DM549 and DS(K)820

Lecture 20: Permutations and Combinations

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Last Time: The Pigeonhole Principle and its Generalization

The Pigeonhole Principle (Theorem 6.2.1)

Let $k \geq 1$ be an integer. When $k + 1$ or more objects are placed into k boxes, there exists at least one box that contains at least two of the objects.

The Generalized Pigeonhole Principle (Theorem 6.2.2)

Let $N, k \geq 1$ be integers. When N or more objects are placed into k boxes, there exists at least one box that contains at least $\lceil N/k \rceil$ of the objects.

Overview of Today's Lecture

Topics today:

- Permutations
- Combinations
- Binomial Coefficients
- Relations between Binomial Coefficients

These topics can be found in Sections 6.3 and 6.4 of Rosen's book.

Towards permutations

Tournament:

- Consider a tournament with five teams.
- Suppose that, in the final ranking, there are no ties.
- How many possibilities are there for the final ranking?
- How many possibilities are there for the top three places of the final ranking?

Permutations

Definition (Permutations)

Let S be a finite set. For any integer r with $0 \leq r \leq |S|$, an *r-permutation* of S is an ordered arrangement of r distinct objects from S.

An $|S|$-permutation of S is simply called a permutation of S.

Computing the number of permutations (Theorem 6.3.1)

Let n and r be integers with $0 \leq r \leq n$. The number of r-permutations of a set with cardinality n is

$$P(n, r) = \frac{n!}{(n-r)!}.$$

Note: $0! = 1$.

Permutations: Recall from Earlier Lecture

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Recall from two lectures ago:

- There are $P(5, 2)$ ways of assigning two new employees to five free offices.
- There are $P(n, m)$ one-to-one functions $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$ when $m \leq n$.

Permutations: Examples

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Examples:

- How many ways are there to arrange 3 out of 6 people in a line?
- Suppose you draw 5 cards one by one from a standard deck of 52 cards. How many sequences are there?

Towards Combinations

Tournament again:

- Consider again a tournament with five teams.
- Suppose again that, in the final ranking, there are no ties.
- Before: How many different final rankings for the top three teams are there?

123	124	125	132	134	135	142	143	145	152	153	154
213	214	215	231	234	235	241	243	245	251	253	254
312	314	315	321	324	325	341	342	345	351	352	354
412	413	415	421	423	425	431	432	435	451	452	453
512	513	514	521	523	524	531	532	534	541	542	543

- Now: How many different sets of teams can constitute the top three teams? (Here, the order does not matter.)
  - 123, 132, 213, 231, 312, 321
  - 124, 142, 214, 241, 412, 421
  - 125, 152, 215, 251, 512, 521
  - 134, 143, 314, 341, 413, 431
  - 135, 153, 315, 351, 513, 531
  - 145, 154, 415, 451, 514, 541
  - 234, 243, 324, 342, 423, 432
  - 235, 253, 325, 352, 523, 532
  - 245, 254, 425, 452, 524, 542
  - 345, 354, 435, 453, 534, 543

Combinations

Definition (Combinations)

Let S be a finite set. For any integer r with $0 \leq r \leq |S|$, an *r-combination* of S is a subset S' of S with $|S'| = r$.

Computing the number of combinations (Theorem 6.3.2)

Let n and r be integers with $0 \leq r \leq n$. The number of r-combinations of a set with cardinality n is

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r! \cdot (n-r)!}.$$

Note: We also denote $C(n, r)$ as

$$\binom{n}{r} \quad \text{(read: "n choose r")}$$

and call it *binomial coefficient*.

Combinations: Examples

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Examples:

- How many possible hands are there in Texas Hold'em (two cards out of 52)?
- How many ways are there to select four out of five different cookies?

Poll: Question 1/3

A test question:

- Suppose someone is trying to answer a test question.
- There are eight possible answers, and each is to be marked as either correct or incorrect.
- The person is unprepared but thinks that probably four of the answers are correct.
- How many ways are there of selecting four out of the eight answers as correct?

