Today

Induction (Section 5.1)
- Recap
- More examples

Sets
- Representations
- Properties

(Section 2.1)

Next time

- Set operations (Section 2.2)

Functions (Section 2.3)

$\boxed{\underline{\text{Ex } 5.1.3}: \quad \underbrace{\sum_{i=0}^{n} 2^i = 2^{n+1}-1}_{P(n)}, \quad n \in \mathbb{N}}$

$$P(0): \underbrace{2^0}_{1} = \underbrace{2^1 - 1}_{2-1=1} \quad \checkmark$$

$$P(1): \underbrace{2^0}+2^1 = 2^2 - 1$$

$$2^1 - 1 + 2^1 = 2 \cdot 2^1 - 1 = 2^2 - 1 \quad \checkmark \qquad P(0) \Rightarrow P(1)$$

$$P(2): \underbrace{2^0 + 2^1}_{2^2-1} + 2^2 = 2^3 - 1$$

$$2^2 - 1 + 2^2 = 2 \cdot 2^2 - 1 = 2^3 - 1 \quad \checkmark \qquad P(1) \Rightarrow P(2)$$

$$P(3): \underbrace{2^0 + 2^1 + 2^2}_{2^3-1} + 2^3 = 2^4 - 1$$

$$2^3 - 1 + 2^3 = 2 \cdot 2^3 - 1 = 2^4 - 1 \quad \checkmark \qquad P(2) \Rightarrow P(3)$$

In general:
Once we have verified $P(k-1)$,
we may use $P(k-1)$ to verify $P(k)$:

$$\underbrace{2^0 + 2^1 + \ldots + 2^{k-1}}_{2^k - 1} + 2^k$$

$$2^k - 1 \quad + 2^k = 2 \cdot 2^k - 1 = 2^{k+1} - 1$$

- and that is exactly what happens in a proof by induction:

1) Prove $P(0)$ — Basis step
2) Prove $\underbrace{P(k-1)}_{\text{Induction hypothesis}} \Rightarrow P(k)$, $k \in \mathbb{Z}^+$ — Inductive step

„Unfolded":

1) $P(0)$
2) $P(0) \Rightarrow P(1)$
   $P(1) \Rightarrow P(2)$
   $P(2) \Rightarrow P(3)$
   $\vdots$

This could also be expressed as

1) Prove $P(0)$
2) Prove $P(k) \Rightarrow P(k+1)$, $k \in \mathbb{N}$

or

1) Prove $P(0)$
2) Prove $P(k-2) \Rightarrow P(k-1)$, $k = 2, 3, 4, \ldots$

or

...

$$\boxed{\sum_{i=0}^{n} 2^i = 2^{n+1}-1, \quad n \in \mathbb{N}}$$

Proof: By induction on n

Basis step: $(k=0)$

$$2^0 = 1 = 2^1 - 1 \qquad P(0)$$

Induction hypothesis: $(k \geq 1)$

$$2^0 + 2^1 + \ldots + 2^{k-1} = 2^k - 1 \qquad P(k-1)$$

Inductive step: $(k \geq 1)$

Direct proof:

$$2^0 + 2^1 + \ldots + 2^{k-1} = 2^k - 1 \qquad P(k-1)$$

$$\Updownarrow$$

$$2^0 + 2^1 + \ldots + 2^{k-1} + 2^k = 2^k - 1 + 2^k$$

$$\Updownarrow \qquad\qquad \Downarrow$$

$$2^0 + 2^1 + \ldots + 2^{k-1} + 2^k = 2 \cdot 2^k - 1$$

$$\Updownarrow$$

$$2^0 + 2^1 + \ldots + 2^{k-1} + 2^k = 2^{k+1} - 1 \qquad P(k)$$

$\square$

All apples have the same color

„Proof": By induction on $n = \#$apples

Basis step: $(k=1)$

Any apple has the same color as itself.

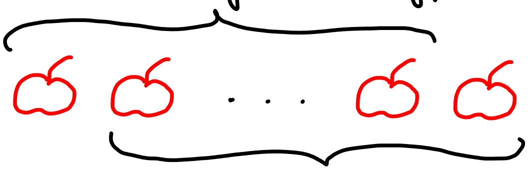
Ind. hyp.: $(k \geq 2)$

In any set of $k-1$ apples, all apples have the same color.

Ind. step: $(k \geq 3)$

Line up k apples:

same color, by ind. hyp.



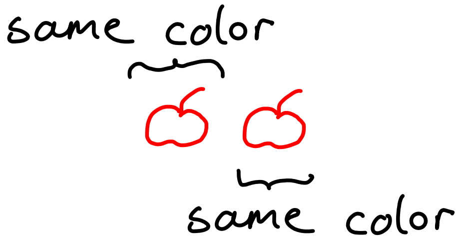
same color, by ind. hyp.

So the first apple has the same color as the next $k-2$ apples which have the same color as the last apple.

