Prof. Eric Swartz NAME: Xida Ren
Math 309

MIDTERM

1.

(a) (10 points) Prove that there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ with $T((1,1)) = (1,2,3)$ and $T((1,2)) = (0,0,1)$.

Proof:

$$T(\begin{bmatrix}1\\2\end{bmatrix}) - T(\begin{bmatrix}1\\1\end{bmatrix}) = T(\begin{bmatrix}0\\1\end{bmatrix}) = \begin{bmatrix}-1\\2\\2\end{bmatrix} \qquad \text{linearity}$$

$$T(\begin{bmatrix}1\\1\end{bmatrix}) - T(\begin{bmatrix}0\\1\end{bmatrix}) = T(\begin{bmatrix}1\\0\end{bmatrix}) = \begin{bmatrix}2\\4\\5\end{bmatrix} \qquad \text{linearity}$$

□

(b) (10 points) If T is as defined in (a), then what is $T((1,0))$?

Proof:

$$T(\begin{bmatrix}1\\0\end{bmatrix}) = \begin{bmatrix}2\\4\\5\end{bmatrix}$$

□

2. Let V be the vector space of all sequences $\{a_n\}$ with entries from $\mathbb{F}$. Define two functions $L, R : V \to V$ by

$$L((a_1, a_2, \dots)) = (a_2, a_3, \dots),$$
$$R((a_1, a_2, \dots)) = (0, a_1, a_2, \dots).$$

(a) (10 points) Prove that L, R are both linear.

Proof:

(1) Additivity

$$\begin{aligned} L((a_1, a_2, \ldots)) + L((b_1, b_2, \ldots)) &= (a_2, a_3, \ldots) + (b_2, b_3, \ldots) \\ &= (a_2 + b_2, a_3 + b_3, \ldots) \\ &= L((a_1 + b_1, a_2 + b_2, \ldots)) \\ &= L((a_1, a_2, \ldots) + (b_1, b_2, \ldots)) \end{aligned}$$

Similarly,

$$\begin{aligned} R((a_1, a_2, \ldots)) + R((b_1, b_2, \ldots)) &= (0, a_1, a_2, \ldots) + (0, b_1, b_2, \ldots) \\ &= (0, a_1 + b_1, a_2 + b_2, \ldots) \\ &= R((a_1 + b_1, a_2 + b_2, \ldots)) \\ &= R((a_1, a_2, \ldots) + (b_1, b_2, \ldots)) \end{aligned}$$

(2) Homogeniety
For $i \geqslant 1$, The ith term of $L(c\{a_i\})$ is ca_{i+1} and the ith term of $cL(\{a_i\})$ is also ca_{i+1}, so the two are the same.
Similarly, for $i \geqslant 2$, the ith term of $R(c\{a_i\}) = R(\{ca_i\})$ is ca_{i-1} and the ith term of $cR(\{a_i\})$ is also ca_{i-1}. For $i = 1$, the 1st term of $R(x_i)$ is 0 for any sequence. So the two are the same.

$\square$

(b) (5 points) Prove that L is surjective but not injective.

Proof: L is surjective: every sequence $\{a_i\}$ can be outputted by L because $L((0, a_1, a_2, \ldots)) = (a_1, a_2, \ldots)$.
L is not injective: $L((0, a_1, a_2, \ldots)) = L((1, a_1, a_2, \ldots)) = (a_1, a_2, \ldots)$. $\square$

(c) (5 points) Prove that R is injective but surjective.

Proof: R is injective because if any two sequences $\{a_i\}, \{b_i\}$ differ at position i, after applying R they must differ at position $i + 1$.

R is not surjective because R cannot output sequences whose first term is not 0. $\square$

3. Let A, B be $n \times n$ matrices. Recall that the *trace* of A is defined by

$$\operatorname{Tr}(A) = \sum_{i=1}^{n} a_{ii},$$

where $A = (a_{ij})$.

(a) (10 points) Prove that $\operatorname{Tr}(A) = \operatorname{Tr}(A^t)$, where A^t is the *transpose* of A.

