RECALL: *adjoint* of T: denoted T^*

T^* is linear and is defined by

$\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ for all

x, y in an inner product space V.

$(\longleftrightarrow$ $n \times n$ matrix A, $A^* = \overline{A}^t)$

T is *normal* if $TT^* = T^*T$.

PROPERTIES OF NORMAL OPERATORS:

THM V: inner product space, T: normal operator

(a) $\|T(x)\| = \|T^*(x)\|$ for all $x \in V$

(b) $T - cI$ is also normal for all $c \in \mathbb{F}$

(c) If x is an eigenvector of T w/ eigenvalue λ, then x is also an eigenvector of T^* w/ eigenvalue $\overline{\lambda}$.

(d) If λ_1, λ_2 are distinct eigenvalues of T w/ corresponding eigenvectors x_1, x_2, then x_1, x_2 are orthogonal.

Pf of (d) Let λ_1, λ_2 be distinct eigenvalues of T w/ corresponding eigenvectors x_1, x_2, where T is a normal operator.

so: $T(x_1) = \lambda_1 x_1$, $T(x_2) = \lambda_2 x_2$.

WANT: $\langle x_1, x_2 \rangle = 0$

so: $$\lambda_1 \langle x_1, x_2 \rangle = \langle \lambda_1 x_1, x_2 \rangle$$
$$= \langle T(x_1), x_2 \rangle$$
$$= \langle x_1, T^*(x_2) \rangle$$
$$= \langle x_1, \overline{\lambda_2} x_2 \rangle$$ since T is normal
$$= \lambda_2 \langle x_1, x_2 \rangle$$

$$\Rightarrow (\lambda_1 - \lambda_2) \langle x_1, x_2 \rangle = 0$$

Since, by assumption, $\lambda_1 \neq \lambda_2$, $\langle x_1, x_2 \rangle = 0$, as desired. $\square$

THM (THE SPECTRAL THM, COMPLEX INNER PRODUCT SPACE)

Let T be a linear operator on a finite-dim'l complex inner product space.

Then, T is normal iff there exists an orthonormal basis for V consisting of eigenvectors of T.

(T: diagonalizable w/ respect to some orthonormal basis)

Pf First, suppose there exists an orthonormal basis β of eigenvectors for T.

Then $[T]_\beta$ is diagonal, and

$$[T^*]_\beta = [T]_\beta^* = \overline{[T_\beta]}^t$$

is diagonal as well.

Diagonal matrices commute, and so

$TT^* = T^*T$, and T is normal.

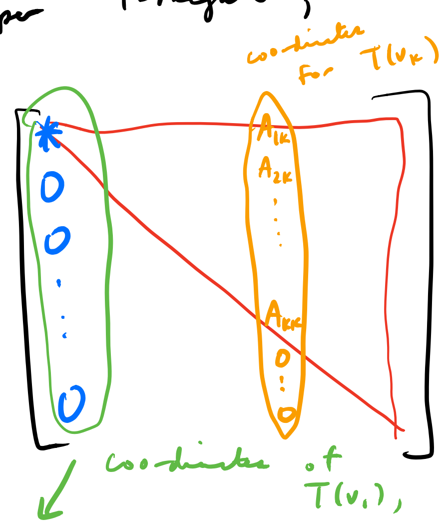
Conversely, suppose T is a normal operator, so $TT^* = T^*T$.

By the Fundamental Thm of Algebra, the characteristic polynomial $\chi_T(t)$ of T splits over $\mathbb{C}$.

By Schur's Thm, there exists an orthonormal basis $\beta = (v_1, \dots, v_n)$ for V such that $[T]_\beta = A$ is upper-triangular.

We proceed now by induction to show each v_i is an eigenvector.

Since A is upper triangular,

$A =$ 

$T(v_1) = \lambda_1 v_1$

So: v_1 is an eigenvector.

Assume that $v_1, \dots, v_{k-1}$ are eigenvectors (for some k), i.e., $T(v_j) = \lambda_j v_j$ for $1 \le j \le k-1$.

We want to show that v_k is an eigenvector for T as well.