Hence, all k apples have the same color

$\square$

Note: The argument in the ind. step does <u>not</u> work for $k=2$:



Thus, we have proven

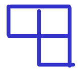
$P(1)$ and

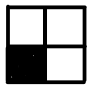
$P(2) \Rightarrow P(3) \Rightarrow P(4) \Rightarrow \cdots$

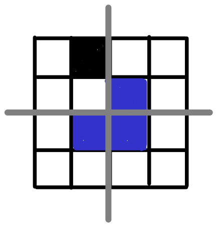
but <u>not</u>

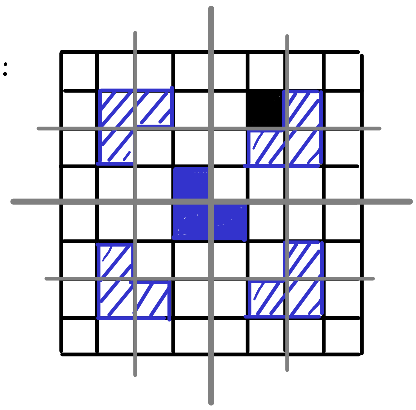
$P(1) \Rightarrow P(2)$

Eks 5.1.14:

For any $n \geq 1$, a $2^n \times 2^n$ chessboard, with one square removed, can be tiled by -pieces

$n = 1$:  ✓

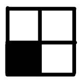
$n = 2$:  ✓

$n = 3$:  ✓

For any $n \geq 1$, a $2^n \times 2^n$ chessboard, with one square removed, can be tiled by -pieces

Proof (by induction on n)

Basis step: $(k=1)$

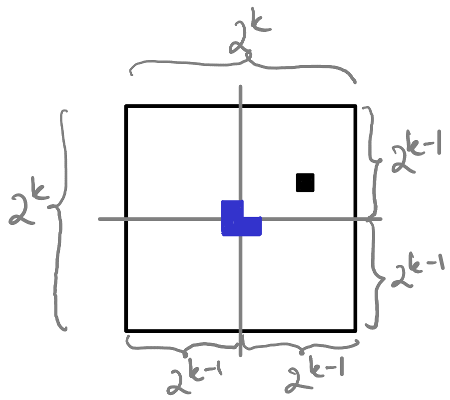


Ind. hyp.: $(k \geq 2)$

A $2^{k-1} \times 2^{k-1}$ chessboard, with one square removed, can be tiled by -pieces

Ind. step: $(k \geq 2)$

A $2^k \times 2^k$ chessboard can be partitioned into four $2^{k-1} \times 2^{k-1}$ boards:



Place a piece covering one square of each of the three $2^{k-1} \times 2^{k-1}$ boards with no square missing

Now, each of the four $2^{k-1} \times 2^{k-1}$ boards has exactly one square that does not need a tile.

Thus, by the ind. hyp., they can each be tiled.

□

Sets Sections 2.1-2.2

(Mængder)

- Representations
  - Enumeration
  - Set builder notation
  - Venn diagrams
- Properties
- Operations

Def. 2.1.1

A set is an unordered collection of distinct elements

$x \in S$: „x is an element of S"

„x is contained in S" („x tilhører S")

$x \notin S$: „x is not an element of S"

Ex: $A = \{1,2,3\} \overset{\text{(unordered)}}{=} \{1,3,2\} \overset{\text{(distinct)}}{=} \{1,2,2,3\}$

$2 \in A$

Representations

Enumeration

Ex: $V = \{a, e, i, o, u\}$

Ex: $B = \{1, \{2,4\}, a\}$

$1 \in B$

$\{2,4\} \in B$

$2 \notin B$

Ex: $\mathbb{N} = \{0, 1, 2, \ldots\}$

Ex: $\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$

Ex: $E = \{0, 2, 4, 6, \ldots\}$

Set builder notation (mængde-bygger-notation)

sometimes written „ : "

Ex: $E = \{ n \in \mathbb{N} \mid 2 \mid n \}$ „divides" („går op i")

„The set of all natural numbers, n, such that 2 divides n"

(„Mængden af naturlige tal, n, hvorom der gælder, at 2 går op i n")

„The set of all natural numbers divisible by 2"

$$E = \{ n \in \mathbb{N} \mid \exists k \in \mathbb{Z} : n = 2k \}$$

$$E = \{ 2k \mid k \in \mathbb{N} \}$$

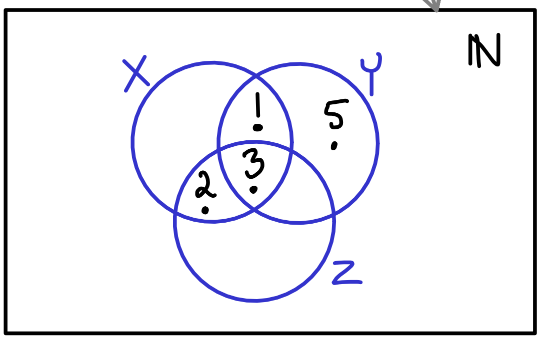
Generally: $\{ \langle \text{element} \rangle \; [\in \langle \text{universe} \rangle] \mid \langle \text{proposition} \rangle \}$

Ex: $\mathbb{Q}^+ = \{ \frac{p}{q} \mid p, q \in \mathbb{Z}^+ \}$

Ex: $\{1, 2, 4, 8, \ldots\} = \{ 2^n \mid n \in \mathbb{N} \}$

Venn diagrams

The universal set
(universet)



Ex: $X = \{1, 2, 3\}$
$Y = \{1, 3, 5\}$
$Z = \{2, 3\}$

Some special types of sets:

The empty set: $\varnothing = \{\}$

Intervals:

$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$
$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$
$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$
$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$

$[a, b)$, $(a, b]$, and (a, b) are sometimes written
$[a, b[$, $]a, b]$, and $]a, b[$

Properties

The size of a set is called its cardinality.

Def. 2.1.4 (kardinalitet)

$|S|$: The cardinality of the set S
elements in S, if S is finite

In a later lecture, we will look at the cardinality of infinite sets.

Ex: $|\{2,4,6\}| = 3$

$|\{2,2,4,6,4\}| = 3$

$|\{2,\{3,4\},5\}| = 3$

$|\{\}| = |\emptyset| = 0$

$|\{\mathbb{Z}^-, \mathbb{Z}^+\}| = 2$