RECALL: Goal is to determine which linear $T: V \to V$ are diagonalizable (V: finite dim'l)

characteristic polynomial: $\chi_T(t) := \det(T - tI)$
($\chi_A(t) := \det(A - tI)$, A matrix)

All eigenvalues of T are zeros of χ_T

LAST TIME: If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of T and v_i is an eigenvector for λ_i for each i, then $\{v_1, \ldots, v_k\}$ is L.I.

CONSEQUENCE: If $\dim(V) = n$, T is a linear operator on V, and T has n distinct eigenvalues, then T is diagonalizable.

This motivates the following definition:

DEF A polynomial $f(t)$ in $P(\mathbb{F})$ $(= \mathbb{F}[t])$ **splits** over $\mathbb{F}$ if there are (not necessarily distinct) scalars $c, a_1, \ldots, a_n \in \mathbb{F}$ such that

$$f(t) = c(t-a_1)(t-a_2) \cdots (t-a_n)$$

Ex $t^2 - 1 = (t-1)(t+1)$ splits over $\mathbb{R}$

$(t^2+1)(t-2)$ does <u>not</u> split over $\mathbb{R}$, since t^2+1 does <u>not</u> split over $\mathbb{R}$

Although $(t^2+1) = (t+i)(t-i)$ splits over $\underline{\mathbb{C}}$.

THM The characteristic polynomial of any diagonalizable linear operator splits.

Pf Suppose T is diagonalizable.
So for some basis β,

$$[T]_\beta = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$

Then $\det(T - tI) = \det(\mathrm{diag}(\lambda_1 - t, \ldots, \lambda_n - t))$

$$= (\lambda_1 - t)(\lambda_2 - t) \cdots (\lambda_n - t)$$

(from HW 7)

$$= (-1)^n (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_n)$$

$\square$

DEF Let λ be an eigenvalue of a linear operator (or matrix) w/ characteristic polynomial $f(t)$. The <u>(algebraic) multiplicity</u> of λ is the largest positive integer k for which $(t - \lambda)^k$ is a factor of $f(t)$.

EX $A = \begin{pmatrix} 5 & 71 & 12 \\ 0 & 5 & 43 \\ 0 & 0 & 12 \end{pmatrix}$

$\chi_A(t) = (-1)^3 (t - 5)^2 (t - 12)$ (HW 8)

$\lambda = 5$ is eigenvalue w/ mult. 2,

$\lambda = 12$ is eigenvalue w/ mult. 1.

DEF Let T be a linear operator on a v.s. V, and let λ be an eigenvalue of T. Define

$$E_\lambda := \{ x \in V : T(x) = \lambda x \}$$
$$= \ker(T - \lambda I)$$

The set E_λ is called the eigenspace of T corresponding to the eigenvalue λ. (Eigenspace of matrix A: eigenspace of L_A)

E_λ is a subspace of V, since it's the kernel of the linear operator $T - \lambda I$.

THM Let be a linear operator on a finite-dim'l v.s. V, and let λ be an eigenvalue of T having multiplicity m. Then $1 \leq \dim(E_\lambda) \leq m$.

Pf Since λ is an eigenvalue, there exists a nonzero eigenvector x, so $x \in E_\lambda$ and $\dim(E_\lambda) \geq 1$.

Let $v_1, \dots, v_\ell$ be a basis for E_λ (so: $\dim(E_\lambda) = \ell$)

and extend to an ordered basis $\beta = (v_1, \dots, v_\ell, v_{\ell+1}, \dots, v_n)$ of V.

Let $A = [T]_\beta$. Since each v_i, $1 \leq i \leq \ell$, is an eigenvector of T w/ eigenvalue λ, we have:

$$A = \begin{array}{cc} & \begin{array}{cc} \ell & n-\ell \end{array} \\ \begin{array}{c} \ell \\ n-\ell \end{array} & \begin{pmatrix} \lambda I_\ell & B \\ O & C \end{pmatrix} \end{array}$$

So: $\chi_T(t) = \chi_A(t) = \det(A - tI)$

$$= \det \begin{pmatrix} (\lambda - t) I_\ell & B \\ O & C - t I_{n-\ell} \end{pmatrix}$$

$$= \det\left((\lambda - t) I_\ell \right) \cdot \det\left(C - t I_{n-\ell} \right)$$

(HW 8)

$$= (t - \lambda)^\ell \cdot g(t) \quad \text{for some polynomial } g(t)$$

Thus $(t - \lambda)^\ell$ is a factor of $\chi_T(t)$, hence mult. m of λ is at least ℓ, i.e.,

$$\dim(E_\lambda) = \ell \leq m. \qquad \square$$

<u>EX</u> Consider the example from last time of $T: \mathbb{R}^2 \to \mathbb{R}^2$ where β is standard basis and $A = [T]_\beta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

$\chi_T(t) = (t-1)^2$, so we consider

E_1, the eigenspace corresponding to $\lambda = 1$.

$$E_1 = \text{Ker}(T - 1 \cdot I) = \text{Ker}(A - I_2)$$

$$E_1 = \text{Ker}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$$

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in E_1$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad x_2 = 0$$

Hence $x = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = x_1 \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{e_1} \in \text{span}(\{e_1\})$

So: $\dim(E_1) = 1$ since

$$E_1 = \text{span}(\{e_1\})$$

so: $\dim(E_1) = 1 < 2 =$ mult of 1 as an eigenvalue

<u>IDEA</u>: We will show that a matrix/linear operator is diagonalizable iff the

dimension of each eigenspace equals the mult. of the corresponding eigenvalue.

LEMMA Let T be a linear operator on V, let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of T. For each i, let $v_i \in E_{\lambda_i}$.

If $v_1 + v_2 + \ldots + v_k = 0$, then

$$v_i = 0 \text{ for all } i.$$

Pf The set of <u>nonzero</u> vectors

say $\{v_{i_1}, \ldots, v_{i_m}\} \subseteq \{v_1, \ldots, v_k\}$

is a set of eigenvectors from distinct eigenvalues and is L.I.,

a contradiction unless the set of nonzero vectors is empty.

Hence each $v_i = 0$. $\square$

THM Let T be a linear operator on V, $\lambda_1, \dots, \lambda_k$ distinct eigenvalues of T. For each $i = 1, 2, \dots, k$, let S_i be a finite L.I. subset of E_{λ_i}. Then $S := S_1 \cup \dots \cup S_k$ is a L.I. subset of V.

Pf Let $S_i = \{ v_{i1}, \dots, v_{in_i} \}$ and

suppose
$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} a_{ij} v_{ij} = 0.$$

Define
$$w_i := \sum_{j=1}^{n_i} a_{ij} v_{ij} \in E_{\lambda_i}.$$

So: $w_1 + \dots + w_k = 0$.

By LEMMA, each $w_i = 0$.

Since each S_i is L.I., each $a_{ij} = 0$, and so S is L.I. $\square$