

# A direct approach to nonuniqueness and failure of compactness for the SQG equation

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## Abstract

We give an alternative proof of the nonuniqueness of weak solutions to the surface quasi-geostrophic equation (SQG) first shown in [3]. Our approach proceeds directly at the level of the scalar field. Furthermore, we prove that every smooth scalar field with compact support that conserves the integral can be realized as a weak limit of solutions to SQG.

## 1 Introduction

In this work, we are concerned with the two-dimensional surface quasi-geostrophic (SQG) equation, which is the following transport equation for a scalar field  $\theta : \mathbb{R} \times \mathbb{T}^2 \rightarrow \mathbb{R}$

$$\partial_t \theta + \nabla \cdot (\theta u) = 0, \quad u := \nabla^\perp \Lambda^{-1} \theta, \quad (1)$$

where  $\Lambda = (-\Delta)^{1/2}$ , and  $\nabla^\perp = (-\partial_2, \partial_1)$ .

This equation is a basic example of an *active scalar* equation, so called because the drift velocity  $u$  depends at every time (nonlocally) on the scalar field  $\theta$  that is being transported. The SQG equation arises as a model in geophysical fluid dynamics, where it has applications to both meteorological and oceanic flows [16, 27]. In this context, the field  $\theta$  represents temperature or surface buoyancy in a certain regime of stratified fluid flow. The equation has been studied extensively in the mathematical literature due to its close analogy with the 3D incompressible Euler equations and the problem of blowup for initially classical solutions, which remains open as it does for the Euler equations. A survey of developments is given in the introduction to [3].

Fundamental to the study of the SQG equation are the following basic conservation laws:

- i For all sufficiently smooth solutions, the *Hamiltonian*  $\frac{1}{2} \int_{\mathbb{T}^2} |\Lambda^{-1/2} \theta(x, t)|^2 dx$  remains constant.
- ii For all sufficiently smooth solutions, the  $L^p$  norms  $\|\theta(t)\|_{L^p(\mathbb{T}^2)}$  remain constant  $1 \leq p \leq \infty$ , as do the integrals  $\int_{\mathbb{T}^2} F(\theta(x, t)) dx$  for any smooth function  $F$ .
- iii For all weak solutions to SQG, the integral  $\int_{\mathbb{T}^2} \theta(x, t) dx$  remains constant.

(To prove (i), multiply (1) by  $\Lambda^{-1} \theta$  and integrate by parts. To prove (ii), use  $\nabla \cdot u = 0$  to check that  $F(\theta)$  satisfies  $\partial_t F(\theta) + \nabla \cdot (F(\theta)u) = 0$ , then integrate. To prove (iii), simply integrate in space.)

Note that, in contrast to (iii), the nonlinear laws (i) and (ii) require that the solution is “sufficiently smooth”. If one expects that turbulent SQG solutions have a dual energy cascade as in the Batchelor-Kraichnan predictions of 2D turbulence [7, 8, 3], then one has motivation to consider weak solutions that are not smooth. A basic open question for the SQG equations is then: What function spaces represent the minimal amount of smoothness required for the conservation laws to hold? This question is exactly the concern of the (generalized) *Onsager conjecture* for the SQG equation. A closely related problem is to find the minimal regularity that guarantees uniqueness for the initial value problem.

Using Hölder spaces as a natural scale to measure spatial regularity, the generalized Onsager conjecture for SQG can be stated as follows. (The space-time regularity below is on a finite time interval.)

- i If  $\theta \in C^0$ , then the conservation of the Hamiltonian holds. However, for any  $0 < \alpha < 1/2$  there exist weak solutions of class  $\Lambda^{-1/2} \theta \in L_t^\infty C_x^\alpha$  that fail to conserve the Hamiltonian.

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- ii Let  $F \in C^\infty(\mathbb{R})$ . Then if  $\theta \in L_t^\infty C_x^\alpha$  for some  $\alpha > 1/3$ , the law  $\int_{\mathbb{T}^2} F(\theta(x, t)) dx \equiv \text{const}$  is satisfied. On the other hand, for any  $\alpha < 1/3$ , there exist solutions  $\theta \in L_t^\infty C_x^\alpha$  that violate this law.

Some remarks about these conjectures are in order:

1. These conjectures generalize the original Onsager conjecture [26], which concerned turbulent dissipation in the incompressible Euler equations and stated that the Hölder exponent  $1/3$  should mark the threshold regularity for conservation of energy for solutions to the incompressible Euler equations. See [15] for discussion of the significance of Onsager's conjecture in turbulence theory.
2. The conjectured threshold exponents are derived from the fact that the conservation law for sufficiently regular solutions has been proven in both cases (i) and (ii). Namely, [22] proves conservation of the Hamiltonian for solutions with  $\theta \in L^3(I \times \mathbb{T}^2)$ , while [1] proves the conservation law (ii) for  $\alpha > 1/3$ . The proofs are variants of the kinematic argument of [9], which proved energy conservation for the Euler equations above Onsager's conjectured threshold.
3. Following the seminal work [13], advances in the method of convex integration have made possible the pursuit of Onsager's conjecture both for the Euler equations and more general fluid equations. In particular, Onsager's conjecture for the 3D Euler equations has been proven in [20] (see also [2, 19]), while the first progress towards the Onsager conjecture (i) for SQG has been made in [3]. See [12, 4] for surveys and [23] for a discussion of generalized Onsager conjectures.
4. To make sense of the conjecture in part (i), it must be noted that the SQG equation is well-defined for  $\theta$  having negative regularity. Namely, for any smooth vector field  $\phi(x)$  on  $\mathbb{T}^2$ , the quadratic form  $\int_{\mathbb{T}^2} \theta \nabla^\perp \Lambda^{-1} \theta \cdot \phi(x) dx$ , initially defined for smooth  $\theta$ , has a unique bounded extension to  $\theta \in H^{-1/2}$ . This fact, which relies on the anti-self-adjointness of the operator  $\nabla^\perp \Lambda^{-1}$ , allows the SQG nonlinearity to be well-defined in  $\mathcal{D}'$  for  $\theta$  of class  $\theta \in L_t^2 H_x^{-1/2}$  (see [3, Definition 1.1]).

The boundedness of the nonlinearity in a negative Sobolev space is key to constructing weak solutions to the SQG equations by compactness methods [28]. These solutions obtained in [28, 24] are known to exist for all time and to have  $L^p$  norms bounded uniformly in  $t$  by the initial data, but it remains unknown whether they are uniquely determined by their initial data.

The [28] construction of weak solutions is closely tied to the phenomenon of **weak compactness** for SQG solutions. Namely weak limits of sequences of solutions to SQG in the space  $L^\infty$  weak-\* must also be solutions to the SQG equations. The proof (see e.g. [22]) relies on the boundedness of the nonlinearity in a negative Sobolev space. The significance of weak compactness in the present context is that weak compactness appears to present an obstruction to attacking Onsager's conjecture and to proving nonuniqueness using convex integration methods when it is present [14].

The main goals of the present paper are to provide an alternative approach to the nonuniqueness for SQG first shown in [3], thus offering a different point of view from which to pursue the Onsager conjecture for SQG, and to establish an “h-principle” result that demonstrates the stark failure of weak compactness for SQG weak solutions. Our main results are the following.

**Theorem 1** (Existence of Weak Solutions). *For any  $0 < \alpha < 3/10$  and  $0 < \beta < 1/4$  there exists nontrivial weak solutions  $\theta$  to SQG with compact support in time such that the potential function  $\Lambda^{-1/2}\theta$  is in the Hölder class  $C_t^0 C_x^\alpha \cap C_t^\beta C_x^0(\mathbb{R} \times \mathbb{T}^2)$ .*

**Theorem 2** (h-principle). *Fix any  $\alpha < 3/10, \beta < 1/4$ , and let  $f$  be a smooth scalar field with compact support in time that satisfies the conservation law  $\int_{\mathbb{T}^2} f(x, t) dx = 0$  as a function of time. Then there exists a sequence of solutions to SQG,  $\{\theta_n\}_{n \in \mathbb{N}}$ , such that each scalar field  $\theta_n$  has compact support in time and a corresponding potential function  $\Lambda^{-1/2}\theta_n$  of Hölder class  $\Lambda^{-1/2}\theta_n \in C_t^\beta C_x^0 \cap C_t^0 C_x^\alpha(\mathbb{R} \times \mathbb{T}^2)$ , and such that  $\Lambda^{-1/2}\theta_n \rightharpoonup \Lambda^{-1/2}f$  in the  $L^\infty(\mathbb{R} \times \mathbb{T}^2)$  weak-\* topology.*

The “h-principle” result stated in Theorem 2 is a new result to this paper, while Theorem 1 was first obtained in the work [3]. (The main result of [3] is stated in terms of  $\Lambda^{-1}\theta$  rather than  $\Lambda^{-1/2}\theta$ , but in terms of spatial regularity the results are equivalent.)

The main contributions of our work are as follows:

1. One of the main ideas of [3] to apply convex integration to the SQG equation is to recast the SQG equation in the following *momentum form* for the unknown  $v = \Lambda^{-1}u = \nabla^\perp\Delta^{-1}\theta$ :

$$\partial_t v + u \cdot \nabla v - (\nabla v)^T \cdot u = -\nabla p, \quad \operatorname{div} v = 0, \quad u = \Lambda v.$$

The authors are then able to perform convex integration at the level of the vector field  $v$  rather than the scalar field  $\theta$ . The scalar field  $\theta$  can then be recovered from  $v$  by setting  $\theta = -\nabla^\perp \cdot v$ .

In this work, we take a different approach, working directly at the level of the scalar field  $\theta$ . To execute this approach, we require the error term in the construction to have the structure of a second order divergence, as opposed to the first order divergence form used in [22]. We also employ the “constant trick” from [10, 29, 22] that the divergence of a function of  $t$  is zero.

Given that the Onsager conjecture for SQG remains open, it is useful to have more than one approach to constructing SQG solutions. Having a second approach gives a separate angle from which to consider the problem and opens the door to considering more general active scalar equations. This approach might also be useful for the still open Onsager conjecture for 2D Euler. Furthermore, it is desirable from a physical point of view to have an approach that works directly at the level of  $\theta$ , since the variable  $\theta$  has a clear physical meaning. We also note there is yet a different approach to convex integration for SQG that has been obtained independently in [5]. This work proceeds at the level of  $\Lambda^{-1}\theta$  and considers the steady state SQG equation.

2. While typically theorems on the failure of compactness can be obtained as essentially a byproduct of a successful convex integration scheme, the argument of [3] is not quite structured towards proving an h-principle result. Our proof of Theorem 2 involves taking a different organizational strategy in the estimates compared to [3]. The proof of the h-principle also involves an additional approximation step compared to similar results in [22, 21], as the convex integration scheme implemented here is required to have compact frequency support.
3. A key idea of [3] is to express the nonlinearity  $u \cdot \nabla v - (\nabla v)^T \cdot u$  as the sum of the divergence of a 2-tensor and the gradient of a scalar function to have good control over high-high to low interactions. Here we show that the desired divergence form can be obtained for the nonlinearity  $\theta\nabla^\perp\Lambda^{-1}\theta$  of SQG, and we extend the derivation so that it applies to general odd multipliers.
4. Finally, we employ a “bilinear microlocal lemma” analogous to [22, Lemma 4.1]. This lemma allows us to obtain a better estimate on the low frequency part of the error compared to the corresponding technique in [3], which relies on the use of sharp time cutoffs to approximate nonlinear phase functions by their initial conditions. Having such an improved estimate is needed for schemes that aim to improve the Hölder regularity. The estimate we obtain is compatible with an ideal case scenario for SQG, so that the only terms now limiting the regularity below the conjectured threshold are the high-frequency interference terms.

The statement of Theorem 2 on the h-principle helps clarify an important point about the method of convex integration, which is that the method appears so far to apply only in cases where an appropriate h-principle implying failure of compactness can be proven [14]. Namely, Theorem 2 shows that even though weak compactness does hold for SQG in  $L^\infty$ , which prevents the schemes of [29, 22] for active scalar equations from applying to SQG, there is still failure of compactness in the space  $\Lambda^{-1/2}\theta \in L^\infty$ , thus permitting the scheme of [3] and the present work to succeed. The statement of Theorem 2 is

modeled off h-principles proven in [22, 21], which also relate weak limits of solutions to conservation laws. In general, “h-principles” in PDE are modeled off the result of Nash [25] that  $C^0$  limits of isometric immersions of closed  $n$ -manifolds into  $\mathbb{R}^{n+2}$  can realize any smooth, short immersion. For further examples of h-principles in fluids, we refer to [6, 11, 2].

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## 2 SQG-Reynolds flows

To state the Main Lemma we will introduce a notion of an SQG-Reynolds flow. The key difference compared to [3] is that we work at the level of the scalar field  $\theta$ , and the error tensor is required to have a double-divergence form. We will consistently employ the summation notation for repeated indices.

**Definition 2.1.** A scalar-valued function  $\theta : \mathbb{R} \times \mathbb{T}^2 \rightarrow \mathbb{R}$  and a symmetric, traceless tensor field  $R^{jl} : \mathbb{R} \times \mathbb{T}^2 \rightarrow \mathbb{R}^{2 \times 2}$  satisfy the **SQG-Reynolds Equations** if

$$\begin{aligned}\partial_t \theta + u^l \nabla_l \theta &= \nabla_j \nabla_l R^{jl} \\ \operatorname{div} u = \nabla_l u^l &= 0 \\ u^l &= T^l \theta,\end{aligned}$$

where the operator  $T$  is given as in (1) by the Fourier multiplier  $m^l(\xi) := i\epsilon^{la}\xi_a/|\xi|$  for  $\xi \in \hat{\mathbb{R}}^2$ . Here  $\epsilon^{la}$  is the Levi-Civita symbol in two dimensions. By convention this symbol is defined as follows

$$\epsilon^{11} = \epsilon^{22} = 0 \quad , \quad \epsilon^{12} = 1 \quad , \quad \epsilon^{21} = -1 \tag{2}$$

Any solution to the SQG-Reynold’s equation,  $(\theta, u, R)$ , is called an **SQG-Reynolds Flow**. The symmetric and traceless tensor field  $R^{jl}$  is called the **stress tensor**.

At times we will write  $(\theta, R)$  to refer to the SQG-Reynolds flow, implying that  $u^l = T^l[\theta]$ .

### 2.1 Frequency and Energy Levels

We will also use a notion of frequency-energy levels similar to those used with the Euler equations in [17] but with the difference that we assume  $\theta$  to be frequency localized similar to [3]. A key point is that the relevant fields are measured relative to the size of the stress tensor  $R^{jl}$ .

**Definition 2.2.** Let  $(\theta, u, R)$  be a solution of the SQG-Reynolds equation,  $\Xi \geq 1$ , and  $\mathcal{D}_u \geq \mathcal{D}_R \geq 0$  be non-negative numbers. Define the **advective derivative** with respect to  $\theta$  to be  $D_t := \partial_t + T^l \theta \nabla_l$ . We say that  $(\theta, u, R)$  has **frequency-energy levels** below  $(\Xi, \mathcal{D}_u, \mathcal{D}_R)$  to order  $L$  in  $C^0$  if  $\theta$  and  $R$  are of class  $C_t^0 C_x^L(\mathbb{R} \times \mathbb{T}^2)$  and the following statements hold

$$\begin{aligned}\operatorname{supp} \hat{\theta} &\subseteq \{\xi : |\xi| \leq \Xi\} \\ \|\nabla_{\vec{a}} \theta\|_{C^0}, \|\nabla_{\vec{a}} u\|_{C^0} &\leq \Xi^{|\vec{a}|} e_u^{1/2} && \text{for all } |\vec{a}| = 0, \dots, L \\ \|\nabla_{\vec{a}} R\|_{C^0} &\leq \Xi^{|\vec{a}|} \mathcal{D}_R && \text{for all } |\vec{a}| = 0, \dots, L \\ \|\nabla_{\vec{a}} D_t \theta\|_{C^0}, \|\nabla_{\vec{a}} D_t u\|_{C^0} &\leq \Xi^{|\vec{a}|} (\Xi e_u^{1/2}) e_u^{1/2} && \text{for all } |\vec{a}| = 0, \dots, L-1 \\ \|\nabla_{\vec{a}} D_t R\|_{C^0} &\leq \Xi^{|\vec{a}|} (\Xi e_u^{1/2}) \mathcal{D}_R && \text{for all } |\vec{a}| = 0, \dots, L-1\end{aligned}$$

The quantity  $(\Xi e_u^{1/2})^{-1}$  is the **natural time scale**, the term  $e_u^{1/2} := \Xi^{1/2} \mathcal{D}_u^{1/2}$ , and  $\nabla$  refers only to derivatives in the spatial variables. We also define a second quantity  $e_R := \Xi \mathcal{D}_R$  that, like  $e_u$ , we think of as having units of “energy density”.

### 3 Main Lemma

**Lemma 3.1** (Main Lemma). *For  $L \geq 2$  there exists a constant  $\hat{C}$  such that the following holds: Given an SQG-Reynolds flow  $(\theta, u, R)$  with frequency and energy levels below  $(\Xi, D_u, D_R)$  and a non-empty closed interval,  $J$ , with  $\text{supp}_t R \subseteq J \subseteq \mathbb{R}$ . Let*

$$N \geq \left( \frac{D_u}{D_R} \right)$$

*Then there exists an SQG-Reynolds flow  $(\dot{\theta}, \dot{u}, \dot{R})$  of the form  $\dot{\theta} = \theta + \Theta, \dot{u} = u + T[\Theta]$  with frequency and energy levels bounded by*

$$(\dot{\Xi}, \dot{D}_u, \dot{D}_R) = (\hat{C}N\Xi, D_R, GD_R), \quad \text{where } G := \left( \frac{D_u^{1/4}}{D_R^{1/4} N^{3/4}} \right)$$

*to order  $L$  in  $C^0$ . Furthermore the new stress  $\dot{R}$  and the correction  $\Theta$  are supported in time in a neighborhood of  $J$*

$$\text{supp}_t \dot{R} \cup \text{supp}_t \Theta \subseteq N(J) := \{t + h : t \in J, |h| \leq 3(\Xi e_u^{1/2})^{-1}\} \quad (3)$$

*Additionally one may arrange that  $\Lambda^{-1/2}\Theta$  has the form  $\Lambda^{-1/2}\Theta = \nabla_i W^i$  that satisfies the following statements*

$$\begin{aligned} \|\nabla_{\vec{a}} \Lambda^{-1/2} \Theta\|_{C^0} &\leq \hat{C}(N\Xi)^{|\vec{a}|} D_R^{1/2}, \quad \text{for } |\vec{a}| \leq 1 \\ \|W\|_{C^0} &\leq \hat{C}(N\Xi)^{-1} D_R^{1/2} \\ \|\partial_t \Lambda^{-1/2} \Theta\|_{C^0} &\leq \hat{C} D_R^{1/2} (N\Xi e_u^{1/2}) \end{aligned}$$

*where  $e_u^{1/2}$  is defined in Definition 2.2.*

### 4 Proof of Main Lemma

#### 4.1 Shape of the Scalar Corrections

Our correction  $\Theta$  is a sum of scalar valued waves  $\Theta_I$  that oscillate at a large frequency  $\lambda$ , similar to the corrections defined in [18, Section 4.2] or [22, Section 5.1].

$$\Theta := \sum_I \Theta_I(x, t), \quad \Theta_I(x, t) := P_\lambda^I(e^{i\lambda\xi_I} \theta_I) = e^{i\lambda\xi_I} (\theta_I + \delta\theta_I). \quad (4)$$

The **frequency parameter**  $\lambda \in 2\pi\mathbb{N}$  is on the order of  $B_\lambda N\Xi$ . More precisely,

$$\lambda \in [B_\lambda N\Xi, 2B_\lambda N\Xi] \cap (2\pi\mathbb{N})$$

where  $B_\lambda \geq 1$  a very large constant associated to  $\lambda$  that is chosen in Section 4.8. The term  $\xi_I := \xi_I(x, t)$  is a non-linear **phase function**,  $\theta_I := \theta_I(x, t)$  is the **amplitude**, and  $\delta\theta_I := \delta\theta_I(x, t)$  is a small correction to ensure that  $\Theta_I$  is compactly supported in frequency space. The second equality for  $\Theta_I$  and the explicit formula for  $\delta\theta_I$  comes from an application of the Microlocal Lemma 4.1 while the other components of the correction are specified below.

#### 4.1.1 The Index Set

The index  $I$  for the wave  $\Theta_I$  is a tuple  $I = (k, f) \in \mathbb{Z} \times F$  where  $F := \{\pm(1, 2), \pm(2, 1)\}$ . Furthermore we define the set  $\mathbb{F} := F/(+, -) = \{(1, 2), (2, 1)\}$ . The index  $k$  specifies the interval of time support for the wave  $\Theta_I = \Theta_{(k, f)}$  and  $f$  represents an initial direction of oscillation. For each index  $I = (k, f) \in \mathbb{Z} \times F$  there is a conjugate index  $\bar{I} = (k, \bar{f}) \in \mathbb{Z} \times F$  such that in the summation (4) each wave  $\Theta_I$  has a conjugate wave  $\Theta_{\bar{I}} := \overline{\Theta}_I$  with an opposite sign phase function  $\xi_{\bar{I}} := -\xi_I$  and matching amplitude  $\theta_{\bar{I}} := \overline{\theta}_I$  so that the overall summation (4) is real-valued. We also define the notation

$$[k] := \begin{cases} 0 & \text{if } k \bmod 2 \equiv 0 \\ 1 & \text{if } k \bmod 2 \equiv 1 \end{cases}$$

#### 4.1.2 Phase functions

For a given SQG-Reynold's flow  $\theta$  and each index  $I = (k, f)$ , the phase functions  $\xi_I$  are solutions to the transport equation

$$(\partial_t + T^l[\theta]\nabla_I)\xi_I = 0, \quad \xi_I(t(I), x) = \hat{\xi}_I(x)$$

where the initial data is  $\hat{\xi}_I(x) = \hat{\xi}_{(k, f)}(x) := J^k f \cdot x$  for  $J :=$  a  $90^\circ$  rotation. The initial time is  $t(I) := k\tau$  where  $\tau$  is the time step from Section 4.1.3. Furthermore, for a small fixed constant  $c_2 > 0$  the time step  $\tau$  in Section 4.1.3 will be sufficiently small to ensure the phase function gradients stay close to their initial data, and stay bounded away from 0

$$|\nabla \xi_I - \nabla \hat{\xi}_I| \leq c_2, \quad |\nabla \xi_I| \geq 1 > 0. \quad (5)$$

The constant  $c_2$  is determined later by calculation in (30). It is taken to be sufficiently small to ensure that the phase gradients remain close to their initial conditions and stay in a neighborhood where the angular frequency localizing operator described in Section 4.1.4 is 1. In particular  $c_2 < 1/4$ .