Poll Everywhere:

Poll: Question 2/3

Assigning cookies:

- I have six different cookies and three guests.
- I want to offer each guest a single cookie (and keep the three remaining ones).
- In how many different ways can I do this?

Poll Everywhere:

Poll: Question 3/3

Bit strings strike back:

- How many bit strings of length eight are there that contain exactly two 1s?

Poll Everywhere:

An Identity

Corollary 6.3.2

Let n, r be integers with $0 \leq r \leq n$. Then it holds that

$$\binom{n}{r} = \binom{n}{n-r}.$$

Two proofs:

- a proof by algebraic manipulations.
- a combinatorial proof.

Another Identity

Corollary 6.4.1

Let $n \geq 0$ be an integer. Then it holds that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Why "Binomial" Coefficients?

Question: How to compute $(x+y)^n$ in general?

The Binomial Theorem (Theorem 6.4.1)

Let x and y be variables and, let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} \cdot x^{n-j} y^j.$$

Note: The right-hand side is the same as

$$\binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1} y + \cdots + \binom{n}{n-1} \cdot x y^{n-1} + \binom{n}{n} \cdot y^n.$$

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
 & & & & + & & & & & & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

Pascal's identity (Theorem 6.4.2)

Let n, k be integers with $1 \leq k \leq n$. Then it holds that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

There is much more to Pascal's Triangle...

1

1 1

1 2 1

1 3 3 3

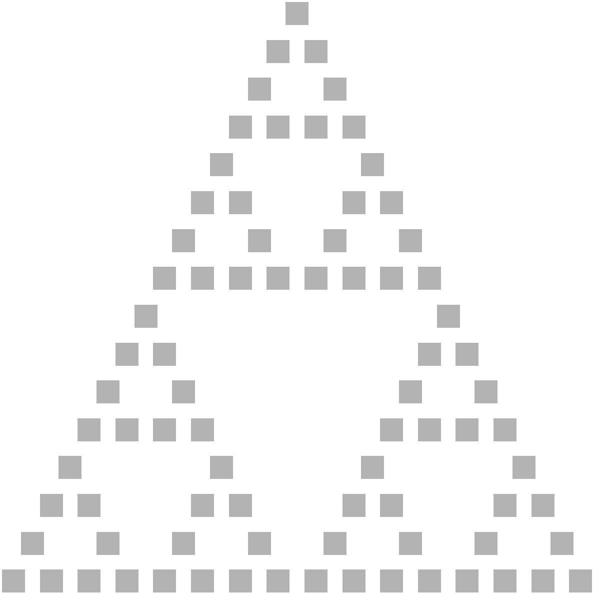
1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

There is much more to Pascal's Triangle...



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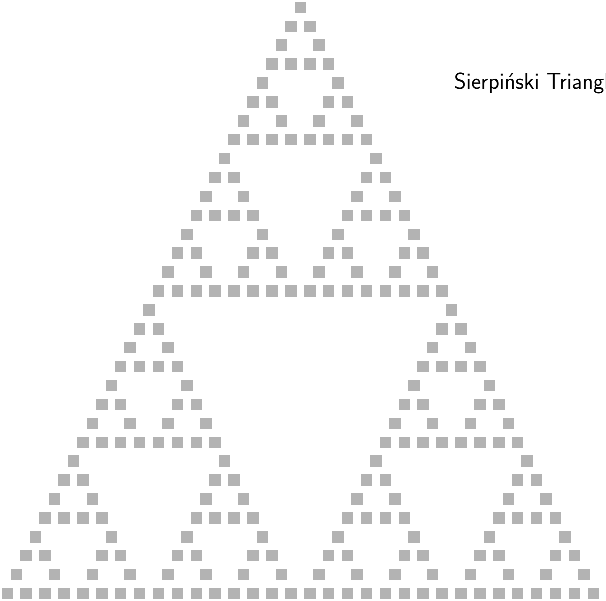

Sierpiński Triangle