Домашно по λСТД

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1.3 Задача

Да се покаже, че $\forall_{l_1,l_2} len(l_1 +\!\!+\ l_2) = len(l_1) + len(l_2)$

Индукция по l_1:

База: $l_1 = []$:

$$\begin{aligned} len([] +\!\!+\ l_2) &\overset{1.5.1}{=} len(l_2) \\ &= 0 + len(l_2) \\ &\overset{1.4.1}{=} len([]) + len(l_2) \end{aligned}$$

ИП: Нека е вярно за $l_1 = l$

Стъпка: Нека $l_1 = (n : l)$:

$$\begin{aligned} len((n : l) +\!\!+\ l_2) &\overset{1.5.2}{=} len(n : (l +\!\!+\ l_2)) \\ &\overset{1.4.2}{=} 1 + len(l + l_2) \\ &\overset{\text{ИП}}{=} 1 + len(l) + len(l_2) \\ &\overset{1.4.2}{=} len((n : l)) + len(l_2) \end{aligned}$$

1.5 Задача

$$\begin{aligned} \Gamma &: \mathcal{P}(U) \to \mathcal{P}(U) \\ \mathbb{\Gamma} &: \mathcal{P}(U) \to \mathcal{P}(U) \to \mathcal{P}(U) \\ \mathbb{\Gamma}(A)(X) &:= A \cup \Gamma(X) \\ \overline{\Gamma} &: \mathcal{P}(U) \to \mathcal{P}(U) \\ \overline{\Gamma}(D) &:= \mu_{\mathbb{\Gamma}(D)} \end{aligned}$$

Нека:

$$\varphi_\Delta := \{X \mid \Delta(X) = X\}$$

Демек:

$$\forall \Delta : \mu_\Delta \in \varphi_\Delta$$

Горните дефиниции важат и за 1.6

$$\begin{aligned} \mathbb{\Gamma}\text{is monotonous} &\iff A \subseteq B \implies \mathbb{\Gamma}(A) \subseteq \mathbb{\Gamma}(B) \\ \mu_{\mathbb{\Gamma}} &= ? \end{aligned}$$

Нека $A \subseteq B$. Тогава:

Знаем, че $\overline{\Gamma}$ е фамилия от монотонни оператори, т.е. $\forall A : \overline{\Gamma}(A)$ е монотонен оператор.

От Кнастер-Тарски(КТ):

$$\mu_\Gamma = \bigcap \{X \subseteq U \mid \Gamma(X) \subseteq X\}$$

Демек, ако $\exists X : \overline{\Gamma}(A)(X) \subseteq X$, т.е. X е "слаба неподвижна"/пренеподвижна точка, то тогава $\mu_{\overline{\Gamma}(A)} \subseteq X$, защото μ_Γ е сечението на всички такива пренеподвижни точки.

Разглеждаме:

$$\begin{aligned} \overline{\Gamma}(A)\left(\mu_{\overline{\Gamma}(B)}\right) &= A \cup \Gamma\left(\mu_{\overline{\Gamma}(B)}\right) \\ &\overset{A \subseteq B}{\subseteq} B \cup \Gamma\left(\mu_{\overline{\Gamma}(B)}\right) \\ &= \overline{\Gamma}(B)\left(\mu_{\overline{\Gamma}(B)}\right) \\ &\overset{\mu_{\overline{\Gamma}(B)}\text{is a fixpoint}}{=} \mu_{\overline{\Gamma}(B)} \\ &\implies \overline{\Gamma}(A)\left(\mu_{\overline{\Gamma}(B)}\right) \subseteq \mu_{\overline{\Gamma}(B)} \\ &\overset{\text{КТ}}{\implies} \mu_{\overline{\Gamma}(A)} \subseteq \mu_{\overline{\Gamma}(B)} \\ &\overset{def}{\implies} \mathbb{\Gamma}(A) \subseteq \mathbb{\Gamma}(B) \end{aligned}$$

Следователно $A \subseteq B \implies \mathbb{\Gamma}(A) \subseteq \mathbb{\Gamma}(B)$
Нека сега намерим $\mu_{\mathbf{\Gamma}}$:
Разглеждаме:
Нека $X \in \varphi_{\overline{\Gamma}(\mu_\Gamma)}$:

$$\begin{aligned}
\overline{\Gamma}(\mu_\Gamma)(X) &\overset{def}{=} \mu_\Gamma \cup \Gamma(X) \\
&\overset{fixpoint}{=} X \\
&\implies \mu_\Gamma \subseteq X
\end{aligned}$$

Обаче $\mu_\Gamma \in \varphi_{\overline{\Gamma}(\mu_\Gamma)}$, понеже:

$$\begin{aligned}
\overline{\Gamma}(\mu_\Gamma)(\mu_\Gamma) &= \mu_\Gamma \cup \Gamma(\mu_\Gamma) \\
&\overset{fixpoint}{=} \mu_\Gamma \cup \mu_\Gamma \\
&\overset{set\ theory}{=} \mu_\Gamma
\end{aligned}$$

