Number Theory and Cryptography

Chapter 4

Chapter Motivation

- ***Number theory*** is the part of mathematics devoted to the study of the integers and their properties.
- Key ideas in number theory include *divisibility* and the *primality* of integers.
- Representations of integers, including binary and hexadecimal representations, are part of number theory.
- Number theory has long been studied because of the beauty of its ideas, its accessibility, and its wealth of open questions.
- We'll use many ideas developed in Chapter 1 about proof methods and proof strategy in our exploration of number theory.
- Mathematicians have long considered number theory to be pure mathematics, but it has important applications to computer science and cryptography studied in Sections 4.5 and 4.6.

Chapter Summary

- Divisibility and Modular Arithmetic
- Integer Representations and Algorithms
- Primes and Greatest Common Divisors
- Solving Congruences
- Applications of Congruences
- Cryptography

Divisibility and Modular Arithmetic

Section 4.1

Section Summary

- Division
- Division Algorithm
- Modular Arithmetic

Division

Definition: If a and b are integers with $a \neq 0$, then *a divides b* if there exists an integer c such that $b = ac$.

- When a divides b we say that *a* is a *factor* or *divisor* of *b* and that *b* is a *multiple* of *a*.
- The notation $a \mid b$ denotes that a divides b.
- If $a \mid b$, then b/a is an integer.
- If a does not divide b, we write $a \nmid b$.

Example: Determine whether $3 \mid 7$ and whether $3 \mid 12$.

Properties of Divisibility

Theorem 1: Let a, b, and c be integers, where $a \neq 0$.

i. If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$;
ii. If $a \mid b$, then $a \mid bm$ for all integers m;
iii. If $a \mid b$ and $b \mid c$, then $a \mid c$.

Proof: *(i)* Suppose $a \mid b$ and $a \mid c$, then it follows that there are integers s and t with $b = as$ and $c = at$. Hence,

$b + c = as + at = a(s + t)$. Hence, $a \mid (b + c)$ ◄

Corollary: If a, b, and c are integers, where $a \neq 0$, such that $a \mid b$ and $a \mid c$, then $a \mid mb + nc$ whenever m and n are integers.

Can you show how it follows easily from from (ii) and (i) of Theorem 1?

Division Algorithm

- When an integer is divided by a positive integer, there is a quotient and a remainder. The statement below is traditionally called the "Division Algorithm," but is really a theorem.

Division Algorithm: If a is an integer and d a positive integer, then there are unique integers q and r, with $0 \leq r < d$, such that $a = dq + r$ (*proved in Section* 5.2).

- d is called the *divisor*.
- a is called the *dividend*.
- q is called the *quotient*.
- r is called the *remainder*.

Definitions of Functions **div** and **mod**

$q = a$ **div** d

$r = a$ **mod** d

Examples:

- What are the quotient and remainder when 101 is divided by 11?

  Solution: The quotient when 101 is divided by 11 is 9 = 101 **div** 11, and the remainder is 2 = 101 **mod** 11.

- What are the quotient and remainder when −11 is divided by 3?

  Solution: The quotient when −11 is divided by 3 is −4 = −11 **div** 3, and the remainder is 1 = −11 **mod** 3.

Congruence Relation

Definition: If a and b are integers and m is a positive integer, then *a is congruent to b modulo m* if m divides $a - b$.

- The notation $a \equiv b \pmod{m}$ says that a is congruent to b modulo m.
- We say that $a \equiv b \pmod{m}$ is a *congruence* and that m is its *modulus*.
- Two integers are congruent mod m if and only if they have the same remainder when divided by m.
- If a is not congruent to b modulo m, we write

$$a \not\equiv b \pmod{m}$$

Example: Determine whether 17 is congruent to 5 modulo 6 and whether 24 and 14 are congruent modulo 6.

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$$a \not\equiv b \pmod{m}$$

Example: Determine whether 17 is congruent to 5 modulo 6 and whether 24 and 14 are congruent modulo 6.

