$\boxed{\text{Last time}}$: Relations (Sections 9.1 and 9.3)

A relation on A is a subset of $A \times A$

Reflexive: $\forall a \in A: aRa$

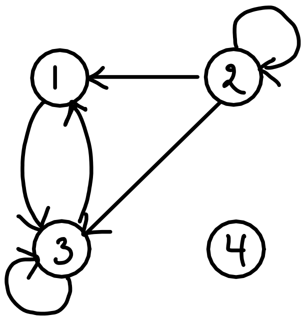
Symmetric: $\forall a, b \in A: (aRb \Rightarrow bRa)$

Antisymmetric: $\forall a, b \in A: (aRb \wedge a \neq b \Rightarrow b \not R a)$

Transitive: $\forall a, b, c \in A: (aRb \wedge bRc \Rightarrow aRc)$

Ex: $R = \{(1,3), (2,1), (2,2), (2,3), (3,1), (3,3)\}$
relation on $\{1,2,3,4\}$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

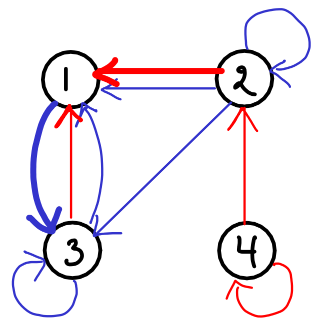


Neither reflexive, symmetric, antisymmetric, nor transitive.

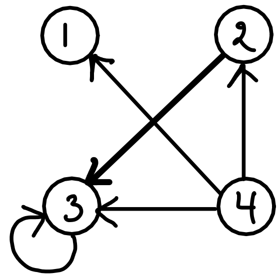
Relations can be composed like functions

Ex: $R = \{(1,3), (2,1), (2,2), (2,3), (3,1), (3,3)\}$

$S = \{(2,1), (3,1), (4,2), (4,4)\}$

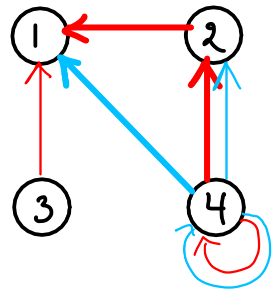


$R \circ S = \{(2,3), (3,3), (4,1), (4,2), (4,3)\}$



$S^2 = S \circ S = \{(4,1), (4,2), (4,4)\}$

$= S^3 = S^4 = \ldots$



Closures of relations (Section 9.4)

Def. 9.4.1:

For any relation R and any property P,
the closure of R w.r.t. P (if it exists)
is the set C satisfying

(1) C has property P

(2) $R \subseteq C$

(3) $\forall$ relation S satisfying (1) and (2): $C \subseteq S$

I.e., C is a smallest possible set containing R and having property P.

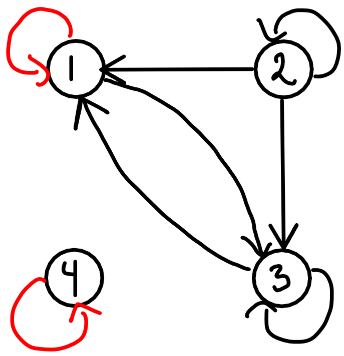
Ex: Relation on $\{1,2,3,4\}$:

$$R = \{(1,3),(2,1),\underline{(2,2)},(2,3),(3,1),\underline{(3,3)}\}$$

The reflexive closure of R is

$$r(R) = \{(1,1),(1,3),(2,1),(2,2),(2,3),(3,1),(3,3),(4,4)\}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



R: relation on A

The reflexive closure of R is

$$r(R) = R \cup \{(a,a) \mid a \in A\}$$

Ex: Relation on $\mathbb{N}$:

$$R_< = \{(a,b) \mid a<b\}$$

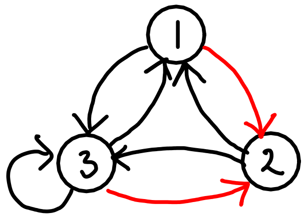
$$r(R_<) = R_< \cup \{(a,a) \mid a \in \mathbb{N}\}$$
$$= \{(a,b) \in \mathbb{N}\times\mathbb{N} \mid a \le b\}$$

Ex: $R = \{(1,3), (2,1), (2,3), (3,1), (3,3)\}$

The symmetric closure of R is

$s(R) = \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2), (3,3)\}$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$



The symmetric closure of a relation R is

$s(R) = R \cup \{(b,a) \mid (a,b) \in R\}$

Ex:

$s(R_<) = R_< \cup R_> = R_{\neq}$

$s($„same parity"$)$ = „same parity"

Antisymmetric closure?

