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Discrete Fourier Transforms (DFT):

* Frequency domain sampling and Reconstruction of Discrete Time Signals
* The Discrete Fourier Transform
* DFT as a linear Transformation
* Properties of DFT:
  - -> Periodicity
  - -> Symmetry
  - -> Multiplication of two DFTs and Circular Convolution
* Additional DFT properties

Frequency domain sampling and reconstruction of discrete time signals

Introduction

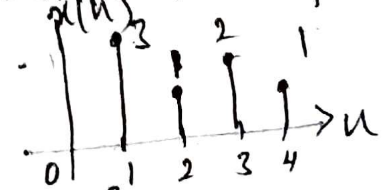
- Today's world is digital almost for all application
- Many applications demands signal processing in frequency domain
- For example, frequency content, periodicity, energy, power spectrum can be better analised in frequency domain than in time domain.
- Hence signals are transformed from time domain to frequency domain. Such transformations can be done from tools like Discrete Fourier Transform (DFT), Fourier Transform (FT)
- Once the required analysis and processing is done in frequency domain, the signals are then transformed back to time domain by using IDFT.

$$x(n) \underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftarrows}} X(k)$$

Analog signals and discrete signals

Analog signals exists at all points

- Analog signals are continuous signals
- Discrete signals are signals where values ... different instants of time and



1.1 Frequency domain Sampling and Reconstruction of discrete time signal.

Need for Frequency domain sampling.

* To perform frequency analysis on a discrete time signal $x(n)$, we convert the time-domain sequence to a frequency domain representation. Such a representation is given by Fourier transform $X(w)$

* The fourier transform $X(w)$ of a discrete time signal $x(n)$ is given as

$$X(w) = \sum_{n=-\infty}^{\infty} x(n) e^{-jwn} \quad \text{- - - - (1).}$$

where w is continuous from 0 to 2π.

* Even though $x(n)$ is a discrete time signal, its spectrum $X(w)$ is continuous in nature hence it cannot be evaluated using digital processor.

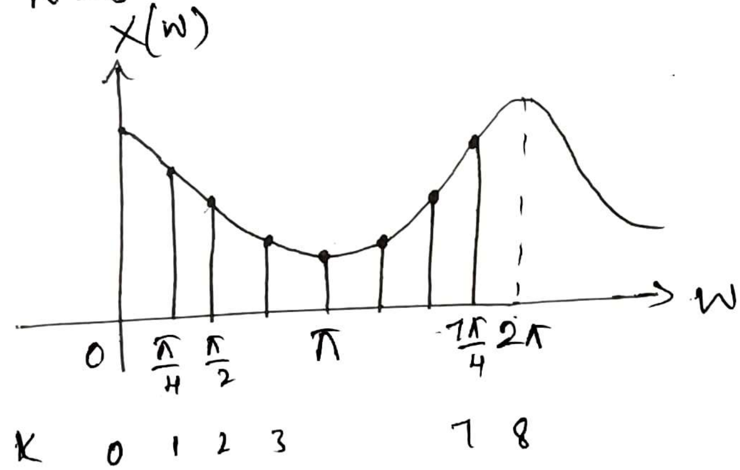
Frequency domain Sampling

* To overcome the above problem, the spectrum $X(w)$ is sampled uniformly. Let N samples are taken from 0 to 2π and hence spacing between the samples $\frac{2\pi}{N}$ and the index of samples b/w 0 to 2π is $K = 0, 1, 2 - - - N-1$

Using $w = \left(\frac{2\pi}{N} K\right)$ in eq (1). results

$$X\left(\frac{2\pi}{N} K\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\left(\frac{2\pi}{N} K\right) n} \quad - - - (2)$$

* Sampling of continuous spectrum uniformly is called frequency domain sampling.
* Figure below shows the samples of $X(w)$ for $N = 8$



$K=0$; $w = \left(\frac{2\pi}{8} \times 0\right) = 0$

$K=1$; $w = \left(\frac{2\pi}{8} \times 1\right) = \frac{\pi}{4}$

$K=2$; $w = \left(\frac{2\pi}{8} \times 2\right) = \frac{\pi}{2}$

$K=3$; $w = \left(\frac{2\pi}{8} \times 3\right) = \frac{3\pi}{4}$

$K=4$; $w = \left(\frac{2\pi}{8} \times 4\right) = \pi$

$K=5$; $w = \left(\frac{2\pi}{8} \times 5\right) = \frac{5\pi}{4}$

$K=6$; $w = \left(\frac{2\pi}{8} \times 6\right) = \frac{3\pi}{2}$

$K=7$; $w = \left(\frac{2\pi}{8} \times 7\right) = \frac{7\pi}{4}$

To determine Minimum value of N

* In eq (2), n varies from $-\infty$ to ∞. Divide the summation into individual summation containing N samples of $x(n)$

$$X\left(\frac{2\pi}{N}k\right) = \cdots + \sum_{n=-N}^{-1} x(n)\, e^{-j\left(\frac{2\pi}{N}k\right)n} + \sum_{n=0}^{N-1} x(n)\, e^{-j\left(\frac{2\pi}{N}k\right)n} + \cdots$$

$$\sum_{n=N}^{2N-1} x(n)\, e^{-j\left(\frac{2\pi}{N}k\right)n} + \cdots \quad (3)$$

* The above summation can be expressed as

$$X\left(\frac{2\pi}{N}k\right) = \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)\, e^{-j\left(\frac{2\pi}{N}k\right)n} \quad (4)$$

* If the index of inner summation is changed from n to $n+lN$ then eq (4) becomes

$$X\left(\frac{2\pi}{N}k\right) = \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n+lN)\, e^{-j\left(\frac{2\pi}{N}k\right)(n+lN)} \quad (5)$$

But $e^{-j\frac{2\pi}{\cancel{N}}k \cdot l\cancel{N}} = \left(e^{-j2\pi}\right)^{kl} = 1$

* Interchange the order of summation in eq(5)

$$\therefore X\left(\frac{2\pi}{N}k\right) = \sum_{n=0} \sum_{l=-\infty} x(n+lN)\, e^{-j\frac{2\pi}{N}\cdots} \quad \ldots (6)$$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi}{N}kn} \quad \ldots (7)$$

where $x_p(n) = \sum_{l=-\infty}^{\infty} x(n+lN)$

$$\sum_{l=-\infty}^{\infty} x(n+lN) = \ldots + x(n-2N) + x(n-N) + x(n) + x(n+N) + x(n+2N) + \ldots$$

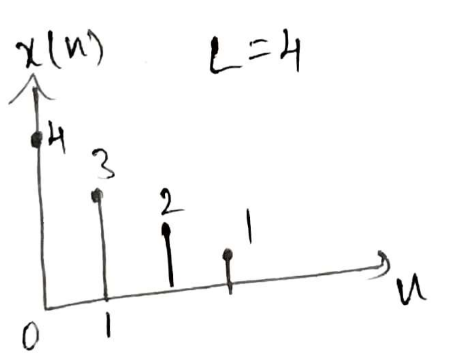
* $x_p(n)$ is a periodic repeatation of $x(n)$ with a period of N samples

