<u>Q</u>: <u>HW 9, #5</u> W: 1-dim'l subspace of $\mathbb{R}^n$

There exists nonzero $w \in W$ w/ all nonnegative entries

Prove that W contains a <u>unique</u> probability vector (sum of entries is 1, all entries nonnegative)

Since $w \neq 0$ and W is 1-dim'l,

$$W = \text{span}(\{w\}).$$

<u>EX</u> $W = \text{span}(\{(1,1,2,3)\})$

If $v \in W$, $v = c(1,1,2,3)$, where $c \in \mathbb{R}$.

$\Rightarrow$ If prob. vector, then

$$1 = c \cdot 1 + c \cdot 1 + c \cdot 2 + c \cdot 3$$

$$= c(1+1+2+3) = 7c$$

$\Rightarrow \quad c = 1/7 \quad \Rightarrow \quad v = \left(\frac{1}{7}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}\right)$

LAST TIME: V v.s., $T: V \to V$ linear

T-invariant subspaces:

W is T-invariant if

$$T(W) \subseteq W$$

T-cyclic subspace generated by v:

$$\text{span}\left(\{v, T(v), T^2(v), \ldots\}\right)$$

We proved:

① If W is a T-invariant subspace and T_W is the restriction of T to W, then $\chi_{T_W}(t)$ divides $\chi_T(t)$.

② V: finite-dim'l, W: T-cyclic subspace generated by v

Suppose $\dim(W) = k$ and

$$a_0 v + a_1 T(v) + a_2 T^2(v) + \dots + a_{k-1} T^{k-1}(v) + T^k(v) = 0.$$

Then: $\chi_{T_W}(t) = (-1)^k \left(a_0 + a_1 t + a_2 t^2 + \dots + a_{k-1} t^{k-1} + t^k \right)$

<u>CAYLEY - HAMILTON THM</u> Let T be a linear operator on a finite dim'l v.s. V w/ characteristic polynomial $\chi_T(t)$.

Then $\chi_T(T) = T_0$, the zero transformation.

That is, T "satisfies" its characteristic polynomial.

<u>EX</u> $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ $T := L_A$

so: $\chi_T = \chi_A$

$$\chi_T(t) = \det(A - tI) = \begin{vmatrix} 1-t & 2 \\ 0 & 3-t \end{vmatrix} = (t-1)(t-3)$$
$$= t^2 - 4t + 3$$

<u>Cayley - Hamilton</u>: $T^2 - 4T + 3I = \underbrace{T_0}_{\text{zero transformation}}$

$$L_A^2 = L_{A^2} \qquad A^2 = \begin{pmatrix} 1 & 8 \\ 0 & 9 \end{pmatrix}$$

$$T^2 - 4T + 3I \longleftrightarrow A^2 - 4A + 3I$$

$$\begin{pmatrix} 1 & 8 \\ 0 & 9 \end{pmatrix} - 4\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-4+3 & 8-8+0 \\ 0-0+0 & 9-12+3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \checkmark$$

<u>Pf of Cayley – Hamilton</u>

To prove this, we'll show that $\chi_T(T)(v) = 0$ for <u>all</u> $v \in V$.

If $v = 0$, then $\chi_T(T)(0) = 0$ since $\chi_T(T)$ is linear.

If $v \neq 0$, then let W be the

T-cyclic subspace generated by v.

V finite dim'l $\Rightarrow$ W finite dim'l,

so let $\dim(W) = k$.

This means there exist $a_0, a_1, \ldots, a_{k-1} \in \mathbb{F}$

such that $a_0 v + a_1 T(v) + \cdots + a_{k-1} T^{k-1}(v) + T^k(v) = 0$

and $\chi_{T_W}(v) = (-1)^k (a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k)$.

Thus $\chi_{T_W}(T) = (-1)^k (a_0 + a_1 T + \cdots + a_{k-1} T^{k-1} + T^k)$

and $\chi_{T_W}(T)(v) = 0$.

We also proved that $\chi_{T_W}(t) \mid \chi_T(t)$,

and so for some polynomial $g(t)$,

$$\chi_T(t) = g(t) \cdot \chi_{T_W}(t).$$

Therefore,

$$\chi_T(T)(v) = g(T) \cdot \chi_{T_W}(T)(v)$$
$$= g(T)(0) = 0.$$

But v was arbitrary, and so

$$\chi_T(T) = T_0, \text{ the zero transformation.} \qquad \square$$

<u>COR</u> If A is an $n \times n$ matrix, then $\chi_A(A) = 0$.

zero matrix

<u>PF</u> HW 10, # 6.

