Addis Ababa science & Technology University
Department of Mathematics

Discrete mathematics-I
CHAPTER-3: Recurrence relation

TALK OUTLINES

3.1 Definition and Examples of recurrence relation

3.2 Modeling using recurrence relation

3.3 Solving recurrence relation

INTRODUCTION

- ✓ **M**any counting problems cannot be solved easily using the methods discussed in Chapter 1.
- ✓ One such problem is: **How many bit strings of length *n* do not contain two consecutive zeros?**
- ✓ The techniques studied in this chapter, together with the basic techniques of Chapter 1, can be used to solve many counting problems.

Motivation

The number of bacteria in a colony doubles every hour. If the colony begins with 5 bacteria, how many will be present in n hours?

Solution:

Let a_n : the number of bacteria at the end of n hours

Then;

$$a_n = 2a_{n-1}, \qquad a_0=5$$

Since bacteria double every hour

Initial condition

By using recursive definition

What we are doing actually?

➡ We find a formula/model for a_n (the relationship between a_n and a_0) – this is called **recurrence relations** between the term of sequence.

3.1 Recurrence Relation

A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

A sequence is call **solution** of a recurrence relation if its term satisfy the recurrence relation.

Recursive definition

$$\begin{cases} a_n = 2a_{n-1} & \longrightarrow \text{Recurrence relation} \\ a_0 = 5 & \longrightarrow \text{Initial condition} \end{cases}$$

- A recursive algorithm provides the solution of a problem of size n in term of the solutions of one or more instances of the same problem in smaller size.

- The initial condition specify the terms that precede the first term where the recurrence relation takes effect.

Example 3.1

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3,4$, and suppose that $a_0 =3$ and $a_1 = 5$.

List the first four term.

$a_0 = 3$

$a_1 = 5$

$a_2 = a_1 - a_0 =$ ______________________

$a_3 =$ ______________________________

What is a_5?

$a_5 =$ ______________________________

Example 3.2

Determine whether $\{a_n\}$ where $a_n = 3n$ for every nonnegative integer n, is a solution of recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$

Suppose that $a_n = 3n$ for every nonnegative integer n,
Then for $n \geq 2$;

$2a_{n-1} - a_{n-2}$ = ____________________

Therefore,

Determine whether $\{a_n\}$ where $a_n = 5$ for every nonnegative integer n, is a solution of recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$

$2a_{n-1} - a_{n-2}$ = ____________________

EXERCISE 3.1

1. Let a_n denotes the *n*th term of a sequence satisfying the given initial condition (s) and the recurrence relation. Compute the first four terms of the sequence.

A $$a_1 = -2, \quad a_n = 3 - \left(a_{n-1}\right)^2 - 2a_{n-1} \quad \text{for } n \geq 2$$

B $$a_0 = 128, \quad a_n = \frac{1}{4} a_{n-1} \quad \text{for } n \geq 1$$

C $$b_1 = 5, \quad b_n = 7 - 2b_{n-1} \quad \text{for } n \geq 2$$

D $$a_1 = 1, \quad a_2 = 2, \quad a_n = a_{n-1} 3 + a_{n-2} \quad \text{for } n \geq 3$$

E $$d_1 = 1, d_2 = 2, d_3 = 3, d_n = d_{n-1} + d_{n-2} + d_{n-3} \quad \text{for } n \geq 4$$

EXERCISE 3.1

2. Is the sequence $\{a_n\}$ a solution of the $a_n = 3 - 2a_{n-1} - a_{n-2}$ if

$a_n = 2n - 1$

3. Is the sequence $\{a_n\}$ a solution of the $a_n = -3a_{n-1} + 6a_{n-2}$ if

$a_n = (-3)^n$

4. Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2} + 2$ if $a_n = n^2 - 3$

3.2 Modeling with Recurrence Relation

- We can use recurrence relation to model a wide variety of problems such as
  - Finding a compound interest
  - Counting a population of rabbits
  - Determining the number of moves in the Tower of Hanoi puzzle
  - Counting bit strings

Example 3.3: Compound interest

Suppose that a person deposits Birr10,000 in a savings account at CBE yielding 11% per year with interest compounded annually. Define recursively the compound amount in the account at the end of *n* years.