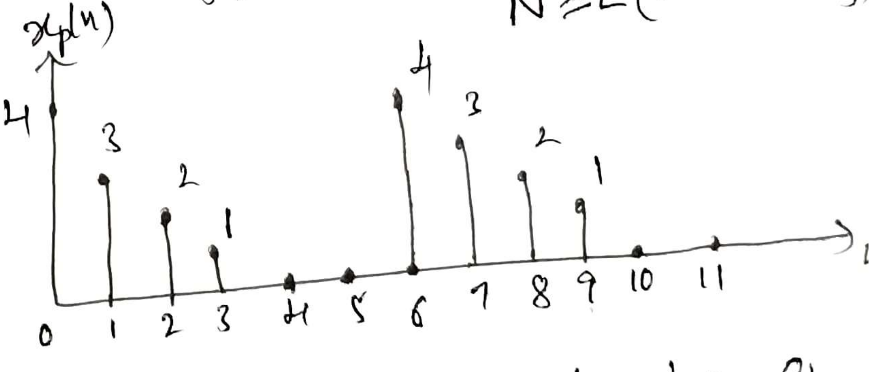
<u>No aliasing</u> and <u>aliasing condition</u>

Let $x(n)$ be a non-periodic signal with L samples. Signal $x_p(n)$ is periodic with N sam

case 1 : $N \geq L$ (No aliasing) $N=6$ $L=4$

$N \geq L$ (No aliasing)

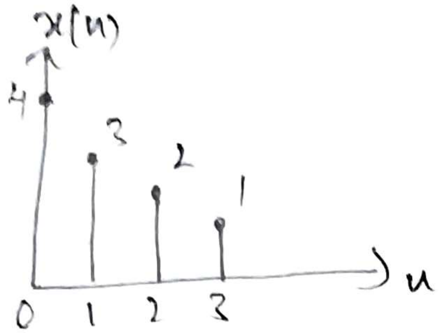




* Since $N \geq L$ in this case, no overlapping of signals Since $N=6$ signal repeats at $n = 6, 12, 18 \ldots$

Case 2 : $N < L$ (Aliasing)

Suppose $N = 3$ and $L = 4$



* Since $N < L$, there is a overlapping of signals at $n = 3, 6, 9$ ...... called as aliasing.

Reconstruction:

The periodic signal $x_p(n)$ can be expressed by discrete Fourier series as

$$x_p(n) = \sum_{K=0}^{N-1} C_K \, e^{j\left(\frac{2\pi}{N}K\right)n} \qquad n = 0, 1, 2 \ldots N-1 \qquad \text{--- (8)}$$

and the fourier coefficient $C_K = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) \, e^{-j\left(\frac{2\pi}{N}K\right)n}$ --- (9)

From eq (7) $\sum_{n=0}^{N-1} x_p(n) \, e^{-j\left(\frac{2\pi}{N}K\right)n} = X\left(\frac{2\pi}{N}K\right)$

$\therefore$ eq (9) becomes

$$C_K = \frac{1}{N} X\left(\frac{2\pi}{N}K\right) \equiv \frac{1}{N} X(K) \qquad \text{--- (10)}$$

Substitute eq (10) in eq (8)

$$x_p(n) = \frac{1}{N}\sum_{K=0}^{N-1} X\left(\frac{2\pi K}{N}\right) e^{j\left(\frac{2\pi}{N}K\right)n} \quad n = 0,1,2 \ldots N-1 \quad \text{--- (11)}$$

Eq (11) gives the expression for time domain sequence $x_p(n)$ from frequency domain samples $X\left(\frac{2\pi}{N}K\right)$

*

1.2 Definition of DFT and IDFT

DFT

Let $x(n)$ be a discrete signal of length L, $X(K)$ be the ~~N-point~~ DFT of $x(n)$

The N-point DFT of $x(n)$ where $N \geq L$ is defined as

$$X\left(\frac{2\pi}{N}K\right) = X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\left(\frac{2\pi}{N}K\right)n} \quad K = 0,1 \ldots N-1$$

The above equation transforms the time domain sequence $x(n)$ to a frequency domain samples $X(K)$ and is called Discrete Fourier Transform (DFT)

IDFT

Let $x(n)$ be a discrete time signal of length L, $X(K)$ be the N-point DFT of $x(n)$

$$\text{IDFT}\{X(K)\} = x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, e^{j\left(\frac{2\pi}{N}K\right)n} \qquad n=0,1,\ldots N-1$$

The above equation allows us to recover the sequence $x(n)$ from frequency domain samples $X(K)$ and is called Inverse Discrete Fourier Transform.

1.3 Twiddle factor W_N

Let $W_N = e^{-j\frac{2\pi}{N}} = \cos\left(\frac{2\pi}{N}\right) - j\sin\left(\frac{2\pi}{N}\right)$

W_N is a complex quantity and is called twiddle factor

Therefore DFT and IDFT becomes

$$\text{DFT};\quad X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn} \qquad K=0,1,2\ldots N-1$$

$$\text{IDFT};\quad x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, W_N^{-Kn} \qquad n=0,1,2\ldots N-1$$

1.3.1 Properties of Twiddle factor

(1) Periodicity: $W_N^{K+N} = W_N^{K}$

Proof: $W_N^{K+N} = e^{-j\left(\frac{2\pi}{N}\right)K+N}$

$$= e^{-j\frac{2\pi}{N}K} \cdot e^{-j\frac{2\pi}{\cancel{N}}\times \cancel{N}}$$

$$W_N^{K+N} = e^{-j\frac{2\pi}{N}K} \cdot 1$$

$$\boxed{W_N^{K+N} = W_N^{K}} \qquad \because W_N = e^{-j\frac{2\pi}{N}}$$

(2) Symmetry: $W_N^{K+\frac{N}{2}} = -W_N^{K}$

Proof: $W_N^{K+\frac{N}{2}} = e^{-j\frac{2\pi}{N}\left(K+\frac{N}{2}\right)}$

$$= e^{-j\frac{2\pi}{N}\cdot K} \cdot e^{-j\frac{2\pi}{\cancel{N}}\times\frac{\cancel{N}}{\cancel{2}}}$$

$$= e^{-j\frac{2\pi}{N}K}(-1)$$

$$\boxed{W_N^{K+\frac{N}{2}} = -W_N^{K}}$$

(3) Conjugation: $(W_N)^* = W_N^{-1}$

Proof: $(W_N)^* = \left(e^{-j\frac{2\pi}{N}}\right)^*$

$$= e^{j\frac{2\pi}{N}} = e^{-\left(-\frac{2\pi}{N}\right)}$$

$$\boxed{(W_N)^* = W_N^{-1}}$$

(4) Modifying the period: $W_N^{2K} = W_{N/2}^{K}$

Proof: $W_N^{2K} = e^{-j\frac{2\pi}{N}\times 2K}$

$= e^{-j\frac{2\pi}{N/2}\cdot K}$

$$\boxed{W_N^{2K} = W_{N/2}^{K}}$$

Numerical Problems on DFT & IDFT

① Find the DFT of the sequence $x(n) = \{0, 1, 2, 3\}$
Plot the magnitude and phase spectrum