Not defined

A relation which is not antisymmetric cannot be made antisymmetric by <u>adding</u> pairs to it.

Ex: $R = \{(1,2), (2,3), (3,4)\}$

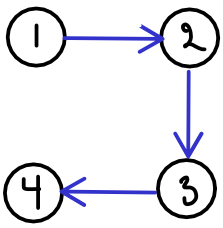
To make R transitive, we need to add R^2:

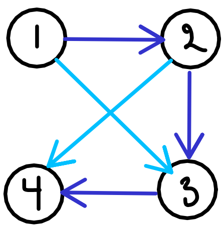
$$R \cup R^2 = \{(1,2), (2,3), (3,4)\} \cup \{(1,3), (2,4)\}$$
$$= \{(1,2), (1,3), (2,3), (2,4), (3,4)\}$$

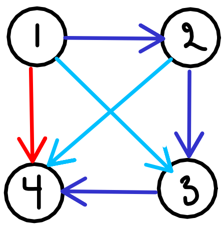
But then we also need to add R^3:

$$R \cup R^2 \cup R^3 = \{(1,2), (1,3), (2,3), (2,4), (3,4)\} \cup \{(1,4)\}$$
$$= \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$$

And now, we are done, since $R^4 = \{\}$







Theorem 9.4.2

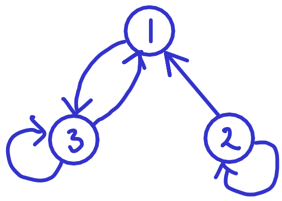
The transitive closure of any relation R is

$$t(R) = R^*$$

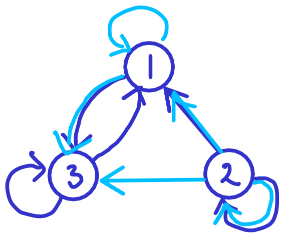
Proof:

- $R \subseteq R^*$
- R^* is transitive (by Theorem 9.4.1):
  For any paths $a \rightsquigarrow b$ and $b \rightsquigarrow c$,
  G_{R^*} also has an edge $a \to c$
- For any transitive relation S s.t. $R \subseteq S$,
  $R^* \subseteq S$

Ex: $R = \{(1,3), (2,1), (2,2), (3,1), (3,3)\}$



$R^2 = \{(1,1), (1,3), (2,1), (2,2), (2,3), (3,3)\}$

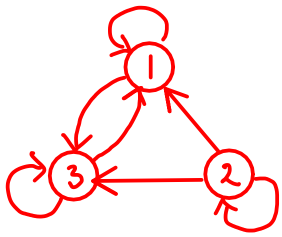


Note: For each path of length > 2, there is a path of length ≤ 2.

Thus,

$t(R) = R \cup R^2$

$= \{(1,1), (1,3), (2,1), (2,2), (2,3), (3,1), (3,3)\}$



Equivalence Relations

Section 9.5

Def 9.5.1

A relation which is
- reflexive
- symmetric
- transitive

is called an equivalence relation

Ex :

✓: $=$, same parity

✗: $\neq$, $\leq$, $|$

Def. 9.5.2

If R is an equivalence relation and $(a,b) \in R$, then a and b are equivalent

Def. 9.5.3

If R is an equivalence relation on A and $a \in A$, then the equivalence class of a is

$$[a]_R = \{ b \mid (a,b) \in R \}$$

$$= \{ b \mid (b,a) \in R \}, \text{ since } R \text{ is symmetric}$$

Ex: Relation on $\mathbb{N}$:

$$\{ (a,b) \mid a \text{ and } b \text{ have the same parity} \}$$

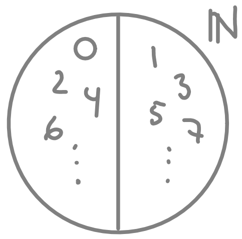
$$[0]_R = \{0,2,4,6,\ldots\} = [2]_R = [4]_R = \ldots$$

$$[1]_R = \{1,3,5,7,\ldots\} = [3]_R = [5]_R = \ldots$$

Note:

$[0]_R \cup [1]_R = \mathbb{N}$ and

$[0]_R \cap [1]_R = \emptyset$,



i.e., the eq classes form a partition of $\mathbb{N}$

Ex: Relation on $\mathbb{N}$:

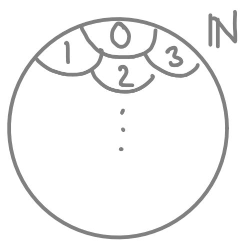
$$\{ (a,b) \mid a = b \}$$

$$[0]_S = \{0\}$$

$$[1]_S = \{1\}$$

$$[2]_S = \{2\}$$

$$\vdots$$



Again, the eq. classes form a partition of $\mathbb{N}$.