Solution:

Let a_n : the amount in the account after *n* years

Then;

a_n = (compound amount at the end of (n -1)th years)

+ (interest for the n^{th} years)

$$a_n = a_{n-1} + 0.11a_{n-1}$$

$$= 1.11\, a_{n-1}$$

With initial condition; $a_0 = 10,000$

Example 3.4: Population of rabbits

A young pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month. Find a recurrence relation for the numbers of pairs of rabbits on the island after n months if all the rabbits alive.

Solution:

Let f_n : the numbers of pairs of rabbits on the island after n months

Then;

f_n = (number of pairs of rabbits in (n -1)th months)

+ (number of newborn pairs of rabbits)

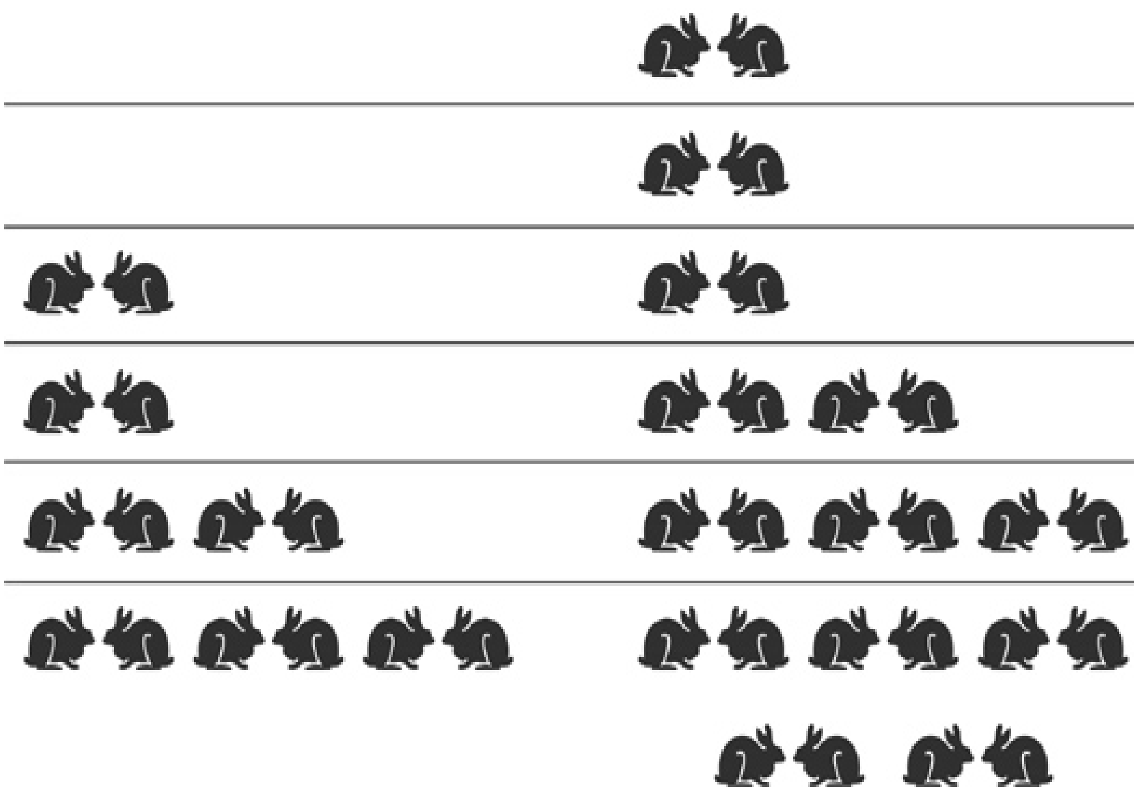
$f_n = f_{n-1} + f_{n-2}$

WHY???

