<u>LAST TIME</u>: $T: V \to V$ linear transformation (linear operator)

If $v \in V$, $v \neq 0$, and $T(v) = \lambda v$ for some $\lambda \in \mathbb{F}$, then

eigenvalue

eigenvector

Generally speaking, how can we find eigenvectors and eigenvalues?

Suppose $T(v) = \lambda v$, $v \in V$, $v \neq 0$, $\lambda \in \mathbb{F}$.

Then $T(v) = \lambda I(v)$ (I: identity)

$$\Rightarrow (T - \lambda I)(v) = 0$$

Since $v \neq 0$, $\dim(\operatorname{Ker}(T - \lambda I)) \geq 1$,

so $T - \lambda I$ is <u>not</u> invertible.

<u>THM</u> Let $T: V \to V$ be a linear operator. The scalar $\lambda \in \mathbb{F}$ is an eigenvalue iff $\det(T - \lambda I) = 0$.

PF As we see from the above, λ : eigenvalue w/ eigenvector v implies $T - \lambda I$ is not invertible $\Rightarrow$ $\det(T - \lambda I) = 0$.

Conversely, if $\det(T - \lambda I) = 0$, then $T - \lambda I$ is <u>not</u> invertible.

So: let $v \in \operatorname{Ker}(T - \lambda I)$, $v \neq 0$.

$$\Rightarrow \quad (T - \lambda I)v = 0$$

$$\Rightarrow \quad T(v) = \lambda v. \qquad \square$$

DEF Let T be a linear operator on V. The polynomial $\chi_T(t) := \det(T - tI)$ is called the <u>characteristic polynomial</u> of T. (Similarly, if $A \in M_n(\mathbb{F})$,

$$\chi_A(t) := \chi_{L_A}(t) \,)$$

NOTE Even though t is a variable that's not in the underlying field $\mathbb{F}$, we can still work w/ the determinant.

<u>Reason</u>: Subtle. $\mathbb{F}[t]$ is an <u>integral domain</u>, can be extended to a <u>field of fractions</u>.

This will be a field, and det is well-defined over that field.

<u>EX</u> Consider V a v.s. over $\mathbb{R}$,

$\beta = (v_1, v_2)$ a basis for V.

Let $T: V \longrightarrow V$ be given by

$$T(v_1) = 3v_1 + 4v_2,$$
$$T(v_2) = v_1 + v_2$$

$$\left(\text{so: } A = [T]_\beta = \begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix} \right)$$

$$A - tI_2 = \begin{pmatrix} 3-t & 1 \\ 4 & 1-t \end{pmatrix}, \text{ and}$$

$$\chi_T(t) = \chi_A(t) = \det(A - tI_2)$$

$$= (3-t)(1-t) - 4$$

$$= t^2 - 4t - 1$$

So: $$t = \frac{4 \pm \sqrt{16+4}}{2} = \frac{4 \pm 2\sqrt{5}}{2}$$

$$= 2 \pm \sqrt{5}$$

(possible) eigenvalues

Eigenvectors? Consider first $\lambda = 2+\sqrt{5}$

Then $A - (2+\sqrt{5})I_2 = \begin{pmatrix} 1-\sqrt{5} & 1 \\ 4 & -1-\sqrt{5} \end{pmatrix}$

Suppose $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \operatorname{Ker}\left(A - (2+\sqrt{5})I_2\right)$

Then $$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \left(A - (2+\sqrt{5})I_2\right)x$$

$$= \begin{pmatrix} (1-\sqrt{5})x_1 + x_2 \\ 4x_1 + (-1-\sqrt{5})x_2 \end{pmatrix}$$

$(1-\sqrt{5})x_1 + x_2 = 0 \Rightarrow$

so: $\quad x_2 = (-1+\sqrt{5})x_1$

$$\left(4x_1 + (-1-\sqrt{5})x_2 = 0 \Rightarrow \text{(same solution)} \right)$$

$$x = \begin{pmatrix} x_1 \\ (-1+\sqrt{5})x_1 \end{pmatrix} = x_1 \cdot \begin{pmatrix} 1 \\ -1+\sqrt{5} \end{pmatrix}$$

Need something nonzero in kernel,

so take $\begin{pmatrix} 1 \\ -1+\sqrt{5} \end{pmatrix}$

CHECK:

$$\begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1+\sqrt{5} \end{pmatrix} = \begin{pmatrix} 2+\sqrt{5} \\ 3+\sqrt{5} \end{pmatrix}$$

$$= (2+\sqrt{5})\begin{pmatrix} 1 \\ -1+\sqrt{5} \end{pmatrix} \checkmark$$

Similarly, we find that $\begin{pmatrix} 1 \\ -1-\sqrt{5} \end{pmatrix}$

is an eigenvector for eigenvalue $2-\sqrt{5}$.