#### 4.1.3 Time cutoffs

Define a family of time cutoffs similar to what is done in [22, Section 5.2]. For a given index  $I = (k, f)$  define the **time step**  $\tau := (\Xi e_u^{1/2})^{-1} b$  where  $b$  is a small fraction defined by  $b := \left( \frac{B_y^{1/2}}{D_R^{1/2} B_\lambda^{3/2} N^{3/2}} \right)^{1/2} b_0$  and  $b_0 \leq 1$  is a uniform constant choosen to make  $\tau$  sufficiently small enough to satisfy the conditions of (5). Note that  $\tau$  is smaller than the natural time scale given in Section 2.2. Additionally, we define time cutoff functions as elements of a rescaled partition of unity in time

$$\phi_k(t) := \phi\left(\frac{t - k\tau}{\tau}\right), \quad \text{where } \sum_{u \in \mathbb{Z}} \phi^2(t - u) = 1$$

By our choice of  $\phi_k$  each scalar correction  $\Theta_I$  will be supported in a time interval  $[k\tau - \frac{2}{3}\tau, k\tau + \frac{2}{3}\tau]$ . The size of the time interval is choosen to optimize the size of the resulting errors between the transprt stress and high frequency interference stress.

#### 4.1.4 Angular Frequency Localizing Operator

The operator  $P_\lambda^I$  localizes to frequencies of order  $\lambda$  in a neighborhood around  $\lambda \nabla \hat{\xi}_I$ . This is identical to the frequency localizing operator used in [22, Section 5.1]. More precisely, for a function  $f \in C^0(\mathbb{T}^2)$  and an index  $I$  we define frequency localizing operators in the following way:

$$P_\lambda^I f := \int_{\mathbb{R}^2} f(x - h) \chi_\lambda^I(h) dh$$

where  $\hat{\chi}_\lambda^I(\xi) := \hat{\chi}_1^I(\lambda^{-1}\xi)$  and we take  $\hat{\chi}_1^I \in C_c^\infty(B_{|\nabla\hat{\xi}_I|/2}(\nabla\hat{\xi}_I))$  to be a smooth bump function such that  $\hat{\chi}_1^I(\xi) = 1$  if  $|\xi - \nabla\hat{\xi}_I| \leq |\nabla\hat{\xi}_I|/4$ . Moreover it follows from this definition that the corrections  $\Theta_I$  and  $\Theta$  have frequency support contained in the annulus of order  $\lambda$ . That is

$$\text{supp } \hat{\Theta}_I \cup \text{supp } \hat{\Theta} \subseteq \{\xi \in \hat{\mathbb{R}}^2 : \lambda/2 \leq |\xi| \leq 2\lambda\}$$

#### 4.1.5 Lifting Function

A lifting function is used in the calculation of the low-frequency stress error in Section 4.6 below. Its purpose is to scale the wave corrections so that they are larger than the low-frequency stress error that is approximately of size  $D_R$ . Let  $\tau$  be the time step from Definition 2.2 and let  $K$  be a large constant to be determined later in the calculation (31). We choose  $e(t) : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$  to be a function such that  $e(t) \geq K D_R$  for all  $t \in \{s + h | s \in J, |h| \leq \epsilon_t + (\Xi e_u^{1/2})^{-1}\}$ , where  $\epsilon_t$  is the flow mollification parameter defined below in (7), such that the bounds

$$\left\| \left( \frac{d}{dt} \right)^r e(t)^{1/2} \right\|_{C^0} \lesssim (\Xi e_u^{1/2})^r D_R^{1/2}, \quad 0 \leq r \leq 2$$

hold, and such that  $e$  has time support  $\text{supp }_t e(t) \subseteq N(J) := \{t + h : t \in J, |h| \leq 3(\Xi e_u^{1/2})^{-1}\}$ . A simple way to construct  $e(t)$  is to define  $N_0(J) = \{t + h : t \in J, |h| \leq 2(\Xi e_u^{1/2})^{-1}\}$ , take the characteristic function of  $N_0(J)$ , mollify in time with a non-negative convolution kernel  $\eta_{(\Xi e_u^{1/2})^{-1}}$  supported in  $\{|t| \leq (1/3)(\Xi e_u^{1/2})^{-1}\}$ , and multiply by  $(K D_R)^{1/2}$  to construct  $e^{1/2}(t)$ :

$$e^{1/2}(t) = (K D_R)^{1/2} \eta_{(\Xi e_u^{1/2})^{-1}} * \mathbf{1}_{N_0(J)}(t).$$

We remark that the last time support condition holds because  $\epsilon_t \leq (\Xi e_u^{1/2})^{-1}$ , which is verified in Section 4.8 below.

#### 4.1.6 Amplitudes

Given an index  $I = (k, f)$ , the amplitude  $\theta_I(x, t)$  has the form

$$\theta_I(x, t) = \lambda^{1/2} \gamma_I(x, t) \phi_k(t) e^{1/2}(t). \quad (6)$$

The factors  $e(t)$  and  $\phi_k(t)$  are the previously defined lifting function and time-cutoff. The functions  $\gamma_I : \mathbb{T}^2 \times \mathbb{R} \rightarrow \mathbb{R}$  are called the **coefficients**, and are specified in Section 4.6. In particular  $\gamma_I$  depends on the choice of initial directions  $\mathbb{F}$  from 4.1.1 and we construct it to be close in absolute value to 1 for the duration of the time interval  $\phi_k(t)$  is supported on.

## 4.2 Shape of the Velocity Corrections

By applying the Microlocal Lemma 4.1 with the convolution operator  $T^l P_\lambda^I$ , we can calculate the drift velocity correction  $U_I := U_I(x, t) : \mathbb{T}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$  from  $\Theta_I$ :

$$U_I^l := T^l \Theta_I = T^l P_\lambda^I [e^{i\lambda\xi_I} \theta_I] = e^{i\lambda\xi_I} (\theta_I m^l(\nabla\xi_I) + \delta u_I^l).$$

We define

$$U_I^l := e^{i\lambda\xi_I} (u_I^l + \delta u_I^l), \quad u_I^l := \theta_I m^l(\nabla\xi_I)$$

with an explicit error term  $\delta u_I^l$  given by the Microlocal Lemma 4.1. Here we have used that  $m^l$  is homogeneous of degree zero to obtain  $m^l(\lambda\nabla\xi_I) = m^l(\nabla\xi_I)$ , and have used that the frequency cutoff in the symbol of  $P_\lambda^I$  is equal to 1 at the point  $\lambda\nabla\xi_I$ , due to the condition  $|\nabla\xi_I - \nabla\hat{\xi}_I| \leq c_2$  in (5).

### 4.3 The Microlocal Lemma

We borrow the following lemma from [22, Section 4], which shows that a convolution operator applied to a highly oscillatory function is multiplication operator to leading order. In all of our applications the kernel  $K^l(h)$  will be a Schwartz function essentially supported on scales of order  $|h| \sim \lambda^{-1}$ . The Fourier-transform of a function  $K^l : \mathbb{R}^2 \rightarrow \mathbb{C}$  is normalized to be  $\hat{K}^l(\xi) = \int_{\mathbb{R}^2} e^{-i\xi \cdot h} K^l(h) dh$ .

**Lemma 4.1** (Microlocal Lemma). *Suppose that  $T^l[\Theta](x) = \int_{\mathbb{R}^2} \Theta(x - h) K^l(h) dh$  is a convolution operator acting on functions  $\Theta : \mathbb{T}^2 \rightarrow \mathbb{C}$  with a kernel  $K^l : \mathbb{R}^2 \rightarrow \mathbb{C}$  in the Schwartz class. Let  $\xi : \mathbb{T}^2 \rightarrow \mathbb{C}$  and  $\theta : \mathbb{T}^2 \rightarrow \mathbb{C}$  be smooth functions and  $\lambda \in \mathbb{Z}$  be an integer. Then for any input of the form  $\Theta = e^{i\lambda\xi(x)}\theta(x)$  we have the formula*

$$T^l[\Theta](x) = e^{i\lambda\xi(x)} \left( \theta(x) \hat{K}^l(\lambda\nabla\xi(x)) + \delta[T\Theta](x) \right)$$

where the error in the amplitude term has the explicit form

$$\begin{aligned} \delta[T^l\Theta](x) &= \int_0^1 dr \frac{d}{dr} \int_{\mathbb{R}^2} e^{-i\lambda\nabla\xi(x) \cdot h} e^{iZ(r,x,h)} \theta(x - rh) K^l(h) dh \\ Z(r,x,h) &= r\lambda \int_0^1 h^a h^b \partial_a \partial_b \xi(x - sh) (1-s) ds \end{aligned}$$

The proof of this lemma is given in [22, Section 4].

### 4.4 Mollification of the Stress Tensor

For the stress tensor,  $R$ , we define its spatial mollification  $R_{\epsilon_x}$  by  $R_{\epsilon_x}^{jl} := \chi_{\epsilon_x} * \chi_{\epsilon_x} * R^{jl}$ , with a Schwartz kernel at length scale  $\epsilon_x$  so that  $\chi_{\epsilon_x}(h) = \epsilon_x^{-2}\chi_1(\epsilon_x^{-1}h)$  where  $\chi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a Schwartz kernel with integral 1 with a vanishing moment condition that is  $\int_{\mathbb{R}^2} h^{\vec{a}} \chi_{\epsilon_x}(h) dh = 0$  for all multi-indices  $1 \leq |\vec{a}| \leq L$ . We next define the mollified stress tensor  $R_\epsilon$  by a mollification of  $R_{\epsilon_x}$  along the coarse scale flow in time:

$$R_\epsilon^{jl} = \eta_{\epsilon_t} *_\Phi R^{jl} := \int_{\mathbb{R}} R_{\epsilon_x}^{jl}(\Phi_s(t, x)) \eta_{\epsilon_t}(s) ds$$

Where  $\eta_{\epsilon_t}(s) = \epsilon_t^{-1}\eta_1(\epsilon_t^{-1}s)$  is a smooth, positive mollifying kernel with compact support in the interval  $|s| \leq \epsilon_t \leq (\Xi e_u^{1/2})^{-1}$  and  $\int_{\mathbb{R}} \eta_{\epsilon_t}(s) ds = 1$ , while  $\Phi_s(t, x)$  is the flow map of  $\partial_t + u \cdot \nabla$ , which takes values in  $\mathbb{R} \times \mathbb{T}^2$  and is defined as the unique solution to the ODE

$$\Phi_s(t, x) = (t + s, \Phi_s^i(t, x)), \quad \Phi_0(t, x) = (t, x), \quad \frac{d}{ds} \Phi_s^i(t, x) = u^i(\Phi_s(t, x)).$$

We choose the mollification parameters to be

$$\epsilon_x = c_0 N^{-\frac{3}{2L}} \Xi^{-1}, \quad \epsilon_t = c_0 N^{-\frac{3}{2}} \Xi^{-3/2} D_R^{-1/2} \tag{7}$$

By this choice of parameters we may obtain bounds that are similar to the bounds given in [17]. Finally, the constant  $c_0$  will only depend on the Schwartz kernels  $\eta_1, \chi_1$ . We state  $C^0$  norm estimates for  $R_\epsilon$  in Section 4.7.1 below.

## 4.5 The Error Terms

Let  $(\theta, u, R)$  be an SQG-Reynolds flow as defined in 2.1. We will construct the new scalar field  $\hat{\theta} := \theta + \Theta$  and  $\hat{u} = T[\hat{\theta}]$  by adding a high frequency correction  $\Theta$ . The new Reynolds stress solves

$$\nabla_j \nabla_l \hat{R}^{jl} = \nabla_j \nabla_l R_\epsilon^{jl} + \nabla_j \nabla_l (R^{jl} - R_\epsilon^{jl}) + (\partial_t \Theta + T^l \theta \nabla_l \Theta + T^l \Theta \nabla_l \theta) + T^l \Theta \nabla_l \Theta$$

If we recall that the corrections are of the form  $\Theta = \sum_I \Theta_I$  then we can decompose the new error into the following terms

$$\begin{aligned} \nabla_j \nabla_l \hat{R}^{jl} &= \nabla_j \nabla_l (R_T^{jl} + R_H^{jl} + R_M^{jl} + R_S^{jl}) \\ \text{where } \nabla_j \nabla_l R_H^{jl} &= \sum_{J,I \in \mathbb{Z} \times F, J \neq \bar{I}} T^l [\Theta_I] \nabla_l \Theta_J + T^l [\Theta_J] \nabla_l \Theta_I \\ \nabla_j \nabla_l R_T^{jl} &= \partial_t \Theta + T^l \theta \nabla_l \Theta + T^l \Theta \nabla_l \theta \\ \nabla_j \nabla_l R_M^{jl} &= \nabla_j \nabla_l (R^{jl} - R_\epsilon^{jl}) \\ \nabla_j \nabla_l R_S^{jl} &= \nabla_j \nabla_l R_\epsilon^{jl} + \nabla_l \sum_{I \in \mathbb{Z} \times F} T^l [\Theta_I] \Theta_{\bar{I}} \end{aligned}$$

We refer to the term  $R_S$  as the **Low frequency stress error**,  $R_T$  as the **Transport stress error**,  $R_H$  as the **High frequency interference stress error**, and  $R_M$  as the **Mollification stress error**. These terms are symmetric  $(2,0)$ -tensors. Let  $\mathcal{R}$  be the second order anti-divergence operator defined in 4.5.1, let  $B_\lambda$  be the bilinear anti-divergence operator defined in (17), and let  $M_{[k]}$  be the constant matrix defined in (23). We can define the stress errors using these tools as

$$\hat{R}^{jl} := R_H^{jl} + R_T^{jl} + R_M^{jl} + R_S^{jl} \quad (8)$$

$$R_H^{jl} := \sum_{J,I \in \mathbb{Z} \times F, J \neq \bar{I}} \mathcal{R}^{jl} [T^l [\Theta_I] \nabla_l \Theta_J + T^l [\Theta_J] \nabla_l \Theta_I] \quad (9)$$

$$R_T^{jl} := \mathcal{R}^{jl} [\partial_t \Theta + T^l \theta \nabla_l \Theta + T^l \Theta \nabla_l \theta] \quad (10)$$

$$R_M^{jl} := R^{jl} - R_\epsilon^{jl} \quad (11)$$

$$R_S^{jl} := \sum_{I=(k,f) \in \mathbb{Z} \times F} B_\lambda^{jl} [\Theta_I, \Theta_{\bar{I}}] - \phi_k^2(t) (e(t) M_{[k]} - R_\epsilon^{jl}) \quad (12)$$

The operator  $\mathcal{R}$  above satisfies  $\nabla_j \nabla_l \mathcal{R}^{jl}[U] = U$  whenever  $U$  has mean zero. Note that the argument of  $\mathcal{R}$  in lines (9) and (10) has mean zero due to the frequency support of  $\partial_t \Theta$  and the fact that the other terms can be written in divergence form. In line (12), we use that  $\nabla_j \nabla_l B_\lambda^{jl} [\Theta_I, \Theta_{\bar{I}}] = \nabla_l [T^l [\Theta_I] \Theta_{\bar{I}} + T^l [\Theta_{\bar{I}}] \Theta_I]$  and that the spatial divergence of a function of  $t$  alone is 0.

In the following sections we define the operators used above as well as other tools used in estimating the stress tensors. To prove the Main Lemma we will prove estimates on each of the stress errors. What we will show is that for the weighted norm  $\hat{H}[\cdot]$  defined in Lemma 4.3,

$$\hat{H}[R_T] \lesssim (B_\lambda N \Xi)^{-3/2} \mathbb{D}_R^{1/2} (\Xi e_u^{1/2}) b^{-1}, \quad \hat{H}[R_S] \lesssim B_\lambda^{-1} N^{-1} \mathbb{D}_R, \quad \hat{H}[R_H] \lesssim b \mathbb{D}_R, \quad \hat{H}[R_M] \leq \frac{G \mathbb{D}_R}{1000}$$

Note that the choice of  $b$  in Section 4.1.3 optimizes between  $R_T$  and  $R_H$ , which are the largest errors.

### 4.5.1 Second Order Anti-Divergence Equation

We use a second order anti-divergence operator to produce a symmetric and traceless  $(2,0)$  tensor for the transport stress error and the high-high interference error. For all smooth, mean-zero  $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ ,

we define the **second order anti-divergence operator** as the operator

$$\mathcal{R} := 2\mathcal{R}_1 - \mathcal{R}_2 \quad (13)$$

where  $\mathcal{R}_1$  and  $\mathcal{R}_2$  are the second order anti-divergece operators  $\mathcal{R}_1^{jl} := \nabla^j \nabla^l \Delta^{-2}$ ,  $\mathcal{R}_2^{jl} := \delta^{jl} \Delta^{-1}$ . We note that  $\mathcal{R}^{jl}[f]$  is a symmetric, traceless tensor that satisfies  $\nabla_j \nabla_l \mathcal{R}^{jl}[f] = f$ .

#### 4.5.2 A Bilinear Anti-Divergence Form

We use a bilinear first order anti-divergence operator to produce a symmetric, traceless (2,0) tensor for the low frequency stress error. The work of this section is motivated by [3, Section 5.4.2].

Consider the problem of solving  $\nabla_j R^{j\ell} = Q^\ell$  for the term

$$Q^\ell(x) := T^\ell[\Theta_I]\Theta_J + T^\ell[\Theta_J]\Theta_I.$$

Although it is not obvious that an anti-divergence for  $Q^\ell$  would have a good form, there is at least hope of finding an anti-divergence for  $Q^\ell$  as  $Q^\ell$  satisfies the necessary and sufficient condition of having integral 0 due to the anti-self-adjointness of  $T^\ell$ , which is equivalent to the multiplier  $m^\ell$  being odd. Since the anti-self-adjointness is most easily seen in frequency space, there is motivation to look at the problem in frequency space<sup>3</sup> to define  $R^{jl}$ .

Recall now that the corrections from Section 4.1 are constructed with compact frequency support in the annulus  $\{\xi : 2^{-1}\lambda \leq |\xi| < 2\lambda\}$ . We start by replacing  $\Theta_I, \Theta_J$  with Schwartz approximations that we will also call  $\Theta_I, \Theta_J$  by an abuse of notation. By using a mollification in frequency space we may assume that the Schwartz approximations maintain frequency support in  $\{\xi : 2^{-1}\lambda \leq |\xi| < 2\lambda\}$ , and that they converge pointwise in physical space while also remaining uniformly bounded there. For these approximations we construct a bilinear anti-divergence as follows. Define a smooth bump function  $\hat{\chi}_{\approx \lambda}$  that is 1 on the annulus  $\{\xi : 2^{-1}\lambda \leq |\xi| < 2\lambda\}$  so that  $\hat{\Theta}_I = \hat{\chi}_{\approx \lambda} \Theta_I$  and  $\hat{\Theta}_J = \hat{\chi}_{\approx \lambda} \Theta_J$ . Recall from Definition 2.1 that  $m$  is the multiplier for  $T$  and take a Fourier transform of  $Q^\ell(x)$ .

$$\begin{aligned} \hat{Q}^\ell(\xi) &= \int_{\hat{\mathbb{R}}^2} m^\ell(\xi - \eta) \hat{\Theta}_I(\xi - \eta) \hat{\Theta}_J(\eta) + \hat{\Theta}_I(\xi - \eta) m^\ell(\eta) \hat{\Theta}_J(\eta) \frac{d\eta}{(2\pi)^2} \\ &= \int_{\hat{\mathbb{R}}^2} m^\ell(\xi - \eta) \hat{\chi}_{\approx \lambda} \hat{\Theta}_I(\xi - \eta) \hat{\Theta}_J(\eta) + \hat{\Theta}_I(\xi - \eta) m^\ell(\eta) \hat{\chi}_{\approx \lambda} \hat{\Theta}_J(\eta) \frac{d\eta}{(2\pi)^2} \\ \text{Define } m_{\approx \lambda}(\cdot) &:= m(\cdot) \hat{\chi}_{\approx \lambda}(\cdot) \\ &= \int_{\hat{\mathbb{R}}^2} [m_{\approx \lambda}^\ell(\xi - \eta) + m_{\approx \lambda}^\ell(-\eta)] \hat{\Theta}_I(\xi - \eta) \hat{\Theta}_J(\eta) \frac{d\eta}{(2\pi)^2} \\ \text{By the oddness of the multiplier } m_{\approx \lambda}^\ell & \\ &= \int_{\hat{\mathbb{R}}^2} [m_{\approx \lambda}^\ell(\xi - \eta) - m_{\approx \lambda}^\ell(-\eta)] \hat{\Theta}_I(\xi - \eta) \hat{\Theta}_J(\eta) \frac{d\eta}{(2\pi)^2} \\ &= i\xi_j \int_{\hat{\mathbb{R}}^2} \int_0^1 -i\nabla^j m_{\approx \lambda}^\ell(u_\sigma) d\sigma \hat{\Theta}_I(\zeta) \hat{\Theta}_J(\eta) \frac{d\eta}{(2\pi)^2} \\ &=: i\xi_j \int_{\hat{\mathbb{R}}^2} \hat{K}_\lambda^{jl}(\zeta, \eta) \hat{\Theta}_I(\zeta) \hat{\Theta}_J(\eta) \frac{d\eta}{(2\pi)^2}, \end{aligned} \quad (14)$$

where we have set

$$\zeta := \xi - \eta, \quad u_\sigma := \sigma(\xi - \eta) - (1 - \sigma)\eta,$$

---

<sup>3</sup>See also [21] for a different anti-divergence operator derived using Taylor expansion in frequency space.

and we have used Taylor's Remainder Theorem in the last equality. In the last line we defined

$$\hat{K}_\lambda^{jl}(\zeta, \eta) := \int_0^1 -i\nabla^j m_{\approx\lambda}^l(u_\sigma) \hat{\chi}_{\approx\lambda}(\zeta) \hat{\chi}_{\approx\lambda}(\eta) d\sigma. \quad (15)$$

Line (14) defines in frequency space a solution  $R^{jl}$  to  $\nabla_j R^{jl} = Q^l$  that is not symmetric, whereas our ultimate goal is in fact to define a solution to  $\nabla_j \nabla_l R^{jl} = \nabla_l Q^l$  that is symmetric. Therefore we proceed by taking the symmetric part of the above solution (noting that  $\nabla_j \nabla_l R_{\text{asym}}^{jl} = 0$  for the anti-symmetric part), and taking an inverse Fourier transform.

