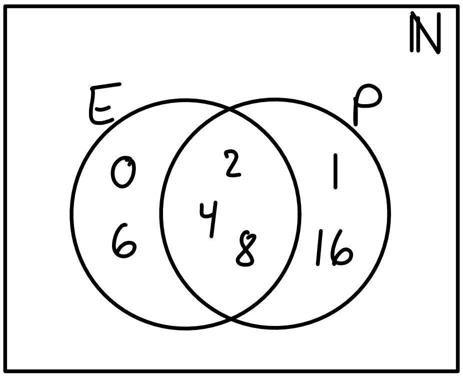
Sets

Sections 2.1-2.2

Ex :

$$E = \{2k \mid k \in \mathbb{N} \wedge k \leq 4\} = \{0, 2, 4, 6, 8\}$$

$$P = \underbrace{\{2^k \mid k \in \mathbb{N} \wedge k \leq 4\}}_{\text{Set builder notation}} = \underbrace{\{1, 2, 4, 8, 16\}}_{\text{enumeration}}$$



Universe

Venn diagram

Cardinality

Ex : $|\{2,4,6\}| = 3$

$|\{2, \{3,4\}, 5\}| = 3$

$|\{\}| = |\emptyset| = 0$

$|\{\mathbb{Z}^-, \mathbb{Z}^+\}| = 2$

Quiz

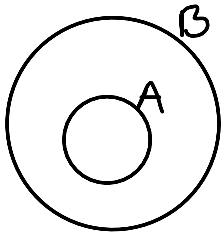
Def 2.1.3: Subset (Delmængde)

$A \subseteq B \iff \forall x : (x \in A \Rightarrow x \in B)$

„A is a subset of B"

„B is a superset of A"

(overmængde)



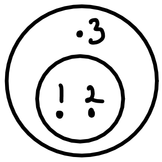
Ex: $\{1,2\} \subseteq \{1,2,3\}$

$\{1,2,3\} \not\subseteq \{1,2\}$

$\{1,2\} \subseteq \{1,2\}$

$\{1,2\} \subseteq \mathbb{N}$

$\mathbb{N} \subseteq \mathbb{Z}$



Theorem 2.1.6:

For any set S,

(i) $S \subseteq S$

(ii) $\emptyset \subseteq S$

Proof:

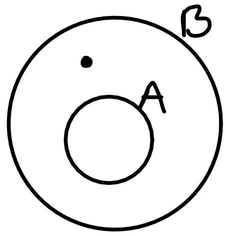
(i): $\forall x : x \in S \Rightarrow x \in S$

(ii): $\forall x : \underbrace{\underbrace{x \in \emptyset}_{F} \Rightarrow x \in S}_{S}$

$\square$

Def.: Proper subset

$A \subset B \iff A \subseteq B \wedge A \neq B$



Ex: $\{1,2\} \subset \{1,2,3\}$

$\{1,2,3\} \not\subset \{1,2,3\}$

$\emptyset \subset \{1,2,3\}$

$\mathbb{N} \subset \mathbb{Z}$

Def. 2.1.6: Power set (potensmængde)

$\mathcal{P}(S) = \{A \mid A \subseteq S\}$

Ex: $\mathcal{P}(\emptyset) = \{\emptyset\}$

Ex: $\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$

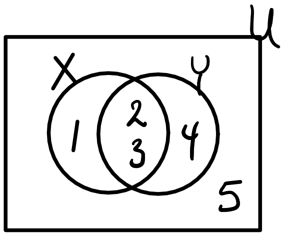
Ex: $\mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

Note that $|\mathcal{P}(S)| = 2^{|S|}$.

This can be proven by induction or by a technique from the last topic of the course, counting.

Operations

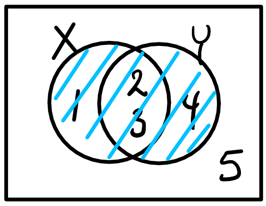
<u>Ex</u>: $X = \{1,2,3\}$
$Y = \{2,3,4\}$
$U = \{1,2,3,4,5\}$



Union (foreningsmængde)

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

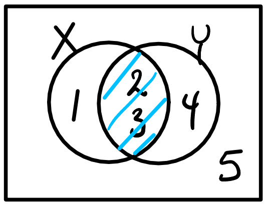
$$X \cup Y = \{1,2,3,4\}$$



Intersection (Fællesmængde / snitmængde)

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

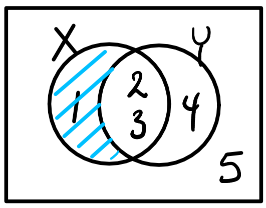
$$X \cap Y = \{2,3\}$$



(Set) difference („A fraregnet B")

$$A - B = A \setminus B = \{x \mid x \in A \wedge x \notin B\}$$

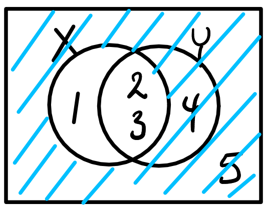
$$X - Y = \{1\}$$



Complement (Komplement)

$$\overline{A} = \{x \in U \mid x \notin A\}$$

$$X = \{4,5\}$$

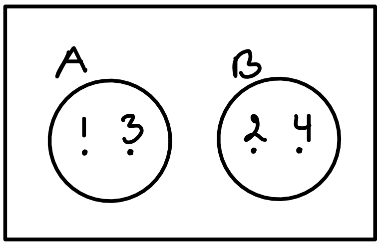


Def:

(disjunkte)

A and B are **disjoint** if $A \cap B = \emptyset$

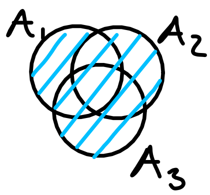
Ex:



Def 2.2.6:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

Ex:



Def 2.2.6:

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

Ex:

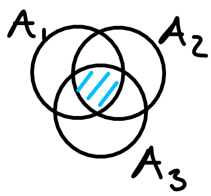


Table 2.2.1: Set identities

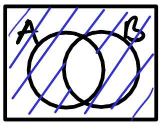
(p 140) Table 2.2.1 ~ Table 1.3.6: (p 29)

$\cup \sim \vee$

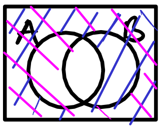
$\cap \sim \wedge$

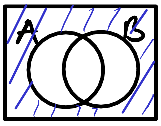
$\overline{} \sim \neg$

De Morgan's Laws:

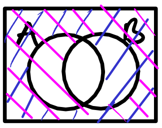


$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$





$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$



Def. 2.1.7: **tuple** (tupel)

An *n-tuple* is an *ordered* collection of n *not necessarily distinct* elements.

Ex: $(1,2)$ is a 2-tuple

$(1,2) \neq (2,1)$

$(1,2,2)$ is a 3-tuple

$(1,2,2) \neq (1,2)$

Def 2.1.8: **Cartesian product** (kartesisk produkt)

$$A \times B = \{ (a,b) \mid a \in A \wedge b \in B \}$$

Ex: $\{1,3,5\} \times \{2,4\} =$

$\{(1,2),(1,4),(3,2),(3,4),(5,2),(5,4)\}$

$\neq \{2,4\} \times \{1,3,5\} =$

$\{(2,1),(2,3),(2,5),(4,1),(4,3),(4,5)\}$

Note: $|A \times B| = |A| \cdot |B|$

More generally:

Def 2.1.9: Cartesian product (kartesisk produkt)

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$$

Ex: $\{1,2\} \times \{3,4,5\} \times \{6,7\} =$

$\{(1,3,6), (1,3,7), (1,4,6), (1,4,7), (1,5,6), (1,5,7),$
$(2,3,6), (2,3,7), (2,4,6), (2,4,7), (2,5,6), (2,5,7)\}$

$$A^n = A \times A \times \ldots \times A$$
$$= \{(a_1, a_2, \ldots, a_n) \mid a_1, a_2, \ldots, a_n \in A\}$$

Ex: $\{1,2\} \times \{1,2\} = \{(1,1), (1,2), (2,1), (2,2)\}$

Quiz

Functions

Section 2.3

Ex:

$f(x) = 2x + 5$

$f(x) = x^2$

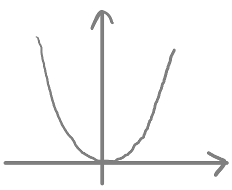
$f(x) = \frac{1}{x}$

Note: A function is not fully defined until we have specified its domain and codomain:

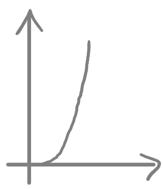
Def 2.3.1: A function $f : \overset{\text{domain}}{X} \to \overset{\text{codomain}}{Y}$ assigns exactly one element in Y to each element in X.

Ex:

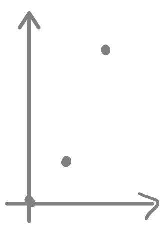
$f : \mathbb{R} \to \mathbb{R}, \; f(x) = x^2$



