Ex $S = \{ \overset{v_1}{(1,2,3,4)}, \overset{v_2}{(1,1,1,1)}, \overset{v_3}{(2,7,12,17)}, \underset{v_4}{(0,0,0,1)} \}$

Let $W := \operatorname{span}(S)$.

Q: Is there $S' \subsetneq S$ such that $\operatorname{span}(S') = W$?

For instance, does $v_4 = a_1 v_1 + a_2 v_2 + a_3 v_3$ for some $a_i \in \mathbb{R}$?

$$\Rightarrow (0,0,0,1) = a_1 (1,2,3,4) + a_2 (1,1,1,1) + a_3 (2,7,12,17)$$

$$\Rightarrow \begin{array}{ll} a_1 + a_2 + 2a_3 = 0 & (1) \\ 2a_1 + a_2 + 7a_3 = 0 & (2) \\ 3a_1 + a_2 + 12a_3 = 0 & (3) \\ 4a_1 + a_2 + 17a_3 = 1 & (4) \end{array}$$

$(2)-(1)$: $a_1 + 5a_3 = 0$

$(4)-(3)$: $a_1 + 5a_3 = 1$

NO SOLUTIONS!

so: $v_4 \notin \operatorname{span}(\{v_1, v_2, v_3\})$

However, this doesn't mean S is minimal (necessarily)!

Indeed, $v_1 = (1,2,3,4)$, $v_2 = (1,1,1,1)$,
$v_3 = (2,7,12,17)$, $v_4 = (0,0,0,1)$,

and: $5v_1 - 3v_2 = (5-3, 10-3, 15-3, 20-3)$

$= (2, 7, 12, 17) = v_3$

$\Rightarrow v_3 \in \text{span}(\{v_1, v_2\})$

so: $W = \text{span}(\{v_1, v_2, v_4\})$

The question is: <u>how</u> do we determine whether or not a spanning set is minimal without doing inordinate calculation?

Notice above that $5v_1 - 3v_2 = v_3$, so:

$$\underline{5}v_1 + (\underline{-3})v_2 + \underline{1}v_3 + 0v_4 = 0,$$

and so the zero vector is a <u>nontrivial</u> linear combination of v_1, v_2, v_3, v_4.

DEF A subset S of a v.s. V is called *linearly dependent* if there exists a finite nonempty set of distinct vectors $\{v_1, \dots, v_n\}$ in S and scalars $a_1, \dots, a_n$ that are not all $\underset{\text{(scalar)}}{0}$ such that

$$a_1 v_1 + a_2 v_2 + \dots + a_n v_n = \underset{\text{(vector)}}{0}.$$

In this case, we also say that the vectors of S are linearly dependent.

NOTE We always have $0v_1 + 0v_2 + \dots + 0v_n = 0$ for any vectors $v_1, \dots, v_n$ (trivial representation of zero vector).

Linear dependence means a nontrivial way to express 0 as a linear combination.

Why is this the "right" definition?

Suppose $v_1, v_2, \ldots, v_m$ are linearly dependent.

Then there exist $a_1, \ldots, a_m \in \mathbb{F}$

such that $a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0$,

and @ least one scalar is nonzero.

WLOG, suppose $a_j \neq 0$.

Then $a_j v_j = (-a_1) v_1 + \cdots + (-a_{j-1}) v_{j-1}$
$+ (-a_{j+1}) v_{j+1} + \cdots + (-a_m) v_m$

$a_j \neq 0$, $\mathbb{F}$ field $\Rightarrow$ a_j has a mult inverse

$$\Rightarrow v_j = a_j^{-1}(a_j v_j) = \sum_{\substack{1 \le k \le m \\ k \neq j}} (-a_k a_j^{-1}) v_k$$

$$\in \text{span}(\{v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m\})$$

So we have essentially proved the following:

<u>THM</u> Let S be a subset of a v.s. V.
If S is linearly dependent, then there exists $v \in S$ such that
$\text{span}(S - \{v\}) = \text{span}(S)$.

DEF A subset S of a v.s. V that is not linearly dependent is called ***linearly independent*** (L.I.)

As in above def: vectors of S are L.I.

Q: Suppose $S = \emptyset$. Is S L.I.?

There does not exist a finite nonempty subset $\{v_1, \dots, v_n\}$ of S such that $a_1 v_1 + \dots + a_n v_n = 0$ for $a_i \in \mathbb{F}$, since there are no nonempty subsets of $S = \emptyset$! Hence S is ***linearly independent***.

