Discrete Mathematics (Math191)

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Chapter 2

Elementary Probability Theory

2.1 Origin Of The Term Probability

- Probability theory originates in seventeenth century when the French mathematician Blaise Pascal determined the odds of winning some popular bets based on the outcome, when a pair of dice is repeatedly rolled.

- In eighteenth century French mathematician Laplace who also studied gambling defined the probability of an event as the number of successful outcomes divided by number of possible outcomes.

2.1 Sample Space and Events

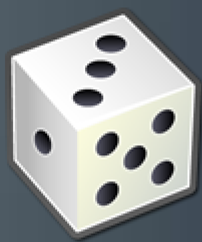
- **Random experiment** is an experiment in which the outcome cannot be determined or predicted exactly in advance.
- **Event:** A collection of one or more outcomes in a statistical experiment. We usually denote events with capital letters A, B, C ...
- **Simple Event:** An event that consists of exactly one outcome in a statistical experiment.
- **Sample Space:** The set of all simple events

Cont….

EXAMPLE : A coin is tossed twice

S= {(H,H),(H,T),(T,H),(T,T)}

E= {(H,H),(T,H)} the event that the second toss was a head



EXAMPLE : Rolling a die

S ={ ⚀ ⚁ ⚂ ⚃ ⚄ ⚅ }

S={1,2,3,4,5,6}

E={1,3,5} the event that an odd number is rolled.

Set Operation

- Union $A \cup B$: an outcome is in $A \cup B$ if it is either in A or in B.
- Intersection $A \cap B$: an outcome is in $A \cap B$ if it is in both A and B.
- Mutually exclusive: A and B are mutually exclusive if $A \cap B = \emptyset$.

Example: Roll die

A: Outcome is below 3

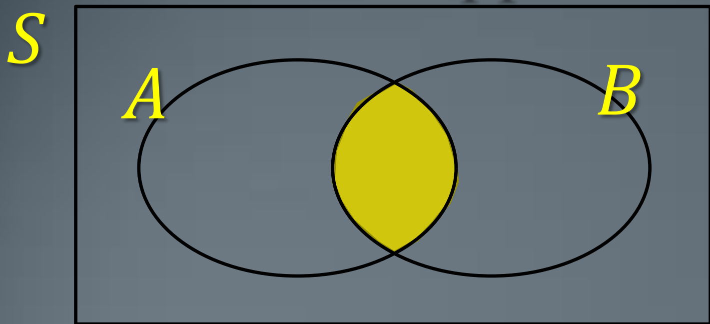
B: Outcome is above 4

Mutually exclusive events

- Complement A^c: outcome that is not in A.

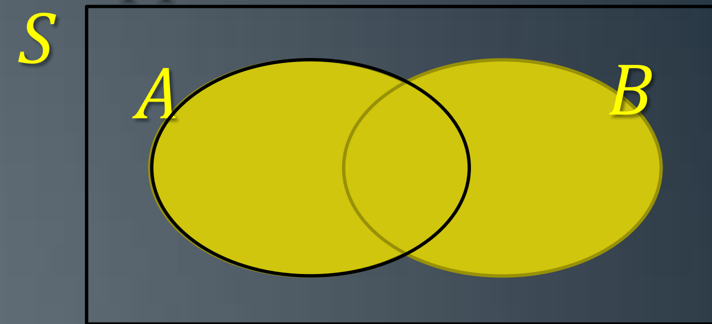
Venn Diagrams

Venn Diagrams allow us to combine events, e.g. “*A* happened and *B* happened”.



The event “*A* and *B*”

Known as the intersection of *A* and *B*.



