Lecture 20

Example:

Tournament:
- Final ranking: $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$
- Top 3: $5 \cdot 4 \cdot 3 = 60$.

The number of r-permutations of a set with n elements ($n \geq r$) is given by the product rule

$$n \cdot (n-1) \cdots (n-r+1) = \frac{n \cdot (n-1) \cdots (n-r+1) \cdot (n-r) \cdots 2 \cdot 1}{(n-r) \cdots 2 \cdot 1}$$

$$= \frac{n!}{(n-r)!}$$

We denote this as $P(n,r)$.

> Recall that that $n!$ (n factorial) is just $n \cdot (n-1) \cdots 2 \cdot 1$.

The number of r-combinations of a set with n elements ($n \geq r$) is given by the division rule:

All r-permutations may be partitioned into subsets $S_1, .., S_c$ such that each subset contains r-permutations of a subset of r elements, i.e.

$$|S_i| = P(r,r) = r!$$

By the division rule, we get

$$C(n,r) = c = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!\, r!}.$$

For $0 \leq r \leq n$, we have $\binom{n}{r} = \binom{n}{n-r}$ since

$$\binom{n}{n-r} = \frac{n!}{(n-r)!\,(n-(n-r))!} = \frac{n!}{(n-r)!\, r!} = \binom{n}{r}. \quad \square$$

Alternatively, observe that there is a bijection between the r-combinations of an n-set, and the $(n-r)$-combinations of an n-set : the bijection sends a subset w/ r elements to its complement subset which has $n-r$ elements. $\square$

For $n \geq 0$, we have $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Indeed, for each $k \in \{0, 1, .., n\}$, $\binom{n}{k}$ is the number of subsets of k elements from an n element set. The sum $\sum_{k=0}^{n} \binom{n}{k}$ is therefore the total number of subsets for an n-element set, and we know that this is 2^n (from previous lecture).

Binomial coefficients:

- $(x+y)^2 = 1 \cdot x^2 + 2xy + 1 \cdot y^2$
- $(x+y)^3 = (x+y)(x^2 + 2xy + y^2)$

$$= x^3 + 2x^2y + xy^2 + yx^2 + 2xy^2 + y^3$$

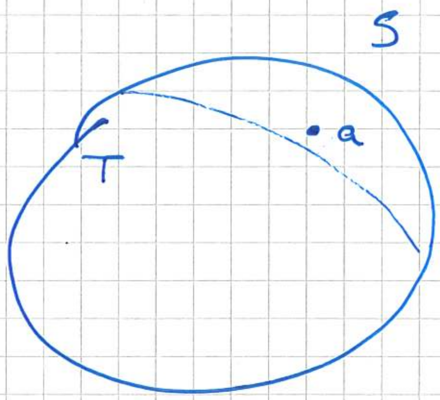
$$= 1 \cdot x^3 + 3x^2y + 3xy^2 + 1 \cdot y^3$$

$$= \binom{3}{0} x^3 + \binom{3}{1} x^2y + \binom{3}{2} xy^2 + \binom{3}{3} y^3$$

the number of xy-strings with 1 y of length 3.

In general, we can expand $(x+y)^n$, and each term is of the form $x^{n-j}y^j$ for $j = 0, 1, .., n$. The number of terms for a specific j is the number of ways of choosing precisely $(n-j)$ x's (or j y's) among xy-strings of length n. This explains the binomial theorem. $\square$

Pascal's identity: for $1 \le k \le n$, we have
$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$



Proof:

If S is an $(n+1)$-element set, then there are $\binom{n+1}{k}$ subsets with k elements. Alternatively, fix an element $a \in S$ and consider the set $T = S \setminus \{a\}$ with n elements. A k-subset of S either contains a (there are $\binom{n}{k-1}$ of those), or it does not contain a (there are $\binom{n}{k}$ of those). By the sum rule, we see that
$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$
$\square$