Soln: $N = 4$

$$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn} \qquad K = 0, 1 \ldots N-1$$

$$X(K) = \sum_{n=0}^{3} x(n)\, W_4^{Kn} \qquad K = 0, 1, 2, 3$$

$$= x(0) W_4^0 + x(1) W_4^K + x(2) W_4^{2K} + x(3) W_4^{3K}$$

$$X(K) = 0 + 1 W_4^K + 2 W_4^{2K} + 3 W_4^{3K}$$

$$X(K) = W_4^K + 2 W_4^{2K} + 3 W_4^{3K}$$

$K=0;\quad X(0) = W_4^0 + 2 W_4^{2\times 0} + 3 W_4^{3\times 0} = 1 + 2 + 3 = 6$

$K=1;\quad X(1) = W_4^1 + 2 W_4^2 + 3 W_4^3 = -j + 2(-1) + 3(j) = -2 + 2j$

$K=2;\quad X(2) = W_4^2 + 2 W_4^4 + 3 W_4^6 = -1 + 2(1) + 3(-1) = -2$

$K=3;\quad X(3) = W_4^3 + 2 W_4^6 + 3 W_4^9 = j + 2(-1) + 3(-j) = -2 - 2j$

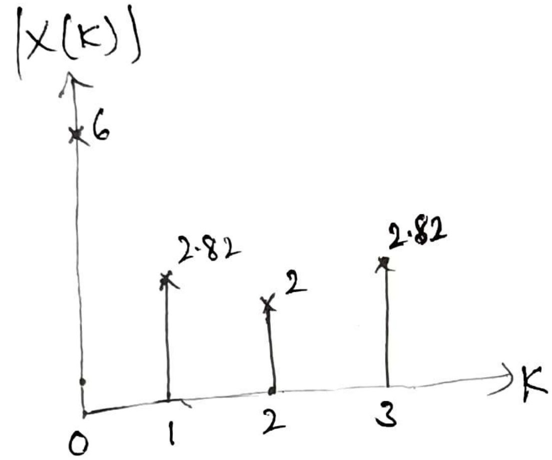
$X(K) = \{6, -2+2j, -2, -2-2j\}$

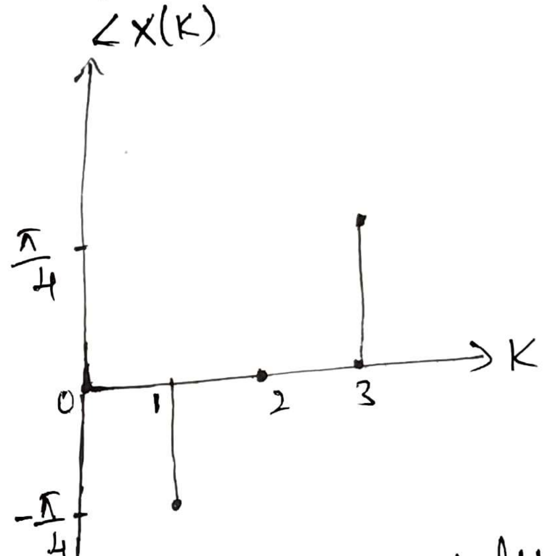
Magnitude $|X(K)| = \{6, \sqrt{(2)^2+(2)^2}, +2, \sqrt{(-2)^2+(-2)^2}\}$

$|X(K)| = \{6, 2.82, +2, 2.82\}$

$\angle X(0) = \tan^{-1}\frac{0}{6} = 0$

Phase $\angle X(K) = \{0, -\frac{\pi}{4}, 0, \frac{\pi}{4}\}$





$\angle X(1) = \tan^{-1}\left(\frac{2}{-2}\right) = -45^\circ$

$\angle X(2) = \tan^{-1}\left(\frac{0}{-2}\right) = 0$

$\angle X(3) = \tan^{-1}\left(\frac{-2}{-2}\right) = 45^\circ$

Note: For $N=4$, twiddle factor values are

$W_4^8 = W_4^4 = W_4^0 = 1$

$W_4^9 = W_4^5 = W_4^1 = -j$

$W_4^{10} = W_4^6 = W_4^2 = -1$

$W_4^{11} = W_4^7 = W_4^3 = +j$

(2) Find the 4-point DFT of the sequence $x(n) = \{1, 2, 3, 4\}$

Soln: $N=4$ $X(K) = \sum_{n=0}^{N-1} x(n) W_N^{Kn}$ $K = 0, 1, \ldots N-1$

$$X(K) = \sum_{n=0} x(n)\, W_4 \qquad K=0,1,2,3$$

$$= x(0)W_4^0 + x(1)W_4^K + x(2)W_4^{2K} + x(3)W_4^{3K}$$

$$X(K) = 1 + 2W_4^K + 3W_4^{2K} + 4W_4^{3K}$$

$K=0;\quad X(0) = 1 + 2W_4^0 + 3W_4^{2\times 0} + 4W_4^{3\times 0\times 1} = 1+2+3+4 = 10$

$K=1;\quad X(1) = 1 + 2W_4^1 + 3W_4^2 + 4W_4^3 = 1+2(-j)+3(-1)+4(+j) = -2+2j$

$K=2;\quad X(2) = 1 + 2W_4^2 + 3W_4^4 + 4W_4^6 = 1+2(-1)+3(1)+4(-1) = -2$

$K=3;\quad X(3) = 1 + 2W_4^3 + 3W_4^6 + 4W_4^9 = 1+2(j)+3(-1)+4(-j) = -2-2j$

$$X(K) = \{10, -2+2j, -2, -2-2j\}$$

(3) Find the IDFT of the sequence

$$X(K) = \{2, 1+j, 0, 1-j\}$$

Solu:

$N=4$

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, W_N^{-Kn} \qquad n=0,1,\dots N-1$$

$$x(n) = \frac{1}{4}\sum_{K=0}^{3} X(K)\, W_4^{-Kn} \qquad n=0,1,2,3$$

$$x(n) = \frac{1}{4}\left[X(0)W_4^0 + X(1)W_4^{-n} + X(2)W_4^{-2n} + X(3)W_4^{-3n}\right]$$

$$x(n) = \frac{1}{4}\left[2 + (1+j)W_4^{-n} + 0 + (1-j)W_4^{-3n}\right]$$

$n=0;\quad x(0) = \frac{1}{4}\left[2 + (1+j)W_4^0 + (1-j)W_4^{-3}\right] = \frac{1}{4}\left[2+1+\cancel{j}+1-\cancel{j}\right]$

$= 1$

$n=1 \qquad x(1) = \frac{1}{4}\left[2 + (1+j)\,W_4^{-1} + (1-j)\,W_4^{-3}\right]$

$$= \frac{1}{4}\left[2 + (1+j)(j) + (1-j)(-j)\right] =$$

$$= \frac{1}{4}\left\{2 + \cancel{j} - 1 - \cancel{j} - 1\right\} = 0$$

$n=2 \qquad x(2) = \frac{1}{4}\left[2 + (1+j)\,W_4^{-2} + (1-j)\,W_4^{-6}\right]$

$$= \frac{1}{4}\left[2 + (1+j)\,{-1} + (1-j)(-1)\right]$$

$$= \frac{1}{4}\left\{2 - 1 - \cancel{j} - 1 + \cancel{j}\right\} = 0$$

$n=3 \qquad x(3) = \frac{1}{4}\left\{2 + (1+j)\,W_4^{-3} + (1-j)\,W_4^{-9}\right\}$

$$= \frac{1}{4}\left[2 + (1+j)(-j) + (1-j)(j)\right].$$

$$= \frac{1}{4}\left\{2 - \cancel{j} + 1 + \cancel{j} + 1\right\} = 1$$

$$x(n) = \{1, 0, 0, 1\}$$

Find the IDFT of the following sequence.

$$X(K) = \{4, -j2, 0, j2\}$$

Solu:

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, W_N^{-Kn} \quad n = 0, 1, 2 .. N-1$$

$$x(n) = \frac{1}{4} \sum_{K=0}^{3} X(K)\, W_4^{-Kn}$$

$$x(n) = \frac{1}{4} \left[X(0) + X(1) W_4^{-n} + X(2) W_4^{-2n} + X(3) W_4^{-3n} \right]$$

$$= \frac{1}{4} \left\{ 4 - j2 W_4^{-n} + 0 + j2 W_4^{-3n} \right]$$

For $n=0$; $x(0) = \frac{1}{4} \left[4 - j2 W_4^{0} + j2 W_4^{0} \right]$

$$= \frac{1}{4} \left[4 - \cancel{j2} + \cancel{j2} \right] = 1$$

For $n=1$; $x(1) = \frac{1}{4} \left[4 - j2 W_4^{-1} + j2 W_4^{-3} \right]$

$$= \frac{1}{4} \left[4 - j2(j) + j2(-j) \right]$$

$$= \frac{1}{4} \left[4 + 2 + 2 \right] = 2$$

For $n=2$; $x(2) = \frac{1}{4} \left[4 - j2 W_4^{-2} + j2 W_4^{-6} \right]$

$$= \frac{1}{4} \left[4 - j2(-1) + j2(-1) \right]$$

$$= \frac{1}{4} \left[4 + j2 - j2 \right] = 1$$

For $n=3$; $x(3) = \frac{1}{4}\left[4 - j2\,W_4^{-3} + j2\,W_4^{-9}\right]$

$= \frac{1}{4}\left[4 - j2(-j) + j2(j)\right]$

$x(3) = \frac{1}{4}\left[4 - 2 - 2\right] = 0$

$\therefore \boxed{x(n) = \{1, 2, 1, 0\}}$

<u>Note</u>:

$W_4^{0} = W_4^{-4} = W_4^{-8} = 1$

$W_4^{0} = 1$ $\quad W_4^{-1} = W_4^{-5} = W_4^{-9} = +j$

$W_4^{-1} = +j$ $\quad W_4^{-2} = W_4^{-6} = W_4^{-10} = -1$

$W_4^{-2} = -1$ $\quad W_4^{-3} = W_4^{-7} = W_4^{-11} = -j$

$W_4^{-3} = -j$

DFT as a linear transformation

* The DFT $X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn} \quad K = 0, 1 \ldots N-1$

can be expressed as

$$X_N = W_N \cdot x_n$$

where

$$X_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}_{N \times 1} \qquad x_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}_{N \times 1}$$

$$W_N = \begin{bmatrix} W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^3 & \cdots & W_N^{N-1} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ W_N^0 & W_N^3 & W_N^6 & \cdots & W_N^{3(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}_{N \times N}$$

For $N = 4$

$$W_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix}$$

* The IDFT of $X(K)$

i.e $x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, W_N^{-Kn}$ $n = 0, 1 \ldots N-1$

can be expressed as $x_N = \frac{1}{N} W_N^{*} X_N$

where W_N^{*} is the conjugate of W_N

For $N=4$

$$W_4^{*} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^{-1} & W_4^{-2} & W_4^{-3} \\ W_4^0 & W_4^{-2} & W_4^{-4} & W_4^{-6} \\ W_4^0 & W_4^{-3} & W_4^{-6} & W_4^{-8} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix}$$

Numerical problems:

(1) Find the DFT of the following sequence using matrix Method of $N=4$

as $x(n) = \cos\left(\frac{n\pi}{4}\right)$ $0 \leq n \leq 3$

$x(0) = \cos(0) = 1$ $x(1) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = 0.707$

$x(2) = \cos\left(\frac{2\pi}{4}\right) = 0$ $x(3) = \cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} = -0.707$

$x(n) = \{1, 0.707, 0, -0.707\}$

$X_4 = W_4\, x_4$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0.707 \\ 0 \\ -0.707 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \cancel{0.707} + 0 - \cancel{0.707} \\ 1 - j0.707 + 0 - j0.707 \\ 1 - \cancel{0.707} + 0 + \cancel{0.707} \\ 1 + j0.707 + 0 + j0.707 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 - j1.414 \\ 1 \\ 1 + j1.414 \end{bmatrix}$$

$$\therefore X(K) = \{1, 1-j1.414, 1, 1+j1.414\}$$

(b) $x(n) = \begin{cases} 1 ; & 0 \leq n \leq 2 \\ 0 ; & \text{otherwise.} \end{cases}$ for $N = 4$

Sol^n:

$N = 4$

$$x(n) = \{1, 1, 1, 0\}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 1 + 1 + 0 \\ \cancel{1} - j - \cancel{1} + 0 \\ 1 - \cancel{1} + \cancel{1} + 0 \\ \cancel{1} + j - \cancel{1} + 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -j \\ 1 \\ j \end{bmatrix}$$

$x(K) = \{3, -j, 1, j\}$

② Find the IDFT of the following sequences using matrix method (linear transformation)

a) $X(K) = \{4, -2j, 0, 2j\}$

Soln: $N = 4$

$$x_N = \frac{1}{4}(W_4)^* X_N$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix}\begin{bmatrix} 4 \\ -2j \\ 0 \\ 2j \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 - \cancel{2j} + \cancel{2j} \\ 4 + 2 + 2 \\ 4 + \cancel{2j} - \cancel{2j} \\ 4 - 2 - 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 8 \\ 4 \\ 0 \end{bmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

$x(n) = \{1, 2, 1, 0\}$

Problems on 8-point DFTs

① Find the 8-point DFT of the sequence

$x(n) = (1, 1, 1, 1, 0, 0, 0, 0)$

Soln: $N = 8$

$W_8^0 = e^{-j\frac{2\pi}{8}\times 0} = 1$

$W_8^1 = e^{-j\frac{2\pi}{8}\times 1} = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

$W_8^2 = e^{-j\frac{2\pi}{8}\times 2} = -j$

$W_8^3 = e^{-j\frac{2\pi}{8}\times 3} = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

$W_8^4 = e^{-j\frac{2\pi}{8}\times 4} = -1$

$W_8^5 = e^{-j\frac{2\pi}{8}\times 5} = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

$W_8^6 = e^{-j\frac{2\pi}{8}\times 6} = j$

$W_8^7 = e^{-j\frac{2\pi}{8}\times 7} = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

Note:

$W_8^{16} = W_8^8 = W_8^0 = 1$

$W_8^{17} = W_8^9 = W_8^1 = 0.707 - j0.707$

$W_8^{18} = W_8^{10} = W_8^2 = -j$

$W_8^{19} = W_8^{11} = W_8^3 = -0.707 - j0.707$

$W_8^{20} = W_8^{12} = W_8^4 = -1$

$W_8^{21} = W_8^{13} = W_8^5 = -0.707 + j0.707$

$W_8^{22} = W_8^{14} = W_8^6 = j$

$W_8^{23} = W_8^{15} = W_8^7 = 0.707 + j0.707$

$$X(k) = \sum_{n=0}^{7} x(n) W_8^{kn} \qquad k = 0, 1 \ldots 7$$

$$= x(0) W_8^{0\times k} + x(1) W_8^{k} + x(2) W_8^{2k} + x(3) W_8^{3k} + x(4) W_8^{4k} + x(5) W_8^{5k} + x(6) W_8^{6k} + x(7) W_8^{7k}$$

$$X(k) = 1 + 1\times W_8^{k} + 1\times W_8^{2k} + 1\times W_8^{3k} + 0 + 0 + 0 + 0$$

$$X(k) = 1 + W_8^{k} + W_8^{2k} + W_8^{3k}$$

$K=0;\quad X(0) = 1 + W_8^{1\times0\times k} + W_8^{2\times0\times k} + W_8^{3\times0\times k}$