This is not a coincidence, according to Thm 9.5.2

In order to prove Theorem 9.5.2, we will first prove Theorem 9.5.1:

Theorem 9.5.1:

If R is an equivalence relation, then

$$\begin{array}{ll} aRb & (i) \\ \Updownarrow & \\ [a]_R = [b]_R & (ii) \\ \Updownarrow & \\ [a]_R \cap [b]_R \neq \varnothing & (iii) \end{array}$$

The fact that (iii) $\Rightarrow$ (ii) is used in the proof of Theorem 9.5.2).

Proof:

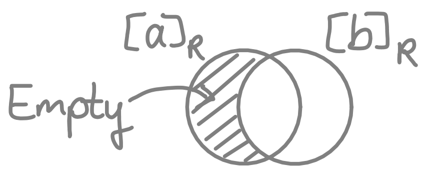
We will prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), giving a direct proof of each implication.

For simplicity we will write $[a]$ and $[b]$ instead of $[a]_R$ and $[b]_R$.

$aRb \Rightarrow [a]=[b]$: (i) $\Rightarrow$ (ii)

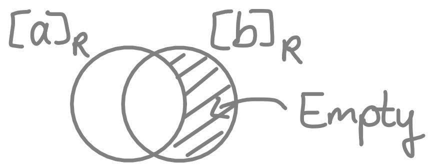
Assume aRb.
Then, bRa, since R is symmetric.

$[a] \subseteq [b]$:

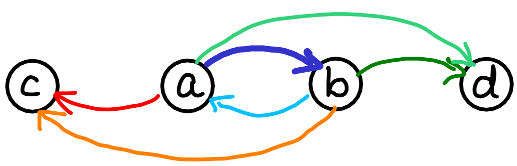


$\Updownarrow$ $c \in [a]$
$\Downarrow$ aRc, by def.
$\Updownarrow$ bRc, since bRa and R is transitive
$c \in [b]$, by def.

$[b] \subseteq [a]$:



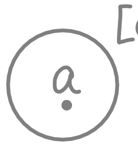
$\Updownarrow$ $d \in [b]$
$\Downarrow$ bRd, by def.
$\Updownarrow$ aRd, since aRb and R is transitive
$d \in [a]$, by def.



Since $[a] \subseteq [b]$ and $[b] \subseteq [a]$, we conclude $[a]=[b]$.

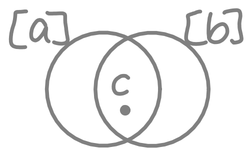
$[a] = [b] \Rightarrow [a] \cap [b] \neq \emptyset$: (ii) $\Rightarrow$ (iii)

$[a] = [b] = [a] \cap [b]$

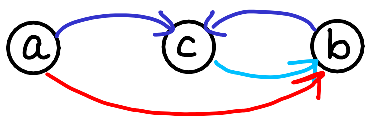


$$
\begin{aligned}
& [a] = [b] \\
\Downarrow\ & [a] \cap [b] = [a] \\
\Downarrow\ & [a] \cap [b] \neq \emptyset, \quad \text{since } a \in [a] \text{ by reflexivity}
\end{aligned}
$$

$[a] \cap [b] \neq \emptyset \Rightarrow aRb$: (iii) $\Rightarrow$ (ii)



$$
\begin{aligned}
& [a] \cap [b] \neq \emptyset \\
\Updownarrow\ & \exists c : \ c \in [a] \wedge c \in [b] \\
\Updownarrow\ & \exists c : \ aRc \wedge bRc, \quad \text{by def.} \\
\Updownarrow\ & \exists c : \ aRc \wedge cRb, \quad \text{by symmetry} \\
\Downarrow\ & aRb \qquad\qquad\qquad\quad, \quad \text{by transitivity}
\end{aligned}
$$



$\square$