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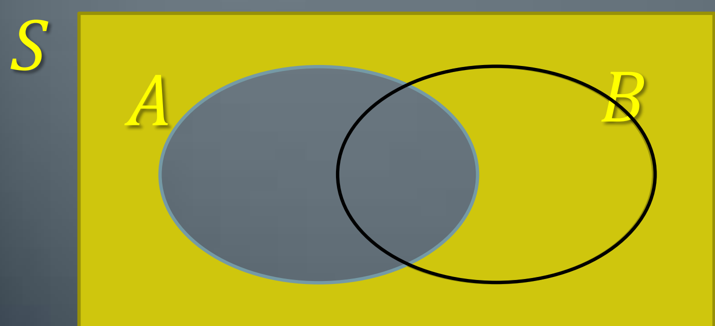
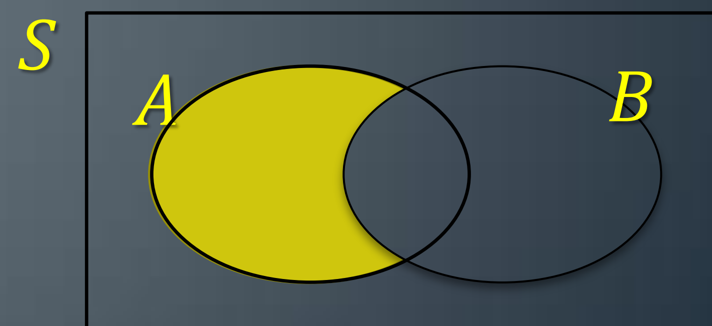


The event “not *A*”

Events that does not belong to *A*.



These can be combined,

e.g. “*A* and not *B*”.

Important Elementary Set Theory Results

- $A \cup B = B \cup A$ and $A \cap B = B \cap A$
- $A \cup (B \cup C) = (A \cup B) \cup C$
- $A \cap (B \cap C) = (A \cap B) \cap C$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $(A^c)^c = A$
- $A \cap S = A$; $A \cup S = S$; $A \cap \emptyset = \emptyset$; and $A \cup A = A$
- $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$

Definition: Probability of an Event *E*.

Suppose that the sample space
$S = \{a_1, a_2, a_3, \ldots a_n\}$
has a finite number, n, of outcomes. Suppose that each of the outcomes is equally likely. Then for any event *E*, *the probability* $P(E)$ *is defined as*

$$P(E) = \frac{n(E)}{n(S)} = \frac{No. of\ outcomes\ in\ E}{Total\ No. of\ outcomes}$$

Probability Of An Event

The probability of event E with sample space S is a number assigned to E that satisfies the following

1. $0 \le p(E) \le 1$
2. $p(S) = 1$
3. For any sequence of events $E_1, E_2 \dots$, which are mutually exclusive $p(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} p(E_n)$.

Theorem: If A is an event in a discrete sample space S, then $P(S)$ equals the sum of the probabilities of the individual outcomes comprising A.

Count…

Example 1: Roll a fair die, we have 6 equally likely outcome {1,2,3,4,5,6}

$$p(\{1\}) = 1/6, p(\{2\}) = 1/6, \ldots \quad p(\{6\}) = 1/6.$$

Example 2: E: The outcome is even. $p(E) = ?$

$$p(E) = p(2,4,6) = p(2) + p(4) + p(6) = 1/2$$

Example 3: Choosing a point from the interval (a, b) at random, that is each point is equally likely to be chosen the probability measure is given by

$$p\big((c,d)\big) = \frac{d-c}{b-a}, \text{ for all interval } (c,d) \subset (a,b)$$

Count…

Properties

1) $p(E) + P(E^c) = 1$

2) Any event E and F (may not mutually exclusive)

$$\mathrm{p}(\mathrm{E} \cup F) = \mathrm{p}(\mathrm{E}) + \mathrm{p}(\mathrm{F}) - \mathrm{p}(E \cap F)$$

- Proof. E and $E^c \cap F$ are disjoint

$$E \cup (E^c \cap F) = E \cup F$$

$$P\big(E \cup (E^c \cap F)\big) = P(E \cup F)$$

$$P(E) + P(E^c \cap F) = P(E \cup F)$$

$$P(E) + P(E^c \cap F) + P(E \cap F) - P(E \cap F) = P(E \cup F)$$

$$P(E) + P[(E^c \cap F) \cup P(E \cap F)] - P(E \cap F) = P(E \cup F)$$

But $[(E^c \cap F) \cup P(E \cap F)] = F$

Therefore $\mathrm{p}(\mathrm{E} \cup F) = \mathrm{p}(\mathrm{E}) + \mathrm{p}(\mathrm{F}) - \mathrm{p}(E \cap F)$.