Define the symmetric part of this kernel as

$$\hat{K}_{\lambda,\text{sym}}^{jl} := \frac{\hat{K}_\lambda^{jl} + \hat{K}_\lambda^{lj}}{2}. \quad (16)$$

It can be shown that  $\hat{K}_{\lambda,\text{sym}}$  is traceless for the specific multiplier of the SQG equation. Now consider

$$\hat{Q}_2^l(\xi) := i\xi_j \int_{\hat{\mathbb{R}}^2} \hat{K}_{\lambda,\text{sym}}^{jl}(\zeta, \eta) \hat{\Theta}_I(\zeta) \hat{\Theta}_J(\eta) \frac{d\eta}{(2\pi)^2}$$

and take the inverse Fourier Transform to obtain

$$\begin{aligned} Q_2^l(x) &= \nabla_j \int_{\hat{\mathbb{R}}^2} e^{i\xi \cdot x} \int_{\hat{\mathbb{R}}^2} \hat{K}_{\lambda,\text{sym}}^{jl}(\zeta, \eta) \hat{\Theta}_I(\zeta) \hat{\Theta}_J(\eta) \frac{d\eta}{(2\pi)^2} \frac{d\xi}{(2\pi)^2} \\ &= \nabla_j \int_{\hat{\mathbb{R}}^2 \times \hat{\mathbb{R}}^2} e^{i(\zeta+\eta) \cdot x} \hat{K}_{\lambda,\text{sym}}^{jl}(\zeta, \eta) \hat{\Theta}_I(\zeta) \hat{\Theta}_J(\eta) \frac{d\eta}{(2\pi)^2} \frac{d\zeta}{(2\pi)^2}. \end{aligned}$$

Observe that  $\hat{K}_{\lambda,\text{sym}}^{jl}$  is compactly supported in frequency and smooth due to the factors of  $\hat{\chi}_{\approx\lambda}$ , making  $\hat{K}_{\lambda,\text{sym}}^{jl}$  a Schwartz function. Thus we can define the corresponding physical space kernel  $K_{\lambda,\text{sym}}^{jl}(h_1, h_2)$  as the inverse Fourier Transform of  $\hat{K}_{\lambda,\text{sym}}^{jl}(\zeta, \eta)$  so that

$$\hat{K}_{\lambda,\text{sym}}^{jl}(\zeta, \eta) = \int_{\hat{\mathbb{R}}^2 \times \hat{\mathbb{R}}^2} e^{-i(\zeta, \eta) \cdot (h_1, h_2)} K_{\lambda,\text{sym}}^{jl}(h_1, h_2) dh_1 dh_2.$$

Then

$$\begin{aligned} Q_2^l(x) &= \nabla_j \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i(\zeta+\eta) \cdot x} \int_{\hat{\mathbb{R}}^2 \times \hat{\mathbb{R}}^2} e^{-i(\zeta, \eta) \cdot (h_1, h_2)} K_{\lambda,\text{sym}}^{jl}(h_1, h_2) dh_1 dh_2 \hat{\Theta}_I(\zeta) \hat{\Theta}_J(\eta) \frac{d\eta}{(2\pi)^2} \frac{d\zeta}{(2\pi)^2} \\ &= \nabla_j \int_{\mathbb{R}^2 \times \mathbb{R}^2} K_{\lambda,\text{sym}}^{jl}(h_1, h_2) \int_{\hat{\mathbb{R}}^2} e^{i\zeta \cdot (x-h_1)} \hat{\Theta}_I(\zeta) \frac{d\zeta}{(2\pi)^2} \int_{\hat{\mathbb{R}}^2} e^{i\eta \cdot (x-h_2)} \hat{\Theta}_J(\eta) \frac{d\eta}{(2\pi)^2} dh_1 dh_2 \\ &= \nabla_j \int_{\mathbb{R}^2 \times \mathbb{R}^2} K_{\lambda,\text{sym}}^{jl}(h_1, h_2) \Theta_I(x-h_1) \Theta_J(x-h_2) dh_1 dh_2 \\ &=: \nabla_j B_\lambda^{jl}[\Theta_I, \Theta_J](x) \end{aligned}$$

Here we define the **Bilinear anti-divergence operator** or **Bilinear form** as

$$B_\lambda^{jl}[F_1, F_2](x) := \int_{\mathbb{R}^2 \times \mathbb{R}^2} K_{\lambda,\text{sym}}^{jl}(h_1, h_2) F_1(x-h_1) F_2(x-h_2) dh_1 dh_2, \quad (17)$$

for scalar fields  $F_1, F_2 \in C^\infty(\mathbb{T}^2)$ . This bi-convolution operator satisfies

$$T^l[\Theta_I] \nabla_l \Theta_J + T^l[\Theta_J] \nabla_l \Theta_I = \nabla_l \nabla_j B_\lambda^{jl}[\Theta_I, \Theta_J](x) \quad (18)$$

for all Schwartz approximations to  $\Theta_I, \Theta_J$  with Fourier support in  $\{|\xi| \sim \lambda\}$ . Since  $\|K_{\lambda,\text{sym}}\|_{L^1(\mathbb{R}^2)}$  is finite, we may pass to the limit in (17) from Schwartz approximations using the dominated convergence theorem to conclude that (18) holds (both in  $\mathcal{D}'$  and classically) for the periodic functions  $\Theta_I, \Theta_J$ . We note that by scaling considerations we have the following inequality for  $h = (h_1, h_2) \in \mathbb{R}^2 \times \mathbb{R}^2$

$$\lambda^m \| |h|^m K_{\lambda,\text{sym}}^{j\ell}(h_1, h_2) \|_{L^1(\mathbb{R}^2 \times \mathbb{R}^2)} \lesssim_m \lambda^{-1}, \quad (19)$$

and we also observe for use in the calculations of Section 4.6.2 that, by (15),

$$\hat{K}_{\lambda,\text{sym}}(\lambda \nabla \xi_I, -\lambda \nabla \xi_I) = -i(2\lambda)^{-1} (\nabla^j m^l + \nabla^l m^j)(\nabla \xi_I) \quad (20)$$

is homogeneous of degree  $-1$  in  $\lambda$  and in  $\nabla \xi_I$ . In the above we have used that  $\nabla m$  is degree  $-1$  homogeneous, and that  $\hat{\chi}_{\approx \lambda}(\lambda \nabla \xi_I) = 1$  while  $\nabla \hat{\chi}_{\approx \lambda}(\lambda \nabla \xi_I) = 0$  due to the condition (5), which keeps the phase gradient within a small distance of its initial conditions.

## 4.6 Low frequency part of the error

The goal of this section is to explain how the wave corrections described in Section 4.1 are used to cancel the Reynolds stress error in the Low frequency stress error. The tools described in this section are similar to ones used in [22]. In this section we will define the coefficient functions  $\gamma$ .

### 4.6.1 The Bilinear Microlocal Lemma

The lemma below is inspired from an analogous version used in [22] and stated in 4.1. This tool tells us that the bilinear convolution operator acts like a multiplicative operator to leading order when it is applied to our highly oscillatory scalar field corrections  $\Theta$ .

**Lemma 4.2** (Bilinear Microlocal Lemma). *For conjugate scalar field corrections  $\Theta_I, \Theta_{\bar{I}}$  as defined in 4.1 and a bilinear form  $B_\lambda$  as defined in (17) we have the following identity*

$$B_\lambda^{j\ell}[\Theta_I, \Theta_{\bar{I}}](x, t) = \theta_I^2(x, t) \hat{K}_{\lambda,\text{sym}}^{j\ell}[\lambda \nabla \xi_I, -\lambda \nabla \xi_I] + \delta B_I^{j\ell}(x, t)$$

where  $\delta B^{j\ell}$  is a small error term that has the explicit form

$$\begin{aligned} \delta B_I^{j\ell}(x, t) &:= \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i\lambda \nabla \xi_I(x) \cdot h_1} e^{-i\lambda \nabla \xi_I(x) \cdot h_2} K_{\lambda,\text{sym}}^{j\ell}[h_1, h_2] \theta_I(x) Y(x, h_1) dh_1 dh_2 \\ &\quad + \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i\lambda \nabla \xi_I(x) \cdot h_1} e^{-i\lambda \nabla \xi_I(x) \cdot h_2} K_{\lambda,\text{sym}}^{j\ell}[h_1, h_2] \theta_I(x) Y(x, h_2) dh_1 dh_2 \\ &\quad + \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i\lambda \nabla \xi_I(x) \cdot h_1} e^{-i\lambda \nabla \xi_I(x) \cdot h_2} K_{\lambda,\text{sym}}^{j\ell}[h_1, h_2] Y(x, h_1) \cdot Y(x, h_2) dh_1 dh_2 \\ Y(x, h) &:= \int_0^1 \frac{d}{dr} e^{iZ(r, x, h)} \theta_I(x - rh) dr \\ Z(r, x, h) &:= r\lambda \int_0^1 h^a h^b \partial_a \partial_b \xi(x - sh)(1-s) ds \end{aligned}$$

*Proof.* The proof is based on the Taylor expansion as in the Microlocal Lemma 4.1. The starting point

for the computation is the approximation

$$\begin{aligned}
B_\lambda^{j\ell}[\Theta_I, \Theta_{\bar{I}}] &= \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i\lambda\xi_I(x-h_1)} e^{-i\lambda\xi_I(x-h_2)} \theta_I(x-h_1) \theta_{\bar{I}}(x-h_2) K_{\lambda, \text{sym}}^{jl}[h_1, h_2] dh_1 dh_2 \\
&= \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left[ e^{i\lambda(\xi_I(x-h_1) - \xi_I(x))} \theta_I(x-h_1) \right] \left[ e^{-i\lambda(\xi_I(x-h_2) - \xi_I(x))} \theta_{\bar{I}}(x-h_2) \right] K_{\lambda, \text{sym}}^{jl}[h_1, h_2] dh_1 dh_2 \\
&\approx \theta_I(x)^2 \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{-i\lambda\nabla\xi_I \cdot h_1} e^{i\lambda\nabla\xi_I \cdot h_2} K_{\lambda, \text{sym}}^{jl}[h_1, h_2] dh_1 dh_2,
\end{aligned}$$

where we recognize that the last integral is exactly  $\hat{K}_{\lambda, \text{sym}}^{jl}[\lambda\nabla\xi_I(x), -\lambda\nabla\xi_I(x)]$ .  $\square$

#### 4.6.2 Explicitly Calculating the Symmetric Multiplier

We explicitly calculate the  $\hat{K}_{\lambda, \text{sym}}^{j\ell}$  term. Starting from  $m^l(p) = i\epsilon^{\ell a} p_a / |p|$  and (20), we have

$$\begin{aligned}
\hat{K}_{\lambda, \text{sym}}^{j\ell}(\lambda\nabla\xi_I, -\lambda\nabla\xi_I) &= -2^{-1}\lambda^{-1}i(\nabla^j m^\ell + \nabla^\ell m^j)(\nabla\xi_I) \\
&= (-2\lambda)^{-1} \left( \frac{\epsilon^{\ell a} \nabla_a \xi_I \nabla^j \xi_I + \epsilon^{ja} \nabla_a \xi_I \nabla^\ell \xi_I}{|\nabla\xi_I|^3} \right) \\
\hat{K}_{\lambda, \text{sym}}(\lambda\nabla\xi_I, -\lambda\nabla\xi_I) &= (2\lambda)^{-1} \left[ \frac{-2\nabla_1 \xi_I \nabla_2 \xi_I}{(\nabla_1 \xi_I)^2 - (\nabla_2 \xi_I)^2} \frac{(\nabla_1 \xi_I)^2 - (\nabla_2 \xi_I)^2}{2\nabla_1 \xi_I \nabla_2 \xi_I} \right] |\nabla\xi_I|^{-3}
\end{aligned} \tag{21}$$

In particular, the result is traceless and homogeneous of degree  $-1$  in  $\lambda$  and the phase gradients.

#### 4.6.3 Solving the Quadratic equation using Linear Phase Functions

Our goal now is to define coefficients  $\gamma_I$  such that the low frequency error term (12) is close to zero.

We will treat the term  $R_\epsilon$  as a lower order term, and will treat the phase gradients  $\nabla\xi_I$  as perturbations of their initial conditions  $\nabla\hat{\xi}_I$ , which are constants. Therefore, employing the approximation

$$B_\lambda[\Theta_I, \Theta_{\bar{I}}] \approx \theta_{\bar{I}}^2 \hat{K}_{\lambda, \text{sym}}^{jl}[\lambda\nabla\xi_I, -\lambda\nabla\xi_I] \approx \theta_{\bar{I}}^2 \hat{K}_{\lambda, \text{sym}}^{jl}[\lambda\nabla\hat{\xi}_I, -\lambda\nabla\hat{\xi}_I]$$

we are motivated by (12) to consider the following linear system

$$\sum_{I \in \{k\} \times \mathbb{F}} \theta_I^2(x, t) \hat{K}_{\lambda, \text{sym}}^{j\ell}[\lambda\nabla\hat{\xi}_I, -\lambda\nabla\hat{\xi}_I] = \phi_k(t) e(t) M_{[k]} \tag{22}$$

In this case, the  $\gamma_I(x, t) = \hat{\gamma}_I$  will be constant coefficients and we assume the linear initial conditions  $\nabla\hat{\xi}_I$  defined in Section 4.1.2. We define a set of constant matrices  $M_{[k]}$  as

$$M_{[k]} := 2^{-1}5^{-3/2} \begin{bmatrix} -8 & 0 \\ 0 & 8 \end{bmatrix} \text{ if } k \equiv 0 \pmod{2}, \quad M_{[k]} := 2^{-1}5^{-3/2} \begin{bmatrix} 8 & 0 \\ 0 & -8 \end{bmatrix} \text{ if } k \equiv 1 \pmod{2}.$$

For any index  $I = (k, f) \in \mathbb{Z} \times \mathbb{F}$  as defined in Section 4.1.1 we can explicitly calculate the left hand

tensor of (22) in terms of

$$\begin{aligned}\hat{K}_{\text{sym}}^{j\ell}(\nabla \hat{\xi}_{(k, J^k(1,2))}, -\nabla \hat{\xi}_{(k, J^k(1,2))}) &= \begin{cases} 2^{-1}5^{-3/2} \begin{bmatrix} -4 & -3 \\ -3 & 4 \end{bmatrix} & \text{if } [k] = 0 \\ 2^{-1}5^{-3/2} \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix} & \text{if } [k] = 1 \end{cases} \\ \hat{K}_{\text{sym}}^{j\ell}(\nabla \hat{\xi}_{(k, J^k(2,1))}, -\nabla \hat{\xi}_{(k, J^k(2,1))}) &= \begin{cases} 2^{-1}5^{-3/2} \begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix} & \text{if } [k] = 0 \\ 2^{-1}5^{-3/2} \begin{bmatrix} 4 & -3 \\ -3 & -4 \end{bmatrix} & \text{if } [k] = 1. \end{cases}\end{aligned}\quad (23)$$

We observe that in either the case  $[k] = 0$  or  $[k] = 1$  the matrix pair is linearly independent and hence crucially spans the two dimensional space of symmetric, traceless  $2 \times 2$  matrices. We also note that

$$\hat{K}_{\lambda, \text{sym}}^{j\ell}[\lambda \nabla \hat{\xi}_I, -\lambda \nabla \hat{\xi}_I] = \lambda^{-1} \hat{K}_{\text{sym}}^{j\ell}[\nabla \hat{\xi}_I, -\nabla \hat{\xi}_I].$$

By applying this identity to (22) and recalling  $\theta_I^2 = \lambda \gamma_I^2 e(t) \phi_k(t)$  from Section 4.1.6 then for any  $k \in \mathbb{Z}$  solving (22) becomes equivalent to solving the following equation of matrices

$$\begin{aligned}M_{[k]} = &\hat{\gamma}_{([k], J^k(1,2))}^2 \hat{K}_{\text{sym}}^{j\ell}(\nabla \hat{\xi}_{(k, J^k(1,2))}, -\nabla \hat{\xi}_{(k, J^k(1,2))}) \\ &+ \hat{\gamma}_{([k], J^k(2,1))}^2 \hat{K}_{\text{sym}}^{j\ell}(\nabla \hat{\xi}_{(k, J^k(2,1))}, -\nabla \hat{\xi}_{(k, J^k(2,1))})\end{aligned}\quad (24)$$

The equation of matrices above is solvable by using constant coefficients  $\hat{\gamma}_{([k], J^k(1,2))}^2 = \hat{\gamma}_{([k], J^k(2,1))}^2 = 1$ .

#### 4.6.4 Solving the Cancellation as a Perturbation of a Linear System

As typical with convex integration schemes we will perform the error cancellation on a mollified version of the stress tensor  $R_\epsilon$  as defined in 4.4. Please refer to [17, Section 18] for detailed discussion of the mollified stress tensor. Consider the term  $\nabla_j \nabla_j R_S^{jl}$  from Section 4.5

$$\nabla_j \nabla_\ell R_S^{jl} = \nabla_j \nabla_l R_\epsilon^{jl} + \nabla_l \sum_{I=(k,f) \in \mathbb{Z} \times F} T^l [\Theta_I] \Theta_{\bar{I}} \quad (25)$$

Let  $e(t)$  be the lifting function defined in Section 4.1.5. Also let  $M_{[k]}$  be the constant matrix defined in (23) and let  $B_\lambda$  be the Bilinear form defined in (17) then (25) is equivalent to

$$(25) = \nabla_j \nabla_\ell \sum_{I=(k,f) \in \mathbb{Z} \times F} B_\lambda^{jl} [\Theta_I, \Theta_{\bar{I}}] - \phi_k^2(t) (e(t) M_{[k]} - R_\epsilon^{jl})$$

If we drop the double divergence from the above expression then we recover the definition of  $R_S$  from (12). We can then apply the Bilinear Microlocal Lemma 4.2 to  $R_S$  to produce

$$R_S = \sum_{I=(k,f) \in \mathbb{Z} \times F} \theta_I^2(x, t) \hat{K}_{\lambda, \text{sym}}^{j\ell}[\lambda \nabla \xi_I, -\lambda \nabla \xi_I] - \sum_k e(t) \phi_k^2(t) \left( M_{[k]} - \frac{R_\epsilon^{jl}}{e(t)} \right) + \sum_I \delta B_I^{jl}(x, t) \quad (26)$$

We will show that on every time interval, indexed by  $k$ , the first two summands add to zero pointwise in  $(x, t)$ . Without loss of generality consider any single time interval indexed by  $k$ . We will prove

$$\sum_{I \in \{k\} \times \mathbb{F}} \theta_I^2(x, t) \hat{K}_{\lambda, \text{sym}}^{j\ell}[\lambda \nabla \xi_I, -\lambda \nabla \xi_I] - e(t) \phi_k^2(t) \left( M_{[k]} - \frac{R_\epsilon^{jl}}{e(t)} \right) = 0 \quad (27)$$

So recall  $\theta_I^2 = \lambda\gamma_I^2 e(t)\phi_k^2(t)$  from Section 4.1.6 and that  $\hat{K}_{\text{sym}}^{j\ell}$  has homogeneity order  $-1$  from (21). By factoring out appropriately can see that the left hand side of (27) vanishes if we can solve

$$\sum_{I \in \{k\} \times \mathbb{F}} \gamma_I^2 \hat{K}_{\lambda, \text{sym}}^{j\ell} [\nabla \xi_I, -\nabla \xi_I] = M_{[k]}^{j\ell} - \varepsilon^{jl}, \quad \text{where } \varepsilon^{jl} := \frac{R_\epsilon^{j\ell}}{e(t)} \quad (28)$$

We view (28) as a perturbation of our solved linear system (22). Let  $\mathring{\mathcal{S}}_{2 \times 2}$  be the space symmetric, traceless  $(2, 0)$  tensors. For parameters  $(p_1, p_2)$  define the linear map  $\mathcal{L}_{(p_1, p_2)} : \mathbb{R} \times \mathbb{R} \rightarrow \mathring{\mathcal{S}}_{2 \times 2}$

$$\mathcal{L}_{(p_1, p_2)}(x_1, x_2) = \left( x_1 \hat{K}_{\lambda, \text{sym}}^{j\ell}[p_1, -p_1] + x_2 \hat{K}_{\lambda, \text{sym}}^{j\ell}[p_2, -p_2] \right)$$

If we set

$$p_1 = \nabla \hat{\xi}_{I_1}, \quad p_2 = \nabla \hat{\xi}_{I_2} \quad I_1 = [k] \times J^k(1, 2), \quad I_2 = [k] \times J^k(2, 1)$$

then by the calculation (24) we know that  $\mathcal{L}_{(\nabla \hat{\xi}_{I_1}, \nabla \hat{\xi}_{I_2})}(\hat{\gamma}_{I_1}^2, \hat{\gamma}_{I_2}^2) = \mathcal{L}_{(\nabla \hat{\xi}_{I_1}, \nabla \hat{\xi}_{I_2})}(1, 1) = M_{[k]}$ . Moreover for the parameters  $\nabla \hat{\xi}_{I_1}$ , and  $\nabla \hat{\xi}_{I_2}$ , we know from the linear independence of the matrices in (23) that  $\mathcal{L}_{(\nabla \hat{\xi}_{I_1}, \nabla \hat{\xi}_{I_2})}$  maps a basis in  $\mathbb{R}^2$  to a basis in  $\mathring{\mathcal{S}}_{2 \times 2}$ . Therefore the map  $\mathcal{L}_{(\nabla \hat{\xi}_{I_1}, \nabla \hat{\xi}_{I_2})}$  is an invertible linear map and by (24)

$$\mathcal{L}_{(\nabla \hat{\xi}_{I_1}, \nabla \hat{\xi}_{I_2})}^{-1}(M_{[k]}) = (1, 1). \quad (29)$$

Now  $\mathcal{L}_{(p_1, p_2)}^{-1}$  is a map that depends smoothly on the parameters  $(p_1, p_2)$ . Thus we may choose a constant  $c_2 > 0$  and sufficiently small in the condition (5) such that

$$|\nabla \xi_I - \nabla \hat{\xi}_I| < c_2 \implies \left\| \mathcal{L}_{(\nabla \xi_{I_1}, \nabla \xi_{I_2})}^{-1} - \mathcal{L}_{(\nabla \hat{\xi}_{I_1}, \nabla \hat{\xi}_{I_2})}^{-1} \right\| < \frac{1}{4\|M_{[k]}\|}. \quad (30)$$

Furthermore when we define our lifting function  $e(t)$  from Section 4.1.5 we may choose a sufficiently large constant  $K$  such that

$$e(t) > K \|R_\epsilon\|_{C^0} \implies \|\varepsilon\|_{L^\infty} < 8^{-1} \left\| \mathcal{L}_{(\nabla \hat{\xi}_{I_1}, \nabla \hat{\xi}_{I_2})}^{-1} \right\|_{op}^{-1} \quad \text{and} \quad \|M_{[k]} - \varepsilon\|_{L^\infty} \leq 2\|M_{[k]}\|_{L^\infty}. \quad (31)$$

We combine these two conditions with (29) to get

$$\begin{aligned} \left\| (1, 1) - \mathcal{L}_{(\nabla \xi_{I_1}, \nabla \xi_{I_2})}^{-1}(M_{[k]} - \varepsilon) \right\|_{L^\infty} &\leq \left\| \mathcal{L}_{(\nabla \hat{\xi}_{I_1}, \nabla \hat{\xi}_{I_2})}^{-1}(M_{[k]}) - \mathcal{L}_{(\nabla \hat{\xi}_{I_1}, \nabla \hat{\xi}_{I_2})}^{-1}(M_{[k]} - \varepsilon) \right\|_{L^\infty} \\ &\quad + \left\| \mathcal{L}_{(\nabla \hat{\xi}_{I_1}, \nabla \hat{\xi}_{I_2})}^{-1}(M_{[k]} - \varepsilon) - \mathcal{L}_{(\nabla \xi_{I_1}, \nabla \xi_{I_2})}^{-1}(M_{[k]} - \varepsilon) \right\|_{L^\infty} \\ &\leq \left\| \mathcal{L}_{(\nabla \hat{\xi}_{I_1}, \nabla \hat{\xi}_{I_2})}^{-1} \right\|_{op} \|\varepsilon\|_{L^\infty} + \left\| \mathcal{L}_{(\nabla \xi_{I_1}, \nabla \xi_{I_2})}^{-1} - \mathcal{L}_{(\nabla \hat{\xi}_{I_1}, \nabla \hat{\xi}_{I_2})}^{-1} \right\|_{op} \|M_{[k]} - \varepsilon\|_{L^\infty} \\ &\leq 1/2. \end{aligned}$$

Let  $(\gamma_{I_1}^2, \gamma_{I_2}^2) = \mathcal{L}_{(\nabla \xi_{I_1}, \nabla \xi_{I_2})}(M_{[k]} - \varepsilon)$ . By our previous calculations we can ensure that  $1/2 < \gamma_{I_1}^2, \gamma_{I_2}^2 < 2$  on the time support of  $\phi_k(t)$ . Therefore there exists  $1/2 < \gamma_{I_1}, \gamma_{I_2} < 2$  that solve (27) pointwise in  $(x, t)$ . By these calculations we can define the **coefficients**

$$\gamma_I = \gamma_I(\nabla \xi_I, \varepsilon) \in (2^{-1}, 2) \quad (32)$$

as the unique, positive functions solving

$$\sum_{I \in \{k\} \times \mathbb{F}} \theta_I^2(x, t) \hat{K}_{\lambda, \text{sym}}^{j\ell}[\lambda \nabla \xi_I, -\lambda \nabla \xi_I] - e(t) \phi_k^2(t) \left( M_{[k]} - \frac{R_\epsilon^{j\ell}}{e(t)} \right) = 0$$

from (26). We give  $C^0$ -norm estimates for  $\gamma_I$  in Section 4.7.1. It follows that (26) reduces to

$$R_S = \sum_{I \in \mathbb{Z} \times \mathbb{F}} \delta B_I^{jl}. \quad (33)$$

## 4.7 Estimates

### 4.7.1 Basic Estimates for the Construction

We state all of the estimates necessary for controlling the wave corrections.

