RECALL: V : v.s. over $\mathbb{F}$, an ordered field
(for us, $\mathbb{F}$: $\mathbb{R}$ or $\mathbb{C}$)

inner product on V : $\langle \, , \rangle : V \times V \longrightarrow \mathbb{F}$
satisfying for all $x, y, z \in V$, $c \in \mathbb{F}$:

linear in 1ST coordinate
(a) $\langle x+z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$

(b) $\langle cx, y \rangle = c \langle x, y \rangle$

(c) $\langle y, x \rangle = \overline{\langle x, y \rangle}$
(bar : complex conjugation)

(d) $\langle x, x \rangle > 0$ if $x \neq 0$

standard inner product (or **dot product**) on $\mathbb{F}^n$:

If $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{F}^n$,

then $\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$.

A vector space V equipped w/ an inner product: **inner product space**

DEF Let $A \in M_{m \times n}(\mathbb{F})$. The *conjugate transpose* (or *adjoint*) of A is the $n \times m$ matrix A^* defined by

$$(A^*)_{ij} := \overline{A_{ji}} \quad \text{for all } i, j.$$

Ex If $A = \begin{pmatrix} 1 & 2+i & 3-i \\ 4 & 5+6i & 7 \end{pmatrix}$,

then $A^* = \begin{pmatrix} 1 & 4 \\ 2-i & 5-6i \\ 3+i & 7 \end{pmatrix}$.

DEF Let $V = M_n(\mathbb{F})$ and define

$$\langle A, B \rangle := \operatorname{tr}(B^* A) \quad \text{for } A, B \in V.$$

This is the *Frobenius inner product*.

Check to see this is an inner product:

(a) $\langle A+B, C\rangle = tr(C^*(A+B))$

$= tr(C^*A + C^*B)$

$= tr(C^*A) + tr(C^*B)$

$= \langle A, C\rangle + \langle B, C\rangle$

(b) $\langle cA, B\rangle = tr(B^*(cA))$

$= tr(c(B^*A))$

$= c \cdot tr(B^*A)$

$= c\langle A, B\rangle$

(c) $\langle B, A\rangle = tr(A^*B)$

$= tr(\overline{A}^t B)$

$= tr(\overline{A}^t (B^t)^t)$

$= tr((B^t\overline{A})^t)$

$= tr(B^t\overline{A})$

$$= \overline{\overline{tr\left(B^t \bar{A}\right)}}$$

$$= \overline{tr\left(\bar{B}^t A\right)}$$

$$= \overline{tr\left(B^* A\right)} = \overline{\langle A, B \rangle}.$$

(d): Suppose $A \neq O$, $A \in M_n(\mathbb{F})$.

$$\langle A, A \rangle = tr\left(A^* A\right)$$

$$= \sum_{i=1}^{n} \left(A^* A\right)_{ii}$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} (A^*)_{ij} A_{ji} \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{A_{ji}} A_{ji}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} |A_{ji}|^2 > 0,$$

since $A \neq O \Rightarrow$ @ least one entry of A is nonzero

THM Let V be an inner-product space. Then, for $x, y, z \in V$ and $c \in \mathbb{F}$, the following are true:

(a) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$

(b) $\langle x, cy \rangle = \bar{c} \langle x, y \rangle$

(c) $\langle x, 0 \rangle = \langle 0, x \rangle = 0$

in V in $\mathbb{F}$

(d) $\langle x, x \rangle = 0$ iff $x = 0$

(e) If $\langle x, y \rangle = \langle x, z \rangle$ for all $x \in V$, then $y = z$.

Pf (a): BOOK

(b), (c): EXERCISES

(d): Suppose $x = 0$.

Then $\langle 0, 0 \rangle = 0$ by (c)

(OR: $\langle 0, 0 \rangle = \langle 0+0, 0 \rangle$

$$= \langle 0,0\rangle + \langle 0,0\rangle$$

$$\Rightarrow \langle 0,0\rangle = 0 \;)$$

Now suppose $\langle x,x\rangle = 0$.

By DEF of inner product, if $x \neq 0$,

then $\langle x,x\rangle > 0$.

Since $\langle x,x\rangle = 0$, we have $x = 0$.

(e) Suppose $\langle x,y\rangle = \langle x,z\rangle$ for <u>all</u> $x \in V$.