Theorem

- Suppose that we have a random experiment with sample space S and probability function P and A and B are events. Then we have the following results:

I. $P(\emptyset) = 0$

II. $P(A^c) = 1 - P(A)$

III. $P(B \cap A^c) = P(B) - P(A \cap B)$

IV. If A subset of B then $P(A) \leq P(B)$.

Example

- $Suppose\ that\ we\ toss\ two\ coins, and\ assume\ they\ are\ equally\ likely.$

  $Let\ A\ =\ \{(H,H),(H,T\ )\}\ and\ B = \ \{(H,H),(T\ ,H)\}$

  $then\ find\ the\ probabilities\ of\ A, B, A^c, B^c, and\ S^c.$

Solution:

- **P(A)=**$\frac{2}{4} = 0.5$
- **P(B)=**$\frac{2}{4} = 0.5$
- **P(**A^c**)=**$1 - P(A) = 0.5$
- **P(**B^c**)=**$1 - P(B) = 0.5$
- $P(S^c) = 1 - P(S) = 1 - 1 = 0 = P(\emptyset)$

2.3 Conditional probability

- **Definition:** If $P(B) > 0$, the conditional probability of A given B, denoted by $P(A|B)$, is

$$P(A|B) = \frac{p(A \cap B)}{P(B)}$$

Example : A box contains black chips and white chips. A person selects two chips without replacement. If the probability of selecting a black chip *and* a white chip is $15/56$, and the probability of selecting a black chip on the first draw is 3/8, find the probability of selecting the white chip on the second draw, *given* that the first chip selected was a black chip.

Count…

Solution

Let B= selecting a black chip and W= selecting a white chip. Then

$$P(W|B) = \frac{p(W \cap B)}{P(B)} = \frac{15/56}{3/8} = \frac{5}{7}$$

BAYE'S THEOREM

- Theorem: Suppose events $F_1, F_2, F_3, \ldots, F_n$ are mutually exclusive and $\bigcup_{i=1}^{n} F_i = S$. Given any event E

$$p(F_i|E) = \frac{P(F_i)P(E|F_i)}{\sum_{j=1}^{n} P\left(F_j\right)P(E|F_j)}$$

Example: A bag X contains 2 white and 3 red balls and a bag Y contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from bag Y.

Solution E_1 : the event of choosing bag X

E_2: the ball is drawn from bag Y

A: ball is red

count…

$$p(E_2|A) = \frac{P(E_2)P(A|E_2)}{\sum_{j=1}^{2} P\left(E_j\right)P(A|E_j)}$$

$$p(E_2|A) = \frac{\frac{1}{2} * \frac{5}{9}}{\frac{1}{2} * \frac{3}{5} + \frac{1}{2} * \frac{5}{9}}$$

$$= \frac{25}{52}$$

2.4 Independent events

Definition:-Two events A and B are said to be independent if $P(A \cap B) = P(A)P(B)$. If in addition, $P(B) > 0$, independence is equivalent to the condition $P(A|B) = P(A)$.

Example: A coin is tossed and a dice is rolled simultaneously. What is the probability of a tail and a 6 coming at the output

Solution: Since $p(T) = 1/2$ and $p(6) = 1/6$

probability of a tail coming in the toss and getting a 6 in dice is $^1/_2 * ^1/_6 = ^1/_{12}$.

2.5 Random variables and expectation

Example: A random variable is a variable whose value depends on the outcome of a random event.

Probability distributions

x	red	green	blue	orange
$P(X = x)$	0.3	0.4	0.1	0.2

A **random variable X represents a single experiment/ trial.** It consists of outcomes with a probability for each.

$$P(X = x)$$

i.e. X is a random variable (capital letter), but x is a particular outcome

"The probability that…

…the outcome of the random variable X…

…was the specific outcome x"

A shorthand for $P(X = x)$ is ! $p(x)$ (note the lowercase p).

It's like saying "the probability that the outcome of my coin throw was heads " ($P(X = heads)$) vs "the probability of heads" ($p(heads)$). In the latter the coin throw was implicit, so we can skip the '$X =$ '.