**Proposition 4.1** (Phase Gradient Estimates). *For the phase gradients  $\nabla \xi_I$  defined in Section 4.1.2 there exists a positive number,  $b_0 \leq 1$  such that the conditions (5) on  $\nabla \xi_I$  hold for  $|t - t(I)| \leq \tau$ , and such that for all  $t \in \mathbb{R}$  with  $|t - t(I)| \leq \tau$ , we have the following estimates*

$$\begin{aligned} \|\nabla_{\vec{a}} D_t^r (\nabla \xi_I)\|_{C^0} &\lesssim_{\vec{a}} N^{(|\vec{a}|+1-L)+/L} \Xi^{|\vec{a}|} (\Xi e_u^{1/2})^r, \quad \text{for all } |\vec{a}| \geq 0, r = 0, 1 \\ \|\nabla_{\vec{a}} D_t^2 (\nabla \xi_I)\|_{C^0} &\lesssim_{\vec{a}} N^{(|\vec{a}|+2-L)+/L} \Xi^{|\vec{a}|} (\Xi e_u^{1/2})^2, \quad \text{for all } |\vec{a}| \geq 0, r = 2 \end{aligned}$$

Moreover the  $r = 1$  bound holds also when using the differential operator  $\nabla_{\vec{a}_1} D_t \nabla_{\vec{a}_2}$  applied to  $\nabla \xi_I$ , where  $|\vec{a}_1| + |\vec{a}_2| = |\vec{a}|$ , and the  $r = 2$  bound holds also when using the differential operator  $\nabla_{\vec{a}_1} D_t \nabla_{\vec{a}_2} D_t \nabla_{\vec{a}_3}$  applied to  $\nabla \xi_I$ , where  $|\vec{a}_1| + |\vec{a}_2| + |\vec{a}_3| = |\vec{a}|$ .

*Proof.* See [17, Sections 17.1-17.3]. We note that the choice of  $b_0$  depends only on the geometric constant  $c_2$  in the first inequality of (5), which in turn implies the second inequality of (5).  $\square$

**Remark 4.1.** Due to  $u$  being frequency localized by assumption one can identify  $u$  with  $u_\epsilon$  from [17] in this proof. Doing this forgoes the need for any mollification factors like what appears in [17], which allows the above estimates to improve to  $\|\nabla_{\vec{a}} D_t^r (\nabla \xi_I)\|_{C^0} \lesssim_{\vec{a}} \Xi^{|\vec{a}|} (\Xi e_u^{1/2})^r$ , for all  $|\vec{a}| \geq 0, r = 0, 1, 2$ .

**Proposition 4.2** (Mollification estimates). *We have the following bounds for the  $R_\epsilon$  from Section 4.4.*

$$\begin{aligned} \|\nabla_{\vec{a}} D_t^r R_\epsilon\|_{C^0} &\lesssim_{\vec{a}} N^{\frac{3}{2L}(|\vec{a}|+r-L)+} \Xi^{|\vec{a}|} (\Xi e_u^{1/2})^r D_R, \quad \text{for all } |\vec{a}| \geq 0, r = 0, 1 \\ \|\nabla_{\vec{a}} D_t^2 R_\epsilon\|_{C^0} &\lesssim_{\vec{a}} N^{\frac{3}{2L}(|\vec{a}|+1-L)+} \Xi^{|\vec{a}|} (\Xi e_u^{1/2}) \epsilon_t^{-1} D_R, \quad \text{for all } |\vec{a}| \geq 0 \\ \|R - R_\epsilon\|_{C^0} &\lesssim \left( (\Xi e_u^{1/2}) \epsilon_t + \epsilon_x^L \Xi^L \right) D_R \end{aligned}$$

*Proof.* We cite the proofs of [17, Proposition 18.5] and [17, Proposition 18.7]. Here we identify the coarse scale flow  $u_\epsilon$  from [17] with  $u_\epsilon := u = T\theta$  because  $u$  is restricted to low frequencies by assumption. Our different choice of mollification factors  $\epsilon_x, \epsilon_t$  in (7) means that the estimates here are the same estimates from [17] but with  $N$  replaced by  $N^{\frac{3}{2}}$ . The bound on  $\|R - R_\epsilon\|_{C^0}$  comes from [17, Section 18.3], which employs the decomposition  $R - R_\epsilon = (R - \eta_{\epsilon_t} *_{\Phi} R) + \eta_{\epsilon_t} *_{\Phi} (R - R_{\epsilon_x})$ . Here  $*_{\Phi}$  denotes mollification along the flow as defined in Section 4.4 and  $R_{\epsilon_x}$  is the spatial mollification of  $R$  in Section 4.4.  $\square$

**Proposition 4.3** (Coefficients of the stress equation estimates). *For components of the amplitude  $\varepsilon$  from (28) and  $\gamma_I$  from (32) the following bounds hold uniformly for  $I \in \mathbb{Z} \times \mathbb{F}$*

$$\begin{aligned} \|\nabla_{\vec{a}} D_t^r \varepsilon\|_{C^0} + \|\nabla_{\vec{a}} D_t^r \gamma_I\|_{C^0} &\lesssim_{\vec{a}} N^{\frac{3}{2L}(|\vec{a}|+1-L)+} \Xi^{|\vec{a}|} (\Xi e_u^{1/2})^r, \quad \text{for all } |\vec{a}| \geq 0, r = 0, 1 \\ \|\nabla_{\vec{a}} D_t^2 \varepsilon\|_{C^0} + \|\nabla_{\vec{a}} D_t^2 \gamma_I\|_{C^0} &\lesssim_{\vec{a}} N^{\frac{3}{2L}(|\vec{a}|+1-L)+} \Xi^{|\vec{a}|} (\Xi e_u^{1/2}) \epsilon_t^{-1}, \quad \text{for all } |\vec{a}| \geq 0 \end{aligned}$$

*Proof.* For the estimates of  $\varepsilon_I$  we cite the proof of [17, Proposition 20.1] but use our estimates  $R_\epsilon$  from Proposition 4.2. It is important to add that our choices of  $\epsilon_t^{-1}$  from (7) and  $N$  from Lemma 3.1 are different from [17] but all of the arguments remain the same. For the bounds on  $\gamma_I$  we cite [17, Proposition 20.2]. We remark that in the first bound, slightly sharper estimates hold if the terms are controlled separately but we have combined them together here.  $\square$

**Proposition 4.4** (Principal amplitude estimates). *For the amplitude  $\theta_I$  from (6) and the induced velocity vector field  $u_I = T\theta_I$  from 4.2 the following bounds hold uniformly for  $I \in \mathbb{Z} \times F$*

$$\begin{aligned} \|\nabla_{\vec{a}} D_t^r \theta_I\|_{C^0} + \|\nabla_{\vec{a}} D_t^r u_I\|_{C^0} &\lesssim_{\vec{a}} \lambda^{1/2} N^{\frac{3}{2L}(|\vec{a}|+1-L)+} \Xi^{|\vec{a}|} D_R^{1/2} \tau^{-r}, \quad \text{for all } |\vec{a}| \geq 0, r = 0, 1 \\ \|\nabla_{\vec{a}} D_t^2 \theta_I\|_{C^0} + \|\nabla_{\vec{a}} D_t^2 u_I\|_{C^0} &\lesssim_{\vec{a}} \lambda^{1/2} B_\lambda^{3/2} N^{\frac{3}{2L}(|\vec{a}|+1-L)+} \Xi^{|\vec{a}|} D_R^{1/2} (\Xi e_u^{1/2}) \epsilon_t^{-1}, \quad \text{for all } |\vec{a}| \geq 0. \end{aligned}$$

*Proof.* First we recall from Section 4.2 that  $u_I^l = \theta_I m^l(\nabla \xi_I)$ . The bounds for  $u_I$  follow similarly to the bounds on  $\theta_I$ , so we restrict attention to estimating  $\theta_I$ . The following calculations are outlined in [17, Propositions 21.1 - 21.4]. They follow the basic scheme that the bounds of  $\theta_I$  are roughly a rescaling of the estimates for  $\gamma_I$ . For example we give a calculation of  $D_t^2 \theta_I$ .

$$\begin{aligned} D_t^2 \theta_I &= D_t^2 \left[ \lambda^{1/2} \gamma_I(x, t) \phi \left( \frac{t - k\tau}{\tau} \right) e^{1/2}(t) \right] \\ &= \lambda^{1/2} \gamma_I(x, t) \partial_t^2 \left[ \phi_k(t) e^{1/2}(t) \right] + \lambda^{1/2} D_t^2 [\gamma_I(x, t)] \phi_k(t) e^{1/2}(t) + 2\lambda^{1/2} D_t [\gamma_I(x, t)] \partial_t \left[ \phi_k(t) e^{1/2}(t) \right] \\ &= A_I + A_{II} + A_{III} \end{aligned}$$

We begin by estimating the first term using our bounds from Section 4.1.3, 4.1.5 and Proposition 4.3. We will also need the observation that  $\Xi e_u^{1/2} \leq \tau^{-1}$  that is due to our definitin of  $b^{-1} \geq 1$  in Section 4.1.3.

$$\|\nabla_{\vec{a}} A_I\|_{C^0} \lesssim \lambda^{1/2} \|\nabla_{\vec{a}} \gamma_I\|_{C^0} \tau^{-2} \mathbb{D}_R^{1/2} \lesssim_{\vec{a}} \lambda^{1/2} N^{\frac{3}{2L}(|\vec{a}|+1-L)+} \Xi^{|\vec{a}|} \tau^{-2} \mathbb{D}_R^{1/2}$$

We can also bound the second term using the estimates from Proposition 4.3

$$\|\nabla_{\vec{a}} A_{II}\|_{C^0} \lesssim_{\vec{a}} \lambda^{1/2} \|\nabla_{\vec{a}} D_t^2 \gamma_I\|_{C^0} \mathbb{D}_R^{1/2} \lesssim_{\vec{a}} \lambda^{1/2} N^{\frac{3}{2L}(|\vec{a}|+1-L)+} \Xi^{|\vec{a}|} (\Xi e_u^{1/2}) \epsilon_t^{-1} \mathbb{D}_R^{1/2}$$

For the third term we use a combination of Proposition 4.3 and the definitions from Sections 4.1.3 and 4.1.5.

$$\|\nabla_{\vec{a}} A_{III}\|_{C^0} \lesssim_{\vec{a}} \lambda^{1/2} \tau^{-1} \mathbb{D}_R^{1/2} \|\nabla_{\vec{a}} D_t \gamma_I\|_{C^0} \lesssim_{\vec{a}} \lambda^{1/2} N^{\frac{3}{2L}(|\vec{a}|+1-L)+} \Xi^{|\vec{a}|} (\Xi e_u^{1/2}) \tau^{-1} \mathbb{D}_R^{1/2}$$

Finally, we compare these bounds by proving that  $\tau^{-1} \lesssim B_\lambda^{3/4} \epsilon_t^{-1}$  and  $\tau^{-2} \lesssim B_\lambda^{3/2} (\Xi e_u^{1/2}) \epsilon_t^{-1}$  with implied constants that only depend on  $c_0$  from the definition  $\epsilon_t$  given in 4.4. We start by proving  $\tau^{-1} \lesssim B_\lambda^{3/4} \epsilon_t^{-1}$

$$\begin{aligned} \tau^{-1} \lesssim B_\lambda^{3/4} \epsilon_t^{-1} &\iff \frac{\mathbb{D}_R^{1/4} B_\lambda^{3/4} N^{3/4}}{\mathbb{D}_u^{1/4}} \Xi e_u^{1/2} \lesssim B_\lambda^{3/4} N^{3/2} \Xi^{3/2} \mathbb{D}_R^{1/2} \iff \\ B_\lambda^{3/4} \Xi^{3/2} \mathbb{D}_R^{1/4} \mathbb{D}_u^{1/4} N^{3/4} &\lesssim B_\lambda^{3/4} N^{3/2} \Xi^{3/2} \mathbb{D}_R^{1/2} \iff \frac{\mathbb{D}_u^{1/4}}{\mathbb{D}_R^{1/4}} \lesssim N^{3/4} \end{aligned} \tag{34}$$

The last inequality holds due to the assumption  $N \geq \frac{\mathbb{D}_u}{\mathbb{D}_R} \geq 1$  in the Main Lemma 3.1. Next we verify  $\tau^{-2} \lesssim B_\lambda^{3/2} \Xi e_u^{1/2} \epsilon_t^{-1}$ .

$$\begin{aligned} \tau^{-2} \lesssim B_\lambda^{3/2} \Xi e_u^{1/2} \epsilon_t^{-1} &\iff \frac{\mathbb{D}_R^{1/2} B_\lambda^{3/2} N^{3/2}}{\mathbb{D}_u^{1/2}} \Xi^2 e_u \lesssim B_\lambda^{3/2} \Xi e_u^{1/2} N^{3/2} \Xi^{3/2} \mathbb{D}_R^{1/2} \iff \\ \mathbb{D}_R^{1/2} N^{3/2} \Xi^3 \mathbb{D}_u^{1/2} &\lesssim \Xi^3 \mathbb{D}_u^{1/2} N^{3/2} \mathbb{D}_R^{1/2} \iff 1 \lesssim 1 \end{aligned}$$

The last inequality holds for a sufficiently small constant,  $c_0$ , taken in the definition of  $\epsilon_t$  of line (7). Now that we have shown  $\tau^{-1} \lesssim B_\lambda^{3/4} \epsilon_t^{-1}$  and  $\tau^{-2} \lesssim B_\lambda^{3/2} (\Xi e_u^{1/2}) \epsilon_t^{-1}$  we may apply this to compare our previously found bounds of  $\nabla_{\vec{a}} A_I$ ,  $\nabla_{\vec{a}} A_{II}$ , and  $\nabla_{\vec{a}} A_{III}$ . Doing so and taking  $B_\lambda \geq 1$  so that  $B_\lambda^{3/2} \geq B_\lambda^{3/4}$  gives the claimed bound on  $\|\nabla_{\vec{a}} D_t^2 \theta_I\|_{C^0}$ .  $\square$

**Proposition 4.5** (Bilinear Microlocal correction estimates). *Let  $L \geq 2$  and assume the following bounds for the coarse-scale velocity*

$$\|\nabla_{\vec{a}} u\|_{C^0} \lesssim_{\vec{a}} N^{(|\vec{a}| - L)_+ / L} \Xi |\vec{a}| e_u^{1/2}, \quad \text{for all } |\vec{a}| \geq 0 \quad (35)$$

For  $\delta B_I^{jl}$  defined in 4.2 and  $D_t = \partial_t + u^i \nabla_i$  the following holds uniformly in  $I \in \mathbb{Z} \times \mathbb{F}$

$$\|\nabla_{\vec{a}} D_t^\rho \delta B_I^{jl}\|_{C^0} \lesssim_{\vec{a}} B_\lambda^{-1} N^{-1} N^{\frac{3}{2L} (|\vec{a}| + 2 - L)_+} \Xi |\vec{a}| D_R \tau^{-\rho}, \quad \text{for all } |\vec{a}| \geq 0, 0 \leq \rho \leq 1. \quad (36)$$

We note that the estimate  $\hat{\mathcal{D}}_R = \frac{D_R}{N}$  obtained in the  $C^0$  bound above is consistent with an SQG scheme aimed at the optimal regularity.

*Proof.* We recall the definition of  $\delta B_I$  from Lemma 4.2

$$\begin{aligned} \delta B_I^{j\ell}(x, t) &= \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i\lambda \nabla \xi_I(x) \cdot h_1} e^{-i\lambda \nabla \xi_I(x) \cdot h_2} K_{\lambda, \text{sym}}^{j\ell}[h_1, h_2] \theta_I(x) Y(x, h_1) dh_1 dh_2 \\ &\quad + \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i\lambda \nabla \xi_I(x) \cdot h_1} e^{-i\lambda \nabla \xi_I(x) \cdot h_2} K_{\lambda, \text{sym}}^{j\ell}[h_1, h_2] \theta_I(x) Y(x, h_2) dh_1 dh_2 \\ &\quad + \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i\lambda \nabla \xi_I(x) \cdot h_1} e^{-i\lambda \nabla \xi_I(x) \cdot h_2} K_{\lambda, \text{sym}}^{j\ell}[h_1, h_2] Y(x, h_1) Y(x, h_2) dh_1 dh_2 \\ &:= A_I + A_{II} + A_{III} \\ Y(x, h) &:= \int_0^1 \frac{d}{dr} e^{iZ(r, x, h)} \theta_I(x - rh) dr \\ Z(r, x, h) &:= r\lambda \int_0^1 h^a h^b \partial_a \partial_b \xi(x - sh)(1 - s) ds \end{aligned}$$

We show the bound for just  $\|\nabla_{\vec{a}} D_t^\rho A_I\|_{C^0}$ , with  $0 \leq \rho \leq 1$ , as the corresponding bounds on  $A_{II}$  and  $A_{III}$  are similar. To aid in the computation, we first decompose  $Y(x, h)$  as

$$\begin{aligned} Y(x, h) &= Y_1(x, h) - Y_2(x, h) \\ Y_1(x, h) &:= \int_0^1 e^{iZ(r, x, h)} \left[ i\lambda \int_0^1 h^a h^b \nabla_a \nabla_b \xi_I(x - sh)(1 - s) ds \right] \theta_I(x - rh) dr \\ Y_2(x, h) &:= \int_0^1 e^{iZ(r, x, h)} \nabla_a \theta_I(x - rh) h^a dr \end{aligned}$$

We return our attention to bounding the integral  $\nabla_{\vec{a}} D_t^\rho A_I$  in  $\nabla_{\vec{a}} D_t^\rho \delta B_I^{jl}$ . Consider

$$A_I = \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i\lambda \nabla \xi_I(x) \cdot h_1} e^{-i\lambda \nabla \xi_I(x) \cdot h_2} K_{\lambda, \text{sym}}^{j\ell}[h_1, h_2] \theta_I(x) Y_1(x, h_1) dh_1 dh_2 \quad (37)$$

$$- \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i\lambda \nabla \xi_I(x) \cdot h_1} e^{-i\lambda \nabla \xi_I(x) \cdot h_2} K_{\lambda, \text{sym}}^{j\ell}[h_1, h_2] \theta_I(x) Y_2(x, h_1) dh_1 dh_2 \quad (38)$$

For the sake of brevity we will just discuss how to show the claimed bounds on  $\nabla_{\vec{a}} D_t^\rho(37)$ . By similar methods one can prove the same bounds on  $\nabla_{\vec{a}} D_t^\rho(38)$  and by linearity we obtain the bounds on  $\nabla_{\vec{a}} D_t^\rho A_I$ . Hence we consider just the term  $\nabla_{\vec{a}} D_t^\rho(37)$ . Using the product rule for  $\nabla_{\vec{a}} D_t^\rho$  we may write

$$\begin{aligned} \nabla_{\vec{a}} D_t^\rho(37) &= \sum_{\substack{\vec{a}_1 + \dots + \vec{a}_6 = \vec{a} \\ \rho_1 + \dots + \rho_6 = \rho}} \int_0^1 dr \int_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla_{\vec{a}_1} D_t^{\rho_1} e^{i\lambda \nabla \xi_I(x) \cdot h_1} \nabla_{\vec{a}_2} D_t^{\rho_2} e^{-i\lambda \nabla \xi_I(x) \cdot h_2} K_{\lambda, \text{sym}}^{j\ell}[h_1, h_2] \\ &\quad \cdot \nabla_{\vec{a}_3} D_t^{\rho_3} e^{iZ(r, x, h_1)} \nabla_{\vec{a}_4} D_t^{\rho_4} iZ(1, x, h_1) \nabla_{\vec{a}_5} D_t^{\rho_5} \theta_I(x - rh_1) \nabla_{\vec{a}_6} D_t^{\rho_6} \theta_I(x) dh_1 dh_2. \end{aligned} \quad (39)$$