Solution:

- $17 \equiv 5 \pmod{6}$ because 6 divides $17 - 5 = 12$.
- $24 \not\equiv 14 \pmod{6}$ since 6 divides $24 - 14 = 10$ is not divisible by 6.

More on Congruences

Theorem 4: Let m be a positive integer. The integers a and b are congruent modulo m if and only if there is an integer k such that $a = b + km$.

Proof:

- If $a \equiv b \pmod{m}$, then (by the definition of congruence) $m \mid a - b$. Hence, there is an integer k such that $a - b = km$ and equivalently $a = b + km$.
- Conversely, if there is an integer k such that $a = b + km$, then $km = a - b$. Hence, $m \mid a - b$ and $a \equiv b \pmod{m}$. ◀

The Relationship between (mod m) and **mod** m Notations

- The use of "mod" in $a \equiv b \pmod{m}$ is different from its use in $a \bmod m = b$.
  - $a \equiv b \pmod{m}$ is a relation on the set of integers.
  - In $a \bmod m = b$, the notation **mod** denotes a function.
- The relationship between these notations is made clear in this theorem.
- **Theorem 3**: Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod{m}$ if and only if $a \bmod m = b \bmod m$.

Congruences of Sums and Products

Theorem 5: Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then

$a + c \equiv b + d \pmod{m}$ and $ac \equiv bd \pmod{m}$

Proof:

- Because $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, by Theorem 4 there are integers s and t with $b = a + sm$ and $d = c + tm$.
- Therefore,
  - $b + d = (a + sm) + (c + tm) = (a + c) + m(s + t)$ and
  - $b\,d = (a + sm)(c + tm) = ac + m(at + cs + stm)$.
- Hence, $a + c \equiv b + d \pmod{m}$ and $ac \equiv bd \pmod{m}$. ◀

Example: Because $7 \equiv 2 \pmod{5}$ and $11 \equiv 1 \pmod{5}$, it follows from Theorem 5 that

$18 = 7 + 11 \equiv 2 + 1 = 3 \pmod{5}$

$77 = 7 \times 11 \equiv 2 \times 1 = 2 \pmod{5}$

Algebraic Manipulation of Congruences

- Multiplying both sides of a valid congruence by an integer preserves validity.

  If $a \equiv b \pmod{m}$ holds then $c \cdot a \equiv c \cdot b \pmod{m}$, where c is any integer, holds by Theorem 5 with $d = c$.

- Adding an integer to both sides of a valid congruence preserves validity.

  If $a \equiv b \pmod{m}$ holds then $c + a \equiv c + b \pmod{m}$, where c is any integer, holds by Theorem 5 with $d = c$.

- However, **dividing** a congruence by an integer **does not** always produce a valid congruence.

  Example: The congruence $14 \equiv 8 \pmod{6}$ holds. However, dividing both sides by 2 does not produce a valid congruence since $14/2 = 7$ and $8/2 = 4$, but $7 \not\equiv 4 \pmod{6}$.

Computing the **mod** *m* Function of Products and Sums

- We use the following corollary to Theorem 5 to compute the remainder of the product or sum of two integers when divided by *m* from the remainders when each is divided by *m*.

Corollary: Let *m* be a positive integer and let *a* and *b* be integers. Then

$$(a + b) \ (\mathbf{mod}\ m) = ((a \ \mathbf{mod}\ m) + (b \ \mathbf{mod}\ m)) \ \mathbf{mod}\ m$$

and

$$ab \ \mathbf{mod}\ m = ((a \ \mathbf{mod}\ m) \ (b \ \mathbf{mod}\ m)) \ \mathbf{mod}\ m.$$

(*proof in text*)

Arithmetic Modulo *m*

Definitions: Let $\mathbf{Z}_m$ be the set of nonnegative integers less than m: $\mathbf{Z}_m = \{0, 1, \ldots, m-1\}$

- The operation $+_m$ is defined as $a +_m b = (a + b)$ **mod** m. This is *addition modulo m*.
- The operation $\cdot_m$ is defined as $a \cdot_m b = (a \times b)$ **mod** m. This is *multiplication modulo m*.
- Using these operations is said to be doing *arithmetic modulo m*.

