Recursive Definitions & Structural Induction

In Week 40, we saw how to define **sequences**, like the **Fibonacci numbers**, recursively:

$$\boxed{\begin{array}{l} f_0 = 0, \quad f_1 = 1 \\ f_n = f_{n-1} + f_{n-2}, \quad n \geq 2 \end{array}} \qquad \begin{array}{l} \text{Basis step} \\ \text{Recursive step} \end{array}$$

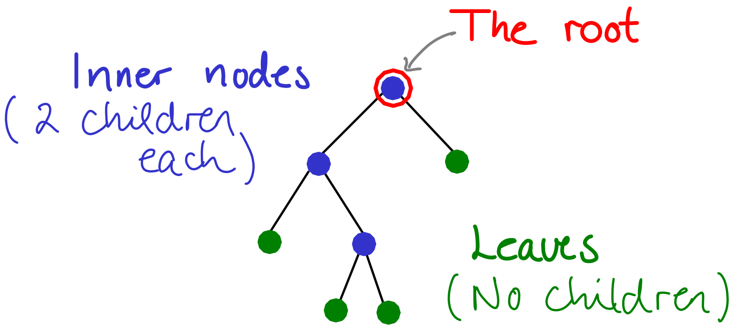
We used **strong induction** to prove that $f_n > \phi^{n-2}$, for $n \geq 3$.

Now, we shall see how **structures and sets** can be defined recursively.
Then, we will see how **structural induction** can be useful for proving properties of recursively defined sets.

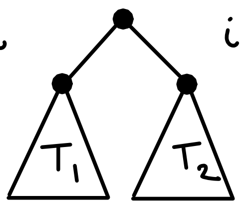
Full binary trees

Trees are a special kind of graphs, i.e., they consist of vertices and edges.
Full binary trees are a special kind of rooted trees in which each vertex has 0 or 2 children.

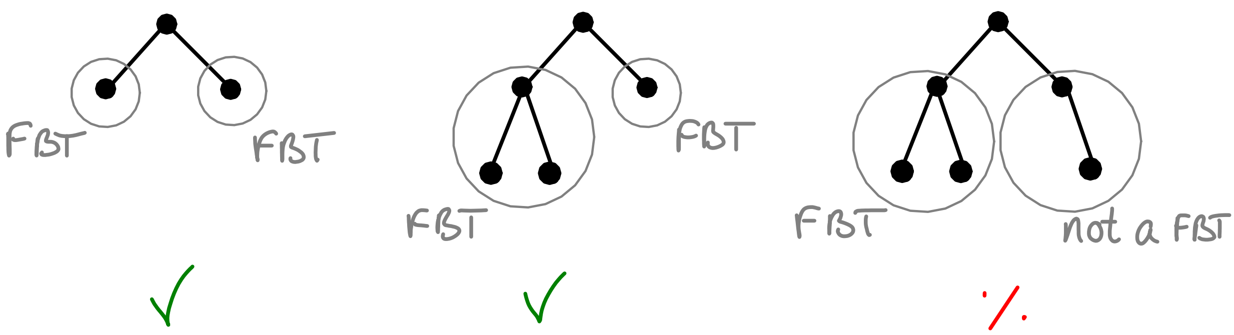
Ex:



Recursive definition of full binary trees (FBTs)

• is a FBT	(Basis step)
If T_1 and T_2 are FBTs, then  is a FBT	(Recursive step)

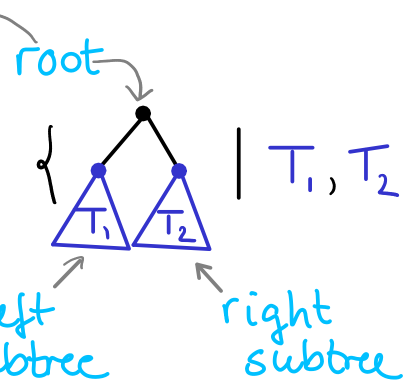
Ex:



An alternative way of phrasing the recursive definition:

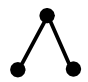
Def. 5.3.5: Full Binary Trees

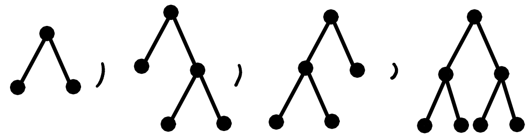
$S_1 = \{\bullet\}$ — root — Basis step

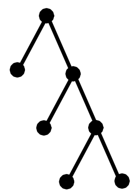
$S_i = S_{i-1} \cup \{$ 

$\mid T_1, T_2 \in S_{i-1}\}, \ i \geq 2$ — Recursive step

Thus,

$S_2 = \{\bullet,$  $\}$

$S_3 = \{\bullet,$  $\}$

$S_4 = \{$ __________ " __________ ,  $, \ldots\}$

$(\ |S_4| = 26\ )$

Structural Induction

Proving $P(S_i)$, for all $i \geq 1$

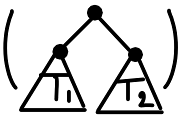
Prove $P(S_1)$

Prove that $P(S_i) \Rightarrow P(S_{i+1})$, for all $i \geq 1$

We will use structural induction to prove a relation between the number of vertices and the height (defined on the next page) of a full binary tree.