We begin with estimates for the first factor in (39). We expand this term using the (combinatorial version of the) Faá di Bruno formula as follows

$$|\nabla_{\vec{a}_1} e^{i\lambda \nabla \xi_I(x) \cdot h_1}| \lesssim_{\vec{a}_1} \sum_{\pi \in \Pi(|\vec{a}_1|)} |h_1|^{\|\pi\|} \lambda^{|\pi|} \prod_{B \in \pi} \left\| \nabla^{|B|} \nabla \xi_I \right\|_{C^0}.$$

In the line above  $\Pi(|\vec{a}_1|)$  is the set of all partitions of the set of positive integers  $\{1, \dots, |\vec{a}_1|\}$ . For a given partition  $\pi \in \Pi(|\vec{a}_1|)$ , we call one subset,  $B \in \pi$ , a block and we let  $|B|$  denote the size of the subset and  $|\pi|$  denote the number of blocks in  $\pi$ . We apply Proposition 4.1 and get

$$|\nabla_{\vec{a}_1} e^{i\lambda \nabla \xi_I(x) \cdot h_1}| \lesssim_{\vec{a}_1} \sum_{\pi \in \Pi(|\vec{a}_1|)} |h_1|^{\|\pi\|} \lambda^{|\pi|} \prod_{B \in \pi} N^{(|B|+1-L)_+/L} \Xi^{|B|}.$$

We now recall an elementary *counting inequality*, which states that for any  $x_i, M \in \mathbb{R}^+, 1 \leq i \leq m$ , we have  $\sum_{i=1}^m (x_i - M)_+ \leq ((\sum_{i=1}^m x_i) - M)_+$ . A proof can be found in [17, Lemma 17.1]. Applying this inequality with  $M = L - 1$ , and using that  $\sum_{B \in \pi} |B| = |\vec{a}_1|$  gives

$$|\nabla_{\vec{a}_1} e^{i\lambda \nabla \xi_I(x) \cdot h_1}| \lesssim_{\vec{a}_1} (1 + |h|^{\|\vec{a}_1\|} \lambda^{|\vec{a}_1|}) N^{(|\vec{a}_1|+1-L)_+/L} \Xi^{|\vec{a}_1|}. \quad (40)$$

Next we treat the advective derivative of the first factor in (39) using a combination of the Leibniz rule, the Faá di Bruno formula, and our estimates from Proposition 4.1

$$\begin{aligned} |\nabla_{\vec{a}_1} D_t e^{i\lambda \nabla \xi_I(x) \cdot h_1}| &\lesssim_{\vec{a}_1} \sum_{\vec{b}_1 + \vec{b}_2 = \vec{a}_1} \sum_{\pi \in \Pi(|\vec{b}_1|)} |h_1|^{\|\pi\|} \lambda^{|\pi|} \left( \prod_{B \in \pi} \left\| \nabla^{|B|} \nabla \xi_I \right\|_{C^0} \right) (|h_1| \lambda \left\| \nabla_{\vec{b}_2} D_t \nabla \xi_I \right\|_{C^0}) \\ &\lesssim_{\vec{a}_1} \sum_{\vec{b}_1 + \vec{b}_2 = \vec{a}_1} \sum_{\pi \in \Pi(|\vec{b}_1|)} |h_1|^{\|\pi\|+1} \lambda^{|\pi|+1} \prod_{B \in \pi} \left( N^{(|B|+1-L)_+/L} \Xi^{|B|} \right) \left( N^{(|b_2|+1-L)_+/L} \Xi^{|b_2|} (\Xi e_u^{1/2}) \right) \end{aligned}$$

In order to simplify this expression we can combine the exponents of the  $N$  factors. We apply the counting inequality with  $M = L - 1$  and use that  $\sum_{B \in \pi} |B| = |\vec{a}_1|$  to get

$$|\nabla_{\vec{a}_1} D_t e^{i\lambda \nabla \xi_I(x) \cdot h_1}| \lesssim_{\vec{a}_1} (|h| \lambda + |h|^{\|\vec{a}_1\|+1} \lambda^{|\vec{a}_1|+1}) N^{(|\vec{a}_1|+1-L)_+/L} \Xi^{|\vec{a}_1|} (\Xi e_u^{1/2}). \quad (41)$$

Combining estimates (40) and (41) gives

$$|\nabla_{\vec{a}_1} D_t^{\rho_1} e^{i\lambda \nabla \xi_I(x) \cdot h_1}| \lesssim_{\vec{a}_1} (1 + |h|^{\|\vec{a}_1\|+\rho_1} \lambda^{|\vec{a}_1|+\rho_1}) N^{(|\vec{a}_1|+1-L)_+/L} \Xi^{|\vec{a}_1|} (\Xi e_u^{1/2})^{\rho_1}, \quad (42)$$

and the same bound holds for the second factor in (39)

$$|\nabla_{\vec{a}_2} D_t^{\rho_2} e^{-i\lambda \nabla \xi_I(x) \cdot h_2}| \lesssim_{\vec{a}_2} (1 + |h|^{\|\vec{a}_2\|+\rho_2} \lambda^{|\vec{a}_2|+\rho_2}) N^{(|\vec{a}_2|+1-L)_+/L} \Xi^{|\vec{a}_2|} (\Xi e_u^{1/2})^{\rho_2}. \quad (43)$$

Next we prove the bounds on fourth factor of (39),  $\nabla_{\vec{a}} D_t^{\rho_4} Z(r, x, h_1)$ , by direct calculation. For the bound on the spatial derivative, we use Proposition 4.1 and  $\Xi = B_\lambda^{-1} N^{-1} \lambda$  to obtain

$$|\nabla_{\vec{a}} Z(r, x, h_1)| \lesssim \lambda |h|^2 \left\| \nabla_{\vec{a}} \nabla^2 \xi_I \right\| \lesssim B_\lambda^{-1} N^{-1} \lambda^2 |h|^2 N^{(|\vec{a}|+2-L)_+/L} \Xi^{|\vec{a}|},$$

where the constant is independent of  $r$ . For the advective derivative bound, we need to approximate the value of  $u^i(x)$  in  $D_t$  by the value of  $u^i$  at a nearby point. For example

$$(\partial_t + u^i(x) \nabla_i) \nabla_a \nabla_b \xi_I(x - sh_1) = D_t \nabla_a \nabla_b \xi_I(x - sh_1) + (u^i(x) - u^i(x - sh_1)) \nabla_i \nabla_a \nabla_b \xi_I(x - sh_1)$$

We can control the second term with Taylor's theorem and the counting inequality with  $M = L - 2$

$$\begin{aligned} (u^i(x) - u^i(x - sh_1)) \nabla_i \nabla_a \nabla_b \xi_I(x - sh_1) &= -sh_1^c \int_0^1 \nabla_c u^i(x - \sigma sh_1) d\sigma \nabla_i \nabla_a \nabla_b \xi_I(x - sh_1) \quad (44) \\ |\nabla_{\vec{a}_4}(44)| &\lesssim_{\vec{a}_4} \sum_{|\vec{b}_1|+|\vec{b}_2|=|\vec{a}_4|} |h_1| \left[ (\Xi e_u^{1/2})^{\vec{b}_1} N^{(|\vec{b}_1|+1-L)_+/L} \right] \Xi^{|\vec{b}_2|+2} N^{(|\vec{b}_2|+3-L)_+/L} \\ |\nabla_{\vec{a}_4}(44)| &\lesssim_{\vec{a}_4} |h_1| \Xi^{|\vec{a}_4|+2} N^{(|\vec{a}_4|+3-L)_+/L} (\Xi e_u^{1/2}). \end{aligned}$$

We use  $N^{(|\vec{a}_4|+3-L)_+/L} \Xi \leq N^{(|\vec{a}_4|+2-L)_+/L} \lambda$  and  $\Xi^{|\vec{a}_4|+1} = B_\lambda^{-1} N^{-1} \lambda \Xi^{|\vec{a}_4|}$  to obtain

$$\begin{aligned} |\nabla_{\vec{a}_4} D_t^{\rho_4} Z(r, x, h_1)| &\lesssim_{\vec{a}_4} \lambda |h_1|^2 \Xi^{|\vec{a}_4|+1} (\Xi e_u^{1/2})^{\rho_4} \cdot \left[ N^{(|\vec{a}_4|+2-L)_+} + \mathbf{1}_{\{\rho_4=1\}} |h_1| N^{(|\vec{a}_4|+3-L)_+/L} \Xi \right] \\ &\lesssim_{\vec{a}_4} B_\lambda^{-1} N^{-1} (|h_1|^2 \lambda^2 + |h_1|^{2+\rho_4} \lambda^{2+\rho_4}) N^{(|\vec{a}_4|+2-L)_+/L} \Xi^{|\vec{a}_4|} (\Xi e_u^{1/2})^{\rho_4}. \quad (45) \end{aligned}$$

Next we compute a bound on the third factor,  $\nabla_{\vec{a}_3} D_t^{\rho_3} e^{iZ(r,x,h_1)}$ , as follows

$$\begin{aligned} \left| \nabla_{\vec{a}_3} e^{iZ(r,x,h_1)} \right| &\lesssim_{\vec{a}_3} \sum_{\pi \in \Pi(|\vec{a}_3|)} \lambda^{|\pi|} |h_1|^{2|\pi|} N^{(|\vec{a}_3|+2-L)_+/L} \Xi^{|\vec{a}_3|+|\pi|} \\ &\lesssim_{\vec{a}_3} (1 + \lambda^{2|\vec{a}_3|} |h|^{2|\vec{a}_3|}) N^{(|\vec{a}_3|+2-L)_+/L} \Xi^{|\vec{a}_3|} \\ \left| \nabla_{\vec{a}_3} D_t e^{iZ(r,x,h_1)} \right| &\lesssim_{\vec{a}_3} \sum_{\vec{b}_1+\vec{b}_2=\vec{a}_3} \left| \nabla_{\vec{b}_1} e^{iZ(r,x,h_1)} \nabla_{\vec{b}_2} D_t Z(r, x, h_1) \right| \\ &\lesssim_{\vec{a}_3} \sum_{\vec{b}_1+\vec{b}_2=\vec{a}_3} \sum_{\pi \in \Pi(|\vec{b}_1|)} \left( \prod_{B \in \pi} \left| \nabla^{|B|} Z(r, x, h_1) \right| \right) \left| \nabla_{\vec{b}_2} D_t Z(r, x, h_1) \right|. \quad (46) \end{aligned}$$

If we apply this calculation, Proposition 4.1, the assumed bounds (35), and the counting inequality with  $M = L - 2$  to (46) then we get

$$\left| \nabla_{\vec{a}_3} D_t e^{iZ(r,x,h_1)} \right| \lesssim_{\vec{a}_3} (|h|^2 \lambda^2 + |h|^{2|\vec{a}_3|+2} \lambda^{2|\vec{a}_3|+2}) N^{(|\vec{a}_3|+2-L)_+/L} \Xi^{|\vec{a}_3|} (\Xi e_u^{1/2}),$$

and then

$$\left| \nabla_{\vec{a}_3} D_t^{\rho_3} e^{iZ(r,x,h_1)} \right| \lesssim_{\vec{a}_3} (1 + |h|^{2|\vec{a}_3|+2} \lambda^{2|\vec{a}_3|+2}) N^{(|\vec{a}_3|+2-L)_+/L} \Xi^{|\vec{a}_3|} (\Xi e_u^{1/2})^{\rho_3}. \quad (47)$$

For the fifth factor in (39) we again approximate  $u^i(x)$  in  $D_t$  by the value  $u^i(x - rh)$  at a nearby point. We use Proposition 4.1, Proposition 4.4, (35), and the counting inequality with  $M = L - 2$  to get

$$\begin{aligned} \left| \nabla_{\vec{a}_5} (\partial_t + u^i(x) \nabla_i)^{\rho_5} \theta_I(x - rh_1) \right| &\leq \|\nabla_{\vec{a}_5} D_t^{\rho_5} \theta_I\|_{C^0} + \left| \nabla_{\vec{a}_5} [(u^i(x) - u^i(x - rh_1)) \nabla_i]^{\rho_5} \theta_I(x - rh_1) \right| \\ &\lesssim_{\vec{a}_5} \Xi^{|\vec{a}_5|} N^{\frac{3}{2L} (|\vec{a}_5|+\rho_5+1-L)_+} (1 + |h| \lambda) \lambda^{1/2} \mathbb{D}_R^{1/2} \tau^{-\rho_5}. \quad (48) \end{aligned}$$

The last factor in (39),  $\nabla_{\vec{a}_6} D_t^{\rho_6} \theta_I$ , can be controlled by a direct application of Proposition 4.4. We can combine this bound with our previously found estimates (42), (43), (47), (45) and (48). Then applying

the counting inequality with  $M = L - 2$  with the Kernel scaling estimate from (19) yields

$$\begin{aligned} \|(39)\|_{C^0} &\lesssim_{\vec{a}} \sum_{\substack{\vec{a}_1 + \dots + \vec{a}_6 = \vec{a} \\ \rho_1 + \dots + \rho_6 = 1}} \int_0^1 dr \int_{\mathbb{R}^2 \times \mathbb{R}^2} dh_1 dh_2 |K_{\lambda, \text{sym}}^{j\ell}[h_1, h_2]| \\ &\cdot (1 + |h|^{\|\vec{a}_1\| + \rho_1} \lambda^{|\vec{a}_1| + \rho_1}) N^{(\|\vec{a}_1\| + 1 - L)_+ / L} \Xi^{|\vec{a}_1|} (\Xi e_u^{1/2})^{\rho_1} \\ &\cdot (1 + |h|^{\|\vec{a}_2\| + \rho_2} \lambda^{|\vec{a}_2| + \rho_2}) N^{(\|\vec{a}_2\| + 1 - L)_+ / L} \Xi^{|\vec{a}_2|} (\Xi e_u^{1/2})^{\rho_2} \\ &\cdot (1 + |h|^{2|\vec{a}_3| + 2} \lambda^{2|\vec{a}_3| + 2}) N^{(\|\vec{a}_3\| + 2 - L)_+ / L} \Xi^{|\vec{a}_3|} (\Xi e_u^{1/2})^{\rho_3} \\ &\cdot (|h|^2 \lambda^2 + |h|^{2+\rho_4} \lambda^{2+\rho_4}) N^{(\|\vec{a}_4\| + 2 - L)_+ / L} \Xi^{|\vec{a}_4|} (\Xi e_u^{1/2})^{\rho_4} B_\lambda^{-1} N^{-1} \\ &\cdot (1 + |h| \lambda) N^{\frac{3}{2L} (\|\vec{a}_5\| + \rho_5 + 1 - L)_+} \Xi^{|\vec{a}_5|} \lambda^{1/2} D_R^{1/2} \tau^{-\rho_5} \\ &\cdot (1 + |h| \lambda) N^{\frac{3}{2L} (\|\vec{a}_6\| + \rho_6 + 1 - L)_+} \Xi^{|\vec{a}_6|} \lambda^{1/2} D_R^{1/2} \tau^{-\rho_6} \\ \|(39)\|_{C^0} &\lesssim B_\lambda^{-1} N^{-1} \Xi^{|\vec{a}|} N^{\frac{3}{2L} (\|\vec{a}\| + 2 - L)_+} D_R \tau^{-\rho}, \end{aligned}$$

where in the last line we used  $(\Xi e_u^{1/2}) \leq \tau^{-1}$ . This bound completes our work for estimating one summand of  $\nabla_{\vec{a}} D_t^\rho A_I$ . The other summand (38) may be handled using the same approach, but in this case the factor  $B_\lambda^{-1} N^{-1}$  arises from the estimates for  $h^a \nabla_a \theta_I(x - rh)$ . By linearity and our estimates of (37) and (38), we get the claimed bound (36) for  $\nabla_{\vec{a}} D_t^\rho A_I$ . The remaining integrals  $A_{II}$  and  $A_{III}$  may be done by an identical calculation. Hence by linearity we have our claimed bound for  $\nabla_{\vec{a}} \delta B_I$ .  $\square$

**Proposition 4.6** (Microlocal correction estimates). *The following bounds hold uniformly for  $I \in \mathbb{Z} \times F$*

$$\begin{aligned} \|\nabla_{\vec{a}} D_t^r \delta \theta_I\|_{C^0} + \|\nabla_{\vec{a}} D_t^r \delta u_I\|_{C^0} &\lesssim_{\vec{a}} \lambda^{1/2} B_\lambda^{-1} N^{-1} N^{\frac{3}{2L} (\|\vec{a}\| + 2 - L)_+} \Xi^{|\vec{a}|} D_R^{1/2} \tau^{-r} \quad \text{for all } |\vec{a}| \geq 0, r = 0, 1 \\ \|\nabla_{\vec{a}} D_t^2 \delta \theta_I\|_{C^0} + \|\nabla_{\vec{a}} D_t^2 \delta u_I\|_{C^0} &\lesssim_{\vec{a}} \lambda^{1/2} B_\lambda^{-1} N^{-1} N^{\frac{3}{2L} (\|\vec{a}\| + 2 - L)_+} \Xi^{|\vec{a}|} D_R^{1/2} (\Xi e_u^{1/2}) \epsilon_t^{-1} \quad \text{for all } |\vec{a}| \geq 0 \end{aligned}$$

*Proof.* For the same reasons as those given in Proposition 4.4 the calculations for  $\delta u_I$  are comparable to those for  $\delta \theta_I$ . Hence it will be enough to discuss the estimates for  $\delta \theta_I$ . These calculations are done via direct calculation (see [22, Lemma 7.5], which uses the same microlocal lemma and an analogous scalar correction). Hence all of our estimates follow from our previous estimates on  $\nabla \xi_I$  from Proposition 4.1 and  $\theta_I$  from Proposition 4.4.  $\square$

**Proposition 4.7** (Correction Estimates). *Let  $U_I = T\Theta_I$ . The following bounds hold uniformly for  $I \in \mathbb{Z} \times F$*

$$\begin{aligned} \|\nabla_{\vec{a}} D_t^r \Theta_I\|_{C^0} + \|\nabla_{\vec{a}} D_t^r U_I\|_{C^0} &\lesssim_{|\vec{a}|} \lambda^{1/2 + |\vec{a}|} D_R^{1/2} \tau^{-r} \quad \text{for all } |\vec{a}| \geq 0, r = 0, 1 \\ \|\nabla_{\vec{a}} D_t^2 \Theta_I\|_{C^0} + \|\nabla_{\vec{a}} D_t^2 U_I\|_{C^0} &\lesssim_{|\vec{a}|} B_\lambda^{3/2} \lambda^{1/2 + |\vec{a}|} D_R^{1/2} (\Xi e_u^{1/2}) \epsilon_t^{-1} \quad \text{for all } |\vec{a}| \geq 0 \end{aligned}$$

*Proof.* Consider  $\Theta_I = e^{i\lambda \xi_I} (\theta_I + \delta \theta_I) := e^{i\lambda \xi_I} \tilde{\theta}_I$ . Observe by Propositions 4.4 and 4.6 that  $\tilde{\theta}_I$  satisfies the same estimates as  $\theta_I$  (but with larger constants) since the bounds for  $\delta \theta_I$  are lower order. The proposition then follows from the formulas

$$D_t \Theta_I = e^{i\lambda \xi_I} D_t \tilde{\theta}_I, \quad D_t^2 \Theta_I = e^{i\lambda \xi_I} D_t^2 \tilde{\theta}_I$$

by appealing to the bounds from Propositions 4.1, 4.4, 4.6. The bounds for  $U_I$  follow similarly.  $\square$

#### 4.7.2 Weighted Norm

For the purpose of a shorter presentation we introduce the following weighted norm that was used in Lemma 4.1 and Lemma 4.2 of [18].

**Lemma 4.3** (Weighted Norm). *Let  $D_t = \partial_t + T^l \theta \nabla_l$  be the advective derivative with respect to  $\theta$ ,  $\tilde{D}_t = \partial_t + T^l \tilde{\theta} \nabla_l$  be the advective derivative with respect to  $\tilde{\theta} = \theta + \Theta$ ,  $\lambda = B_\lambda N \Xi$  as defined in Section 4.1, and define  $\hat{e}_u^{1/2} := (\lambda D_R)^{1/2}$ . For each advective derivative  $\tilde{D}_t \in \{D_t, \tilde{D}_t\}$  define the weighted norm  $\hat{H} \in \{H, \tilde{H}\}$  by*

$$\hat{H}[F] := \max_{0 \leq r \leq 1} \max_{0 \leq |\vec{a}|+r \leq L} \frac{\|\nabla_{\vec{a}} \tilde{D}_t^r F\|_{C^0}}{\lambda^{|\vec{a}|} (\lambda \hat{e}_u^{1/2})^r}$$

Furthermore  $\hat{H}[F]$  satisfies the triangle inequality  $\hat{H}[F+G] \leq \hat{H}[F] + \hat{H}[G]$  and the product rule  $\hat{H}[FG] \leq \hat{H}[F] \cdot \hat{H}[G]$ . The weighted norms are also comparable  $\hat{H}[F] \lesssim H[F] \lesssim \tilde{H}[F]$ .

*Proof.* The proof is identical to the proof of [18, Lemma 4.2] but with  $T^l[\Theta] = U^l$  identified with  $\tilde{V}^l$ ,  $T^l[\theta] = u^l$  identified with both  $v^l$  and  $v_\epsilon^l$ , and  $\hat{e}_u^{1/2}$  identified with  $e_\phi^{1/2}$ .  $\square$

#### 4.7.3 The Second Order Anti-Divergence Operator Bound

The Lemma below is needed for applying the order -2, second order anti-divergence operator from Section 4.5.1 to frequency localized scalar functions. This estimate is from [18, Lemma 4.3].

