DM549/DS(K)820/MM537/DM547

Lecture 3: More on Quantifiers

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Last Time: Propositional Logic

Definitions:

- tautology, contradiction, contingency,
- logically equivalent.

Important logical equivalences:

- the Distributive Law,
- De Morgan's Laws,
- contraposition,
- implication only using $\land$, $\lor$, $\neg$,
- bi-implication as two implications.

Last Time: Propositional Functions and Quantifiers

Important definitions:

- propositional functions,
- the universal quantifier,
- the existential quantifier,
- the uniqueness quantifier (much less common than the other two!).

More on Quantifiers

Remarks:

- We say that the quantifier *binds* variables x.
- In the above statements, we call D the *domain* (domæne) or universe (univers).
- We also say that we *quantify over* (kvantificerer over) D.
- When clear from the context, the domain is sometimes left out.
- Some authors leave out the colon.
- How to memorize?
  - for ∀ll,
  - there ∃xists,
  - "!" looks a bit like "1".
- Quantifiers have a *higher* preference (i.e., they are evaluated earlier) than the operators $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, $\oplus$.

Poll everywhere

Quantification over Restricted Domains

Let $P(x)$ and $Q(x)$ both be propositional functions.

We can **restrict** the domain to values x satisfying $Q(x)$ with the following notation:

- $\forall x \in D, Q(x) : P(x) \equiv \forall x \in D : (Q(x) \Rightarrow P(x))$,
- $\exists x \in D, Q(x) : P(x) \equiv \exists x \in D : (Q(x) \wedge P(x))$,
- $\exists! x \in D, Q(x) : P(x) \equiv \exists! x \in D : (Q(x) \wedge P(x))$.

Remarks:

- Read: "... x in D with Q(x) ...".
- Notice the difference in how the quantification with restricted domain can be translated into a quantification without restricted domain.

Negating Quantified Statements

De Morgan's Laws for Quantifiers (Table 1.4.2)

$$\neg\forall x \in D : P(x) \equiv \exists x \in D : \neg P(x), \quad \neg\exists x \in D : P(x) \equiv \forall x \in D : \neg P(x)$$

Proof idea:

- Interpret the quantified statements as conjunction/disjunction,
- apply De Morgan's (regular) laws,
- interpret disjunction/conjunction as quantified statement.

Interpretation: Can pull "$\neg$" to the right, but, by doing so, we "flip" quantifiers ($\forall$ changes to $\exists$ and $\exists$ changes to $\forall$).

Poll everywhere

Nested Quantifiers

Note:

- A propositional function may also have two (or more!) variables.
- By adding a quantifier in front (and binding one of the variables), one obtains a propositional function with only one variable.
- By adding another quantifier in front (and binding another variable), one obtains a *proposition* (with no variables!).
- The resulting proposition has two *nested* quantifiers.

The Order of Quantifiers

The Order of Quantifiers matters:

- $\forall x : \exists y : P(x, y)$ does not imply $\exists y : \forall x : P(x, y)$ in general.
- $\exists x : \forall y : P(x, y)$ implies $\forall y : \exists x : P(x, y)$.

...but not always:

- $\forall x : \forall y : P(x, y) \equiv \forall y : \forall x : P(x, y)$.
- $\exists x : \exists y : P(x, y) \equiv \exists y : \exists x : P(x, y)$.
  - Also write $\forall x, y : P(x, y)$ and $\exists x, y : P(x, y)$.

In general: We may exchange consecutive quantifiers if and only if they are of the same type!

Formulate Precisely

Joke:

- Person 1: “Someone steals a car every fifteen seconds.”
- Person 2: “We have to find that person and stop them.”

Seriously:

- If you express a mathematical statement in natural language, make sure it is unambiguous!
- When writing down as formal logic, such ambiguities cannot happen.
  - (This is why we are learning about this topic!)