$= 1 + 1 + 1 + 1 = 4$

$K=1;\quad X(1) = 1 + W_8^{1} + W_8^{2} + W_8^{3}$

$= 1 + (0.707 - j0.707) + (-j) + (-0.707 - j0.707)$

$X(1) = 1 - j2.414$

$K=2;\quad X(2) = 1 + W_8^{2} + W_8^{4} + W_8^{6}$

$= \cancel{1} + (-\cancel{j}) + (\cancel{-1}) + \cancel{j}$

$X(2) = 0$

$K=3;\quad X(3) = 1 + W_8^{3} + W_8^{6} + W_8^{9}$

$= 1 + (-0.707 - j0.707) + j + (0.707 - j0.707)$

$= 1 - j0.414$

$K=4;\quad X(4) = 1 + W_8^{4} + W_8^{8} + W_8^{12}$

$= 1 + (-1) + 1 + (-1)$

$X(4) = 0$

$K=5\quad X(5) = 1 + W_8^{5} + W_8^{10} + W_8^{15}$

$= 1 + (-0.707 + j0.707) + (-j) + (0.707 + j0.707)$

$X(5) = 1 + j0.414$

$K=6\quad X(6) = 1 + W_8^{6} + W_8^{12} + W_8^{18}$

$= 1 + j + (-1) + (-j)$

$X(6) = 0$

$K=7$; $X(7) = 1 + W_8 + W_8 + W_8$

$$X(7) = 1 + 0.707 + j0.707 + j + (-0.707 + j0.707)$$

$$X(7) = 1 + j2.414$$

$$X(K) = \{4, 1-j2.414, 0, 1-j0.414, 0, 1+j0.414, 0, 1+j2.414\}$$

Using matrix method

$$X_8 = W_8 \, x_8$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{\sqrt2}-j\frac{1}{\sqrt2} & -j & -\frac{1}{\sqrt2}-j\frac{1}{\sqrt2} & -1 & -\frac{1}{\sqrt2}+j\frac{1}{\sqrt2} & j & \frac{1}{\sqrt2}+j\frac{1}{\sqrt2} \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -\frac{1}{\sqrt2}-j\frac{1}{\sqrt2} & j & \frac{1}{\sqrt2}-j\frac{1}{\sqrt2} & -1 & \frac{1}{\sqrt2}+j\frac{1}{\sqrt2} & -j & -\frac{1}{\sqrt2}+j\frac{1}{\sqrt2} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\frac{1}{\sqrt2}+j\frac{1}{\sqrt2} & -j & \frac{1}{\sqrt2}+j\frac{1}{\sqrt2} & -1 & \frac{1}{\sqrt2}-j\frac{1}{\sqrt2} & j & -\frac{1}{\sqrt2}-j\frac{1}{\sqrt2} \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & \frac{1}{\sqrt2}+j\frac{1}{\sqrt2} & j & -\frac{1}{\sqrt2}+j\frac{1}{\sqrt2} & -1 & -\frac{1}{\sqrt2}-j\frac{1}{\sqrt2} & -j & \frac{1}{\sqrt2}-j\frac{1}{\sqrt2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 1-j2.414 \\ 0 \\ 1-j0.414 \\ 0 \\ 1+j0.414 \\ 0 \\ 1+j0.414 \end{bmatrix}$$

b) Find the DFT of $x(n) = (-1)^n$ for $n \le 0 \le 7$

Soln: Here $N = 8$ $\quad x(n) = \{1, -1, 1, -1, 1, -1, 1, -1\}$

$$\therefore X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & j & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & j & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & j & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \\ 1 & j & -1 & -j & +1 & j & -1 & -j \\ 1 & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & j & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 8 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore X(k) = \{0, 0, 0, 0, 8, 0, 0, 0\}$$

c) Find the IDFT of $X(K)$ given.

$$X(k) = \{255,\ 48.63 + j166.05,\ -51 + j102,\ -78.63 + j46.05,\ -85,\ -78.63 - j46.05,\ -51 - j102,\ 48.63 - j166.05\}$$

$$x_N = \frac{1}{8} W_8^{*} X_N$$

$$x_8 = \frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{\sqrt2}+j\frac{1}{\sqrt2} & j & -\frac{1}{\sqrt2}+j\frac{1}{\sqrt2} & -1 & -\frac{1}{\sqrt2}-j\frac{1}{\sqrt2} & -j & \frac{1}{\sqrt2}-j\frac{1}{\sqrt2} \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & -\frac{1}{\sqrt2}+j\frac{1}{\sqrt2} & -j & \frac{1}{\sqrt2}+j\frac{1}{\sqrt2} & -1 & \frac{1}{\sqrt2}-j\frac{1}{\sqrt2} & j & -\frac{1}{\sqrt2}-j\frac{1}{\sqrt2} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\frac{1}{\sqrt2}-j\frac{1}{\sqrt2} & j & \frac{1}{\sqrt2}-j\frac{1}{\sqrt2} & -1 & \frac{1}{\sqrt2}+j\frac{1}{\sqrt2} & -j & -\frac{1}{\sqrt2}+j\frac{1}{\sqrt2} \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & \frac{1}{\sqrt2}-j\frac{1}{\sqrt2} & -j & -\frac{1}{\sqrt2}-j\frac{1}{\sqrt2} & -1 & -\frac{1}{\sqrt2}+j\frac{1}{\sqrt2} & j & \frac{1}{\sqrt2}+j\frac{1}{\sqrt2} \end{bmatrix} \begin{bmatrix} 255 \\ 48.63+j166.05 \\ -51+j102 \\ -78.63+j46.05 \\ -85 \\ -78.63-j46.05 \\ -51-j102 \\ 48.63-j166.05 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 8 \\ 16 \\ 24 \\ 32 \\ 40 \\ 48 \\ 56 \\ 72 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$$

3. Find the 5-point DFT of the discrete time signal $x(n) = \{1, 2, 3, 1\}$

Soln: $N = 5 \quad L = 4$

we should append one zero to sequence $x(n)$ to make $N = 4$

$\therefore x(n) = \{1, 2, 3, 1, 0\}$

Soln: $X(K) = \sum_{n=0}^{4} x(n) W_5^{Kn}$

$X ~~(K) = x(0)W_5^0 + x(1)W_5^1 + x(2)W_5^2 +~~$

$W_8^{10} = W_5^5 = W_5^0 = 1$

$W_8^{11} = W_5^6 = W_5^1 = 0.309 - j0.951$

$W_5^{12} = W_5^7 = W_5^2 = -0.809 - j0.887$

$W_5^{13} = W_5^8 = W_5^3 = -0.809 + j0.587$

$W_5^{14} = W_8^9 = W_5^4 = 0.309 + j0.951$

$$X(K) = x(0)W_5^0 + x(1)W_5^K + x(2)W_5^{2K} + x(3)W_5^{3K} + x(4)W_5^{4K}$$

$$= 1 + 2W_5^K + 3W_5^{2K} + 1 \times W_5^{3K} + 0$$

$$X(K) = 1 + 2W_5^K + 3W_5^{2K} + W_5^{3K}$$

$K = 0$; $\quad X(0) = 1 + 2 + 3 + 1 = 7$

$K = 1$; $\quad X(1) = 1 + 2W_5^1 + 3W_5^2 + W_5^3 = 1 + 2(0.309 - j0.951) + 3(-0.809 - j0.587) + (-0.809 + j0.587)$

$X(1) = -1.618 - j3.076$

Problems on ... point DFT

Find the N point DFT of the following sequence

1. $x(n) = \delta(n)$

We know that $X(K) = \sum_{n=0}^{N-1} x(n) W_N^{Kn} \quad K = 0, \dots, N-1$

$$= \sum_{n=0}^{N-1} \delta(n) W_N^{Kn}$$

$$= \delta(0) W_N^{0} + \delta(1) W_N^{1} + \delta(2) W_N^{2} \dots$$

We know that $\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$

$$X(K) = 1 + 0 + 0$$

$$\boxed{X(K) = 1}$$

$DFT\{\delta(n)\} = 1 \qquad IDFT(1) = \delta(n)$

2. $x(n) = \delta(n - n_0)$

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{Kn}$$

$$= \sum_{n=0}^{N-1} \delta(n - n_0) W_N^{Kn} \quad K = 0, 1, \dots, N-1$$

By definition $\delta(n - n_0) = \begin{cases} 1 & n = n_0 \\ 0 & \text{otherwise} \end{cases}$

$$X(K) = 1 \cdot W_N^{Kn_0}$$

$DFT\{\delta(n - n_0)\} = W_N^{Kn_0} \qquad IDFT\{W_N^{Kn_0}\} = \delta(n - n_0)$