THM Let S be a subset of a v.s. V. If S is L.I. and S' is any proper subset of S, then S' is L.I.. Moreover,

$$\text{span}(S') \subsetneq \text{span}(S).$$

Pf (HW 3, #6)

The proof of the following is similar:

THM Let S be a L. I. subset of v.s. V and let $v \in V$, $v \notin S$. Then $S \cup \{v\}$ is linearly dependent iff $v \in \text{span}(S)$.

READ §1.6!

BASES AND DIMENSION

DEF A basis B for a v.s. V is a linearly independent set of vectors that generates V, i.e., $\text{span}(B) = V$. (The vectors in B form a basis for V.)

IDEA: A basis is the bare minimum needed to generate V.

EX span $(\phi) = \{0\}$ and ϕ is L.I.,
so ϕ is a basis for the zero vector space.

EX $e_1 = (\underbrace{1, 0, \ldots, 0}_{n \text{ coordinates}})$, $e_2 = (0, 1, 0, \ldots, 0)$, $\ldots$,
$e_n = (0, \ldots, 0, 1)$

Then $B = \{e_1, \ldots, e_n\}$ is the <u>standard basis</u> of $\mathbb{F}^n$.

THM Let V be a v.s. and $B = \{v_1, \ldots, v_n\}$ be a subset of V.
Then B is a basis iff every $v \in V$ can be expressed <u>uniquely</u> as a linear combination

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n,$$

where each $a_i \in \mathbb{F}$.

Pf Suppose first that B is a basis (B: spanning set, L.I.) and let $v \in V$.

Since $\operatorname{span}(B) = V$, there exist $a_i \in \mathbb{F}$

such that $v = \sum_{i=1}^{n} a_i v_i$.

Suppose now that

$$v = \underbrace{\sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} b_i v_i}.$$

This means $\sum_{i=1}^{n} (a_i - b_i) v_i = 0$,

and, since B is a L.I. set,

this must be the trivial representation

of 0, i.e., for all i, $a_i - b_i = 0$

$\Rightarrow a_i = b_i$ for all i,

i.e., the expression

$$v = \sum_{i=1}^{n} a_i v_i \text{ is unique.}$$

Now suppose that B is a set of

vectors that has the property that

every $v \in V$ can be expressed

uniquely as $v = \sum_{i=1}^{n} a_i v_i$, $a_i \in \mathbb{F}$.

First, since each $v \in V$ can expressed this way, span $(\beta) = V$.

Since $0 \in V$, 0 can be expressed uniquely as $0 = \sum_{i=1}^{n} a_i v_i$.

Moreover, since $\sum_{i=1}^{n} 0 v_i = 0$, we have $\sum_{i=1}^{n} a_i v_i = 0$ only if each $a_i = 0$. This means B is a L.I. set, and hence B is a basis. $\square$

<u>EX</u> Consider $(1,1,1)$, $(1,-1,1)$, $(1,1,-1)$ in $\mathbb{R}^3$.

One can check that these vectors span $\mathbb{R}^3$ and are linearly independent, so they form a basis.

For instance, if $(r_1, r_2, r_3) \in \mathbb{R}^3$, then:

$$(r_1, r_2, r_3) = \left(\frac{r_2+r_3}{2}\right)\cdot(1,1,1) + \left(\frac{r_1-r_2}{2}\right)\cdot(1,-1,1) + \left(\frac{r_1-r_3}{2}\right)\cdot(1,1,-1)$$

Moreover, if $a_1(1,1,1) + a_2(1,-1,1) + a_3(1,1,-1) = (0,0,0)$

then:

$$a_1 + a_2 + a_3 = 0 \quad (1)$$
$$a_1 - a_2 + a_3 = 0 \quad (2)$$
$$a_1 + a_2 - a_3 = 0 \quad (3)$$

Then $(2)+(3)$: $2a_1 = 0 \Rightarrow a_1 = 0$

This implies in (2) that $a_2 = a_3$

and then $a_2 = a_3 = 0$ from (1).

$\Rightarrow$ only way to express $(0,0,0)$ as a linear combination is w/ all coeffs 0

$\Rightarrow$ L. I.

So, the above expression of (r_1, r_2, r_3) as a linear combination is unique

<u>EX</u> $P_n(\mathbb{F})$: set of all polynomials w/ degree $\leq n$ and coeffs in $\mathbb{F}$

Standard basis of $P_n(\mathbb{F})$ is $\{1, x, \dots, x^n\}$

<u>EX</u> $\{1, x, x^2, x^3, \dots\}$ is a basis for

$$\mathbb{F}[x] = P(\mathbb{F})$$

(all polynomials w/ coeffs in $\mathbb{F}$)

<u>NEXT TIME</u>:

<u>THM</u> If a v.s. V is generated by a finite set S, then some subset of S is a <u>basis</u> for V.

Hence V has a finite basis.