**Lemma 4.4.** *Suppose  $Q: \mathbb{R}^2 \rightarrow \mathbb{R}$  is Schwartz and satisfies the following bounds*

$$\max_{0 \leq |\vec{a}| \leq |\vec{b}| \leq L} \lambda^{-(|\vec{b}| - |\vec{a}|)} \|h^{\vec{a}} \nabla_{\vec{b}} Q\|_{L^1(\mathbb{R}^2)} \leq \Lambda^{-1}$$

for some real number  $\Lambda^{-1} \geq 0$ . Let  $H$  be the weighted norm,  $\hat{e}_u^{1/2}$  be the energy quantity, and  $D_t$  be the advective derivative as defined in Lemma 4.3. Then for any smooth  $U$  on  $\mathbb{T}^3$  we have

$$H[Q * U] \lesssim \Lambda^{-1} \left( \|U\|_{C^0} + (\lambda \hat{e}_u^{1/2})^{-1} \|D_t U\|_{C^0} \right)$$

*Proof.* The proof is identical to [18, Lemma 4.3 ]. We present a table where terms used in the proof of the [18, Lemma 4.3 ] are listed in the top row and must be replaced by the corresponding term in the bottom row for doing the analogous proof here.

$\bar{H}[\cdot]$	$\bar{D}_t$	$e_v$	$e_\varphi$	$N \geq (e_v/e_\varphi)^{1/2}$
$H[\cdot]$	$D_t$	$e_u$	$\hat{e}_u$	$N \geq (e_u/\hat{e}_u)^{1/2}$

The last column of the table is the conditions  $N \geq \frac{e_v^{1/2}}{e_\varphi^{1/2}}$  that appears in [18, Lemma 4.3 ] and must be replaced by the conditions  $N \geq \frac{e_u^{1/2}}{\hat{e}_u^{1/2}}$  for the proof here. The condition  $N \geq \frac{e_u^{1/2}}{\hat{e}_u^{1/2}}$  follows from the assumption  $N \geq \mathbb{D}_u/\mathbb{D}_R$  made in the Main Lemma 3.1 and by taking  $B_\lambda \geq 1$  in the definition  $\lambda = B_\lambda N \Xi$ .  $\square$

**Corollary 4.1.** *Given the second order anti-divergence operator  $\mathcal{R}^{jl}: C^0(\mathbb{T}^2) \rightarrow C^0(\mathbb{T}^2)$  defined in (13), a frequency projection operator  $P_\lambda$  to frequencies  $\{\xi \subseteq \hat{\mathbb{R}}^2 : \lambda/10 \leq |\xi| \leq 10\lambda\}$ , and any smooth  $U$  on  $\mathbb{R}^2$  we have*

$$H[\mathcal{R}^{jl} P_\lambda[U]] \lesssim \lambda^{-2} \left( \|U\|_{C^0} + (\lambda \hat{e}_u^{1/2})^{-1} \|D_t U\|_{C^0} \right)$$

*Proof.* The result follows immediately from 4.4 by taking  $\Lambda^{-1} = C\lambda^{-2}$ . We can define  $\Lambda^{-1}$  in this way due to the frequency support of  $\mathcal{R}^{jl}$ .  $\square$

#### 4.7.4 Estimating the Stress Errors

Recall from (8) that the new stress  $\hat{R}$  will be decomposed into four terms. We begin with estimating the term  $R_T$ .

#### 4.7.5 Transport Stress Estimate

**Proposition 4.8** (Transport Stress Error). *There exists a symmetric tensor,  $R_T^{jl}$ , as defined in (10) that satisfies the estimate*

$$H[R_T] \lesssim (B_\lambda N \Xi)^{-3/2} D_R^{1/2} (\Xi e_u^{1/2}) b^{-1}$$

*Proof.* Recall from the discussion in Section 4.5 that

$$\nabla_j \nabla_l R_T^{jl} = D_t \Theta + T^l [\Theta] \nabla_l \theta \quad (49)$$

Since the right hand side of this identity has mean-zero on  $\mathbb{T}^2$  there is a well defined stress term

$$R_T := \mathcal{R}^{jl} [\partial_t \Theta + T^l \theta \nabla_l \Theta + T^l \Theta \nabla_l \theta]$$

as stated in Definition (10). The frequency support of  $\theta$  and  $u$  is  $\text{supp } \hat{\theta} \cup \text{supp } \hat{u} \subseteq B_\Xi(0)$  whereas  $\Theta$  has frequency support in an annulus of size  $\lambda$  as explained in Section 4.1.4. Therefore the argument of  $\mathcal{R}$  above has frequency support in  $\{\xi \in \hat{\mathbb{R}}^2 : \lambda/3 \leq |\xi| \leq 3\lambda\}$  and we may replace  $\mathcal{R}$  by  $\mathcal{R}P_{\approx \lambda}$  for the appropriate frequency-localization operator  $P_{\approx \lambda}$ . Corollary 4.1 then gives

$$H[R_T] \lesssim \lambda^{-2} \left( \|(49)\|_{C^0} + (\lambda \hat{e}_u^{1/2})^{-1} \|D_t(49)\|_{C^0} \right)$$

The term  $\|(49)\|_{C^0}$  equals  $\|D_t \Theta + T^l [\Theta] \nabla_l \theta\|_{C^0}$ . By Proposition 4.7 and Definition 2.2

$$\|(49)\|_{C^0} \lesssim (\lambda \mathbb{D}_R)^{1/2} \tau^{-1} + (\lambda \mathbb{D}_R)^{1/2} (\Xi e_u^{1/2}) \lesssim (\lambda \mathbb{D}_R)^{1/2} \tau^{-1} \quad (50)$$

Additionally, we can estimate the term  $\|D_t(49)\|_{C^0}$  as

$$\|D_t(49)\|_{C^0} = \|D_t^2 \Theta + D_t U^l \nabla_l \theta + U^l D_t \nabla_l \theta\|_{C^0} \quad (51)$$

We can control this quantity using Proposition 4.7, Definition 2.2, and a bound for  $D_t \nabla_l \theta$  that follows from the SQG-Reynold's equation  $\theta$ . From the SQG-Reynold's equation we have

$$D_t \nabla_l \theta = \nabla_l \nabla_a \nabla_b R^{ab} - (\nabla_l u^k) (\nabla_k \theta)$$

and by Definition 2.2

$$\|D_t \nabla_l \theta\|_{C^0} \lesssim \Xi^3 \mathbb{D}_R + (\Xi e_u^{1/2})^2 \lesssim (\Xi e_u^{1/2})^2$$

Using this bound with Proposition 4.7 and Definition 2.2 we can see that

$$\|(51)\|_{C^0} \lesssim B_\lambda^{3/2} (\lambda \mathbb{D}_R)^{1/2} (\Xi e_u^{1/2}) \epsilon_t^{-1} + (\lambda \mathbb{D}_R)^{1/2} \tau^{-1} (\Xi e_u^{1/2}) + N^{\frac{3}{2L}(1-L)+} (\lambda \mathbb{D}_R)^{1/2} (\Xi e_u^{1/2})^2$$

By our definitions of  $\epsilon_t, \tau$  stated in (7) and Section 4.1.3 respectively it follows that

$$(\Xi e_u^{1/2})^2 \leq (\Xi e_u^{1/2}) \tau^{-1} \leq B_\lambda^{3/2} (\Xi e_u^{1/2}) \epsilon_t^{-1}$$

where  $\tau^{-1} \leq \epsilon_t^{-1}$  follows from (34). Next we show that  $B_\lambda^{3/2} (\Xi e_u^{1/2}) \epsilon_t^{-1} \lesssim B_\lambda^{-3/4} \lambda \hat{e}_u^{1/2} \tau^{-1}$ .

$$\begin{aligned} B_\lambda^{3/2} (\Xi e_u^{1/2}) \epsilon_t^{-1} \lesssim B_\lambda^{-3/4} \lambda \hat{e}_u^{1/2} &\iff B_\lambda^{3/2} (\Xi e_u^{1/2}) N^{\frac{3}{2L}} \Xi^{3/2} \mathbb{D}_R^{1/2} \lesssim B_\lambda^{-3/4} (B_\lambda N \Xi)^{3/2} \mathbb{D}_R^{1/2} (\Xi e_u^{1/2}) b^{-1} \\ &\iff 1 \lesssim B_\lambda^{-3/4} \left( \frac{\mathbb{D}_R^{1/2} B_\lambda^{3/2} N^{3/2}}{\mathbb{D}_u^{1/2}} \right)^{1/2} \iff \frac{\mathbb{D}_u^{1/4}}{\mathbb{D}_R^{1/4}} \lesssim N^{3/4} \end{aligned}$$

The last inequality holds due to the assumption  $N \geq \mathbb{D}_u/\mathbb{D}_R$  in the Main Lemma 3.1. In summary, we have shown that

$$\|(51)\|_{C^0} \lesssim (\lambda \mathbb{D}_R)^{1/2} B_\lambda^{-3/4} (\lambda \hat{e}_u^{1/2}) \tau^{-1}.$$

By combining our estimates for (50) and (51) we conclude that

$$H[R_T] \lesssim \lambda^{-2} (\lambda \mathbb{D}_R)^{1/2} \tau^{-1} + B_\lambda^{-3/4} \lambda^{-2} (\lambda \mathbb{D}_R)^{1/2} \tau^{-1} \lesssim \lambda^{-3/2} \mathbb{D}_R^{1/2} (\Xi e_u^{1/2}) b^{-1}.$$

□

#### 4.7.6 Hi Frequency Interference Stress Estimate

**Proposition 4.9.** *There exists a symmetric tensor,  $R_H^{jl}$ , as defined in (9) that satisfies the estimate*

$$H[R_H] \lesssim b \mathbb{D}_R$$

*Proof.* Recall that  $R_H^{jl} := \mathcal{R}^{jl} \left[ \sum_{J \neq I} U_I^l \nabla_l \Theta_J + U_J^l \nabla_l \Theta_I \right]$  as stated in Definition 9. The argument of  $\mathcal{R}$  above has frequency support in  $\{\xi \subseteq \hat{\mathbb{R}}^2 : \lambda/6 \leq |\xi| \leq 6\lambda\}$  due to the localizations in frequency and angle of the scalar corrections and velocity corrections defined in Section 4.1.4. In particular,  $\mathcal{R}$  may be replaced by  $\mathcal{R}P_{\approx \lambda}$  so that Corollary 4.1 applies.

Since  $R_H^{jl}$  is given by a finite sum then by linearity it will be enough to show the claimed bound for a single summand. Let a single summand be  $R_{H(I,J)}^{jl} := \mathcal{R}^{jl} [U_I^l \nabla_l \Theta_J + U_J^l \nabla_l \Theta_I]$ . By Corollary 4.1 this quantity is bonded by

$$H[R_{H(I,J)}^{jl}] \lesssim \lambda^{-2} \left( \|(53)\|_{C^0} + (\lambda \hat{e}_u^{1/2})^{-1} \|D_t(53)\|_{C^0} \right) \quad (52)$$

$$\nabla_j \nabla_l R_{H(I,J)}^{jl} = U_I^l \nabla_l \Theta_J + U_J^l \nabla_l \Theta_I \quad (53)$$

We begin with estimating  $\|(53)\|_{C^0}$ . Recall from the definitions given in Sections 4.2 and 4.1 respectively  $U_I^l = e^{i\lambda\xi_I} (u_I^l + \delta u_I^l)$  where  $u_I^l = \theta_I m^l(\lambda \nabla \xi_I)$ , and  $\Theta_I = e^{i\lambda\xi_I} (\theta_I + \delta \theta_I)$ . It follows that

$$U_I^l \nabla_l \Theta_J + U_J^l \nabla_l \Theta_I = \lambda e^{i\lambda(\xi_I + \xi_J)} [\nabla_l \xi_J u_I^l \theta_J + \nabla_l \xi_I u_J^l \theta_I] \quad (54)$$

$$+ \lambda e^{i\lambda(\xi_I + \xi_J)} [\nabla_l \xi_J u_I^l \delta \theta_J + \nabla_l \xi_I u_J^l \delta \theta_I] \quad (55)$$

$$+ \lambda e^{i\lambda(\xi_I + \xi_J)} [\nabla_l \xi_J \delta u_I^l (\theta_J + \delta \theta_J) + \nabla_l \xi_I \delta u_J^l (\theta_I + \delta \theta_I)] \quad (56)$$

$$+ e^{i\lambda(\xi_I + \xi_J)} [(u_I + \delta u_I)^l \nabla_l (\theta_J + \delta \theta_J) + (u_J + \delta u_J)^l \nabla_l (\theta_I + \delta \theta_I)] \quad (57)$$

By Propositions 4.4 and 4.6 we see that  $\|(55)\|_{C^0}, \|(56)\|_{C^0} \lesssim \lambda \Xi \mathbb{D}_R$  and  $\|(57)\|_{C^0} \lesssim \lambda \Xi \mathbb{D}_R$ . We will see that these terms are lower order in  $\lambda$  relative to the leading order term in  $\lambda$ , which is  $\|(54)\|_{C^0}$ . We consider the term (54) and we calculate the amplitude of this wave,  $\nabla_l \xi_J u_I^l \theta_J + \nabla_l \xi_I u_J^l \theta_I$ , by taking  $m^l(\xi) = i\varepsilon^{la} \xi_a / |\xi|$  for  $\xi \in \hat{\mathbb{R}}^2$  from Definition 2.1.

$$\nabla_l \xi_J u_I^l \theta_J + \nabla_l \xi_I u_J^l \theta_I = i \theta_I \theta_J \varepsilon^{la} (\nabla_a \xi_I |\nabla \xi_I|^{-1} \nabla_l \xi_J + \nabla_a \xi_J |\nabla \xi_J|^{-1} \nabla_l \xi_I) \quad (58)$$

By the anti-symmetric properties of  $\varepsilon^{li}$  we see that  $\varepsilon^{la} (\nabla_a \xi_I \nabla_l \xi_J + \nabla_l \xi_I \nabla_a \xi_J) = 0$  and so we can add a scalar multiple of this identity into the previous line to obtain

$$(58) = i \theta_I \theta_J \varepsilon^{la} \left( \nabla_a \xi_I \nabla_l \xi_J (|\nabla \xi_I|^{-1} - |\nabla \hat{\xi}_I|^{-1}) + \nabla_a \xi_J \nabla_l \xi_I (|\nabla \xi_J|^{-1} - |\nabla \hat{\xi}_J|^{-1}) \right) \quad (59)$$

In (59) we have introduced  $\hat{\xi}_I, \hat{\xi}_J$  that are the initial conditions to  $\xi_I, \xi_J$  respectively as defined in Section 4.1.2. Furthermore  $|\nabla \hat{\xi}_I|^{-1} = |\nabla \hat{\xi}_J|^{-1} = 5^{-1/2}$ . In order to control (59) we will use

$\left| |\nabla \xi_I|^{-1} - |\nabla \hat{\xi}_I|^{-1} \right|, \left| |\nabla \xi_I|^{-1} - |\nabla \hat{\xi}_J|^{-1} \right| \leq b$ . This fact comes from a direct calculation of  $\xi_I$  that also holds for  $\xi_J$ . To start we show that  $\|D_t(|\nabla \xi_I|^{-1})\|_{C^0} \lesssim (\Xi e_u^{1/2})$ .

$$D_t(|\nabla \xi_I|^{-1}) = -|\nabla \xi_I|^{-3} \nabla^a \xi_I D_t \nabla_a \xi_I \implies \|D_t(|\nabla \xi_I|^{-1})\|_{C^0} \lesssim c_3^{-2} (\Xi e_u^{1/2}) \quad (60)$$

Where the inequality uses Proposition 4.1 and  $c_3 > 0$  is a small constant that we fix for condition (5). Then

$$|\nabla \xi_I|^{-1} - |\nabla \hat{\xi}_I|^{-1} \leq \int_0^\tau \|D_t(|\nabla \xi_I|^{-1})\|_{C^0} ds \lesssim \tau (\Xi e_u^{1/2}) = b \quad (61)$$

We can apply this bound along with the estimates from Propositions 4.1 and 4.4 to (59) to conclude that  $\|(59)\|_{C^0} \lesssim \lambda b D_R$ ,  $\|(54)\|_{C^0} \lesssim \lambda^2 b \mathbb{D}_R$ , and finally

$$\lambda^{-2} \|(53)\|_{C^0} \lesssim b \mathbb{D}_R \quad (62)$$

We also estimate the term  $\|D_t(53)\|_{C^0}$  by direct calculation. For the sake of brevity we will only present calculations for estimating the advective derivative of the first term (54), ad the other terms are similar and lower order. We observe that the advective derivative of (54) is equivalent to

$$D_t(54) = \lambda e^{i\lambda(\xi_I + \xi_J)} [(D_t \nabla_l \xi_J) u_I^l \theta_J + (D_t \nabla_l \xi_I) u_J^l \theta_I] \quad (63)$$

$$+ \lambda e^{i\lambda(\xi_I + \xi_J)} [\nabla_l \xi_J (D_t u_I^l) \theta_J + \nabla_l \xi_I (D_t u_J^l) \theta_I] \quad (64)$$

$$+ \lambda e^{i\lambda(\xi_I + \xi_J)} [\nabla_l \xi_J u_I^l (D_t \theta_J) + \nabla_l \xi_I u_J^l (D_t \theta_I)] \quad (65)$$

We directly estimate each term in  $C^0$ -norm using Propositions 4.1 and 4.4 to obtain

$$\|D_t(54)\|_{C^0} \lesssim \lambda^2 \mathbb{D}_R \left( (\Xi e_u^{1/2}) + \tau^{-1} \right) \lesssim \lambda^2 \mathbb{D}_R \tau^{-1}$$

All that remains is to verify that  $\lambda^{-2} \left( \lambda \hat{e}_u^{1/2} \right)^{-1} \lambda^2 \mathbb{D}_R \tau^{-1} \lesssim b \mathbb{D}_R$ . This is equivalent to checking  $\tau^{-1} b^{-1} \lesssim \lambda \hat{e}_u^{1/2}$  via the calcuation

$$\tau^{-1} b^{-1} \lesssim \lambda \hat{e}_u^{1/2} \iff \Xi^{3/2} \mathbb{D}_u^{1/2} \left( \frac{\mathbb{D}_R^{1/2} B_\lambda^{3/2} N^{3/2}}{\mathbb{D}_u^{1/2}} \right) \lesssim (B_\lambda N \Xi)^{3/2} \mathbb{D}_R^{1/2} \iff 1 \lesssim 1$$

As a result we have

$$\lambda^{-2} \left( \lambda \hat{e}_u^{1/2} \right)^{-1} \|D_t(54)\|_{C^0} \lesssim b \mathbb{D}_R$$

By similar direct calculations of the other terms  $D_t(55), D_t(56), D_t(57)$  it can be shown that

$$\lambda^{-2} \left( \lambda \hat{e}_u^{1/2} \right)^{-1} \|D_t(53)\|_{C^0} \lesssim b \mathbb{D}_R \quad (66)$$

and by combining (62) and (66) with (52) we conclude the claimed bound on  $H[R_H]$ .  $\square$

#### 4.7.7 Low Frequency Stress Estimate

**Proposition 4.10.** *There exists a symmetric tensor,  $R_S^{jl}$  as defined in (12) that satisfies the estimate*

$$H[R_S] \lesssim B_\lambda^{-1} N^{-1} \mathbb{D}_R$$

*Proof.* Recall from Section 4.5 that the Low Frequency Stress Error is defined as

$$R_S^{jl} := \sum_{I=(k,f) \in \mathbb{Z} \times \mathbb{F}} B_\lambda^{jl} [\Theta_I, \Theta_I] - \phi_k^2(t) (e(t) M_{[k]} - R_\epsilon^{jl}) = \sum_{I \in \mathbb{Z} \times \mathbb{F}} \delta B_I^{jl}$$

where the last equality holds due to the calculations of Section 4.6 and is stated in (33) and where  $\delta B_I^{jl}$  is defined in Lemma 4.2. Therefore it is enough to compute  $H \left[ \sum_{I \in \mathbb{Z} \times \mathbb{F}} \delta B_I^{jl} \right]$  directly. Due to the partition of unity in time  $\{\phi_k^2(t)\}_{k \in \mathbb{Z}}$  that appears in the definition of  $\theta_I$  and in the definition of  $\delta B_I^{jl}$ ,  $R_S$  is a locally finite sum in time. Hence for  $0 \leq |\vec{a}| \leq L$  and  $r = 0, 1$

$$\left\| \nabla_{\vec{a}} D_t^r \sum_{I \in \mathbb{Z} \times \mathbb{F}} \delta B_I^{jl} \right\|_{C^0} \lesssim \sup_{I \in \mathbb{Z} \times \mathbb{F}} \left\| \nabla_{\vec{a}} D_t^r \delta B_I^{jl} \right\|_{C^0} \lesssim_{\vec{a}} B_\lambda^{-1} N^{-1} N^{\frac{3}{2L}(|\vec{a}|+1-L)_+} \Xi^{|\vec{a}|} \mathbb{D}_R \tau^{-r}.$$

In the last inequality we used Proposition 4.5. We remark that the coarse scale velocity assumptions of (35) are satisfied by the flow  $u = T[\theta]$  with the assumed bounds by the frequency and energy levels in Definition 2.2. Next we show the inequality  $\tau^{-1} \lesssim B_\lambda^{-3/4} \lambda e_u^{\diamond 1/2}$  holds due to the following calculation

$$\begin{aligned} \tau^{-1} \leq B_\lambda^{-3/4} \lambda e_u^{\diamond 1/2} &\iff \Xi^{3/2} \mathbb{D}_u^{1/2} \left( \frac{\mathbb{D}_R^{1/2} B_\lambda^{3/2} N^{3/2}}{\mathbb{D}_u^{1/2}} \right)^{1/2} \lesssim B_\lambda^{-3/4} (B_\lambda N \Xi)^{3/2} \mathbb{D}_R^{1/2} \\ &\iff \frac{\mathbb{D}_u^{1/4}}{\mathbb{D}_R^{1/4}} \lesssim N^{3/4}. \end{aligned}$$

The last inequality holds due to our assumption on  $N \geq \mathbb{D}_u / \mathbb{D}_R$  from the Main Lemma 3.1. By using the inequality  $N^{\frac{3}{2L}(|\vec{a}|+1-L)_+} \Xi^{|\vec{a}|} \leq \lambda^{|\vec{a}|}$  for  $0 \leq |\vec{a}| \leq L$  and the inequality  $\tau^{-1} \lesssim \lambda e_u^{\diamond 1/2}$ , we have that

$$H[R_S] = H \left[ \sum_{I \in \mathbb{Z} \times \mathbb{F}} \delta B_I^{jl} \right] \lesssim B_\lambda^{-1} N^{-1} \mathbb{D}_R$$

□

#### 4.7.8 Mollification Stress Estimate

**Proposition 4.11.** *There exists a symmetric tensor,  $R_M^{jl}$  as defined in (11) and there exists a constant  $B_\lambda \geq 1$  sufficiently large so that the following estimate is satisfied*

$$\hat{H}[R_M] \leq \frac{GD_R}{1000}, \quad G := \frac{\mathbb{D}_u^{1/4}}{\mathbb{D}_R^{1/4} N^{3/4}}$$

where  $G$  is the constant defined in the Main Lemma 3.1.