Probability Distributions vs Probability Functions

There are two ways to write the mapping from outcomes to probabilities:

As a function:

The "{" means we have a 'piecewise function'. This just simply means we choose the function from a list depending on the input.

$$p(x) = \begin{cases} 0.1x, & x = 1,2,3,4 \\ 0, & otherwise \end{cases}$$

e.g. if x =3, then the probability is $0.1 \times 3 = 0.3$

Advantages of functional form:

Can have a rule/expression based on the outcome. it would be impossible to list the probability for every outcome. More compact.

As a table:

x	1	2	3	4
$p(x)$	0.1	0.2	0.3	0.4

Advantages of table form:

Probability for each outcome more explicit.

Example

The random variable X represents the number of heads when three coins are tossed.

Underlying Sample Space

{ HHH,
HHT,
HTT,
HTH,
THH,
THT,
TTH,
TTT }

Distribution as a Table

Num heads x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Distribution as a Function

$$P(X = x) = \begin{cases} \frac{1}{8} & x = 0,3 \\ \frac{3}{8} & x = 1,2 \\ 0 & otherwise \end{cases}$$

The Binomial distribution

- Many real problems (experiments) have two possible outcomes, for instance, a person may be HIV-Positive or HIV-Negative, a seed may germinate or not, the sex of a new born bay may be a girl or a boy, etc.

- Technically, the two outcomes are called Success and Failure.

- Experiments or trials whose outcomes can be classified as either a "success" or as a "failure" are called Bernoulli trails.

The Binomial distribution

✐ You can model a random variable X with a binomial distribution $B(n, p)$ if

- there are a fixed number of trials, n,
- there are two possible outcomes: 'success' and 'failure',
- there is a fixed probability of success, p
- the trials are independent of each other

If $X \sim B(n, p)$ then:

$$P(X = r) = \binom{n}{r} p^r (1 - p)^{n-r}$$

The Binomial distribution

Theorem: The probability of exactly r successes in n independent Bernoulli trials, with probability of success p and probability of failure $q = 1 - p$, is

$$P(X = r) = C(n, r)p^r q^{n-r}$$

Exercise: A fair coin is flipped 4 times. Let X be the number of heads appearing out of the four trials. Calculate the following probabilities that 2 heads will appear

The Binomial distribution

Example: The probability that a randomly chosen member of a reading group is left-handed is 0.15.

A random sample of 20 members of the group is taken.

a. Suggest a suitable model for the random variable X, the number of members in the sample who are left-handed. Justify your choice.

b. Use your model to calculate the probability that:

i. exactly 7 of the members in the sample are left-handed

ii. fewer than two of the members in the sample are left-handed.

Count…

a. The random variable can take two values, left-handed or right-handed.

There are a fixed number of trials, 20, and a fixed probability of success: 0.15.

Assuming each member in the sample is independent, a suitable model is $X \sim B(20, 0.15)$.

b. i $P(X = 7) = (270) \times (0.15)7(0.85)13$

$= 0.016\ 01\ldots$

$= 0.0160$ (3 s.f.)

ii $P(X < 2) = P(X = 0) + P(X = 1)$

$= 0.038\ 75\ldots + 0.136\ 79\ldots$

$= 0.176$ (3 s.f.)

Expected Values of discrete random variable

Definition: Let X be a discrete random variable with possible values $x_1, x_2, \ldots$ and probability mass function (pmf) $p(x)$. The expected value and variance of X are calculated by

$$E(X) = \mu = \sum_i x_i p(x_i)$$

$$Var(X) = \sigma^2 = E(X^2) - E(X)^2$$

$$= \sum x^2\, p(x) - \mu^2$$

Count…

- **Example:** Flip a coin 3 times. Let X be the number of heads. Find $E(X)$.

Solution:

$$E(X) = \frac{1}{8}(X(HHH) + X(HHT) + X(HTH) + X(THH) + X(TTH) + X(THT) + X(HTT) + X(TTT))$$

$$= \frac{1}{8}(3 + 2 + 2 + 2 + 1 + 1 + 1 + 0)$$

$$= \frac{12}{8}$$

$$= \frac{3}{2}$$

Count…

Theorem: let X and Y are random variable on the sample space S and a and b are real number, then

i. $E(X+Y) = E(X) + E(Y)$

ii. $E(aX+b) = aE(X) + b$

End of Chapter 2
Thank you!