DM549/DS(K)820/MM537/DM547

Lecture 4: Introduction to Proofs

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Last Time: More on Quantifiers

Aspects:

- quantification over restricted domains
- negating quantified statements
- nested quantifiers

Negating Statements with Nested Quantification

We already know how to do this:

- First negate outer quantification, "flipping" it and moving the negation inwards,
- then do the same with second quantifier,
- etc.
- Or directly: Flip all quantifiers and move the negation all the way inwards.

More Quantifiers

Rule of Thumb

number of quantifier flips (as you read from left to right)

$$\approx$$

complexity of proposition

Even and Odd Numbers

Definition (Definition 1.7.1)

For a number $n \in \mathbb{Z}$ is called

- *even* (lige) if and only if $\exists k \in \mathbb{Z} : n = 2k$.
- *odd* (ulige) if and only if $\exists k \in \mathbb{Z} : n = 2k + 1$.

Further, two numbers $n, n' \in \mathbb{Z}$ are said to have the same *parity* (paritet) if and only if they are both even or both odd.

Remark: In definitions, one often uses "if" instead of "if and only if" (but means the same).

Proving Propositions on Even and Odd Numbers

Example 1.7.1

Let $n \in \mathbb{Z}$. Then

$$n \text{ is odd} \;\Rightarrow\; n^2 \text{ is odd}.$$

Proof Method: Direct Proof:

- We use that $((p \Rightarrow p_1) \wedge (p_1 \Rightarrow p_2) \wedge \cdots \wedge (p_n \Rightarrow q)) \Rightarrow (p \Rightarrow q)$.
- We also just write $p \Rightarrow p_1 \Rightarrow p_2 \Rightarrow \cdots \Rightarrow p_n \Rightarrow q$.

Example 1.7.9

Let $n \in \mathbb{Z}$. Then

$$n^2 \text{ is odd} \;\Rightarrow\; n \text{ is odd}.$$

Q: Can we use a direct proof?
– Not immediately clear. (What is the parity of $\sqrt{2k+1}$?)

Proof Method: Proof by Contraposition:

- We use that $(p \Rightarrow q) \equiv (\neg q \Rightarrow \neg p)$.

Combining the Two Propositions

Example 1.7.1 (restated)

Let $n \in \mathbb{Z}$. Then

$$n \text{ is odd } \Rightarrow n^2 \text{ is odd.}$$

Example 1.7.9 (restated)

Let $n \in \mathbb{Z}$. Then

$$n^2 \text{ is odd } \Rightarrow n \text{ is odd.}$$

Corollary

Let $n \in \mathbb{Z}$. Then

$$n^2 \text{ is odd } \Leftrightarrow n \text{ is odd.}$$

Another Proof Technique

Proposition

There are two people in this room that were born on the same weekday.

Proof Method: Proof by Contradiction:

- We use that $(\neg p \Rightarrow F) \Rightarrow p$.

(**Remark:** Equivalently, we could use contraposition to show $T \Rightarrow p$.)

Recall: $\mathbb{Q} = \{\frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{Z}^+\}$ is the set of rational numbers.

Example 1.7.11

It holds that $\sqrt{2} \notin \mathbb{Q}$.

No proof here, but classical proof by contradiction.

Overview of Proof Methods

Proof methods:

- Direct Proof:
  - We use that $((p \Rightarrow p_1) \wedge (p_1 \Rightarrow p_2) \wedge \cdots \wedge (p_n \Rightarrow q)) \Rightarrow (p \Rightarrow q)$.
  - We also just write $p \Rightarrow p_1 \Rightarrow p_2 \Rightarrow \cdots \Rightarrow p_n \Rightarrow q$.
- Proof by Contraposition:
  - We use that $(p \Rightarrow q) \equiv (\neg q \Rightarrow \neg p)$.
- Proof by Contradiction:
  - We use that $(\neg p \Rightarrow F) \Rightarrow p$.

Other "Proof" Methods

Examples (not serious!)**:**

- Proof by intimidation: "Trivial." ("Don't be stupid; of course it's true!")
- Proof by omission: "The proof is left as an exercise to the reader."
  - cf. Fermat's Last Theorem
- ...many more can be found on the internet.
  - e.g., `https://users.cs.northwestern.edu/~riesbeck/proofs.html`

Another example

Proposition

Let $n \in \mathbb{Z}$. Then
$$\left\lfloor \frac{n+1}{2} \right\rfloor \geq \frac{n}{2}.$$

Recall: $\lfloor \cdot \rfloor$ rounds down; $\lceil \cdot \rceil$ rounds up.

Proof method:

- Distinguish two cases.
- In each case, give a direct proof.

Overview of "Tricks"

Examples:

- Use that $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$.
  - See example on slide 7.
- Use that $p_1 \Leftrightarrow p_2 \Leftrightarrow p_3 \equiv p_1 \Rightarrow p_2 \Rightarrow p_3 \Rightarrow p_1$
  - Can be used in Exercise 1.7.43.
- Make a case distinction.
  - See example on slide 11.

Existence Proofs

Question

Let $n \in \mathbb{Z}$ with $n \geq 2$. Can an $n \times n$ chess board be covered by domino bricks (each of which covers two squares of the chess board)?

No domino bricks may overlap or stick out over the edge.

Constructive existence proof: We have proven existence by giving a feasible covering.

Proposition (restated)

There are two people in this room that were born on the same weekday.

Non-constructive existence proof: We have not found the weekday, but we have proven that it exists.

More on Chess

Question

Let $n \in \mathbb{Z}$ with $n \geq 2$. Can an $n \times n$ chess board be covered by domino bricks (each of which covers two squares of the chess board), after removing two diagonally opposite squares?

No domino bricks may overlap or stick out over the edge.

Morale: To solve a Mathematical problem, it may help to

- first look at simpler special cases.
- draw suitable pictures.

Test 1

Don't forget that the test has been opened last Friday.