Следователно μ_Γ хем е $\in \varphi_{\overline{\Gamma}(\mu_\Gamma)}$, хем е $\subseteq X, \forall X \in \varphi_{\overline{\Gamma}(\mu_\Gamma)}$.
Следователно $\mu_\Gamma = \mu_{\overline{\Gamma}(\mu_\Gamma)}$.
Разглеждаме (ново) $X \in \varphi_{\mathbf{\Gamma}}$:
Искаме $\mathbb{\Gamma}(X) = X \implies \mu_\Gamma \subseteq X$

$$\begin{aligned}
X &= \mathbb{\Gamma}(X) \\
&\overset{def}{=} \mu_{\overline{\Gamma}(X)} \\
&\overset{\substack{\exists Y \\ def}}{=} X \cup \Gamma(Y) \\
&\overset{fixpoint}{=} Y \\
&\implies X = Y \\
&\implies \Gamma(Y) \subseteq Y \\
&\overset{\text{КТ}}{\implies} \mu_\Gamma \subseteq Y \\
&\overset{Y=X}{\implies} \mu_\Gamma \subseteq X
\end{aligned}$$

Следователно μ_Γ хем е $\in \varphi_{\mathbf{\Gamma}}$, хем е $\subseteq X, \forall X \in \varphi_{\mathbf{\Gamma}}$.
Следователно $\mu_\Gamma = \mu_{\mathbf{\Gamma}}$.

1.6 Задача

$$\overline{\Gamma}\left(\overline{\Gamma}(D)\right) \overset{?}{=} \overline{\Gamma}(D)$$

Разглеждаме с цел $\overline{\Gamma}(D) \subseteq \overline{\Gamma}\left(\overline{\Gamma}(D)\right)$:

$$\begin{aligned}
\mathbb{\Gamma}(D)(\overline{\Gamma}(D)) &:= D \cup \Gamma(\overline{\Gamma}(D)) \\
&\overset{set\ theory}{\implies} D \subseteq \mathbb{\Gamma}(D)(\overline{\Gamma}(D)) \overset{fixpoint}{=} \overline{\Gamma}(D) \\
&\implies D \subseteq \mathbb{\Gamma}(D) \\
&\overset{\overline{\Gamma}\ monotonous}{\implies} \overline{\Gamma}(D) \subseteq \overline{\Gamma}\left(\overline{\Gamma}(D)\right)
\end{aligned}$$

Разглеждаме с цел $\overline{\Gamma}\left(\overline{\Gamma}(D)\right) \subseteq \overline{\Gamma}(D)$:

$$\begin{aligned}
\mathbb{\Gamma}\left(\overline{\Gamma}(D)\right)\left(\overline{\Gamma}(D)\right) &\overset{def}{=} \overline{\Gamma}(D) \cup \Gamma\left(\overline{\Gamma}(D)\right) \\
\Gamma\left(\overline{\Gamma}(D)\right) &\subseteq D \cup \Gamma\left(\overline{\Gamma}(D)\right) \\
&\overset{def}{=} \mathbb{\Gamma}(D)\left(\overline{\Gamma}(D)\right) \\
&\overset{fixpoint}{=} \overline{\Gamma}(D) \\
&\implies \overline{\Gamma}(D) \cup \Gamma\left(\overline{\Gamma}(D)\right) \overset{\subseteq}{=} \overline{\Gamma}(D) \\
&\implies \mathbb{\Gamma}\left(\overline{\Gamma}(D)\right)\left(\overline{\Gamma}(D)\right) = \overline{\Gamma}(D) \\
&\overset{def}{\implies} \overline{\Gamma}(D) \in \varphi_{\mathbf{\Gamma}\left(\overline{\Gamma}(D)\right)} \\
&\overset{\text{КТ}}{\implies} \mu_{\mathbf{\Gamma}\left(\overline{\Gamma}(D)\right)} \subseteq \overline{\Gamma}(D) \\
&\overset{def}{=} \overline{\Gamma}\left(\overline{\Gamma}(D)\right) \subseteq \overline{\Gamma}(D)
\end{aligned}$$

2.12 Задача

Да се докаже, че релацията $\leq$ е частична наредба, т.е. че е рефлексивна, транзитивна и антисиметрична. Упътване: покажете, че $M \leq N \iff Sub\,(M) \subseteq Sub\,(N)$...

" $\implies$ " $M \leq N$: Индукция по N

1. $N \equiv x \in V$:

$$\begin{aligned} M \leq x &\implies M \in Sub\,(x) = \{x\} \\ &\implies Sub\,(M) = Sub\,(N) \end{aligned}$$

2. $N \equiv N_1N_2$: $M \leq N_1N_2 \implies M \in Sub(N_1N_2) = \{N_1N_2\} \cup Sub(N_{1)} \cup Sub(N_2)$
   (а) $M \equiv N_1N_2 \implies Sub\,(M) = Sub\,(N)$
   (б) $M \in Sub(N_1) \implies Sub\,(M) \subseteq Sub(N_1) \subseteq Sub\,(N)$
   (в) $M \in Sub(N_2) \implies Sub\,(M) \subseteq Sub(N_2) \subseteq Sub\,(N)$
3. $N \equiv \lambda_x N_1$: $M \leq \lambda_x N_1 \implies M \in \{\lambda_x N_1\} \cup Sub(N_1)$
   (а) $M \equiv \lambda_x N_1 \implies Sub\,(M) = Sub\,(N)$
   (б) $M \in Sub(N_1) \implies Sub\,(M) \subseteq Sub(N_1) \subseteq Sub\,(N)$

" $\impliedby$ " $Sub\,(M) \subseteq Sub\,(N)$: Индукция по M

$$\begin{aligned} M \in Sub\,(M) &\overset{\subseteq}{\implies} M \in Sub\,(N) \\ &\overset{def}{\implies} M \leq N \end{aligned}$$