Example: Find $7 +_{11} 9$ and $7 \cdot_{11} 9$.

Solution: Using the definitions above:

- $7 +_{11} 9 = (7 + 9)$ **mod** $11 = 16$ **mod** $11 = 5$
- $7 \cdot_{11} 9 = (7 \cdot 9)$ **mod** $11 = 63$ **mod** $11 = 8$

Arithmetic Modulo *m*

- The operations $+_m$ and $\cdot_m$ satisfy many of the same properties as ordinary addition and multiplication.
  - *Closure*: If a and b belong to $\mathbf{Z}_m$, then $a +_m b$ and $a \cdot_m b$ belong to $\mathbf{Z}_m$.
  - *Associativity*: If a, b, and c belong to $\mathbf{Z}_m$, then $(a +_m b) +_m c = a +_m (b +_m c)$ and $(a \cdot_m b) \cdot_m c = a \cdot_m (b \cdot_m c)$.
  - *Commutativity*: If a and b belong to $\mathbf{Z}_m$, then $a +_m b = b +_m a$ and $a \cdot_m b = b \cdot_m a$.
  - *Identity elements*: The elements 0 and 1 are identity elements for addition and multiplication modulo m, respectively.
    - If a belongs to $\mathbf{Z}_m$, then $a +_m 0 = a$ and $a \cdot_m 1 = a$.

continued →

Arithmetic Modulo *m*

- *Additive inverses*: If $a \neq 0$ belongs to $\mathbf{Z}_m$, then $m - a$ is the additive inverse of a modulo m, and o is its own additive inverse.
  - $a +_m (m - a) = 0$ and $0 +_m 0 = 0$
- *Distributivity*: If a, b, and c belong to $\mathbf{Z}_m$, then
  - $a \cdot_m (b +_m c) = (a \cdot_m b) +_m (a \cdot_m c)$ and $(a +_m b) \cdot_m c = (a \cdot_m c) +_m (b \cdot_m c)$.
- **Multiplicative inverses** have not been included since they **do not always exist.** For example, there is no multiplicative inverse of 2 modulo 6.

Integer Representations and Algorithms

Section 4.2

Section Summary

- Integer Representations
  - Base *b* Expansions
  - Binary Expansions
  - Octal Expansions
  - Hexadecimal Expansions
- Base Conversion Algorithm
- Algorithms for Integer Operations

Representations of Integers

- In the modern world, we use *decimal,* or *base* 10, *notation* to represent integers. For example when we write 965, we mean $9 \cdot 10^2 + 6 \cdot 10^1 + 5 \cdot 10^0$.
- We can represent numbers using any base *b*, where *b* is a positive integer greater than 1.
- The bases *b* = 2 (*binary*), *b* = 8 (*octal*) , and *b*= 16 (*hexadecimal*) are important for computing and communications.

HANDS AND FEET

* Tribes from Papua New Guinea have at least 900 different counting systems

* Many tribes count past their fingers, so they don't use base 10

* One tribe counts toes after fingers, giving them a base 20 system

* The word for 10 is *two hands*

* 15 is *two hands and one foot*

* 20 is *one man*

(Next colour figures from *"Go Figure: A totally cool book about numbers"*, by Johnny Ball)

