NORMAL AND SELF-ADJOINT (HERMITIAN) OPERATORS

RECALL: If A is an $n \times n$ matrix w/ entries in $\mathbb{F}$ $(= \mathbb{C}, \mathbb{R})$, then

$$A^* = \overline{A}^t = \overline{(A^t)}$$

Given a linear operator T on an inner product space V, T^* is the linear operator (if it exists, and it does when V is finite-dim'l) such that, for all $x, y \in V$,

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle$$

With respect to an orthonormal basis β,

$$[T^*]_\beta = [T]_\beta^*$$

adjoint

conjugate transpose

LEMMA Let T be a linear operator on a finite-dim'l inner product space V. If T has an eigenvector w/ eigenvalue λ, then T^* has an eigenvector w/ eigenvalue $\bar{\lambda}$.

Pf Let v be an eigenvector for T w/ eigenvalue λ.

Then, for any $x \in V$,

$$0 = \langle 0, x \rangle = \langle (T - \lambda I)(v), x \rangle$$
$$= \langle v, (T - \lambda I)^*(x) \rangle$$
$$= \langle v, (T^* - \bar{\lambda} I)(x) \rangle,$$

and so v is orthogonal to $\operatorname{Im}(T^* - \bar{\lambda} I)$.

Since $v \neq 0$, this means $T^* - \bar{\lambda} I$ is not surjective (since $v \neq 0$, $\langle v, v \rangle \neq 0$, and so $v \notin \operatorname{Im}(T^* - \bar{\lambda} I)$).

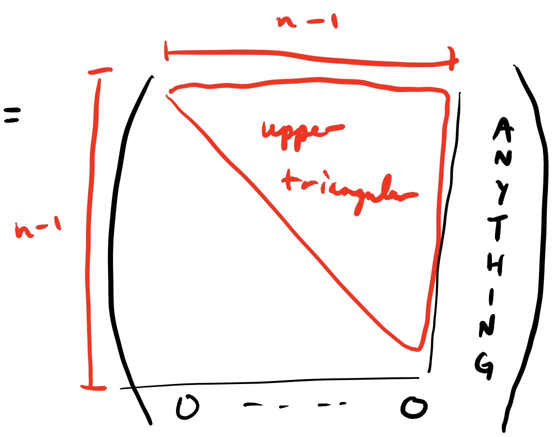
$\Rightarrow \operatorname{Ker}(T^* - \bar{\lambda} I) \neq \{0\}$, and any nonzero vector in $\operatorname{Ker}(T^* - \bar{\lambda} I)$ is an eigenvector of T^* w/ eigenvalue $\bar{\lambda}$. $\square$

THM (Schur) Let T be a linear operator on a finite-dim'l inner product space V.

Suppose the characteristic polynomial χ_T of T splits (factors into degree 1 terms). Then, there exists an orthonormal basis β for V such that the matrix $[T]_\beta$ is upper triangular.

<u>Pf</u> The proof is by induction on $n = \dim(V)$. The strategy is to find a T-invariant $(n-1)$-dim'l subspace. Then:

$[T]_\beta =$



When $n = 1$, every matrix is upper triangular.

Assume the result holds for linear operators on $(n-1)$-dim'l inner product spaces for operators whose characteristic polynomial splits.

We know that T^* has a (unit) eigenvector z w/ eigenvalue λ.

Thus $T^*(z) = \lambda z$ and $\|z\| = 1$.

Let $W = \text{span}(\{z\})$.

<u>RECALL</u>:

$$W^\perp := \{ y \in V : \langle y, w \rangle = 0 \text{ for all } w \in W \}$$

subspace of V

We will show that $W^\perp$ is, in fact, the desired $(n-1)$-dim'l T-invariant subspace.

