RECALL:

THM (GRAM-SCHMIDT ORTHOGONALIZATION)

V: inner product space

$S = \{w_1, \dots, w_n\}$ is a L.I. subset of V.

Define $S' := \{v_1, \dots, v_n\}$ recursively, where:

$v_1 := w_1$, and, when $2 \le k \le n$,

$$v_k := w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j .$$

Then, S' is an orthogonal set of nonzero vectors s.t. $\mathrm{span}(S') = \mathrm{span}(S)$.

IDEA: Assume $\{v_1, \dots, v_{k-1}\}$ is an orthogonal set.

Let $1 \le i \le k-1$.

$$\langle v_k, v_i \rangle = \left\langle w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\langle v_j, v_j \rangle} v_j ,\ v_i \right\rangle$$

$$= \langle w_k, v_i \rangle - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\langle v_j, v_j \rangle} \underbrace{\langle v_j, v_i \rangle}_{}$$

Since $\{v_1, \dots, v_{k-1}\}$ is orthogonal, only time $\langle v_j, v_i \rangle \neq 0$ is when $i = j$

$$= \langle w_k, v_i \rangle - \frac{\langle w_k, v_i \rangle}{\langle v_i, v_i \rangle} \cdot \langle v_i, v_i \rangle = 0$$

So: $\{v_1, \ldots, v_k\}$ is an orthogonal set ...

THE ADJOINT OF A LINEAR OPERATOR

For a matrix, $A^* := \overline{A}^t \quad (= \overline{A^t})$.

In some sense, the adjoint operation $(*)$ acts like complex conjugation.

First, a preliminary result:

THM Let V be a finite-dim'l inner-product space over $\mathbb{F}$, and let $g : V \to \mathbb{F}$ be a linear transformation. Then, there exists a unique $y \in V$ such that $g(x) = \langle x, y \rangle$ for all $x \in V$.

PF First, we choose an orthonormal basis

$$\beta = \{v_1, \ldots, v_n\} \text{ for } V.$$

Given: $g : V \to \mathbb{F}$ linear.

We define $y := \sum_{i=1}^{n} \overline{g(v_i)}\, v_i$.

For each $v_j \in \beta$,

$$\langle v_j, y \rangle = \left\langle v_j, \sum_{i=1}^{n} \overline{g(v_i)}\, v_i \right\rangle$$

$$= \sum_{i=1}^{n} \langle v_j, \overline{g(v_i)}\, v_i \rangle$$

$$= \sum_{i=1}^{n} g(v_i) \underbrace{\langle v_j, v_i \rangle}_{\substack{\text{orthonormal basis means} \\ \langle v_j, v_i \rangle = \delta_{ij}}}$$

$$= g(v_j) \langle v_j, v_j \rangle = g(v_j)$$

Since $\langle \cdot, \cdot \rangle$ is linear in first coordinate, the map $\langle \cdot, y \rangle : V \to \mathbb{F}$ given by $x \longmapsto \langle x, y \rangle$ is linear.

Since this linear map agrees w/ g on β, $g(x) = \langle x, y \rangle$ for <u>all</u> $x \in V$.

For uniqueness, if $\langle x, y \rangle = \langle x, y' \rangle$ for all $x \in V$, then $y = y'$! $\square$

THM Let V be a finite-dim'l inner product space, and let T be a linear operator on V. Then, there exists a unique function $T^* : V \longrightarrow V$ such that

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle$$

for all $x, y \in V$. Furthermore, T^* is linear.

Pf Let $y \in V$. Define $g : V \longrightarrow \mathbb{F}$ by

$$g(x) = \langle T(x), y \rangle$$

for all $x \in V$.

If $a, b \in \mathbb{F}$ and $u, v \in V$,

$$\begin{aligned} g(au + bv) &= \langle T(au+bv), y \rangle \\ &= \langle aT(u) + bT(v), y \rangle \\ &= a\langle T(u), y \rangle + b\langle T(v), y \rangle \\ &= a\, g(u) + b\, g(v), \end{aligned}$$

and so g is linear.

so: $g(x) := \langle T(x), y \rangle$ is a linear map from V to $\mathbb{F}$.

By the previous THM, there is a unique vector $y' \in V$ such that

$$g(x) = \langle x, y' \rangle \quad \text{for all } x \in V.$$

Define $T^*: V \longrightarrow V$ by $T^*(y) := y'$.

