Estimating error on projections

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Let $\tilde{\rho}$ and $\tilde{\sigma}$ be two approximate representations of a finite group G, corresponding to the exact representations ρ and σ. We have $\|\tilde{\rho}_g - \rho_g\|_{\mathrm{FRO}} \leq \varepsilon_\rho$ and $\|\tilde{\sigma}_g - \sigma_g\|_{\mathrm{FRO}} \leq \varepsilon_\sigma$, valid for all $g \in G$.

We also define the *condition number* γ_ρ, such that $\|\tilde{\rho}_g\|_2 \cong \|\rho_g\|_2 \leqslant \gamma_\rho$ for all $g \in G$. When ρ_g is unitary, $\gamma_\rho = 1$; otherwise, it is given by the condition number of the matrix A such that $A\rho_g A^{-1}$ is a unitary representation (TODO: prove that γ_ρ does not depend on A).

The group G has a decomposition as a cartesian product of sets $G = T_n \times \cdots \times T_1$, with $|G| = \prod_i |T_i|$. Given a matrix X, we compute the approximation projection $\tilde{X} = \tilde{X}_n$ using the recursion:

$$\tilde{X}_i = \sum_{t \in T_i} \tilde{\rho}_t \tilde{X}_{i-1} \tilde{\sigma}_{t^{-1}}, \qquad \tilde{X}_0 = X. \tag{1}$$

We now want to compute the error associated with $\tilde{X}_i$, compared to the exact projection $X_i = \sum_t \rho_t X_{i-1} \sigma_t$. We write $\xi_i = \|X_i - \tilde{X}_i\|_{\mathrm{FRO}}$, and set $\xi_0 = 0$. Now, we bound:

$$\begin{aligned}
\xi_i = \|\tilde{X}_i - X_i\|_F \;&\cong\; \sum_t \|(\rho_t - \tilde{\rho}_t) X_{i-1} \sigma_{t^{-1}}\|_F + \|\rho_t(\tilde{X}_{i-1} - X_{i-1})\sigma_{t^{-1}}\|_F + \|\rho_t X_{i-1}(\sigma_t - \sigma_{t^{-1}})\|_F \\
&\leqslant\; \sum_t \|(\rho_t - \tilde{\rho}_t)\|_F \cdot \|X_{i-1}\sigma_{t^{-1}}\|_2 + \|\rho_t\|_2 \cdot \|\tilde{X}_{i-1} - X_{i-1}\|_F \cdot \|\sigma_{t^{-1}}\|_2 + \\
&\quad\; + \|\rho_t X_{i-1}\|_2 \|\sigma_t - \sigma_{t^{-1}}\|_F \\
&\leqslant\; \sum_t \varepsilon_\rho \cdot \|X_{i-1}\|_2 \cdot \gamma_\sigma + \gamma_\rho \xi_{i-1} \gamma_\sigma + \gamma_\rho \cdot \|X_{i-1}\| \cdot \varepsilon_\sigma.
\end{aligned}$$

Now, we assume $\|X_{i-1}\|_2 \leqslant \mu = \|X\|_2$ (TODO: prove). Then

$$\xi_i \leqslant |T_i| (\varepsilon_\rho \mu \gamma_\sigma + \gamma_\rho \xi_{i-1} \gamma_\sigma + \gamma_\rho \mu \varepsilon_\sigma). \tag{2}$$

This gives terrible error bounds, of the order of $\varepsilon_n = 10^{-1}$ for $G = S_6$ and both ρ and σ being the dimension 16 irreducible representation corresponding to the partition $6 = 3 + 2 + 1$ in the standard Specht basis. While its coefficients are in $\{-1, 0, +1\}$, we assumed an error on the image coefficients of the order of the machine precision.

The true error $\|\tilde{X} - X\|_F$ is of the order of 10^{-15}. We had estimated $\varepsilon_\rho = \varepsilon_\sigma \cong 10^{-14}$, which is pretty pessimistic due to the internal architecture; the true value is around 10^{-16}. The nonunitary of the representation is $\gamma_\rho = \gamma_\sigma \cong 9$.

In the error computation, most of the error comes from the nonunitary in the $\gamma_\rho \xi_{i-1} \gamma_\sigma$ term. Replacing $\gamma_\rho \xi_{i-1} \gamma_\sigma \rightarrow \xi_{i-1}$ gives a bound $\xi_i \simeq 10^{-8}$.