Q: "Closed under addition and multiplication" in practical terms?

Take +.

Formally, if + is a binary on a set S, then S is <u>closed</u> under addition.

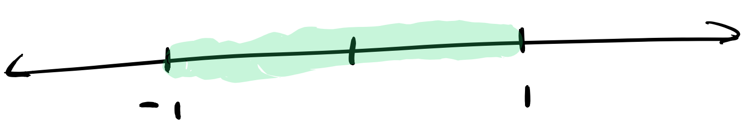
We view addition as a function that takes <u>pairs</u> of elements of S and gives as output another element of S

$$+ : S \times S \longrightarrow S$$

$$+ : (s_1, s_2) \longmapsto \text{"} s_1 + s_2 \text{"}$$

When is something <u>not</u> closed?

<u>EX</u> $S := \{ x \in \mathbb{R} : |x| \leq 1 \}$



S is <u>not</u> closed under addition!

get something <u>not</u> in S as output

For example, $0.6 \in S$, $0.8 \in S$

but $0.6 + 0.8 = 1.4 \notin S$

On the other hand, S <u>is</u> closed under multiplication!

<u>VECTOR SPACES</u>

<u>DEF</u> A <u>vector space</u> V over a field $\mathbb{F}$ consists of a set on which two operations (addition and scalar multiplication) are defined such that for any $x, y \in V$, there is a unique element $x + y \in V$, and for each
(sum)
$a \in \mathbb{F}$ and $x \in V$, there is a unique element
(scalar)
$ax \in V$, such that the following hold:
(product of a and x)

(VS 1) (COMMUTATIVITY OF ADDITION)
For all $x, y \in V$, $x + y = y + x$

(VS 2) (ASSOCIATIVITY OF ADDITION)
For all $x, y, z \in V$,
$$(x+y) + z = x + (y+z)$$

(VS 3) (ZERO VECTOR) There is an element $0 \in V$ such that $x + 0 = x$ for <u>all</u> $x \in V$.

(VS 4) (INVERSES) For each $x \in V$, there is $y \in V$ such that $x + y = 0$.

(VS 5) (MULTIPLICATION BY 1)
For all $x \in V$, $1x = x$.

(VS 6) (SCALAR MULT. IS WELL-DEFINED)
For all $a, b \in \mathbb{F}$, $x \in V$,

$$(ab)x = a(bx)$$

(VS 7) (DISTRIBUTIVE LAW: VECTOR SUM)
For all $a \in \mathbb{F}$, $x, y \in V$,

$$a(x+y) = ax + ay$$

(VS 8) (DISTRIBUTIVE LAW: SCALAR SUM)
For all $a, b \in \mathbb{F}$, $x \in V$,

$$(a+b)x = ax + bx$$

elements of $\mathbb{F}$: <u>scalars</u>

elements of V : <u>vectors</u>

<u>NOTE</u> ① This definition says nothing about what vectors <u>do</u>, only about what they <u>are</u>! <u>Abstraction</u>.

② There is $0 \in \mathbb{F}$, $0 \in V$

technically different

We'll see from context which we mean, but the two "behave" similarly.

<u>EX</u> Let $\mathbb{F}$ be <u>any</u> field, $V = \{0\}$, where $0 + 0 = 0$ and $a0 = 0$ for <u>all</u> $a \in \mathbb{F}$.

(Comm. of add.?)

(VS1) $0 + 0 = 0 + 0$ ✓

(Assoc. of add.?)

(VS2) $(0+0)+0 = 0 + 0 = 0$

$= 0 + 0 = 0 + (0+0)$ ✓

(VS3) Zero vector? 0 ✓

(VS4) Inverses? $0+0=0$ ✓

(VS5) $1 \cdot 0 = 0$ ✓

(VS6) $(ab)0 = 0 = a0 = a(b0)$ ✓

(VS7) $a(0+0) = a0 = 0$
$= 0+0 = a0 + a0$ ✓

(VS8) $(a+b)0 = 0 = 0+0$
$= a0 + b0$

So: V is a vector space
(<u>zero vector space</u>)

<u>EX</u> Let $\mathbb{F}$ be any field and for some fixed $n \in \mathbb{N}$, define

$$\mathbb{F}^n := \{ (a_1, a_2, \dots, a_n) : a_i \in \mathbb{F} \}$$

Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{F}^n$

Then $x + y := (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$, and, if $a \in \mathbb{F}$,

$$ax := (ax_1, ax_2, \dots, ax_n)$$

Since each coordinate is an element of $\mathbb{F}$, $\mathbb{F}^n$ is <u>closed</u> under these operations.

(VS1) (Comm of addition)

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

$$= (y_1 + x_1, \dots, y_n + x_n) \quad \text{(add. in } \mathbb{F} \text{ is comm.)}$$

$$= y + x \quad \checkmark$$

(VS2) (assoc of addition)

$$(x+y)+z = ((x_1 + y_1) + z_1, \dots, (x_n + y_n) + z_n)$$

$$= (x_1 + (y_1 + z_1), \dots, x_n + (y_n + z_n)) \quad \text{(add. assoc. in } \mathbb{F})$$

$$= x + (y+z)$$

(VS3) <u>Zero vector</u>: $0 = (0, 0, \dots, 0)$

(VS4) <u>Inverses?</u> If $x \in \mathbb{F}^n$,

then $x = (x_1, x_2, \dots, x_n)$

Inverse is $-x := (-x_1, -x_2, \dots, -x_n)$

every element of $\mathbb{F}$ has an additive inverse

(VS5) $1x = (1x_1, 1x_2, \dots, 1x_n)$

$= (x_1, \dots, x_n) = x$ ✓

(VS6) $(ab)x \stackrel{?}{=} a(bx)$

Follows from assoc. of multiplication in $\mathbb{F}$

(VS7), (VS8): Follow from distributive law in $\mathbb{F}^n$.

READ §1.1 - 1.2 !

<u>EX</u> V: set of unit vectors in $\mathbb{R}^3$

scalars: $\mathbb{R}$

Is V a v.s. over $\mathbb{R}$?

<u>NO</u>! Very much <u>not</u> closed under addition or scalar mult.

For instance, $(1,0,0) \in V$

$$(1,0,0) + (1,0,0) = (2,0,0) \notin V$$

$$5 \cdot (1,0,0) = (5,0,0) \notin V$$

THM (Cancellation Law for Vector Addition)

If $x, y, z \in V$ and $x + z = y + z$,

then $x = y$.

Pf Suppose $x, y, z \in V$, $x + z = y + z$

$$\begin{aligned}
x &= x + 0 && \text{(Zero vector)} \\
&= x + (z + (-z)) && \text{(Additive inverse)} \\
&= (x + z) + (-z) && \text{(Assoc. of +)} \\
&= (y + z) + (-z) \\
&= y + (z + (-z)) && \text{(Assoc.)} \\
&= y + 0 && \text{(Add inverse)} \\
&= y && \text{(Zero vector)}
\end{aligned}$$

$\square$

FROM HW: (1) Zero vector 0 is unique

(2) Additive inverses of vectors are unique

THM In any v.s. V:

(a) $0x = 0$ for all $x \in V$

$\uparrow$ in $\mathbb{F}$ $\quad$ $\uparrow$ in V

(b) $(-a)x = -(ax) = a(-x)$

(c) $a0 = 0$ for all $a \in \mathbb{F}$.

$\uparrow$ in V $\quad$ $\uparrow$ in V

IDEA: Proofs are very similar to those we did for fields

NEXT TIME: §1.3