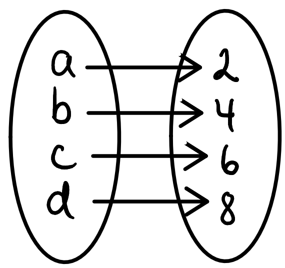
Cardinality

Section 2.5

Def. 2.5.1

$|A| = |B| \iff \exists$ bijection $f: A \to B$

Ex: $|\{a,b,c,d\}| = |\{2,4,6,8\}|$



Ex: $|\mathbb{Z}^+| = |E^+|$:

$$
\begin{aligned}
1 &\longrightarrow 2\\
2 &\longrightarrow 4\\
3 &\longrightarrow 6\\
4 &\longrightarrow 8\\
&\vdots \qquad \vdots
\end{aligned}
$$

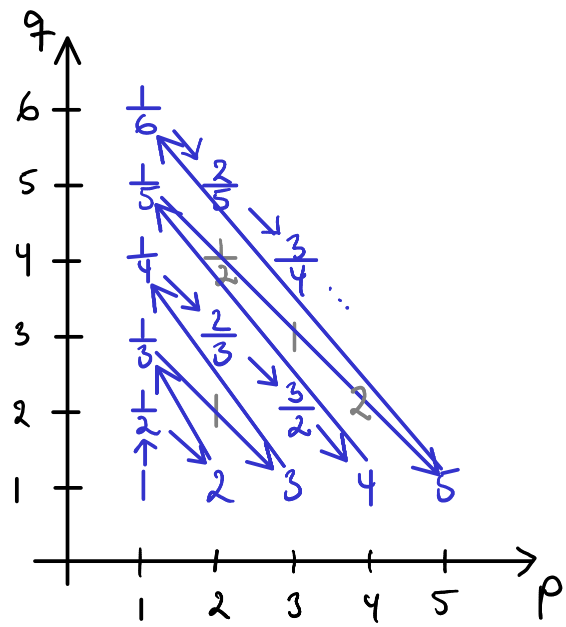
$E^+ = \{2n \mid n \in \mathbb{Z}^+\}$

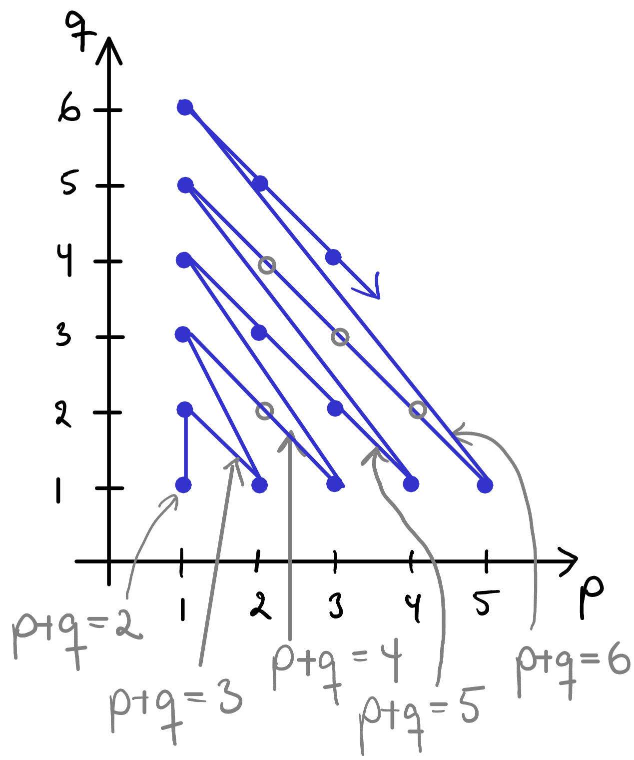
$|\mathbb{Z}^+| = |\mathbb{Z}|$:

$$
\begin{aligned}
1 &\longrightarrow 0\\
2 &\longrightarrow -1\\
3 &\longrightarrow 1\\
4 &\longrightarrow -2\\
5 &\longrightarrow 2\\
&\vdots \qquad \vdots
\end{aligned}
$$

		Cardinality	Ex
Countable (Tællelig)	Finite (endelig)	# elements	$\{2,4,6,8\}$
	Countably infinite (tællelig uendelig)	$\aleph_0$	$E^+, \mathbb{Z}^+, \mathbb{Z}$
	Uncountable (overtællelig)		

Ex 2.5.4: $\mathbb{Q}^+ = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}^+ \right\}$





$1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, 4, \frac{1}{5}, 5, \ldots$

Thus, $\mathbb{Q}^+$ is countable, i.e., $|\mathbb{Q}^+| = \aleph_0$.

$\mathbb{Q}$ is also countable:

Enumerate $\mathbb{Q}^-$ like $\mathbb{Q}^+$.
Start with 0 and then merge the enumerations of $\mathbb{Q}^+$ and $\mathbb{Q}^-$, like we did with $\mathbb{Z}^+$ and $\mathbb{Z}^-$:

$0, -1, 1, -\frac{1}{2}, \frac{1}{2}, -2, 2, -\frac{1}{3}, \frac{1}{3}, -3, 3, \ldots$

Ex 2.5.5: $\mathbb{R}$ is uncountable

Proof

$\underbrace{\text{Assume for the sake of contradiction}}_{\text{(Antag til modstrid)}}$ that $\mathbb{R}$ is countable.

