LAST TIME: A : $n \times n$ matrix

$\rho_i(A) :=$ sum of abs. value of entries of row i of A

$$r_i := \rho_i(A) - |A_{ii}| = \sum_{j \neq i} |A_{ij}|$$

i^{TH} Gerschgorin Disk:

$$C_i := \{ z \in \mathbb{C} : |z - A_{ii}| \leq r_i \}$$

THM (Gerschgorin's Disk Thm) Let $A \in M_n(\mathbb{C})$. Then every eigenvalue of A is contained in some Gerschgorin disk.

Pf Let λ be an eigenvalue of A w/ corresponding eigenvector $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$.

Since $Av = \lambda v$, for each i we have:

$$\sum_{j=1}^{n} A_{ij} v_j = \lambda v_i$$

Since v is an eigenvector, $v \neq 0$.

We choose v_k to be the entry in vector rep'n of v that has <u>largest</u> absolute value.

Note that:

$$|\lambda v_k - A_{kk} v_k| = \left| \sum_{j=1}^{n} A_{kj} v_j - A_{kk} v_k \right|$$

$$= \left| \sum_{j \neq k} A_{kj} v_j \right|$$

$$\leq \sum_{j \neq k} |A_{kj}| \, |v_j|$$

$$\leq \sum_{j \neq k} |A_{kj}| \, |v_k|$$

$$= |v_k| \cdot \sum_{j \neq k} |A_{kj}| = |v_k| \cdot r_k$$

$$\Rightarrow |v_k| \, |\lambda - A_{kk}| = |\lambda v_k - A_{kk} v_k| \leq |v_k| \cdot r_k$$

$$\Rightarrow |\lambda - A_{kk}| \leq r_k$$

$$\Rightarrow \lambda \text{ is in } C_k. \qquad \square$$

RECALL: We used this to show that, if A is a transition matrix, then $|\lambda| \le 1$ for **all** eigenvalues λ.

EX

$$A = \begin{pmatrix} 5 & 2 & 3i \\ 1 & 2 & 3 \\ 1+i & 2 & 4 \end{pmatrix}$$

$\rho_1(A) = 5 + 2 + 3 = 10$

$\rho_2(A) = 6$

$\rho_3(A) = \sqrt{1+1} + 2 + 4 = \sqrt{2} + 6$

$r_1 = 10 - 5 = 5$

$r_2 = 6 - 2 = 4$

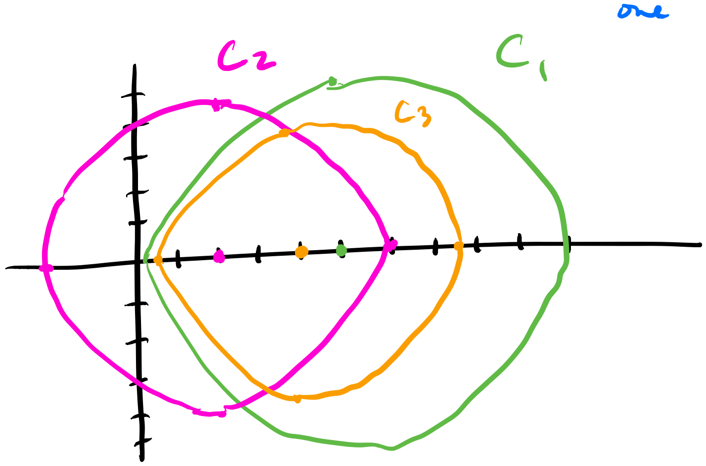
$r_3 = \sqrt{2} + 6 - 4 = \sqrt{2} + 2$

$C_1 = \{ z \in \mathbb{C} : |z - 5| \le 5 \}$

$C_2 = \{ z \in \mathbb{C} : |z - 2| \le 4 \}$

$C_3 = \{ z \in \mathbb{C} : |z - 4| \le \sqrt{2} + 2 \}$

every eigenvalue of A is in one of these three disks



RECALL: **transition matrix**: all entries nonnegative, all columns sum to 1

(**regular**: some power of transition matrix A has **all** entries nonzero)

THM Every transition matrix has 1 as an eigenvalue.

Pf If A is a transition matrix, then each column of A sums to 1.

