Lecture 18 Combinatorics 1

PRODUCT RULE

Example (License plates)

$L = \{A, B, C, \dots, Z\}$

$N = \{0, 1, 2, \dots, 99\,999\}$

The number of distinct license plates is

$|L \times L \times N| = 26 \cdot 26 \cdot 10\,000$

$= 67\,600\,000$

Example (Subsets)

Let $S = \{s_1, s_2, \dots, s_n\}$ be a finite set, $|S| = n$.

The power set $P(S)$ is the collection of all subsets of S.

Define a function $\varphi : P(S) \to \{0,1\}^n$ given by

$$\varphi(T) = (\varphi(T)_1, \varphi(T)_2, \dots, \varphi(T)_n)$$

where $T \subseteq S$ is a subset, and

$$\varphi(T)_i = \begin{cases} 1 & \text{if } s_i \in T, \\ 0 & \text{if } s_i \notin T \end{cases} \quad \text{for all } i = 1, \dots, n.$$

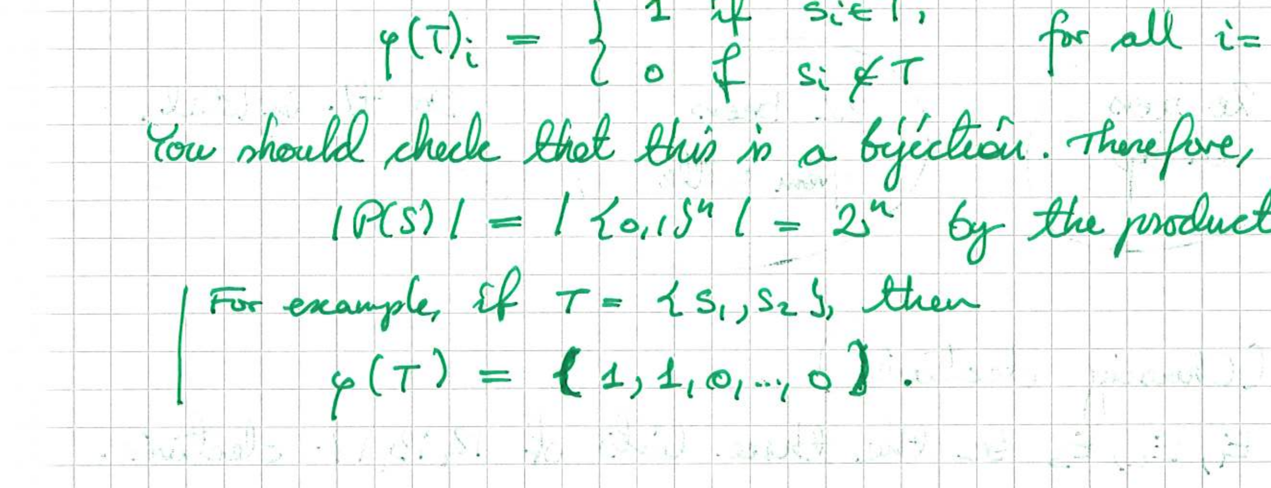
You should check that this is a bijection. Therefore,

$|P(S)| = |\{0,1\}^n| = 2^n$ by the product rule.

For example, if $T = \{s_1, s_2\}$, then

$\varphi(T) = (1, 1, 0, \dots, 0)$.

Example (assigning offices)



So there are $|\{1,2,3,4,5\} \times \{1,2,3,4\}| = 5 \cdot 4 = 20$ such ways.

Example (Injective functions)

Consider finite sets $M = \{m_1, m_2, \dots, m_k\}$ and $N = \{n_1, n_2, \dots, n_l\}$.

If $k > l$, then there are <u>no</u> injective functions $f : M \longrightarrow N$.

If $k \leq l$, then there are

$$l \text{ options for } f(m_1),$$
$$l-1 \text{ options for } f(m_2),$$
$$\vdots$$
$$l-k+1 \text{ options for } f(m_k).$$

So the number of injective functions $f : M \to N$ is

$l \cdot (l-1) \cdots (l-k+1)$ by the product rule.

SUM RULE

<u>Example</u>

20 beers on tap

22 beers from bottle

So 42 in total.

<u>Example</u> (Choosing electives)

Let E_1, E_2, E_3 be the three lists of 12, 13, 17 electives.

There are

$$|E_1 \cup E_2 \cup E_3| = |E_1| + |E_2| + |E_3| = 12 + 13 + 17 = 42 \text{ electives.}$$

Example (Menu)

M: 66 options on the menu

N: 64 options that are not vegan.

V: vegan options.

Then, $M = V \cup N$ and $66 = |M| = |V| + |N| = |V| + 64$,

so $|V| = 66 - 64 = 2$ vegan options.

Example (Passwords)

Let P_6, P_7, P_8 be the sets of possible passwords of lengths 6, 7, 8.

The number of strings of length 6 consisting of lower-case English letters and digits is $(26+10)^6$, and just English letters is 26^6.

So $|P_6| = (26+10)^6 - 26^6 = 1\ 867\ 866\ 560$.

Similarly,

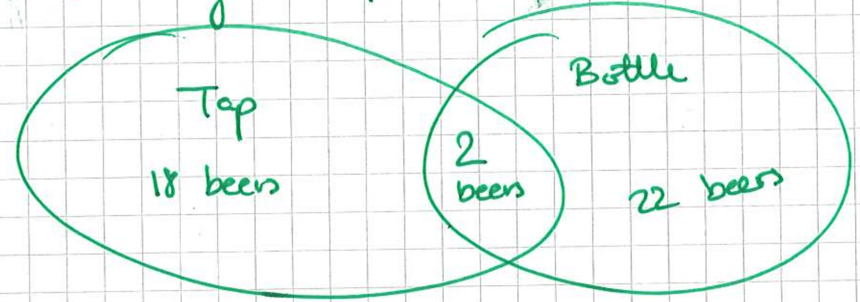
$$|P_7| = (26+10)^7 - (26)^7 = 70\ 332\ 353\ 920$$

$$|P_8| = (26+10)^8 - (26)^8 = 2\ 612\ 282\ 842\ 880.$$

Total number is $|P_6| + |P_7| + |P_8|$.

SUBTRACTION RULE

Example (Ordering beers)



Example (Bitstrings)

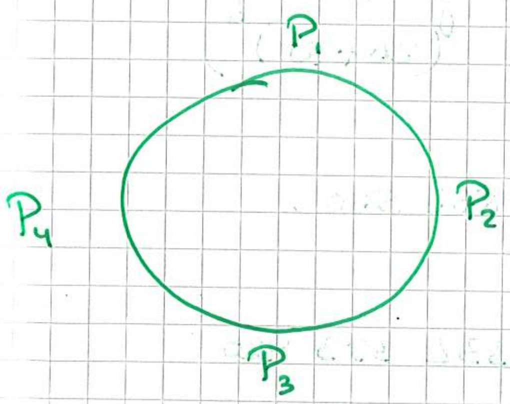
B_1: bitstrings of length 8 starting with 1, $|B_1| = 2^7$

B_2: bitstrings of length 8 ending with 00, $|B_2| = 2^6$.

Note $|B_1 \cap B_2| = 2^5$, so $|B_1 \cup B_2| = 2^7 + 2^6 - 2^5 = 160$.

Example (Seatings)

By the product rule there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways of seating 4 people.



Two seatings are considered the same if one is a rotation of the other.

The set of all seatings is partitioned into groups of 4 where two seatings are considered the same if and only if they are in the same partition set. By the division rule, there are $\frac{24}{4} = 6$ distinct seatings.