LINEAR TRANSFORMATIONS

DEF Let V, W be v.s. over the same field $\mathbb{F}$. A function $T: V \to W$ is a *linear transformation* if, for all $x, y \in V$ and $c \in \mathbb{F}$,

(a) $T(x+y) = T(x) + T(y)$

(b) $T(cx) = cT(x)$

EQUIVALENTLY: For all $x, y \in V$, $a, b \in \mathbb{F}$,

$$T(ax+by) = aT(x) + bT(y)$$

Can choose $a = b = 1$ and $a = c$, $b = 0$, respectively, to satisfy the definition.

IDEA: *operation-preserving*

$$\begin{tikzcd}
\underline{\underline{V}} & & \underline{\underline{W}} \\
v, w \arrow[rr, "\text{Apply } T"] \arrow[d, "\text{Add/mult by scalars}"'] & & T(v), T(w) \arrow[d, "\text{Add/mult by scalars}"] \\
av + bw \arrow[rr, "\text{Apply } T"] & \text{same if a L.T.} & \boxed{\begin{matrix} aT(v) + bT(w) \\ \shortparallel \\ T(av+bw) \end{matrix}}
\end{tikzcd}$$

Ex Projection

If $m < n$, then define $T : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ by

$$T : (x_1, x_2, \dots, x_m, x_{m+1}, \dots, x_n) \longmapsto (x_1, \dots, x_m)$$

Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, $a, b \in \mathbb{F}$

$$\begin{aligned} T(ax+by) &= T\big((ax_1+by_1, ax_2+by_2, \dots, ax_n+by_n)\big) \\ &= (ax_1+by_1, \dots, ax_m+by_m) \\ &= a(x_1, \dots, x_m) + b(y_1, \dots, y_m) \\ &= aT(x) + bT(y) \quad \checkmark \end{aligned}$$

Ex Let V be a v.s. over $\mathbb{F}$,

$\{0\}$ the zero v.s. over $\mathbb{F}$.

$T_0 : V \to \{0\}$ by $T_0(v) = 0$.

Then
$$\begin{aligned} T_0(ax+by) &= 0 \\ &= 0 + 0 \\ &= a0 + b0 \\ &= aT_0(x) + bT_0(y) \quad \checkmark \end{aligned}$$

Ex Let V be a v.s. over $\mathbb{F}$,

$$id_V : \mathbb{F} \to \mathbb{F} \quad \text{by} \quad id_V(v) = v.$$

(Book: I_V)

Then $id_V(ax+by) = ax+by = a\, id_V(x) + b\, id_V(y)$ ✓

Ex Let $V = \mathbb{R}^2$ as a v.s. over $\mathbb{R}$,

and define $\rho_\alpha : V \longrightarrow V$ by

$$\rho_\alpha(\underbrace{(r, \theta)}_{\text{polar coordinates}}) = (r, \theta + \alpha)$$

In rectangular coordinates,

$$\rho_\alpha(\underbrace{(x,y)}_{\text{rectangular coordinates}}) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

is a L.T. (CHECK!)

Ex Consider $P_n(\mathbb{R})$, $P_{n-1}(\mathbb{R})$ (polynomials w/ coeff. in $\mathbb{R}$)

Define $D : P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ by

$$D(f) = \frac{df}{dx}$$

$$D(af+bg) = \frac{d(af+bg)}{dx}$$

$$= a\frac{df}{dx} + b\,{}^{dg}\!/_{dx}$$

$$= aD(f) + bD(g)$$ ✓

DEF Let V, W be a v.s., $T: V \to W$ a linear transformation. We define the <u>kernel</u> (or <u>null space</u>) $\ker(T)$ $(= N(T))$ to be the set of all vectors $x \in V$ such that $T(x) = 0$.

$$\mathrm{Ker}(T) = \{x \in V : T(x) = 0\}$$
$$(\,\overset{\shortparallel}{N(T)}\,)$$

The <u>image</u> (or <u>range</u>) $\mathrm{Im}(T)$ $(=R(T))$ of T is the subset of W consisting of all images (under T) of vectors in V, i.e.,

$$\mathrm{Im}(T) = \{T(x) : x \in V\}$$
$$(\,\overset{\shortparallel}{R(T)}\,)$$

EX Projection

If $T : \mathbb{F}^n \to \mathbb{F}^m$ is the projection map as above,

$$\operatorname{Ker}(T) = \{(\underbrace{0, \dots, 0}_{1^{st}\ m \text{ coordinates}}, x_{m+1}, \dots, x_n) : x_i \in \mathbb{F}\}$$

$$\operatorname{Im}(T) = \mathbb{F}^m$$

EX $D : P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$

(differentiation as above)

$$\operatorname{Ker}(D) = \{c : c \in \mathbb{R}\}$$

$$\operatorname{Im}(D) = P_{n-1}(\mathbb{R})$$

Kernel, Image: vector spaces!

