LAST TIME: L.T. $\longrightarrow$ matrices
(w/ respect to ordered bases of finite-dim'l v.s.)

What about the other direction?

DEF Let $A$ be an $m \times n$ matrix w/ entries from $\mathbb{F}$. Define by

$$L_A : \mathbb{F}^n \to \mathbb{F}^m$$

the function $L_A(x) = Ax$ ($x$: column vector in $\mathbb{F}^n$)

$L_A$: left-multiplication transformation

THM $L_A : \mathbb{F}^n \to \mathbb{F}^m$ is linear

Moreover, if $\mathbb{F}^n$ has the standard ordered basis $\beta_n$ and $\mathbb{F}^m$ also has the standard ordered basis $\beta_m$, Then:

(a) $[L_A]_{\beta_n}^{\beta_m} = A$

(b) $L_A = L_B$ iff $A = B$

(c) $L_{A+B} = L_A + L_B$ ;

for $a \in \mathbb{F}$, $L_{aA} = a L_A$

(d) If $T: \mathbb{F}^n \to \mathbb{F}^m$ is linear, there is a <u>unique</u> $m \times n$ matrix $A$ s.t. $T = L_A$ $\left( = [T]_{\beta_n}^{\beta_m} \right)$

(e) If $E$ is an $n \times p$ matrix, then

$$L_{AE} = L_A L_E$$

(f) If $m = n$, $L_{I_n} = \mathrm{id}_{\mathbb{F}^n}$ $\left( = I_{\mathbb{F}^n} \right)$

<u>Pf</u> First, $L_A(ax+by) = A(ax+by)$

$$= aAx + bAy$$

$$= a L_A(x) + b L_A(y)$$

(a) $L_A(e_j) = Ae_j$ is column $j$ of $A$ for each $j$

Hence $[L_A]_{\beta_n}^{\beta_m} = A$.

(b) Suppose $L_A = L_B$.

$$\Rightarrow \quad [L_A]_{\beta_n}^{\beta_m} = [L_B]_{\beta_n}^{\beta_m}$$

$$\Rightarrow \quad A = B$$

Conversely, $A = B$

$$\Rightarrow \quad Ax = Bx \quad \text{for all } x \in \mathbb{F}^n$$

$$\Rightarrow \quad L_A = L_B$$

(other proofs are similar) $\square$

<u>THM</u> Matrix multiplication is associative:

$$A(BC) = (AB)C$$

<u>Pf</u> $A \longleftrightarrow L_A$, $B \longleftrightarrow L_B$, $C \longleftrightarrow L_C$, etc.

$$\begin{aligned} L_{A(BC)} = L_A L_{BC} &= L_A (L_B L_C) \\ &= (L_A L_B) L_C \quad \left(\begin{matrix}\text{function comp.}\\ \text{is assoc.}\end{matrix}\right) \\ &= L_{AB} L_C \\ &= L_{(AB)C} \end{aligned}$$

Since $L_{A(BC)} = L_{(AB)C}$, $A(BC) = (AB)C$.

$\square$

NOTE The associativity of matrix mult. can be proved directly, but it's messy.

NOTE Matrix mult. is not commutative!

Ex

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 7 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 4 & 6 \end{bmatrix}$$

## INVERTIBLE LINEAR TRANSFORMATIONS

DEF Let $T: V \to W$ be a linear transformation. We say that the linear transformation $S : W \to V$ is an <u>inverse</u> of $T$ if

$$TS = id_W \ (= I_W) \quad \text{and}$$
$$ST = id_V \ (= I_V).$$

If $T$ has an inverse, then it's <u>invertible.</u>

<u>NOTE</u> Suppose $S, U$ are inverses of $T$. Then:

$$S = S I_W = S(TU) = (ST)U$$
$$= I_V U = U$$

Hence we may talk about <u>the</u> inverse of $T$ if it has an inverse, and we denote the inverse of $T$ by $T^{-1}$.

Since inverses are unique (if they exist),

and $(TS)(S^{-1}T^{-1}) = I$,

this means $(TS)^{-1} = S^{-1}T^{-1}$.

<u>LEMMA</u> $T: V \to W$ has an inverse

$S: W \to V$ iff $T$ is a bijection

<u>Pf</u> HW 1, # 6 (Consider $T$ as a function ...)

<u>LEMMA</u> If $T: V \to W$ is linear

and $T$ is invertible, then

$T^{-1}: W \to V$ is linear.

<u>Pf</u> Let $w_1, w_2 \in W$.

Since $T$ is a bijection, there exist

unique $v_1, v_2 \in V$ such that

$T(v_1) = w_1$, $T(v_2) = w_2$

$(T^{-1}(w_1) = v_1 , \quad T^{-1}(w_2) = v_2)$.

Let $a, b \in \mathbb{F}$.

$$T^{-1}(aw_1 + bw_2) = T^{-1}(a\,T(v_1) + b\,T(v_2))$$
$$= T^{-1}(T(av_1 + bv_2))$$
$$= av_1 + bv_2$$
$$= a\,T^{-1}(w_1) + b\,T^{-1}(w_2).$$

$\square$

Similarly, an $n \times n$ matrix is invertible iff there is an $n \times n$ matrix $B$ such that $AB = BA = I$.

