LAST TIME: $T: V \to W$ linear transformation

$$\text{nullity}(T) := \dim(\text{Ker}(T))$$
$$\text{rank}(T) := \dim(\text{Im}(T))$$

THM (Dimension Thm / "Rank-Nullity")

Let V, W be v.s., $T: V \to W$ linear.

If V is finite-dim'l, then

$$\text{nullity}(T) + \text{rank}(T) = \dim(V).$$

Pf FROM LAST TIME: $\dim(V) = n$

$$\dim(\text{Ker}(T)) = k \leq n$$

$\{v_1, \ldots, v_k\}$: basis for $\text{Ker}(T)$

Replacement Thm: extended to a basis

$$\{v_1, \ldots, v_k\ v_{k+1}, \ldots, v_n\}$$

WANT: $\{T(v_{k+1}), \ldots, T(v_n)\}$ is a basis for $\text{Im}(T)$

We showed $\text{span}(\{T(v_{k+1}), \ldots, T(v_n)\}) = \text{Im}(T)$.

Still need to show it's a L.I. set.

Now suppose $\sum_{i=k+1}^{n} b_i \, T(v_i) = 0$

L.T. $\Rightarrow \quad T\left(\sum_{i=k+1}^{n} b_i v_i\right) = 0$

$\Rightarrow \quad \sum_{i=k+1}^{n} b_i v_i \in \mathrm{Ker}(T) = \mathrm{span}(\{v_1, \ldots, v_k\})$

So: there are $(-b_1), \ldots, (-b_k) \in \mathbb{F}$

such that $\sum_{i=k+1}^{n} b_i v_i = \sum_{i=1}^{k} (-b_i) v_i$

$$\Rightarrow \quad \sum_{i=1}^{n} b_i v_i = 0$$

But $\{v_1, \ldots, v_n\}$ is basis, so L.I.

$\Rightarrow \quad b_i = 0$ for all i

$\Rightarrow \quad \{T(v_{k+1}), \ldots, T(v_n)\}$ is L.I.

and hence a basis for $\mathrm{Im}(T)$

Therefore, $\dim(\mathrm{Im}(T)) = n - k$, and so

$\mathrm{nullity}(T) + \mathrm{rank}(T) = \dim(V)$. $\square$

THM Let V, W be v.s. over $\mathbb{F}$ and suppose $\{v_1, \dots, v_n\}$ is a basis for V. For any $w_1, w_2, \dots, w_n \in W$ (not necessarily distinct!), there exists exactly one linear transformation $T: V \to W$ such that $T(v_i) = w_i$ for each i.

(In other words, a L.T. is entirely determined by the image of a basis.)

Pf Let $T: V \to W$ be a function such that $T(v_i) = w_i$ for all i.

EXISTENCE: Let $x \in V$. Then $x = \sum_{i=1}^{n} a_i v_i$, where the a_i are unique.

So, we define

$$T(x) := \sum_{i=1}^{n} a_i w_i$$

Let $y = \sum_{i=1}^{n} b_i v_i \in V$, $c, d \in \mathbb{F}$.

$$T(cx+dy) = T\left(\sum_{i=1}^{n} (ca_i + db_i) v_i\right)$$

$$= \sum_{i=1}^{n} (ca_i + db_i) w_i$$

$$= c \sum_{i=1}^{n} a_i w_i + d \sum_{i=1}^{n} b_i w_i$$

$$= c\,T(x) + d\,T(y),$$

and so T is linear. ✓

<u>UNIQUENESS</u>: Assume $U(v_i) = w_i$ is linear and T is as above

$$\left(T\left(\sum_{i=1}^{n} a_i v_i\right) = \sum_{i=1}^{n} a_i w_i \right)$$

Let $x \in V$ and $x = \sum_{i=1}^{n} a_i v_i$

$$U(x) = U\left(\sum_{i=1}^{n} a_i v_i\right) = \sum_{i=1}^{n} a_i U(v_i)$$

(linear)

$$= \sum_{i=1}^{n} a_i w_i = T(x).$$

$\square$

COORDINATES AND MATRICES

DEF Let V be a finite-dim'l v.s.
An ordered basis is a basis endowed w/ a specific order (i.e. a sequence).

EX Standard basis for $\mathbb{F}^n$ over $\mathbb{F}$ can be given the order

$$(e_1, e_2, \ldots, e_n), \quad \text{where}$$

$$e_i := (0, 0, \ldots, 0, \underset{\substack{\uparrow \\ i^{\text{TH}} \text{ coordinate}}}{1}, 0, \ldots, 0)$$

In principle, we could order these differently. For instance, (e_1, e_2, e_3, e_4) and (e_3, e_2, e_4, e_1) consist of the same basis vectors but are different ordered bases.

