HW 6, #5: V, W nonzero v.s. over $\mathbb{F}$,

$\mathcal{B}$: basis for V

Prove for any function $f : \mathcal{B} \to V$

that there exists exactly one L.T.

T such that $T(x) = f(x)$ for all $x \in \mathcal{B}$.

Finite-dim'l case: done already

Infinite-dim'l case?

Two things: (1) existence of T

(2) uniqueness of T

For (2): consider two L.T. T_1, T_2 w/

this property and prove they're equal

(1): Want to define T as a L.T.

that agrees w/ f.

Let $v \in V$. Since $\mathcal{B}$ is a basis,

$v = \text{---}$

So: define $T(v) := \text{---}$

Then show this function is linear.

RECALL:

DEF <u>k^{TH} exterior product</u> $\Lambda^k V$

(V: v.s. over $\mathbb{F}$)

$\Lambda^k V$ is itself a v.s. over $\mathbb{F}$,

equipped w/ <u>exterior multiplication</u> map

$$\psi : V^k \to \Lambda^k V$$

<u>notation</u>: $\psi(v_1, \ldots, v_k)$ is denoted by

$$\underbrace{v_1 \wedge v_2 \wedge \cdots \wedge v_k}_{\text{k-vector}}$$

<u>k-vector</u>

(or: pure/simple <u>k-wedge</u>)

The pair $(\Lambda^k V, \psi)$ satisfies:

(1) ψ is alternating and k-linear:

<u>k-linear</u>: For all i,

$$v_1 \wedge v_2 \wedge \dots \wedge (au_i + bv_i) \wedge \dots \wedge v_k$$

$$= a(v_1 \wedge v_2 \wedge \dots \wedge u_i \wedge \dots \wedge v_k)$$

$$+ b(v_1 \wedge \dots \wedge v_i \wedge \dots \wedge v_k)$$

<u>Alternating</u>: $v_1 \wedge \dots \wedge v \wedge \dots \wedge v \wedge \dots v_k = 0$

same

<u>RECALL</u>: If $\mathrm{char}(\mathbb{F}) > 2$ (we assume it to be), then

$$v_1 \wedge v_2 \wedge \dots \wedge v_j \wedge \dots \wedge v_i \wedge \dots \wedge v_k$$

$$= -(v_1 \wedge v_2 \wedge \dots \wedge v_i \wedge \dots \wedge v_j \wedge \dots \wedge v_k)$$

(2) Given a basis $\beta = (v_1, \dots, v_n)$ of V, the collection

$$\beta_k = \{v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{i_k} : 1 \le i_1 < i_2 < \dots < i_k \le n\}$$

is a <u>basis</u> for $\Lambda^k V$.

Each basis vector in β_k $\longleftrightarrow$ subset $\{i_1, \dots, i_k\}$ of $\{1, \dots, n\}$ of size k

$$\Rightarrow \quad \dim \Lambda^k V = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

<u>EX</u> Consider $\Lambda^2 V$, where $\dim(V) = 2$.

Let $\beta = (v_1, v_2)$ be a basis for V.

What is a basis for $\Lambda^2 V$?

Let $v \wedge w \in \Lambda^2 V$, where $v, w \in V$.

Since β is a basis,

$$v = a_1 v_1 + a_2 v_2, \qquad w = b_1 v_1 + b_2 v_2$$

So:

$$v \wedge w = (a v_1 + a_2 v_2) \wedge (b_1 v_1 + b_2 v_2)$$

$$= \big(a_1 v_1 \wedge (b v_1 + b_2 v_2)\big) + \big(a_2 v_2 \wedge (b_1 v_1 + b_2 v_2)\big)$$

(2-linear)

$$= (a_1 v_1 \wedge b_1 v_1) + (a_1 v_1 \wedge b_2 v_2)$$

$$+ \underbrace{(a_2 v_2 \wedge b_1 v_1)}_{= 0} + (a_2 v_2 \wedge b_2 v_2)$$

$$= a_1 b_1 (v_1 \wedge v_1) + a_1 b_2 (v_1 \wedge v_2) + a_2 b_1 (v_2 \wedge v_1) + \underbrace{a_2 b_2 (v_2 \wedge v_2)}_{= 0}$$

alternating

$$= a_1 b_2 (v_1 \wedge v_2) + a_2 b_1 (v_2 \wedge v_1)$$

$$= a_1 b_2 (v_1 \wedge v_2) - a_2 b_1 (v_1 \wedge v_2) \leftarrow \text{alternating}$$

$$= (a_1 b_2 - a_2 b_1)(v_1 \wedge v_2)$$

NOTE Not every element of $\Lambda^k V$ is necessarily a k-vector! For instance, if $k = 2$, $\dim(V) = 4$, $\beta = (v_1, v_2, v_3, v_4)$ is a basis for V, then

$$\underbrace{(v_1 \wedge v_2) + (v_3 \wedge v_4)}_{\text{not a } k\text{-vector}} \in \Lambda^k V$$

