<u>THM</u> If a vector space V is generated by a <u>finite</u> set S $(V = \mathrm{span}(S))$, then some subset of S is a <u>basis</u> for V. Hence V has a finite basis.

<u>RECALL</u>: B is a <u>basis</u> for V if $\mathrm{span}(B) = V$ <u>and</u> B is <u>linearly independent</u> (L.I.)

<u>Pf</u> Since S is finite, <u>every</u> subset of S is finite.

Define $D := \{ d \in \mathbb{Z} : |T| = d$ for some subset $T \subseteq S$ w/ $\mathrm{span}(T) = V \}$

Since $\mathrm{span}(S) = V$, $S \subseteq S$, $|S| \in D$, so D is nonempty.

Moreover, if $n \in D$, then $n \geq 0$, so D is a nonempty subset of $\mathbb{Z}$ that is bounded below.

This means D has a least element,
say l.

Since $l \in D$, there exists $B \subseteq S$
such that $|B| = l$ and $\text{span}(B) = V$.

Suppose B is a linearly dependent set.
Since ϕ is L.I., this means $B \neq \phi$,
and B is finite, so WLOG
$B = \{v_1, v_2, \ldots, v_l\}$, and, since
B is linearly dependent, there exist
$a_i \in \mathbb{F}$ (not <u>all</u> 0) such that

$$\sum_{i=1}^{l} a_i v_i = 0$$

<u>RECALL</u>: If B is linearly dependent,
there exists $v \in B$ such that

$$\text{span}(B - \{v\}) = \text{span}(B).$$

However, this means $\text{span}(B - \{v\}) = V$,
and $|B - \{v\}| = l - 1$.

This means $l-1 \in D$.

But $l-1 < l$, the <u>least</u> element of D, a contradiction.

Hence B must be L.I., and hence B is a (finite) basis for V, as desired. $\square$

<u>**NOTE**</u> (1) Compare w/ book proof!

"induction"

BOOK: "bottom-up," building successively bigger L.I. sets until one is a spanning set

"WOP" (more or less)

THIS ONE: "top-down," maintaining "spanning" until as small as possible & showing that this implies L.I.

(2) Another feature of this proof: it showed that <u>any</u> subset $T \subseteq S$ such that $\operatorname{span}(T) = V$ and $|T|$ minimal is a basis!

Moreover, the <u>minimum</u> of $|T|$ such that $\text{span}(T) = V$ was determined <u>before</u> we picked out a set of that size!

With this in mind, we want to show that the size of a (finite) basis is an <u>invariant</u> of such a v.s.

<u>THM</u> (Replacement Theorem)
Let V be a v.s. that is generated by a set S containing exactly n elements (i.e., $\text{span}(S) = V$ and $|S| = n$), and let L be a linearly independent subset of V w/ $|L| = m$.
Then, $m \leq n$, and there exists $S' \subseteq S$ such that $|S'| = n - m$ and $\text{span}(L \cup S') = V$.

<u>Pf</u> We proceed by induction on m.

If $m = 0$, then $L = \phi$,

and $|S| = n = n - 0$, $S \cup \phi = S$,

so $|\phi \cup S| = |S| = n$

and $\operatorname{span}(\phi \cup S) = V$.

Moreover, $0 \leq n$.

Assume now that the result is true for all linearly independent subsets of V of size $(m-1)$, $m \geq 1$.

Let L be a L.I. subset of V w/ $|L| = m$. Hence $L = \{v_1, v_2, \dots, v_m\}$.

Since L is L.I., so is

$$\tilde{L} = \{v_1, v_2, \dots, v_{m-1}\}$$

$(\tilde{L} = \phi$ when $m = 1)$

Using the inductive hypothesis on $\tilde{L}$,

there exist $u_m, u_{m+1}, \ldots, u_n \in S$

such that, if $\{u_m, \ldots, u_n\} = \tilde{S}$,

then $|\tilde{L} \cup \tilde{S}| = n$, $\operatorname{span}(\tilde{L} \cup \tilde{S}) = V$,

and $n \geq m-1$.

Suppose $n = m-1$.

This means $\tilde{S} = \phi$ and so $\operatorname{span}(\tilde{L}) = V$.

Now, $v_m \in V = \operatorname{Span}(\tilde{L}) = \operatorname{span}(\{v_1, \ldots, v_{m-1}\})$,

so $v_m = \sum_{i=1}^{m-1} a_i v_i$ for some $a_i \in \mathbb{F}$

$\Rightarrow \sum_{i=1}^{m} a_i v_i = 0$ where $a_m = -1$

$\Rightarrow L$ is linearly dependent $\Rightarrow \Leftarrow$

Hence $n \geq m-1$ and $n \neq m-1 \Rightarrow n \geq m$.

Now, $\operatorname{span}(\tilde{L} \cup \tilde{S}) = V$ and $v_m \in V$,

so there exist coefficients $a_1, a_2, \ldots, a_n$

such that $\sum_{i=1}^{m-1} a_i v_i + \sum_{i=m}^{n} a_i u_i = v_m$

If every $a_i = 0$ when $i \geq m$,

then $v_m \in \text{span}(\tilde{L})$, which is (again)

a contradiction to L being L.I.

WLOG, $a_k \neq 0$, where $k \geq m$.

This means:

$$\sum_{i=1}^{m-1} a_i v_i + \sum_{\substack{m \leq i \leq n \\ i \neq k}} a_i u_i + a_k u_k = v_m$$

Since $\mathbb{F}$ is a field, $a_k \neq 0$, $a_k^{-1} \in \mathbb{F}$.

$$\Rightarrow \quad u_k = \sum_{i=1}^{m-1} \left(a_k^{-1}(-a_i)\right) v_i + a_k^{-1} v_m + \sum_{\substack{m \leq i \leq n \\ i \neq k}} \left(a_k^{-1}(-a_i)\right) u_i$$

Let $S' = \tilde{S} \setminus \{u_k\}$.

Since $\tilde{L} \subseteq \text{span}(L \cup S')$

and $\tilde{S} \subseteq \text{span}(L \cup S')$,

$$V = \text{span}(\tilde{L} \cup \tilde{S}) \subseteq \text{span}(L \cup S') \subseteq V$$

$$\Rightarrow \quad \text{span}(L \cup S') = V,$$

and the result follows by induction. $\square$

COR Let V be a vector space that has a finite basis. Then every basis for V contains the same number of vectors.

Pf Let B_1, B_2 be (finite) bases for V.

WLOG, $|B_1| \leq |B_2|$.

However since B_2 is L.I. and $\text{span}(B_1) = V$, by the Replacement Thm, $|B_2| \leq |B_1|$

$$\Rightarrow \quad |B_1| = |B_2|. \qquad \square$$

NOTES Suppose V is a v.s. w/ basis B, $|B| = d$.

(1) No subset S w/ $|S| < d$ can span V

(2) Every subset T w/ $|T| > d$ must be linearly dependent

(3) We saw before that every minimal spanning set of V (in S) had the same size. This shows that every (maximal) L.I. set of size d is a basis.

(4) Every L.I. set can be extended to a basis

(5) Every spanning set contains a basis.

NEXT TIME: dimension, Lagrange Interpolation

READ: § 1.7