Relations

Sections 9.1, 9.3 - 9.6

Properties   Representations

Generalization of functions

Used in, e.g., databases

Ex:

(student number, name, education)   tertiary relation

(student number, course)

(parent, child)

$<$

$=$

binary relations

We will only work with <u>binary</u> relations:

> **Def. 9.1.1**
>
> A, B : sets
>
> A relation from A to B is a subset of $A \times B$

Ex above:

- A: set of valid student numbers
  B: set of courses, or
- $A = B = \mathbb{Z}$, or...

Relations are a generalization of functions:

- $f: A \to B$ assigns one element in B to each element in A
- A relation from A to B relation no, one, or more elements in B to each element in A

We will only consider relations with $A = B$:

Def. 9.1.2:

A: set

A relation on A is a subset of $A \times A$

Ex: A relation on $\mathbb{Z}$

$$R_{abs} = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid |a| = |b|\}$$
$$= \{(0,0), (-1,-1), (-1,1), (1,-1), (1,1), (-2,-2), \ldots\}$$

$(a,b) \in R$ is read „a is related to b via R" and may also be written aRb.

$\boxed{\text{Representations}}$ of relations (Section 9.3)

- Set representation
  - Enumeration
  - Set builder notation

  } Examples above
- Matrices
- Graphs

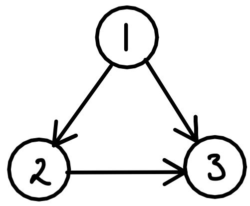
Ex:

$S = \{1,2,3\}$

$R_< = \{ (a,b) \in S \times S \mid a<b \}$

$$M_{R_<} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$G_{R_<}$:

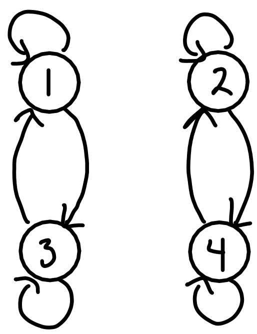


Ex:

Relation on $\{1,2,3,4\}$:

$R_p = \{ (a,b) \mid$ a and b have the same parity $\}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$



$\boxed{\text{Properties}}$ of relations (Section 9.1)

Reflexive, (anti)symmetric, transitive

Def. 9.1.3

R: relation on A

R is reflexive, if
$\forall a \in A: (a,a) \in R$

Ex:

✓: $\leq$, $=$, $|$, $\{(1,1),(2,2),(3,3)\}$ on $\{1,2,3\}$
same parity, same absolute value

✗: $<$, $\neq$, $\{(1,1),(2,1),(2,2)\}$ on $\{1,2,3\}$
parent of

How do you tell from the matrix or graph repr. whether a relation is reflexive?

Def. 9.1.4

R: relation on A

R is symmetric, if

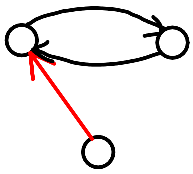
$$\forall a,b \in A: \big((a,b)\in R \Rightarrow (b,a)\in R\big)$$

Ex:

✓: $=$, $\neq$, same parity, same absolute value

✗: $<$, $\leq$, $|$, parent of

$\{(1,2),(2,1),(3,1)\}$



Def. 9.1.4:

R: relation on A

R is antisymmetric, if

$$\forall a,b \in A: \big((a,b)\in R \Rightarrow (b,a)\notin R \vee a=b\big)$$

$$\Updownarrow$$

$$\forall a,b \in A: \big((a,b)\in R \wedge (b,a)\in R \Rightarrow a=b\big)$$ def. in textbook

$$\Updownarrow$$

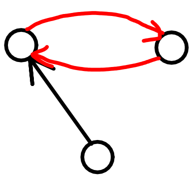
$$\forall a,b \in A: \big((a,b)\in R \wedge a\neq b \Rightarrow (b,a)\notin R\big)$$

Ex:

✓: $=$, $<$, $\leq$, $|$, parent of

✗: $\neq$, same parity, same absolute value

$\{(1,2),(2,1),(3,1)\}$



Def. 9.1.5:

R: relation on A

R is transitive, if

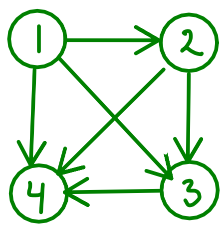
$\forall a,b,c \in A: \big((a,b) \in R \wedge (b,c) \in R \Rightarrow (a,c) \in R \big)$

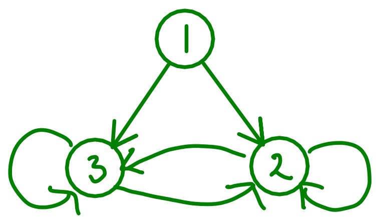
Ex:

✓: $\leq$, $<$, $=$,

$\{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$,

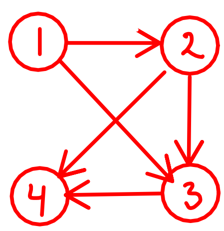
$\{(1,2),(1,3),(2,2),(3,2),(3,3)\}$

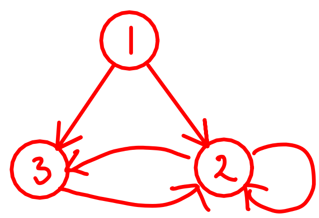




✗: $\neq$, $\{(1,2),(1,3),(2,3),(2,4),(3,4)\}$,

$\{(1,2),(1,3),(2,2),(3,2)\}$





Combinations of relations

Relations are sets and can be combined via set operations:

Ex: $R = \{(1,2), (3,1)\}$, $S = \{(1,2), (1,3), (2,2), (2,3)\}$

$$R \cap S = \{(1,2)\}$$
$$R \cup S = \{(1,2), (1,3), (2,2), (2,3), (3,1)\}$$
$$R - S = \{(3,1)\}$$
$$R \oplus S = \{(1,3), (2,2), (2,3), (3,1)\}$$

Relations are also a generalization of functions and can be composed like functions:

Def. 9.1.6

R: relation from A to B

S: relation from B to C

$$S \circ R = \{(a,c) \mid \exists b \in B : (a,b) \in R \land (b,c) \in S\}$$

Notice similarity:

$$(g \circ f)(x) = g(f(x))$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

Ex: $R = \{(1,2), (3,1)\}$, $S = \{(1,2), (1,3), (2,2), (2,3)\}$

$$S \circ R = \{(1,2), (1,3), (3,2), (3,3)\}$$

$$R \circ S = \{(1,1), (2,1)\}$$

$S = \{(1,2), (1,3), (2,2), (2,3)\}$, $R = \{(1,2), (3,1)\}$

$R = \{(1,2), (3,1)\}$, $R = \{(1,2), (3,1)\}$

$$R^2 = R \circ R = \{(3,2)\}$$

$$\boxed{\begin{array}{l} \underline{\text{Def 9.1.7}} \\ R^1 = R \\ R^{n+1} = R^n \circ R, \quad \text{for } n \geq 1 \end{array}}$$

$$R^3 = R^2 \circ R = \{\} = R^4 = R^5 = \dots$$

$S = \{(1,2), (1,3), (2,2), (2,3)\}$, $S = \{(1,2), (1,3), (2,2), (2,3)\}$

$$S^2 = S \circ S = \{(1,2), (1,3), (2,2), (2,3)\} = S$$
$$= S^3 = S^4 = \dots$$

Def 9.4.3:

$$R^* = \bigcup_{i=1}^{\infty} R^i \quad = R \cup R^2 \cup R^3 \cup \cdots$$

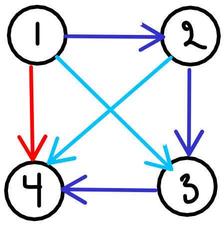
is called the connectivity relation

Ex: $R = \{(1,2), (2,3), (3,4)\}$

$R^2 = \{(1,3), (2,4)\}$

$R^3 = \{(1,4)\}$

$R^4 = \{\} = R^5 = R^6 = \ldots$



$$R^* = R \cup R^2 \cup R^3$$
$$= \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$$

Theorem 9.4.1:

$(a,b) \in R^k \iff \exists$ a,b-path of length k in G_R

Proof: Induction on k

Basis: $(a,b) \in R \iff a \to b$

Ind. hyp.: $(a,b) \in R^{k-1} \iff \underbrace{a \rightsquigarrow b}_{k-1 \text{ edges}}$

Ind. step:

$$
\begin{aligned}
& (a,b) \in R^k \\
\text{(def. of } R^k\text{)} \quad \Updownarrow \quad & \\
& \exists c:\ (a,c) \in R^{k-1} \ \wedge\ (c,b) \in R \\
\text{(ind. hyp.)} \quad \Updownarrow \quad & \\
& \exists c:\ \underbrace{a \rightsquigarrow c}_{k-1 \text{ edges}} \ \wedge\ c \to b \\
\Updownarrow \quad & \\
& \underbrace{a \rightsquigarrow b}_{k \text{ edges}}
\end{aligned}
$$

$\square$

R: relation on a finite set A

$$R^* = \bigcup_{i=1}^{|A|} R^i$$

<u>Proof</u>:

Let $n = |A|$

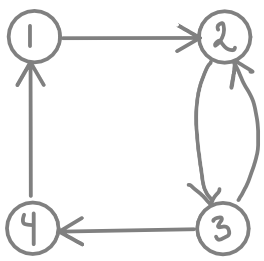
Then G_R has exactly n vertices.

Thus, any path of length $\ell > n$ in G_R contains a cycle.

Hence, for any path of length $\ell > n$ from a vertex a to a vertex b, there is also a path of length $\ell' \leq n$ from a to b.

$\square$

Ex:



Path 1, 2, 3, 2, 3, 4 $\rightarrow$

Path 1, 2, 3, 4