LAST TIME: V, W are finite-dim'l v.s. over $\mathbb{F}$

B: ordered basis for V

C: ordered basis for W

$T: V \longrightarrow W$ linear

$B = (v_1, \dots, v_n)$, $C = (w_1, \dots, w_m)$

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$$

If $A = (a_{ij})$, then $A = [T]_B^C$.

If x has coordinates $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ w.r.t (w/ respect to) B,

$$\left(x = \sum_{i=1}^{n} x_i v_i \right)$$

then the coordinates of $T(x)$ w.r.t C

are $A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

EX Consider the L.T. $D: P_4(\mathbb{R}) \to P_3(\mathbb{R})$

given by $D(f) = df/dx$,

and give both $P_4(\mathbb{R})$, $P_3(\mathbb{R})$ the standard ordered bases:

$$B_4 = (1, x, x^2, x^3, x^4),$$
$$B_3 = (1, x, x^2, x^3)$$

$$D(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$$
$$D(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$$
$$D(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$$
$$D(x^3) = 3x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2 + 0 \cdot x^3$$
$$D(x^4) = 4x^3 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 4x^3$$

Then $A = [D]_{B_4}^{B_3} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

Consider $18x^3 + \frac{3}{2}x + \sqrt{2} \in P_4(\mathbb{R})$

Coordinates: $\begin{bmatrix} \sqrt{2} \\ 3/2 \\ 0 \\ 18 \\ 0 \end{bmatrix}$

$$A \begin{bmatrix} \sqrt{2} \\ 3/2 \\ 0 \\ 18 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 0 \\ 54 \\ 0 \end{bmatrix} \longleftrightarrow 54x^2 + 3/2$$

DEF Let $f, g : V \to W$ be functions, where V, W are v.s. over $\mathbb{F}$, $a \in \mathbb{F}$. Define

$f + g : V \to W$ by

$$(f+g)(v) := f(v) + g(v)$$

and $af : V \to W$ by

$$(af)(v) := a f(v)$$

for all $v \in V$.

LEMMA Let V, W be v.s. over $\mathbb{F}$. If $T, U : V \to W$ are linear transformations and $a, b \in \mathbb{F}$, then $aT + bU : V \to W$ is linear.

Pf Let $x, y \in V$, $c, d \in \mathbb{F}$.

$$(aT + bU)(cx + dy) = (aT)(cx+dy) + (bU)(cx+dy)$$
$$= aT(cx+dy) + bU(cx+dy)$$
$$= a\Big(cT(x) + dT(y)\Big) + b\Big(cU(x) + dU(y)\Big)$$
$$= c\Big(aT(x) + bU(x)\Big) + d\Big(aT(y) + bU(y)\Big)$$
$$= c\,(aT+bU)(x) + d\,(aT+bU)(y). \quad \square$$

DEF $\mathcal{L}(V, W) := \{ T : V \to W : T \text{ linear} \}$

When $V = W$, we write $\mathcal{L}(V)$

THM $\mathcal{L}(V, W)$ is a v.s. over $\mathbb{F}$ w/ operations defined above.

Pf The last LEMMA proves closure under $+$, scalar mult.

Let $S, T, U \in \mathcal{L}(V, W)$, $a, b \in \mathbb{F}$, $x \in V$.

(VS1) $(S+T)(x) = S(x) + T(x)$

$= T(x) + S(x)$

$= (T+S)(x)$

so: $S+T = T+S$

(VS2) similar

(VS3) $T_0 : V \to W$ by $T_0(x) = 0$

Then $(S+T_0)(x) = S(x) + T_0(x)$

$= S(x) + 0$

$= S(x)$

so $S + T_0 = S$

(VS4) Given $T : V \to W$, we have

$(T + (-1)T)(x) = T(x) + (-1)T(x)$

$= (1 + (-1))\, T(x)$

$= 0\, T(x)$

$= 0$

$= T_0(x)$,

and so $T + (-1)T = T_0$.

(VS 5) $\quad (1T)(x) = 1 \cdot T(x) = T(x)$

(VS 6) $\quad (a(bT))(x) = a\big((bT)(x)\big)$

$$= a\,(b\,T(x))$$

$$= (ab)\,T(x)$$

$$= ((ab)T)(x)$$

so $\quad a(bT) = (ab)T$. $\square$

(VS 7), (VS 8): similar.