**DEF** Two vector spaces $V, W$ are <u>isomorphic</u> if there exists an invertible linear transformation $T: V \to W$.

# The map $T$: <u>isomorphism</u>

<u>IDEA</u>: Structure-preserving bijection

$$\begin{tikzcd}
\underline{\underline{V}} & \underline{\underline{W}} \\
x, y \arrow[r, "\text{Apply } T"] \arrow[d, "\substack{\text{Mult by}\\ \text{scalars,}\\ \text{add}}"'] & T(x), T(y) \arrow[d, "\substack{\text{mult by}\\ \text{scalars,}\\ \text{add}}"] \\
ax+by \arrow[r, "\text{Apply } T"] & \boxed{\begin{array}{c} aT(x)+bT(y) \\ \shortparallel \\ T(ax+by) \end{array}}
\end{tikzcd}$$

Same if $T$ is an isomorphism

<u>THM</u> Let $V, W$ be two finite-dim'l v.s. over $\mathbb{F}$. Then $V, W$ are isomorphic iff $\dim(V) = \dim(W)$.

<u>Pf</u> Suppose first that $V, W$ are isomorphic, and let $\dim(V) = n$, let $V$ have basis $B = \{v_1, \dots, v_n\}$.

Since we have an isomorphism, there exists $T: V \to W$ that is linear and a bijection.

Since $T$ is surjective, $\mathrm{Im}(T) = W$.

Since $T$ is injective, $\mathrm{Ker}(T) = \{0\}$.

By Rank-Nullity,

$$\dim(\mathrm{Im}(T)) + \dim(\mathrm{Ker}(T)) = \dim(V)$$

so: $\dim(W) + 0 = \dim(V) = n$,

and so $\dim(W) = \dim(V)$.

Conversely, suppose $W, V$ are v.s. over $\mathbb{F}$, and $\dim(W) = \dim(V) = n$.

Let $B = (v_1, \dots, v_n)$ : basis for $V$

$C = (w_1, \dots, w_n)$ : basis for $W$.

Define a function $T: V \to W$ where
$$T(v_i) = w_i \quad \text{for all } i.$$

We have shown previously that there is a unique L.T. $T$ such that
$$T(v_i) = w_i \quad \text{for all } i.$$

$$\operatorname{span}(\{T(v_1), \dots, T(v_n)\})$$
$$= \operatorname{span}(\{w_1, \dots, w_n\}) = W,$$

so $\operatorname{Im}(T) = W$ and $T$ is surjective.

If $x = \sum_{i=1}^{n} a_i v_i \in \operatorname{Ker}(T)$,

then $T(x) = 0$, i.e.,
$$0 = T\left(\sum_{i=1}^{n} a_i v_i\right) = \sum_{i=1}^{n} a_i T(v_i)$$
$$= \sum_{i=1}^{n} a_i w_i.$$

$\Rightarrow$ $a_i = 0$ for each $i$

(since $C = \{w_1, \dots, w_n\}$ is a basis)

But then $x = 0 \Rightarrow \mathrm{Ker}(T) = \{0\}$

$\Rightarrow T$ is injective

Therefore, $T$ is an isomorphism. $\square$

**COR** If $V$ is a v.s. over $\mathbb{F}$ and $\dim(V) = n$, then $V$ is isomorphic to $\mathbb{F}^n$.

**DEF** Let $B$ be an ordered basis for an $n$-dim'l v.s. $V$ over $\mathbb{F}$.

The standard representation of $V$ with respect to $B$ is the function $\phi_B : V \to \mathbb{F}^n$ defined by

$$\phi_B(x) = [x]_B \quad \text{for all } x \in V.$$

**NOTE** Not difficult to show $\phi_B$ is an isomorphism.

# CHANGE OF BASIS

We already showed that, if $B$ and $C$ are two ordered bases of v.s. $V$, then the matrix $Q = [I_V]_B^C$ changes bases, i.e., for any $v \in V$,

$$[v]_C = Q\,[v]_B$$

Note that, since $I_V$ is invertible, $Q$ is invertible.

<u>Q</u> : What if we want to change the matrix of a <u>linear</u> transformation from one basis to another?

Let $T: V \longrightarrow V$ be linear.

$B, C$ : ordered bases of $V$

Consider the change of coordinates matrix

$Q = [I_V]_B^C$ that changes B

coordinates to C coordinates.

so: $$Q\,[T]_B = [I]_B^C\,[T]_B^B$$

$$= [I\,T]_B^C$$

$$= [T]_B^C$$

$$= [T\,I]_B^C$$

$$= [T]_C^C\,[I]_B^C$$

$$= [T]_C\,Q$$

So:

**THM** Let $T: V \to V$ be linear,

$B, C$: ordered bases of $V$,

$Q$: changes coordinates from $B$ to $C$

Then: $$[T]_B = Q^{-1}[T]_C\,Q$$

$$\left([T]_C = Q\,[T]_B\,Q^{-1}\right).$$

NEXT TIME: § 2.5, 2.6