 \wedge \cdots \wedge v_n))$$
$$= c\,T_*(v_1 \wedge \cdots \wedge v_n)$$

$$= c \cdot d\,(v_1 \wedge \cdots \wedge v_n)$$
$$= d \cdot c\,(v_1 \wedge \cdots \wedge v_n)$$
$$= dy$$

Therefore, there exists a scalar $d \in \mathbb{F}$ such that $T_*(x) = dx$ for <u>all</u> $x \in \Lambda^n V$.

<u>DEF</u> Let $\dim(V) = n$. If $T: V \to V$ is linear, then the <u>determinant</u> of T, $\det(T) \in \mathbb{F}$, is the unique scalar such that
$$T_*(x) = \det(T) \cdot x$$
for <u>all</u> $x \in \Lambda^n V$.

Is this really the determinant we're used to??

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\dim(V) = 2$, basis (v_1, v_2)

determinant we're used to: $ad - bc$

$L_A : V \to V$

$L_A(v_1) = av_1 + cv_2$

$L_A(v_2) = bv_1 + dv_2$

Basis for $\Lambda^2 V$: $v_1 \wedge v_2$

so: $(L_A)_* : \Lambda^2 V \longrightarrow \Lambda^2 V$

$$(L_A)_*(v_1 \wedge v_2) = L_A(v_1) \wedge L_A(v_2)$$

$$= (av_1 + cv_2) \wedge (bv_1 + dv_2)$$

$$= (av_1 \wedge (bv_1 + dv_2)) + (cv_2 \wedge (bv_1 + dv_2))$$

$$= (av_1 \wedge bv_1) + (av_1 \wedge dv_2) + (cv_2 \wedge bv_1) + (cv_2 \wedge dv_2)$$

$$= \underbrace{ab(v_1 \wedge v_1)}_{=0} + ad(v_1 \wedge v_2) + bc(v_2 \wedge v_1) + \underbrace{cd(v_2 \wedge v_2)}_{=0}$$

$$= ad(v_1 \wedge v_2) - bc(v_1 \wedge v_2)$$

since we switched $v_2 \wedge v_1$ to $v_1 \wedge v_2$

$$= (ad - bc)(v_1 \wedge v_2).$$

$\uparrow$ $\det(L_x)$

PROP The determinant of T is independent of the choice of basis.

PF Let $(v_1, \dots, v_n)$, $(w_1, \dots, w_n)$ be (ordered) bases.

Let $T_*(v_1 \wedge \dots \wedge v_n) = \det(T)(v_1 \wedge \dots \wedge v_n)$,

so $\det$ is defined in terms of first basis $(v_1, \dots, v_n)$.

$v_1 \wedge \dots \wedge v_n$ is a basis for $\Lambda^n V$,

and so is $w_1 \wedge \dots \wedge w_n$,

so: $w_1 \wedge \dots \wedge w_n = c\,(v_1 \wedge \dots \wedge v_n)$,

for some $c \in \mathbb{F}$, $c \neq 0$.

Thus
$$\begin{aligned} T_*(w_1 \wedge \dots \wedge w_n) &= c\,T_*(v_1 \wedge \dots \wedge v_n) \\ &= c \cdot \det(T) \cdot (v_1 \wedge \dots \wedge v_n) \\ &= \det(T) \cdot (w_1 \wedge \dots \wedge w_n). \quad \square \end{aligned}$$

THM If $S, T : V \to V$ are linear maps, then

$$\det(TS) = \det(T)\det(S).$$

Pf Let $x \in \Lambda^n V$

Then $\det(TS) \cdot x = (TS)_*(x)$

$$= T_* S_* (x)$$

$$= T_* (\det(S) \cdot x)$$

$$= \det(T) \cdot \det(S) \cdot x \qquad \square$$

THM If $\dim(V) = n$ and $T : V \to V$ is linear, then T is invertible iff $\det(T) \neq 0$.

Pf Suppose $\det(T) \neq 0$.

Let $(v_1, \dots, v_n)$ be a basis for V.

Then $T(v_1) \wedge \dots \wedge T(v_n) = T_*(v_1 \wedge \dots \wedge v_n)$

$$= \det(T)\,(v_1 \wedge \cdots \wedge v_n) \neq 0$$

Since $T(v_1) \wedge \cdots \wedge T(v_n) \neq 0$,

$(T(v_1), \ldots, T(v_n))$ is L.I.

$$\Rightarrow \dim(\operatorname{Im}(T)) = n$$

$\Rightarrow$ T bijection $\Rightarrow$ T invertible

Conversely, suppose T is invertible.

Then $(T(v_1), \ldots, T(v_n))$ is basis for V

$\Rightarrow$ $(T(v_1), \ldots, T(v_n))$ is L.I.

$$\Rightarrow T(v_1) \wedge \cdots \wedge T(v_n) \neq 0$$

$$\Rightarrow \det(T) \cdot (v_1 \wedge \cdots \wedge v_n)$$

$$= T_*(v_1 \wedge \cdots \wedge v_n)$$

$$= T(v_1) \wedge \cdots \wedge T(v_n) \neq 0$$

$$\Rightarrow \det(T) \neq 0 \qquad \square$$

COR (1) $\det(I_V) = 1$

(2) If $T : V \to V$ is invertible,

then $\det(T^{-1}) = \dfrac{1}{\det(T)}$.

PF (1): $(I_V)_*(v_1 \wedge \cdots \wedge v_n) = I_V(v_1) \wedge \cdots \wedge I_V(v_n)$

$= v_1 \wedge \cdots \wedge v_n$,

so $\det(I_V) = 1$.

(2) $1 = \det(I_V) = \det(T T^{-1})$

$= \det(T) \cdot \det(T^{-1})$. $\square$

NEXT TIME: A proof that any function "like" the determinant is the determinant.