Q: Do we have to prove theorems/corollaries that have been discussed in class on HW/tests?

A: **HW**: Yes, sometimes I'll leave some things for you to prove.

Tests: Midterm: open book, open notes. After grading once, you can turn it in again (like HW) and earn back up to half points you missed.

Final: Closed book, closed notes; you will have a sheet of practice problems ahead of time

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Q: (From last time)

"Almost a vector space," where only Distributive Law(s) fail?

$\underline{A}$: yes.

$\underline{\text{Ex}}$ $V = \mathbb{R}^2$, scalars: $\mathbb{R}$

vector addition: defined as usual, so

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2)$$

$\underline{\text{scalar mult}}$: For every $r \in \mathbb{R}$, $v \in V$,

define $rv := v$.

(VS1) - (VS 4): only about addition, and we didn't alter that ✓

(VS 5): $1v = v$ ✓

(VS 6):
$$\begin{array}{ccc} a(bv) & \stackrel{?}{=} & (ab)v \\ \shortparallel & & \shortparallel \\ av & & v \\ \shortparallel & & \\ v & & \end{array}$$
✓

(VS 7):
$$\begin{array}{ccc} a(v+w) & \stackrel{?}{=} & av + aw \\ \shortparallel & & \shortparallel \\ v+w & & v+w \end{array}$$
✓

(VS 8): $(a+b)v \overset{?}{=} av + bv$

$$\begin{array}{ccc} (a+b)v & \overset{?}{=} & av+bv \\ \| & & \| \\ v & & v+v \end{array}$$

NO!

This doesn't work, so only (VS 8) fails.

LAST TIME: Does every v.s. have a basis?

AXIOM OF CHOICE: (One version)

$\mathcal{A}$: collection of disjoint sets. There exists a function f defined on $\mathcal{A}$ such that $f(A) \in A$ for all A in $\mathcal{A}$.

partial order: "$\leq$"

CHAIN: collection $\mathcal{C}$ such that if $A, B \in \mathcal{C}$, then either $A \leq B$ or $B \leq A$

The following is equivalent to the Axiom of Choice:

ZORN'S LEMMA Suppose P is a partially ordered set that has the property that every chain in P has an upper bound in P. Then, the set P contains a maximal element.

(m maximal : there does not exist a such that $m \leq a$)

NOTE If Axiom of Choice is assumed, then you can prove Zorn's Lemma. If Zorn's Lemma is assumed, then you can prove Axiom of Choice.

NOTE The Maximal Principle described in BOOK is an easy consequence of Zorn's Lemma: the partially ordered set is some family of sets $\mathcal{F}$, partial order is $\subseteq$.

DEF Let S be a subset of a v.s. V. A maximal linearly independent subset of S is a subset B satisfying:

(a) B is L.I.

(b) The <u>only</u> L.I. subset of S that contains B is B itself.

<u>THM</u> Let V be a v.s., S a subset such that S generates V.

If B is a maximal L.I. subset of S, then B is a basis for V.

<u>PF</u> Assume B is a max'l L.I. subset of S.

Since B is L.I., this amounts to showing that $\text{span}(B) = V$.

Suppose $\text{span}(B) \subsetneq V$. Since $V = \text{span}(S)$, if $S \subseteq \text{span}(B)$, then $V = \text{span}(S) \subseteq \text{span}(B) \subsetneq V$, a contradiction.

So $\text{span}(B) \subsetneq V \Rightarrow S \not\subseteq \text{span}(B)$, and hence there is $u \in S$ such that $u \notin \operatorname{span}(B)$.

We know from a THM:

$u \notin \operatorname{span}(B)$, B L.I. $\Rightarrow$ $B \cup \{u\}$ is L.I.

But $B \subsetneq B \cup \{u\} \subseteq S$, a contradiction to maximality of B.

Therefore, $\operatorname{span}(B) = V$, and B is a basis. $\square$

<u>THM</u> Assume the Axiom of Choice / Zorn's Lemma.

Let S be a L.I. subset of a v.s. V. There exists a max'l L.I. subset of V that contains S.

<u>Pf</u> Let $\tilde{\mathcal{F}}$ be the family of all L.I. subsets of V containing S.

$\mathcal{F}$ is partially ordered under inclusion: $\subseteq$

GOAL: Use Zorn's Lemma to prove existence of max'l element.

Consider a chain $\mathcal{C}$ in $\mathcal{F}$

(RECALL: for all A, B in $\mathcal{C}$, either $A \subseteq B$ or $B \subseteq A$;

so: "$\cdots \subseteq A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$")

Let $$U := \bigcup_{A \in \mathcal{C}} A$$

We claim that U is an upper bound for $\mathcal{C}$ in $\mathcal{F}$.

To show this, we must show that:

(1) $U \in \mathcal{F}$

(2) U is an upper bound for $\mathcal{C}$, i.e., $B \in \mathcal{C} \Rightarrow B \subseteq U$.

(2) is easier: $B \in \mathcal{C} \Rightarrow B \subseteq \bigcup_{A \in \mathcal{C}} A$

since $B \in \mathcal{C}$; hence $B \subseteq \mathcal{U}$.

We must now show that $\mathcal{U} \in \mathcal{F}$, i.e.,
we must show that $\mathcal{U}$ is L.I.
and $S \subseteq \mathcal{U}$.

Since $S \subseteq B \in \mathcal{C}$, $S \subseteq \bigcup_{A \in \mathcal{C}} A = \mathcal{U}$.

Let $v_1, v_2, \ldots, v_m \in \mathcal{U}$.

Since each $v_i \in \mathcal{U}$, there exists
A_i in $\mathcal{C}$ such that $v_i \in A_i$.

Since $\mathcal{C}$ is a chain and m is
finite, there is one A_k such
that $A_i \subseteq A_k$ for all i, $1 \leq i \leq m$.

This means $v_i \in A_k$ for all i,
and, since A_k is L.I., there
is no nontrivial linear combination
of the v_i that is the zero vector.

However, $\{v_1, \dots, v_m\}$ was arbitrary,

so U is L.I.

$\Rightarrow U \in \mathcal{F}$.

This means $\tilde{\mathcal{F}}$ is a partially ordered set where every chain has an upper bound <u>in</u> $\mathcal{F}$. By Zorn's Lemma, $\tilde{\mathcal{F}}$ has a <u>max'l</u> element M, and so M is a max'l L.I. subset of V containing S. $\square$

<u>COR</u> Every vector space has a basis.

<u>NOTE</u> In fact, if "Every v.s. has a basis" is <u>assumed</u> (as an axiom), then the Axiom of Choice can be proven! Like Zorn's Lemma, it's equivalent to Axiom of Choice.

FACT: Every basis of an infinite-dim'l v.s. has the same cardinality.

(use: HW 4, #6)

NEXT TIME: linear transformations

READ: §2.1