Head and Shoulders

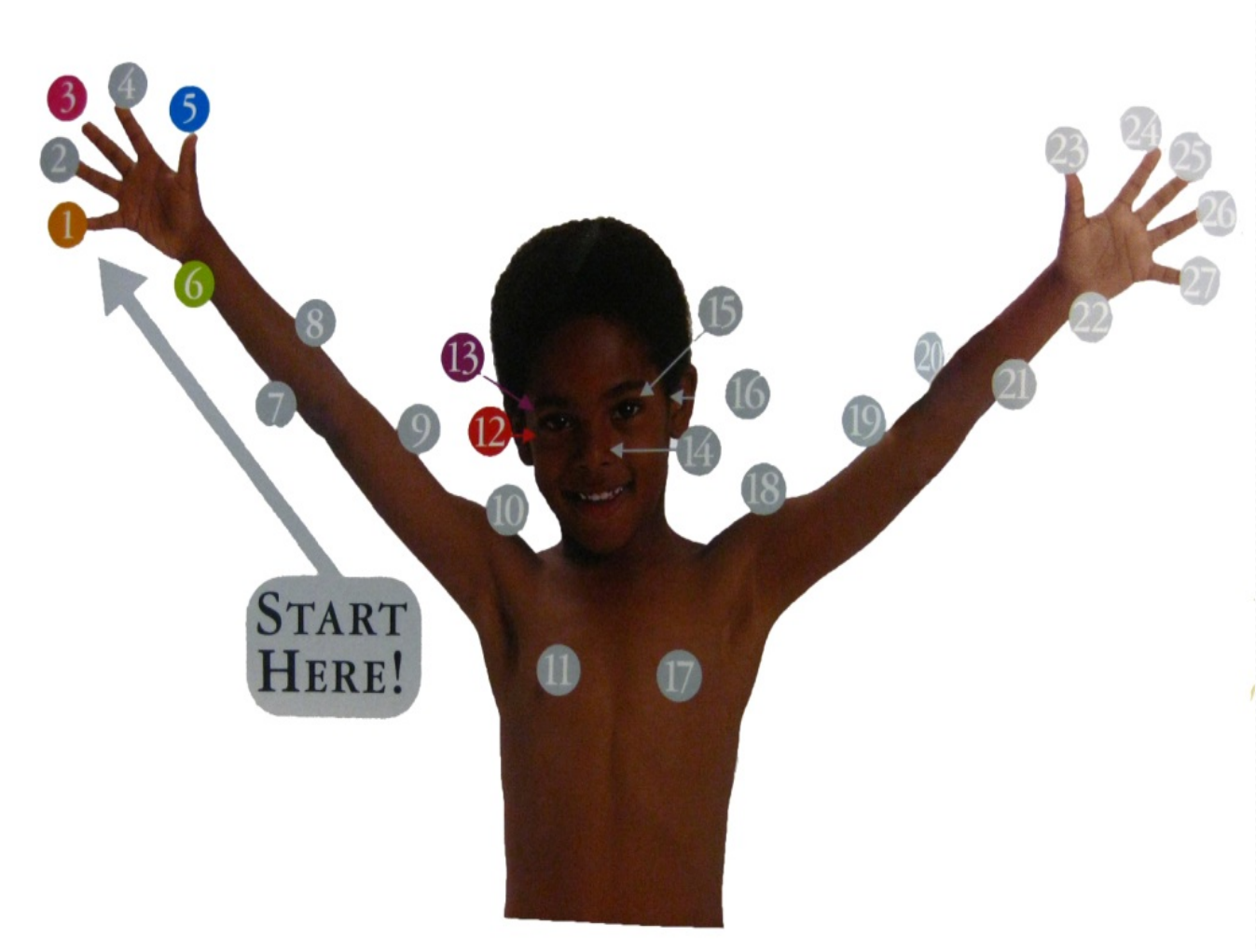
* In some parts of Papua New Guinea, people start counting on the little finger and then cross the arm, body, and other arm

