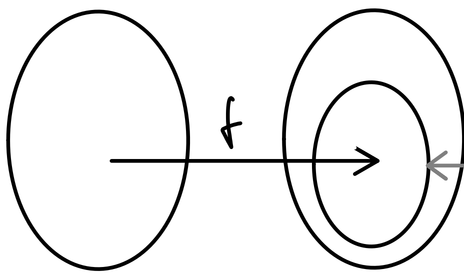
Functions

Section 2.3

Domain, Dom(f)
(Definitionsmængde, Dm(f))

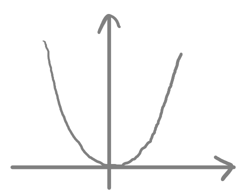
Codomain
(Sekundærmængde)



Image/range, Im(f)
(Værdimængde/billedmængde, Vm(f))

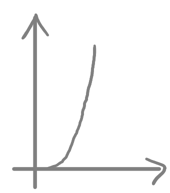
Ex:

$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$



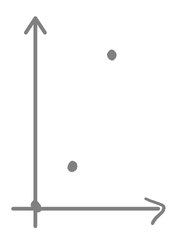
$\text{Im}(f) = \mathbb{R}^+ \cup \{0\}$

$g : \mathbb{R}^+ \to \mathbb{R}, \quad g(x) = x^2$



$\text{Im}(f) = \mathbb{R}^+$

$h : \mathbb{N} \to \mathbb{N}, \quad h(x) = x^2$



$\text{Im}(f) = \{0, 1, 4, 9, \dots\}$

Def 2.3.5:

$f: X \to Y$ is **injective / one-to-one** (injektiv / en-til-en), if

$$\forall x_1, x_2 \in \mathrm{Dom}(f): \big(f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \big)$$

$$\Updownarrow$$

$$\forall x_1, x_2 \in \mathrm{Dom}(f): \big(x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \big)$$

An injective function is an **injection**

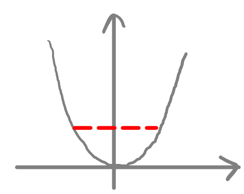
Ex (revisited)

$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$

$g: \mathbb{R}^+ \to \mathbb{R}, \quad g(x) = x^2$

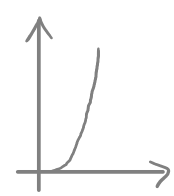
$h: \mathbb{N} \to \mathbb{N}, \quad h(x) = x^2$

f is not injective, since, e.g., $f(-1) = f(1)$.



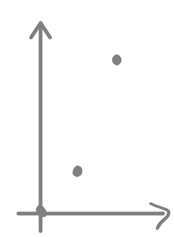
g is injective:

$$g(x_1) = g(x_2)$$
$$\Updownarrow$$
$$x_1^2 = x_2^2$$
$$\Updownarrow$$
$$x_1 = x_2 \quad \vee \quad x_1 = -x_2$$
$$\Updownarrow$$
$$x_1 = x_2, \quad \text{since} \quad x_1, x_2 = \mathbb{R}^+$$



Similarly:

h is injective



None of these functions are surjective...

Def. 2.3.7:

$f: X \to Y$ is **surjective / onto** (surjektiv / på), if

$\forall y \in Y : \exists x \in X : f(x) = y$

$\Updownarrow$

$\operatorname{Im}(f) = Y$

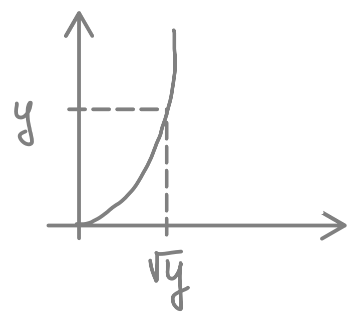
A surjective function is a **surjection**

Ex:

$f: \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = x^2$ is surjective:

For any $y \in \mathbb{R}^+$,

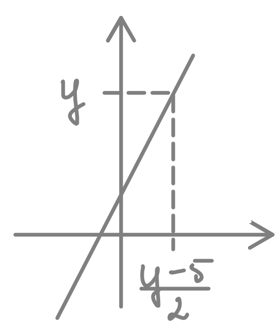
$$f(x) = y$$
$$\Updownarrow$$
$$x^2 = y$$
$$\Uparrow$$
$$x = \sqrt{y} \in \mathbb{R}^+ = \operatorname{Dom}(f)$$



$g: \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 5$ is surjective

For any $y \in \mathbb{R}^+$,

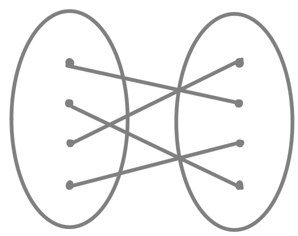
$$f(x) = y$$
$$\Updownarrow$$
$$2x + 5 = y$$
$$\Updownarrow$$
$$x = \frac{y-5}{2} \in \mathbb{R} = \operatorname{Dom}(f)$$



Def 2.3.8:

Bijective (bijektiv): Injective and surjective

Bijection / one-to-one correspondence (bijektion / en-til-en-korrespondance): A bijective function



Intuitively:

Injective: Each y-value is „hit" by at most one x-value
Surjective: Each y-value is „hit" by at least one x-value
Bijective: Each y-value is „hit" by exactly one x-value

The functions in the previous example are bijective, and therefore they are invertible:

Def. 2.3.9

For a bijection $f: X \to Y$, the inverse of f is the function $f^{-1}: Y \to X$ such that

$$\forall x \in X: \quad f(x) = y \iff f^{-1}(y) = x$$

Ex: $g: \mathbb{R} \to \mathbb{R}, \quad g(x) = 2x+5$

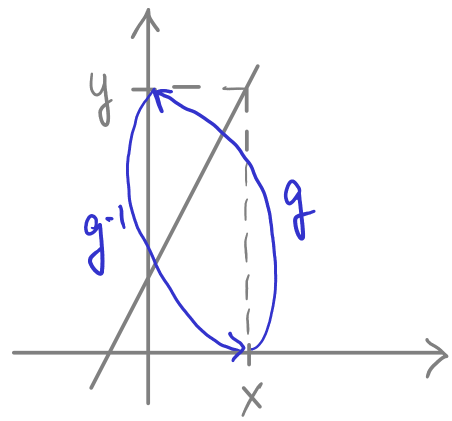
$$2x+5 = y$$
$$\Updownarrow$$
$$x = \frac{y-5}{2}$$

Thus,

$$g^{-1}(y) = \frac{y-5}{2}$$

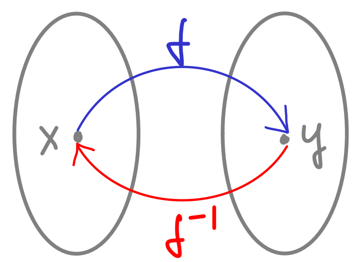
Or, renaming the variable,

$$g^{-1}(x) = \frac{x-5}{2}$$



Note:

$$\forall x \in X: \quad f^{-1}(f(x)) = x$$
$$\forall y \in Y: \quad f(f^{-1}(y)) = y$$



Thus, we can double-check the above solution:

$$g(g^{-1}(x)) = g\left(\frac{x-5}{2}\right) = 2\,\frac{x-5}{2} + 5 = x-5+5 = x$$
$$g^{-1}(g(x)) = g^{-1}(2x+5) = \frac{2x+5-5}{2} = x$$

Functions may be combined into new functions:

