DM549/DS(K)820/MM537/DM547

Lecture 5: Proofs by Induction

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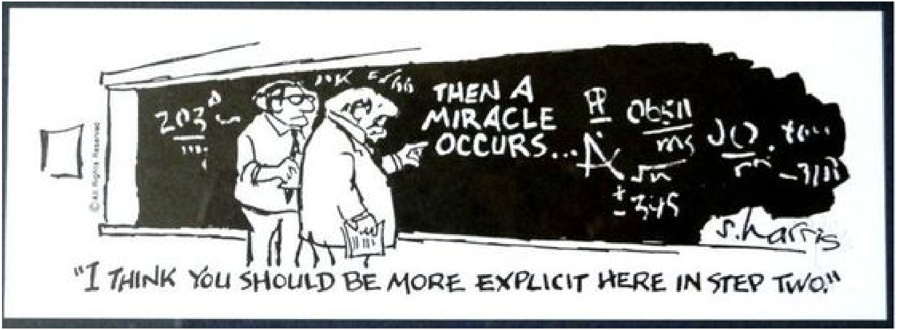
Last Time: Proofs



Last Time: Proof Methods

Overview:

- Direct Proof:
  - We use that $((p \Rightarrow p_1) \wedge (p_1 \Rightarrow p_2) \wedge \cdots \wedge (p_n \Rightarrow q)) \Rightarrow (p \Rightarrow q)$.
  - We also just write $p \Rightarrow p_1 \Rightarrow p_2 \Rightarrow \cdots \Rightarrow p_n \Rightarrow q$.
  - Example: Proof that n odd $\Rightarrow n^2$ odd for all n.
- Proof by Contraposition:
  - We use that $(p \Rightarrow q) \equiv (\neg q \Rightarrow \neg p)$.
  - Example: Proof that n^2 odd $\Rightarrow n$ odd for all n.
- Proof by Contradiction:
  - We use that $(\neg p \Rightarrow F) \Rightarrow p$.
  - Example: Proof that there are two people in this room that were born on the same weekday.

Last Time: Some Tricks

Examples:

- Use that $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$.
  - Example: Proof that n odd $\Leftrightarrow n^2$ odd for all n.
- Use that $p_1 \Leftrightarrow p_2 \Leftrightarrow p_3 \equiv p_1 \Rightarrow p_2 \Rightarrow p_3 \Rightarrow p_1$
  - Example: Exercise 1.7.43 (Sheet 4).
- Make a case distinction.
  - Example: Proof that $\lfloor (n+1)/2 \rfloor \geq n/2$ for all n.

Last Time: Problem Solving

Take-Aways: To solve a Mathematical problem, it may help to...

- first look at simpler special cases.
- draw suitable pictures.

Some Terminology

Types of Propositions:

- theorem (sætning): important proposition that is proven to be true.
- conjecture (påstand): unproven important proposition that is believed to be true.
- lemma (lemma): auxiliary theorem.
- corollary (korollar): proposition that immediately follows from a theorem.

Introducing Another Proof Method

Theorem (Example 5.1.4)

For all $n \in \mathbb{N}$, it holds that

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1.$$

Proof idea: Proof by induction:

- First show $\sum_{i=0}^{0} 2^i = 2^1 - 1$. (basis step)
- Then show that

$$\forall k \in \mathbb{Z}^+ : \left(\sum_{i=0}^{k-1} 2^i = 2^k - 1 \Rightarrow \sum_{i=0}^{k} 2^i = 2^{k+1} - 1 \right).$$

(inductive step)

An Interpretation with Dominoes

Interpretation:

- Imagine an infinite row of dominoes, say, one for each positive integer.
- Suppose you have to explain to someone not familiar with the concept why *all* of the dominoes have fallen over.
- Possible explanation:
  - Somebody has knocked over the first domino.
    (Corresponds to basis step.)
  - If a domino falls over, then also the next domino falls over.
    (Corresponds to inductive step.)

Induction: A Recipe

Recipe 1 for Proofs by (Simple) Induction

To show that $P(n)$ holds for all $n \geq m$, prove:

- Basis step: Prove that $P(m)$ holds.
- Inductive step: Prove that

  $$\underbrace{P(k)}_{\text{inductive hypothesis}} \Rightarrow P(k+1)$$

  for all $k \geq m$.

Some vocabulary:

- (Mathematical) induction (induktion)
- basis step (basisskridt)
- inductive step (induktionsskridt)
- inductive hypothesis (induktionshypotese)

Some sources use slightly different terms.

Induction: Another Recipe

Recipe 2 for Proofs by (Simple) Induction

To show that $P(n)$ holds for all $n \geq m$, prove:

- Basis step: Prove that $P(m)$ holds.
- Inductive step: Prove that
$$P(k-1) \Rightarrow P(k)$$
for all $k \geq m+1$.

Note: It does not matter which recipe you use.

Another Proof by Induction

Theorem (Example 5.1.6)

For all $n \in \mathbb{Z}$ with $n \geq 4$, it holds that

$$2^n < n!\,.$$

Yet Another Proof by Induction

Fact

The sum of angles of a triangle is 180°.

Definition

Let $n \in \mathbb{Z}$ with $n \geq 3$. Then an n-gon is called *convex* if any line segment connecting two of its vertices only runs through the interior and boundary of the n-gon.

Theorem

For all $n \in \mathbb{Z}$ with $n \geq 3$, it holds that for all convex n-gons the sum of its angles is $(n-2) \cdot 180^\circ$.