3. $x(n) = \delta(n+n_0)$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{N-1} \delta(n+n_0)\, W_N^{kn}$$

By definition. $\delta(n+n_0) = \begin{cases} 1 & n=-n_0 \\ 0 & \text{elsewhere} \end{cases}$.

$$\therefore X(k) = 1 \cdot W_N^{-kn_0}$$

$$DFT[\delta(n+n_0)] = W_N^{-kn_0} \qquad IDFT\left[W_N^{-kn_0}\right] = \delta(n+n_0)$$

4. $x(n) = \begin{cases} 1 & 0 \leq n \leq l-1 \\ 0 & \text{elsewhere.} \end{cases}$ where $l \leq N$ or $x(n) = 1,\ 0 \leq n \leq N-1$

or

$$x(n) = \begin{cases} (1)^n & 0 \leq n \leq N-1 \\ 0 & \text{elsewhere} \end{cases}$$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{N-1} 1 \cdot W_N^{kn}$$

$$= \sum_{n=0}^{N-1} (W_N^{k})^n$$

when $k=0$ $\quad \sum_{n=0}^{N-1} (W_N^k)^n = 1+1+1 - - - 1 = N$

when $k \neq 0$ $\quad \sum_{n=0}^{N-1} (W_N^k)^n = \dfrac{1-(W_N^k)^N}{1-W_N^k} = \dfrac{1-(W_N^N)^k}{1-W_N^k}$

$$= \frac{1-1}{1-W_N^k}$$

$X(k) = 0$

$$\therefore X(k) = \begin{cases} N & \text{when } k=0 \\ 0 & \text{when } k \neq 0 \end{cases}$$

For example when $x(n) = \{1,1,1,1,1,1,1,1\}$ $N=8$

DFT of $x(n)$ is $X(K) = \{8,0,0,0,0,0,0,0\}$

5. $x(n) = \cos \frac{2\pi K_0 n}{N}$; $0 \leq n \leq N-1$, Find N-point DFT

We know that $\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$

$$\cos \frac{2\pi K_0 n}{N} = \frac{e^{\frac{j2\pi K_0 n}{N}} + e^{-\frac{j2\pi K_0 n}{N}}}{2}$$

$$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn}$$

$$= \sum_{n=0}^{N-1} \frac{e^{\frac{j2\pi K_0 n}{N}} + e^{-\frac{j2\pi K_0 n}{N}}}{2}\, W_N^{Kn}$$

$$= \frac{1}{2} \sum_{n=0}^{N-1} \left(W_N^{-K_0 n} + W_N^{K_0 n}\right) W_N^{Kn}$$

$$= \frac{1}{2} \left[\sum_{n=0}^{N-1} W_N^{(K-K_0)n} + \sum_{n=0}^{N-1} W_N^{(K+K_0)n}\right]$$

W.K.T $\sum_{n=0}^{N-1} W_N^{Kn} = N\,\delta(K)$

$$= \frac{1}{2}\left[N\,\delta(K-K_0) + N\,\delta(K+K_0)\right]$$

$$= \frac{N}{2}\left[\delta(K-K_0) + \delta(K+K_0)\right]$$

$x(n) = \sin \frac{\quad}{N} \qquad 0 \le n \le N-1$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\sin \frac{2\pi k_0 n}{N} = \frac{e^{\frac{j2\pi k_0 n}{N}} - e^{-\frac{j2\pi k_0 n}{N}}}{2j}$$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{N-1} \left[\frac{e^{\frac{j2\pi k_0 n}{N}} - e^{-\frac{j2\pi k_0 n}{N}}}{2j} \right] W_N^{kn}$$

$$= \frac{1}{2j} \sum_{n=0}^{N-1} \left[W_N^{-k_0 n} - W_N^{k_0 n} \right] W_N^{kn}$$

$$X(k) = \frac{1}{2j} \sum_{n=0}^{N-1} \left\{ W_N^{(k-k_0)n} - W_N^{(k+k_0)n} \right\}$$

$$X(k) = \frac{1}{2j} \left\{ \sum_{n=0}^{N-1} W_N^{(k-k_0)n} - \sum_{n=0}^{N-1} W_N^{(k+k_0)n} \right\}$$

$$= \frac{1}{2j} \left[N\,\delta(k-k_0) - N\,\delta(k+k_0) \right]$$

$$= \frac{N}{2j} \left[\delta(k-k_0) - \delta(k+k_0) \right]$$

7. $x(n) = e^{j\omega m n} \qquad 0 \le n \le N-1$

$$\omega = \frac{2\pi}{N}$$

$$x(n) = e^{j\frac{2\pi}{N} mn}$$

$$x(n) = W_N^{-mn}$$

$$X(k) = \sum_{n=0}^{N-1} W_N^{-mn} \cdot W_N^{kn}$$

$$X(K) = \sum_{n=0} \quad W_N$$

W.K.T. $\sum_{n=0}^{N-1} W_N^{kn} = N\delta(k)$

$$\therefore X(K) = N \cdot \delta(K-m)$$

$$X(K) = \begin{cases} N & ; K = m \\ 0 & ; \text{elsewhere.} \end{cases}$$

8) $x(n) = e^{-j\omega mn}$

$$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{N-1} e^{-j\omega mn} \cdot W_N^{kn}$$

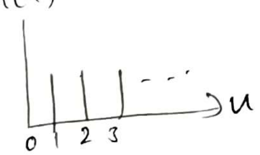
$\omega = \frac{2\pi}{N}$

$$= \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}mn} \cdot W_N^{kn} = \sum_{n=0}^{N-1} W_N^{mn} \cdot W_N^{kn}$$

$$\sum_{n=0}^{N-1} \left(W_N^{m+k}\right)^n = N\,\delta(m+k)$$

ie $X(K) = \begin{cases} N & ; K=-m \\ 0 & \text{elsew } K\neq-m \end{cases}$

9) $x(n) = u(n) - u(n-n_0) \qquad 0 \le n \le N-1$



$$x(n) = \begin{cases} 1 & 0 < n \le n_0 - 1 \\ 0 & \text{otherwise} \end{cases}$$

$$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{N-1} 1 \cdot W_N^{kn}$$

$$= \sum_{n=0}^{N-1} \left(W_N^{k}\right)^n$$

$$X(K) = \frac{1-\left(W_N^{k}\right)^{n_0}}{1-W_N^{k}}.$$

10. $x(n) = ...$

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$= \sum_{n=0}^{N-1} e^{j2\pi mn/N} \cdot W_N^{kn}$$

$$= \sum_{n=0}^{N-1} W_N^{-mn} \cdot W_N^{kn}$$

$$= \sum_{n=0}^{N-1} W_N^{(K-m)n}$$

$$X(K) = N \cdot \delta(K-m) \quad \text{ie} \quad X(K) = \begin{cases} N ; & K = m \\ 0 & K \neq m \end{cases}$$

11. $x(n) = a^n$

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$= \sum_{n=0}^{N-1} a^n W_N^{kn}$$

$$= \sum_{n=0}^{N-1} (a W_N^k)^n$$

$$= \frac{1-(aW_N^k)^N}{1-aW_N^k}$$

$$X(K) = \frac{1-a^N \cdot W_N^{K}}{1-aW_N^k} = \frac{1-a^N}{1-aW_N^k}$$

12. Compute the N-point DFT of the sequence

$x(n) = an, \quad 0 \leq n \leq N-1$

$$X(K) = \sum_{n=0}^{N-1} an \cdot W_N^{kn}$$

$$= a \sum_{n=0}^{N-1} n \cdot W_N^{kn}$$

Let $S = \sum_{n=0}^{N-1} b^n = \frac{b^N - 1}{b-1} \quad b \neq 1$

Differentiating both the sides of ... with respect to b, we get.

