RECALL: inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$

$x, y, z \in V$; $c \in \mathbb{F}$

(a) $\langle x+z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$

(b) $\langle cx, y \rangle = c \langle x, y \rangle$

(c) $\langle y, x \rangle = \overline{\langle x, y \rangle}$

(d) If $x \neq 0$, then $\langle x, x \rangle > 0$.

LAST TIME:

- $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$
- $\langle x, cy \rangle = \bar{c} \langle x, y \rangle$
- $\langle x, 0 \rangle = \langle 0, x \rangle = 0$
- $\langle x, x \rangle = 0$ iff $x = 0$
- If $\langle x, y \rangle = \langle x, z \rangle$ for all $x \in V$, then $y = z$.

norm of x: $\|x\| := \sqrt{\langle x, x \rangle}$

THM The following are true:

(a) $\|cx\| = |c| \cdot \|x\|$

(b) $\|x\| = 0 \Leftrightarrow x = 0$; in any case, $\|x\| \geq 0$

(c) (Cauchy-Schwarz Inequality)

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$$

(d) (Triangle Inequality)

$$\|x+y\| \leq \|x\| + \|y\|$$

Pf of (d)

$$(\|x\| + \|y\|)^2 = \|x\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2$$

$$\geq \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2 \quad \left(\text{Cauchy-Schwarz}\right)$$

If $\langle x,y\rangle = a+bi$,

then $|\langle x,y\rangle| = \sqrt{a^2+b^2}$

$\geq \sqrt{a^2} = |a|$

$\geq a$

$2\,\mathrm{Re}(\langle x,y\rangle) = 2a$

$\langle x,y\rangle + \overline{\langle x,y\rangle} = (a+bi) + (a-bi) = 2a$

$$\geq \|x\|^2 + 2\underbrace{\mathrm{Re}(\langle x,y\rangle)}_{\text{real part of } \langle x,y\rangle} + \|y\|^2$$

$$= \langle x,x\rangle + \left(\langle x,y\rangle + \overline{\langle x,y\rangle}\right) + \langle y,y\rangle$$

$$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$$

$$= \langle x+y, x+y\rangle = \|x+y\|^2$$

Therefore, $\|x+y\| \leq \|x\| + \|y\|$. $\square$

GRAM-SCHMIDT ORTHOGONALIZATION

DEF Let V be an inner product space. Vectors $x, y \in V$ are <u>orthogonal</u> (or <u>perpendicular</u>) if $\langle x, y \rangle = 0$.

A subset $S \subseteq V$ is <u>orthogonal</u> if $\langle x, y \rangle = 0$ for any distinct $x, y \in S$.

<u>unit vector</u>: an $x \in V$ such that $\|x\| = 1$

A subset $S \subseteq V$ is <u>orthonormal</u> if S is orthogonal and S consists only of unit vectors.

GOAL: Find an orthonormal <u>basis</u> for a v.s.

Suppose $S = \{ v_1, \dots, v_k \}$ is orthogonal.

Then: $\langle v_i, v_j \rangle = 0$ when $i \neq j$.

$\longrightarrow$ replace S by

$$S' = \left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \ldots, \frac{v_k}{\|v_k\|} \right\}$$

So: If $i \neq j$, $\left\langle \frac{v_i}{\|v_i\|}, \frac{v_j}{\|v_j\|} \right\rangle = \frac{1}{\|v_i\|} \cdot \overline{\frac{1}{\|v_j\|}} \langle v_i, v_j \rangle = 0$,

so it's still an orthogonal set.

But now: $\left\langle \frac{v_i}{\|v_i\|}, \frac{v_i}{\|v_i\|} \right\rangle = \frac{1}{\|v_i\|} \cdot \overline{\frac{1}{\|v_i\|}} \langle v_i, v_i \rangle$

$$= \frac{1}{\|v_i\|^2} \cdot \|v_i\|^2 = 1 \checkmark$$

So S' is orthonormal.

<u>DEF</u> Let V be an inner product space. A subset β of V is an <u>orthonormal basis</u> for V if β is an ordered basis that is an orthonormal set.