THM Let $T : V \to W$ be a linear transformation. Then $\operatorname{Ker}(T)$ is a subspace of V, $\operatorname{Im}(T)$ is a subspace of W.

Pf Given $T: V \to W$ a L.T.

Let $x, y \in Ker(T)$, $a, b \in \mathbb{F}$.

First, $Ker(T) \neq \emptyset$, since $T(0) = 0$.

$$\left(T(0) = T(0+0) = T(0) + T(0) \Rightarrow T(0) = 0 \right)$$

Then $T(ax+by) = aT(x) + bT(y)$ (linear)

$= a0 + b0$ ($x, y \in Ker(T)$)

$= 0$

$\Rightarrow ax + by \in Ker(T)$

Hence $Ker(T) \neq \emptyset$, and it's closed under both addition / scalar mult.

Therefore, $Ker(T)$ is a subspace of V.

Let $v, w \in Im(T)$, $a, b \in \mathbb{F}$.

First, since $T(\underset{\text{in } V}{0}) = \underset{\text{in } W}{0}$, $0 \in Im(T)$,

so $Im(T) \neq \emptyset$.

Since $v, w \in Im(T)$, there exist $x, y \in V$

such that $T(x)=v$, $T(y)=w$.

So: $$av+bw = aT(x)+bT(y)$$
$$= T(ax+by)$$
$$\Rightarrow av+bw \in Im(T).$$

Therefore, since $Im(T) \neq \emptyset$ and $Im(T)$ is closed under vector addition and scalar mult., $Im(T)$ is a subspace of W. $\square$

<u>LEMMA</u> Let $T: V \to W$ be a L.T. Then T is injective iff $Ker(T) = \{0\}$.

<u>Pf</u> Suppose first that T is injective. We know that $T(0)=0$.

Suppose $v \in Ker(T)$.
Then $T(v) = 0 = T(0)$
$\Rightarrow v = 0$ (injectivity)
Hence $Ker(T) = \{0\}$.

Now, conversely, suppose $Ker(T) = \{0\}$.

Suppose $v, w \in V$, $T(v) = T(w)$.

Then $T(v-w) = T(v) - T(w) = 0$

$\Rightarrow v - w \in \mathrm{Ker}(T) \Rightarrow v - w = 0$

$\Rightarrow v = w$, and hence

T is injective. $\square$

<u>DEF</u> Let V, W be v.s., $T: V \to W$ be linear.

If $\mathrm{Ker}(T)$, $\mathrm{Im}(T)$ are finite-dim'l,

then we define the <u>nullity</u> of T,

denoted $\mathrm{nullity}(T)$, and the <u>rank</u>

of T, denoted $\mathrm{rank}(T)$, to be:

$$\mathrm{nullity}(T) := \dim(\mathrm{Ker}(T))$$
$$\mathrm{rank}(T) := \dim(\mathrm{Im}(T)).$$

<u>THM</u> (Dimension Thm / "Rank-Nullity")

Let V, W be v.s., $T: V \to W$ lin.

If V is finite dim'l, then

$$\mathrm{nullity}(T) + \mathrm{rank}(T) = \dim(V).$$

Pf Since $\dim(V)$ is finite, we let
$\dim(V) = n$.

Since $Ker(T)$ is a subspace of V,
we may let $\dim(Ker(T)) = k \leq n$.

Let $\{v_1, \ldots, v_k\}$ be a basis for
$Ker(T)$.

This is L.I. subset of V, and,
by Replacement Thm, we may extend
this to a basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$
of V.

It suffices to show $\{T(v_{k+1}), \ldots, T(v_n)\}$
is a basis for $Im(T)$,
since then $\dim(Im(T)) = n-k$.

First, we'll show $span(\{T(v_{k+1}), \ldots T(v_n)\})$
$= Im(T)$.

Let $w \in Im(T)$. By definition of image,
there exists $v \in V$ such that
$T(v) = w$.

Since $v \in V$ and $\{v_1, \dots, v_n\}$ is a basis,

there exist $a_1, \dots, a_n \in \mathbb{F}$ such that

$$v = a_1 v_1 + \dots + a_n v_n = \sum_{i=1}^{n} a_i v_i$$

Hence
$$w = T(v) = T\left(\sum_{i=1}^{n} a_i v_i\right)$$
$$= \sum_{i=1}^{n} a_i \, T(v_i)$$
$$= \sum_{i=1}^{k} a_i \underbrace{T(v_i)}_{\substack{v_i \text{ in } \ker(T) \\ \text{when } 1 \le i \le k}} + \sum_{i=k+1}^{n} a_i \, T(v_i)$$
$$= 0 + \sum_{i=k+1}^{n} a_i \, T(v_i)$$
$$= \sum_{i=k+1}^{n} a_i \, T(v_i) \in \operatorname{span}(\{T(v_{k+1}), \dots, T(v_n)\})$$

<u>NEXT TIME</u>: We'll finish the proof.

<u>READ</u>: §2.2.