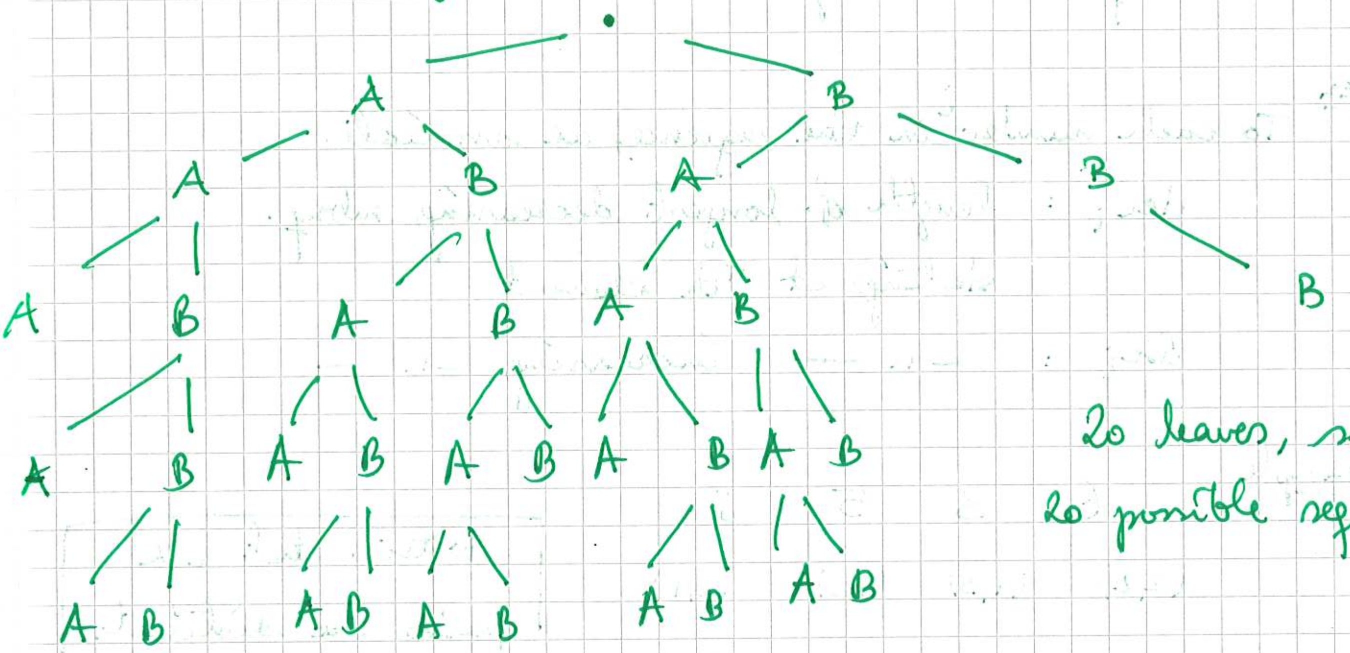
Lecture 19. Pigeonhole principle

Example (Tree diagrams)

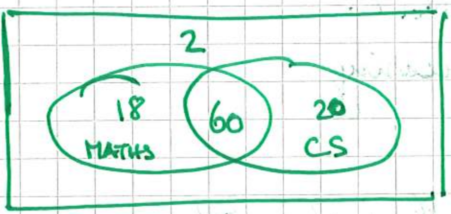


20 leaves, so 20 possible sequences.

Poll 1

$$|\{0\}\times\{0,1\}^6\times\{1\}| + |\{1\}\times\{0,1\}^6\times\{0\}| = 2^6 + 2^6 = 128.$$

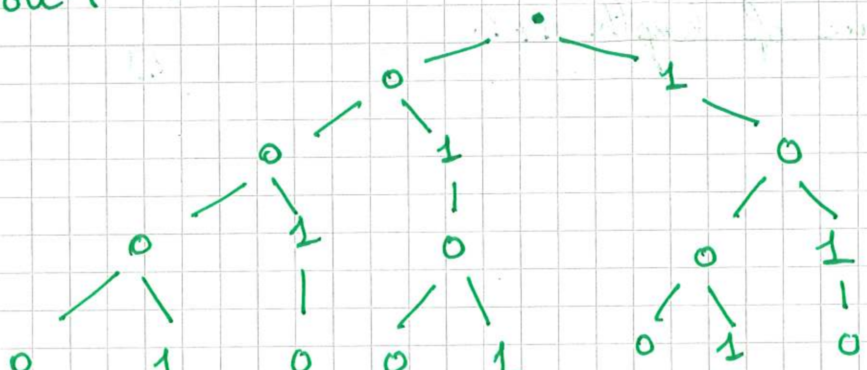
Poll 2



Poll 3

$$\frac{5!}{5} = 24 \quad \text{(Division rule)}$$

Poll 4



8 leaves.

Claim Any sequence of n^2+1 ~~elements~~ distinct numbers contains a strictly increasing or decreasing subsequence of length $n+1$.

IDEA

To each number i in the sequence, we associate

dec_i : length of longest decreasing subseq. starting at i'th element

inc_i : — " — increasing — " —

Example

6	3	5	8
(2,2)	(1,3)	(1,2)	(1,1)

Notice that all pairs are distinct.

→ If two pairs were equal we could extend the increasing or decreasing sequence.

Proof

For a sequence of n^2+1 numbers, construct the associated sequence $(\text{dec}_1, \text{inc}_1), (\text{dec}_2, \text{inc}_2), \ldots, (\text{dec}_{n^2+1}, \text{inc}_{n^2+1})$.

If there were no decreasing or increasing subsequences of length $n+1$, then

$\text{dec}_i \leq n$ and $\text{inc}_i \leq n$ for all $i = 1, \ldots, n^2+1$.

But there are at most n^2 possible values of such pairs, so two of them must be equal by the Pigeonhole principle.

This is not possible. Therefore, there exists an increasing or decreasing subsequence of length $n+1$.

$\square$

Claim For every integer $n \geq 1$, there exists an integer $k \geq 1$ s.t. $k \cdot n$ consists only of 0's and 1's.

Proof:

Consider the sequence

$$\alpha_1 = 1, \ \alpha_2 = 11, \ \alpha_3 = 111, \ \dots, \ \alpha_{n+1} = \underbrace{1 \cdots 1}_{n+1}$$

Two of the α_i's must have the same remainder modulo n (by Pigeonhole principle). Suppose this is α_i and α_j with $\alpha_i < \alpha_j$. Then,

$$\alpha_i \equiv \alpha_j \pmod{n}$$

so there exists $k \geq 1$ s.t. $n \cdot k = \alpha_j - \alpha_i$ and this consists only of 0s and 1s. $\square$

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