With initial condition; $f_1 = 1$ and $f_2 = 1$; $n \geq 3$

[Problem develop by Leonardo Pisano (Liber abaci, 13th century) – lead to Fibonacci number]

Example 3.4: Population of rabbits



Reproducing pairs (at least two months old)	Young pairs (less than two months old)	Month	Reproducing pairs	Young pairs	Total pairs
		1	0	1	1
		2	0	1	1
		3	1	1	2
		4	1	2	3
		5	2	3	5
		6	3	5	8

Example 3.5: Tower of Hanoi

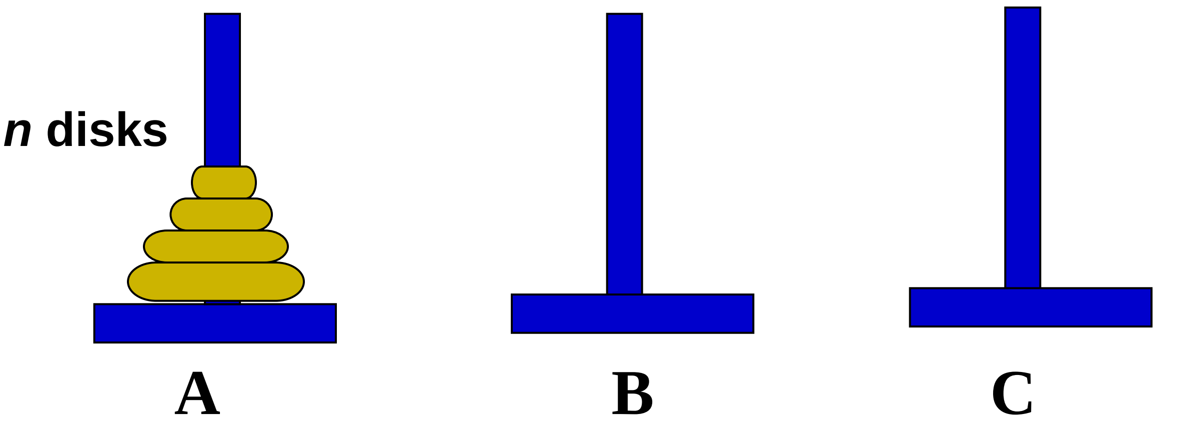
- Popular Puzzle invented by Edouard Lucas (French Mathematician, late 19^{th} century)

RULES OF PUZZLE:

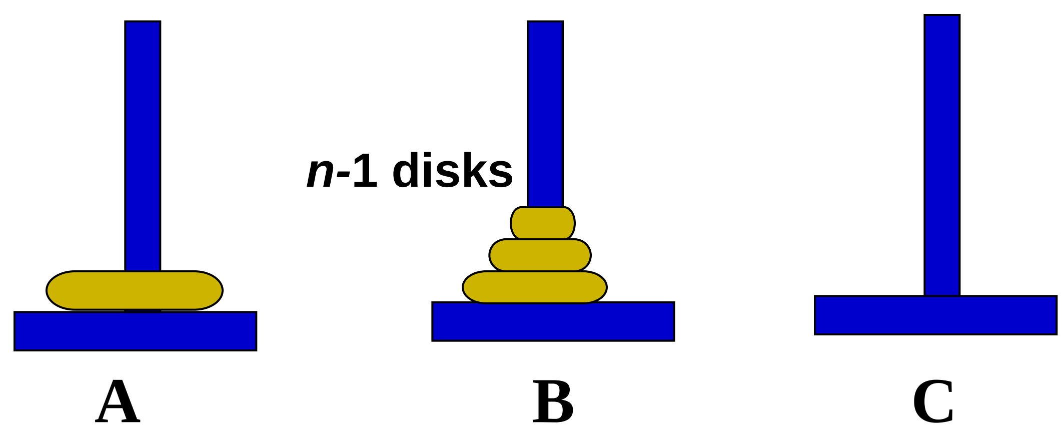
- Suppose we have 3 pegs labeled A, B, C and a set of disks of different sizes.
- These disks are arranged from the largest disk at the bottom to the smallest disk at the top (1 to *n* disks) on peg A.
- The goal: to have all disks on peg C in order of size, with the largest on the bottom.
- Only one disk is allow to move at a time from a peg to another.
- Each peg must always be arranged from the largest at the bottom to the smallest at the top

