DETERMINANTS

READ §4.1–4.5!

We will be proceeding differently.

The definition you're "used to":

DEF Let A be an $n \times n$ matrix w/ entries in $\mathbb{F}$.

If $n = 1$ and $A = (a)$, then $\det(A) = a$

$(\det : M_1(\mathbb{F}) \to \mathbb{F})$

If $n = 2$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det : M_2(\mathbb{F}) \to \mathbb{F}$ given by

$$\det(A) = ad - bc.$$

In general, we define the determinant recursively by

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(\tilde{A}_{1j}),$$

where $A = (a_{ij})$ and $\tilde{A}_{1j}$ is the matrix defined by deleting row 1, column j of A.

EX $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

determinant of $\begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}$

$$\det(A) = 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \overbrace{\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(-3) - 2(-6) + 3(-3)$$

$$= -3 + 12 - 9 = 0$$

NOTE This definition is dependent on the matrix A (as opposed the underlying linear transformation). If we view A as corresponding to the linear transformation L_A, then the determinant depends on the choice of basis (when, actually, it does not depend on choice of basis).

Also, it's very difficult to work with.

Our goal is to provide a basis-free definition of a determinant of a linear transformation.

DEF Let V, W be v.s. over $\mathbb{F}$. A map

$$F : \underbrace{V \times V \times \cdots \times V}_{\substack{k \text{ copies} \\ \text{in } k \text{ coordinates}}} \longrightarrow W$$

is <u>k-linear</u> if it is linear in each coordinate, i.e., if

$$F(v_1, \ldots, v_{i-1}, au_i + bv_i, v_{i+1}, \ldots, v_k)$$

$$= a\,F(v_1, \ldots, v_{i-1}, u_i, v_{i+1}, \ldots, v_k)$$

$$+ b\,F(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_k)$$

for <u>all</u> i.

DEF A map $F : \overbrace{\underbrace{V \times V \times \cdots \times V}_{k \text{ copies}}}^{V^k} \longrightarrow W$ is <u>alternating</u> if, whenever two distinct co-ordinates are the same vector, then the output is 0, i.e.,

$$F(\ldots, \underbrace{v, \ldots, v}_{\text{same}}, \ldots) = 0.$$

<u>LEMMA</u> Suppose $F : V^k \to W$ is k-linear.
Then F is alternating iff interchanging two co-ordinates changes the sign of F, i.e.,

switched from original position

$$F(v_1, v_2, \ldots, v_{i-1}, v_j, v_{i+1}, \ldots, v_{j-1}, v_i, v_{j+1}, \ldots, v_k)$$

$$= -F(v_1, \ldots, v_k),$$

assuming $\operatorname{char}(\mathbb{F}) > 2$.

<u>Pf</u> Assume F is k-linear and alternating.
Then: $0 = F(\ldots, \underbrace{v_i + v_j}_{i^{\text{TH}}\text{ coordinate}}, \ldots, \underbrace{v_i + v_j}_{j^{\text{TH}}\text{ coordinate}}, \ldots)$

$$= F(\ldots, v_i + v_j, \ldots, v_i, \ldots)$$
$$+ F(\ldots, v_i + v_j, \ldots, v_j, \ldots)$$

$$= F(\ldots, v_i, \ldots, v_i, \ldots) + F(\ldots, v_j, \ldots, v_i, \ldots)$$
$$+ F(\ldots, v_i, \ldots, v_j, \ldots) + F(\ldots, v_j, \ldots, v_j, \ldots)$$

$$= 0 + F(\ldots, v_j, \ldots, v_i, \ldots)$$
$$+ F(\ldots, v_i, \ldots, v_j, \ldots) + 0, \quad \text{(alternating)}$$

so

$$F(\ldots, v_j, \ldots, v_i, \ldots) = -F(\ldots, v_i, \ldots, v_j, \ldots)$$

On the other hand, suppose F is k-linear and inter-changing entries flips the sign.

Then $F(\ldots, v, \ldots, v, \ldots) = -F(\ldots, v, \ldots, v, \ldots)$

$$\Rightarrow \quad 2 \cdot F(\ldots, v, \ldots, v, \ldots) = 0$$

$$\Rightarrow \quad F(\ldots, v, \ldots, v, \ldots) = 0$$

(assuming $\mathrm{char}(\mathbb{F}) > 2$). $\square$

PROP The set $\mathcal{L}(V^k, W)$ of k-linear maps from V^k to W is a v.s. over $\mathbb{F}$.

Pf EXERCISE. $\square$

PROP The set $\mathcal{L}^{Alt}(V^k, W)$ of all alternating k-linear maps from V^k to W is a subspace of $\mathcal{L}(V^k, W)$.

Pf The zero map (i.e., every k-tuple is sent to 0) is in $\mathcal{L}^{Alt}(V^k, W)$, so $\mathcal{L}^{Alt}(V^k, W) \neq \emptyset$.

Let $a, b \in \mathbb{F}$, $F, G \in \mathcal{L}^{Alt}(V^k, W)$.

We know that $aF + bG$ is k-linear, since $\mathcal{L}(V^k, W)$ is a v.s.

We must show that $aF + bG \in \mathcal{L}^{Alt}(V^k, W)$, i.e., we must show that $aF+bG$ is <u>alternating</u>.

$$\underbrace{(aF+bG)}(\ldots, v, \ldots, v, \ldots)$$

$$= aF(\ldots, v, \ldots, v, \ldots) + bG(\ldots, v, \ldots, v, \ldots)$$

$$= a0 + b0 \quad \left(\begin{matrix}F, G \text{ both} \\ \text{alternating}\end{matrix}\right)$$

$$= 0 + 0 = 0,$$

so $aF + bG \in \mathcal{L}^{Alt}(V^k, W)$. $\square$

<u>DEF</u> The <u>k^{TH} exterior (or wedge) product</u>

$$\Lambda^k V \qquad (V: \text{v.s.})$$

is a vector space equipped w/ a map

$$\psi: V^k \longrightarrow \Lambda^k V$$

called <u>exterior multiplication</u>.

$\psi(v_1, v_2, \ldots, v_k)$ is denoted by $v_1 \wedge v_2 \wedge \ldots \wedge v_k$

and is called a <u>pure/simple k-wedge</u> (or <u>k-blade</u> or <u>k-vector</u>).

The pair $(\Lambda^k V, \psi)$ satisfies:

(1) ψ is <u>alternating k-linear</u>, so

$$v_1 \wedge \dots \wedge (a u_i + b v_i) \wedge \dots \wedge v_k$$

$$= a(v_1 \wedge \dots \wedge u_i \wedge \dots \wedge v_k) + b(v_1 \wedge \dots \wedge v_i \wedge \dots \wedge v_k)$$

<u>and</u>

$$v_1 \wedge \dots \wedge v \wedge \dots \wedge v \wedge \dots \wedge v_k = 0$$

same

(2) Given a basis $\beta = (v_1, \dots, v_n)$ of V, the collection

$$\beta_k = \{ v_{i_1} \wedge \dots \wedge v_{i_k} : 1 \le i_1 < i_2 < \dots < i_k \le n \}$$

is a basis for $\Lambda^k V$.