Since the vector y' is unique by the previous THM, T^* is unique.

It remains to show that T^* is linear.

For $x, u, v \in V$, $a, b \in \mathbb{F}$,

$$\begin{aligned} \langle x, T^*(au+bv) \rangle &= \langle T(x), au+bv \rangle \\ &= \bar{a} \langle T(x), u \rangle + \bar{b} \langle T(x), v \rangle \\ &= \bar{a} \langle x, T^*(u) \rangle + \bar{b} \langle x, T^*(v) \rangle \\ &= \langle x, a T^*(u) + b T^*(v) \rangle \end{aligned}$$

Since x is arbitrary, this holds for <u>all</u> $x \in V$, and so

$$T^*(au+bv) = aT^*(u) + bT^*(v),$$

and T^* is linear. $\square$

<u>DEF</u> The T^* defined in the previous THM is the <u>adjoint</u> of T.

When V is <u>infinite</u>-dim'l, the adjoint of a linear operator T is defined to be the function T^* such that

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle \text{ for all}$$

$x, y \in V$, <u>provided it exists</u>.

<u>THM</u> Let V be a finite-dim'l inner product space and let β be an orthonormal basis for V. If T is a linear operator on V, then

$$[T^*]_\beta = ([T]_\beta)^*.$$

Pf Let $A = [T]_\beta$, $B = [T^*]_\beta$,

where $\beta = \{v_1, \dots, v_n\}$.

Column j of $B = [T^*]_\beta$ is $[T^*(v_j)]_\beta$,

and, since $T^*(v_j) = \underbrace{\sum_{i=1}^{n} \langle T^*(v_j), v_i \rangle v_i}_{\text{b/c } \beta \text{ is orthonormal}}$

$$\left(\text{so: } B = [T^*]_\beta = \begin{array}{c} j^{\text{TH}} \\ \downarrow \\ \left(\begin{array}{ccc} & \langle T^*(v_j), v_1 \rangle & \\ & \langle T^*(v_j), v_2 \rangle & \\ & \vdots & \\ & \langle T^*(v_j), v_n \rangle & \end{array} \right) \end{array} \right)$$

$$\begin{aligned} B_{ij} &= \langle T^*(v_j), v_i \rangle \\ &= \overline{\langle v_i, T^*(v_j) \rangle} \\ &= \overline{\langle T(v_i), v_j \rangle} \end{aligned}$$

$$\left(\text{By similar reasoning, } A = [T]_\beta = \begin{array}{c} i^{\text{TH}} \\ \downarrow \\ \left(\begin{array}{ccc} & \langle T(v_i), v_1 \rangle & \\ & \langle T(v_i), v_2 \rangle & \\ & \vdots & \\ & \langle T(v_i), v_n \rangle & \end{array} \right) \end{array} \right)$$

so: $B_{ij} = \overline{\langle T(v_i), v_j \rangle} = \overline{A_{ji}}$

$$= (A^*)_{ij}$$

Therefore, $[T^*]_\beta = B = A^* = ([T]_\beta)^*$. $\square$

<u>COR</u> If $A \in M_n(\mathbb{F})$, then

$$L_{A^*} = (L_A)^*.$$

<u>PROPERTIES OF ADJOINTS</u>

<u>THM</u> Let V be an inner product space, T, U linear operators on V. Then,

(a) $(T+U)^* = T^* + U^*$

(b) $(cT)^* = \bar{c}\, T^*$ for any $c \in \mathbb{F}$

(c) $(TU)^* = U^* T^*$

(d) $T^{**} = T$

(e) $I^* = I$

Pf of (a) Let T, U be linear operators on V.

Then, for any $x, y \in V$,

$$\langle x, (T+U)^*(y) \rangle = \langle (T+U)(x), y \rangle$$

$$= \langle T(x) + U(x), y \rangle$$

$$= \langle T(x), y \rangle + \langle U(x), y \rangle$$

$$= \langle x, T^*(y) \rangle + \langle x, U^*(y) \rangle$$

$$= \langle x, (T^* + U^*)(y) \rangle$$

Therefore, $(T+U)^* = T^* + U^*$. $\square$

COR Let A, B be $n \times n$ matrices. Then,

(a) $(A+B)^* = A^* + B^*$

(b) $(cA)^* = \bar{c} A^*$ for all $c \in \mathbb{F}$

(c) $(AB)^* = B^* A^*$

(d) $A^{**} = A$

(e) $I^* = I$