<u>THM</u> Let V, W be finite-dim'l v.s. over $\mathbb{F}$ w/ ordered bases B, C, resp., and let $S, T : V \to W$ be linear, $c, d \in \mathbb{F}$. Then

$$[cS + dT]_B^C = c\,[S]_B^C + d\,[T]_B^C$$

<u>Pf</u> Let $B = (v_1, \dots, v_n)$, $C = (w_1, \dots, w_m)$

If $S(v_j) = \sum_{i=1}^m a_{ij} w_i$, $\quad T(v_j) = \sum_{i=1}^m b_{ij} w_i$, then $[S]_B^C = (a_{ij})$, $[T]_B^C = (b_{ij})$,

and so the i,j-entry of

$c[S]_B^C + d[T]_B^C$ is $ca_{ij} + db_{ij}$.

On the other hand,

$$(cS + dT)(v_j) = cS(v_j) + dT(v_j)$$

$$= c\sum_{i=1}^{m} a_{ij}w_i + d\sum_{i=1}^{m} b_{ij}w_i$$

$$= \sum_{i=1}^{m} (ca_{ij} + db_{ij})w_i$$

The result follows. $\square$

LEMMA The composition of two linear transformations (when the composition is well-defined!) is also a linear transformation.

Pf Let $S: U \to V$, $T: V \to W$

U, V, W : v.s. over $\mathbb{F}$

Denote $T \circ S$ by TS and let

$x, y \in U, \qquad a, b \in \mathbb{F}.$

$$
\begin{aligned}
TS(ax+by) &= T(S(ax+by)) \\
&= T(a\,S(x) + b\,S(y)) \\
&= a\,T(S(x)) + b\,T(S(y)) \\
&= a\,TS(x) + b\,TS(y). \quad \square
\end{aligned}
$$

How do these operations relate to matrices?

Let U have basis $B_1 = (u_1, \dots, u_\ell)$,
V have basis $B_2 = (v_1, \dots, v_m)$,
W have basis $B_3 = (w_1, \dots, w_n)$.

Suppose $S: U \to V$ has matrix

$$[S]_{B_1}^{B_2} = (a_{ij}),$$

$T: V \to W$ has matrix $[T]_{B_2}^{B_3} = (b_{ij})$

This means, for each j,

$$S(u_j) = \sum_{i=1}^{m} a_{ij} v_i,$$

and, for each j,

$$T(v_j) = \sum_{i=1}^{n} b_{ij} w_i$$

So: $$TS(u_j) = T(S(u_j)) = T\left(\sum_{i=1}^{m} a_{ij} v_i\right)$$

$$= \sum_{i=1}^{m} a_{ij} T(v_i)$$

$$= \sum_{i=1}^{m} a_{ij} \left(\sum_{k=1}^{n} b_{ki} w_k\right)$$

$$= \sum_{k=1}^{n} \boxed{\left(\sum_{i=1}^{m} b_{ki} a_{ij}\right)} w_k$$

$$[T]_{B_2}^{B_3} = \begin{array}{rc} & \\ \text{row } k & \begin{bmatrix} & & & \\ \underbrace{b_{k1} \quad b_{k2} \quad \cdots \quad b_{km}} \\ & & & \end{bmatrix} \end{array} \qquad [S]_{B_1}^{B_2} = \begin{array}{c} \text{column } j \\ \begin{bmatrix} & a_{1j} & \\ & a_{2j} & \\ & \vdots & \\ & a_{mj} & \end{bmatrix} \end{array}$$

This is why matrix multiplication works the way it does. So:

THM If U has basis B_1, V has basis B_2, W has basis B_3,
$S : U \to V$ linear, $T : V \to W$ linear,

then $$[TS]_{B_1}^{B_3} = [T]_{B_2}^{B_3} \cdot [S]_{B_1}^{B_2}$$

matrix mult.

COR If $S, T \in \mathcal{L}(V)$ and V has ordered basis B, then
$$[ST]_B = [S]_B [T]_B .$$

DEF <u>Kronecker delta</u> $$\delta_{ij} := \begin{cases} 1, & \text{if } i=j \\ 0, & \text{otherwise} \end{cases}$$

Consider V, an n-dim'l v.s. over $\mathbb{F}$ w/ ordered basis B.

$$I_n := [\mathrm{id}_V]_B = (\delta_{ij})$$

I_n : $n \times n$ identity matrix

I_1 : $\begin{bmatrix} 1 \end{bmatrix}$

I_2 : $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

I_4 : $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

So far : L.T. $\longrightarrow$ matrices

What about other direction?

NEXT TIME. Read §2.4