Testing for irreducibility

by Denis Rosset

Let G be a compact group. Let $\rho: G \to \mathcal{U}(\mathbb{F}^d)$ be a unitary representation over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$; equipping $V = \mathbb{F}^d$ with the Euclidean basis, we can write the images ρ_g using matrices.

Our problem is to determine whether ρ is irreducible over $\mathbb{F}$. We use the following test.

1 The test

Let X be a matrix sample from the Gaussian Unitary/Orthogonal Ensemble.

In the real case, we have $X_{ij} \sim \mathcal{N}(0,1)$. In the complex case, we have $X = R + \sqrt{-1} I$ with $R_{ij} \sim \mathcal{N}(0,1)$ and $I_{ij} \sim \mathcal{N}(0,1)$.

Then we compute its projection on the space of matrices that commute with ρ:

$$\hat{X} = \int_G \mathrm{d}\mu(g)\, \rho_g X \rho_g^{-1}. \tag{1}$$

Finally, we set $\mu = \operatorname{tr} \hat{X} / d$, and $\Delta = \|\hat{X} - \mu \mathbb{1}_d\|_F$.

Proposition 1. *If ρ is irreducible, we have $\Delta = 0$. If ρ is reducible, then the probability of having $\Delta \leqslant \delta$ is upper bounded by ε:*

$$\varepsilon = P(\Delta \leqslant \delta), \qquad \varepsilon \leqslant \begin{cases} \operatorname{erf}(\delta / \sqrt{2}), & \mathbb{F} = \mathbb{R}, \\ 1 - \exp[-\delta^2 / 2], & \mathbb{F} = \mathbb{C}. \end{cases} \tag{2}$$

2 Proof

2.1 Simplified case

We first consider the case where ρ is the sum of two inequivalent representations σ^1 and σ^2. Then there exists a matrix Q such that

$$Q \rho_g Q^\dagger = \begin{pmatrix} \sigma_g^1 & \\ & \sigma_g^2 \end{pmatrix}, \tag{3}$$

where σ^1 and σ^2 are two representations of G. By Schur's lemma, $\hat{X}$ has the form

$$\hat{X} = Q \underbrace{\begin{pmatrix} \lambda_1 \mathbb{1}_{d_1} & \\ & \lambda_2 \mathbb{1}_{d_2} \end{pmatrix}}_{\Lambda} Q^\dagger, \tag{4}$$

where $d_1 = \dim \sigma_1$ and $d_2 = \dim \sigma_2$, and $d = d_1 + d_2$. Now, the matrix $\hat{X}$ obtained during the projection is the one that commutes with ρ and minimizes the Frobenius norm $\|X - \overline{X}\|_F$, and thus

$$\lambda_1 = \frac{1}{d_1} \sum_{i=1}^{d_1} \hat{X}_{ii}, \qquad \lambda_2 = \frac{1}{d_2} \sum_{i=d_1+1}^{d} \hat{X}_{ii}. \tag{5}$$

We have furthermore $\mu = \frac{\operatorname{tr}\hat{X}}{d} = \frac{\operatorname{tr}\Lambda}{d} = \frac{\lambda_1 d_1 + \lambda_2 d_2}{d}$.

In the real case, λ_1 is the average of d_1 independent variables of distribution $\mathcal{N}(0,1)$, and λ_2 is the average of d_2 independent variables of distribution $\mathcal{N}(0,1)$. Thus $\lambda_1 \sim \mathcal{N}(0, 1/\sqrt{d_1})$, $\lambda_2 \sim \mathcal{N}(0, 1/\sqrt{d_2})$, while

$$\lambda_1 - \lambda_2 \sim \mathcal{N}\left(0, \sqrt{\frac{d}{d_1 d_2}}\right), \tag{6}$$

and

$$P(|\lambda_1 - \lambda_2| \leqslant \epsilon) = \operatorname{erf}\left(\frac{\epsilon}{\sqrt{2}\sigma}\right), \qquad \sigma = \sqrt{d/d_1 d_2}. \tag{7}$$