Def. 2.3.3

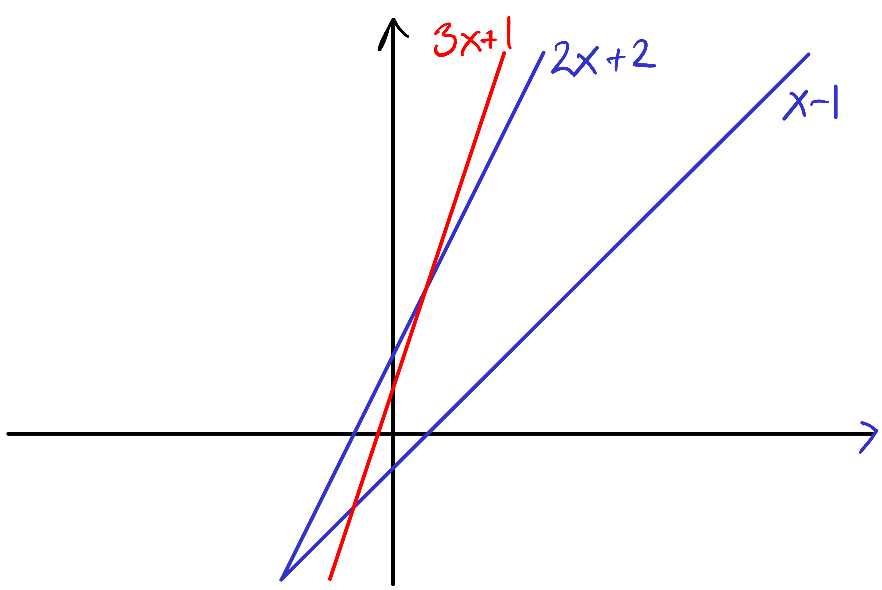
Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$

$$(f+g)(x) = f(x) + g(x)$$
$$(f \cdot g)(x) = f(x) \cdot g(x)$$

Ex:

$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = 2x+2$
$g: \mathbb{R} \to \mathbb{R}, \quad g(x) = x-1$

$$(f+g)(x) = 2x+2 + x-1 = 3x+1$$
$$(f \cdot g)(x) = (2x+2)(x-1) = 2x^2-2$$



Def. 2.3.10: Function composition

Let $f: A \to B$, $g: B \to C$

$$(g \circ f)(x) = g(f(x))$$

Ex: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2x+2$

$g: \mathbb{R} \to \mathbb{R}$, $g(x) = x-1$

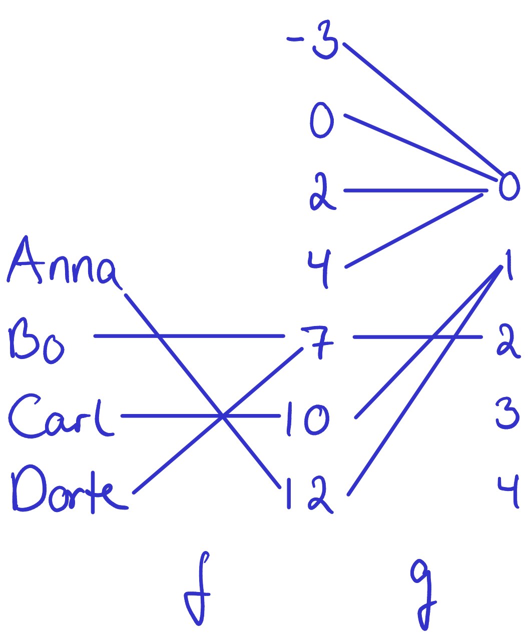
Here, $A = B = C$, so both $f \circ g$ and $g \circ f$ are defined.

$$(f \circ g)(x) = f(g(x)) = f(x-1) = 2(x-1)+2 = 2x$$

$$(g \circ f)(x) = g(f(x)) = g(2x+2) = 2x+2-1 = 2x+1$$

Ex: f: name $\to$ grade

g: grade $\to$ number



How many students got the same grade as Dorte?

$$(g \circ f)(\text{Dorte}) = g(f(\text{Dorte})) = g(7) = \underline{\underline{2}}$$

Def 2.3.6

A function is

(voksende)

- increasing if $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$
- strictly increasing if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
  (strengt voksende)

(aftagende)

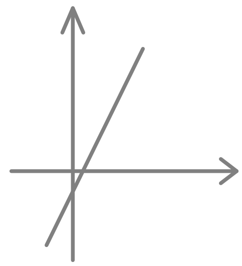
- decreasing if $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$
- strictly decreasing if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$
  (strengt aftagende)

- monotone if it is increasing or decreasing
  (monoton)

Note: A continuous function is injective, iff it is strictly increasing or decreasing.