$g : \mathbb{R}^+ \to \mathbb{R}, \; g(x) = x^2$



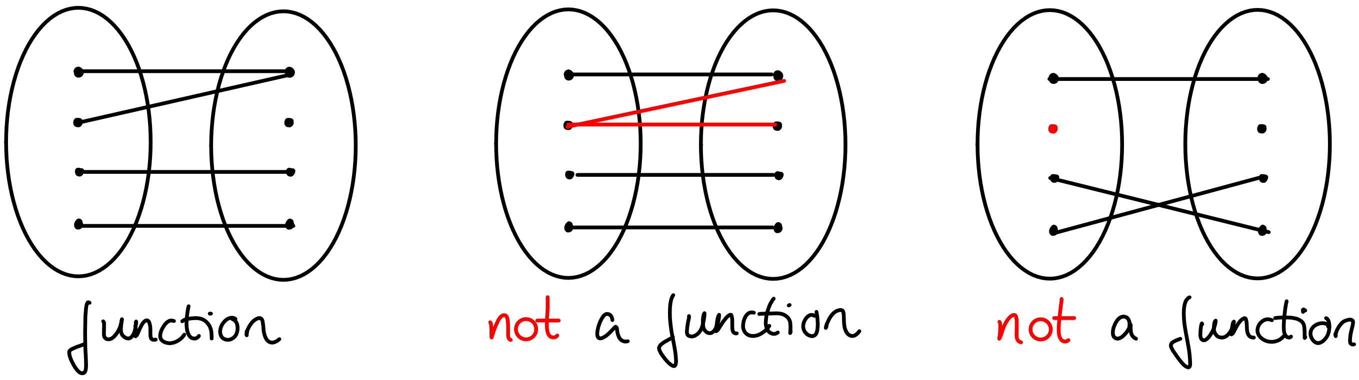
$h : \mathbb{N} \to \mathbb{N}, \; h(x) = x^2$



Def 2.3.1: (Again)

A function $f: X \to Y$ assigns exactly one element in Y to each element in X.

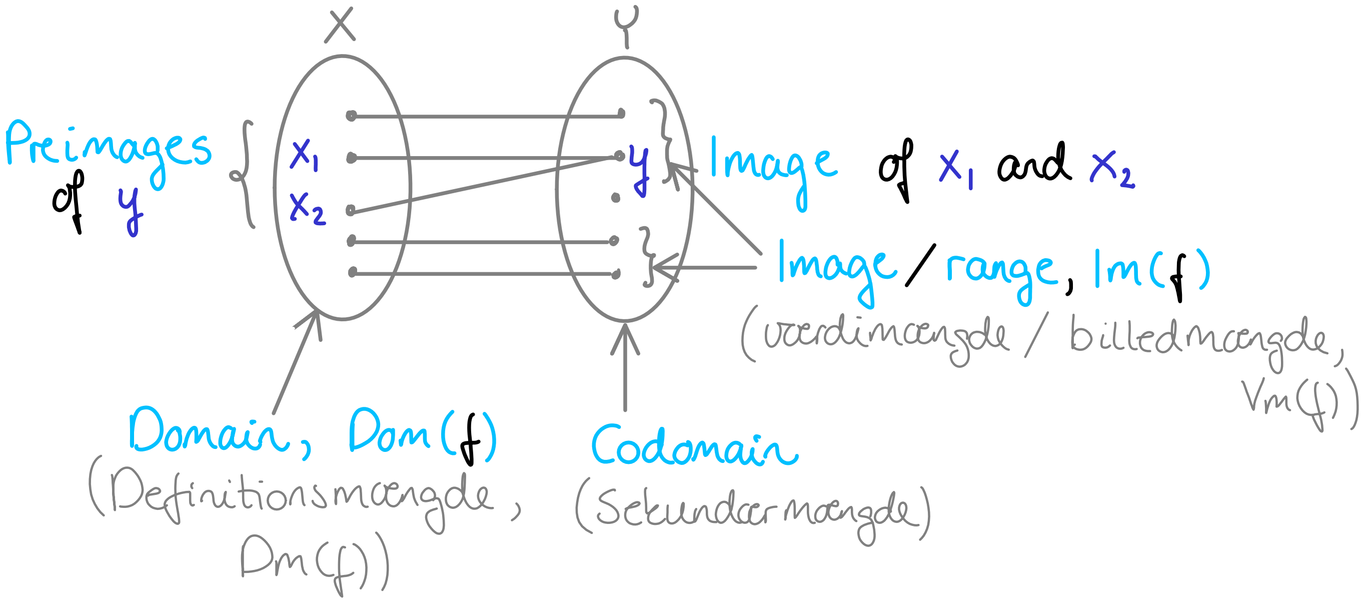
The word „exactly" is important:



Def 2.3.2 + 2.3.4 :

$f : X \to Y$

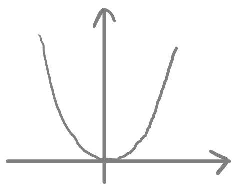
f maps X to Y



$$\mathrm{Im}(f) = \{ y \mid \exists x \in X : f(x) = y \} = \{ f(x) \mid x \in X \}$$

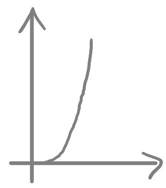
Ex :

$f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$



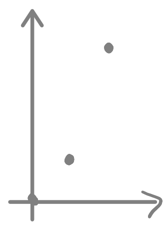
$\mathrm{Im}(f) = \mathbb{R}^+ \cup \{0\}$

$g : \mathbb{R}^+ \to \mathbb{R},\ g(x) = x^2$



$\mathrm{Im}(g) = \mathbb{R}^+$

$h : \mathbb{N} \to \mathbb{N},\ h(x) = x^2$



$\mathrm{Im}(h) = \{0, 1, 4, \dots\}$