<u>EX</u> · Standard basis of $\mathbb{F}^n$, where $\langle \cdot, \cdot \rangle$ is the dot product

$\beta = (e_1, \ldots, e_n)$, where $e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ i^{th} coordinate.

• $\left\{ \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right) \right\}$ is an orthonormal basis for $\mathbb{R}^2$

<u>THM</u> Let V be an inner product space and $S = \{v_1, \dots, v_k\}$ be an orthogonal subset of V consisting of nonzero vectors. If $y \in \operatorname{span}(S)$, then

$$y = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i$$

"coordinate w/ respect to v_i"

<u>PF</u> Since $y \in \operatorname{span}(S)$, $y = \sum_{i=1}^{k} y_i v_i$, where each $y_i \in \mathbb{F}$.

$$\langle y, v_j \rangle = \left\langle \sum_{i=1}^{k} y_i v_i, v_j \right\rangle$$

$$= \sum_{i=1}^{k} y_i \langle v_i, v_j \rangle$$

since $\langle v_i, v_j \rangle = 0$ if $i \neq j$:

$$= y_j \langle v_j, v_j \rangle = y_j \cdot \|v_j\|^2$$

$$\Rightarrow \quad y_j = \frac{\langle y, v_j \rangle}{\|v_j\|^2}$$

so: $$y = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i \qquad \square$$

COR If $S = \{v_1, \ldots, v_k\}$ is an ortho<u>normal</u> subset of V and $y \in \operatorname{span}(S)$, then $$y = \sum_{i=1}^{k} \langle y, v_i \rangle v_i .$$

COR If V is an inner product space and S is a subset of nonzero orthogonal vectors, then S is L. I.

Pf Suppose $\sum_{i=1}^{k} a_i v_i = 0$, where $v_1, \ldots, v_k \in S$.

For each j, $1 \le j \le k$,

$$0 = \langle 0, v_j \rangle = \left\langle \sum_{i=1}^{k} a_i v_i , v_j \right\rangle$$

$$= \sum_{i=1}^{k} a_i \langle v_i, v_j \rangle$$

$$= a_j \langle v_j, v_j \rangle$$

Since $v_j \neq 0$, $\langle v_j, v_j \rangle \neq 0$

$\Rightarrow a_j = 0$ for each j

Thus, S is L.I. $\square$

<u>THM</u> (GRAM-SCHMIDT ORTHOGONALIZATION)

Let V be an inner product space and $S = \{w_1, \ldots, w_n\}$ be a L.I. subset of V. Define $S' := \{v_1, \ldots, v_n\}$, where $v_1 := w_1$ and, for $2 \leq k \leq n$,

$$v_k := w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j .$$

Then, S' is an orthogonal set of nonzero vectors such that $\mathrm{span}(S') = \mathrm{span}(S)$.

Pf : NEXT TIME

Ex Let $w_1 = (1,1,1)$, $w_2 = (2,3,4)$,

$w_3 = (1,0,0)$

Then $\{w_1, w_2, w_3\}$ is a basis for $\mathbb{R}^3$.

Let $v_1 := w_1 = (1,1,1)$.

$$v_2 := w_2 - \sum_{j=1}^{2-1} \frac{\langle w_2, v_j \rangle}{\|v_j\|^2} v_j$$

$$= w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$= (2,3,4) - \frac{9}{3}(1,1,1)$$

$$= (-1, 0, 1)$$

So: $v_1 = (1,1,1)$, $v_2 = (-1,0,1)$

$$v_3 := w_3 - \frac{\langle w_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle w_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$$

$$= (1,0,0) - \frac{1}{3}(1,1,1) - \frac{(-1)}{2}(-1,0,1)$$

$$= \left(\frac{1}{6}, -\frac{1}{3}, \frac{1}{6} \right)$$

Thus $\{v_1, v_2, v_3\} = \{ (1,1,1), (-1,0,1), (\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}) \}$

is an orthogonal basis, and

$$\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\} = \left\{ \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right), \left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \left(\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6} \right) \right\}$$

is an ortho<u>normal</u> basis.