*Proof.* We follow the calculations outlined in [17, Section 18.3]. Recall from Lemma 4.3 that  $\hat{H}[R_M] \leq C_0 H[R_M]$  for some positive constant  $C_0$  and from (11) and Section 4.4,  $R_M := R - R_\epsilon$ .

We begin with the 0th derivative bound from Proposition 4.2:

$$C_0 \|R_M\|_{C^0} = C_0 \|R - R_\epsilon\|_{C^0} \leq C \mathbb{D}_R \left( (\Xi e_u^{1/2}) \epsilon_t + \epsilon_x^L \Xi^L \right)$$

for some positive constant  $C$  independent of  $B_\lambda$ . By our choice of mollification parameters in (7)

$$\epsilon_x = c_0 N^{-\frac{3}{2L}} \Xi^{-1}, \quad \epsilon_t = c_0 N^{-3/2} \Xi^{-3/2} \mathbb{D}_R^{-1/2}$$

and by taking the constant  $c_0 > 0$  sufficiently small we obtain the bound  $C_0 \|R_M\|_{C^0} \leq \frac{\mathbb{D}_u^{1/2} \mathbb{D}_R^{1/2}}{1000 N^{3/2}}$ . We observe that  $\frac{\mathbb{D}_u^{1/2} \mathbb{D}_R^{1/2}}{1000 N^{3/2}} \leq \frac{G \mathbb{D}_R}{1000}$  if and only if  $\frac{\mathbb{D}_u^{1/4}}{\mathbb{D}_R^{1/4}} \leq N^{3/4}$ . This second condition holds because of the condition  $N \geq \mathbb{D}_u/\mathbb{D}_R$  in the Main Lemma 3.1 so we are able to conclude that

$$C_0 \|R_M\|_{C^0} \leq \frac{G \mathbb{D}_R}{1000}. \quad (67)$$

Our next step is to show estimates for  $C_0 \|\nabla_{\vec{a}} R_M\|_{C^0}$  in the case  $1 \leq |\vec{a}| \leq L$ . In this case we use Proposition 4.2 and the frequency and energy levels assumed in the Main Lemma 3.1 to get

$$C_0 \|\nabla_{\vec{a}} R_M\|_{C^0} \leq C_0 (\|\nabla_{\vec{a}} R\|_{C^0} + \|\nabla_{\vec{a}} R_\epsilon\|_{C^0}) \leq C \Xi^{|\vec{a}|} \mathbb{D}_R$$

for some positive constant  $C$  that is independent of  $B_\lambda$ . It remains to check  $C \Xi^{|\vec{a}|} \mathbb{D}_R \leq \frac{\lambda^{|\vec{a}|} G \mathbb{D}_R}{1000}$ . This is equivalent to checking  $C \frac{\Xi^{|\vec{a}|}}{\lambda^{|\vec{a}|}} \leq \frac{G}{1000}$ . By the definition of  $\lambda \geq B_\lambda N \Xi$  in Section 4.1, the assumption  $|\vec{a}| \geq 1$ , and by taking a constant  $B_\lambda \geq 1$  sufficiently large we have that

$$C \frac{\Xi^{|\vec{a}|}}{\lambda^{|\vec{a}|}} \leq \frac{C}{B_\lambda N} \leq \frac{C}{B_\lambda} \frac{1}{N} \leq \frac{1}{1000 N} \leq \frac{G}{1000}$$

Where in the last inequality we have used the condition on  $N \geq 1$  from the Main Lemma 3.1 and the fact that  $(\mathbb{D}_u/\mathbb{D}_R)^{1/4} \leq 1$ . We have now shown

$$C_0 \frac{\|\nabla_{\vec{a}} R_M\|_{C^0}}{\lambda^{|\vec{a}|}} \leq \frac{G \mathbb{D}_R}{1000}, \quad \text{for } 0 \leq |\vec{a}| \leq L \quad (68)$$

Lastly, we want to show an estimate for  $C_0 \|\nabla_{\vec{a}} D_t R_M\|_{C^0}$  for  $0 \leq |\vec{a}| \leq L-1$ . By Proposition 4.2 and Definition 2.2

$$C_0 \|\nabla_{\vec{a}} D_t R_M\|_{C^0} \leq \|\nabla_{\vec{a}} D_t R\|_{C^0} + \|\nabla_{\vec{a}} D_t R_\epsilon\|_{C^0} \leq C \Xi^{|\vec{a}|} \mathbb{D}_R (\Xi e_u^{1/2})$$

for  $0 \leq |\vec{a}| \leq L-1$  and some positive constant  $C$  independent of  $B_\lambda$ . The  $N \geq \mathbb{D}_u/\mathbb{D}_R$  assumption implies  $e_u^{1/2} \leq \hat{e}_u^{1/2}$ . Using this bound, the previous bound, and  $\lambda \geq B_\lambda N \Xi$  shows

$$C_0 \frac{\|\nabla_{\vec{a}} D_t R_M\|_{C^0}}{\lambda^{|\vec{a}|} (\lambda \hat{e}_u^{1/2})} \leq C \frac{\Xi^{|\vec{a}|} \mathbb{D}_R (\Xi e_u^{1/2})}{\lambda^{|\vec{a}|} (\lambda \hat{e}_u^{1/2})} \leq C \mathbb{D}_R \frac{\Xi}{\lambda} = \frac{C}{B_\lambda} \frac{\mathbb{D}_R}{N}$$

We observe that if  $B_\lambda$  is a sufficiently large constant then  $\frac{C}{B_\lambda} \leq \frac{1}{1000}$  and  $\frac{\mathbb{D}_R}{N} \leq G \mathbb{D}_R$  because of the condition  $N \geq \mathbb{D}_u/\mathbb{D}_R$  from the Main Lemma 3.1. Therefore we have the estimate

$$C_0 \frac{\|\nabla_{\vec{a}} D_t R_M\|_{C^0}}{\lambda^{|\vec{a}|} (\lambda \hat{e}_u^{1/2})} \leq \frac{G \mathbb{D}_R}{1000}, \quad \text{for } 0 \leq |\vec{a}| \leq L-1 \quad (69)$$

By our three estimates (67), (68), and (69) we can conclude the claimed bound

$$\overset{*}{H}[R_M] \leq C_0 H[R_M] \leq \frac{G \mathbb{D}_R}{1000}$$

□

## 4.8 Verifying the Conclusions of the Main Lemma

In this section we verify the conclusions of the Main Lemma 3.1. We begin by checking that the new SQG-Reynolds flow  $(\hat{\theta}, \hat{u}, \hat{R})$  has frequency and energy levels below  $(\hat{\Xi}, \hat{\mathcal{D}}_u, \hat{\mathcal{D}}_R)$ . First consider the new Reynold's Stress  $\hat{R}$ . By Lemma 4.3  $\hat{H}[\hat{R}] \leq CH[\hat{R}]$  for some positive constant  $C$  that is independent of  $B_\lambda$ . By Propositions 4.9, 4.8, 4.10, and 4.11

$$\begin{aligned}\hat{H}[\hat{R}] &\leq C(H[R_H] + H[R_T] + H[R_S]) + \hat{H}[R_M] \\ &\leq C \left( b\mathcal{D}_R + (B_\lambda N \Xi)^{-3/2} \mathcal{D}_R^{1/2} (\Xi e_u^{1/2}) b^{-1} + B_\lambda^{-1} N^{-1} \mathcal{D}_R \right) + \frac{G\mathcal{D}_R}{1000}\end{aligned}$$

Recalling the definitions from Section 4.1.3 and Main Lemma 3.1

$$b = \left( \frac{\mathcal{D}_u^{1/2}}{\mathcal{D}_R^{1/2} B_\lambda^{3/2} N^{3/2}} \right)^{1/2} b_0, \quad G = \frac{\mathcal{D}_u^{1/4}}{\mathcal{D}_R^{1/4} N^{3/4}}$$

and by taking  $B_\lambda \gg 1$  sufficiently large the last sum will be small enough to ensure that

$$\hat{H}[\hat{R}] \leq \frac{G\mathcal{D}_R}{10}$$

By the definition of  $\hat{H}[\cdot]$  this shows  $\hat{R}$  satisfies the bounds of Definition 2.2 for the new frequency-energy levels  $(\hat{\Xi}, \hat{\mathcal{D}}_u, \hat{\mathcal{D}}_R)$ . Additionally we check that  $\hat{\theta}, \hat{u}$  satisfy the desired bounds. We present only the calculations for  $\hat{\theta}$  as the  $\hat{u}$  calculations are completely identical. For  $0 \leq |\vec{a}| \leq L$  and  $r = 0, 1$

$$\|\nabla_{\vec{a}} D_t^r \hat{\theta}\|_{C^0} \leq \|\nabla_{\vec{a}} D_t^r \theta\|_{C^0} + \|\nabla_{\vec{a}} D_t^r \Theta\|_{C^0} \lesssim N^{\frac{3}{2L}(|\vec{a}|+1-L)_+} \Xi^{|\vec{a}|} \mathcal{D}_R^{1/2} \tau^{-r} + \lambda^{|\vec{a}|+1/2} \mathcal{D}_R^{1/2} \tau^{-r}$$

where the last inequalities are due to Propositions 4.4, 4.7. We observe that for  $L \geq 2$  we have  $N^{\frac{3}{2L}(|\vec{a}|+1-L)_+} \Xi^{|\vec{a}|} \leq \lambda^{|\vec{a}|} \leq \Xi^{|\vec{a}|}$ . Additionally  $\lambda^{1/2} \mathcal{D}_R^{1/2} \leq \hat{e}_u^{1/2}$  and we verify  $\tau^{-1} \leq \Xi \hat{e}_u^{1/2}$  below

$$\begin{aligned}\tau^{-1} \leq \Xi \hat{e}_u^{1/2} &\iff b^{-1} \Xi^{3/2} \mathcal{D}_u^{1/2} \leq (\hat{C} N \Xi)^{3/2} \mathcal{D}_R^{1/2} \iff \\ b_0^{-1} \frac{\mathcal{D}_u^{1/4} B_\lambda^{3/4} N^{3/4}}{\mathcal{D}_R^{1/4}} \mathcal{D}_u^{1/2} &\leq (\hat{C} N)^{3/2} \mathcal{D}_R^{1/2} \iff \frac{\mathcal{D}_u^{1/4}}{\mathcal{D}_R^{1/4}} B_\lambda^{3/4} &\leq b_0 \hat{C}^{3/2} N^{3/4}\end{aligned}$$

Here the last inequality holds due to our choice of  $N \geq \mathcal{D}_u/\mathcal{D}_R$  in the Main Lemma 3.1 and by taking  $\hat{C} \gg B_\lambda$ . By these inequalities we have that for  $0 \leq |\vec{a}| \leq L$  and  $r = 0, 1$

$$\|\nabla_{\vec{a}} D_t^r \hat{\theta}\|_{C^0} \lesssim \lambda^{|\vec{a}|+1/2} \mathcal{D}_R^{1/2} \tau^{-r} \leq \Xi^{|\vec{a}|} \hat{e}_u^{1/2} \left( \Xi \hat{e}_u^{1/2} \right)^r,$$

where again the last inequality holds for a sufficiently large choice of constant  $\hat{C}$ . We have now shown the claimed frequency and energy levels  $(\hat{\theta}, \hat{u}, \hat{R}) \leq (\hat{\Xi}, \hat{\mathcal{D}}_u, \hat{\mathcal{D}}_R)$ . Next we will show the claimed bounds of  $\Lambda^{-1/2} \Theta$  and  $W^i$  where  $\Lambda^{-1/2} \Theta = \nabla_i W^i$ . Recall from Section 4.1.4 that the scalar correction  $\Theta$  has a frequency supp in the frequency band of scale  $\lambda$ . The compact frequency supports with standard Littlewood-Paley theory and Proposition 4.7 gives the following estimate for  $|\vec{a}| \leq 1$

$$\|\nabla_{\vec{a}} \Lambda^{-1/2} \Theta\|_{C^0} \lesssim \lambda^{|\vec{a}|-1/2} \|\Theta\|_{C^0} \lesssim \lambda^{|\vec{a}|} \mathcal{D}_R^{1/2}$$

By taking  $\hat{C} \gg B_\lambda$  sufficiently large we get the claimed estimate from the Main Lemma 3.1.

$$\|\nabla_{\vec{a}} \Lambda^{-1/2} \Theta\|_{C^0} \leq \hat{C}(N \Xi)^{|\vec{a}|} \mathcal{D}_R^{1/2}$$

Next let  $\mathcal{R}^i := (-\Delta)^{-1}\nabla^i$  be a first order anti-divergence operator and let  $P_\lambda$  be an operator that localizes to frequency  $\{|\xi| \sim \lambda\}$  such that  $\Theta = P_\lambda\Theta$ . Define  $W^i := \mathcal{R}^i P_\lambda \Lambda^{-1/2} \Theta$ . Then

$$\|W^i\|_{C^0} = \left\| \mathcal{R}^i P_\lambda \Lambda^{-1/2} \Theta \right\|_{C^0} \lesssim \lambda^{-1} \left\| \Lambda^{-1/2} \Theta \right\|_{C^0} \lesssim \lambda^{-1} D_R^{1/2}.$$

For  $|\vec{a}| \leq 1$ , we then have

$$\lambda \left\| \nabla_{\vec{a}} W^i \right\|_{C^0} + \left\| \nabla_{\vec{a}} \Lambda^{-1/2} \Theta \right\|_{C^0} \lesssim \lambda^{|\vec{a}|} D_R^{1/2}$$

By using the definition of  $\lambda \geq B_\lambda N \Xi$  from Section 4.1 and by taking  $\hat{C} \gg B_\lambda$  sufficiently large this shows the claimed estimate from the Main Lemma 3.1

$$\left\| \nabla_{\vec{a}} W^i \right\|_{C^0} \leq \hat{C}(N\Xi) D_R^{1/2}$$

Next we consider  $\left\| \partial_t \Lambda^{-1/2} \Theta \right\|_{C^0}$ . We begin with the term  $\partial_t \Lambda^{-1/2} \Theta$

$$\partial_t \Lambda^{-1/2} \Theta = \Lambda^{-1/2} D_t \Theta - \Lambda^{-1/2} (u^l \nabla_l \Theta) \quad (70)$$

Consider the first term (70). We observe that  $D_t \Theta$  is also supported in frequency at scales  $\lambda$  due to the how  $u = T\theta$  and  $\hat{\theta}$  is supported at scales  $\Xi \ll \lambda$  as stated in Definition 2.2. By the frequency support and Proposition 4.7

$$\left\| \Lambda^{-1/2} D_t \Theta \right\|_{C^0} \lesssim \lambda^{-1/2} \|D_t \Theta\|_{C^0} \lesssim \lambda^{-1/2} \lambda^{1/2} \mathbb{D}_R^{1/2} \tau^{-1} = \mathbb{D}_R^{1/2} \tau^{-1}$$

For the second term in (70) we again use the fact that the function  $u^l \nabla_l \Theta$  is supported in frequency at scales  $\lambda$  to obtain

$$\left\| \Lambda^{-1/2} (u^l \nabla_l \Theta) \right\|_{C^0} \lesssim \lambda^{-1/2} \|u\|_{C^0} \|\nabla_l \Theta\|_{C^0} \lesssim e_u^{1/2} \lambda \mathbb{D}_R^{1/2}.$$

In total we have shown

$$\left\| \partial_t \Lambda^{-1/2} \Theta \right\|_{C^0} \lesssim \mathbb{D}_R^{1/2} (\tau^{-1} + \lambda e_u^{1/2}).$$

We now compare the time scales and we verify that  $\tau^{-1} \lesssim B_\lambda^{-1/4} \lambda e_u^{1/2}$  with the calculation below:

$$\tau^{-1} \lesssim B_\lambda^{-1/4} \lambda e_u^{1/2} \iff b^{-1} \lesssim B_\lambda^{3/4} N \iff b_0^{-1} \frac{\mathbb{D}_R^{1/4} B_\lambda^{3/4} N^{3/4}}{\mathbb{D}_u^{1/4}} \lesssim B_\lambda^{3/4} N \iff \frac{\mathbb{D}_R^{1/4}}{\mathbb{D}_u^{1/4}} \lesssim N^{1/4}.$$

Where the last inequality holds because  $\mathbb{D}_R/\mathbb{D}_u \leq 1$  and  $N \geq 1$ . By this calculation we can obtain the claimed estimate by taking a constant  $\hat{C} \gg B_\lambda$  sufficiently large to get

$$\left\| \partial_t \Lambda^{-1/2} \Theta \right\|_{C^0} \lesssim \mathbb{D}_R^{1/2} \lambda e_u^{1/2} \leq \hat{C} \mathbb{D}_R^{1/2} (N \Xi e_u^{1/2})$$

All that remains to check is the time support of condition (3). This containment follows from the time support of the lifting function  $e(t)$  defined in Section 4.1.5 and by verifying that  $\epsilon_t \ll (\Xi e_u^{1/2})^{-1}$ , which in turn follows from the definition of  $\epsilon_t = c_0 N^{-3/2} \Xi^{-3/2} \mathbb{D}_R^{1/2}$  in (7) and the condition  $N \geq \mathbb{D}_u/\mathbb{D}_R$  from the Main Lemma 3.1.

## 5 Main Lemma Implies the Main Theorem

In this section we prove the Main Theorem 3.1 using the Main Lemma 3.1. Before applying the Main Lemma 3.1 we fix the parameter  $L = 2$ . Given  $0 < \alpha < 3/10$ ,  $0 < \beta < 1/4$ , we will produce by iteration of the Main Lemma a sequence of solutions  $(\theta_{(k)}, R_{(k)})$  to the SQG-Reynold's system such that  $R_{(k)} \rightarrow 0$  uniformly in  $C^0(\mathbb{R} \times \mathbb{T}^2)$ , the time support of  $\theta_{(k)}$  is contained in the interval  $J_{(k)}$ , and the sequence of functions  $\{\Lambda^{-1/2}\theta_{(k)}\}_{k \in \mathbb{N}}$  converges in  $C_t^\beta C_x^0 \cap C_t^0 C_x^\alpha(\mathbb{R} \times \mathbb{T}^2)$ . We remark that the non-triviality of our constructed solutions will be a consequence of the h-principle proven in Section 6.

### 5.1 The base case $k = 0$

Fix  $\alpha, \beta$  be such that  $0 < \alpha < 3/10$ ,  $0 < \beta < 1/4$ . For stage  $k = 0$  we assume that  $(\theta_{(0)}, R_{(0)})$  are any given smooth, compactly supported SQG-Reynolds flow on  $\mathbb{R} \times \mathbb{T}^2$  (possibly 0). We assume also that the support of the spatial Fourier transform  $\text{supp } \hat{\theta}$  is bounded in frequency space, uniformly in time.

Let  $J_{(0)}$  be a nonempty time interval containing the support of  $(\theta_{(0)}, R_{(0)})$ . We choose a triple of parameters  $(\Xi_{(0)}, D_{u(0)}, D_{R(0)})$  that are admissible according to Definition 2.2 of the frequency and energy levels. Namely, take  $D_{R(0)}$  to be any number  $D_{R(0)} \geq \|R_{(0)}\|_{C^0}$ , and set  $D_{u(0)}$  to equal  $ZD_{R(0)}$ , where  $Z = Z_{\alpha, \beta} \gg 1$  is chosen to satisfy conditions (73) and (75) below. Finally, take  $\Xi_{(0)}$  sufficiently large so that the remaining conditions in Definition 2.2 are satisfied.

### 5.2 Choosing the parameters

We iterate the Main Lemma 3.1 to produce SQG-Reynolds solutions  $(\theta_{(k)}, R_{(k)})$  that are bounded above by the corresponding frequency and energy levels  $(\Xi_{(k)}, D_{u(k)}, D_{R(k)})$ . We also impose the ansatz

$$D_{R(k+1)} = \frac{D_{R(k)}}{Z}, \quad \text{for all } k \geq 0. \quad (71)$$

The parameter  $Z = Z_{\alpha, \beta}$  is chosen sufficiently large to guarantee the conditions (73), (75).

To achieve this ansatz we must choose the frequency growth parameter  $N$  to be

$$N = Z^{5/3}. \quad (72)$$

Note that by the ansatz, after the  $k$ th application of the Main Lemma 3.1  $Z = \frac{D_{R(k-1)}}{D_{R(k)}} = \frac{D_{u(k)}}{D_{R(k)}}$  and it follows that after the  $k + 1$ th application of the Main Lemma 3.1

$$D_{R(k+1)} = G_{(k)} D_{R(k)} = \frac{Z^{1/4}}{N^{3/4}} D_{R(k)} = \frac{D_{R(k)}}{Z}.$$

Hence by our choice of  $N$  we can maintain the ansatz for all  $k \geq 0$ .

### 5.3 Regularity of the Solution

In this section we will show the convergence of  $\{\Lambda^{-1/2}\theta_{(k)}\}_{k \in \mathbb{N}}$  in  $C_t^\beta C_x^0 \cap C_t^0 C_x^\alpha(\mathbb{R} \times \mathbb{T}^2)$  for the given  $\alpha, \beta$ . As a consequence, we have strong convergence of  $\theta_{(k)}$  in  $L_t^2 H^{-1/2}$ , which implies weak convergence of the nonlinearity  $\nabla_l[\theta_{(k)} T^l[\theta]] \rightharpoonup \nabla_l[\theta T^l[\theta]]$  in  $\mathcal{D}'(\mathbb{R} \times \mathbb{T}^2)$ . Since  $\nabla_j \nabla_l R_{(k)}^{jl} \rightharpoonup 0$ , the limiting scalar field  $\theta$  is therefore a weak solution to the SQG equation.