Example 3.5: Tower of Hanoi

ILLUSTRATION:



Initial Position in the Tower of Hanoi



Intermediate Position in the Tower of Hanoi

SOLUTION

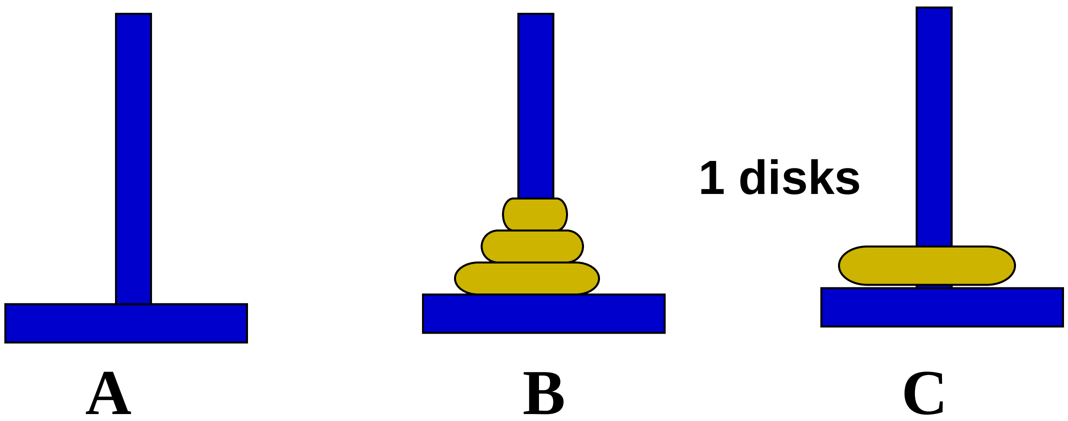
Let H_n :
the numbers of moves with *n* disks

transfer the top of *n*-1 disks to peg B

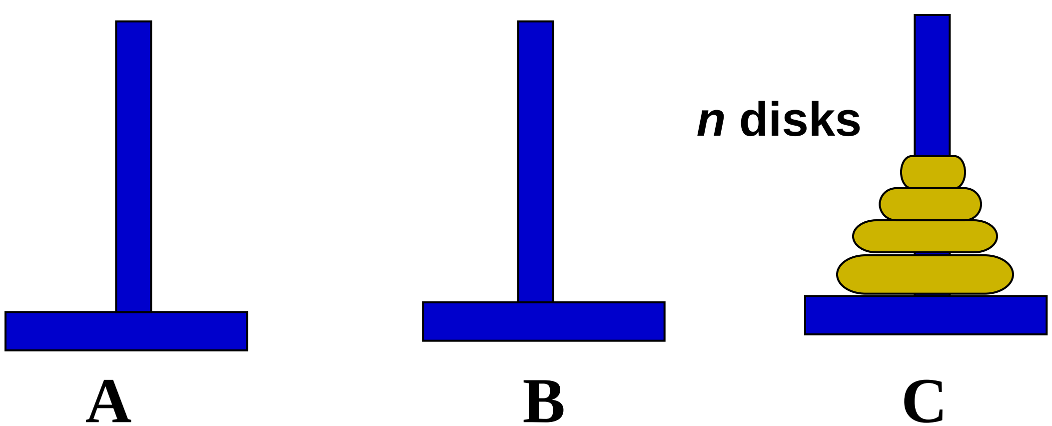
$\boxed{H_{n-1} \text{ moves}}$

Example 3.5: Tower of Hanoi

ILLUSTRATION:



Intermediate Position in the Tower of Hanoi



Last Position in the Tower of Hanoi

SOLUTION

Moves the largest disk to peg C

1 move

transfer the top of n-1 disks to peg B

H_{n-1} moves

Example 3.5: Tower of Hanoi

Solution (continue):

Thus the number of moves is given by:

$$H_n = 2H_{n-1} + 1 \quad ; \quad H_1 = 1$$

Where:

transfer the top of n-1 disks to peg B		Moves the largest disk to peg C		transfer the top of n-1 disks to peg C
H_{n-1} moves	+	1 move	+	H_{n-1} moves

Example 3.6: Bit Strings

Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive 0s. How many bit strings are there of six length?