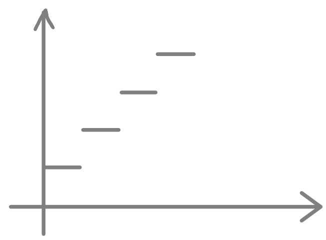
Ex:

$f(x) = 2x - 1$



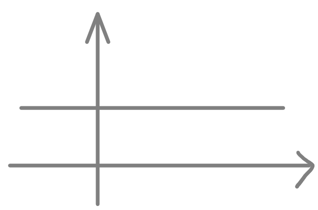
Strictly increasing

$f(x) = \lceil x \rceil$



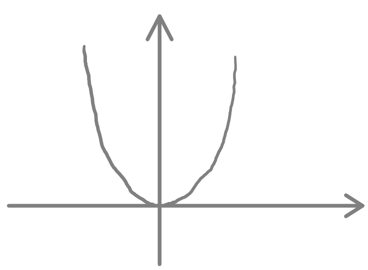
Increasing

$f(x) = 3$



Increasing and decreasing

$f(x) = x^2$



Not monotone

Cardinality

Section 2.5

$|A|$: Kardinaliteten af A

#elementer, hvis A er endelig

Ex:

$$|\{a,b,c,d\}| = 4$$

$$|\{2,4,6,8\}| = 4$$

$$E^+ = \{2n \mid n \in \mathbb{Z}^+\}$$

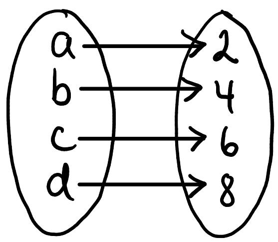
$$|E^+| = ?$$

$$|\mathbb{Z}^+| = ?$$

Def. 2.5.1

$$|A| = |B| \iff \exists \text{ bijection } f : A \to B$$

Ex: $|\{a,b,c,d\}| = |\{2,4,6,8\}|$



Ex: $|\mathbb{Z}^+| = |E^+|$

$$\begin{array}{ccc} 1 & \longrightarrow & 2 \\ 2 & \longrightarrow & 4 \\ 3 & \longrightarrow & 6 \\ 4 & \longrightarrow & 8 \\ \vdots & & \vdots \end{array}$$

Bijection:

$f : \mathbb{Z}^+ \to E^+$, $\quad f(n) = 2n$

So what is the cardinality of $\mathbb{Z}^+$ and E^+?

The Hebrew letter aleph

<u>Def 2.5.3</u>

The cardinality of $\mathbb{Z}^+$ is called $\aleph_0$. null / naught / zero

A set is **countable** (tællelig) if it is **finite** or has **cardinality** $\aleph_0$.
A set with **cardinality** $\aleph_0$ is called **countably infinite** (tælleligt uendelig).

If a set is **not countable**, it is **uncountable** (overtællelig).

