Summations

(Summing the terms of a sequence)

We will look at
Geometric series and
Arithmetic series

Geometric Series

(Geometriske rækker)

Ex: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

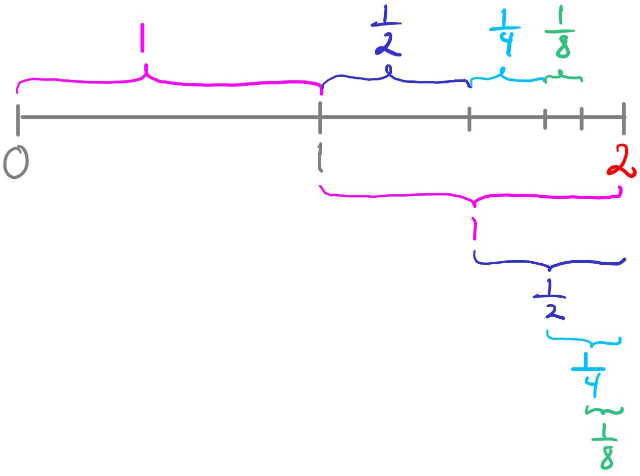
$n=0:\quad 1 = 2 - 1$

$n=1:\quad 1 + \frac{1}{2} = 2 - \frac{1}{2}$

$n=2:\quad 1 + \frac{1}{2} + \frac{1}{4} = 2 - \frac{1}{4}$

$n=3:\quad 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 2 - \frac{1}{8}$

$\vdots$



$$\boxed{\sum_{i=0}^{n} \left(\tfrac{1}{2}\right)^i = 2 - \left(\tfrac{1}{2}\right)^n}$$

Proven in an exercise
(Week 39)

$$\sum_{i=1}^{n} \left(\tfrac{1}{2}\right)^i = 1 - \left(\tfrac{1}{2}\right)^n$$

$$\sum_{i=0}^{\infty} \left(\tfrac{1}{2}\right)^i = 2$$

$$\sum_{i=1}^{\infty} \left(\tfrac{1}{2}\right)^i = 1$$

Ex: $1 + 2 + 4 + 8 + \ldots$

$$\boxed{\sum_{i=0}^{n} 2^i = 2^{n+1} - 1}$$

Proven in a lecture (Week 38)

Illustration:

	Binary:
2^0	1
2^1	10
2^2	100
2^3	1000
$2^4 - 1$	1111 = 10000 − 1

Ex: $1 + 3 + 9 + 27 + \dots$

$$\boxed{\sum_{i=0}^{n} 3^i = \frac{3^{n+1}-1}{2}}$$

Illustration:

	Ternary
$2 \cdot 3^0$	2
$2 \cdot 3^1$	20
$2 \cdot 3^2$	200
$2 \cdot 3^3$	2000
$3^4 - 1$	2222 $= 10000 - 1$

Thus,

$$2 \cdot (3^0 + 3^1 + 3^2 + 3^3) = 3^4 - 1$$

which is equivalent to

$$3^0 + 3^1 + 3^2 + 3^3 = \frac{3^4 - 1}{2}$$

Ex: $1 + 4 + 16 + 64 + \dots$

$$\boxed{\sum_{i=0}^{n} 4^i = \frac{4^{n+1}-1}{3}}$$

Illustration (as before):

$$3 \cdot (4^0 + 4^1 + 4^2 + 4^3) = 4^4 - 1$$

$$\Updownarrow$$

$$4^0 + 4^1 + 4^2 + 4^3 = \frac{4^4-1}{3}$$

In general: (Not only for integers)

$$(r-1)\sum_{i=0}^{n} r^i = (r-1)(1+r+r^2+\ldots+r^n)$$
$$= r(1+r+r^2+\ldots+r^n) - (1+r+r^2+\ldots+r^n)$$
$$= \cancel{r+r^2+r^3+\ldots+}\,r^{n+1} - 1 \cancel{-r-r^2-\ldots-r^n}$$
$$= r^{n+1}-1$$

$\Downarrow$

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}, \quad \text{for } r \neq 1$$

