Lecture 21

Example (permutations with repetition)

There are 26^8 different strings of length 8 of ~~distinct~~ English upper-case letters.

Example (combinations with repetition)

Choosing fruits: Apples (A), Oranges (O), Pears (P).

$\#A \mid \#O \mid \#P$

4A: $\ast\ast\ast\ast\,|\,|$ 4O: $|\,\ast\ast\ast\ast\,|$ 4P: $|\,|\,\ast\ast\ast\ast$

3A, 1O: $\ast\ast\ast\,|\,\ast\,|$ 3A, 1P: $\ast\ast\ast\,|\,|\,\ast$ 3O, 1A: $\ast\,|\,\ast\ast\ast\,|$

3O, 1P: $|\,\ast\ast\ast\,|\,\ast$ 3P, 1A: $\ast\,|\,|\,\ast\ast\ast$ 3P, 1O: $|\,\ast\,|\,\ast\ast\ast$

2A, 2O: $\ast\ast\,|\,\ast\ast\,|$ 2A, 2P: $\ast\ast\,|\,|\,\ast\ast$ 2O, 2P: $|\,\ast\ast\,|\,\ast\ast$

2A, 1O, 1P: $\ast\ast\,|\,\ast\,|\,\ast$ 2O, 1A, 1P: $\ast\,|\,\ast\ast\,|\,\ast$ 2P, 1A, 1O: $\ast\,|\,\ast\,|\,\ast\ast$.

There are as many ways as there are strings consisting of 4 $\ast$ and 2 $|$, i.e.

$$\binom{6}{2} = \binom{6}{4} = \frac{6!}{2!\,4!} = 15 .$$

Alternative definitión: If S has n elements, then

an r-combinatión is an assignment of variables $x_1, x_2, .., x_n \in \{0,1\}$ s.t. $\sum_{i=1}^{n} x_i = r$.

An r-combination with repetitión is an assignment of variables $x_1, x_2, .., x_n \in \{0,1,..,r\}$ s.t. $\sum_{i=1}^{n} x_i = r$.

An r-combinatión with repetitión corresponds to $(n-1+r)$ strings with r $*$'s and $n-1$ $|$'s :

the number of $*$'s in the i'th compartment is the number of objects of the i'th type (this is the value of x_i).

Therefore, the number of r-combinatións with repetitión is

$$\binom{n-1+r}{r}.$$

Example (bills) there are $n=5$ bills and I have $r=5$ bills in my pockets, so there are $\binom{9}{5}$ possibilities.

Permutatións with indistinguishable objects

- UNCOPYRIGHTABLE : all unique letters, so any permutation will produce a distinct string : $P(15,15) = 15!$
- COOKIE : $P(6,6) = 6!$ permutatións, but 2 O's with $P(2,2) = 2$ internal permutations, so $\frac{6!}{2} = 6 \cdot 5 \cdot 4 \cdot 3 = 360$ different permutatións.
- INDISTINGUISHABILITY (6: I, 2: N, 2: S, 2: T, 8 distinct letters) $\frac{20!}{6!\,2!\,2!\,2!}$ different permutatións.

Example (dealing cards)

The number of ways to deal is

$$C(52,5) \cdot C(47,5) \cdot C(42,5) \cdot C(37,5)$$

$$= \frac{52!}{\cancel{47!}\,5!} \; \frac{\cancel{47!}}{\cancel{42!}\,5!} \; \frac{\cancel{42!}}{\cancel{37!}\,5!} = \frac{\cancel{37!}}{32!\,5!}$$

$$= \frac{52!}{5!\,5!\,5!\,5!\,32!}, \quad \text{by the product rule.}$$

Note that this may be viewed as permutations of 52 cards with 4 types each containing 5 indistinguishable objects, and a 5th type with 32 objects.

Example (Santa's problem)

This corresponds to 10-combinations of a set of 8 elements with repetition, so

$$C(8-1+10,\, 10) = \frac{17!}{10!\,7!}.$$

Example (4 distinguishable balls into 3 indistinguishable bins)

So there are 14 such ways.

One bin filled: $\{\{A,B,C,D\}\}$

Two bins: $\{\{A,B,C\},\{D\}\}$, $\{\{A,B,D\},\{C\}\}$

$\{\{A,C,D\},\{B\}\}$, $\{\{B,C,D\},\{A\}\}$

$\{\{A,B\},\{C,D\}\}$, $\{\{A,C\},\{B,D\}\}$

$\{\{A,D\},\{B,C\}\}$,

Three bins: $\{\{A,B\},\{C\},\{D\}\}$, $\{\{A,C\},\{B\},\{D\}\}$, $\{\{A,D\},\{B\},\{C\}\}$

$\{\{B,D\},\{A\},\{C\}\}$, $\{\{C,D\},\{A\},\{B\}\}$, $\{\{B,C\},\{A\},\{D\}\}$.

Example (Books into boxes)

Indistinguishable objects into indistinguishable boxes.

Total number of 9 ways.

6,

5, 1

4,2   4,1,1

3,3   3,2,1 , 3, 1,1,1

2,2,2   2,2,1,1