So: if $\gamma = \left(\begin{pmatrix} 1 \\ -1+\sqrt{5} \end{pmatrix}, \begin{pmatrix} 1 \\ -1-\sqrt{5} \end{pmatrix} \right)$,

then $[T]_\gamma = \begin{pmatrix} 2+\sqrt{5} & 0 \\ 0 & 2-\sqrt{5} \end{pmatrix}$.

Indeed, in general:

<u>THM</u> Let T be a linear operator on V, λ an eigenvalue of T.

Then $v \in V$ is an eigenvector iff $v \neq 0$ and $v \in \operatorname{Ker}(T - \lambda I)$.

<u>THM</u> Let $T: V \to V$, $\dim(V) = n$.

(i) The degree of χ_T, the characteristic polynomial, is n and leading coefficient is $(-1)^n$.

(ii) T has $\leq n$ distinct eigenvalues

Pf EXERCISE. $\square$

DIAGONALIZABILITY

Q: Are all linear operators / matrices diagonalizable?

NO! Ex 1 If $T: \mathbb{R}^2 \to \mathbb{R}^2$ has matrix $A = [T]_\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

w/ respect to the standard basis β, then $\chi_T(t) = \chi_A(t) = t^2 + 1$

$$= (t-i)(t+i)$$

$$A - iI = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}$$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -ix_1 + x_2 \\ -x_1 - ix_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so: $x_2 = i x_1$

$\Rightarrow$ eigenvector $\begin{pmatrix} 1 \\ i \end{pmatrix} \notin \mathbb{R}^2$

While this matrix is diagonalizable over $\mathbb{C}$, it is <u>not</u> diagonalizable over $\mathbb{R}$.

<u>Ex 2</u> Let $T: \mathbb{F}^2 \to \mathbb{F}^2$ have matrix

$$A = [T]_\beta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

w/ respect to standard basis β.

Then $\chi_T(t) = \chi_A(t) = (1-t)^2 - 0 \cdot 1$

$$= (t-1)^2$$

<u>If</u> T is diagonalizable w/ respect to some basis γ, then

$$[T]_\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

But, if Q changes basis from β to γ, then:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = [T]_\beta = Q^{-1}[T]_\gamma Q$$
$$= Q^{-1} I_2 Q = I_2$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow\Leftarrow$

(For instance, $T(e_2) = e_1 + e_2 \neq e_2$, so $T \neq I$)

However, we often can say something definitive.

<u>THM</u> Let T be a linear operator on V and let $\lambda_1, \ldots, \lambda_k$ be <u>distinct</u> eigenvalues of T w/ (distinct) eigenvectors $v_1, \ldots, v_k$, resp. Then $\{v_1, \ldots, v_k\}$ is L.I.

Before we prove this, a corollary:

<u>COR</u> Let T be a linear operator on an n-dim'l v.s. V.
If T has n distinct eigenvalues, then T is diagonalizable.

<u>Pf of THM</u>: We proceed by induction on k.

When $k=1$, $v_1 \neq 0$, so $\{v_1\}$ is L.I. ✓

Now assume that, if $\lambda_1, \dots, \lambda_{k-1}$ are distinct eigenvalues w/ eigenvectors $v_1, \dots, v_{k-1}$, then $\{v_1, \dots, v_{k-1}\}$ is L.I.

Consider now k distinct eigenvalues $\lambda_1, \dots, \lambda_k$ w/ eigenvectors $v_1, \dots, v_k$.

Suppose $a_1 v_1 + a_2 v_2 + \dots + a_k v_k = 0$.

This means:

$$0 = T(0) = T(a_1 v_1 + \dots + a_k v_k)$$

$$= a_1 \lambda_1 v_1 + \cdots + a_k \lambda_k v_k \quad \#1$$

Also, $\sum_{i=1}^{k} a_i v_i = 0 \Rightarrow \sum_{i=1}^{k} a_i \lambda_k v_i = 0 \quad \#2$

Subtracting #2 from #1,

$$a_1(\lambda_1 - \lambda_k)v_1 + \cdots + a_{k-1}(\lambda_{k-1} - \lambda_k)v_{k-1} = 0$$

Since $\{v_1, \ldots, v_{k-1}\}$ is L.I. by ind. hyp.,

for each i, $a_i(\lambda_i - \lambda_k) = 0$.

Since $\lambda_i \neq \lambda_k$ for $1 \leq i \leq k-1$,

so $a_i = 0$ for $1 \leq i \leq k-1$.

This means $a_k v_k = 0 \Rightarrow a_k = 0$

Hence $\{v_1, \ldots, v_k\}$ is L.I. $\square$