If $y \in W^\perp$ and $x = cz \in W$, then

$$\begin{aligned}
\langle T(y), x \rangle &= \langle T(y), cz \rangle \\
&= \langle y, T^*(cz) \rangle \\
&= \langle y, c\,T^*(z) \rangle \\
&= \langle y, c\lambda z \rangle \\
&= \overline{c\lambda} \langle y, z \rangle
\end{aligned}$$

$$= \overline{c\lambda} \cdot 0 = 0,$$

and hence $T(y) \in W^{\perp}$.

So: $W^{\perp}$ is T-invariant.

By HW 11, #5, $\dim(W^{\perp}) = \dim(V) - \dim(W)$
$= n - 1.$

As we have proved, the characteristic polynomial $\chi_{T_{W^{\perp}}}(t)$ of the restriction of T to $W^{\perp}$ divides $\chi_T(t)$, and so $\chi_{T_{W^{\perp}}}(t)$ splits.

By inductive hypothesis, $W^{\perp}$ has an orthonormal basis γ such that $[T_{W^{\perp}}]_\gamma$ is upper triangular.

Therefore, if $\beta = \gamma \cup \{z\}$, then

$$[T]_\beta = \left(\begin{array}{c|c} [T_{W^{\perp}}]_\gamma & [T(z)]_\beta \\ \hline 0 \; - - - - - \; 0 & \end{array}\right),$$

and so $[T]_\beta$ is upper triangular. $\square$

Suppose β is an orthonormal basis for V that consists of eigenvectors of a linear operator T. Thus $[T]_\beta$ is diagonal.

This means $[T^*]_\beta = [T]_\beta^*$

$$= \overline{[T_\beta]^t}$$

is diagonal, too!

Now, diagonal matrices commute, so, if V has an orthonormal basis of eigenvectors of T, then $T^*T = TT^*$.

This motivates the following:

<u>DEF</u> Let V be an inner product space, T a linear operator on V. Then T is <u>normal</u> if $TT^* = T^*T$.

($n\times n$ real or complex matrix A is <u>normal</u> if $AA^* = A^*A$)

Note that T is normal iff $[T]_\beta$ is normal, where β is an orthonormal basis.

PROPERTIES OF NORMAL OPERATORS:

<u>THM</u> Let V be an inner product space, T a normal operator ($TT^* = T^*T$).

Then: (a) $\|T(x)\| = \|T^*(x)\|$ for all $x \in V$.

(b) $T - cI$ is normal for all $c \in \mathbb{F}$.

(c) If x is an eigenvector of the normal operator T w/ eigenvalue λ, then x is an eigenvector of T^* w/ eigenvalue $\bar{\lambda}$

(so $T(x) = \lambda x \Rightarrow T^*(x) = \bar{\lambda} x$)

(d) If λ_1, λ_2 are distinct eigenvalues of the normal operator T w/ corresponding eigenvectors x_1, x_2, then x_1, x_2 are orthogonal (so: $\langle x_1, x_2 \rangle = 0$).

Pf of (c): Suppose $T(x) = \lambda x$.

Then $(T - \lambda I)(x) = 0$, so

$$\| (T - \lambda I)(x) \| = 0.$$

$$\Rightarrow \quad 0 = \langle (T - \lambda I)(x), (T - \lambda I)(x) \rangle$$

$$= \langle (T - \lambda I)^* (T - \lambda I)(x), x \rangle$$

$$= \langle (T - \lambda I)(T - \lambda I)^* (x), x \rangle$$

normal! (part (b); easy exercise)

$$= \langle (T - \lambda I)^* (x), (T - \lambda I)^* (x) \rangle$$

$$= \| (T - \lambda I)^* (x) \|$$

$$= \| (T^* - \bar{\lambda} I)(x) \|$$

$$\Rightarrow \quad (T^* - \bar{\lambda} I)(x) = 0$$

$$\Rightarrow \quad T^*(x) = \bar{\lambda} x. \qquad \square$$

NEXT TIME: Pf of (d), SPECTRAL THM