Lecture 17

A matrix is a rectangular array of numbers, e.g.

$$A = \underbrace{\begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}}_{3 \text{ columns}} \Big\} \text{ 2 rows}$$

A is 2×3 matrix and $a_{2,1} = 1$, $a_{1,3} = 4$.

Sum of matrices is defined entrywise:

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 2+1 & 3+1 & 4+0 \\ 1+0 & 0+1 & 2+0 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 4 \\ 1 & 1 & 2 \end{bmatrix}$$

This is only defined if A and B have the same sizes.

The product of matrices is more complicated:
Consider the system of equations

$$\begin{aligned} 2x_1 + 3x_2 \phantom{{}+x_3} &= 15 \\ 7x_2 + x_3 &= 1 \\ x_1 + 5x_2 - x_3 &= -7 \end{aligned}$$

We rewrite the left-hand side

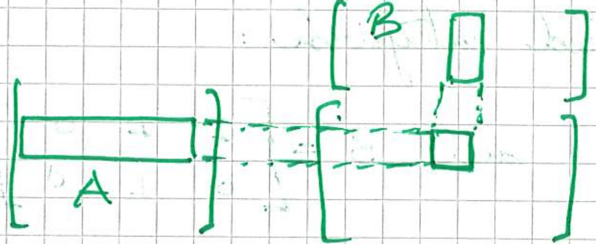
$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 7 & 1 \\ 1 & 5 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 0 \cdot x_3 \\ 0 \cdot x_1 + 7x_2 + x_3 \\ x_1 + 5x_2 - x_3 \end{bmatrix}$$

Another example

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\cdot1+3\cdot1+4\cdot2 & \boxed{2\cdot2+3\cdot1} & 3\cdot1+4\cdot1 \\ 1\cdot1+2\cdot2 & 1\cdot2 & 2\cdot1 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & \boxed{7} & 7 \\ 5 & 2 & 2 \end{bmatrix}$$

In general, it looks like



The order of multiplication is important:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } B \cdot A = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$$

There is an identity matrix $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

- $\begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{3 \text{ rows}} = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}$

- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}}_{2 \text{ columns}}$

The transpose:

- $$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 4 & 2 \end{bmatrix}$$

Rows become columns, columns become rows.

- $$\begin{bmatrix} 2 & 3 & 0 \\ 4 & 5 & 7 \\ 1 & -3 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 5 & -3 \\ 0 & 7 & 2 \end{bmatrix}.$$

For Boolean matrices, we interpret "0" as False, and "1" as True.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Boolean products are relevant for iterations of relations:

R:

$$\begin{tikzcd} 1 \arrow[r] \arrow[dr] & 2 \arrow[d] \\ 4 & 3 \arrow[l] \end{tikzcd}$$

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Observe that

$$A \odot A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and this is exactly the matrix representation for the square relation R^2.
Similarly, we can represent R^3 using $A \odot A \odot A = A^{[3]}$.

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