LINEAR COMBINATIONS AND SYSTEMS OF LINEAR EQUATIONS

Consider a vector space V.

How do we "build" subspaces?

Or, more accurately, what structure do subspaces have?

v.s. over $\mathbb{R}$

EX Let $V = \mathbb{R}^3$, the set of ordered triples of elements of $\mathbb{R}$.

Pick a vector in $\mathbb{R}^3$, say $v = (1,2,3)$.

How do we create a subspace W of V containing v?

We could take $W = \mathbb{R}^3$, but this is overkill. Let's try to make W as small as possible.

Can we get away w/ $W = \{(1,2,3)\}$?

<u>NO</u>. It's not closed under vector addition:

$$(1,2,3) + (1,2,3) = (2,4,6) \notin \{(1,2,3)\}$$

What about $W := \{n(1,2,3) : n \in \underbrace{\mathbb{N}}_{\text{positive integers}}\}$?

Now W is closed under addition...

but no <u>zero vector</u>!

Also, no inverses for vectors.

What about $W := \{n(1,2,3) : n \in \mathbb{Z}\}$?

Not closed under scalar mult.!

Certainly $\{r(1,2,3) : r \in \mathbb{R}\} \subseteq W$.

Conversely, $U := \{r(1,2,3) : r \in \mathbb{R}\}$ <u>is</u> a subspace.

$$0(1,2,3) = (0,0,0) \in U$$

Suppose $u \in U$, $r \in \mathbb{R}$.

Then $u = s(1,2,3)$, where $s \in \mathbb{R}$

$$\Rightarrow \quad ru = r\big(s(1,2,3)\big)$$
$$= (rs)\cdot(1,2,3) \in U$$

Suppose $u, v \in U$.

Then $u = s(1,2,3)$, $v = t(1,2,3)$

$$u + v = (s+t)\cdot(1,2,3) \in U.$$

So U <u>is</u> a subspace of V.

Geometrically, $\{r(1,2,3) : r \in \mathbb{R}\}$ is the "one-dimensional" subspace which consists of the line through the origin and $(1,2,3)$.

Generally, if W is a subspace of V (v.s. over $\mathbb{F}$) and $v \in W$, then

$$\operatorname{span}(\{v\}) := \{av : a \in \mathbb{F}\} \subseteq W$$

Ex $V = \mathbb{R}^3$ v.s. over $\mathbb{R}$,

two vectors: $v = (1,2,3)$, $w = (0,0,1)$

What can we say about a subspace containing both?

Certainly, from previous example,

$$\text{span}(\{v\}) = \{a(1,2,3) : a \in \mathbb{R}\} \subseteq W$$
$$\text{span}(\{w\}) = \{b(0,0,1) : b \in \mathbb{R}\} \subseteq W$$

BUT W must also be closed under vector addition!

For instance,

$$2 \cdot (1,2,3) + 1 \cdot (0,0,1) = (2,4,7) \in W$$

Thus

$$\{a(1,2,3) + b(0,0,1) : a, b \in \mathbb{R}\} \subseteq W$$

Indeed, generally if V is a v.s. over $\mathbb{F}$,

$$\text{span}(\{v,w\}) := \{av + bw : a, b \in \mathbb{F}\}$$

is always contained in any subspace

containing both v, w.

Moreover, span$(\{v,w\})$ is actually a subspace!

<u>DEF</u> Let V be a v.s. over a field $\mathbb{F}$ and S a nonempty subset of V.
A vector $v \in V$ is called a <u>linear combination</u> of vectors of S if there exist a finite number of vectors $u_1, \dots, u_n \in S$ and scalars $a_1, \dots, a_n \in \mathbb{F}$ such that $v = a_1 u_1 + a_2 u_2 + \dots + a_n u_n$.

v: <u>linear combination</u> of $u_1, \dots, u_n$;

$a_1, \dots, a_n$: coefficients of the linear combination

The <u>span</u> of S, denoted span(S), is the set consisting of all linear combinations of the vectors in S.

For convenience, span$(\phi) := \{0\}$.

