ERO (ECO): (1) Interchange two rows [columns]

(2) Mult row [column] by nonzero constant

(3) Add mult of any row [column] to any other row [column]

Elementary matrix: matrix E s.t. mult by E transforms a matrix into what it would be after ERO (ECO)

ERO: EA (mult on left)

ECO: AE (mult on right)

THM The rank of a matrix is the dimension of the subspace generated by its columns.

EX $A :=$

$$\begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Since $v_4 = v_1 + v_2 + v_3$ and v_1, v_2, v_3 are L.I., $\text{rank}\left(\substack{\text{subspace generated by}\\ \text{columns}}\right) = 3$

$$\Rightarrow \text{rank}(A) = 3$$

THM Let A be an $m \times n$ matrix of rank r. Then $r \leq \min\{m, n\}$, and, by means of a finite # of ERO/ECO, A can be transformed into the matrix

$$D = \begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix},$$

where the O_i are zero matrices.

EX Both ERO, ECO are necessary @ times.

If $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$, then we

Can use ERO as follows:

$$\xrightarrow{R_2 \mapsto R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \xrightarrow{R_1 \mapsto R_1 - R_2} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

Need ECO to get to $\left(\underbrace{\boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}}_{I_2} \underbrace{\boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}}}_{\text{zero matrix}} \right)$

Similarly, $\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$ needs ERO's to get to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

<u>SKETCH OF PROOF</u>: $A = O \iff \text{rank}(A) = 0$

Assume $\text{rank}(A) > 0$. So $A \neq O$.

We proceed by induction on m, the # of rows.

<u>$m = 1$</u>: (1) $A \neq O$, so we can use ECO to get a nonzero entry in

first column (1,1 - entry)
by inter-changing rows if necessary.

(2) Use ECO to make the first column "1":

so now $\begin{pmatrix} 1 & \text{———} \end{pmatrix}$

(3) We subtract multiples of column 1 from all other columns to zero them out, getting $\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$ ✓

Assume true for $m = k$.

<u>$m = k+1$</u>: Get something nonzero in top left (1,1 - entry) w/ inter-changes of rows and/or columns, scale to 1,1 - entry to "1" via either an ERO or ECO,

Then eliminate rest row of 1 and rest of column 1 w/ ECO's, ERO's, resp.

so:

$$\left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & k \times n & \\ 0 & & & \end{array}\right)$$

By induction, we can reduce the $k \times n$ submatrix to desired form (ind. hyp.)

$\square$

COR Let A be $m \times n$, rank r.

Then there exist invertible matrices B, which is $m \times m$, and C, which is $n \times n$, such that

$$D = BAC,$$

where $D = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$.

IDEA: B: product of EROs

C: product of ECOs.

COR $A : m \times n$

(a) $\operatorname{rank}(A^t) = \operatorname{rank}(A)$

If $A = (a_{ij})$, $A^t = (b_{ij})$,

then $b_{ij} = a_{ji}$.

(b) The rank of A is the dimension of the space generated by the rows.

(c) The rows and columns generate subspaces of the same dimension, and these dimensions both equal the rank of the matrix.

Pf (of (a)): From above, $D = BAC$,

where, if $r = \text{rank}(A)$,

$$D = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

$$D^t = (BAC)^t = C^t A^t B^t \text{ (why?)}$$

$\text{rank}(D^t) = r$, and so

$\text{rank}(A^t) = r$.

Rest follows by considering A^t. $\square$

THM $T: U \to V$ linear, $S: V \to W$ linear.

Then $\text{rank}(ST) \leq \min\{\text{rank}(S), \text{rank}(T)\}$

In particular, for matrices A, B:

$$\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$$

Pf EXERCISE.

THM Every invertible matrix is the product of elementary matrices.

Pf A invertible $\Rightarrow$ A is $n \times n$ for some n,

$$\dim(\mathrm{Ker}(L_A)) = 0$$

$$\Rightarrow \mathrm{rank}(A) = n$$

Thus there are matrices B, C s.t. B is the product of elem. matrices from ERO, C is the product of elem. matrices from ECO, and

$$BAC = I_n$$

Since elementary matrices are invertible,

$$A = B^{-1}(BAC)C^{-1}$$
$$= B^{-1}C^{-1}$$

Since the inverse of an elementary matrix is another elementary matrix of the same type,

B^{-1} : product of elem. matrices

C^{-1} : " " " "

$\Rightarrow$ A : " " " " $\square$

<u>NOTES</u> (1) When $\operatorname{rank}(A) = n$, $A: n\times n$,

then proof of above THM

transforming A into $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$

$= I_n$

can be modified so that <u>only</u>

ERO's (or <u>only</u> ECO's)

are used.

(2) This gives us an algorithm

for finding the inverse of

an invertible matrix A :

A invertible $\Rightarrow$ $A = E_1 E_2 \cdots E_d$,

where each E_i is an

elementary matrix.

So: $A^{-1} = E_d^{-1} E_{d-1}^{-1} \cdots E_2^{-1} E_1^{-1}$

So: Assume we apply ERO/ECO's corresponding to $E_d^{-1}, \ldots, E_1^{-1}$ and we get the identity.

Then the product of $E_d^{-1} \cdots E_1^{-1}$ is the inverse of A, which can be "tracked" by simultaneously applying these matrices to I.

<u>EX</u> $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 1 & 6 & 9 \end{pmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 4 & 5 & 0 & 1 & 0 \\ 1 & 6 & 9 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & 4 & 6 & -1 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & 0 & -1/2 \\ 0 & 1 & 0 & -1 & 3/2 & -1/2 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right]$$

$$\Rightarrow \quad A^{-1} = \begin{pmatrix} 3/2 & 0 & -1/2 \\ -1 & 3/2 & -1/2 \\ 1/2 & -1 & 1/2 \end{pmatrix}$$

SYSTEMS OF EQUATIONS

$$a_{11} x_1 + \cdots + a_{1n} x_n = b_1$$
$$a_{21} x_1 + \cdots + a_{2n} x_n = b_2$$
$$\vdots$$
$$a_{m1} x_1 + \cdots + a_{mn} x_n = b_m$$

$A = (a_{ij})$, $\quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $\quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

<u>coefficient matrix</u>

$$\Rightarrow \quad Ax = b$$

if there exists a solution: <u>consistent</u>

otherwise: <u>inconsistent</u>

<u>IDEA</u>: homogeneous equation:

$$Ax = 0$$

Solutions: $x \in \operatorname{Ker}(L_A)$

so, for instance, if $m < n$,

then $Ax = 0$ <u>always</u> has

a nonzero solution

(Rank-Nullity)

<u>THM</u> If K is the solution set

of system $Ax = b$, then,

if s is <u>any</u> solution,

$$K = s + \operatorname{Ker}(L_A)$$

$$= \{ s + v : v \in \operatorname{Ker}(L_A) \}.$$

<u>IDEA</u>: Suppose $Aw = b$, $As = b$

$$\Rightarrow A(w - s) = 0$$

$$\Rightarrow w - s \in \operatorname{Ker}(L_A)$$

Conversely, if $z \in \operatorname{Ker}(L_A)$,

then $A(s+z) = As + 0 = As = b$. $\square$

If is invertible, $n \times n$, then

<u>unique</u> sol'n is $A^{-1}b$;

Conversely, if solution is unique,

then $\ker(L_A) = \{0\}$

$\Rightarrow$ A is invertible.