Projects for COS2000, Semester II, 2019/20

Project A1: Population evolution of a coupled Logistic system

Our first venture into population dynamics was the Logistic map:

$$x_{n+1} = \alpha x_n(1 - x_n), \quad n = 0, 1, 2, \cdots,$$

where n labels time or generation, α is the system parameter, and $0 \le x_n \le 1$ is the *normalized* population. x_0 is the initial population when we begin our observation.

In this project, we explore a simple model consisting of two *coupled* Logistic maps, that is used to examine the effects of spatial heterogeneity on population dynamics.

1. A. Hastings, *Complex interactions between dispersal and dynamics: Lessons from coupled logistic equations*, Evology, 74(5) (1993) 1362-1372
2. M. Gyllenberg, G. Söderbacka and S. Ericsson, *Does migration stabilize local population dynamics? Analysis of a discrete metapopulation model*, Mathematical Biosciences, 118 (1993) 25-49

Not everything we know about the Logistic map can be transferred to this coupled maps system. As you proceed with this project, you should notice the *similarities* and *differences*.

The model

The environment in this model consists of two patches between which the individuals diffuse. The two populations are labeled by x and y respectively, and the coupled evolution equations are given by

$$x_{n+1} = (1-\beta)L(x_n) + \beta L(y_n), \qquad y_{n+1} = \beta L(x_n) + (1-\beta)L(y_n),$$

for $n = 0, 1, 2, \cdots$, and $L(x) = \alpha x(1-x)$ is the Logistic map. We refer to β ($0 \le \beta \le 0.5$) as the *coupling parameter*.

1. For $\alpha = 2.9$:

   Known: For this system parameter value, the Logistic map has two fixed points, 0 and $(\alpha - 1)/\alpha$.

   To Verify: For all β values, and *almost* all initial population pairs (x_0, y_0), both populations tend to the nonzero fixed point value $(\alpha - 1)/\alpha$. The fixed point is said to be *stable*.

[Obviously, one cannot carry out this verification for *all* β values. So do your computation for a number of β values, and some chosen (x_0, y_0) for each of those β values.]

2. For $\alpha = 3.2$:

   Known: For this system parameter value, the Logistic map has a (stable) period-2 orbit (i.e. cycling through 2 points repeatedly), and two (unstable) fixed points: 0 and $(\alpha - 1)/\alpha = (3.2 - 1.0)/3.2 = 0.6875$.
   [For this value of α, with *almost all* initial population x_0, the evolution through the Logistic map will lead to the period-2 points. That's why we say the period-2 orbit is *stable*. But if $x_0 = 0$ or $x_0 = 0.6875$ is chosen, the system will still stay at those fixed point values respectively – but any slight departure (e.g. 0.00000001 or 0.68750001) will take the system to the period-2 points. The fixed points are *unstable* in this region of the system parameter.]

   To explore:

   (a) For $\beta = 0.01$, and using different initial populations $(x_0, y_0) = (0.1, 0.1)$, $(0.1, 0.2)$, $0.1, 0.3)$, $(0.1, 0.4)$, and $(0.1, 0.5)$, what long-term behavior(s) do you see in the two populations?

   (b) For $\beta = 0.06$, repeat (a) above and comment on what you see.

3. For $\alpha = 3.5$:

   Known: For this system parameter value, the Logistic map has a stable period-4 orbit (i.e. cycling through 4 points repeatedly).

   To explore:
   Repeat (a) and (b) of item 2. above, with $\beta = 0.01$, 0.015, and 0.040.

4. For $\alpha = 3.7$:

   Known: The Logistic map exhibits chaos for this parameter value.

   To explore:
   Repeat (a) and (b) of item 2. above, with $\beta = 0.01$, 0.015, and 0.040.

Suggestions:

◇ Discard some number of the iterations as transients, and analyze/plot your data only after the transients.

◇ It is useful to consider plotting x_n vs. y_n, and try to understand what the patterns in this plot are trying to tell you.

February 3, 2020