Solve yourself

Solution:

Let a_n : number of bit strings of length n that do not have two consecutive 0s

Then;

a_n = (number of bit strings of length n-1 that do not have two consecutive 0s)
+ (number of bit strings of length n-2 that do not have two consecutive 0s)

$$a_n = a_{n-1} + a_{n-2}\ ; \qquad n \geq 3$$

With initial condition;

$a_1 = 2$, both strings of length 1 do not have consecutive 0s (0 and 1)

$a_2 = 3$, the valid strings only 01, 10 and 11

Example 3.7: Codeword enumeration

A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. For instance, 1230407869 is valid, whereas 120987045608 is not valid. Let a_n be the number of valid *n*-digit codewords. Find a recurrence relation for a_n.

Solution:

Let a_n : the number of valid *n*-digit codewords

Then;

a_n = (number of valid *n*-digit codewords obtained by adding a digit other than 0 at the end of a valid string of length $n - 1$)

\+ (number of valid *n*-digit codewords obtained by adding a digit 0 at the end of an invalid string of length $n - 1$)

$$a_n = 9a_{n-1} + \left(10^{n-1} - a_{n-1}\right) = 8a_{n-1} + 10^{n-1}$$

With initial condition;

$a_1 = 9$, there are 10 one-digit strings and only one the string 0 not valid

Example 3.7: Codeword enumeration

A valid codeword of length *n* can be formed by:

A) Adding a digit other than 0 at the end of a valid string of length *n* – 1

- This can be done in nine ways. Hence, a valid string with *n* digits can be formed in this manner in $9a_{n-1}$ ways.

B) Adding a digit 0 at the end of an invalid string of length *n* – 1

- This produces a string with an even number of 0 digits because the invalid string of length *n* – 1 has an odd number of 0 digits. The number of ways that this can be done equals the number of invalid (*n* – 1)-digit strings. Because there are 10^{n-1} strings of length *n* – 1, and a_{n-1} are valid, then there are $10^{n-1} - a_{n-1}$ invalid (*n* – 1)-digit strings.

Thus;

$$a_n = A + B = 9a_{n-1} + \left(10^{n-1} - a_{n-1}\right) = 8a_{n-1} + 10^{n-1},\ a_1 = 9,\ n \geq 2$$

EXERCISE 3.1

6. Define recursively : 1, 4, 7, 10, 13, …

7. Beti deposits BIRR 1500 in a local savings bank at an annual interest rate of 8% compounded annually. Define recursively the compound amount a_n she will have in her account at the end of n years. How much will be in her account after 3 years?

8. There are n students at a class. Each person shakes hands with everybody else exactly one. Define recursively the number of handshakes that occur.

EXERCISE 3.1

9. Find a recurrence relation and initial condition for the number of ways to climb *n* stairs if the person climbing the stairs can take one, two or three stairs at a time. How many ways can this person climb a building with eight stairs?

10. Find a recurrence relation and initial condition for the number of bit strings of length *n* that contain three consecutive 0s. How many bit strings of length seven contain three consecutive 0s?

11. Find a recurrence relation for the number of bit sequences of length *n* with an even number of 0s?

3.3 SOLVING RECURRENCE RELATION

- Solve a recurrence relation using iterative method
- Solve a linear homogeneous recurrence relation with constant coefficients
- Solve a linear nonhomogeneous recurrence relation with constant coefficients

Iterative Method

- **Solving** the recurrence relation for a function *f* means finding an **explicit formula** for *f* (*n*). The **iterative method** of solving it involves two steps:
  - Apply the recurrence formula iteratively and look for the **pattern** to predict an explicit formula
    - ❖ Forward: start from a_0 to a_n
    - ❖ Backward : start from a_n to a_0
  - Use **Induction** to prove that the formula does indeed hold for every possible value of the integer *n*.

