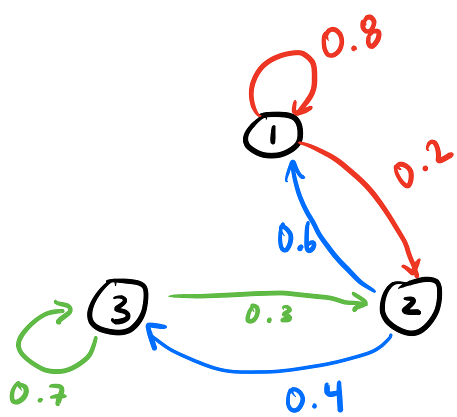
RECALL: **Transition matrix**: • square $(n \times n)$
• all entries are real, nonnegative
• all columns sum to 1

DEF A transition matrix is called **regular** if **some** power of the matrix contains only positive entries.

LAST TIME: We had an example w/ graph on 6 vertices. While we had a transition matrix, it was **not** regular, since every power had a 0 somewhere.

EX



$$M = \begin{pmatrix} 0.8 & 0.6 & 0 \\ 0.2 & 0 & 0.3 \\ 0 & 0.4 & 0.7 \end{pmatrix}$$

M itself has 0's for entries

However : $M^2 = \begin{pmatrix} 0.76 & 0.48 & 0.18 \\ 0.16 & 0.24 & 0.21 \\ 0.08 & 0.28 & 0.61 \end{pmatrix}$

Since M^2 has all positive entries,

M is a <u>regular</u> transition matrix.

Let's analyze this system.

Suppose we start on ①, ②, ③ w/

probabilities p_1, p_2, p_3, resp., so $p_1 + p_2 + p_3 = 1$.

<u>probability vector</u> : $\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = w$

Where we "eventually" end up:

$$\lim_{n \to \infty} (M^n w)$$

$$M - tI = \begin{pmatrix} 0.8-t & 0.6 & 0 \\ 0.2 & -t & 0.3 \\ 0 & 0.4 & 0.7-t \end{pmatrix}$$

$$\begin{aligned} \chi_M(t) &= \det(M - tI) \\ &= (0.8-t)(-t)(0.7-t) + 0 + 0 \\ &\quad - 0 - 0.12(0.8-t) - 0.12(0.7-t) \\ &= -t^3 + 1.5t^2 - 0.32t - 0.18 \end{aligned}$$

$$\chi_M(1) = -1 + 1.5 - 0.32 - 0.18 = 0,$$

so 1 is an eigenvalue

synthetic division:

$$\begin{array}{c|cccc} & -1 & 1.5 & -0.32 & -0.18 \\ 1 & & -1 & 0.5 & 0.18 \\ \hline & -1 & 0.5 & 0.18 & 0 \end{array}$$

pull out "−" sign

$$\chi_M(t) = -(t-1)(t^2 - 0.5t - 0.18)$$

Other eigenvalues: $t = \dfrac{0.5 \pm \sqrt{0.25 + 0.72}}{2}$

$= \dfrac{0.5 \pm \sqrt{0.97}}{2}$

<u>Find an eigenvector for $\lambda = 1$</u>:

If $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in E_1$ (Eigenspace for $\lambda = 1$)

then $x \in \mathrm{Ker}(M - 1 \cdot I)$

so:

$$\begin{pmatrix} -0.2 & 0.6 & 0 \\ 0.2 & -1 & 0.3 \\ 0 & 0.4 & -0.3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} -0.2x_1 + 0.6x_2 \\ 0.2x_1 - x_2 + 0.3x_3 \\ 0.4x_2 - 0.3x_3 \end{pmatrix}$$

so: $-0.2x_1 + 0.6x_2 = 0 \Rightarrow x_1 = 3x_2$

$$0.4x_2 - 0.3x_3 = 0 \implies x_3 = \tfrac{4}{3}x_2$$

so $$x = \begin{pmatrix} 3x_2 \\ x_2 \\ \tfrac{4}{3}x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \\ 4/3 \end{pmatrix}$$

We can proceed similarly for $\lambda = \dfrac{0.5 \pm \sqrt{0.97}}{2}$, and we end up w/ C.O.B. matrix:

$$Q = \begin{pmatrix} 3 & -5.5 - 5\sqrt{0.97} & -5.5 + 5\sqrt{0.97} \\ 1 & 1 & 1 \\ 4/3 & 4.5 + 5\sqrt{0.97} & 4.5 - 5\sqrt{0.97} \end{pmatrix}$$

Also:

$$Q^{-1} = \begin{pmatrix} \frac{3}{16} & \frac{3}{16} & \frac{3}{16} \\ & \text{(stuff)} & \end{pmatrix}$$

so: $$Q^{-1} M Q = \operatorname{diag}\left(1, \frac{0.5+\sqrt{0.97}}{2}, \frac{0.5-\sqrt{0.97}}{2}\right)$$

$$Q^{-1} M^n Q = \operatorname{diag}\left(1^n, \left(\frac{0.5+\sqrt{0.97}}{2}\right)^n, \left(\frac{0.5-\sqrt{0.97}}{2}\right)^n\right)$$

