Coursework (5) for *Introductory Lectures on Optimization*

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Excercise 1. Prove the following theorem:
for any $\boldsymbol{x}_0 \in \mathrm{dom}\, f$, all vectors $\boldsymbol{g} \in \partial f(\boldsymbol{x}_0)$ are supporting to the level set $\mathcal{L}_f(f(\boldsymbol{x}_0))$:

$$\langle \boldsymbol{g},\ \boldsymbol{x}_0 - \boldsymbol{x} \rangle \geq 0, \quad \forall \boldsymbol{x} \in \mathcal{L}_f(f(\boldsymbol{x}_0)) \equiv \{\boldsymbol{x} \in \mathrm{dom}\, f : f(\boldsymbol{x}) \leq f(\boldsymbol{x}_0)\}.$$

Proof of Excercise 1: According to the definition of subdifferential, we have

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}_0) + \langle \boldsymbol{g}, \boldsymbol{x} - \boldsymbol{x}_0 \rangle$$

Then, for any $\boldsymbol{x} \in \mathcal{L}_f(f(\boldsymbol{x}_0))$, we have

$$f(\boldsymbol{x}) \leq f(\boldsymbol{x}_0)$$

Therefore,

$$\langle \boldsymbol{g}, \boldsymbol{x} - \boldsymbol{x}_0 \rangle \leq f(\boldsymbol{x}) - f(\boldsymbol{x}_0) \leq 0.$$

$$\langle \boldsymbol{g}, \boldsymbol{x}_0 - \boldsymbol{x} \rangle \geq 0.$$

Therefore, for any $\boldsymbol{x}_0 \in \mathrm{dom}\, f$, all vectors $\boldsymbol{g} \in \partial f(\boldsymbol{x}_0)$ are supporting to the level set $\mathcal{L}_f(f(\boldsymbol{x}_0))$. □

Excercise 2. Prove the following theorem:
let $Q \subseteq \mathrm{dom}\, f$ be a closed convex set, $\boldsymbol{x}_0 \in Q$ and

$$\boldsymbol{x}^* = \mathrm{argmin}\{f(\boldsymbol{x}) | \boldsymbol{x} \in Q\}.$$

Then for any $g \in \partial f(\boldsymbol{x}_0)$ we have $\langle \boldsymbol{g},\ \boldsymbol{x}_0 - \boldsymbol{x}^* \rangle \geq 0$.

Proof of Excercise 2: According to the definition of subdifferential, we have

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}_0) + \langle \boldsymbol{g}, \boldsymbol{x} - \boldsymbol{x}_0 \rangle$$

Then, for any $\boldsymbol{x} = \boldsymbol{x}^* \in Q$, we have

$$f(\boldsymbol{x}^*) \leq f(\boldsymbol{x_0})$$

Therefore,

$$\langle \boldsymbol{g}, \boldsymbol{x}^* - \boldsymbol{x}_0 \rangle \leq f(\boldsymbol{x}^*) - f(\boldsymbol{x}_0) \leq 0$$

$$\langle \boldsymbol{g}, \boldsymbol{x}_0 - \boldsymbol{x}^* \rangle \geq 0$$

□

Excercise 3. Prove the following theorem:
let f be closed and convex. Assume that it is differentiable on its domain. Then $\partial f(\boldsymbol{x}) = \{\nabla f(\boldsymbol{x})\}$ for any $\boldsymbol{x} \in \text{int}(\text{dom } f)$.

Proof of Excercise 3: It is obvious that $\partial f(\boldsymbol{x}) \in \{\nabla f(\boldsymbol{x})\}$ for any $\boldsymbol{x} \in \text{int}(\text{dom } f)$.

Now we prove that there is no other vector in $\nabla f(\boldsymbol{x})$ except $\partial f(\boldsymbol{x})$.

Assume that there is a vector $\boldsymbol{g} \in \partial f(\boldsymbol{x})$ and $\boldsymbol{g} \neq \nabla f(\boldsymbol{x})$.

According to the definition of subdifferential, we have

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}_0) + \langle \boldsymbol{g}, \boldsymbol{x} - \boldsymbol{x}_0 \rangle$$

Let $\boldsymbol{x} = \boldsymbol{x}_0 + at(\nabla f(\boldsymbol{x}_0) - \boldsymbol{g})$, where $a \in \mathbb{R}$ and $t \in (0, 1)$.

Therefore,

$$f(\boldsymbol{x}_0 + at(\boldsymbol{x} - \boldsymbol{x}_0)) = f(\boldsymbol{x}_0) + at\langle \nabla f(\boldsymbol{x}_0), \nabla f(\boldsymbol{x}_0) - \boldsymbol{g} \rangle + o(t) \geq f(\boldsymbol{x}_0) + at\langle \nabla f(\boldsymbol{x}_0), \nabla f(\boldsymbol{x}_0) - \boldsymbol{g} \rangle$$

Which implies that

$$\begin{aligned}
& at\|\nabla f(\boldsymbol{x}_0) - \boldsymbol{g}\|^2 + o(t) \geq 0 \\
& \Leftrightarrow a\|\nabla f(\boldsymbol{x}_0) - \boldsymbol{g}\|^2 + \frac{o(t)}{t} \geq 0 \\
& \Rightarrow \lim_{t \to 0} a\|\nabla f(\boldsymbol{x}_0) - \boldsymbol{g}\|^2 + \frac{o(t)}{t} \geq 0 \\
& \Rightarrow a\|\nabla f(\boldsymbol{x}_0) - \boldsymbol{g}\|^2 \geq 0
\end{aligned}$$

Since $a \in \mathbb{R}$ and $a \neq 0$, we have

$$\|\nabla f(\boldsymbol{x}_0) - \boldsymbol{g}\| = 0$$

Which implies that $\boldsymbol{g} = \nabla f(\boldsymbol{x}_0)$. It is a contradiction.

Therefore, $\partial f(\boldsymbol{x}) = \{\nabla f(\boldsymbol{x})\}$ for any $\boldsymbol{x} \in \text{int}(\text{dom } f)$. $\square$