We first calculate the  $C_x^\alpha(\mathbb{T}^2)$  regularity. Define the partial sum

$$\Lambda^{-1/2}\theta_{(K)} := \sum_{k=0}^K \Lambda^{-1/2}\Theta_{(k)}$$

where  $\Theta_{(k)}$  is the scalar correction from the  $k$ th application of the Main Lemma 3.1. From this definition it follows that

$$\left\| \Lambda^{-1/2} \theta_{(K)} \right\|_{C_x^\alpha} \leq \sum_{k=0}^K \left\| \Lambda^{-1/2} \Theta_{(k)} \right\|_{C_x^\alpha}$$

and it will be enough to show a bound on  $\left\| \Lambda^{-1/2} \Theta_{(k+1)} \right\|_{C_x^\alpha}$  that decays exponentially in  $k$ . We may interpolate between the estimates for  $|\vec{a}| = 0, 1$  derivatives given at the end of the  $k+1$ th application of the Main Lemma 3.1 to obtain the following bound for  $\alpha \in (0, 1)$

$$\left\| \Lambda^{-1/2} \Theta_{(k+1)} \right\|_{C_x^\alpha} \leq \hat{C} (N \Xi_{(k)})^\alpha \mathbb{D}_{R(k)}^{1/2}$$

Define the upperbound quantity

$$E_{\alpha(k+1)} := (N \Xi_{(k)})^\alpha \mathbb{D}_{R(k)}^{1/2}$$

so that  $\left\| \Lambda^{-1/2} \Theta_{(k+1)} \right\|_{C_x^\alpha} \leq \hat{C} E_{\alpha(k+1)}$ . We will show that this upperbound  $E_{\alpha(k+1)}$  decays exponentially in  $k$  for an appropriate parameter  $\alpha$ . We apply the ansatz (71), our choice of  $N = Z^{5/3}$  from (72), and the frequency and energy levels from the Main Lemma 3.1

$$E_{\alpha(k+1)} = Z^{5\alpha/3} (\hat{C} N \Xi_{(k-1)})^\alpha Z^{-1/2} \mathbb{D}_{R(k-1)}^{1/2} = \hat{C}^\alpha Z^{\frac{10\alpha-3}{6}} (N \Xi_{(k-1)})^\alpha \mathbb{D}_{R(k-1)}^{1/2} = \hat{C}^\alpha Z^{\frac{10\alpha-3}{6}} E_{\alpha(k)}$$

Now we observe that  $E_{\alpha(k+1)} < E_{\alpha(k)}$  if and only if  $\hat{C}^\alpha Z^{\frac{10\alpha-3}{6}} = \left( \hat{C}^{\frac{6\alpha}{10\alpha-3}} Z \right)^{\frac{10\alpha-3}{6}} < 1$ . For a given  $0 < \alpha < 3/10$  we choose the parameter  $Z \gg 1$  sufficiently large so that

$$\left( \hat{C}^{\frac{6\alpha}{10\alpha-3}} Z \right) < 1. \quad (73)$$

It follows that, with  $Z$  as above, the upperbound quantity  $E_{\alpha(k+1)}$  decays exponentially in  $k$ . Hence for the function  $\Lambda^{-1/2} \theta := \lim_{K \rightarrow \infty} \Lambda^{-1/2} \theta_{(K)}$

$$\max_t \left\| \Lambda^{-1/2} \theta \right\|_{C_x^\alpha} \leq \sum_{k=0}^{\infty} \max_t \left\| \Lambda^{-1/2} \Theta_{(k)} \right\|_{C_x^\alpha} < \infty$$

for the given  $\alpha < 3/10$ .

Next we establish the  $C_t^\beta C_x^0$  regularity. By interpolating the estimates at the end of  $k+1$ th application of the Main Theorem 3.1 again we obtain the bound

$$\left\| \Lambda^{-1/2} \Theta_{(k+1)} \right\|_{C_t^\beta C_x^0} \leq \hat{C} \mathbb{D}_{R(k)}^{1/2} \left( (N \Xi_{(k)}) e_{u(k)}^{1/2} \right)^\beta$$

and we define the upperbound quantity

$$E_{\beta(k+1)} := \mathbb{D}_{R(k)}^{1/2} \left( N \Xi_{(k)} e_{u(k)}^{1/2} \right)^\beta$$

In order to show that this quantity decays as exponentially in  $k$  we will need the identity

$$e_{u(k)}^{1/2} = \hat{C}^{1/2} Z^{1/3} e_{u(k-1)}^{1/2}. \quad (74)$$

This identity can be directly computed using the Main Lemma 3.1, our choice of  $N = Z^{5/3}$  from (72), ansatz (71), and the identity  $\mathbb{D}_{u(k)} = \frac{\mathbb{D}_{u(k-1)}}{Z}$  that comes from combining the ansatz (71) with the Main Lemma 3.1

$$e_{u(k)}^{1/2} = \Xi_{(k)}^{1/2} \mathbb{D}_{u(k)}^{1/2} = (\hat{C} N \Xi_{(k-1)})^{1/2} \frac{\mathbb{D}_{u(k-1)}}{Z^{1/2}} = \hat{C}^{1/2} N^{1/2} Z^{-1/2} e_{u(k-1)}^{1/2} = \hat{C}^{1/2} Z^{1/3} e_{u(k-1)}^{1/2}.$$

We apply (74) and calculate  $E_{\beta(k+1)}$  as

$$\begin{aligned} E_{\beta(k+1)} &= \frac{\mathbb{D}_{R(k-1)}^{1/2}}{Z^{1/2}} \left( Z^{5/3} (\widehat{C} N \Xi_{(k-1)}) \widehat{C}^{1/2} Z^{1/3} e_{u(k-1)}^{1/2} \right)^\beta \\ &= \widehat{C}^{\frac{3\beta}{2}} Z^{2\beta - \frac{1}{2}} \mathbb{D}_{R(k-1)}^{1/2} \left( N \Xi_{(k-1)} e_{u(k-1)}^{1/2} \right)^\beta \\ &= \widehat{C}^{\frac{3\beta}{2}} Z^{\frac{4\beta-1}{2}} E_{\beta(k)}. \end{aligned}$$

We observe that the upperbound quantity  $E_{\beta(k+1)}$  decays exponentially in  $k$  if and only if

$$\widehat{C}^{\frac{3\beta}{2}} Z^{\frac{4\beta-1}{2}} = \left( \widehat{C}^{\frac{3\beta}{4\beta-1}} Z \right)^{\frac{4\beta-1}{2}} < 1.$$

For a given  $0 < \beta < 1/4$  we can choose a parameter  $Z \gg 1$  sufficiently large so that

$$\left( \widehat{C}^{\frac{3\beta}{4\beta-1}} Z \right)^{\frac{4\beta-1}{2}} < 1. \quad (75)$$

From this inequality, it follows that  $E_{\beta(k+1)}$  decays as exponentially in  $k$  and

$$\left\| \Lambda^{-1/2} \theta \right\|_{C_t^\beta C_x^0} \leq \sum_{k=0}^{\infty} \left\| \Lambda^{-1/2} \Theta_{(k)} \right\|_{C_t^\beta C_x^0} < \infty,$$

for the given  $\beta < 1/4$ .

Since  $C_t^\beta C_x^0$  and  $C_t^0 C_x^\alpha$  are Banach spaces, we have shown that  $\Lambda^{-1/2} \theta \in C_t^\beta C_x^0 \cap C_t^0 C_x^\alpha$ .

#### 5.4 Compact Support in Time of the Solution

In this section we will discuss how the limiting SQG solution,  $\theta := \lim_{k \rightarrow \infty} \theta_{(k)}$ , has compact support in time. By (3) it is enough to show that the series  $\sum_{k=0}^{\infty} \left( \Xi_{(k)} e_{u(k)}^{1/2} \right)^{-1}$  converges. By the choice of  $N = Z^{5/3}$  from (72) and the identity (74), we have

$$\left( \Xi_{(k)} e_{u(k)}^{1/2} \right)^{-1} = \widehat{C}^{-3/2} Z^{-2} \left( \Xi_{(k-1)} e_{u(k-1)}^{1/2} \right)^{-1}$$

We note that  $\widehat{C}^{-3/2} Z^{-2} < 1$ , and so  $\sum_{k=0}^{\infty} \left( \Xi_{(k)} e_{u(k)}^{1/2} \right)^{-1} < \infty$ .

## 6 An h-principle

We present the proof to Theorem 2 here.

Let  $f$  be a given smooth scalar field with compact support that satisfies  $\int_{\mathbb{T}^2} f(x, t) dx = 0$  for all  $t$ , and let  $0 < \alpha < 3/10, 0 < \beta < 1/4$  be given. We want to construct a sequence of weak solutions  $\theta_n$  to the SQG equation of class  $\Lambda^{-1/2} \theta_n \in C_t^0 C_x^\alpha \cap C_t^\beta C_x^0$  with such that  $\Lambda^{-1/2} \theta_n \rightharpoonup \Lambda^{-1/2} f$  in  $L_{t,x}^\infty$  weak-\*.

The first step is to approximate  $f$  by scalar fields  $f_n, n \in \mathbb{N}$ , that have compact frequency support. Let  $\eta_n(h) = 2^{2n} \eta(2^n h)$  be a standard mollifying kernel in the spatial variables with compact frequency support  $\text{supp } \hat{\eta}_n \subseteq \{|\xi| \leq 2^n\}$ . Set  $f_n = \eta_n * f$ . With this choice, we have that

$$\sup_n \|\nabla_{\vec{a}} \partial_t^r f_n\|_{C^0} \lesssim \|\nabla_{\vec{a}} \partial_t^r f\|_{C^0}, \text{ for } 0 \leq |\vec{a}|, r, \text{ and } \lim_{n \rightarrow \infty} \left\| \Lambda^{-1/2} (f_n - f) \right\|_{C^0} = 0. \quad (76)$$

The time support of  $f_n$  is also contained in the time support of  $f$ .

Now observe that the scalar fields  $f_n$  can be realized as SQG-Reynolds flows. Namely, consider  $\partial_t f_n + T^l f_n \nabla_l f_n$  and observe that this function has mean-0 over  $\mathbb{T}^2$  due to the non-linear term  $T^l f_n \nabla_l f_n = \nabla_l [f_n T^l f_n]$  being the divergence of a smooth vector field and the assumption that  $\int_{\mathbb{T}^2} f(x, t) dx = 0$  in Theorem 2. Therefore we may apply the second-order anti-divergence operator  $\mathcal{R}$  defined in (13). Specifically, we define symmetric and traceless tensors  $R_n$  such that

$$R_n := \mathcal{R} [\partial_t f_n + T^l f_n \nabla_l f_n], \text{ which implies } \partial_t f_n + T^l f_n \nabla_l f_n = \nabla_j \nabla_l R_n^{jl}.$$

The pair  $(f_n, R_n)$  is thus a smooth SQG-Reynolds flow with frequency support  $\text{supp } \hat{f} \subseteq \{|\xi| \leq 2^n\}$ . Observe also that by (76), we have a uniform bound on the errors  $R_n$ :

$$\sup_n \|R_n\|_{C^0} \leq \hat{\mathbf{D}}_R < \infty \quad \text{for some } \hat{\mathbf{D}}_R > 0.$$

Starting with  $(f_n, R_n)$ , we now construct a weak solution  $\theta_n$  to SQG close to  $f_n$  in the appropriate weak topology. The SQG solution  $\theta_n$  is constructed by iterating the Main Lemma 3.1 using an initial choice frequency and energy levels  $(\Xi_{n(0)}, \mathbf{D}_{R,n(0)}, \mathbf{D}_{u,n(0)})$  chosen as follows. Take  $\mathbf{D}_{R,n(0)} = \hat{\mathbf{D}}_R > 0$  and  $\mathbf{D}_{u,n(0)} = Z\mathbf{D}_{R,n(0)}$  where  $Z = Z_{\alpha,\beta} \gg 1$  is as in Section 5. Take  $\Xi_{n(0)}$  sufficiently large such that the SQG-Reynolds flow  $(\theta_{n(0)}, R_{n(0)}) := (f_n, R_n)$  satisfies Definition 2.2, and such that

$$\lim_{n \rightarrow \infty} \Xi_{n(0)} = \infty. \tag{77}$$

In particular, the frequency support condition will be satisfied for  $\Xi_{n(0)} \geq 2^n$ .

Repeated application of the Main Lemma using the parameter choices of Section 5 yields a sequence of SQG-Reynolds flows  $(\theta_{n(k)}, R_{n(k)})$  such that  $\theta_{n(k)}$  converges as  $k \rightarrow \infty$  to an SQG solution  $\theta_n$  with  $\Lambda^{-1/2} \theta_n \in C_t^\beta C_x^0 \cap C_t^0 C_x^\alpha$ . We claim that the solutions  $\theta_n$  satisfy  $\Lambda^{-1/2} \theta_n \rightharpoonup \Lambda^{-1/2} f$  in  $L_{t,x}^\infty$  weak-\*.

Let  $\epsilon > 0$  and a test function  $\varphi \in L^1(\mathbb{R} \times \mathbb{T}^2)$  be given. We will show that there exists a large index  $M$ , depending on  $\varphi$  and  $\epsilon$ , such that for all  $n \geq M$

$$\left| \int_{\mathbb{R} \times \mathbb{T}^2} \Lambda^{-1/2} (\theta_n - f) \varphi dx dt \right| < 3\epsilon.$$

First, observe by (76) and Hölder that for all  $n \geq M$  sufficiently large we have

$$\left| \int_{\mathbb{R} \times \mathbb{T}^2} \Lambda^{-1/2} (f_n - f) \varphi dx dt \right| < \epsilon. \tag{78}$$

Next, take a smooth approximation of compact support  $\tilde{\varphi} \in C_c^\infty(\mathbb{R} \times \mathbb{T}^2)$  with

$$\|\varphi - \tilde{\varphi}\|_{L^1(\mathbb{R} \times \mathbb{T}^2)} < \frac{\epsilon}{2\hat{C}\hat{\mathbf{D}}_R^{1/2}}, \tag{79}$$

where  $\hat{C}$  is the constant from the Main Lemma 3.1. It now suffices to estimate the integral

$$\left| \int_{\mathbb{R} \times \mathbb{T}^2} \Lambda^{-1/2} (\theta_n - f_n) \varphi dx dt \right| \leq \left| \int_{\mathbb{R} \times \mathbb{T}^2} \Lambda^{-1/2} (\theta_n - f_n) \tilde{\varphi} dx dt \right| \tag{80}$$

$$+ \left| \int_{\mathbb{R} \times \mathbb{T}^2} \Lambda^{-1/2} (\theta_n - f_n) (\varphi - \tilde{\varphi}) dx dt \right|. \tag{81}$$

Let  $\Theta_{n(k)}$  be the corrections defined so that  $\theta_n - f_n = \sum_{k \geq 0} \Theta_{n(k)}$  and the integral (80) is

$$(80) = \left| \int_{\mathbb{R} \times \mathbb{T}^2} \sum_{k \geq 0} \Lambda^{-1/2} \Theta_{n(k)} \tilde{\varphi} dx dt \right|.$$

From the Main Lemma 3.1,  $\Lambda^{-1/2}\Theta_{n(k)} = \nabla_i W_{n(k)}^i$ ,  $\|W_{n(k)}\|_{C^0} \leq \widehat{C} (N\Xi_{n(k)})^{-1} \mathbb{D}_{R,n(k)}^{1/2}$ , and

$$(80) \leq \sum_{k \geq 0} \left| \int_{\mathbb{R} \times \mathbb{T}^2} \nabla_i W_{n(k)}^i \tilde{\varphi} dx dt \right| \leq \sum_{k \geq 0} \|W_{n(k)}\|_{C^0} \|\nabla \tilde{\varphi}\|_{L^1} \leq \|\nabla \tilde{\varphi}\|_{L^1} \sum_{k \geq 0} \widehat{C} (N\Xi_{n(k)})^{-1} \mathbb{D}_{R,n(k)}^{1/2} \\ \leq \frac{2\widehat{C}\mathbb{D}_{R,n(0)}^{1/2}}{N\Xi_{n(0)}} \|\nabla \tilde{\varphi}\|_{L^1} \leq \frac{\widehat{C}\mathbb{D}_R^{1/2}}{\Xi_{n(0)}} \|\nabla \tilde{\varphi}\|_{L^1} < \epsilon.$$

Here we used that  $Z \gg 1$  and  $\Xi_{n(k+1)}^{-1} \mathbb{D}_{R,n(k+1)}^{1/2} \leq Z^{-\frac{5}{3}-\frac{1}{2}} \Xi_{n(k)}^{-1} \mathbb{D}_{R,n(k)}^{1/2}$  to sum the geometric series, and the last inequality holds by (77) for all  $n \geq M$  sufficiently large. To bound (81), we now observe

$$(81) \leq \|\Lambda^{-1/2}(\theta_n - f_n)\|_{C^0} \|\varphi - \tilde{\varphi}\|_{L^1}.$$

Next we show a bound on  $\|\Lambda^{-1/2}(\theta_n - f_n)\|_{C^0}$ . To start  $\|\Lambda^{-1/2}(\theta_n - f_n)\|_{C^0} \leq \sum_{k \geq 0} \|\Lambda^{-1/2}\Theta_{n(k)}\|_{C^0}$ . Recall from the Main Lemma 3.1 that  $\|\Lambda^{-1/2}\Theta_{n(k)}\|_{C^0} \leq \widehat{C}\mathbb{D}_{R,n(k)}^{1/2}$  and we use the ansatz from (71) to get  $\mathbb{D}_{R,n(k)} = \frac{\mathbb{D}_{R,n(0)}}{Z^k}$ . From this equation, it follows that

$$\|\Lambda^{-1/2}(\theta_n - f_n)\|_{C^0} \leq \sum_{k \geq 0} \widehat{C}\mathbb{D}_{R,n(0)}^{1/2} Z^{-k/2} \leq 2\widehat{C}\mathbb{D}_{R,n(0)}^{1/2} = 2\widehat{C}\mathbb{D}_R^{1/2},$$

and therefore

$$(81) \leq 2\widehat{C}\mathbb{D}_R^{1/2} \|\varphi - \tilde{\varphi}\|_{L^1} < \epsilon.$$

The last inequality is due to our choice of  $\tilde{\varphi}$  in (79). By combining the estimates of (78), (80) and (81), we have now shown that there exists an  $M$  such that for all  $n \geq M$

$$\left| \int_{\mathbb{R} \times \mathbb{T}^2} \Lambda^{-1/2}(\theta_n - f) \varphi dx dt \right| < 3\epsilon,$$

and we can conclude the claim that  $\Lambda^{-1/2}\theta_n \rightharpoonup \Lambda^{-1/2}f$  in the  $L^\infty$  weak-\* topology.

## References

- [1] I. Akramov and E. Wiedemann. Renormalization of active scalar equations. *Nonlinear Analysis*, 179:254–269, 2019.
- [2] T. Buckmaster, C. de Lellis, L. Székelyhidi, Jr., and V. Vicol. Onsager’s conjecture for admissible weak solutions. *Comm. Pure Appl. Math.*, 72(2):229–274, 2019.
- [3] T. Buckmaster, S. Shkoller, and V. Vicol. Nonuniqueness of weak solutions to the SQG equation. *Comm. Pure Appl. Math.*, 72(9):1809–1874, 2019.
- [4] T. Buckmaster and V. Vicol. Convex integration and phenomenologies in turbulence. *EMS Surv. Math. Sci.*, 6(1):173–263, 2019.
- [5] X. Cheng, H. Kwon, and D. Li. Non-uniqueness of steady state weak solutions to the surface quasi-geostrophic equations. *In preparation*.
- [6] A. Choffrut. h-Principles for the Incompressible Euler Equations. *Arch. Ration. Mech. Anal.*, 210(1):133–163, 2013.

- [7] P. Constantin. Scaling exponents for active scalars. *J. Statist. Phys.*, 90(3-4):571–595, 1998.
- [8] P. Constantin. Energy spectrum of quasigeostrophic turbulence. *Physical review letters*, 89(18):184501, 2002.
- [9] P. Constantin, W. E, and E. S. Titi. Onsager’s conjecture on the energy conservation for solutions of Euler’s equation. *Comm. Math. Phys.*, 165(1):207–209, 1994.
- [10] D. Cordoba, D. Faraco, and F. Gancedo. Lack of uniqueness for weak solutions of the incompressible porous media equation. *Arch. Ration. Mech. Anal.* 200, 3:725–746, 2011.
- [11] S. Daneri and L. Székelyhidi, Jr. Non-uniqueness and h-principle for Hölder-continuous weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.*, 224(2):471–514, 2017.
- [12] C. De Lellis. The Onsager theorem. In *Surveys in differential geometry 2017. Celebrating the 50th anniversary of the Journal of Differential Geometry*, volume 22 of *Surv. Differ. Geom.*, pages 71–101. Int. Press, Somerville, MA, 2018.
- [13] C. De Lellis and L. Székelyhidi. Dissipative continuous Euler flows. *Invent. Math.*, 193(2):377–407, 2013.
- [14] C. De Lellis and L. Székelyhidi, Jr. The  $h$ -principle and the equations of fluid dynamics. *Bull. Amer. Math. Soc. (N.S.)*, 49(3):347–375, 2012.
- [15] C. De Lellis and L. Székelyhidi, Jr. Dissipative Euler flows and Onsager’s conjecture. *J. Eur. Math. Soc. (JEMS)*, 16(7):1467–1505, 2014.
- [16] I. M. Held, R. T. Pierrehumbert, S. T. Garner, and K. L. Swanson. Surface quasi-geostrophic dynamics. *Journal of Fluid Mechanics*, 282:1–20, 1995.
- [17] P. Isett. *Hölder continuous Euler flows in three dimensions with compact support in time*, volume 196 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2017.
- [18] P. Isett. Nonuniqueness and existence of continuous, globally dissipative Euler flows. *arXiv preprint arXiv:1710.11186*, 2017.
- [19] P. Isett. On the endpoint regularity in Onsager’s conjecture. *arXiv preprint arXiv:1706.01549*, 2017.
- [20] P. Isett. A proof of Onsager’s conjecture. *Ann. of Math. (2)*, 188(3):871–963, 2018.
- [21] P. Isett and S.-J. Oh. On nonperiodic Euler flows with Hölder regularity. *Arch. Ration. Mech. Anal.*, 221(2):725–804, 2016.
- [22] P. Isett and V. Vicol. Hölder continuous solutions of active scalar equations. *Ann. PDE*, 1(1):Art. 2, 77, 2015.
- [23] S. Klainerman. On Nash’s unique contribution to analysis in just three of his papers. *Bulletin of the American Mathematical Society*, 54(2):283–305, 2017.
- [24] F. Marchand. Existence and regularity of weak solutions to the quasi-geostrophic equations in the spaces  $L^p$  or  $\dot{H}^{-1/2}$ . *Comm. Math. Phys.*, 277(1):45–67, 2008.
- [25] J. Nash.  $C^1$  isometric embeddings i, ii. *Ann. Math.*, 60:383–396, 1954.
- [26] L. Onsager. Statistical hydrodynamics. *Nuovo Cimento (9)*, 6 Supplemento(2 (Convegno Internazionale di Meccanica Statistica)):279–287, 1949.

- [27] J. Pedlosky. *Geophysical fluid dynamics*. Springer Science & Business Media, 1982.
- [28] S. G. Resnick. *Dynamical problems in non-linear advective partial differential equations*. ProQuest LLC, Ann Arbor, MI, 1995. Thesis (Ph.D.)–The University of Chicago.
- [29] R. Shvydkoy. Convex integration for a class of active scalar equations. *J. Amer. Math. Soc.*, 24(4):1159–1174, 2011.