Math 309 – Intermediate Linear Algebra   Homework 5   Xida Ren

(Due Friday, March 2)

Each problem will be graded out of 10 points.

1. Consider the complex numbers $\mathbb{C}$ as a vector space over $\mathbb{R}$. Define $T : \mathbb{C} \to \mathbb{C}$ by $T(z) = \overline{z}$, where $\overline{z}$ is the complex conjugate of z. Prove that T is linear and compute $[T]_B$, where B is the standard ordered basis $(1, i)$. Is T still a linear transformation when $\mathbb{C}$ is viewed as a vector space over $\mathbb{C}$?

T is linear:

1. Additivity:

   Additivity is satisfied: $T(a+bi)+T(c+di) = a-bi+c-di = (a+c)-(b+d)i = T(a+bi+c+di)$

2. Homogeneity:

   $T(c(a+bi)) = T(ca+cbi) = ca - cbi = c(a-bi) = cT(a+bi)$

Standard matrix:

In B-representation, $a+bi = \begin{bmatrix} a \\ b \end{bmatrix}$. So

$$[T]_B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Linear transformation for $\mathbb{C}$ over $\mathbb{C}$: no: if c is a complex number, we nolonger have homogeneity:

$$\begin{aligned}(x+yi)T(a+bi) &= (x+yi)(a-bi) \\ &= ax + ayi - bxi + by \\ &= ax + by + (ay - bx)i\end{aligned}$$

Now,

$$\begin{aligned}T((x+yi)(a+bi)) &= T(ax + ayi + bxi - by) \\ &= T((ax - by) + (ay + bx)i) \\ &= (ax - by) - (ay + bx)i\end{aligned}$$

which is different from $ax + by + (ay - bx)i$.

2. Let $T : \mathbb{F}^n \to \mathbb{F}$ be a linear transformation. If $\{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{F}^n$, prove that there exist scalars $a_1, \ldots, a_n$ such that, if $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$, then

$$T(x) = \sum_{i=1}^{n} a_i x_i.$$

Do these scalars need to be unique?

They need to be unique.

If there are two sets of scalars a_i and b_i, consider $\sum_{i=1}^n a_i e_i$ and $\sum_{i=1}^n b_i e_i$. They should be the same, because they both give us $T(x)$.

Now, subtract the two sums and regroup the terms to get $0 = \sum_{i=1}^n (a_i - b_i)e_i$. If any a_i is different from any b_i, there will be nonzero coefficients, which means that $\{e_1, \ldots, e_n\}$ is not a basis because a basis must be linearly independent.

A function $f : V \to W$ is called *additive* if $f(x + y) = f(x) + f(y)$ for all $x, y \in V$.

3. If $\mathbb{R}$ is viewed as a vector space over $\mathbb{Q}$, prove that any additive map from $\mathbb{R}$ to $\mathbb{R}$ is also a linear transformation.

Let T denote our additive map. Because we already have additivity, we only need to prove that T is homogeneous, that it preserves scalar multiplication.

Let $r \in \mathbb{R}$ be a vector in $\mathbb{R}$ over $\mathbb{Q}$. Let $c = \frac{a}{b}$ be a scalar, where $c \in \mathbb{Q}$ and $a, b \in \mathbb{Z}$. By symmetry, assume that b is positive. We split the proof into two cases: where a is positive, and where a is negative.

For positive a we have:

$$
\begin{aligned}
cr &= \frac{a}{b}r && \text{original equation} \\
bcr &= ar && \text{multiply both sides with } b \neq 0 \\
\sum_{i=1}^{b} cr &= \sum_{j=1}^{a} r && \text{expand integer multiplication to summation} \\
T\left(\sum_{i=1}^{b} cr\right) &= T\left(\sum_{j=1}^{a} r\right) && \\
\sum_{i=1}^{b} T(cr) &= \sum_{j=1}^{a} T(r) && T \text{ additive} \\
bT(cr) &= aT(r) && \text{contract summation to integer multiplication} \\
T(cr) &= \frac{a}{b}T(r) && \text{multiply both sides by } b^{-1}
\end{aligned}
$$

Finally, we have to proveve that $T(-r) = -T(r)$ to cover the negative-a case. Simply use additivity:

$$
\begin{aligned}
T(-r) &= T(0 - r) + T(r) - T(r) && \text{witchcraft} \\
&= T(-r + r) - T(r) && \text{additivity} \\
&= T(0) - T(r) && \text{definition of additive inverse} \\
&= -T(r) && \text{see below}
\end{aligned}
$$

We can do the last step because $T(0 + 0) = T(0) \implies T(0) + T(0) = T(0) \implies T(0) = 0$.

So for negative c, we can just extract the negative sign, apply homogeneity for positive scalars, and put the sign back in.

4. Let V be a vector space and W a subspace of V.

(a) Define the mapping $\rho : V \to V/W$ by $\rho(v) = v + W$. Prove that ρ is a linear transformation.

