Coursework (5) for *Introductory Lectures on Optimization*

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Excercise 1. Prove the following theorem:
Let $\|\cdot\|$ be a vector norm in $\mathbb{R}^n$, then

$$\partial\|\cdot\| = \left\{V(\boldsymbol{x}) \triangleq \left\{\boldsymbol{v}\in\mathbb{R}^n \middle| \langle \boldsymbol{v},\ \boldsymbol{x}\rangle = \|\boldsymbol{x}\|, \|\boldsymbol{v}\|_* \le 1\right\}\right\},$$

where $\|\boldsymbol{v}\|_*$ is the dual norm of $\|\cdot\|$, defined as

$$\|\boldsymbol{v}\|_* \triangleq \sup_{\|\boldsymbol{u}\|\le 1} \langle \boldsymbol{v},\ \boldsymbol{u}\rangle.$$

Proof of Excercise 1:

1. Prove that $\partial\|\cdot\| \subseteq \left\{V(\boldsymbol{x}) \triangleq \left\{\boldsymbol{v}\in\mathbb{R}^n \middle| \langle \boldsymbol{v},\ \boldsymbol{x}\rangle = \|\boldsymbol{x}\|, \|\boldsymbol{v}\|_* \le 1\right\}\right\}$

   Let $\boldsymbol{v}\in\partial\|\boldsymbol{x}\|$, then we have

   $$\|\boldsymbol{y}\| \ge \|\boldsymbol{x}\| + \langle \boldsymbol{v}, \boldsymbol{y}-\boldsymbol{x}\rangle, \forall \boldsymbol{y}\in\mathbb{R}^n$$

   If $\boldsymbol{y}=0$, then we have

   $$\|\boldsymbol{x}\| \le \langle \boldsymbol{v}, \boldsymbol{x}\rangle$$

   If $\boldsymbol{y}=2\boldsymbol{x}$, then we have

   $$\begin{aligned}&2\|\boldsymbol{x}\| \ge \|\boldsymbol{x}\| + \langle \boldsymbol{v}, \boldsymbol{x}\rangle\\ &\Rightarrow \|\boldsymbol{x}\| \ge \langle \boldsymbol{v}, \boldsymbol{x}\rangle\end{aligned}$$

   Therefore, we have $\|\boldsymbol{x}\| = \langle \boldsymbol{v}, \boldsymbol{x}\rangle$.

   Since

   $$\|\boldsymbol{v}\|_* \triangleq \sup_{\|\boldsymbol{u}\|\le 1} \langle \boldsymbol{v},\ \boldsymbol{u}\rangle.$$

   Then we have $1 \ge \|\boldsymbol{u}\| \ge \|\boldsymbol{x}\| + \langle \boldsymbol{v}, \boldsymbol{u}-\boldsymbol{x}\rangle$. Which implies that $1 \ge \langle \boldsymbol{v}, \boldsymbol{x}\rangle + \langle \boldsymbol{v}, \boldsymbol{u}\rangle - \langle \boldsymbol{v}, \boldsymbol{x}\rangle = \langle \boldsymbol{v}, \boldsymbol{u}\rangle$.

   Therefore, we have $\|\boldsymbol{v}\|_* \le 1$, which implies that $\boldsymbol{v}\in V(\boldsymbol{x})$.

2. Prove that $\left\{V(\boldsymbol{x}) \triangleq \left\{\boldsymbol{v} \in \mathbb{R}^n \middle| \langle \boldsymbol{v},\ \boldsymbol{x}\rangle = \|\boldsymbol{x}\|, \|\boldsymbol{v}\|_* \leq 1\right\}\right\} \subseteq \partial \|\cdot\|$

   Let $\boldsymbol{v} \in V(\boldsymbol{x})$, then we have

$$\begin{aligned} & \|\boldsymbol{y}\| \geq \|\boldsymbol{x}\| + \langle \boldsymbol{v}, \boldsymbol{y} - \boldsymbol{x}\rangle \\ & \Leftrightarrow \|\boldsymbol{y}\| \geq \langle \boldsymbol{v}, \boldsymbol{x}\rangle + \langle \boldsymbol{v}, \boldsymbol{y} - \boldsymbol{x}\rangle \\ & \Leftrightarrow \|\boldsymbol{y}\| \geq \langle \boldsymbol{v}, \boldsymbol{y}\rangle \end{aligned}$$

   Since $\|\boldsymbol{v}\|_* \leq 1$, we have

$$\langle \boldsymbol{v}, \boldsymbol{y}\rangle \leq \|\boldsymbol{v}\|_* \|\boldsymbol{y}\| \leq \|\boldsymbol{y}\|$$

   Therefore, we have $\boldsymbol{v} \in \partial \|\boldsymbol{x}\|$, which implies that $\left\{V(\boldsymbol{x}) \triangleq \left\{\boldsymbol{v} \in \mathbb{R}^n \middle| \langle \boldsymbol{v},\ \boldsymbol{x}\rangle = \|\boldsymbol{x}\|, \|\boldsymbol{v}\|_* \leq 1\right\}\right\} \subseteq \partial \|\cdot\|$.

3. Therefore, we have $\partial \|\cdot\| = \left\{V(\boldsymbol{x}) \triangleq \left\{\boldsymbol{v} \in \mathbb{R}^n \middle| \langle \boldsymbol{v},\ \boldsymbol{x}\rangle = \|\boldsymbol{x}\|, \|\boldsymbol{v}\|_* \leq 1\right\}\right\}$.