<u>WANT</u>: $y = z$

<u>IDEA</u>: For any $x \in V$, since $\langle x,y\rangle = \langle x,z\rangle$,

$$0 = \langle x,y\rangle - \langle x,z\rangle$$

$$= \langle x,y\rangle + \langle x,\boxed{-1}\, z\rangle$$

$$= \langle x, y-z\rangle$$

Let $x = y-z$.

Then $\langle y-z,y\rangle = \langle y-z,z\rangle$, so

$$\langle y-z,y-z\rangle = \langle y-z,y\rangle - \langle y-z,z\rangle = 0$$

$$\Rightarrow y-z = 0 \text{ by (d)}$$

$\Rightarrow \quad y = z . \qquad \square$

<u>DEF</u> Let V be an inner product space. For $x \in V$, we define the <u>norm</u> (or <u>length</u>) of x by

$$\|x\| := \sqrt{\langle x, x \rangle}$$

<u>EX</u> If $V = \mathbb{F}^n$, $\langle \cdot , \cdot \rangle$ is the dot product, and $x = (x_1, \ldots, x_n)$,

then $\|x\| = \sqrt{\langle x, x \rangle}$

$$= \sqrt{\sum_{i=1}^{n} x_i \overline{x_i}}$$

$$= \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

<u>THM</u> Let V be an inner product space over $\mathbb{F}$. Then, for all $x, y \in V$ and all $c \in \mathbb{F}$, the following are true:

(a) $\|cx\| = |c| \cdot \|x\|$

(b) $\|x\| = 0$ iff $x = 0$; in any case,
$\|x\| \geq 0$

(c) (Cauchy - Schwarz Inequality)
$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

(d) (Triangle Inequality)
$$\|x + y\| \leq \|x\| + \|y\|$$

<u>Pf</u> (a): $\|cx\| = \sqrt{\langle cx, cx \rangle}$

$$= \sqrt{c \langle x, cx \rangle}$$

$$= \sqrt{c\bar{c} \langle x, x \rangle}$$

$$= \sqrt{|c|^2 \langle x, x \rangle}$$

$$= |c| \sqrt{\langle x, x \rangle} = |c| \cdot \|x\|$$

(b) $\|x\| = 0 \iff \sqrt{\langle x, x \rangle} = 0$

$\iff \langle x, x \rangle = 0 \iff x = 0$

If $x \neq 0$, then $\langle x, x \rangle > 0$

$$\Rightarrow \quad \|x\| = \sqrt{\langle x, x \rangle} > 0.$$

(c) **WANT:** $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$

IDEA: $\langle x - y, x - y \rangle = \langle x, x \rangle - \langle y, x \rangle - \langle x, y \rangle + \langle y, y \rangle$

If $y = 0$, then

$$|\langle x, 0 \rangle| = 0 = \|x\| \cdot 0 = \|x\| \cdot \|0\| \quad \checkmark$$

in $\mathbb{F}$ in V

Suppose $y \neq 0$. Notice by (b), for any $c \in \mathbb{F}$,

$$0 \leq \|x - cy\|^2 = \langle x - cy, x - cy \rangle$$

$$= \langle x, x \rangle - \bar{c} \langle x, y \rangle - c \langle y, x \rangle + c\bar{c} \langle y, y \rangle$$

Let $c := \dfrac{\langle x, y \rangle}{\langle y, y \rangle}$ (ok, since $y \neq 0$)

so:

$$\langle x,x\rangle - \frac{\overline{\langle x,y\rangle}}{\langle y,y\rangle}\langle x,y\rangle - \frac{\langle x,y\rangle}{\langle y,y\rangle}\overset{\overline{\langle x,y\rangle}}{\overset{\shortparallel}{\langle y,x\rangle}} + \frac{\langle x,y\rangle\overline{\langle x,y\rangle}}{\langle y,y\rangle} \geq 0$$

$$\Rightarrow \quad \langle x,x\rangle - \frac{\overline{\langle x,y\rangle}\langle x,y\rangle}{\langle y,y\rangle} \geq 0$$

$$\langle x,x\rangle\langle y,y\rangle \geq \overline{\langle x,y\rangle}\langle x,y\rangle$$

$$\|x\|^2\,\|y\|^2 \geq |\langle x,y\rangle|^2$$

$$\Rightarrow \quad |\langle x,y\rangle| \leq \|x\|\cdot\|y\| \quad \square$$