> In hw3, we defined the operations $(v+W)+(u+W) = ((u+v)+W)$ and $c(v+W) = (cv+W)$. By definition, we have additivity $(\rho(u+v) = ((u+v)+W) = \rho(u)+\rho(v))$ and homogeneity $(\rho(av) = (av + W) = a\rho(v))$, so ρ is a linear transformation.

(b) If V is finite-dimensional, how do $\dim(V)$, $\dim(W)$, and $\dim(V/W)$ relate?

> $\dim(V) = \dim(W) + \dim(V/W)$. V/W can be seen as the result of collapsing V along $W \in V$.

5. Let V be an n-dimensional vector space over $\mathbb{F}$ with ordered basis B. Define $T : V \to \mathbb{F}^n$ by $T(x) = [x]_B$. Prove that T is linear.

> T **is additive:** let $[x]_B = \begin{bmatrix} x_i \end{bmatrix}$, $[y]_B = \begin{bmatrix} y_i \end{bmatrix}$, and $B = \{b_i\}$. We have $x = \sum_{i=1}^n x_i b_i$ and $y = \sum_{i=1}^n y_i b_i$.
>
> Now consider the sum $x + y$. It is equal to $\sum_{i=1}^n (x_i + y_i) b_i$. So the coefficients are $[x_i + y_i]$, and they are unique because otherwise, if there are two different sets of coefficients $[c_i]$ and $[c'_i]$ that could generate $x + y$ after multiplying with the $\{b_i\}$, we can say $\sum_{i=1}^n (c_i - c'_i) b_i = 0$ and conclude that the b_i are linearly dependant, which is impossible.
>
> T **is homogeneous:** let $x = \sum_i^n x_i b_i$. $cx = c \sum_i^n c x_i b_i$. Extracting the coefficients gives $[cx]_B = c[x]_B$. Again, by the same reasoning, there can be no alternative representation unless B is not a basis.

6. Let V and W be vector spaces such that $\dim(V) = \dim(W)$, and let $T : V \to W$ be linear. Prove that there exist ordered bases B (for V) and C (for W) such that $[T]_B^C$ is a diagonal matrix.

Proof. Take any basis $B = \{b_1, \ldots, b_n\}$ of V let C be a basis of the image of B. There is a matrix $M = [T]_B^C$ representing T with input in terms of B and output in terms of C.

Now, we modify M with elementary row/column operations to make it into a diagonal matrix. For each row operation we modify C accordingly to make sure that (1) C is still a basis and (2) $M = [T]_B^C$ or, equivalently, $\forall v \in V : M[v]_B = [T(v)]_C$, still holds.

Consider the column vector $w \in \mathbb{F}^{\dim(W)}, w = M[v]_B$.

- When we swap rows i and j of M, the corresponding columns of w also swap. So to make sure $M[v]_B = [T(v)]_C$ we have to swap the corresponding vectors in C. C remains a basis, because the vectors are unchanged.
- When we multilpy row i of M by the nonzero scalar $c \in F$, we will have to similarly multiply vector i in C by c^{-1}, so that the scalars cancel out when we multiply each C-vector by w components to restore $T(v)$. C remains a basis, because origonal C is in the span of the transformed C, and linear independance is preserved because if $a_i' \cdot cc_i + \sum_{k \neq i} a_k c_k = 0$ and some of the a_k are not zero, claim by C wasn't an independent set to begin with by setting a_i with ca_i' and showing that $\sum_k a_k c_k = 0$.
- When row i is modified by setting $w_i' = w_i + w_j$, the change has to be inverted in C by letting the new $c_j' = c_j - c_i$. This way,

$$\begin{aligned} w_i' c_i + w_j c_j' + \sum_{k \neq i,j} w_k c_k &= (w_i + w_j)c_i + w_j(c_j - c_i) + \sum_{k \neq i,j} w_k c_k \\ &= w_i c_i + w_j c_i - w_j c_i + w_j c_j \sum_{k \neq i,j} w_k c_k \\ &= \sum_k w_k c_k \end{aligned}$$

  Span of C is preserved because the new old c_j can be restored by adding c_i to the new c_j' because $c_j' + c_i = c_j - c_i + c_i = c_j$.

  Independence of C is preserved because if

$$a_i' c_i + a_j' c_j' + \sum_{k \neq i,j} a_k' c_k = 0$$

  then

$$\begin{aligned} a_i' c_i + a_j'(c_j - c_i) + \sum_{k \neq i,j} a_k' c_k &= 0 \\ a_i' c_i + a_j' c_j - a_j' c_i + \sum_{k \neq i,j} a_k' c_k &= 0 \\ (a_i' - a_j')c_i + a_j' c_j + \sum_{k \neq i,j} a_k' c_k &= 0 \end{aligned}$$

  but now by linear independence of the c_k, $(a_i' - a_j') = 0, a_j = 0$, and each other a_k is also 0. So the new C is linearly independent.

  So as we perform row operations on M, the basis C could be modified to preserve $[T(v)]_C = M[v]_B$ while still being a basis.

Could I have written C as a "vector of vectors" and did something differnt but easier involving inverting EROs ? $\square$