<u>RECALL</u>: Linear operator $T: V \to V$ ($\longleftrightarrow$ matrix A)
is diagonalizable when we have a basis of eigenvectors

If $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues of T w/ resp. eigenvectors $v_1, v_2, \dots, v_k$, then $\{v_1, \dots, v_k\}$ are L. I.

Furthermore, if S_i is a L. I. subset of the eigenspace E_{λ_i} (for each i), then

$$S := S_1 \cup \dots \cup S_k$$

is L. I.

<u>THM</u> Let T be a linear operator on a finite-dim'l v.s. V such that the characteristic polynomial $\chi_T(t)$ splits, and let $\lambda_1, \dots, \lambda_k$ be the distinct eigenvalues of T.

Then: (i) T is diagonalizable iff the multiplicity of λ_i equals $\dim(E_{\lambda_i})$

for all i.

(ii) If T is diagonalizable and β_i is an ordered basis for E_{λ_i} for each i, then $\beta = (\beta_1, \dots, \beta_k)$ is an ordered basis for V consisting of eigenvectors of T.

(Here: $\beta = (\beta_1, \dots, \beta_k)$ means, if $\beta_i = (v_{i1}, \dots, v_{in_i})$, then $\beta = (v_{11}, \dots, v_{1n_1}, v_{2,1}, \dots, v_{kn_k})$)

Pf Let m_i denote the mult. of λ_i, $d_i := \dim(E_{\lambda_i})$, $n = \dim(V)$.

Suppose first T is diagonalizable, and let β be a basis of eigenvectors.

For each i, let $\beta_i := \beta \cap E_{\lambda_i}$ and let $n_i := |\beta_i|$.

Since β_i is a L.I. subset of E_{λ_i},

$n_i \le d_i \ (= \dim(E_{\lambda_i}))$,

and we know $d_i \le m_i$ ($=$ mult of λ_i).

Since $|\beta| = n$, $\sum_{i=1}^{k} n_i = n$.

On the other hand, we know that

$$\sum_{i=1}^{k} m_i = n, \text{ since } \chi_T \text{ splits.}$$

Thus: $$n = \sum_{i=1}^{k} n_i \le \sum_{i=1}^{k} d_i \le \sum_{i=1}^{k} m_i = n$$

$$\Rightarrow \sum_{i=1}^{k} (m_i - n_i) = \sum_{i=1}^{k} (m_i - d_i) = 0$$

Since $m_i - d_i \ge 0$ for all i,

we conclude $n_i = m_i = d_i$ for all i.

This shows that each β_i is a basis for E_{λ_i}.

Conversely, suppose $m_i = d_i$ for each i.

For each i, let β_i be a basis for E_{λ_i}, and let

$$\beta = (\beta_1, \ldots, \beta_k).$$

By the result we mentioned @ beginning of lecture, β is L.I.

Since $d_i = m_i$ for all i,

$$\sum_{i=1}^{k} d_i = \sum_{i=1}^{k} m_i = n,$$

so $|\beta| = n$, β is a basis, and T is diagonalizable.

(NOTE: This also shows (ii).) $\square$

TEST FOR DIAGONALIZATION:

(1) See if χ_T splits

(2) If it does split, for each eigenvalue λ of T, check if $\text{mult } \lambda = n - \text{rank}(T - \lambda I)$

$(\longleftrightarrow \text{mult } \lambda = \dim(\ker(T - \lambda I)))$

DEF **direct sum**

$W_1, \dots, W_k$: subspaces of V

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k = \bigoplus_{i=1}^{k} W_i$$

means: (1) $V = \sum_{i=1}^{k} W_i = \{ w_1 + \cdots + w_k : w_i \in W_i \}$

(2) $W_j \cap \sum_{i \neq j} W_i = \{0\}$

for each j

THM A linear operator T on a finite-dim'l v.s. V is diagonalizable iff V is the direct sum of the eigenspaces of T.

Pf Let $\lambda_1, \dots, \lambda_k$ be the distinct eigenvalues

First, suppose T is diagonalizable.

Then $V = \sum_{i=1}^{k} E_{\lambda_i}$ by previous THM.

