RECALL: <u>adjoint T^*</u>: $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$

for all $x, y \in V$

<u>normal</u>: $TT^* = T^*T$

<u>THE SPECTRAL THM</u> (COMPLEX INNER PRODUCT SPACES)

T: linear operator on a finite-dim'l complex inner product space V

T is normal $\iff$ there exists an orthonormal basis for V consisting of eigenvectors for T.

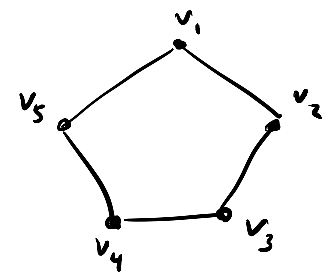
<u>Hermitian</u> (or <u>self-adjoint</u>): $T = T^*$

We proved that: If T is a Hermitian operator on a finite-dim'l inner product space V, then:

(a) Every <u>eigenvalue</u> for T is <u>real</u>.

(b) If V is a <u>real</u> inner product space, then χ_T <u>splits</u> over $\mathbb{R}$.

EX Consider again the <u>adjacency matrix</u> (or incidence matrix) for the graph $\Gamma = C_5$



$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Since $A = A^*$, χ_A <u>splits</u> over $\mathbb{R}$

and all eigenvalues of A are real.

How many ways to "walk" from v_1 to v_i in $\underline{\underline{2}}$ steps?

Now, $$A^{2} = \begin{pmatrix} 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{pmatrix}$$

so: $$A^2 + A = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix} = I + \underbrace{J}_{\text{all 1's matrix}}$$

so: $A^2 + A - I = J$

$$A^2 = J + I - A$$

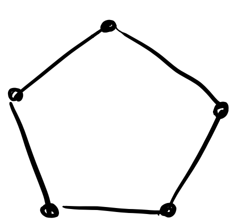
We know from the Spectral Thm (Complex) that, since $A^*A = AA^*$ (since $A = A^*$), there is an orthonormal basis of eigenvectors, say $\beta = (v_1, v_2, v_3, v_4, v_5)$

Corresp. eigenvalues: $\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \lambda_5$

$$A^2 v_i = J v_i + I v_i - A v_i$$

$$\lambda_i^2 v_i = J v_i + v_i - \lambda_i v_i$$

OBSERVATION: $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector for A w/ eigenvalue 2



Each vertex has exactly $\underline{\underline{2}}$ neighbors.

so: $\beta = (v_1, v_2, v_3, v_4, v_5)$

Let $v_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow A v_1 = 2 v_1$

Otherwise, for all $i \geq 2$, $\langle v_1, v_i \rangle = 0$

(orthonormal basis) $v_i = (a_1, a_2, a_3, a_4, a_5)$

<u>standard inner product</u>: $\sum_{j=1}^{5} \frac{1}{\sqrt{5}} 1 \cdot a_j = \frac{1}{\sqrt{5}} \sum_{j=1}^{5} a_j$

$= 0$

$$\Rightarrow \sum_{j=1}^{5} a_j = 0$$

so, if $J = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$,

then $J v_i = \begin{pmatrix} \sum_{j=1}^{5} a_j \\ \sum_{j=1}^{5} a_j \\ \vdots \\ \sum_{j=1}^{5} a_j \end{pmatrix} = 0.$

$$\lambda_i^2 v_i = J v_i + v_i - \lambda_i v_i = v_i - \lambda_i v_i$$

so: $(\lambda_i^2 + \lambda_i - 1) v_i = 0$

v_i: eigenvector, so $v_i \neq 0$

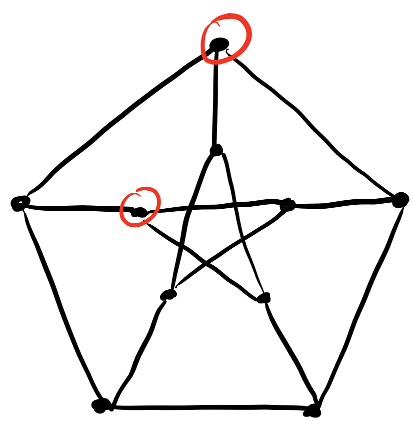
$$\Rightarrow \quad \lambda_i^2 + \lambda_i - 1 = 0$$

$$\Rightarrow \quad \lambda_i = \frac{-1 \pm \sqrt{5}}{2}$$

So: eigenvalues of A are $\underset{\text{mult } 1}{2}$, $\underset{\text{mult. } 2}{\frac{-1+\sqrt{5}}{2}}$, $\underset{\text{mult. } 2}{\frac{-1-\sqrt{5}}{2}}$.

For example, if Γ were the <u>Petersen graph</u>,

then Γ :



you could calculate the eigenvalues in much the same way.

(Here, $A^2 + A = J + 2I$)

<u>FURTHER READING</u> : <u>strongly regular graph</u>

Γ is (v, k, λ, μ) - strongly regular if:

(1) Γ has v vertices

(2) Every vertex of Γ has <u>exactly</u> k neighbors

(3) If $v_i \sim v_j$ (there is an edge from v_i to v_j), then there are <u>exactly</u> λ vertices v_k such that $v_k \sim v_i$, $v_k \sim v_j$.

(4) If $v_i \not\sim v_j$, then there are exactly μ vertices v_k such that $v_k \sim v_i$, $v_k \sim v_j$.

<u>So</u>: C_5 : $(5, 2, 0, 1)$ - strongly regular

Petersen: $(10, 3, 0, 1)$ - strongly regular

Similar techniques work well for <u>strongly regular graphs</u>

THE SPECTRAL THM (REAL INNER PRODUCT SPACES)

Let T be a linear operator on a finite-dim'l real inner product space V.

Then,

T is Hermitian (self-adjoint, $T=T^*$) iff there exists an orthonormal basis β of V consisting of eigenvectors for T.

Pf Suppose first that T is Hermitian. $(T=T^*)$

By the LEMMA, χ_T splits, so there exists an orthonormal basis β such that $[T]_\beta$ is upper triangular by Schur's Thm.

But $[T]_\beta^* = [T^*]_\beta = [T]_\beta$,

so $[T]_\beta^*$ is also upper triangular.

$$\Rightarrow \quad [T]_\beta = [T]_\beta^* = \overline{[T]_\beta}^t \underset{\uparrow}{=} [T]_\beta^t$$

real inner product space

so $[T]_\beta$ is diagonal,

i.e., β is an orthonormal basis of eigenvectors for T.

Conversely, suppose β is an orthonormal basis of eigenvectors for T.

Then $[T]_\beta$ is diagonal.

Since V is a real inner product space, each eigenvector and each eigenvalue for T are real, and so $[T]_\beta$ is real

$$\Rightarrow \quad [T]_\beta^* = \overline{[T]_\beta}^t$$

$$= [T]_\beta^t \quad \text{(real)}$$

$$= [T]_\beta \quad \text{(diagonal)}$$

and thus $T = T^*$ and T is Hermitian. $\square$

<u>NEXT TIME</u> : Jordan Canonical Form