LAST TIME: V: v.s. over $\mathbb{F}$

$$V^* := \mathcal{L}(V, \mathbb{F}) \quad \text{dual space}$$

If $\dim(V) = n$, then $\dim(V^*) = n$.

WANT: identify $V^{**} \longleftrightarrow V$

Given $x \in V$, we define $\hat{x} : V^* \to \mathbb{F}$

by $\hat{x}(f) := f(x)$ for all $x \in V^*$

We saw that $\hat{x} \in V^{**}$

THM Let V be a finite-dim'l v.s. over $\mathbb{F}$.

The map $\psi : V \to V^{**}$ given by

$$\psi(x) = \hat{x}$$

is an isomorphism.

Pf Let $x, y \in V$, $c, d \in \mathbb{F}$.

$\psi(cx + dy) = \widehat{cx + dy}$, so if $f \in V^*$,

$$\begin{aligned}\psi(cx+dy)(f) &= \widehat{cx+dy}\,(f)\\ &= f(cx+dy)\\ &= c\,f(x)+d\,f(y)\\ &= c\,\hat{x}(f)+d\,\hat{y}(f)\\ &= \big(c\,\psi(x)+d\,\psi(y)\big)(f),\end{aligned}$$

and so ψ is linear.

Suppose $x \in \operatorname{Ker}(\psi)$.

Then $\hat{x}(f) = 0$ for all $f \in V^*$, i.e.,

$f(x) = 0$ for all $f \in V^*$.

If $B = (v_1, \ldots, v_n)$ is a basis for V,

then, in particular, $f_j(x) = 0$ for

each j, where f_j is the j^{th} coordinate

map, (If $[v]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, then $f_j(v) = a_j$)

i.e., if $x = \sum_{i=1}^{n} x_i v_i$, then $f_j(x) = x_j$

$\Rightarrow x_j = 0$ for all j $\Rightarrow x = 0$

$\Rightarrow \quad \mathrm{Ker}(\psi) = \{0\} \Rightarrow \psi$ is injective.

Finally, since ψ is linear,

$$\dim(\mathrm{Ker}(\psi)) + \dim(\mathrm{Im}(\psi)) = \dim(V)$$

$$\Rightarrow \quad \dim(\mathrm{Im}(\psi)) = \dim(V)$$

Since $\dim(V^{**}) = \dim(V^*) = \dim(V)$

$$= \dim(\mathrm{Im}(\psi)),$$

ψ is surjective, and, therefore,

$\psi : V \to V^{**}$ is an isomorphism. $\square$

<u>COR</u> Let V be a finite-dim'l v.s. over $\mathbb{F}$. Every ordered basis for V^* is the dual basis for some basis for V.

<u>RECALL</u>: $B = (v_1, \ldots, v_n)$ is a basis for V

<u>dual basis</u>: $B^* = (f_1, \ldots, f_n)$, where f_i is the i^{TH} coordinate map.

Pf Let $(g_1, \dots, g_n)$ be a basis for V^*

Consider the dual basis in V^{**}:

$$(\hat{x}_1, \dots, \hat{x}_n) \text{ for some } \hat{x}_i \in V^{**},$$

i.e., $\hat{x}_i(g_j) = \delta_{ij}$

$$\Rightarrow \quad g_j(x_i) = \hat{x}_i(g_j) = \delta_{ij},$$

and so $(g_1, \dots, g_n)$ is the dual basis of $(x_1, \dots, x_n)$. $\square$

ELEMENTARY ROW OPERATIONS

DEF Let A be an $m \times n$ matrix.

Any one of the following operations on the rows [columns] of A is called an elementary row operation (ERO) [elementary column operation (ECO)]:

(1) interchange any two rows [columns] of A

(2) multiplying any row [column] of A by a nonzero scalar

(3) adding any scalar multiple of a row [column] of to any other row [column]

DEF An <u>elementary matrix</u> is a matrix obtained by performing an ERO or ECO on I_n. It is Type 1, 2, or 3 according to whether the ERO/ECO is type 1, 2, or 3.

EX If we switch rows 2 & 4 of I_4, we get:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

If we multiply row 3 by scalar a:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If we add a times row 1 to row 4:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & 0 & 0 & 1 \end{bmatrix}$$

<u>THM</u> Let $A \in M_{m \times n}(\mathbb{F})$ and suppose B is obtained from A by performing an ERO. Then there exists an $m \times m$ elementary matrix E st $B = EA$.

In fact, E is obtained from I_m by performing the same ERO as that which was performed on A to obtain B. Conversely, if E is an $m \times m$ elementary matrix, then EA is the matrix obtained by performing the same ERO as that which produces E from I.

NOTE ... ECO ... $n \times n$ ... $B = AE$ ... etc.

IDEA: (1) You could just check in each case.

(2) Each ECO really corresponds to a change of basis (ERO: transpose), and an elementary matrix is just a change of basis matrix.

THM Elementary matrices are invertible, and the inverse of each elementary matrix has the same type.

Pf (1) E: elementary matrix

$$\underbrace{EI}_{\text{performing an ERO on } I} = E$$

② **Reverse** ERO (also an ERO of the same type) w/ E':

$$E'E = E'(EI) = I$$

$E'E = I$, and E is invertible $\square$

READ: §3.2

THE RANK OF A MATRIX AND MATRIX INVERSES

DEF If $A \in M_{m\times n}(\mathbb{F})$, we define the **rank** of A, denoted $\mathrm{rank}(A)$ or $\mathrm{rk}(A)$, to be the rank of the linear transformation

$$L_A : \mathbb{F}^n \to \mathbb{F}^m.$$

THM Let $T: V \to W$ be a L.T. b/w finite dim'l spaces V, W w/ ordered bases β, γ, resp., then

$$\operatorname{rank}(T) = \operatorname{rank}([T]_\beta^\gamma)$$

Pf EXERCISE.

THM Let A be an $m \times n$ matrix. If P, Q are invertible $m \times m$, $n \times n$ matrices, resp., then:

(a) $\operatorname{rank}(AQ) = \operatorname{rank}(A)$,

(b) $\operatorname{rank}(PA) = \operatorname{rank}(A)$,

and, therefore,

(c) $\operatorname{rank}(PAQ) = \operatorname{rank}(A)$.

Pf

$$\begin{aligned}\operatorname{rank}(L_{AQ}) &= \operatorname{rank}(L_A L_Q) \\ &= L_A L_Q(\mathbb{F}^n)\end{aligned}$$

$$= L_A(\mathbb{F}^n) \quad \text{(L_Q: invertible)}$$

$$= \text{rank}(L_A)$$

$$\Rightarrow \quad \text{rank}(AQ) = \text{rank}(A).$$

Rest of proof: similar. $\square$

COR Elementary row [column] operations on a matrix are rank-preserving.

Pf ERO $\longleftrightarrow$ left mult. by an elementary matrix E, which is invertible. $\square$

THM The rank of a matrix is the dimension of the subspace generated by its columns.

Pf Let $A = (a_{ij})$, $m \times n$.

$\beta = (v_1, \dots, v_n)$ is a basis for V,

$\gamma = (w_1, \dots, w_m)$ is a basis for W.

$T : V \to W$, $[T]_\beta^\gamma = A$.

Then $\{T(v_j)\}$ is a basis

for $\mathrm{Im}(T)$

$\Rightarrow$ $\mathrm{rank}(A) = \mathrm{rank}(T)$

$= \dim(\mathrm{span}\{T(v_1), \ldots, T(v_n)\})$

On the other hand,

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$$

(corresponds to column j of A) $\square$