MIT calculus (18.01) 23/08/...

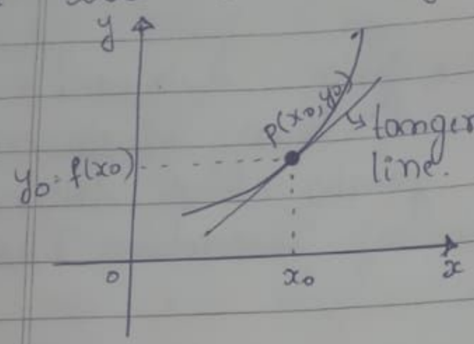
Lecture 1

Rate of Change

Differentiation

Geometric Interpretation of differentiation :-

Q: Find the tangent line to $y = f(x)$ at $P = (x_0, y_0)$



Tangent line :

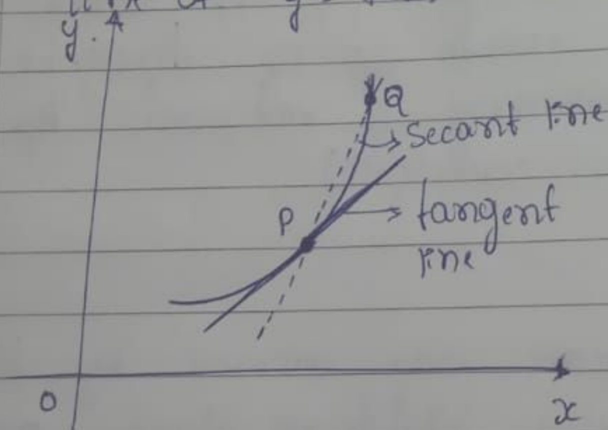
→ It is the line that satisfies equation

$$y - y_0 = m(x - x_0)$$

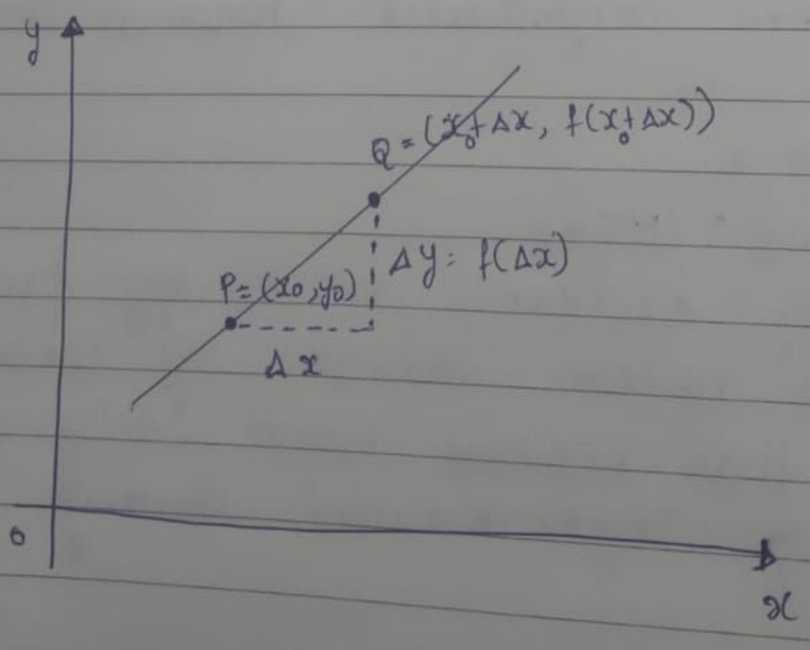
where, $m = \{\text{slope of tangent line}\}$

$= f'(x_0)$

Hence Derivatives : $f'(x_0)$, slope of tangent line at $y = f(x)$ at point $P = (x_0, y_0)$.



→ The tangent line = limit secant lines PQ as $Q \to P$



Slope $= m = \lim_{\Delta x \to 0} \dfrac{\Delta F}{\Delta x}$

slope of secant line

$$\boxed{\Rightarrow f'(x_0) = m = \lim_{\Delta x \to 0} \underbrace{\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}}_{\text{Difference Quotient}}}$$

