JORDAN CANONICAL FORM

Let T be a linear operator on a finite-dim'l v.s. V over $\mathbb{F}$, and suppose the characteristic polynomial $\chi_T(t)$ splits. (For example, if V is a v.s. over $\mathbb{C}$, then $\chi_T(t)$ always splits.)

As we have seen, there's not always a basis β such that $[T]_\beta$ is diagonal.

HOWEVER, when $\chi_T(t)$ splits, there's always a "nice" representation of T, which we call the Jordan Canonical Form (or Jordan Normal Form).

How does T look in this form?

In this form,

$$[T]_\beta = \begin{pmatrix} A_1 & & & O \\ & A_2 & & \\ & & \ddots & \\ O & & & A_k \end{pmatrix}$$ (Block diagonal)

and each matrix A_i is either of the form (λ_i) (if A_i is 1×1)

or $$\begin{pmatrix} \lambda_i & 1 & & O \\ & \lambda_i & 1 & \\ & & \ddots & \ddots \\ & & & 1 \\ O & & & \lambda_i \end{pmatrix}$$

Such an A_i: <u>Jordan block</u> corresponding to the eigenvalue λ_i of T

Such a basis that puts T in this form: <u>Jordan Canonical Basis</u>

As it turns out, the Jordan Canonical Form (or JCF) for each operator T (w/ a χ_T that splits) is <u>unique</u> <u>up</u> <u>to</u>

the ordering of the Jordan blocks.

However, it's not (always) immediate from the characteristic polynomial what the JCF will be!

<u>EX</u> Suppose $\chi_T(t) = \underbrace{t^2}_{2}\underbrace{(t-1)^4}_{4}\underbrace{(t-2)}_{1}\underbrace{(t-3)}_{1}$

The sizes of the Jordan blocks are constrained by the mult. of each eigenvalue, but that's pretty much it.

The JCF could be:
$$\begin{pmatrix} \boxed{0} &&&&&&& \\ & \boxed{0} &&&&&& \\ && \boxed{1} &&&&& \\ &&& \boxed{1} &&&& \\ &&&& \boxed{1} &&& \\ &&&&& \boxed{1} && \\ &&&&&& \boxed{2} & \\ &&&&&&& \boxed{3} \end{pmatrix}$$

<u>OR</u>:
$$\begin{pmatrix} 0 & 1 &&&&&& \\ & 0 &&&&&& \\ && 1 & 1 &&&& \\ &&& 1 & 1 &&& \\ &&&& 1 & 1 && \\ &&&&& 1 && \\ &&&&&& \boxed{2} & \\ &&&&&&& \boxed{3} \end{pmatrix}$$

OR :

$$\begin{pmatrix} 0 & & & & & & & \\ & 0 & & & & & & \\ & & 1 & 1 & & & & \\ & & & 1 & 1 & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 2 & \\ & & & & & & & 3 \end{pmatrix}$$

OR :

$$\begin{pmatrix} 0 & 1 & & & & & & \\ & 0 & & & & & & \\ & & 1 & 1 & & & & \\ & & & 1 & & & & \\ & & & & 1 & 1 & & \\ & & & & & 1 & & \\ & & & & & & 2 & \\ & & & & & & & 3 \end{pmatrix}, \text{ etc. !}$$

Q : How do we determine the JCF ?

Consider the matrix $[T]_\beta = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$

which is in JCF (w/ one Jordan Block)

w/r.t. the basis $\beta = \{v_1, v_2, v_3\}$

Consider $\boxed{T(v_1)} = 3v_1$, and so v_1 is an eigenvector.

$$\Rightarrow \boxed{(T-3I)(v_1) = 0}$$

Consider $\boxed{T(v_2)} = v_1 + 3v_2$

$$\Rightarrow (T-3I)(v_2) = v_1$$

$$\Rightarrow \boxed{(T-3I)^2(v_2) = (T-3I)(v_1) = 0}$$

Similarly, $T(v_3) = v_2 + 3v_3$

$$\Rightarrow (T-3I)(v_3) = v_2$$

$$\Rightarrow \boxed{(T-3I)^3(v_3) = 0}$$

so: If v lies in a Jordan canonical basis of a linear operator T and is associated w/ Jordan block w/ diagonal entries λ, then

$$(T-\lambda I)^m(v) = 0$$

for <u>some</u> $m \in \mathbb{N}$.