Example 3.3.1 (a): Iterative Method

Using the iterative method, predict a solution for the following recurrence relation with the given initial condition.

$$S_n = 2S_{n-1} \quad ; \quad S_0 = 1$$

Solution: (FORWARD)

$$\begin{aligned}
S_0 &= 1 \\
S_1 &= 2S_0 \\
S_2 &= 2S_1 = 2\left(2S_0\right) = 2^2 S_0 \\
S_3 &= 2S_2 = 2\left(2^2 S_0\right) = 2^3 S_0 \\
&\vdots \\
S_n &= 2S_{n-1} = 2\left(2^{n-1} S_0\right) = 2^n S_0 = 2^n \quad ; \quad n \geq 1
\end{aligned}$$

Solution: (BACKWARD)

$$\begin{aligned}
S_n &= 2S_{n-1} \\
&= 2^2 S_{n-2} \\
&= 2^3 S_{n-3} \\
&\vdots \\
&= 2^n S_0 = 2^n
\end{aligned}$$

Example 3.3.2: Compound interest

Suppose that a person deposits Birr10,000 in a savings account at a CBE yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Solution:

Let a_n : the amount in the account after n years

Then;

a_n = (compound amount at the end of (n -1)th years)

+ (interest for the nth years)

$$a_n = a_{n-1} + 0.11a_{n-1}$$

$$= 1.11\, a_{n-1} \qquad , n \geq 1$$

With initial condition; $a_0 = 10,000$

Example 3.3.2: Compound interest

Solution (continue):

By iterative approach:

PROVE BY INDUCTION !

$$
\begin{aligned}
a_0 &= 10000 \\
a_1 &= (1.11)\,a_0 \\
a_2 &= (1.11)\,a_1 = (1.11)^2\,a_0 \\
a_3 &= (1.11)\,a_2 = (1.11)^3\,a_0 \\
&\vdots \\
a_n &= (1.11)\,a_{n-1} = (1.11)^n\,a_0 = (1.11)^n\,(10000)
\end{aligned}
$$

Thus, after 30 years the amount in the account will be:

$$a_{30} = (1.11)^{30}\,(10000) = \underline{\qquad\qquad\qquad}$$

=Birr 228,923

Example 3.3: Tower of Hanoi

The number of moves is given by:

$$H_n = 2H_{n-1} + 1 \quad ; \quad H_1 = 1$$

PROVE BY INDUCTION !

By iterative approach:

$$\begin{aligned}
H_n &= 2H_{n-1} + 1 \\
&= 2\left(2H_{n-2} + 1\right) + 1 = 2^2 H_{n-2} + 2 + 1 \\
&= 2^2\left(2H_{n-3} + 1\right) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1 \\
&\qquad \vdots \\
&= 2^{n-1} H_1 + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1 \\
&= 2^n - 1
\end{aligned}$$

EXERCISE 3.3.1

1. Using iterative method, predict a solution to each of the following recurrence relation. Verify the solutions using induction

   - $a_n = a_{n-1} + 4n \quad ; \quad a_0 = 0 \quad ; \quad n \geq 1$

   - $b_n = b_{n-1} + n^2 \quad ; \quad a_1 = 1 \quad ; \quad n \geq 2$

2. There are *n* students at a class. Each person shakes hands with everybody else exactly one. Define recursively the number of handshakes that occur. Solve this recurrence relation using iterative method and prove it using induction.

3. If a deposit of birr 100 is made on the first day of each month into an account that pays 6% interest per year compounded monthly, show that the amount in the account after 18 years is birr 38929.

Linear Homogeneous Recurrence Relations with constant coefficients

A linear homogeneous recurrence relation of degree *k* with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

Where $c_1, c_2, \dots, c_k$ are real numbers and $c_k \neq 0$

Linear : The RHS is the sum of previous terms of the sequence each multiplied by a function of *n*.