$$\sum_{n=0}^{N-1} n b^{n-1} = \frac{(b-1)Nb^{N-1} - (b^N-1)1}{(b-1)^2}$$

$$\sum_{n=0}^{N-1} n b^{n} = b\left[\frac{Nb^N - Nb^{N-1} - b^N + 1}{(b-1)^2}\right]$$

$$\sum_{n=0}^{N-1} n b^{n} = b\left[\frac{b^N(N-1) - Nb^{N-1} + 1}{(b-1)^2}\right]$$

Letting $b = W_N^{kn}$ in the above equation.

$$\sum_{n=0}^{N-1} n W_N^{kn} = W_N^{k}\left[\frac{W_N^{kN}(N-1) - N W_N^{k(N-1)} + 1}{(W_N^k-1)^2}\right]$$

$$= W_N^{k}\left[\frac{1(N-\cancel{1}) - N \cdot 1 \times W_N^{-k} \cancel{+1}}{(W_N^k-1)^2}\right]$$

$$= \frac{W_N^{k}\left[N - N W_N^{-k}\right]}{(W_N^k-1)^2}$$

$$= \frac{N W_N^{k}\left[1 - W_N^{-k}\right]}{(W_N^k-1)^2}$$

$$\sum_{n=0}^{N-1} n W_N^{kn} = \frac{N(W_N^k-1)}{(W_N^k-1)^2} = \frac{N}{W_N^k-1}$$

$$x(k) = \frac{N}{W_N^k-1} \qquad k \neq 0$$

when $k=0$ $\quad x(0) = a\sum_{n=0}^{N-1} n W_N^0 = a\frac{N(N-1)}{2}$

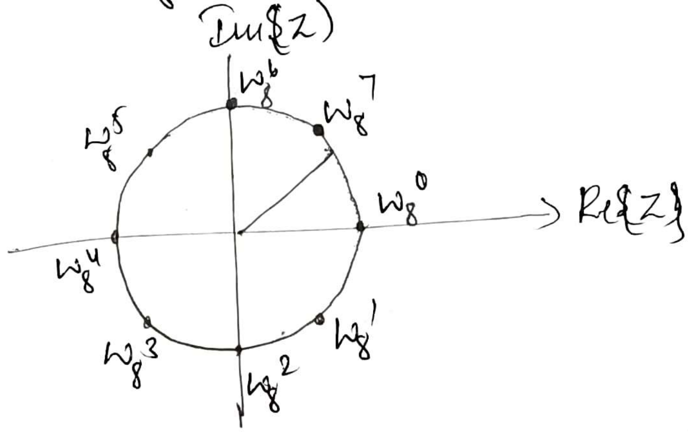
$$\therefore \quad X(K) = \begin{cases} \ldots \\ \dfrac{aN}{W_N^{K}-1} \; ; \; K \neq 0 \end{cases}$$

Problems on DFTs.

13. Compute the 8-point DFT of the sequence $x(n)$ given below.

$$x(n) = (1, 1, 1, 1, 0, 0, 0, 0)$$

The complex basis functions (twiddle factors W_8^n for $0 \le n \le 7$) lie on a circle of unit radius as shown below.



Sequence W_8^n for $0 \le n \le 8$

Since $N = 8$, we get $W_8 = e^{-j\frac{2\pi}{8}K}$.

Thus
$$W_8^0 = 1$$
$$W_8^1 = e^{-j\pi/4} = 0.707 - j0.707$$
$$W_8^2 = e^{-j\pi/2} = -j$$
$$W_8^3 = e^{-j\frac{3\pi}{4}} = -0.707 - j0.707$$
$$W_8^4 = -W_8^0 = -1$$
$$W_8^5 = -W_8^1 = -0.707 + j0.707$$
$$W_8^6 = -W_8^2 = j$$
$$W_8^7 = -W_8^3 = 0.707 + j0.707$$

$$X(k) = \mathrm{DFT}\{x(n)\}$$

$$= \sum_{n=0}^{7} x(n)\, W_8^{kn}$$

$$X(k) = x(0)W_8^{0} + x(1)W_8^{k} + x(2)W_8^{2k} + x(3)W_8^{3k} + x(4)W_8^{4k} + x(5)W_8^{5k} + x(6)W_8^{6k} + x(7)W_8^{7k}$$

$$X(k) = \cancel{W_8}\,1 + W_8^{k} + W_8^{2k} + W_8^{3k}$$

$$X(0) = 1 + W_8^0 + W_8^0 + W_8^0 = 1+1+1+1 = 4$$

$$X(1) = 1 + W_8^1 + W_8^2 + W_8^3$$

$$= 1 + 0.707 - j0.707 - j - 0.707 - j0.707$$

$$= 1 - j2.414$$

$$X(2) = 1 + W_8^2 + W_8^4 + W_8^6 = 0$$

$$X(3) = 1 + W_8^3 + W_8^6 + W_8^1 = 1 - j0.414$$

$$X(4) = 1 + W_8^4 + W_8^0 + W_8^4 = 0$$

$$X(5) = 1 + W_8^5 + W_8^2 + W_8^7 = 1 + j0.414$$

$$X(6) = 1 + W_8^6 + W_8^4 + W_8^2 = 0$$

$$X(7) = 1 + W_8^7 + W_8^6 + W_8^5 = 1 + j2.414$$

Please note the periodic property: $W_N^a = W_N^{a+N}$, where a is any integer

14) Find the IDFT of the 4-point sequence

$X(k) = (4, -j2, 0, j2)$ ~~using DFT~~

$$W_4^{-0} = (W_4^0)^* = 1$$

$$W_4^{-1} = (W_4^1)^* = j$$

$$W_4^{-2} = (W_4^2)^* = -1$$

$$W_4^{-3} = (W_4^3)^* = -j$$

$$x_n = \frac{1}{N}(W_N^*) X_N$$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 4 \\ -j2 \\ 0 \\ j2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4-\cancel{j2}+0+\cancel{j2} \\ 4+2+0+2 \\ 4+\cancel{j2}+0-\cancel{j2} \\ 4-2+0-2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 \\ 8 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$x(n) = (1, 2, 1, 0)$$

15. Consider a signal of length equal to 4 defined by

$$x(n) = (1, 2, 3, 1)$$

a. Compute the 4-point DFT by solving explicitly the 4-by-4 system of linear equations defined by the inverse DFT formula

b. verify the result of part (a) by fining $X(k)$ using the defining equations for DFT

<u>Ans:</u>

we have $\text{IDFT}\{X(k)\} = x(n)$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

$$\Rightarrow \frac{1}{N}\sum_{K=0} X(K)\, e^{\,\cdots} = x(n)$$

$$\sum_{K=0}^{N-1} X(K)\, e^{\frac{j2\pi nK}{N}} = N\, x(n)$$

Since $N=4$

$$\sum_{K=0}^{N-1} X(K)\, e^{\frac{j2\pi nK}{4}} = 4\, x(n) \qquad n=0,1,2,3$$

Hence we get the following linear equation:

$\cancel{X(0)\, e^{j2\pi}}$

$$X(0)+X(1)+X(2)+X(3) = 4\,x(0) = 4$$

$$X(0)+X(1)e^{j\pi/2}+X(2)e^{j\pi}+X(3)e^{j3\pi/2} = 4\,x(1) = 8$$

$$X(0)+X(1)e^{j\pi}+X(2)e^{j2\pi}+X(3)e^{j3\pi} = 4\,x(2) = 12$$

$$X(0)+X(1)e^{j\frac{3\pi}{2}}+X(2)e^{j3\pi}+X(3)e^{j9\pi/2} = 4\,x(3) = 4$$

Putting the above set of equations in matrix form we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 12 \\ 4 \end{bmatrix}$$

