DM549 and DS(K)820

Lecture 18: Simple Counting Techniques

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Counting

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227, 228, 229, 230, 231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 251, 252, 253, 254, 255, 256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270, 271, 272, 273, 274, 275, 276, 277, 278, 279, 280, 281, 282, 283, 284, 285, 286, 287, 288, 289, 290, 291, 292, 293, 294, 295, 296, 297, 298, 299, 300, 301, 302, 303, 304, 305, 306, 307, 308, 309, 310, 311, 312, ...

OK, what is it really?

- We want to count the number of objects with a certain property.
- Equivalently: We want to determine the cardinality of a finite set, where the set is given not explicitly.
- Examples (we will learn solutions to all of them):
  - How many different start hands are there in your favorite card game?
  - How many ways are there to split $3k$ people into groups of size 3 (where $k \in \mathbb{N}$)?
  - What is the number of surjective (onto) functions $f : A \to B$?
- Solutions can get difficult, but there are many tricks that will make your life easier.
- Branch of Mathematics: Combinatorics.

Application 1: Computing Probabilites

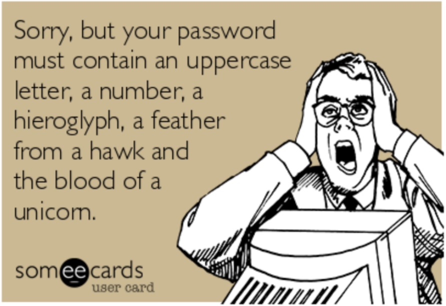
- Recall from high school: Probability of an event $\mathcal{E}$ is

$$\frac{\#\text{favorable elementary events w.r.t. } \mathcal{E}}{\#\text{elementary events}}.$$

- Example: Probability of rolling an even number with a single die roll is $3/6 = 1/2$.
- To compute probability with the above formula in general, we have to *count* both the number of favorable elementary and all elementary events.

Application 2: Password Security

- Number of possible passwords ≈ security of a password (if all passwords are picked with equal probability).
- But what is total number of passwords?
- Rules can be quite difficult:



Application 3: Running Time of Algorithms

- Apart from its correctness, *running time* is usually second most interesting property of an algorithm.
- To analyze running time, we have to *count* number of steps that the algorithm executes.
- We will not analyze running times explicitly here, but techniques from here will be useful for that.

Today

Simple Counting Techniques:

- Product Rule,
- Sum Rule,
- Subtraction Rule,
- Division Rule,
- Tree Diagrams.

All are in Chapter 6.1 in Rosen's book.

An Example Towards the First Rule

Baking cookies:

- I have seven cookie cutters available: star, circle, heart, dog, Christmas tree, gingerbread, snow man.
- I can color them in five different ways: no color, yellow, blue, green, red.
- Any two cookies need to be cut out by different cutters or have a different color.
- How many cookies can I make?

The Product Rule

The Product Rule (for two sets)

For any finite sets S_1, S_2 it holds that $|S_1 \times S_2| = |S_1| \cdot |S_2|$

The Product Rule

For any finite sets $S_1, S_2, \ldots, S_n$, it holds that

$$\underbrace{\left| \bigtimes_{i=1}^{n} S_i \right|}_{|S_1 \times S_2 \times \cdots \times S_n|} = \underbrace{\prod_{i=1}^{n} |S_i|}_{|S_1| \cdot |S_2| \cdot \cdots \cdot |S_n|} .$$

The Product Rule: More Examples

The Product Rule

For any finite sets $S_1, S_2, \ldots, S_n$, it holds that

$$\underbrace{\left| \bigtimes_{i=1}^{n} S_i \right|}_{|S_1 \times S_2 \times \cdots \times S_n|} = \underbrace{\prod_{i=1}^{n} |S_i|}_{|S_1| \cdot |S_2| \cdot \cdots \cdot |S_n|} .$$

License plates (simplified rules):

- License plates consist of two English upper-case letters and a number between 0 and 99,999.
- How many distinct license plates can be formed?

Number of subsets:

- Consider some finite set S.
- How many subsets of S are there?

The Product Rule: Even More Examples

The Product Rule

For any finite sets $S_1, S_2, \ldots, S_n$, it holds that

$$\underbrace{\left|\bigtimes_{i=1}^{n} S_i\right|}_{|S_1 \times S_2 \times \cdots \times S_n|} = \underbrace{\prod_{i=1}^{n} |S_i|}_{|S_1| \cdot |S_2| \cdot \cdots \cdot |S_n|} .$$

Assigning offices:

- Suppose a company has two new employees and five free offices.
- How many ways are there of assigning the two employees two distinct offices out of the free ones?