Then, by Exercise 2.5.16, $R = \{ r \in \mathbb{R} \mid 0 < r < 1 \}$ is also countable, since $R \subseteq \mathbb{R}$.

And $R_{01} = \{ r \in R \mid \text{all digits of } r \text{ are } 0\text{'s or } 1\text{'s} \}$ is also countable.

Hence, the numbers in R_{01} can be enumerated:

$$
\begin{aligned}
r_1 &= 0, d_{11} d_{12} d_{13} \cdots \\
r_2 &= 0, d_{21} d_{22} d_{23} \cdots \\
r_3 &= 0, d_{31} d_{32} d_{33} \cdots \\
&\vdots
\end{aligned}
$$

But no matter which enumeration we choose, there exists a number $s \in R_{01}$ which is not part of the enumeration:

$s = 0, d_1 d_2 d_3 \cdots$, where

$$d_i = \begin{cases} 0, & \text{if } d_{ii} = 1 \\ 1, & \text{if } d_{ii} = 0 \end{cases}$$

This contradicts the assumption that R_{01} is countable. $\square$

We proved:

$$\mathbb{R} \text{ countable} \Rightarrow R \text{ countable} \Rightarrow \underbrace{R_{01} \text{ countable}}_{\text{False}}$$

		Cardinality	Ex
Countable	Finite	# elements	$\{2, 4, 6, 8\}$
	Countably infinite	$\aleph_0$	$\mathbb{Z}, E^+, \mathbb{Q}$
	Uncountable	$\aleph_1, \aleph_2, \ldots$	$\mathbb{R}$

Recursive Definitions

Section 5.3

Ex: Rabbits on an island

A pair of newborn rabbits are put on a rabbit-deserted island.

From the age of two months, a pair of rabbits produce another pair of rabbits every month.

No rabbits die.

At the beginning of the n'th month, there are f_n rabbit pairs on the island.

n:	0	1	2	3	4	5	6	7	8	9	10	11	12	...
f_n:	0	1	1	2	3	5	8	13	21	34	55	89	144	...

Fibonacci Numbers

$f_0 = 0$

$f_1 = 1$

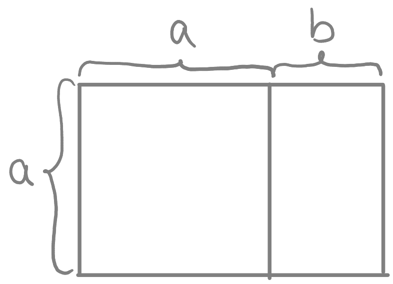
$f_n = f_{n-1} + f_{n-2}$, for $n \geq 2$

Strong induction   Section 5.2

Ex 5.3.4: The Fibonacci numbers grow exponentially:

$\boxed{f_n > \varphi^{n-2}, \quad n \geq 3}$, where $\varphi = \frac{1+\sqrt{5}}{2} \approx 1{,}618$

is called the golden ratio (gyldne snit):



$$\frac{a+b}{a} = \frac{a}{b} \iff \frac{a}{b} = \varphi$$

φ has more interesting properties.
One of them will be useful in the proof of
$f_n > \varphi^{n-2}$:

$$\varphi^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+5+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2} = 1 + \frac{1+\sqrt{5}}{2}$$

$\Downarrow$

$$\varphi^2 = 1 + \varphi \quad (*)$$

Let's check the inequality $f_n > \varphi^{n-2}$ for the first few values of n:

n=3:

$$f_3 = 2$$

$$\varphi^{3-2} = \varphi \approx 1.6 < f_3 \qquad (**)$$

n=4:

$$f_4 = 3$$

$$\varphi^{4-2} = \varphi^2 = 1+\varphi \approx 2.6 < f_4 \qquad (***)$$

n=5:

$$\begin{aligned} f_5 &= f_3 + f_4 \\ &> \varphi + \varphi^2, \quad \text{by } (**) \text{ and } (***) \\ &= \varphi(1+\varphi) \\ &= \varphi \cdot \varphi^2, \quad \text{by } (*) \\ &= \varphi^3 \qquad (****) \end{aligned}$$

n=6:

$$\begin{aligned} f_6 &= f_4 + f_5 \\ &> \varphi^2 + \varphi^3, \quad \text{by } (***) \text{ and } (****) \\ &= \varphi^2(1+\varphi) \\ &= \varphi^2 \cdot \varphi^2, \quad \text{by } (*) \\ &= \varphi^4 \end{aligned}$$

Lets try to generalize this.

$$\boxed{f_n > \varphi^{n-2}, \quad \text{for } n \geqslant 3}$$

Proof: (strong induction on n)

Basis step ($n=3$ and $n=4$):