$\cancel{x(0)}$ $\cancel{x(1)}$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 8 \\ 12 \\ 4 \end{bmatrix}$$

Since $W_N^{-1} = \frac{1}{N} W_N^{*}$

$\frac{1}{N}$

$$\begin{bmatrix} X(0)\\ X(1)\\ X(2)\\ X(3)\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1&1&1&1\\1&-j&-1&j\\1&-1&1&-1\\1&j&-1&-j\end{bmatrix}\begin{bmatrix} \\ 8\\ 12\\ 4\end{bmatrix}$$

$$\begin{bmatrix} X(0)\\ X(1)\\ X(2)\\ X(3)\end{bmatrix} = \begin{bmatrix} 7\\ -2-j\\ 1\\ -2+j\end{bmatrix}$$

Verification

$$X(K) = \sum_{n=0}^{3} x(n)\, W_4^{kn}$$

$$= x(0) + x(1)\, W_4^{k} + x(2)\, W_4^{2k} + x(3)\, W_4^{3k}$$

$$X(K) = 1 + 2W_4^{k} + 3W_4^{2k} + W_4^{3k} \qquad 0 \leq K \leq 3$$

$$W_4^0 = 1 \qquad W_4^1 = -j \qquad W_4^2 = -1 \qquad W_4^3 = j$$

$$X(0) = 1+2+3+1 = \underline{7}$$

$$X(1) = 1 + 2W_4^1 + 3W_4^2 + 3W_4^3 = \underline{-2-j}$$

$$X(2) = 1 + 2W_4^2 + 3W_4^4 + W_4^6$$

$$= 1 + 2W_4^2 + 3W_4^0 + W_4^2 = \underline{1}$$

$$X(3) = 1 + 2W_4^3 + 3W_4^6 + W_4^9$$

$$= 1 + 2W_4^3 + 3W_4^2 + W_4^1 = \underline{-2+j}$$

16 Compute the IDFT of the sequence $x(n) = (2, 2, 1, 1$

<u>Soln</u> $N = 4$

$$W_4^0 = 1 \qquad W_4^1 = -j \qquad W_4^2 = -1 \qquad W_4^3 = j$$

$$X(K) = \sum^{N-1} x(n)\, W_N^{kn}$$

$X(K) = 2 + 2W_4 \ldots$

$K=0 \quad X(0) = 2+2+1+1 = 6$

$K=1 \quad X(1) = 2 + 2W_4^1 + W_4^2 + W_4^3$

$= 2 + 2(-j) + (-1) + j$

$= 1 - j$

$K=2 \quad X(2) = 2 + 2W_4^2 + W_4^4 + W_4^6$

$= 2 + 2(-1) + 1 + (-1)$

$= 0$

$K=3 \quad X(3) = 2 + 2W_4^3 + W_4^6 + W_4^9$

$= 2 + 2j + (-1) + (-j)$

$= 2 + 2j - 1 - j$

$X(3) = 1 + j$

$X(K) = (6, 1-j, 0, 1+j)$

17. Find the DFT of the sequence

$x(n) = 6 + \sin\left(\frac{2\pi n}{4}\right) \; ; \; 0 \le n \le 3$

Soln:

i) $x(0) = 6, \quad x(1) = 7 \quad x(2) = 6, \quad x(3) = 5$

$x(n) = (6, 7, 6, 5) \qquad N=4$

ii) $X(K) = \sum_{n=0}^{N-1} x(n) W_N^{Kn}$

$= x(0)W_4^0 + x(1)W_4^K + x(2)W_4^{2K} + x(3)W_4^{3K}$

$X(K) = 6 + 7W_4^K + 6W_4^{2K} + 5W_4^{3K}$

$X(0) = 6 + 7 + 6 + 5 = 24$

$X(1) = 6 + 7W_4^1 + 6W_4^2 + 5W_4^3$

$\ldots 5(j) = 6 - 7j - 6 + 5j = -2j$

$K = 2$

$$X(2) = 6 + 7W_4^{2} + 6W_4^{4} + 5W_4^{6}$$

$$= 6 + 7(-1) + 6(1) + 5(-1)$$

$$X(2) = 0$$

$K = 3$

$$X(3) = 6 + 7W_4^{3} + 6W_4^{6} + 5W_4^{9}$$

$$= 6 + 7(j) + 6(-1) + 5(-j)$$

$$= 2j$$

$$X(K) = \{24, -2j, 0, 2j\}$$

18. Find the IDFT of the sequence

$$X(K) = (2, 1+j, 0, 1-j)$$

$$x(n) = \frac{1}{N}\sum_{K=0}^{N} X(K)\, W_N^{-Kn}$$

$$= \frac{1}{4}\left[X(0)W_4^{-0\times n} + X(1)W_4^{-n} + X(2)W_4^{-2n} + X(3)W_4^{-3n}\right]$$

$$= \frac{1}{4}\left[2 + (1+j)W_4^{-n} + 0 + (1-j)W_4^{-3n}\right]$$

$$W_4^{-0} = (W_4^{0})^{*} = 1$$

$$W_4^{-1} = (W_4^{1})^{*} = j$$

$$W_4^{-2} = (W_4^{2})^{*} = -1$$

$$W_4^{-3} = (W_4^{3})^{*} = -j$$

$$x(0) = \frac{1}{4}\left[2 + (1+j)\cdot 1 + 0 + (1-j)\right]$$

$$x(0) = 1$$

$$x(1) = \frac{1}{4}\left[2 + (1+j)W_4^{-1} + (1-j)W_4^{-3}\right]$$

$$= \frac{1}{4}\left[2 + (1+j)j + (1-j)(-j)\right] = 0$$

$x(2) = \frac{1}{4}\left[2 + (1+j)W_4^{-2} + (1-j)W_4^{-6}\right]$ $\quad W_4^{-6} = W_4^{-2}$

$= \frac{1}{4}\left[2 + (1+j)\cdot -1 + (1-j)\cdot -1\right] = 0$

$x(3) = \frac{1}{4}\left[2 + (1+j)W_4^{-3} + (1-j)W_4^{-9}\right]$

$= \frac{1}{4}\left[2 + (1+j)W_4^{-3} + (1-j)W_4^{-1}\right]$

$= \frac{1}{4}\left[2 + (1+j)(-j) + (1-j)(j)\right] = 1$

$x(n) = (1, 0, 0, 1)$

19. Compute the 4-point DFT of the sequence $x(n) = (1, 2, 1, 0)$ using ~~the~~ linear transformation of DFT

Soln: $N = 4 \quad W_4^4 = W_4^0 = 1 \quad W_4^9 = W_4^5 = W_4^1 = -j \quad W_4^6 = W_4^2 = -1 \quad W_4^7 = W_4^3 = j$

$$X_N = W_N \cdot x_N$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \end{bmatrix} = \begin{bmatrix} 1+2+1+0 \\ 1-2j-1+0 \\ 1-2+1+0 \end{bmatrix} = \begin{bmatrix} 4 \\ -j2 \\ 0 \end{bmatrix}$$

20. Find the 4 point DFT of the following sequence

$$x(n) = \delta(n) + \delta(n-1) + \delta(n-2)$$

Soln: $N=4$

$$x(n) = \{1, 1, 1, 0\}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1+0 \\ \cancel{1}-j\cancel{-1}+0 \\ 1-1+1+0 \\ \cancel{1}+j-\cancel{1}+0 \end{bmatrix} = \begin{bmatrix} 3 \\ -j \\ 0 \\ j \end{bmatrix}$$