$\square$

Excercise 2. Write down the subdifferentials of following functions.

1. $f(\boldsymbol{x}) = |\boldsymbol{x}|, \boldsymbol{x} \in \mathbb{R}^1$.
2. $f(\boldsymbol{x}) = \sum_{i=1}^{m} |\langle \boldsymbol{a}_i,\ \boldsymbol{x}\rangle - \boldsymbol{b}_i|$.
3. $f(\boldsymbol{x}) = \max_{1\le i\le n} \boldsymbol{x}^{(i)}$.
4. $f(\boldsymbol{x}) = \|\boldsymbol{x}\|$.
5. $f(\boldsymbol{x}) = \|\boldsymbol{x}\|_1 = \sum_{i=1}^{n} |\boldsymbol{x}^{(i)}|$.

Solution of Excercise 2:

1. $f(\boldsymbol{x}) = |\boldsymbol{x}|, \boldsymbol{x} \in \mathbb{R}^1$.
$$\partial f(\boldsymbol{x}) = \begin{cases} \{1\} & \boldsymbol{x} > 0 \\ \{-1\} & \boldsymbol{x} < 0 \\ [-1, 1] & \boldsymbol{x} = 0 \end{cases}$$
2. $f(\boldsymbol{x}) = \sum_{i=1}^{m} |\langle \boldsymbol{a}_i,\ \boldsymbol{x}\rangle - \boldsymbol{b}_i|$.
$$\partial f(\boldsymbol{x}) = \left\{ \sum_{i=1}^{m} s_i \boldsymbol{a}_i \mid s_i \in \begin{cases} \{-1\}, & \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle < \boldsymbol{b}_i, \\ [-1, 1], & \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle = \boldsymbol{b}_i, \\ \{1\}, & \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle > \boldsymbol{b}_i. \end{cases} \right\}$$
3. $f(\boldsymbol{x}) = \max_{1\le i\le n} \boldsymbol{x}^{(i)}$.
$$\partial f(\boldsymbol{x}) = \operatorname{conv}\{\boldsymbol{e}_i \mid \boldsymbol{x}^{(i)} = f(\boldsymbol{x})\}$$
4. $f(\boldsymbol{x}) = \|\boldsymbol{x}\|$.
$$\partial f(\boldsymbol{x}) = \{\boldsymbol{v} \mid \langle \boldsymbol{v}, \boldsymbol{x}\rangle = \|\boldsymbol{x}\|, \|\boldsymbol{v}\|_* \le 1\}$$
$$\|\boldsymbol{v}\|_* = \sup_{\|\boldsymbol{u}\|\le 1} \langle \boldsymbol{v},\ \boldsymbol{u}\rangle$$
5. $f(\boldsymbol{x}) = \|\boldsymbol{x}\|_1 = \sum_{i=1}^{n} |\boldsymbol{x}^{(i)}|$.
$$\partial f(\boldsymbol{x}) = \left\{ \sum_{i=1}^{n} \boldsymbol{s_i} \boldsymbol{e}_i \mid \boldsymbol{s_i} \in \begin{cases} \{-1\}, & \boldsymbol{x}^{(i)} < 0, \\ [-1, 1], & \boldsymbol{x}^{(i)} = 0, \\ \{1\}, & \boldsymbol{x}^{(i)} > 0. \end{cases} \right\}$$

$\square$

Excercise 3. Please write down three sequences and prove that they satisfy the following conditions:

$$h_k > 0, h_k \to 0, \sum_{k=0}^{\infty} h_k = \infty.$$

Solution of Excercise 3:

1. $h_k = \frac{1}{k}$
   - $h_k > 0$: For all $k \geq 1$, $\frac{1}{k}$ is positive
   - $h_k \to 0$: As $k \to \infty$, $\frac{1}{k} \to 0$
   - $\sum_{k=0}^{\infty} h_k = \sum_{k=1}^{\infty} \frac{1}{k}$ is the harmonic series, which diverges to ∞
2. $h_k = \frac{1}{k^{\frac{1}{2}}}$
   - $h_k > 0$: For all $k \geq 1$, $\frac{1}{k^{\frac{1}{2}}}$ is positive
   - $h_k \to 0$: As $k \to \infty$, $\frac{1}{k^{\frac{1}{2}}} \to 0$
   - $\sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{2}}}$ is a p-series with $p = \frac{1}{2}$, which diverges to ∞
3. $h_k = \frac{1}{k^{\frac{1}{3}}}$
   - $h_k > 0$: For all $k \geq 1$, $\frac{1}{k^{\frac{1}{3}}}$ is positive
   - $h_k \to 0$: As $k \to \infty$, $\frac{1}{k^{\frac{1}{3}}} \to 0$
   - $\sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{3}}}$ is a p-series with $p = \frac{1}{3}$, which diverges to ∞

Proof why $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges to ∞ when $p \leq 1$:

Since

$$\int_1^{\infty} \frac{1}{x} dx = ln(x)|_1^{\infty} = \infty$$

Then we have $\sum_{k=0}^{\infty} h_k = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges to ∞.

Since

$$\frac{1}{k^p} \geq \frac{1}{k} \quad \text{for all } p \leq 1$$

Then we have $\sum_{k=0}^{\infty} h_k = \sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges to ∞ for all $p \leq 1$.

$\square$