THM The span of any <u>subset</u> S of a v.s. V (over $\mathbb{F}$) is a subspace of V. Moreover, any subspace of V that contains S must also contain the span of S.

Pf $\text{span}(\phi) = \{0\}$, which is a subspace of V (<u>and</u> is contained in <u>every</u> subspace of V), so we may assume $S \neq \phi$.

Let $v \in S$. Then $v = 1v \in \text{span}(S)$, so $\text{span}(S) \neq \phi$

Let $x, y \in \text{span}(S)$.

So: $x = \sum_{i=1}^{m} a_i x_i$, $y = \sum_{i=1}^{n} b_i y_i$, where $x_i, y_i \in S$.

Then:

$$x + y = a_1 x_1 + \dots + a_m x_m + b_1 y_1 + \dots + b_n y_n \in \text{span}(S)$$

NOTE It could be that $x_2 = y_1$, for instance, or $x_1 = x_m = y_3$.

We never specified that the vectors in the sum were distinct, so this is OK!

Moreover, for any $c \in \mathbb{F}$,

$$cx = c\left(\sum_{i=1}^{m} a_i x_i\right) = \sum_{i=1}^{m} c(a_i x_i)$$

$$= \sum_{i=1}^{m} (ca_i) x_i \in \operatorname{span}(S)$$

Therefore, $\operatorname{span}(S)$ is a subspace.

Now, let W be a subspace of V such that $S \subseteq W$.

Let $x \in \operatorname{span}(S)$. Then, for some $n \in \mathbb{N}$, there exist $x_1, \dots, x_n \in S$ and $a_1, \dots, a_n \in \mathbb{F}$ such that

$$x = \sum_{i=1}^{n} a_i x_i .$$

Since W is a subspace,

each $a_i x_i \in W$ (closed under scalar mult)

A brief inductive argument:

$a_1 x_1 \in W$ ✓

Assume $a_1 x_1 + \cdots + a_k x_k \in W$ for some $k \in \mathbb{N}$.

Then $a_1 x_1 + \cdots + a_k x_k + a_{k+1} x_{k+1}$

$$= \underbrace{(a_1 x_1 + \cdots + a_k x_k)}_{\in W \text{ by ind. hyp}} + \underbrace{(a_{k+1} x_{k+1})}_{\in W \text{ by above}}$$

$\in W$ (closed under addition)

Therefore, $\text{span}(S) \subseteq W$. $\square$

<u>REMARK</u> $\text{span}(S)$ does <u>not</u> depend on the order in which the elements of S are listed, i.e.,

$$\text{span}(\{v, w\}) = \text{span}(\{w, v\}), \text{ etc.}$$

DEF A set S is said to span (respectively, generate) the v.s. V if $\text{span}(S) = V$, i.e., if every vector of V is a linear combination of elements of S.

If $\text{span}(S) = V$, then S is a spanning set (resp., generating set) for V.

Q: Do spanning sets always exist?

YES. $\text{span}(V) = V$, so such a set always exists!

READ §1.5!

LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

When $\mathbb{F}$ is an infinite field (such as $\mathbb{R}$), if V is a vector space over $\mathbb{F}$ and W is a subspace (that is not the zero subspace!), then it is clear that W is infinite.

(Take $w \neq 0 \in W$. Then $\text{span}(\{w\})$ is infinite!)

So: While $\text{span}(W) = W$, it is likely that we can find a (much) smaller set S such that $\text{span}(S) = W$?

How do we tell that we have such a set? And how do we know it's <u>minimal</u>?

<u>EX</u> Consider $v_1 = (1,2,3,4)$, $v_2 = (1,1,1,1)$,

$v_3 = (2, 7, 12, 17)$, $v_4 = (0, 0, 0, 1)$

in $\mathbb{R}^4$ (v.s. over $\mathbb{R}$).

$S = \{v_1, v_2, v_3, v_4\}$, and let

$W = \text{span}(S)$.

Can we find a smaller set $S' \subseteq S$ such that $\underline{W = \text{span}(S')}$??