Suppose $v \in E_{\lambda_j} \cap \sum_{i \neq j} E_{\lambda_i}$

This means $(-v_j) = \sum_{i \neq j} v_i$, where $v_i \in E_{\lambda_i}$

$$\Rightarrow \sum_{i=1}^{k} v_i = 0$$

$\Rightarrow$ each $v_i = 0$ (LEMMA from last time)

$\Rightarrow v = 0$

Hence $V = \bigoplus_{i=1}^{k} E_{\lambda_i}$

Conversely, suppose $V = \bigoplus_{i=1}^{k} E_{\lambda_i}$.

Then $\sum_{i=1}^{k} \dim(E_{\lambda_i}) = \dim(V) = \sum_{i=1}^{k} \operatorname{mult}(\lambda_i)$

$\Rightarrow \dim(E_{\lambda_i}) = \operatorname{mult}(\lambda_i)$,

so, by previous THM,

T is diagonalizable. $\square$

MATRIX LIMITS AND MARKOV CHAINS

DEF Let $L, A_1, A_2, \ldots$ be $n \times p$ matrices w/ entries in $\mathbb{C}$.

The sequence $A_1, A_2, \ldots$ is said to <u>converge</u> to the $n \times p$ matrix L, called the limit of the sequence, if

$$\lim_{m \to \infty} (A_m)_{ij} = L_{ij}$$

for all $1 \leq i \leq n$, $1 \leq j \leq p$.

NOTATION: $\lim_{m \to \infty} A_m = L$

Ex If

$$A_m = \begin{pmatrix} \frac{2m^2+m}{5m^2+43m-187} & 5 \\ \left(\frac{1}{2}\right)^m & \left(1+\frac{1}{m}\right)^m \end{pmatrix},$$

then $\lim_{m \to \infty} A_m = \begin{pmatrix} \frac{2}{5} & 5 \\ 0 & e \end{pmatrix}$.

Since limits of matrices are essentially limits of sequences in each entry/coordinate, we can exploit known properties of limits: sums, products by constant scalar, etc.

__EX__ if $\lim_{m \to \infty} A_m = L$ and $c \in \mathbb{C}$,

then $\lim_{m \to \infty} (c A_m) = c \lim_{m \to \infty} A_m = cL$.

In particular:

__THM__ If $\lim_{m \to \infty} A_m = L$ and Q is an invertible matrix, then

$$\lim_{m \to \infty} \left(Q A_m Q^{-1} \right) = Q L Q^{-1}.$$

DEF <u>open unit disk</u> $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$

RECALL: If $z = a + bi$,
$|z| = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}}$.

THM Let A be a square matrix w/ complex entries. Then $\lim_{m \to \infty} A^m$ exists iff both of the following conditions hold:

(i) Every eigenvalue of A is contained in $\mathbb{D} \cup \{1\}$.

(ii) If 1 is an eigenvalue of A, then the dimension of the eigenspace corresponding to 1 equals the mult. of 1 as an eigenvalue of A.

IDEA: $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is <u>bad</u> for the purpose of taking limits, since $B^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$,

and so $\lim_{m \to \infty} B^m$ DNE.

Failing condition (ii) leads to a situation like this.

<u>Easier</u>:

<u>THM</u> Let $A \in M_n(\mathbb{C})$ satisfy:

(i) Every eigenvalue of A is contained in $\mathbb{D} \cup \{1\}$

(ii) A is diagonalizable.

Then $\lim_{m \to \infty} A^m$ exists.

<u>IDEA</u>: If $QAQ^{-1} = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, then:

$$QA^mQ^{-1} = (QAQ^{-1})^m = \operatorname{diag}(\lambda_1^m, \dots, \lambda_n^m)$$

$$\Rightarrow \quad A^m = Q^{-1} \operatorname{diag}(\lambda_1^m, \dots, \lambda_n^m)\, Q.$$

If $\lambda_i \in \mathbb{D} \cup \{1\}$, then

$$\lim_{m\to\infty} \lambda_i^m = \begin{cases} 0, & \lambda_i \in \mathbb{D} \\ 1, & \text{if } \lambda_i = 1 \end{cases}$$

Result follows.