LAST TIME: Let $T: V \to V$ be a linear operator. A subspace W of V is a T-invariant subspace if

$$T(W) \subseteq W, \quad \text{i.e., if}$$

$$T(w) \in W \quad \text{for all } w \in W.$$

It's actually not particularly difficult to create such subspaces.

Start w/ $v \in V$, and consider

$$W := \operatorname{span}(\{v, T(v), T^2(v), \dots\})$$

W is the T-cyclic subspace of V generated by v

LEMMA Let $T: V \to V$ be linear, $v \in V$, and W the T-cyclic subspace of V generated by v. Then, W is T-invariant.

PF Let $w \in W$, and let $T^0 := I$, so $T^0(v) = v$.

We haven't specified V or W is finite-dim'l

If $w = 0$, then $T(w) = 0 \in W$ ✓

If $w \neq 0$, there exist $n_1 < n_2 < \dots < n_k \in \mathbb{N} \cup \{0\}$ such that $w = \sum_{i=1}^{k} a_i T^{n_i}(v)$, where $a_i \in \mathbb{F}$.

Then
$$T(w) = T\left(\sum_{i=1}^{k} a_i T^{n_i}(v)\right)$$
$$= \sum_{i=1}^{k} a_i T\left(T^{n_i}(v)\right)$$
$$= \sum_{i=1}^{k} a_i T^{n_i+1}(v) \in W,$$

and so W is T-invariant. □

THM Let V be a v.s., $0 \neq v \in V$, $T: V \to V$ linear.

Assume there exists $n \in \mathbb{N}$ such that

$$T^n(v) \in \operatorname{span}(\{v, T(v), \dots, T^{n-1}(v)\})$$

and assume further that n is the smallest such positive integer.

If $W := \text{span}(\{v, T(v), \ldots, T^{n-1}(v)\})$,

then W is the T-cyclic subspace of V generated by v and $\dim(W) = n$.

Moreover, $(v, T(v), \ldots, T^{n-1}(v))$ is a basis for W.

<u>Pf</u> Let U be the T-cyclic subspace generated by v. Clearly, $W \subseteq U$.

We know, by assumption, $T^n(v) \in W$, so assume $T^k(v) \in W$ for some $k \geq n$.

Then $T^{k+1}(v) = T(T^k(v)) \in T(W)$

If $x \in W$, then $x = \sum_{i=0}^{n-1} a_i T^i(v)$,

and so: $T(x) = T\left(\sum_{i=0}^{n-1} a_i T^i(v)\right)$

$$= \sum_{i=0}^{n-1} a_i T^{i+1}(v)$$

$$= \underbrace{\sum_{i=0}^{n-2} a_i T^{i+1}(v)}_{\substack{\in W, \text{ since} \\ v,\, T(v), \ldots,\, T^{n-1}(v) \in W}} + \underbrace{a_n T^n(v)}_{\substack{\in W, \text{ since} \\ T^n(v) \in W}}$$

Thus $T(x) \in W \Rightarrow W$ is T-invariant.

Hence, by induction on k, $T^k(v) \in W$ for all $k \geq n$, and so $U \subseteq W$.

Hence $U = W$.

Moreover, n is the smallest integer m such that $T^m(v) \in \text{span}(\{v, \ldots, T^{m-1}(v)\})$, so $\{v, \ldots, T^{n-1}(v)\}$ is L.I.

Therefore, $\dim(W) = n$ and $(v, \ldots, T^{n-1}(v))$ is a basis for W. $\square$

THM Let T be a linear operator on a finite-dim'l v.s. V, and let W be a T-invariant subspace of V.

If T_W is the restriction of T to W,

$$\left(\begin{array}{l} T_W : W \to W \\ T_W(w) := T(w) \end{array} \right)$$

Then the characteristic polynomial of T_W divides the characteristic polynomial of T.

Pf Let $\gamma = (v_1, \dots, v_k)$ be a basis for W and extend γ to a basis

$\beta = (v_1, \dots, v_k, v_{k+1}, \dots, v_n)$ of V.

Let $A = [T]_\beta$, $B_1 = [T_W]_\gamma$.

Note that the $i^{\underline{TH}}$ column of A is $[T(v_i)]_\beta$

Since W is T-invariant, when $1 \leq i \leq k$,

$T(v_i) = T_W(v_i)$, and so:

W is T-invariant

$$A = \begin{pmatrix} B_1 & B_2 \\ O & B_3 \end{pmatrix}$$

Thus $\chi_T(t) = \chi_A(t) = \det(A - tI)$

$$= \det\left(\begin{pmatrix} B_1 - tI & B_2 \\ O & B_3 - tI \end{pmatrix}\right)$$

$$= \det(B_1 - tI) \cdot \det(B_3 - tI)$$

(HW 8, #3)

$$= \chi_{T_W}(t) \cdot g(t),$$

where $g(t)$ is some polynomial.

So, $\chi_{T_W}(t) \mid \chi_T(t)$, as desired. $\square$

THM Let T be a linear operator on a finite-dim'l v.s. V, and let W be the T-cyclic subspace of V generated by v, where $v \in V$, $v \neq 0$.

If $\dim(W) = k$ and

$$a_0 v + a_1 T(v) + \dots + a_{k-1} T^{k-1}(v) + T^k(v) = 0,$$

then $\chi_{T_W}(t) = (-1)^k (a_0 + a_1 t + \dots + a_{k-1} t^{k-1} + t^k)$.

Pf Since V is finite-dim'l, there exists $m \in \mathbb{N}$ such that $T^m(v) \in \operatorname{span}(\{v, \dots, T^{m-1}(v)\})$

$$(m \leq \dim(V) + 1).$$

If k is the least such integer, then we proved $\dim(W) = k$ and $\beta = (v, T(v), \dots, T^{k-1}(v))$ is a basis for W.

Since $T^k(v) \in \operatorname{span}(\{v, \dots, T^{k-1}(v)\})$, there exist <u>unique</u> $a_0, \dots, a_{k-1} \in \mathbb{F}$

such that $a_0 v + a_1 T(v) + \dots + a_{k-1} T^{k-1}(v) + T^k(v) = 0$.

$T^k(v) = -a_0 v - a_1 T(v) - \dots - a_{k-1} T^{k-1}(v)$

Consider $[T_W]_\beta$.

$$[v]_\beta = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad [T(v)]_\beta = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$[T^i(v)]_\beta = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} \\ \\ \\ \leftarrow i+1^{\text{st}} \text{ row} \\ \\ \\ \\ \end{array},$$

since $\beta = (v, \dots, T^{k-1}(v))$.

$$[T_W]_\beta = \left([T(v)]_\beta \;\; [T(T(v))]_\beta \;\; \cdots \;\; [T(T^{k-1}(v))]_\beta \right)$$

$$= \begin{pmatrix} 0 & 0 & & 0 & -a_0 \\ 1 & 0 & & 0 & -a_1 \\ 0 & 1 & & 0 & -a_2 \\ 0 & 0 & \cdots & 0 & -a_3 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 1 & -a_{k-1} \end{pmatrix}$$

EXERCISE: Prove the characteristic polynomial of the above matrix is

$$(-1)^k(a_0 + a_1 t + \dots + a_{k-1} t^{k-1} + t^k).$$

$\square$

CAYLEY-HAMILTON THEOREM Let T be a linear operator on a finite-dim'l v.s. V w/ characteristic polynomial $\chi_T(t)$.

Then, $\chi_T(T) = T_0$, the zero transformation that sends every vector in V to 0.

That is, T "satisfies" its own characteristic polynomial.

Pf: NEXT TIME.