Examples :-

1) Find slope of tangent line of $f(x) = 1/x$

$\Rightarrow$ consider $\frac{\Delta f}{\Delta x} = \frac{\left(1/(x_0+\Delta x)\right) - \left(1/x_0\right)}{\Delta x}$

$$= \frac{1}{\Delta x}\left(\frac{x_0 - x_0 - \Delta x}{x_0(x_0+\Delta x)}\right)$$

$$= \frac{1}{\Delta x}\left(\frac{-\Delta x}{x_0(x_0+\Delta x)}\right)$$

$$= \frac{-1}{x_0(x_0+\Delta x)}$$

$$\text{slope} = m = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

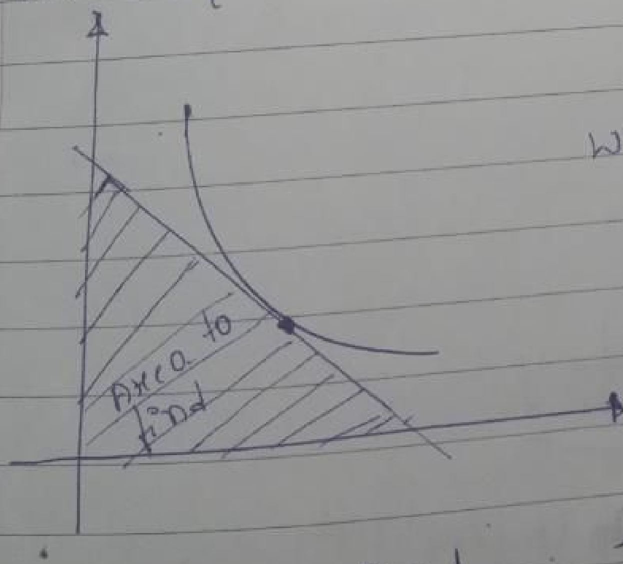
$$= \lim_{\Delta x \to 0} \left(\frac{-1}{x_0(x_0+\Delta x)}\right)$$

$$\boxed{\text{slope} = m = \frac{-1}{x_0^2}}$$

2) Find the area of triangles enclosed by the axes and the tangent to $y = 1/x$.

WKT $m = \frac{-1}{x_0^2}$

$\Rightarrow$



WKT tangent line is

$$y - y_0 = m(x - x_0)$$

$$\Rightarrow y - y_0 = \frac{-1}{x_0^2}(x - x_0) \rightarrow ①$$

Find x intercept by substituting $y = 0$

$$0 - \frac{1}{x_0} = \frac{-1}{x_0^2}(x - x_0)$$

$$-\frac{1}{x_0} = \frac{-x}{x^2} + \frac{1}{x_0}$$

$$\Rightarrow \frac{x}{x_0^2} = \frac{2}{x_0}$$

$$\Rightarrow \boxed{x = 2x_0}$$

Find y intercept by putting $x = 0$ in ①

$$y - y_0 = \frac{-1}{x_0^2}(0 - x_0)$$

$$\Rightarrow y - y_0 = \frac{x_0}{x_0^2}$$

$$\Rightarrow y - y_0 = \frac{1}{x_0}$$

$$\Rightarrow y - \frac{1}{x_0} = \frac{1}{x_0}$$

$$\Rightarrow y = \frac{2}{x_0}$$

$$\Rightarrow \boxed{y = 2y_0}$$

Area $= \frac{1}{2}$ (base × height)

$$= \frac{1}{2}(2x_0 \times 2y_0)$$

$$= 2x_0 y_0$$

$$= 2x_0\left(\frac{1}{x_0}\right)$$

$$= \underline{\underline{2}}$$

Notations :-

$$f' = \underbrace{\frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}f = \frac{d}{dx}y}_{}$$

Newton's notation

Leibniz (It is implicitly understood)

3) Find slope of tangent line to $f(x) = x^n$, $n = 1, 2, 3$

$$\Rightarrow \frac{d}{dx}(x^n) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

From Binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$= \binom{n}{0} x^{n-0} y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \dots$$

$$\dots + \binom{n}{n} x^{n-n} y^n$$

$$= (1)\, x^n + n x^{n-1} y + \left(\frac{n!}{(n-2)!\,2} x^{n-2} y^2 + \dots + (1)\, y^n \right) \longrightarrow \text{call as } O((y)^2) \text{ (terms with order } y^2, y^3 \dots)$$

$$= x^n + n x^{n-1} y + O((y)^2)$$

$$\Rightarrow \frac{d}{dx}(x^n) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left((x+\Delta x)^n - x^n \right)$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left(\cancel{x^n} + n x^{n-1} \Delta x + O((\Delta x)^2) - \cancel{x^n} \right)$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left(n x^{n-1} \Delta x + O((\Delta x)^2) \right)$$

$$= \lim_{\Delta x \to 0} n x^{n-1} + O(\Delta x)$$

$$= n x^{n-1} + 0$$

$$\boxed{\frac{d}{dx}(x^n) = n x^{n-1}}$$