$f_3 = 2$

$\varphi^{3-2} = \varphi \approx 1.6 < f_3$

$f_4 = 3$

$\varphi^{4-2} = \varphi^2 = 1+\varphi \approx 2.6 < f_4$

Induction hypothesis ($n=k-1$ and $n=k$, $k \geqslant 4$):

$f_{k-1} > \varphi^{k-3}$ and $f_k > \varphi^{k-2}$

Inductive step ($n=k+1$, $k \geqslant 4$):

$$\begin{aligned} f_{k+1} &= f_{k-1} + f_k \\ &> \varphi^{k-3} + \varphi^{k-2}, && \text{by the ind. hyp.} \\ &= \varphi^{k-3}(1+\varphi) \\ &= \varphi^{k-3} \cdot \varphi^2, && \text{by } (*) \\ &= \varphi^{k-1} \end{aligned}$$

$\square$

Letting $P(n) \equiv f_n > \varphi^{n-2}$, we just proved

$$P(3) \wedge P(4)$$

$$P(3) \wedge P(4) \Rightarrow P(5)$$

$$P(4) \wedge P(5) \Rightarrow P(6)$$

$$P(5) \wedge P(6) \Rightarrow P(7)$$

$$\vdots$$

Thus, we used a more general formulation of the principle of induction:

Strong induktion (Section 5.2)

Prove $P(n)$, $n \geq m$:

Basis step:

Prove $P(m), P(m+1), \dots, P(m+\ell)$, $\ell \geq 0$

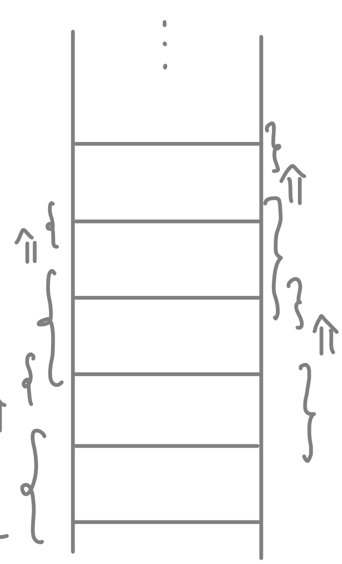
Inductive step:

Prove $P(m) \wedge P(m+1) \wedge \cdots \wedge P(k) \Rightarrow P(k+1)$, $k \geq m+\ell$

Earlier we compared induction to a domino effect. One could also think of an infinitely high ladder:

- In the basis step, we prove that steps $m, m+1, \dots, m+\ell$ are reachable.
- In the inductive step, we prove that if step $m, m+1, \dots, k$ are reachable, then step $k+1$ is also reachable.

Fibonacci example:



Basis: the two lowest steps are reachable

$f_8 > \varphi^6$

$f_7 > \varphi^5$

$f_6 > \varphi^4$

$f_5 > \varphi^3$

$f_4 > \varphi^2$

$f_3 > \varphi$

Ex: Any integer $n \geq 4$ can be written as a sum of 2's and 5's, i.e.,

$$\forall n \in \mathbb{N}, n \geq 4 : \underbrace{\exists\, a, b \in \mathbb{N} : n = 2a + 5b}_{P(n)}$$

Let's first check the smallest values of n:

$$\begin{aligned}
4 &= 2 \cdot 2 + 5 \cdot 0 \\
5 &= 2 \cdot 0 + 5 \cdot 1 \quad +2 \\
6 &= 2 \cdot 3 + 5 \cdot 0 \quad +2 \\
7 &= 2 \cdot 1 + 5 \cdot 1 \quad +2 \\
8 &= 2 \cdot 4 + 5 \cdot 0 \quad +2 \\
9 &= 2 \cdot 2 + 5 \cdot 1
\end{aligned}$$

$$\boxed{\forall n \in \mathbb{N},\ n \geq 4 \ : \ \exists\, a, b \in \mathbb{N} : \ n = 2a + 5b}$$

<u>Proof</u>: (strong induction on n)

Basis step ($n=4$ and $n=5$):

$$4 = 2 \cdot 2 + 5 \cdot 0$$
$$5 = 2 \cdot 0 + 5 \cdot 1$$

Induction hypothesis ($n = k-1$ and $n = k$, $k \geq 5$):

$$k-1 = 2a + 5b, \quad a, b \in \mathbb{N}$$

Inductive step ($n = k+1$, $k \geq 5$):

$$\begin{aligned} k+1 &= (k-1) + 2 \\ &= 2a + 5b + 2, \quad a, b \in \mathbb{N} \\ &= 2(a+1) + 5b, \quad a+1, b \in \mathbb{N} \end{aligned}$$

$\square$

We proved

$P(4)$ and $P(5)$

$P(4) \Rightarrow P(6) \Rightarrow P(8) \Rightarrow P(10) \Rightarrow \ldots$

$P(5) \Rightarrow P(7) \Rightarrow P(9) \Rightarrow P(11) \Rightarrow \ldots$

Thus, if we had only proven $P(4)$ in the basis step, we had proven the statement only for <u>even</u> integers $n \geq 4$.