<u>DEF</u> T: linear operator on V

λ: scalar

A nonzero vector $x \in V$ is called a <u>generalized eigenvector of T corresponding to λ</u> if $(T - \lambda I)^m (x) = 0$

for some $m \in \mathbb{N}$.

<u>DEF</u> T: linear operator on V

λ: eigenvalue of T

The <u>generalized eigenspace of T</u> (corresp. to λ), denoted K_λ, is the subset of V defined by

$$K_\lambda := \{ x \in V : (T - \lambda I)^m (x) = 0 \text{ for } \underline{\text{some}}\ m \in \mathbb{N} \}$$

For instance, the generalized eigenspace always contains the eigenspace.

FACT: If $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of T (where χ_T splits), then
$$V = \bigoplus_{i=1}^{k} K_{\lambda_i} .$$

FACT: Each K_λ is a T-invariant subspace

IDEA FOR FINDING JCF: Find *cycles of generalized eigenvectors*

DEF T: linear operator on V

x: gen. eigenvector corresp. to eigenvalue λ

If r is the smallest integer such that $(T-\lambda I)^r(x) = 0$, then the ordered set

$$\{ (T-\lambda I)^{r-1}(x), (T-\lambda I)^{r-2}(x), \ldots, (T-\lambda I)^2(x), (T-\lambda I)(x), x \}$$

is the cycle of generalized eigenvectors of T corresp. to λ.

EX Consider $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

Then $\chi_A(t) = (t-1)^3(t-2)$

so: $\dim(K_2) = 1$, so $\underset{\text{generalized eigenspace}}{K_2} = \underset{\text{eigenspace}}{E_2}$.

Indeed, $A - 2I = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \Rightarrow (A-2I)(x) = \begin{pmatrix} -x_1 + x_2 + x_3 + x_4 \\ -x_2 + x_3 + x_4 \\ -x_3 + 2x_4 \\ 0 \end{pmatrix}$

If $x \in \ker(A-2I)$,

Then: $x_3 = 2x_4$

$x_2 = x_3 + x_4 = 3x_4$

$x_1 = x_2 + x_3 + x_4 = 6x_4$

So: $x = x_4 \begin{pmatrix} 6 \\ 3 \\ 2 \\ 1 \end{pmatrix}$

What about K_1?

$$A - I = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$(A - I)(x) = \begin{pmatrix} x_2 + x_3 + x_4 \\ x_3 + x_4 \\ 2x_4 \\ x_4 \end{pmatrix}$$

If $x \in \ker(A - I)$, then

$x_4 = 0 \Rightarrow x_3 = 0 \Rightarrow x_2 = 0$

so: $\mathrm{Ker}(A - I) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

Since $\dim(\mathrm{Ker}(A-I)) = 1$, there is only <u>one</u> cycle of generalized eigenvectors.

Immediately, this means that the JCF for A is:

$$\begin{pmatrix} \boxed{\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & \boxed{2} \end{pmatrix}$$

Let's finish by actually finding the right basis.

$$(A-I)^2 = \begin{pmatrix} 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$x \in \mathrm{Ker}((A-I)^2) \implies \begin{pmatrix} x_3 + 4x_4 \\ 3x_4 \\ 2x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad x_4 = 0, \quad x_3 = 0$$

Thus $\ker\left((A-I)^2\right) = \operatorname{span}\left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \right\}$

eigenvector

generalized eigenvector

$$(A-I)^3 = \begin{pmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

If $x \in \ker\left((A-I)^3\right)$, then $\begin{pmatrix} 6x_4 \\ 3x_4 \\ 2x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \quad x_4 = 0$$

<u>THUS</u>: $v = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \in \ker\left((A-I)^3\right)$,

$$v \notin \ker\left((A-I)^2\right)$$

WANT: $\left\{ (A-I)^2 v, \ (A-I)v, \ v \right\}$

so: $(A-I)(v) = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$,

$(A-I)^2(v) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Let $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 3 \\ 2 \\ 1 \end{pmatrix} \right\}$.

If S is COB matrix $\begin{pmatrix} 1 & 1 & 0 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,

then $S^{-1}AS = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.