Math 332 Homework 4

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Solve the following problems. Each problem is 4 points.

1. Find the Huffman code for the codeword $A, E, G, H, I, L, O, P, R, S$, with frequencies $7, 6, 2, 3, 5, 4, 6, 3, 3, 4$, respectively.

   The Huffman code I found is as following (with letters listed in order of frequency, largest to smallest): $A : 000, O : 010, E : 011, I : 100, S : 110, L : 111, R : 0011, P : 0010, H : 1010, G : 1011$. Note, this Huffman code is not unique (I assigned the nodes with smaller values 1 and greater values 0 when distributing the 1,0s).

2. Two people play a game on a graph G, alternatively picking vertices. Player 1 starts at any vertex. Each subsequent choice must be adjacent to the preceding choice (of the other player) and not used before. Thus together they follow a path. The last player who moves wins. Prove that the second player has a winning strategy if G has a perfect matching, and otherwise the first player has a winning strategy.

   If G has a perfect matching M, every vertex is saturated by M. The second player can follow this strategy and always win: after the first player chooses a vertex a, the second player follows the edge in M that connects a to a new vertex b (every vertex is saturated by M so some edge like this exists). The second player must then choose a vertex c adjacent to b. b is already incident to an edge in M, so the edge connecting b and c cannot be in M. c is saturated, however, so it must be incident to a different edge in M. Now, player 2 can follow that edge and choose the next endpoint. If player 2 keeps repeating this approach, following edges in M, there will always be another edge to follow (because every vertex must be saturated by M and every vertex player 1 chooses remains unsatured). Therefore, player 2 will choose the final vertex and win.

   If G does not have a perfect matching, it still has a maximum matching M. Player 1 can start by choosing some vertex not saturated by M (which exists since M is not perfect). Now, player 2 can either choose a vertex saturated on unsaturated by M. If the new vertex is unsaturated, there exists a path from one unsaturated vertex to another, which is an M-augmenting path, thus, by Berge's theorem, contradicting M being a maximum matching. If player 2 chooses a saturated vertex, player 1 simply follows the incident edge in M to a new vertex. Thus, player 1 will choose the final vertex and win.

3. Prove that if $\frac{e(H)}{n(H)} \leq d$ for every subgraph H of a graph G, then G has an orientation with outdegree at most d at each vertex. (Hint: Apply Hall's Theorem to an X.Y-bigraph where $X = E(G)$ and Y consists of d copies of $V(G)$.)

   Create an X, Y bipartite graph where X is a set of vertices for which each vertex represents an edge in G and Y consists of d copies of $V(G)$. Hall's theorem says that if $|N(S)| \geq |S|$ for all subsets S of X, the bipartite graph has a matching which saturates X. The condition that $\frac{e(H)}{n(H)} \leq d$ for every subgraph H of a graph G can be altered so $e(H) \leq d \cdot n(H)$. Returning to our bipartite graph, let