Proof: We know that

$$\operatorname{Tr}(A) = \sum_{i=1}^{n} a_{ii}$$

Because A^T is defined by $a_{ij}^T = a_{ji}$, we have

$$\operatorname{Tr}(A) = \sum_{i=1}^{n} a_{ii}^T = \sum_{i=1}^{n} a_{ii} =$$

Which is the same as the trace of A. □

(b) (10 points) Prove that $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$.

Proof:

$$\begin{aligned}
\operatorname{Tr}(AB) &= \sum_{i=1}^{n} (AB)_{ii} && \text{definition of trace} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji} && \text{def. of product} \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij} && \text{rearrangement} \\
&= \sum_{j=1}^{n} (BA)_{jj} && \text{def. of product} \\
&= \operatorname{Tr}(BA) && \text{def. of trace}
\end{aligned}$$

□

4. (20 points) If A is an $m \times n$ matrix and B is an $n \times p$ matrix, prove that

$$(AB)^t = B^t A^t,$$

where M^t denotes the *transpose* of the matrix M.

Proof: We prove that for all i, j in bounds, entry ij of $(AB)^T$ is the same as entry ij of $B^T A^T$.

Entry ij of $(AB)^T$ is entry ji of AB. Entry ji of AB is the dot product of the jth row of A and the ith column of B, by the definition of matrix products.

Entry ij of the product $B^T A^T$ is the product of the ith row of B^T and the jth column of A^T. Because transposition swaps the first and second indices, rows of M^T are the columns of M and vice versa. So row i of B^T is actually column i of B and column j of A^T is actually row j of A. So entry ij of $B^T A^T$ is just the product of row j of A and column i of B, which is the same as entry ij of $(AB)^T$.

Symbolically,

$$\begin{aligned}
(AB)^T_{ij} &= (AB)_{ji} && \text{def. of transposition} \\
&= \sum_{k=1}^{n} A_{jk} B_{ki} && \text{def. of product} \\
&= \sum_{k=1}^{n} B^T_{ik} A^T_{kj} && \text{swap indices for transpose} \\
&= (B^T A^T)_{ij} && \text{def. of product}
\end{aligned}$$

□

5. (20 points) Let β be a subset of an infinite-dimensional vector space V. Prove that β is a basis for V if and only if for each nonzero vector in V, there exist unique vectors $u_1, \ldots, u_n$ in β and and unique nonzero scalars $c_1, \ldots, c_n$ such that $v = c_1u_1 + c_2u_2 + \cdots + c_nu_n$.

Only if:

Proof: Suppose β is a basis.

A basis has to span V, and it has to be linearly independent.

By definition of spanning, for each $v \in V$, v can be expressed as some finite linear combination of n vectors in $b_i \in \beta$, with coefficients $c_1 \ldots c_n$. So

$$v = \sum_{i=1}^{n} c_i b_i$$

Now we prove that the scalars c_i are unique.

Suppose there are two ways to get v as finite linear combinations of vectors in β. Let

$$B = \{b_1 \ldots b_n\}$$

be the union of the β-vectors used in each of the ways.

For the first way, there will be coefficients c_i such that

$$\sum_{i=1}^{n} c_i b_i = v$$

. Similarly, there exists d_i s.t.

$$\sum_{i=1}^{n} d_i b_i = v$$

.

Now, if the c_i and the d_i are different, for some i, $c_i - d_i \neq 0$.

But that means that because $\sum_{i=1}^{n}(d_i - c_i)b_i = v - v = 0$, B is linearly dependant, which means that β couldn't have been a basis. □

If:

Proof: Suppose that for each nonzero vector in V, there exist unique vectors $u_1, \ldots, u_n$ in β and and unique nonzero scalars $c_1, \ldots, c_n$ such that $v = c_1u_1 + c_2u_2 + \cdots + c_nu_n$. We prove that β is a basis: a linearly independant spanning set.

Because each $v \in V$ can be expressed as a linear combination of vectors in β, we already know that it spans V.

Suppose that β is not linearly independant. Then some $u_k \in \beta$ can be expressed as a linear combination of the other vectors in β. Then $u_k \in V$ can be expressed as two different linear combinations of vectors in β: $1 \cdot u_k$ and however u_k is expressed in terms of the other vectors, violating our assumption. □