Thus
$$A^t \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \text{sum of row } 1 \\ \text{sum of row } 2 \\ \vdots \\ \text{sum of row } n \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Since 1 is an eigenvalue of A^t, 1 is also an eigenvalue of A. $\square$

FACT 1: If A is a regular $n \times n$ transition matrix, then the multiplicity of the eigenvalue 1 is 1.

(To prove this, we need facts about "Jordan Canonical Form" from Ch 7)

FACT 2: Moreover, proven in book (see 5.18 - 5.19, pp 298 - 300) that if λ is an eigenvalue of a regular transition matrix A and $|\lambda| = 1$, then $\lambda = 1$.

FACT 3: $\lim\limits_{m \to \infty} A^m$ exists for a regular transition matrix A

(Why: A has eigenvalue 1 w/ mult 1, all other eigenvalues λ have $|\lambda| < 1$)

THM Let A be an $n \times n$ regular transition matrix. If $L := \lim\limits_{m \to \infty} A^m$, then the columns of L are identical and equal to the unique probability vector v that is an eigenvector for A corresp. to eigenvalue 1. Moreover, if w is any probability vector, then $\lim_{m\to\infty}(A^m w) = v$.

Pf Since $\lim_{n\to\infty} A^m = L$,

$$AL = A \lim_{m\to\infty} A^m = \lim_{m\to\infty} A^{m+1} = L,$$

so each column of L is an eigenvector for A corresp. to eigenvalue 1.

Let $\mathbb{1} := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

Since $A^t \mathbb{1} = \mathbb{1}$, $(A^t)^m \mathbb{1} = \mathbb{1}$ for all $m \in \mathbb{N}$.

Hence $$\mathbb{1} = \lim_{m\to\infty} (A^t)^m \mathbb{1} = \lim_{m\to\infty} (A^m)^t \mathbb{1}$$
$$= L^t \mathbb{1}$$

$\Rightarrow$ Thus each row of L^t sums to 1, and hence each <u>column</u> of L sums to 1, i.e., each column of L is the unique probability vector v that is an eigenvector for A w/ eigenvalue 1.

Finally, if w is a prob. vector, then

$$\lim_{m\to\infty} (A^m w) = \left(\lim_{m\to\infty} A^m\right) w$$

$$= Lw$$

Since $w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$ is a prob. vector, $\sum_{i=1}^{n} w_i = 1$.

Each column of L equals v, where $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

so: $L = \begin{pmatrix} v_1 & v_1 & \cdots & v_1 \\ v_2 & v_2 & \cdots & v_2 \\ \vdots & \vdots & & \vdots \\ v_n & v_n & \cdots & v_n \end{pmatrix}$

Thus $Lw = \begin{pmatrix} \sum_{i=1}^{n} v_1 w_i \\ \sum_{i=1}^{n} v_2 w_i \\ \vdots \\ \sum_{i=1}^{n} v_n w_i \end{pmatrix} = \begin{pmatrix} v_1 \cdot \sum_{i=1}^{n} w_i \\ v_2 \cdot \sum_{i=1}^{n} w_i \\ \vdots \\ v_n \cdot \sum_{i=1}^{n} w_i \end{pmatrix}$

$$= \begin{pmatrix} v_1 \cdot 1 \\ v_2 \cdot 1 \\ \vdots \\ v_n \cdot 1 \end{pmatrix} = v. \qquad \square$$

<u>DEF</u> v in this situation is called the <u>fixed probability vector</u> or <u>stationary vector</u>.

INVARIANT SUBSPACES AND THE CAYLEY-HAMILTON THEOREM

<u>IDEA</u>: Eigenvectors are useful b/c $T(v) = \lambda v$, i.e., $T(v) \in \text{span}(v)$.

We can consider such subspaces more generally.

DEF Let T be a linear operator on a v.s. V. A subspace W of V is a <u>T-invariant subspace</u> of V if $T(W) \subseteq W$, i.e., if $T(w) \in W$ for all $w \in W$.

EX

- $\{0\}$
- V
- E_λ for any eigenvalue λ
- $\mathrm{Ker}(T)$
- Since $T: V \to V$, $T(V) = \mathrm{Im}(T)$ is T-invariant