Since T is upper triangular,

$$T(v_k) = A_{1k} v_1 + A_{2k} v_2 + \dots + A_{kk} v_k.$$

On the other hand, when $j < k$,

$$A_{jk} = \underbrace{\langle T(v_k), v_j \rangle}_{\text{(b/c } \beta \text{ is an orthonormal basis)}}$$

$$= \langle v_k, T^*(v_j) \rangle$$

$$= \langle v_k, \overline{\lambda_j} v_j \rangle \quad \left(\begin{array}{c}\text{PROPERTIES OF } T^* \\ \text{when } T \text{ is normal}\end{array}\right)$$

$$= \lambda_j \underbrace{\langle v_k, v_j \rangle}_{\text{basis is orthonormal}} = 0$$

SO: $T(v_k) = A_{kk} v_k$, and hence

v_k is an eigenvector for T.

By induction, $\beta = (v_1, \dots, v_n)$ is an orthonormal basis of eigenvectors of T. $\square$

<u>DEF</u> Let T be a linear operator on V, an inner product space.

We say that T is <u>Hermitian</u> (or <u>self-adjoint</u>) if $T = T^*$.

An $n \times n$ real or complex matrix A is <u>Hermitian</u> (or <u>self-adjoint</u>) if $A = A^* = \overline{A}^t$.

<u>EX</u> Let A be the <u>incidence matrix</u> of a graph Γ. Γ consists of <u>vertices</u> $v_1, \dots, v_n$ and <u>undirected</u> <u>edges</u> $\{v_i, v_j\}$. When $\{v_i, v_j\}$ is an

edge, we write $v_i \sim v_j$.

<u>Incidence matrix</u> A:

$$A_{ij} = \begin{cases} 1, & \text{when } v_i \sim v_j \\ 0, & \text{otherwise} \end{cases}$$

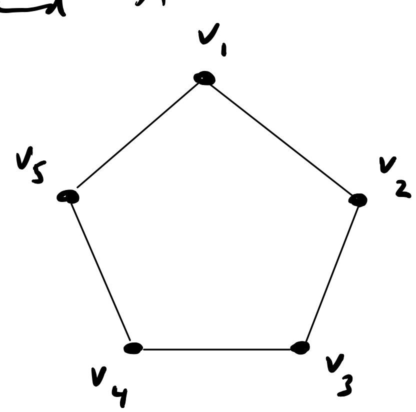
Graphs here will be <u>simple</u>, which means there are no "multiple edges" and no "loops" (so $v_i \not\sim v_i$).

In this case A is a real matrix, so $A = \overline{A}$.

Moreover, $v_i \sim v_j$ iff $v_j \sim v_i$ (since edges are undirected), so $A = A^t$.

Hence $A = A^*$, and A is Hermitian.

For instance, if $\Gamma = C_5$:



$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Indeed, $A^* = A$.

LEMMA Let T be a Hermitian operator on a finite-dim'l inner product space V. (V could be a real or complex inner product space.)

Then, (a) Every eigenvalue of T is real.

(b) Suppose now that V is a real inner product space. Then, the characteristic polynomial $\chi_T(t)$ splits over $\mathbb{R}$.

Pf (a) Suppose $T = T^*$ and let x be an eigenvector of T w/ eigenvalue λ, i.e., $T(x) = \lambda x$, where $x \neq 0$.

Since $T = T^*$,

$$TT^* = T^2 = T^*T,$$

so T is <u>normal</u>.

Thus x is an eigenvector for T^* w/ eigenvalue $\bar{\lambda}$, and so:

$$\lambda x = T(x) = T^*(x) = \bar{\lambda} x,$$

so $\lambda = \bar{\lambda}$ (since $x \neq 0$), and hence $\lambda \in \mathbb{R}$.

(b) Since T is Hermitian, it's normal, so there exists an orthonormal basis β of eigenvectors $v_1, \dots, v_n$ of T (over $\mathbb{C}$).

By (a), if $T(v_i) = \lambda_i v_i$, then $\lambda_i \in \mathbb{R}$.

Thus $[T]_\beta = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, and

so $\chi_T(t) = \chi_{[T]_\beta}(t) = (-1)^n (t - \lambda_1) \cdots (t - \lambda_n)$, and so χ_T <u>splits</u> over $\mathbb{R}$. $\square$