Homogeneous : No terms occur that are not multiplies of the a_j s.

Degree *k* : a_n is expressed in terms of the previous *k* terms of the sequence

Constant coefficients : $c_1, c_2, \dots, c_k$

Recurrence relation : with *k* initial condition

$$a_0 = C_0, \quad a_1 = C_1, \quad \dots \quad a_{k-1} = C_{k-1}$$

Example 3.3.4: Linear Homogeneous RR

Linear Homogeneous Recurrence Relation

$P_n = (1.11)\, P_{n-1}$ degree one

$f_n = f_{n-1} + f_{n-2}$ degree two

$a_n = a_{n-5}$ degree five

1. Often occur in modeling of problems

2. Can be systematically solved

Not Linear Homogeneous Recurrence Relation

$H_n = 2H_{n-1} + 1$

$B_n = nB_{n-1}$

$a_n = a_{n-1} + a^2_{n-2}$

Solving Linear Homogeneous Recurrence Relations with constant coefficients

- OUR AIM – look for the solutions of the form $a_n = r^n$, where r is constant.

- Note that, $a_n = r^n$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ iff $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}$.

- When both sides of this equation are divided by r^{n-k} and the RHS is subtracted from the left, we obtain a characteristic equation: $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_{k-1} r - c_k = 0$

- The solution of this equation are called the characteristics roots.

Solving Linear Homogeneous Recurrence Relations with constant coefficients

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristics equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_{k-1} r - c_k = 0$ has k distinct roots $r_1, r_2, \ldots, r_k$.

Then a sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ iff $a_n = \alpha_1 r_1{}^n + \alpha_2 r_2{}^n + \ldots + \alpha_k r_k{}^n$ for $n \geq 0$ and $\alpha_1, \alpha_2, \ldots, \alpha_k$ constant.

If the characteristic equation has several (m) repeated roots, then the solution of the recurrence relation is given by:

$$a_n = \alpha_1 r_1{}^n + \alpha_2 n r_1{}^n + \ldots + \alpha_m n^m r_1{}^n + \ldots + \alpha_k r_k{}^n$$

Example 3.3.5: Second-Order LHRRWCCs

Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ where $a_0 = 4$ and $a_1 = 7$.

Solution:

1) Find the general solution of the recurrence relation
   - the characteristic equation is given by $r^2 - 5r + 6 = 0$
   - the characteristic roots are 2 and 3
   - thus, the general solution is $a_n = A \cdot 2^n + B \cdot 3^n$

2) Find the constant values, *A*,*B* and C using the initial conditions
   - Solving the linear system: $$a_0 = A + B = 4$$ $$a_1 = 2A + 3B = 7$$: $A = 5$, $B = -1$

3) Thus the so $a_n = 5 \cdot 2^n - 3^n, n \geq 0$ nce relation and initial conditions is .

Example 3.3.6: Higher-Order LHRRWCCs

Solve the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ where $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

Solution:

1) Find the general solution of the recurrence relation
   - the characteristic equation is given by $r^3 - 6r^2 + 11r - 6 = 0$
   - the characteristic roots are 1, 2 and 3
   - thus, the general solution is $a_n = A \cdot 1^n + B \cdot 2^n + C \cdot 3^n$

2) Find the constant values, *A* and *B* using the initial conditions
   - Solving the linear system: ($A = 1$, $B = -1$, $C = 2$)

$$a_0 = A + B + C = 2, \quad a_1 = A + 2B + 3C = 5, \quad a_2 = A + 4B + 9C = 15$$

3) Thus the unique solution to the recurrence relation and initial conditions is $a_n = 1 - 2^n + 2 \cdot 3^n, n \geq 0$.

Example 3.3.7: LHRRWCCs with multiple roots

Suppose that the roots of the characteristic equation of a linear homogenous recurrence relation are 2, 2, 2, 5, 5, and 9. What is the form of general solution?