In the complex case, $\lambda_{1,2} = \lambda_{1,2}^R + \sqrt{-1}\lambda_{1,2}^I$ where $\lambda_1^{R,I} \sim \mathcal{N}(0, 1/\sqrt{d_1})$ and $\lambda_2^{R,I} \sim \mathcal{N}(0, 1/\sqrt{d_2})$. Thus

$$\operatorname{Re}\lambda_1 - \lambda_2, \operatorname{Im}\lambda_1 - \lambda_2 \sim \mathcal{N}\left(0, \sqrt{\frac{d}{d_1 d_2}}\right) \tag{8}$$

and

$$P(|\lambda_1 - \lambda_2| \leqslant \epsilon) = P((\operatorname{Re}\lambda_1 - \lambda_2)^2 + (\operatorname{Im}\lambda_1 - \lambda_2)^2 \leqslant \epsilon^2) = P(x^2 + y^2 \leqslant \epsilon^2/\sigma^2) \tag{9}$$

where $x, y \sim \mathcal{N}(0,1)$. Now, $x^2 + y^2$ is distributed according to the χ^2 distribution, which has CDF $f(z) = 1 - e^{-z/2}$, and thus

$$P(|\lambda_1 - \lambda_2| \leqslant \epsilon) = 1 - \exp[-\epsilon^2/2\sigma^2]. \tag{10}$$

Let us come back to the main computation. In the case of irreducible ρ, we have $\lambda_1 = \lambda_2$ by Schur's lemma; and then $\Delta = 0$ follows. Now, in the case of reducible ρ, we compute

$$\begin{aligned}
\Delta &= \|\hat{X} - \mu\mathbb{1}_d\|_F \\
&= \|Q\Lambda Q^\dagger - \mu\,\mathbb{1}_d\,\|_F \\
&= \|\Lambda - \mu\mathbb{1}_d\|_F \\
&= \sqrt{d_1|\lambda_1 - \mu|^2 + d_2|\lambda_2 - \mu|^2} \\
&= \sqrt{d_1|d\lambda_1 - d_1\lambda_1 - d_2\lambda_2|^2 + d_2|d\lambda_2 - d_1\lambda_1 - d_2\lambda_2|^2}/d \\
&= \sqrt{d_1|d_2\lambda_1 - d_2\lambda_2|^2 + d_2|d_1\lambda_2 - d_1\lambda_1|^2}/d \\
&= \sqrt{d_1 d_2^2|\lambda_1 - \lambda_2|^2 + d_1^2 d_2|\lambda_1 - \lambda_2|^2}/d \\
&= \sqrt{d_1 d_2(d_1 + d_2)|\lambda_1 - \lambda_2|^2}/d \\
&= \sqrt{\frac{d_1 d_2}{d}}|\lambda_1 - \lambda_2| = \frac{|\lambda_1 - \lambda_2|}{\sigma}
\end{aligned}$$

Then $P(\Delta \leqslant \delta) = \operatorname{erf}(\delta/\sqrt{2})$ in the real case, and $P(\Delta \leqslant \delta) = 1 - \exp[-\delta^2]$ in the complex case.

2.2 General case

In the case where σ^1 and σ^2 are equivalent, the off-diagonal blocks can now be non-zero, which cannot decrease Δ. Our bound thus still holds.

In the case where ρ is not unitary, we have $\Delta = \|\hat{X} - \mu\mathbb{1}_d\|_F = \|A(\hat{X}_U - \mu\mathbb{1}_d)A^{-1}\|_F$ where A is a change of basis matrix such that $A^{-1}\rho_g A$ is unitary; and $\hat{X}_U$ is the projection on the commutant of that unitary representation. Let Δ_U be the random variable arising out of the test for unitary representations. Then $\Delta \leqslant \operatorname{cond}(A)\Delta_U$.

We have not considered formally the case where ρ splits into more than two subrepresentations; we believe our bound still holds, as δ and d are all small.