<u>CAUTION</u> The following "argument" is nonsense:

$$\cancel{\chi_A(t) = \det(A - tI),}$$

$$\cancel{\text{so } \chi_A(A) = \det(A - AI) = \det(0) = 0}$$

What's wrong: tI: multiply each <u>entry</u> of I by t

AI: <u>just</u> matrix mult, not "entrywise mult"

THE MINIMAL POLYNOMIAL

The Cayley-Hamilton Thm tells us that a linear operator T on a finite dim'l v.s. "satisfies" its own characteristic polynomial, χ_T, i.e., $\chi_T(T) = T_0$.

Hence there is a polynomial of <u>minimal degree</u> that T satisfies.

<u>EX</u> $I : \mathbb{F}^n \to \mathbb{F}^n$, the identity map.

If β is the standard basis,

$$[I]_\beta = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} = I_n,$$

and so $\chi_I(t) = (-1)^n (t-1)^n$

However, $I - I = 0$,

so I "satisfies" the polynomial $t-1$.

DEF Let T be a linear operator on a finite dim'l v.s. A polynomial $m(t)$ is called a <u>minimal polynomial</u> of T if $m(t)$ is a <u>monic</u> polynomial (leading coeff: 1) of least degree such that

$$m(T) = T_0.$$

<u>ASIDE</u>:

<u>DIVISION ALGORITHM FOR POLYNOMIALS w/ COEFFICIENTS FROM A FIELD</u>

Let $f(t)$, $g(t)$ be polynomials w/ coefficients in field $\mathbb{F}$.

If $g(t) \neq 0$ and $\deg(g(t)) = n \in \mathbb{N} \cup \{0\}$, then there exist polynomials $q(t)$, $r(t)$ w/ coeffs in $\mathbb{F}$ such that

$\deg(r(t)) < \deg(g(t))$

(we can view 0 as having degree "less than zero")

and $f(t) = q(t)\, g(t) + r(t)$.

<u>EX</u> Field : $\mathbb{R}$

$f(t) = 3t^5 - 4t^4 + 3t^3 + t^2 + 1$

$g(t) = 12t^4 + 5$

$\frac{1}{4} t \,(12t^4) = 3t^5$

so we can cancel off leading coeff.

We can do this until remainder has degree less than 4 ($= \deg(g(t))$ here)

$$
\begin{array}{r l}
 & \tfrac{1}{4}t - \tfrac{1}{3} \\
12t^4 + 5 \,\big) & 3t^5 - 4t^4 + 3t^3 + t^2 + 0t + 1 \\
 & -\left(3t^5 \qquad\qquad\qquad\qquad + \tfrac{5}{4}t\right) \\
\hline
 & -4t^4 + 3t^3 + t^2 - \tfrac{5}{4}t + 1 \\
 & -\left(-4t^4 \qquad\qquad\qquad\qquad - \tfrac{5}{3}\right) \\
\hline
 & 3t^3 + t^2 - \tfrac{5}{4}t + \tfrac{8}{3}
\end{array}
$$

so: $q(t) = \frac{1}{4}t - 1/3$

$r(t) = 3t^3 + t^2 - 5/4\, t + 8/3$

<u>THM</u> Let $m(T)$ be a minimal polynomial for $T : V \to V$, V: finite dim'l.

(a) If $f(t)$ is a polynomial such that $f(T) = T_0$, then $m(t)$ divides $f(t)$.

(b) The minimal polynomial is unique

NOTE The minimal polynomial of T will be denoted $m_T(t)$.

Pf (a) Suppose $f(t)$ is a polynomial such that $f(T) = T_0$.

By Division Alg. for Polynomials, there exist polynomials $r(t)$, $q(t)$ such that $f(t) = q(t)\,m(t) + r(t)$ and $\deg(r(t)) < \deg(m(t))$.

$$\Rightarrow \quad T_0 = f(T)$$

$$= \underbrace{q(T)\,m(T)}_{T_0} + r(T)$$

$$= r(T)$$

So: T "satisfies" $r(t)$.

Since $\deg(r(t)) < \deg(m(t))$ and the degree of $m(t)$ is minimal, it must be that $r(t) = 0$.

$\Rightarrow$ $f(t) = q(t)\, m(t)$, and so

$$m(t) \mid f(t).$$

(b) Suppose $m_1(t)$, $m_2(t)$ are minimal polynomials for T.

Then $m_2(t) \mid m_1(t)$ by (a),

so $m_1(t) = q(t)\, m_2(t)$.

Since $\deg(m_1(t)) = \deg(m_2(t))$,

$q(t)$ is a constant c

$$\Rightarrow \quad m_1(t) = c \cdot m_2(t)$$

Since $m_1(t)$, $m_2(t)$ are <u>monic</u>,

$c = 1$, and so $m_1(t) = m_2(t)$. $\square$