Solution:

Thus the general solution to the recurrence relation is

$$
\begin{aligned}
a_n &= A2^n + Bn2^n + Cn^2 2^n + D5^n + En5^n + F\cdot 9^n \\
&= \left(A + Bn + Cn^2\right)2^n + \left(D + En\right)5^n + F\cdot 9^n
\end{aligned}
$$

EXERCISE 3.3.2

1. What is the solution of the following recurrence relation

- $a_n = a_{n-1} + 2a_{n-2} \quad ; \quad a_0 = 2, a_1 = 7$

- $a_n = 6a_{n-1} - 9a_{n-2} \quad ; \quad a_0 = 1, a_1 = 3$

- $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3} \quad ; \quad a_0 = 1, a_1 = -2, a_2 = -1$

2. Find an explicit formula for Fibonacci numbers.

$$F_n = F_{n-1} + F_{n-2} \quad ; \quad F_1 = 1, F_2 = 1$$

Linear NonHomogenous Recurrence Relations with constant coefficients

- A **Linear Nonhomogenous recurrence relation with constant coefficients** (LNHRRWCCs), that is a recurrence relation of the form $a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k} + F(n)$.

- Where
  - $c_1, c_2, \dots, c_k$ are real numbers
  - $F(n)$ is a function not identically zero depending only n
  - $c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k}$ is called the **associated homogenous recurrence relation**

- <u>Example:</u>

$$a_n = 3a_{n-1} + 2n \qquad a_n = a_{n-1} + a_{n-2} + n^2 - n + 1 \qquad a_n = 2a_{n-1} + n3^n$$

Solving Linear NonHomogenous Recurrence Relations with constant coefficients

Let

$a_n^{(h)}$ be the general solution of $c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k}$

And

$a_n^{(p)}$ be the particular solution of LNHRRWCCs,

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k} + F(n)$$

Then the general solution of LNHRRWCCs is given by

$$a_n = a_n^{(h)} + a_n^{(p)}$$

There is no general method to find the particular solution.

THEOREM

Suppose that $\{a_n\}$ satisfy the linear nonhomogenous recurrence relation

$$a_n = c_1 a_{n-1} + c_k a_{n-k} + \ldots + F(n) \text{ where } c_1, c_{2,\ldots,} c_k \in \Re$$

And

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots b_1 n + b_0) s^n \text{ where } b_0, b_1, \ldots, b_t, s \in \Re$$

(i)When s is **not a root** of the characteristic equation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \ldots p_1 n + p_0) s^n$$

(ii) When s is **a root** of the characteristic equation , there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \ldots p_1 n + p_0) s^n$$

Example 3.3.8: LNHRRWCCs

Find all the solutions of recurrence relation $a_n = 3a_{n-1} + 2n$

Solution:

1) Find the general solution of the associated linear homogenous (ALH) equation.
   - the ALH equation is given by $a_n = 3a_{n-1}$
   - thus, the general solution is $a_n^{(h)} = A \cdot 3^n$

2) Find the particular solution
   - Since $F(n) = 2n$ is polynomial with degree 1, then the trial solution linear function: $p_n = cn + d$ (c and d are constant)
   - The equation $a_n = 3a_{n-1} + 2n$ then become $cn + d = 3\left(c\left(n-1\right) + d\right) + 2n = 0$
   - Solve for c and d will gives $c = -1$ and $d = -3/2$
   - Thus, the particular solution is $a_n^{(p)} = -n - \frac{3}{2}$

3) Thus the unique solution to the recurrence relation is $a_n = A \cdot 3^n - n - \frac{3}{2}$

EXERCISE 3.3.3

1. What is the solution of the following recurrence relation

- $a_n = 5a_{n-1} - 6a_{n-2} + 8n^2 \quad ; \quad a_0 = 4, a_1 = 7$

- $a_n = 5a_{n-1} - 6a_{n-2} + 7^n \quad ; \quad a_0 = 4, a_1 = 7$

2. What form does a particular solution of linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ when

$$F(n) = 3^n, \quad F(n) = n3^n, \quad F(n) = n^2 2^n$$
$$F(n) = 4(n+1)3^n, \quad F(n) = (n^2+1)3^n$$