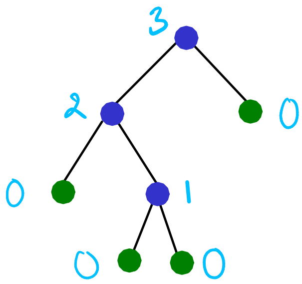
The height, h, of a rooted tree is defined as:

$h(\bullet) = 0$ — Basis step

$h\Big($ 

$\Big) = 1 + \max\{h(T_1), h(T_2)\}$ — Recursive step

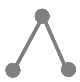
Ex:

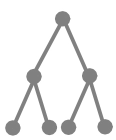


height ($h=3$): # edges in longest path from the root to a leaf

How many vertices can we fit in a full binary tree of a given height?

$h=0$: $\bullet$ $\quad 1 = 2^0 \quad = 2^1 - 1$

$h=1$:  $\quad 3 = 2^0 + 2^1 \quad = 2^2 - 1$

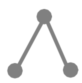
$h=2$:  $\quad 7 = 2^0 + 2^1 + 2^2 = 2^3 - 1$

$n(T)$: # vertices in T

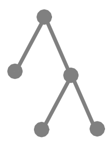
Theorem 5.3.2

For any full binary tree, T, $n(T) \leq 2^{h(T)+1} - 1$

Ex:



$\left.\begin{matrix} n = 3 \\ h = 1 \end{matrix}\right\} \Rightarrow 2^{h+1} - 1 = 3 = n$



$\left.\begin{matrix} n = 5 \\ h = 2 \end{matrix}\right\} \Rightarrow 2^{h+1} - 1 = 7 > n$

Proof by structural induction:

Basis: $S_1 = \{\bullet\}$

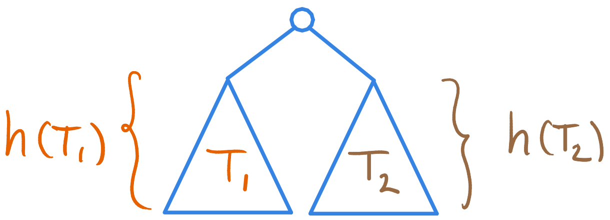
$\left.\begin{matrix} n = 1 \\ h = 0 \end{matrix}\right\} \Rightarrow 2^{h+1} - 1 = 2 - 1 = 1 = n$

Induction hypothesis $(i \geq 1)$:

$$n(T) \leq 2^{h(T)+1} - 1, \quad \text{for all } T \in S_i$$

Inductive step $(i \geq 1)$:

Let $T \in S_{i+1} - S_i$:



$h(T) = 1 + \max\{h(T_1), h(T_2)\}$, by Def. 5.3.6

$T_1, T_2 \in S_i$, by Def. 5.3.5

$$
\begin{aligned}
n(T) &= 1 + n(T_1) + n(T_2) \\
&\leq 1 + \left(2^{h(T_1)+1} - 1\right) + \left(2^{h(T_2)+1} - 1\right), && \text{by the ind. hyp.} \\
&= 2^{h(T_1)+1} + 2^{h(T_2)+1} - 1 \\
&\leq 2^{h(T)} + 2^{h(T)} - 1, && \text{by Def. 5.3.6} \\
&= 2 \cdot 2^{h(T)} - 1 \\
&= 2^{h(T)+1} - 1
\end{aligned}
$$

$\square$

Sequences (Section 2.4)

Ex: (Fibonacci) $0, 1, 1, 2, 3, 5, 8, \dots$

$f_0 = 0, \quad f_1 = 1$

$f_{n+1} = f_n + f_{n-1}, \quad n \geq 2$

Sequence (Følge)

Ordered set or

Function from (a subset of) $\mathbb{N}$

Ex: (Harmonic) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

$f : \mathbb{N}^+ \to \mathbb{Q}^+, \quad f(n) = \frac{1}{n}$

or

$a_n = \frac{1}{n}, \quad n \geq 1$

$\{a_n\} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

Conflict with set notation

Arithmetic Progressions

(Equidistant terms)

Ex: $1, 3, 5, 7, \ldots$

$$a_n = 1 + 2n, \quad n \in \mathbb{N}$$

or

$$a_0 = 1$$
$$a_n = a_{n-1} + 2, \quad n \geq 1$$

Arithmetic progression

$$a, \ a+d, \ a+2d, \ \ldots$$

or

$$a_n = \underbrace{a}_{\text{Initial term}} + n\underbrace{d}_{\text{Common difference}}, \quad n \in \mathbb{N}$$

Ex. above: $a = 1$, $d = 2$

Geometric Progression

(Exponentially increasing or decreasing terms)

Ex: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

$$a_n = \frac{1}{2^n}, \quad n \in \mathbb{N}$$

or

$$a_0 = 1$$
$$a_n = \frac{1}{2} \cdot a_{n-1}, \quad n \geq 1$$

Geometric Progression

$$a, ar, ar^2, \ldots$$

or

$$a_n = \underbrace{a}_{\text{Initial term}}\,\underbrace{r}_{\text{Common ratio}}{}^n, \quad n \in \mathbb{N}$$

Ex above: $a = 1$, $r = \frac{1}{2}$