LAST TIME:

THM Let $T: V \to V$ be linear,
B, C be ordered bases for V,
and Q the matrix that changes coordinates
from B to C.

Then $[T]_B = Q^{-1}[T]_C Q$

(Alternatively: $[T]_C = Q[T]_B Q^{-1}$)

This motivates the following:

DEF Let A, M be matrices in $M_n(\mathbb{F})$.
We say that M is similar to A
if there exists an invertible matrix
Q such that $M = Q^{-1} A Q$.

EX Consider $D: P_2(\mathbb{R}) \to P_2(\mathbb{R})$
by $D(f) = \frac{df}{dx}$

$B = (1, x, x^2)$, $C = (1+x+x^2, 1-x, -x+x^2)$

Suppose Q changes from C to B.

$$Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$1 = \frac{1}{3}(1+x+x^2) + \frac{2}{3}(1-x) - \frac{1}{3}(-x+x^2)$$

$$x = \frac{1}{3}(1+x+x^2) - \frac{1}{3}(1-x) - \frac{1}{3}(-x+x^2)$$

$$x^2 = \frac{1}{3}(1+x+x^2) - \frac{1}{3}(1-x) + \frac{2}{3}(-x+x^2)$$

so: $$Q^{-1} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 2/3 & -1/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}$$

$B = (1, x, x^2)$ $\qquad D(f) = df/dx$

$$[D]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[D]_C = Q^{-1} [D]_B \, Q$$

$$= \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 2/3 & -1/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/3 & 2/3 \\ 0 & 2/3 & -2/3 \\ 0 & -1/3 & -2/3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1/3 & 1/3 \\ 0 & -2/3 & -4/3 \\ -1 & 1/3 & -1/3 \end{bmatrix}$$

Consider $f = 1(1+x+x^2) + 2(1-x) + 3(-x+x^2)$

$$\begin{bmatrix} 1 & -1/3 & 1/3 \\ 0 & -2/3 & -4/3 \\ -1 & 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -16/3 \\ -4/3 \end{bmatrix}$$

$$D(f) = \tfrac{4}{3}\left(1 + x + x^2\right) - \tfrac{16}{3}\left(1 - x\right) - \tfrac{4}{3}\left(-x + x^2\right)$$

CHECK:

$$\begin{array}{cccc} 1 & + x & + x^2 \\ 2 & - 2x & \\ & - 3x & + 3x^2 \\ \hline 3 & - 4x & + 4x^2 \end{array}$$

$$\frac{d}{dx}\left(3 - 4x + 4x^2\right) = -4 + 8x$$

$$\begin{array}{cccc} \tfrac{4}{3} & + \tfrac{4}{3}x & + \tfrac{4}{3}x^2 \\ -\tfrac{16}{3} & + \tfrac{16}{3}x & \\ & + \tfrac{4}{3}x & - \tfrac{4}{3}x^2 \\ \hline -4 & + 8x & \end{array} \quad \checkmark$$

DUAL SPACES

We're now going to consider linear transformations from a v.s. V to its field of scalars $\mathbb{F}$

($\mathbb{F}$ has dimension 1 when viewed as v.s. over $\mathbb{F}$)

DEF A linear transformation $f: V \to \mathbb{F}$ is called a <u>linear functional</u>, and the dual space of V is the v.s. $\mathcal{L}(V, \mathbb{F})$ of linear functionals, denoted V^*.

We'll prove the following result, which will specialize to $\mathcal{L}(V, \mathbb{F})$:

THM Let V, W be finite-dim'l v.s. over $\mathbb{F}$ of dimensions n, m, resp., and let β, γ be ordered bases for V, W, resp. Then the function $\Phi : \mathcal{L}(V, W) \to M_{m \times n}(\mathbb{F})$ defined by $\Phi(T) = [T]_\beta^\gamma$ for $T \in \mathcal{L}(V, W)$ is an isomorphism.

PF First, $\Phi(aS + bT) = [aS + bT]_\beta^\gamma$

$$= a[S]_\beta^\gamma + b[T]_\beta^\gamma$$

$$= a\,\Phi(S) + b\,\Phi(T),$$

and so Φ is linear.

Suppose $\Phi(S) = \Phi(T)$.

Then $[S]_\beta^\gamma = [T]_\beta^\gamma \Rightarrow S = T$

("left-multiplication transformation"),

and so Φ is injective.

Finally, if $A \in M_{m \times n}(\mathbb{F})$, $A = (a_{ij})$,

$\beta = (v_1, \ldots, v_n)$, $\gamma = (w_1, \ldots, w_m)$,

then there is a <u>unique</u> linear trans.

$T : V \to W$ such that

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i ,$$

and so $\Phi(T) = A$, and Φ

is surjective.