Given an ordered basis $B = (v_1, \dots, v_n)$,

we can express any $v \in V$ uniquely

as $$v = \sum_{i=1}^{n} a_i v_i$$

DEF The coordinates of v relative to the ordered basis B are

$$[v]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

EX $V = P_5(\mathbb{R})$, standard ordered basis

$$B = (1, x, x^2, x^3, x^4, x^5).$$

$$[3x^4 + 6x + \pi]_B = \begin{bmatrix} \pi \\ 6 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

If we chose a different basis, say

$$B' = (2,\ x^2+x,\ x^2-x,\ x^3,\ x^4,\ x^5)$$

$$[3x^4+6x+\pi]_{B'} = \begin{bmatrix} \pi/2 \\ 3 \\ -3 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

$$3x^4+6x+\pi$$

$$= \tfrac{\pi}{2}(2) + 3(x^2+x) - 3(x^2-x)$$

$$+ 0x^3 + 3x^4 + 0x^5$$

Suppose we want to <u>change</u> co-ordinates from B to B'.

Let $B = (e_0, e_1, \ldots, e_5)$

$B' = (v_0, v_1, \ldots, v_5)$

For any $f \in P_5(\mathbb{R})$, we can write

$$f(x) = \sum_{i=0}^{5} a_i e_i = \sum_{i=0}^{5} b_i v_i,$$

where the a_i, b_i are unique.

How we write the e_i in terms of the v_i?

$$B = (1, x, x^2, x^3, x^4, x^5)$$

$$B' = (2, x^2+x, x^2-x, x^3, x^4, x^5)$$

$$e_0 = 1 = \tfrac{1}{2}\cdot 2 = \tfrac{1}{2} v_0 = \tfrac{1}{2} v_0 + 0 v_1 + 0 v_2 + \dots + 0 v_5$$

$$e_1 = \tfrac{1}{2}(x^2+x) - \tfrac{1}{2}(x^2-x) = \tfrac{1}{2} v_1 - \tfrac{1}{2} v_2$$

$$e_2 = \tfrac{1}{2}(x^2+x) + \tfrac{1}{2}(x^2+x) = \tfrac{1}{2} v_1 + \tfrac{1}{2} v_2$$

$$e_3 = v_3, \quad e_4 = v_4, \quad e_5 = v_5$$

$$A = \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Indeed, $A \begin{bmatrix} \pi \\ 6 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} \pi/2 \\ 3 \\ -3 \\ 0 \\ 3 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

so $\left[1 + x + x^2 + x^3 + x^4 + x^5\right]_{B'}$

$$= \tfrac{1}{2}(2) + 1(x^2+x) + 0(x^2-x) + 1x^3 + 1x^4 + 1x^5$$

This is the convenient way to change bases.

<u>In general</u>: consider $T: V \longrightarrow W$, where V has ordered basis $B = (v_1, \dots, v_n)$

and W has ordered basis

$$C = (w_1, \dots, w_m)$$

(ABOVE: T was the identity map,

V : $P_5(\mathbb{R})$ w/ basis $B = (e_0, \dots, e_5)$,

W : $P_5(\mathbb{R})$ w/ basis B')

For each v_j, there are unique scalars $a_{1j}, a_{2j}, \dots, a_{mj}$ such that

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$$

DEF Using the above notation, we call the $m \times n$ matrix A defined by

$$A = (a_{ij})$$

$$(A_{ij} = a_{ij})$$

the matrix representation of T in the ordered bases B and C and write $A = [T]_B^C$.

If $V = W$ and $B = C$, then write $A = [T]_B$.

PROP Suppose V has ordered basis

$$B = (v_1, \dots, v_n)$$

and W has ordered basis

$$C = (w_1, \dots, w_m).$$

If $T : V \to W$ is linear and x has coordinates $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ w/ respect to B, then the coordinates of $T(x)$ w/ respect to C are $A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, where $A = [T]_B^C$.

<u>Pf</u> Suppose V, W, B, C, T, x, A are as in the statement, where $A = (a_{ij})$.

Then
$$T(x) = T\left(\sum_{j=1}^{n} x_j v_j\right)$$
$$= \sum_{j=1}^{n} x_j \, T(v_j) \qquad \text{(linear)}$$
$$= \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{m} a_{ij} w_i\right)$$
$$= \sum_{i=1}^{m} \boxed{\left(\sum_{j=1}^{n} a_{ij} x_j\right)} w_i$$

Since row i of $A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is $\displaystyle\sum_{j=1}^{n} a_{ij} x_j$, the result follows. $\square$

<u>READ</u>: §2.3