Q: Should we prove all the properties of a v.s. on HW when asked to determine / prove something is a vector space?

A: If it's not a v.s., of course just show where it fails (same thing goes for fields, HW2).

At least once: go through all parts of a definition, or @ least mention all parts.

DEF A vector space is called <u>finite-dimensional</u> if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis of V is called the <u>dimension</u> of V and is denoted by $\dim(V)$. A vector space that is <u>not</u> finite-dim'l is <u>infinite-dimensional</u>.

We now now have the tools to prove an "intuitively obvious" statement:

THM Let V be a finite-dim'l v.s., W a subspace of V.

Then W is also finite-dim'l, and $\dim(W) \le \dim(V)$.

Furthermore, $\dim(W) = \dim(V) \Rightarrow W = V$.

Pf Let B be a basis for V, $|B| = n$.

Let $B' \subseteq W$ be a L.I. set.

Since $B' \subseteq W$, $B' \subseteq V$, and so B' is a L.I. set in $\underline{V}$.

Since $\operatorname{span}(B) = V$ and $|B| = n$, by Replacement Thm, we have
$$|B'| \le |B| = n.$$

Thus, any basis of W has size $\le n$, and hence $\dim(W) \le \dim(V)$.

In particular, W is finite dim'l.

Suppose $\dim(W) = \dim(V)$.

Then W has a basis B' of size n.

Since B' is a L.I. set of size n in V, by Replacement Thm,

$$\operatorname{span}(B') = V \Rightarrow W = V. \quad \square$$

LAGRANGE INTERPOLATION

Let $c_0, c_1, \dots, c_n$ be distinct scalars in an infinite field $\mathbb{F}$.

(think: $\mathbb{R}$ or $\mathbb{C}$)

Define
$$f_i(x) := \frac{(x-c_0)(x-c_1)\cdots(x-c_{i-1})(x-c_{i+1})\cdots(x-c_n)}{(c_i-c_0)(c_i-c_1)\cdots(c_i-c_{i-1})(c_i-c_{i+1})\cdots(c_i-c_n)}$$

$$= \prod_{\substack{0 \le k \le n \\ k \ne i}} \frac{x-c_k}{c_i-c_k}$$

These are the Lagrange polynomials (associated w/ $c_0, c_1, \dots, c_n$).

Each f_i has degree n and hence is in $P_n(\mathbb{F})$.

Moreover, $f_i(c_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

Suppose for some $a_i \in \mathbb{F}$ we have

$$\sum_{i=0}^{n} a_i f_i = 0$$

"zero polynomial" that is zero @ all inputs

Noting that $\sum_{i=0}^{n} a_i f_i(c_j) = a_j f_j(c_j) = a_j$

$\Rightarrow a_j = 0$ (since polynomial is identically 0)

This is true for $\exists$ j, so the f_i are L.I. !

Since $\dim(P_n(\mathbb{F})) = n+1$,

Standard basis: $\{1, x, x^2, \dots, x^n\}$

$B = \{f_0, f_1, \dots, f_{n+1}\}$ is a <u>basis</u>.

Thus if $g \in P_n(\mathbb{F})$, there exist $b_0, b_1, \dots, b_n \in \mathbb{F}$ such that

$$g = \sum_{i=0}^{n} b_i f_i$$

Note that $g(c_j) = \sum_{i=0}^{n} b_i f_i(c_j)$

$$= b_j f_j(c_j)$$

$$= b_j,$$

and so g is the <u>unique</u> polynomial in $P_n(\mathbb{F})$ such that $g(c_j) = b_j$ for each j.

<u>MOREOVER</u>: if $f \in P_n(\mathbb{F})$ and $f(c_i) = 0$ for $n+1$ <u>distinct</u> scalars $c_0, \ldots, c_n$ in $\mathbb{F}$, then f is the <u>zero function</u>

MAXIMAL LINEARLY INDEPENDENT SETS

We've seen that every v.s. w/ finite spanning set contains a finite basis.

Q: Must every v.s. have a basis?

The answer is extremely subtle; it depends on which axioms you believe!

AXIOM OF CHOICE Let $\mathcal{A}$ be a family of nonempty, disjoint sets. Then, there exists a set that consists of exactly one element from each set A in $\mathcal{A}$.

EQUIVALENTLY: There exists a function ("choice function") f defined on $\mathcal{A}$ such that $f(A) \in A$ for all A in $\mathcal{A}$.

(So: you can choose a single element from each set)

Assuming the axioms of Zermelo-Fraenkel set theory (ZF), one can either take the Axiom of Choice to be true as an axiom (ZFC), or one can choose the negation of Axiom of Choice to be taken as an axiom ($ZF\neg C$), and, either way, there are "models" satisfying the axioms that are consistent.

(Consistent: no contradictions; there is no statement that is both true and false)

ZFC (assuming the Axiom of Choice is true) is the "standard" set of axioms used today, and most are comfortable accepting the Axiom of Choice as an axiom. We will assume the Axiom of Choice.

(There are some strange consequences, like the Banach – Tarski Paradox.)

DEF Let S be a set. A partial order on S is a binary relation $\leq$ satisfying:

(i) (REFLEXIVITY) For all $a \in S$, $a \leq a$.

(ii) (ANTISYMMETRY) If $a \leq b$ and $b \leq a$, then $a = b$.

(iii) (TRANSITIVITY) If $a \leq b$ and $b \leq c$, then $a \leq c$.

<u>NOTE</u> Not all elements need to be comparable! Think: sets.

<u>EX</u> $\{1, 2, 3\}$, $\{3, 74\}$ are not comparable under the partial order "$\subseteq$"

<u>DEF</u> A <u>chain</u> is a collection $\mathcal{C}$ such that, if $a, b \in \mathcal{C}$, either $a \leq b$ or $b \leq a$.

<u>THINK</u>: $\cdots \leq a_0 \leq a_1 \leq a_2 \leq \cdots$

The following is equivalent to the AXIOM OF CHOICE:

ZORN'S LEMMA Suppose P is a partially ordered set that has the property that every chain in P has an upper bound in P. Then, the set P contains a *maximal* element.

(If m is *maximal*, then there does *not* exist $a \in P$ such that $m \leq a$.)

READ §2.1