Number of one-to-one functions:

- Consider functions $f : M \to N$ that are injective (one-to-one) where M and N are finite sets.
- How many such functions are there (depending on M and N)?

An Example Towards the Second Rule

Ordering beers:

- Suppose a bar has 20 beers on tap and 22 beers from bottles.
- No beer that is both available on tap and from the bottle.
- How many different beers can I order?

The Sum Rule

The Sum Rule (for two sets)

For any finite sets S_1, S_2 with $S_1 \cap S_2 = \emptyset$, it holds that $|S_1 \cup S_2| = |S_1| + |S_2|$.

The Sum Rule

For any finite sets $S_1, S_2, \ldots, S_n$ where $S_i \cap S_j = \emptyset$ for any $i \neq j$, it holds that

$$\underbrace{\left|\bigcup_{i=1}^{n} S_i\right|}_{|S_1 \cup S_2 \cup \cdots \cup S_n|} = \underbrace{\sum_{i=1}^{n} |S_i|}_{|S_1|+|S_2|+\cdots+|S_n|} .$$

The Sum Rule: More Examples

The Sum Rule

For any finite sets $S_1, S_2, \ldots, S_n$ where $S_i \cap S_j = \emptyset$ for any $i \neq j$, it holds that

$$\underbrace{\left|\bigcup_{i=1}^{n} S_i\right|}_{|S_1 \cup S_2 \cup \cdots \cup S_n|} = \underbrace{\sum_{i=1}^{n} |S_i|}_{|S_1|+|S_2|+\cdots+|S_n|} .$$

Choosing an elective:

- You have to choose an elective from one of three lists.
- The list have 12, 13, and 17 electives, and no two lists share an elective.
- How many electives can you choose?

Ordering food at a restaurant:

- The menu has 66 options, but 64 of them are not vegan.
- Assuming you are vegan (and eat everything that is vegan), how many dishes can you order?

...arguably not extremely interesting just by itself.

A More Complicated Example

Passwords:

- You want to delete your account from a social network.
- To do so, you have to enter your password.
- You remember your password had length 6, 7, or 8. Each character was a lower-case English letter or a digit (no unicorn blood).
- You also know your password contained at least one digit.
  - Examples: `l0vemath`, `gnrpf45`, `123456`.
- How many passwords do you have to try?

Poll Everywhere

Auditorium:

- The seats in an auditorium shall be labelled.
- Each label consists of an English upper-case letter followed by a number from $\{0, 1, \ldots, 99\}$.
  - Example: A4, H0, K99.
- How many seats can be labelled differently?

Poll Everywhere:

Poll Everywhere

Auditorium at superstitious university:

- The seats in an auditorium shall be labelled.
- Each label consists of an English upper-case letter followed by a number from $\{0, 1, \ldots, 99\}$.
- Only difference: Neither the number 13 nor the digit 4 may appear.
    - Example: A7, H0, K99.
- How many seats can be labelled differently?

Poll Everywhere:

An Example Towards the Third Rule

Ordering beers, more complicated version:

- Suppose a bar has 20 beers on tap and 24 beers from bottles.
- Two of the beers are available both on tap and from the bottle, the rest is not.
- How many different beers can I order?

The Subtraction Rule

The Subtraction Rule

For any finite sets S_1, S_2, it holds that $|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|$

Note:

- The intuition is: The elements in $S_1 \cap S_2$ have been counted twice in $|S_1| + |S_2|$, so their cardinality has to be subtracted again.
- There is a generalization called "inclusion-exclusion principle".

The Subtraction Rule: Another Example

The Subtraction Rule

For any finite sets S_1, S_2, it holds that $|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|$

Bit Strings:

- How many bit strings of length eight start with a 1 bit or end with the two bits 00?
  - Examples: `10010110`, `11100000`, `00000000`.

An Example Towards the Last Rule

Counting cows:

- You are hiking past a meadow in which there are cows only (all of which have four legs).
- You count 572 legs in the meadow.
- How many cows are there in the meadow?

The Division Rule

The Division Rule

Suppose A is a finite set with $A = B_1 \cup B_2 \cup \cdots \cup B_n$ where

- $|B_i| = d$ for all i and
- $B_i \cap B_j = \emptyset$ for all $i \neq j$.

Then $n = |A|/d$.

The Division Rule: Example

The Division Rule

Suppose A is a finite set with $A = B_1 \cup B_2 \cup \cdots \cup B_n$ where

- $B_i \cap B_j = \emptyset$ for all $i \neq j$.
- $|B_i| = d$ for all i and

Then $n = |A|/d$.

Seating a group:

- Suppose four people are to be seated on a circular table.
- Two seatings are considered the same if everyone has the same left neighbor and the same right neighbor in both seatings.
- How many seatings that are considered different are there?