RECALL: Let $f(t)$ be a polynomial w/ coeffs in $\mathbb{F}$.

Let V be a v.s. over $\mathbb{F}$ and $T: V \to V$ linear.

If $f(t) = a_0 + a_1 t + \dots + a_n t^n$,

$$f(T) := a_0 I + a_1 T + \dots + a_n T^n$$

$f(T): V \to V$, linear, and

$$f(T)v = a_0 v + a_1 T(v) + a_2 T^2(v) + \dots + a_n T^n(v).$$

CAYLEY-HAMILTON THM: If V is finite dim'l and T is a linear operator on V, then $\chi_T(T) = T_0$, where χ_T is the characteristic polynomial of T and $T_0: V \to V$ is the zero transformation, defined by $T_0(v) = 0$ for all $v \in V$.

A <u>minimal polynomial</u> $m(t)$ for $T: V \to V$ linear is a monic (leading coeff $= 1$) polynomial of <u>least</u> degree such that

$$m(T) = T_0 .$$

We showed that <u>the</u> minimal polynomial $m_T(t)$ for T is unique, and, if

$f(T) = T_0$ for some polynomial $f(t)$,

then $m_T(t) \mid f(t)$.

<u>DEF</u> Let $A \in M_n(\mathbb{F})$. The <u>minimal polynomial</u> $m_A(t)$ is the monic polynomial of least degree such that $m_A(A) = \underbrace{O}_{\text{all zeros matrix}}$

<u>THM</u> Let V be finite-dim'l v.s., $T: V \to V$ linear, β an ordered basis for V. Then $m_T(t) = m_{[T]_\beta}(t)$.

(Equivalently, $m_A(t) = m_{L_A}(t)$.)

PF EXERCISE.

THM If T is a linear operator on a finite-dim'l v.s. V, then $\lambda \in \mathbb{F}$ is an eigenvalue of T iff

$$m_T(\lambda) = 0.$$

NOTE This shows that the characteristic polynomial and the minimal polynomial have the same zeros.

PF First, since $m_T(t) \mid \chi_T(t)$, then

$$\chi_T(t) = q(t) \cdot m_T(t), \text{ where}$$

$q(t)$ is a polynomial w/ coeff in $\mathbb{F}$.

If λ is a zero of $m_T(t)$, then

$$\chi_T(\lambda) = q(\lambda) \cdot m_T(\lambda) = q(\lambda) \cdot 0 = 0,$$

i.e., λ a root of $m_T(t)$

$\Rightarrow$ λ is an eigenvalue

Now, suppose λ is an eigenvalue of T,

and let v be an eigenvector

corresponding to λ.

HW 10, # 3 : If $g(t)$ is a polynomial

w/ coeffs in $\mathbb{F}$ and λ is an

eigenvalue of T w/ eigenvector x,

then $g(T)(x) = g(\lambda)x$, i.e.,

$g(\lambda)$ is an eigenvalue of $g(T)$

w/ eigenvector x.

By this HW problem,

$$0 = T_0(v) = m_T(T)(v) = m_T(\lambda)(v)$$

Since v is a nonzero vector, this

means $m_T(\lambda) = 0$,

and so every eigenvalue of T is a zero of the min'l polynomial $m_T(t)$. $\square$

COR Let T be a linear operator on a finite-dim'l v.s. V.

Suppose $\chi_T(t)$ factors as

$$\chi_T(t) = (\lambda_1 - t)^{n_1} (\lambda_2 - t)^{n_2} \cdots (\lambda_k - t)^{n_k},$$

where the $n_i \in \mathbb{N}$ and the λ_i are the distinct eigenvalues of T.

Then, there exist integers $m_1, \ldots, m_k$ such that $1 \le m_i \le n_i$ for each i

and $m_T(t) = (t - \lambda_1)^{m_1} \cdots (t - \lambda_k)^{m_k}$.

EX Consider J_n, where $J_n \in M_n(\mathbb{F})$ is the all 1's matrix

so: $$J_n = \begin{array}{cc} & n \\ n & \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & & & & 1 \\ 1 & & & & \vdots \\ \vdots & & & & \\ 1 & 1 & \cdots & & 1 \end{pmatrix} \end{array}$$

Let $J := J_n$.

$$J^2 = \begin{pmatrix} 1 & \cdots & 1 \\ & & \\ & & \end{pmatrix} \begin{pmatrix} 1 & & \\ 1 & & \\ \vdots & & \\ 1 & & \end{pmatrix}$$

$$= \begin{pmatrix} n & n & \cdots & n \\ n & n & \cdots & n \\ \vdots & & & \\ n & n & \cdots & n \end{pmatrix} = nJ$$

So: $J^2 = nJ \Rightarrow J^2 - nJ = 0$

Thus $m_J(t) \mid t^2 - nt$.
$= t(t-n)$

$\Rightarrow$ every zero of $m_J(t)$ is either 0, n

$\Rightarrow$ every eigenvalue of J is either 0 or n.

INNER PRODUCTS AND NORMS

We can give additional structure to a v.s. by defining a type of "vector multiplication," which we can use to define distances, lengths of vectors, etc.

DEF Let V be a v.s. over $\mathbb{F}$.

($\mathbb{F}$: here, always be $\mathbb{R}$ or $\mathbb{C}$; need an "ordered" field)

An <u>inner product</u> on V is a function $\langle \, , \, \rangle : V \times V \longrightarrow \mathbb{F}$

that assigns to each ordered pair of vectors in V a unique scalar in $\mathbb{F}$. (For $x, y \in V$, this scalar is denoted $\langle x, y \rangle$.)

For all $x, y, z \in V$ and $c \in \mathbb{F}$,

the following must hold:

Linear in 1^{st} coordinate

(a) $\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$

(b) $\langle cx, y\rangle = c\langle x, y\rangle$

(c) $\langle y, x\rangle = \overline{\langle x, y\rangle}$, where the bar denotes complex conjugation

(if we want to do something similar in a field of a different characteristic, then we would use "field automorphism" here)

(d) $\langle x, x\rangle > 0$ if $x \neq 0$

<u>NOTE</u> If $\mathbb{F} = \mathbb{R}$, then (c) becomes "$\langle y, x\rangle = \langle x, y\rangle$"

<u>DEF</u> The <u>standard inner product</u> (or <u>dot product</u>) on $\mathbb{F}^n$:

if $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{F}^n$,

then $\langle x, y\rangle := \sum_{i=1}^{n} x_i \overline{y_i}$

EXERCISE: Prove that the standard inner product really is an inner product.

Ex Let $V = C([a,b])$, the set of continuous, real-valued functions defined on $[a,b]$, where $a < b \in \mathbb{R}$.

For $f, g \in V$, define

$$\langle f, g \rangle := \int_a^b f(t)\, g(t)\, dt.$$

(a), (b): (linear in 1^{st} word.) easy

(c): $\left(\langle g, f \rangle = \overline{\langle f, g \rangle} = \langle f, g \rangle \in \mathbb{R} \right)$

easy

(d): $\left(\text{If } f \neq 0, \text{ then } \langle f, f \rangle > 0 \right)$

Suppose $f \neq 0$.

Then f^2 is bounded away from 0 on some subinterval of $[a,b]$ (since f is cts.)

Hence $\langle f, f \rangle = \int_a^b (f(t))^2 \, dt$

$$\geq 0 .$$

DEF A vector space V over $\mathbb{F}$ endowed w/ a specific inner product is called an <u>inner product space</u>.

$\mathbb{F} = \mathbb{C}$: <u>complex inner product space</u>

$\mathbb{F} = \mathbb{R}$: <u>real inner product space</u>