		Cardinality	Ex
Countable	Finite	# elements	$\{2,4,6,8\}$
	Countably infinite	$\aleph_0$	$\mathbb{Z}^+, E^+$
	Uncountable		

Ex 2.5.3: $|\mathbb{Z}^+| = |\mathbb{Z}|$

$$\begin{array}{ccc} \mathbb{Z}^+ & & \mathbb{Z} \\ & & \\ \left.\begin{array}{ccc} 1 & \longrightarrow & 0 \\ 2 & \longrightarrow & -1 \\ 3 & \longrightarrow & 1 \\ 4 & \longrightarrow & -2 \\ 5 & \longrightarrow & 2 \\ 6 & \longrightarrow & -3 \\ 7 & \longrightarrow & 3 \\ & \vdots & \end{array}\right\} & \text{Infinite enumeration of } \mathbb{Z} & \end{array}$$

$0, -1, 1, -2, 2, -3, 3, \ldots$

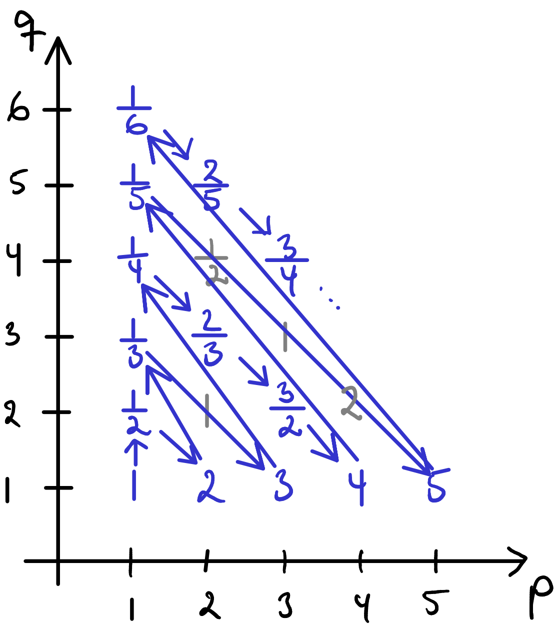
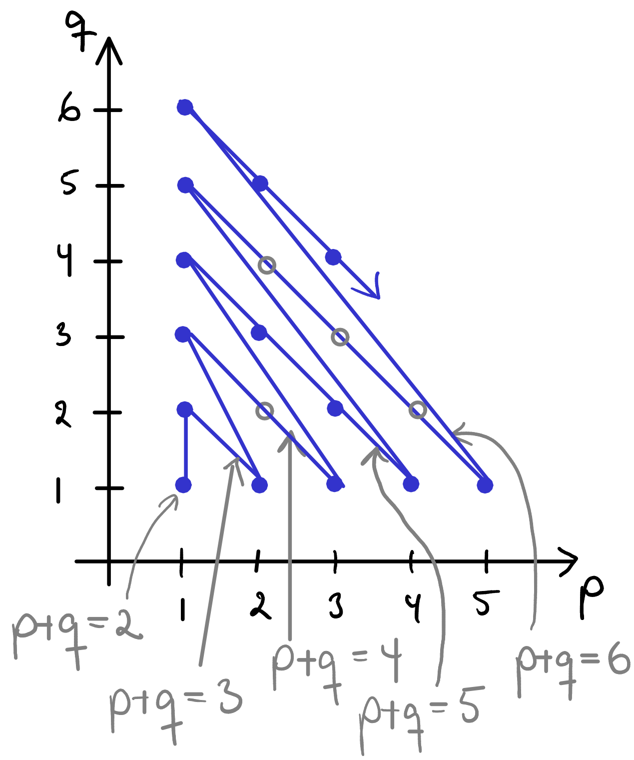
$f: \mathbb{Z}^+ \to \mathbb{Z}, \quad f(n) = (-1)^n \cdot \lfloor \frac{n}{2} \rfloor$

(bijection between $\mathbb{Z}^+$ and $\mathbb{Z}$ as illustrated above)

Thus, $|\mathbb{Z}| = \aleph_0$

Or, $\mathbb{Z}$ is countably infinite

Ex 2.5.4: $\mathbb{Q}^+ = \left\{ \frac{p}{q} \,\middle|\, p, q \in \mathbb{Z}^+ \right\}$

$1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, 4, \frac{1}{5}, 5, \ldots$

Thus, $\mathbb{Q}^+$ is countable, i.e., $|\mathbb{Q}^+| = \aleph_0$.