Lecture 5

On proofs:

- Goldbach's conjecture: Every even positive integer is the sum of two primes.

- Gödel's incompleteness theorems:
  There are true mathematical statements that cannot be proven.

Induction

Claim: For all $n \in \mathbb{N}$, we have
$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1 .$$

Recall:

- For $k, l \in \mathbb{N}$, we write
$$2^k \cdot 2^l = \underbrace{2 \cdot 2 \cdots 2}_{k} \cdot \underbrace{2 \cdot 2 \cdots 2}_{l} = 2^{k+l}$$
In particular, $2^{k+1} = 2^k \cdot 2$.

- We use Sigma for notation
$$\sum_{i=0}^{n} 2^i = 2^0 + 2^1 + 2^2 + \cdots + 2^n .$$

For small values of n, we can simply compute:

$n=0$: $\sum_{i=0}^{0} 2^0 = 2^0 = 1 = 2^1 - 1$ ✓ $P(0)$

$n=1$: $\sum_{i=0}^{1} 2^0 = 2^0 + 2^1 = 3 = 2^2 - 1$ ✓ $P(1)$

$n=2$: $\sum_{i=0}^{2} 2^i = 2^0 + 2^1 + 2^2 = 7 = 2^3 - 1$ ✓ $P(2)$

Let $P(n)$ be the proposition $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

A proof by induction involves 2 steps:

1) verify $P(0)$ (basis step)
2) verify the implication
$P(k) \Longrightarrow P(k+1)$ (induction step)

If we can do this, we obtain a string of implications

$$P(0) \Rightarrow P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow \dots \Rightarrow P(k) \Rightarrow P(k+1) \Rightarrow \dots$$

and we can conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

Proof of claim: We aim to do a proof by induction.

For the basis step, observe that

$$P(0): \quad \sum_{i=0}^{0} 2^i = 2^0 = 2^1 - 1 \quad \checkmark$$

For the induction step, we first write the induction hypothesis: $P(k)$: $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$.

By using the induction hypothesis, we aim to verify $P(k+1)$.

Observe that

$$\sum_{i=0}^{k+1} 2^i = \underbrace{\sum_{i=0}^{k} 2^i}_{} + 2^{k+1} \qquad \text{(by def of sum)}$$

$$\overset{!}{=} 2^{k+1} - 1 + 2^{k+1} \qquad \text{(by induction hypothesis)}$$

$$= 2 \cdot 2^{k+1} - 1$$

$$= 2^{k+2} - 1. \quad \checkmark \qquad P(k+1)$$

This is what we wanted to show. We conclude that the claim holds for all $n \in \mathbb{N}$. $\square$

Claim For all $n \in \mathbb{N}$ with $n \geq 4$ we have $2^n < n!$

Observe that

$\underline{n=1}$: $2^1 = 2$ and $1! = 1$

$\underline{n=2}$: $2^2 = 4$ and $2! = 2 \cdot 1 = 2$

$\underline{n=3}$: $2^3 = 8$ and $3! = 3 \cdot 2 \cdot 1 = 6$

So the claim is <u>not</u> true for $n = 1, 2, 3$.

$\underline{n=4}$: $\left.\begin{aligned} 2^4 &= 2\cdot2\cdot2\cdot2 = 16 \\ 4! &= 4\cdot3\cdot2\cdot1 = 24 \end{aligned}\right\}$ $16 < 24$ ✓

$\underline{n=5}$: $\left.\begin{aligned} 2^5 &= 2\cdot2\cdot2\cdot2\cdot2 = 32 \\ 5! &= 5\cdot4\cdot3\cdot2\cdot1 = 120 \end{aligned}\right\}$ $32 < 120$ ✓

$\underline{n=6}$: $\left.\begin{aligned} 2^6 &= 2\cdot2^5 = 64 \\ 6! &= 6\cdot5! = 720 \end{aligned}\right\}$ $64 < 720$ ✓

Let us do a proof by induction:

<u>Basis step</u> ($n=4$):

$$2^4 = 16 < 24 = 4! \quad ✓$$

For the <u>induction hypothesis</u>, we assume the claim holds for some $k \geq 4$, so

$$2^k < k!$$

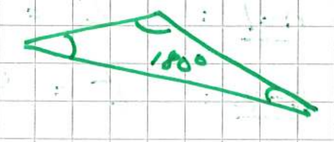
For the <u>induction step</u>, we calculate

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &< 2 \cdot k! \quad \text{(induction hypothesis)} \\ &< (k+1) \cdot k! \\ &= (k+1)! \end{aligned}$$

and this means $2^{k+1} < (k+1)!$ as we wanted. $\square$

A geometric example:

Recall, (e.g. from high school) that the sum of the angles in a triangle is $180°$.

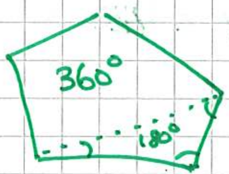


<u>Claim</u> For all $n \in \mathbb{N}$ with $n \geq 3$, for a convex n-gon the sum of its angles is $(n-2) \cdot 180°$.

<u>$n=4$</u>: (~~rectangle~~)

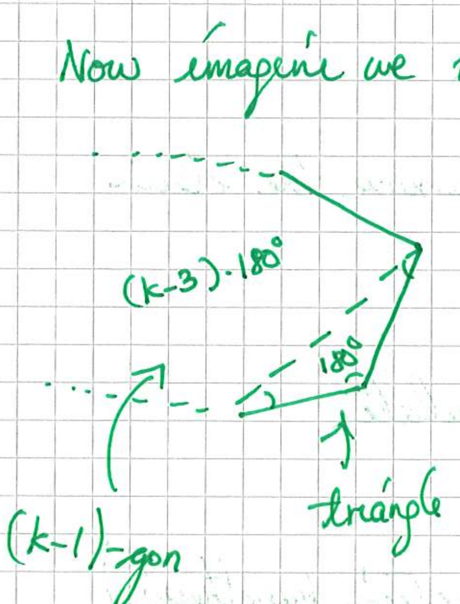
$180° + 180° = 360°$
$(= 2 \cdot 180°)$ ✓

$n=5$:



$360° + 180° = 540°$. ✓
$(= 3 \cdot 180°)$

Now imagine we have a k-gon for some $k \geq 3$



We can draw a triangle inside the n-gon: the large piece has an angle sum of

$(k-3) \cdot 180°$ (induction hypothesis)

and the small one is a triangle. So the total angle sum must be

$$(k-3) \cdot 180° + 180° = (k-2) \cdot 180°$$

and this is what we wanted to show. □