Therefore Φ is an isomorphism. $\square$

<u>COR</u> If $V^* = \mathcal{L}(V, \mathbb{F})$ and V is

finite-dim'l, then $\dim(V^*) = \dim(V)$.

Pf $\mathcal{L}(V, \mathbb{F})$ is isomorphic to $M_{1\times n}(\mathbb{F})$, where, $n = \dim(V)$.

So:

$$\dim(V^*) = \dim(M_{1\times n}(\mathbb{F})) = \dim(\mathbb{F}^n) = \dim(V).$$

$\square$

NOTE Since $\dim(V^*) = \dim(V) = n$, V^* is isomorphic to V!

What is a basis for V^*?

Let $B = (v_1, \ldots, v_n)$ be an ordered basis for V.

If $x \in V$, then $x = \sum_{i=1}^{n} a_i v_i$,

i.e., $$[x]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

We define $f_j : V \longrightarrow \mathbb{F}$ by $f_j(x) = a_j$.

In this case, f_j is the j^{TH} coordinate function w/ respect to the basis $\mathcal{B}$.

THM Let V be a finite-dim'l v.s over $\mathbb{F}$ w/ ordered basis $\mathcal{B} = (v_1, \dots, v_n)$. If f_i is the i^{TH} coo-dinate function w/ respect to $\mathcal{B}$, then $\mathcal{B}^* = \{f_1, \dots, f_n\}$ is an ordered basis for V^* and

$$f = \sum_{i=1}^{n} f(v_i)\, f_i$$

for any $f \in V^*$.

Pf Since $\dim(V^*) = \dim(V) = n$, we only need to show $\mathcal{B}^*$ is L.I. to show it's a basis.

Suppose $\sum_{i=1}^{n} a_i f_i = 0$, $a_i \in \mathbb{F}$.

If $x \in V$, then $\sum_{i=1}^{n} a_i f_i(x) = 0$.

In particular, $\sum_{i=1}^{n} a_i f_i(v_j) = 0$

for each j.

But $\sum_{i=1}^{n} a_i f_i(v_j) = a_j$

$\Rightarrow a_j = 0$ for all j,

and so B^* is L.I.

Now, assume $f \in V^*$. Then f is completely determined by $\{f(v_1), \dots, f(v_n)\}$ (linear transformation).

If $g := \sum_{i=1}^{n} f(v_i) f_i$, then

$$g(v_j) = \sum_{i=1}^{n} f(v_i) f_i(v_j) = \sum_{i=1}^{n} f(v_i) \delta_{ij}$$
$$= f(v_j).$$

Hence $f = g = \sum_{i=1}^{n} f(v_i) f_i$,

and the result follows. $\square$

<u>DEF</u> The ordered basis B^* defined above is called the <u>dual basis</u> of B. It satisfies

$$f_i(v_j) = \delta_{ij} \quad \text{for all } i, j.$$

<u>THM</u> Let V, W be v.s. over $\mathbb{F}$ w/ ordered bases B, C, resp.

Given any linear $T : V \to W$, the mapping $T^t : W^* \longrightarrow V^*$ defined by $T^t(g) = gT$ for all $g \in W^*$ is linear and

$$[T^t]_{C^*}^{B^*} = \left([T]_B^C\right)^t$$

matrix transpose.

<u>Pf</u> First, $V \xrightarrow{T} W \xrightarrow{g} \mathbb{F}$, so $gT : V \to \mathbb{F}$ is linear

and $T^t(g) = gT \in V^*$.

Thus T^t is well-defined.

Let $f, g \in W^*$, $a, b \in \mathbb{F}$.

$$\begin{aligned} T^t(af + bg) &= (af + bg)T \\ &= afT + bgT \\ &= a\,T^t(f) + b\,T^t(g), \end{aligned}$$

and so T^t is linear.

(That $[T^t]_{C^*}^{B^*} = \left([T]_B^C\right)^t$ is a standard computation; see p 121). $\square$

At this point, we want to justify calling V^* the <u>dual</u> space of V, and so we want a natural identification between V^{**} and V.

Given $x \in V$, we define $\hat{x} : V^* \to \mathbb{F}$

by $\hat{x}(f) := f(x)$ for all $f \in V^*$

LEMMA For all $x \in V$, $\hat{x} \in V^{**}$.

Pf

$$\begin{aligned} \hat{x}(af+bg) &= (af+bg)(x) \\ &= af(x) + bg(x) \\ &= a\hat{x}(f) + b\hat{x}(g). \quad \square \end{aligned}$$

THM Let V be a finite-dim'l v.s. over $\mathbb{F}$. The map $\psi : V \to V^{**}$ given by $\psi(x) = \hat{x}$ is an isomorphism.

Pf: NEXT TIME

READ: § 3.1
(We won't cover 2.7 in class, but read if you're interested)