HW 4, #6: "Generalized Replacement Thm"

Let B be a basis for v.s. V,

L: linearly independent subset of V

PROVE: there exists subset $B' \subseteq B$ such that

$L \cup B'$ is a basis for V.

IDEA: We'll model our proof on the proof that every v.s. has a basis.

PF Let $\mathcal{F}$ be the family of linearly independent subsets of $L \cup B$ that contain L.

$\mathcal{F} \neq \emptyset$ since $L \in \mathcal{F}$.

Let $\mathcal{C}$ be a chain in $\mathcal{F}$:

for any $X, Y \in \mathcal{C}$, either $X \subseteq Y$ or $Y \subseteq X$.

Define $$U := \bigcup_{A \in \mathcal{C}} A.$$

Clearly, $X \subseteq U$ for all $X \in \mathcal{C}$.

We need to show that $\mathcal{U}$ is in $\mathcal{F}$.

Since for any $X \in \mathcal{C}$, $L \subseteq X$,

we have $L \subseteq \mathcal{U}$.

Let $v \in \mathcal{U}$. Then $v \in X$ for some X in $\mathcal{F}$,

and, since $X \in \mathcal{F}$, $X \subseteq L \cup B$

$\Rightarrow$ $v \in L \cup B$

$\Rightarrow$ $\mathcal{U} \subseteq L \cup B$.

Suppose $\mathcal{U}$ is <u>not</u> linearly independent.

Then there exist vectors $v_1, \dots, v_k \in \mathcal{U}$

such that $\sum_{i=1}^{k} a_i v_i = 0$ and

not all $a_i = 0$.

By definition, $\mathcal{U} = \bigcup_{A \in \mathcal{C}} A$, so there

exist sets $A_1, A_2, \dots, A_k$ such that

$v_i \in A_i$ for each i.

Since $\mathcal{C}$ is a chain and k is finite,

there exists j such that $A_i \subseteq A_j$

for <u>all</u> i.

In particular, this means $v_i \in A_j$ for all i.

But $A_j \in \mathcal{C}$ so A_j is in $\tilde{\mathcal{F}}$, i.e., A_j is linearly independent, a contradiction to $\sum_{i=1}^{k} a_i v_i = 0$ and not all $a_i = 0$.

Hence no such vectors exist and so U is L.I. $\Rightarrow$ $U \in \tilde{\mathcal{F}}$.

Hence every chain in $\tilde{\mathcal{F}}$ has an upper bound in $\tilde{\mathcal{F}}$. By Zorn's Lemma, $\tilde{\mathcal{F}}$ has a maximal element, say M. Since $M \in \tilde{\mathcal{F}}$,

$$L \subseteq M \subseteq L \cup B,$$

and M is a <u>maximal</u> linearly independent set in $\tilde{\mathcal{F}}$, and so M is a basis of V, as desired. $\square$

HW 5, #1 : $\mathbb{C}$: complex numbers, v.s. over $\underline{\mathbb{R}}$

$T : \mathbb{C} \to \mathbb{C}$ by

$T(z) = \bar{z}$ (complex conjugation)

PROVE :
- T is linear
- Compute $[T]_B$, where $B = (1, i)$
- Is T still linear when $\mathbb{C}$ is viewed as v.s. over $\underline{\mathbb{C}}$?

For any $z \in \mathbb{C}$, $z = a + bi$, where $a, b \in \mathbb{R}$.

$$T(z) = a - bi$$

(Do the calculation to show it's linear; also for matrix)

We know that addition is still the same ✓

mult by an element of $\mathbb{R}$ is same ✓

NOW : When v.s. over $\mathbb{C}$, you're allowed to multiply by complex scalar.

Does $T(xz) = x\,T(z)$, where $x \in \mathbb{C}$?

HW 6, # 1 $\quad B: (1, x, x^2, x^3)$ of $P_3(\mathbb{R})$

$C: (f_0, f_1, f_2, f_3)$ (Lagrange Polynomials)

constants $c_i = i$ for each i.

$c_0 = 0, \; c_1 = 1, \; c_2 = 2, \; c_3 = 3$

so: $$f_0 = \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)}$$

$$f_1 = \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} \quad \text{etc.}$$

Change of coordinates: How do you write f_i wrt to B?

What about each of $1, x, x^2, x^3$ wrt to C?

"Easy": f_i wrt B

Other direction: <u>inverse</u> of this matrix

<u>HW 6, #6</u>: Let V be a nonzero v.s., W: proper subspace of V.

Prove that there exists a <u>nonzero</u> linear functional $f \in V^*$ $(f : V \to \mathbb{F})$ such that $f(x) = 0$ for all $x \in W$.

<u>HINT</u>: Use #5 on this HW.

<u>#5</u>: V, W nonzero v.s. over $\mathbb{F}$, B: basis for V.

Prove that for any function $f : B \to W$ there exists <u>exactly</u> one L.T. T such that $T(x) = f(x)$ for all $x \in B$.

(V: <u>not</u> assumed to be finite here)

<u>#6</u>:

W: proper subspace of V.

Let B' be a basis for W

B' : L.I. subset of V, so extend

B' to a basis B of V (Why?)

Now: define $f : B \to \mathbb{F}$ in a natural

way so that we use #5

to extend this f to a

linear functional $f : V \to \mathbb{F}$

that has the desired properties.