RECALL: $T : V \to V$ linear, $\dim(V) = n$

Define determinant of T as follows:

$$T_* : \Lambda^n V \to \Lambda^n V$$

$$T_* (v_1 \wedge v_2 \wedge \dots \wedge v_n) := T(v_1) \wedge T(v_2) \wedge \dots \wedge T(v_n)$$

T_* is a linear map and $\dim(\Lambda^n V) = 1$, so there is a constant $\det(T)$ such that

$$T_* (v_1 \wedge \dots \wedge v_n) = \det(T) \cdot (v_1 \wedge \dots \wedge v_n)$$

for **all** $v_1 \wedge \dots \wedge v_n \in \Lambda^n V$.

BOOK:

DEF A function $\delta : M_n(\mathbb{F}) \to \mathbb{F}$ is **n-linear** if it is a linear function of each **row** of an $n \times n$ matrix.

$\left(\longleftrightarrow \quad \delta : V^n \longrightarrow \mathbb{F} \quad \underline{n\text{-linear}} \right)$

Ex If $\delta : M_2(\mathbb{R}) \to \mathbb{R}$ is 2-linear,

then $$\delta\left(\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\right) = 2 \cdot \delta\left(\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\right)$$

DEF An n-linear function $\delta : M_n(\mathbb{F}) \to \mathbb{F}$ is <u>alternating</u> if $\delta(A) = 0$ whenever two rows are same

$\left(\longleftrightarrow \quad \delta : V^n \to \mathbb{F} \quad \underline{\text{alternating}} \right)$

THM If $\delta : M_n(\mathbb{F}) \to \mathbb{F}$ is an alternating, n-linear map such that $\delta(I_n) = 1$, then $\delta(A) = \det(A)$ for every $A \in M_n(\mathbb{F})$.

Pf Each row of a matrix in $M_n(\mathbb{F})$ is an element of $\mathbb{F}^n = V$, so it is equivalent to view δ

as an alternating, n-linear map

from $V^n \longrightarrow \mathbb{F}$.

So: we can view $\delta : V^n \longrightarrow \mathbb{F}$,

so δ induces

$$\hat{\delta} : \Lambda^n V \longrightarrow \mathbb{F}$$

$$\left(\hat{\delta}(r_1 \wedge \cdots \wedge r_n) := \delta(r_1, \ldots, r_n), \text{ where the } r_i \text{ are rows of } A \right)$$

<u>HW 7, #6</u>: $\hat{\delta}$ is <u>linear</u>

Since $\delta(I) = 1$, if $(e_1, \ldots, e_n)$ is

the standard basis, then

$$\hat{\delta}(e_1 \wedge \cdots \wedge e_n) = 1.$$

Moreover, since $\dim(\Lambda^n V) = 1$,

$\hat{\delta}$ is an <u>isomorphism</u>.

An isomorphism is determined entirely

by its image on a basis; thus $\hat{\delta}$ (and hence δ) is uniquely determined. $\square$

NOTE The book shows that the definition of the determinant you're used to is such an alternating, n-linear map, so the definitions coincide.

EIGENVECTORS AND EIGENVALUES

Given a linear transformation $T: V \to V$, we want to find nonzero vectors $v \in V$ such that $T(v) = \lambda v$ for some $\lambda \in \mathbb{F}$.

Ideally, we want a <u>basis</u> of such vectors: suppose $\beta = (v_1, \ldots, v_n)$ is a basis for V and $T(v_i) = \lambda_i v_i$ for each i.

This makes T particularly easy to work w/ using this basis: if $v = \sum_{i=1}^{n} a_i v_i$, then $T(v) = \sum_{i=1}^{n} a_i \lambda_i v_i$, and

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & & O \\ & \lambda_2 & & \\ & & \ddots & \\ O & & & \lambda_n \end{pmatrix}$$

In this case, T is called <u>diagonalizable</u>, and we sometimes write $[T]_\beta = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Motivating example:

<u>EX</u> Let $A = \begin{pmatrix} 7/2 & 3 \\ -3/2 & -1 \end{pmatrix}$.

What is A^{1000}?

If $Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, then $Q^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$,

and $$QAQ^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Notice that $$(QAQ^{-1})^2 = QAQ^{-1}QAQ^{-1}$$
$$= QA^2Q^{-1},$$

and it is easy to show by induction that $(QAQ^{-1})^k = QA^kQ^{-1}$ for all $k \in \mathbb{N}$. So:

$$A^k = Q^{-1}(QAQ^{-1})^k Q$$
$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2^k & 0 \\ 0 & (\frac{1}{2})^k \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} - (\frac{1}{2})^k & 2^{k+1} - (\frac{1}{2})^{k-1} \\ -(2^k) + (1/2)^k & -(2^k) + (\frac{1}{2})^{k-1} \end{pmatrix}$$

so: $A^{1000} = \begin{pmatrix} 2^{1001} - \frac{1}{2^{1000}} & 2^{1001} - \frac{1}{2^{999}} \\ -(2^{1000}) + \frac{1}{2^{1000}} & -(2^{1000}) + \frac{1}{2^{999}} \end{pmatrix}$

This may seem like a contrived example, but being able to do this is <u>extremely</u> useful in applications.

With this in mind, we have the following definition:

<u>DEF</u> Let T be a linear operator on a space V $(T: V \to V)$. A nonzero vector $v \in V$ is called an <u>eigenvector</u> of T if there exists a scalar $\lambda \in \mathbb{F}$ such that $T(v) = \lambda v$. The scalar λ is called the <u>eigenvalue</u> corresponding to the eigenvector v.

Similarly, if $A \in M_n(\mathbb{F})$, a nonzero $v \in V$ is an eigenvector of A if v is an eigenvector of L_A, i.e., if $Av = \lambda v$ for some $\lambda \in \mathbb{F}$, in which case λ is the eigenvalue corresponding to eigenvector v.

Given this terminology, the following is easy to prove:

THM A linear operator T on a finite-dim'l v.s. is diagonalizable iff there exists an ordered basis β for V consisting of eigenvectors of T.

so: diagonalization $\longleftrightarrow$ finding a basis of eigenvectors

NOTE The concept of eigenvalue/eigenvector makes sense in infinite-dim'l v.s. as well.

EX (BOOK) Let $C^\infty(\mathbb{R})$ denote the set of all functions $f: \mathbb{R} \to \mathbb{R}$ having derivatives of all orders ("infinitely differentiable")

(EXAMPLES: polynomials, sine, cosine, exponential functions, etc. ---)

$C^\infty(\mathbb{R})$ is a subspace of the v.s. of all functions from $\mathbb{R}$ to $\mathbb{R}$, $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

Let $D: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be the differentiation operator, i.e.,

$$D(f) = \frac{df}{dx} = f'.$$

D is linear.

What are its eigenvectors and eigenvalues?

Suppose $D(f) = \lambda f$, i.e., $f' = \lambda f$,

where $\lambda \in \mathbb{R}$.

This is a first-order differential equation whose solutions are of the form $f(x) = c e^{\lambda x}$ for some $c \in \mathbb{R}$.

Thus <u>every</u> real number $\lambda \in \mathbb{R}$ is an eigenvalue of D w/ eigenvector $e^{\lambda x}$.

(Note that, when $\lambda = 0$, the eigenvectors are simply the nonzero constant functions.)