Here: basis for $\Lambda^2 V$ is:

$$(v_1 \wedge v_2,\ v_1 \wedge v_3,\ v_1 \wedge v_4,\ v_2 \wedge v_3,\ v_2 \wedge v_4,\ v_3 \wedge v_4)$$

Ex If $k > n = \dim(V)$, then, for any $u_1 \wedge \dots \wedge u_k \in \Lambda^k V$, there is a linear dependence among the u_i, i.e., for some j,

$$u_j = \sum_{i \neq j} a_i u_i$$

Thus: $u_1 \wedge u_2 \wedge \dots \wedge u_j \wedge \dots \wedge u_k$

$$= u_1 \wedge u_2 \wedge \dots \wedge \left(\sum_{i \neq j} a_i u_i\right) \wedge \dots \wedge u_k$$

$$= a_i \sum_{i \neq j} \left(u_1 \wedge u_2 \wedge \dots \wedge u_i \wedge \dots \wedge \underbrace{u_i}_{j^{\text{th}} \text{ co-ordinate}} \wedge \dots \wedge u_k\right)$$

$= 0$, and hence $\Lambda^k V = \{0\}$ when $k > \dim(V)$.

PROP The k-vector $v_1 \wedge \dots \wedge v_k$ equals 0 in $\Lambda^k V$ iff $\{v_1, \dots, v_k\}$ is linearly dependent in V.

Pf We just saw that "linear dependence" $\Rightarrow$ "equals 0" (same proof)

Now suppose $\{v_1, \dots, v_k\}$ is L.I.

We can extend this set to a basis

$$\beta = (v_1, \dots, v_k, v_{k+1}, \dots, v_n) \text{ of } V.$$

Since $v_1 \wedge \dots \wedge v_k$ is a basis vector of β_k, a basis of $\Lambda^k V$,

$$v_1 \wedge \dots \wedge v_k \neq 0. \qquad \square$$

PROP Let $\dim(V) = n$ and $\beta = (v_1, \dots, v_n)$ be any ordered basis of V.

Then $\dim(\Lambda^n V) = 1$ and

$(v_1 \wedge v_2 \wedge \dots \wedge v_n)$ is a basis for $\Lambda^n V$.

Pf Let $u_1 \wedge u_2 \wedge \dots \wedge u_n \in \Lambda^n V$, where

$u_j \in V$ for each j.

Then each $u_j = \sum_{i=1}^{n} a_{ij} v_i$, so:

$u_1 \wedge u_2 \wedge \cdots \wedge u_n$

$$= \left(\sum_{i=1}^{n} a_{i1} v_i\right) \wedge \left(\sum_{i=1}^{n} a_{i2} v_i\right) \wedge \cdots \wedge \left(\sum_{i=1}^{n} a_{in} v_i\right)$$

We expand n-linearly.

Since alternating, all n-vectors w/ repeated coordinates are 0, so the only nonzero terms are of the form

$$a_{i_1 1} a_{i_2 2} \cdots a_{i_n n} \left(v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_n} \right),$$

where $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$ as a set.

We can interchange the v_{i_j} repeatedly, changing the sign of the n-vector each time, until this vector is

$c_{i_1 i_2 \cdots i_n} (v_1 \wedge v_2 \wedge \cdots v_n)$, where

$c_{i_1 i_2 \cdots i_n} = \pm a_{i_1 1} a_{i_2 2} \cdots a_{i_n n}$, and so

$$u_1 \wedge \cdots \wedge u_n = \sum_{\substack{(i_1, i_2, \ldots, i_n) \\ \{i_1, \ldots, i_n\} = \{1, \ldots, n\}}} c_{i_1 i_2 \cdots i_n} (v_1 \wedge \cdots \wedge v_n)$$

$$\in \operatorname{span}(v_1 \wedge \cdots \wedge v_n). \quad \square$$

<u>NEXT TIME</u>: Define the determinant of $T: V \to V$, where $\dim(V) = n$, in terms of $\Lambda^n V$.