Thus, we obtain

$$\boxed{\textbf{Theorem 2.4.1} \quad \sum_{i=0}^{n} r^i = \begin{cases} \dfrac{r^{n+1}-1}{r-1}, & \text{for } r \in \mathbb{R}-\{1\} \\ n+1, & \text{for } r=1 \end{cases}}$$

(Note: We could also have proven this using induction)

$$\boxed{\sum_{i=1}^{\infty} r^i = \frac{1}{1-r}, \quad \text{for } r<1}$$

Ex:

$$\sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^i = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$$

Arithmetic Series (Aritmetiske rækker)

Ex: $1+2+3+4+\dots$ (Example 5.1.1)

$$\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}} :$$

Even n:

$$\sum_{i=1}^{n} i = 1+2+\dots+\frac{n}{2}+\left(\frac{n}{2}+1\right)+\dots+(n-1)+n = (n+1)\frac{n}{2}$$

$n+1$

$n+1$

$n+1$

Odd n:

$n+1$

$n+1$

$n+1$

$$\begin{aligned}
\sum_{i=1}^{n} i &= 1+2+\dots+\frac{n-1}{2}+\frac{n+1}{2}+\frac{n+3}{2}+\dots+(n-1)+n \\
&= \frac{n-1}{2}(n+1)+\frac{n+1}{2} \\
&= (n-1)\cdot\frac{n+1}{2}+\frac{n+1}{2} \\
&= \frac{n(n+1)}{2}
\end{aligned}$$

In general:

$$\boxed{\sum_{i=0}^{n} (a+di) = (n+1)a + \frac{n(n+1)}{2} d}$$:

$$\sum_{i=0}^{n} (a+di) = \sum_{i=0}^{n} a + d \sum_{i=0}^{n} i$$
$$= (n+1)a + d\,\frac{n(n+1)}{2}$$

Ex: $1+3+5+\ldots+2k-1$

$$\boxed{\sum_{i=0}^{k-1} (1+2i) = k^2}$$:

$$\sum_{i=0}^{k-1} (1+2i) = k \cdot 1 + \frac{(k-1)k}{2} \cdot 2$$
$$= k + (k-1)k$$
$$= k + k^2 - k$$
$$= k^2$$

Sums also show up when calculating the number of iterations of a loop to determine the running time of an algorithm.

In the below examples, T denotes the number of iterations of the inner loop.

Ex:

```
For i:=1 to 20
    For j:=1 to 10
        ...
```

$$T = 20 \cdot 10 \left(= \sum_{i=1}^{20} 10 \right) = 200$$

Ex:

```
For i:=1 to 20
    For j:=1 to i
        ...
```

$$T = \sum_{i=1}^{20} i = \frac{20 \cdot 21}{2} = 210$$

Ex:

```
For i:=11 to 20
    For j=1 to i
        ...
```

$$T = \sum_{i=11}^{20} i = \sum_{i=1}^{20} i - \sum_{i=1}^{10} i = 210 - \frac{10 \cdot 11}{2} = 155$$

or

$$T = \sum_{i=11}^{20} i = \sum_{i=0}^{9} (11+i) = 10 \cdot 11 + \frac{9 \cdot 10}{2} = 155$$

Ex:

For $i := 1$ to 20

For $j := 1$ to i

For $k := 1$ to j

...

$$T = \sum_{i=1}^{20} \sum_{j=1}^{i} j$$

$$= \sum_{i=1}^{20} \frac{i(i+1)}{2}$$

$$= \frac{1}{2}\left(\sum_{i=1}^{20} i^2 + \sum_{i=1}^{20} i \right)$$

$$= \frac{1}{2}\left(\frac{20 \cdot 21 \cdot 41}{6} + \frac{20 \cdot 21}{2} \right),$$ by Table 2.4.2 (p. 180)

$$= 1540$$

Notation:

$$\sum_{i=1}^{n} i = \sum_{1 \le i \le n} i = \sum_{i \in A} i,$$ where $A = \{1, 2, 3, \ldots, n\}$

$$\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdot \ldots \cdot a_n$$