$$\lim_{n\to\infty} Q^{-1} M^n Q = \operatorname{diag}(1, 0, 0)$$

so, if $L := \lim_{n\to\infty} M^n$, then

$$L = Q \operatorname{diag}(1,0,0)\, Q^{-1}$$

$$= Q \begin{pmatrix} \frac{3}{16} & \frac{3}{16} & \frac{3}{16} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & & \\ 1 & (\text{stuff}) & \\ 4/3 & & \end{pmatrix} \begin{pmatrix} \frac{3}{16} & \frac{3}{16} & \frac{3}{16} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{9}{16} & 9/16 & 9/16 \\ \frac{3}{16} & \frac{3}{16} & 3/16 \\ 4/16 & 4/16 & 4/16 \end{pmatrix}$$

probability vector that is an eigenvector for $\lambda = 1$

Also,

$$\lim_{m\to\infty} (M^m w) = Lw = L\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{9}{16}(p_1+p_2+p_3) \\ \frac{3}{16}(p_1+p_2+p_3) \\ 4/16\,(p_1+p_2+p_3) \end{pmatrix} = \begin{pmatrix} 9/16 \\ 3/16 \\ 4/16 \end{pmatrix}$$

Regardless of the starting probabilities, you end up in state 1 $9/16$ of time, state 2 $3/16$ of time, and state 3 $4/16$ of time!

(**If** the system runs for a "large" number of steps.)

Why does something like this happen?

<u>One idea</u>: Suppose $L = \lim_{m\to\infty} A^m$.

Then $AL = A \lim_{m\to\infty} A^m = \lim_{m\to\infty} A^{m+1} = L$

Each column of L is an eigenvector of A w/ eigenvalue 1.

This isn't an accident. We need a little theory.

DEF Let $A \in M_n(\mathbb{C})$, $A = (A_{ij})$

$$\rho_i(A) := \sum_{j=1}^{n} |A_{ij}|$$ (sum of abs. value of each entry in row i)

$$\nu_j(A) := \sum_{i=1}^{n} |A_{ij}|$$ (sum of abs. value of each entry in column j)

row sum: $\rho(A) := \max_{1 \le i \le n} \{\rho_i(A)\}$

column sum: $\nu(A) := \max_{1 \le j \le n} \{\nu_j(A)\}$

Ex $A = \begin{pmatrix} i & 1 \\ 0 & 1-i \end{pmatrix}$

$$\rho_1(A) = |i| + |1| = 2$$

$$\rho_2(A) = |0| + |1-i| = \sqrt{2}$$

$$\nu_1(A) = 1$$

$$\nu_2(A) = 1+\sqrt{2}$$

$$\rho(A) = 2, \qquad \nu(A) = 1+\sqrt{2}$$

DEF For an $n \times n$ matrix A, define the i^{TH} Gerschgorin disk C_i to be the disk in the complex plane w/ center A_{ii} and radius

$$r_i := \rho_i(A) - |A_{ii}| = \sum_{j \neq i} |A_{ij}|$$

$$C_i := \{ z \in \mathbb{C} : |z - A_{ii}| \leq r_i \}$$

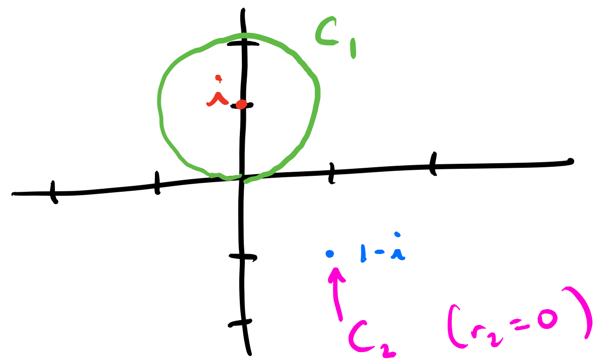
EX Going back to the last example:

$$A = \begin{pmatrix} i & 1 \\ 0 & 1-i \end{pmatrix} \qquad \begin{aligned} A_{11} &= i \\ A_{22} &= 1-i \end{aligned}$$

$$r_1 = \sum_{j \neq 1} |A_{1j}| = |A_{12}| = 1$$

$$r_2 = \sum_{j \neq 2} |A_{2j}| = |A_{21}| = 0$$

so :



__THM__ (Gerschgorin's Disk Thm; also called Gershgorin's Circle Thm)

Let $A \in M_n(\mathbb{C})$. Then every eigenvalue of A is contained in a Gerschgorin disk.

__Pf__ : Next time.

__COR__ Let λ be any eigenvalue of $A \in M_n(\mathbb{C})$. Then

$$|\lambda| \leq \min \{ \rho(A), \nu(A) \}.$$

<u>Pf</u> By Gerschgorin's Disk Thm,

$$|\lambda - A_{kk}| \leq r_k \quad \text{for some } k,$$

so: $|\lambda| = |(\lambda - A_{kk}) + A_{kk}| \leq |\lambda - A_{kk}| + |A_{kk}|$

$$\leq r_k + |A_{kk}| = \rho_k(A) \leq \rho(A)$$

Each eigenvalue of A is an eigenvalue of A^t (same char. poly; HW)

and $\rho(A^t) = \nu(A)$.

Hence $|\lambda| \leq \min \{ \rho(A), \nu(A) \}$. $\square$

<u>COR</u> If A is a transition matrix and λ is an eigenvalue of A, then $|\lambda| \leq 1$.