SUBSPACES

DEF A subset W of a vector space (v.s.) over a field $\mathbb{F}$ is called a subspace of V if W is a v.s. over $\mathbb{F}$ w/ operations of addition and scalar mult. defined on V.

EX Given a v.s. V, V itself and $\{0\}$ are always subspaces of V

$\{0\}$: zero subspace

In some sense, these are "trivial" subspaces.

When, exactly, is a subset W of V a subspace?

First (and this holds generally), because properties (VS1) (COMM OF ADD), (VS2) (ASSOC OF ADD.), (VS5) (MULT BY 1), (VS6) (SCALAR MULT WELL-DEFINED), and (VS7), (VS8) (DIST LAWS) hold for **all** vectors in V and $W \subseteq V$, these automatically hold for W as well.

<u>EX</u> Let $w_1, w_2 \in W$

$w_1 + w_2 \in V$

$w_1 + w_2 = w_2 + w_1$ ((VS1) for V)

so: $w_1 + w_2 = w_2 + w_1$ ✓

It remains to show:

(1) $W \neq \phi$

(2) (Closure under $+$) If $x, y \in W$, then $x + y \in W$.

(3) (Closure under scalar mult.) If $x \in W$ and $c \in \mathbb{F}$, then $cx \in W$.

(4) W has a zero vector

(5) Each vector in W has an additive inverse.

GOOD NEWS: Suppose $W \neq \emptyset$ and W is closed under scalar mult.

Since there exists $w \in W$ and $\underbrace{0 \in \mathbb{F}}_{\text{scalar}}$,

and we know $\underbrace{0}_{\text{zero vector}} = 0w \in W$,

the additive identity / zero vector 0 (of V) is in W, and so W contains a zero vector.

Furthermore, $\underbrace{-v = (-1)v}_{\text{Dist Law and inverses are unique}} \in W$ whenever $v \in W$, and so each vector v has an inverse in W.

Notice we've "proved" the following:

THM Let V be a v.s. and W a subset of V. Then W is a subspace of V iff the following three conditions hold for the operations defined in V:

(a) $W \neq \phi$ (nonempty)

(b) If $x, y \in W$, then $x+y \in W$ (closed under $+$)

(c) If $x \in W$, $c \in \mathbb{F}$, then $cx \in W$
(closed under scalar mult.)

<u>NOTE</u> Compare this to THM 1.3!
This is slightly stronger since $W \neq \phi$ could be easier to check than $0 \in W$ (although essentially equivalent)

<u>EX</u> $\mathbb{F}$: field, $V = \mathbb{F}^n$

$$W := \left\{ (x_1, \ldots, x_n) \in V : \sum_{i=1}^{n} x_i = 0 \right\}$$

First, since $\sum_{i=1}^{n} 0 = 0 + \cdots + 0 = 0$,
$(0, 0, \ldots, 0) \in W$, and $W \neq \phi$.

Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$,
$x, y \in W$, so: $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 0$

Thus $x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$

$$\sum_{i=1}^{n} (x_i + y_i) = \left(\sum_{i=1}^{n} x_i\right) + \left(\sum_{i=1}^{n} y_i\right)$$

(Assoc, Comm of +)

$$= 0 + 0 = 0$$

$$\Rightarrow x + y \in W$$

Finally, if $c \in \mathbb{F}$,

$cx = (cx_1, cx_2, \ldots, cx_n)$, and

$$\sum_{i=1}^{n} cx_i = c\left(\sum_{i=1}^{n} x_i\right) \quad \text{(Dist Law)}$$

$$= c \cdot 0 = 0$$

$$\Rightarrow cx \in W$$

Therefore, W is a <u>subspace</u> of V.

<u>Q</u>: If $0 \in \mathbb{F}$, $v \in V$, why is $0v = 0$?

$$\underset{\text{zero scalar}}{0}\, v = \underset{\text{zero vector}}{0}$$

$$0v + 0v = (0+0)v \quad \text{(DIST. SCALARS (VS8))}$$
$$= 0v$$

$$0v = 0v + 0$$
$$= 0v + (0v + -(0v))$$
$$= (0v + 0v) + -(0v)$$
$$= (0+0)v + -(0v)$$
$$= 0v + -(0v)$$
$$= 0 \quad \checkmark$$

<u>Ex</u> Let $V = M_{n\times n}(\mathbb{F})$ ($n\times n$ matrices w/ entries in $\mathbb{F}$)

$(= M_n(\mathbb{F}))$

and let $W = \{A \in V : A^t = A\}$

$\underline{\text{symmetric matrices}}$

$$(A^t)_{ij} = (A)_{ji}$$

<u>SUB EX</u> $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$, W

but $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \notin W$

First, if 0 is the all 0's matrix,

then $0^t = 0$, and so $0 \in W$

and $W \neq \emptyset$

Let $A, B \in W$, so $A^t = A$, $B^t = B$

Let $(A)_{ij} = a_{ij}$, $(B)_{ij} = b_{ij}$

Thus $(A+B)_{ij} = a_{ij} + b_{ij}$

$= a_{ji} + b_{ji}$ (symm. matrices)

$= (A+B)_{ji}$,

and so $(A+B)^t = A+B \Rightarrow A+B \in W$

Similarly, if $c \in \mathbb{F}$, $A \in W$,

$(cA)_{ij} = ca_{ij}$

$= c a_{ji}$ (symm. matrices)

$= (cA)_{ji}$

and so $(cA)^t = cA \Rightarrow cA \in W$

Hence W is a subspace of V.

THM Any intersection of subspaces of a v.s. V is a subspace of V.

Pf Let U, W be subspaces of V.

Since U, W are subspaces,

$0 \in U, \quad 0 \in W \Rightarrow 0 \in U \cap W$

$\Rightarrow U \cap W \neq \emptyset$

Let $x, y \in U \cap W, \quad c \in \mathbb{F}$

So: $x, y \in U$, U subspace

$\Rightarrow x+y, \ cx \in U$

Similarly, $x, y \in W$, W subspace

$\Rightarrow x+y, \ cx \in W$

$\Rightarrow x+y, \ cx \in U \cap W$

$\Rightarrow U \cap W$ subspace. $\square$

READ § 1.4!