* The Faiwol tribe counts 27 body parts as numbers

* word for 14 is *nose*

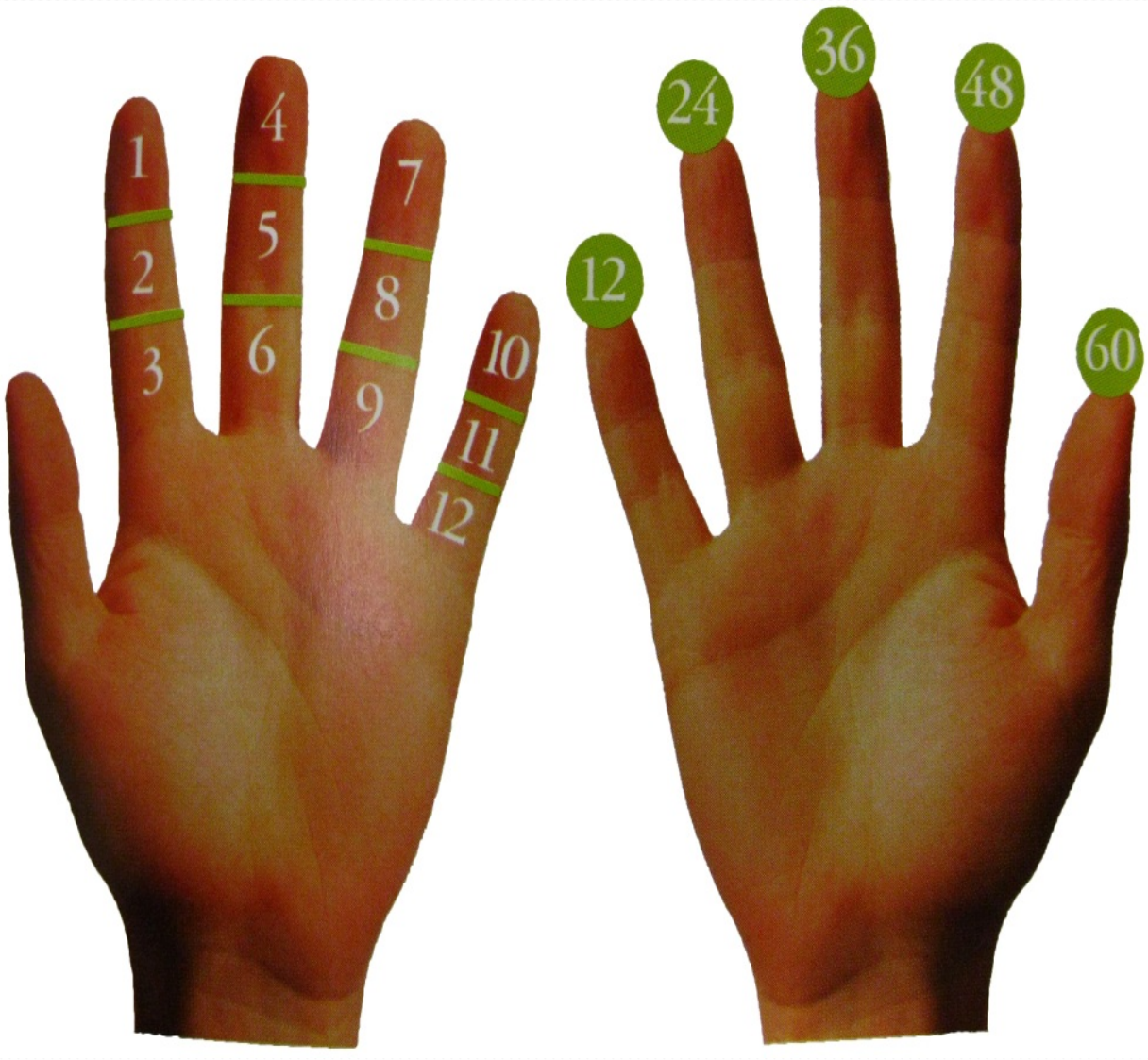
* for numbers > 27, they add *one man*

* 40 would be *one man and right eye*



BABYLONIANS

- Lived in present day Irak, 6,000 years ago
- Counted in base 60
- Babylonians invented minutes and seconds, which we still count in sixties today

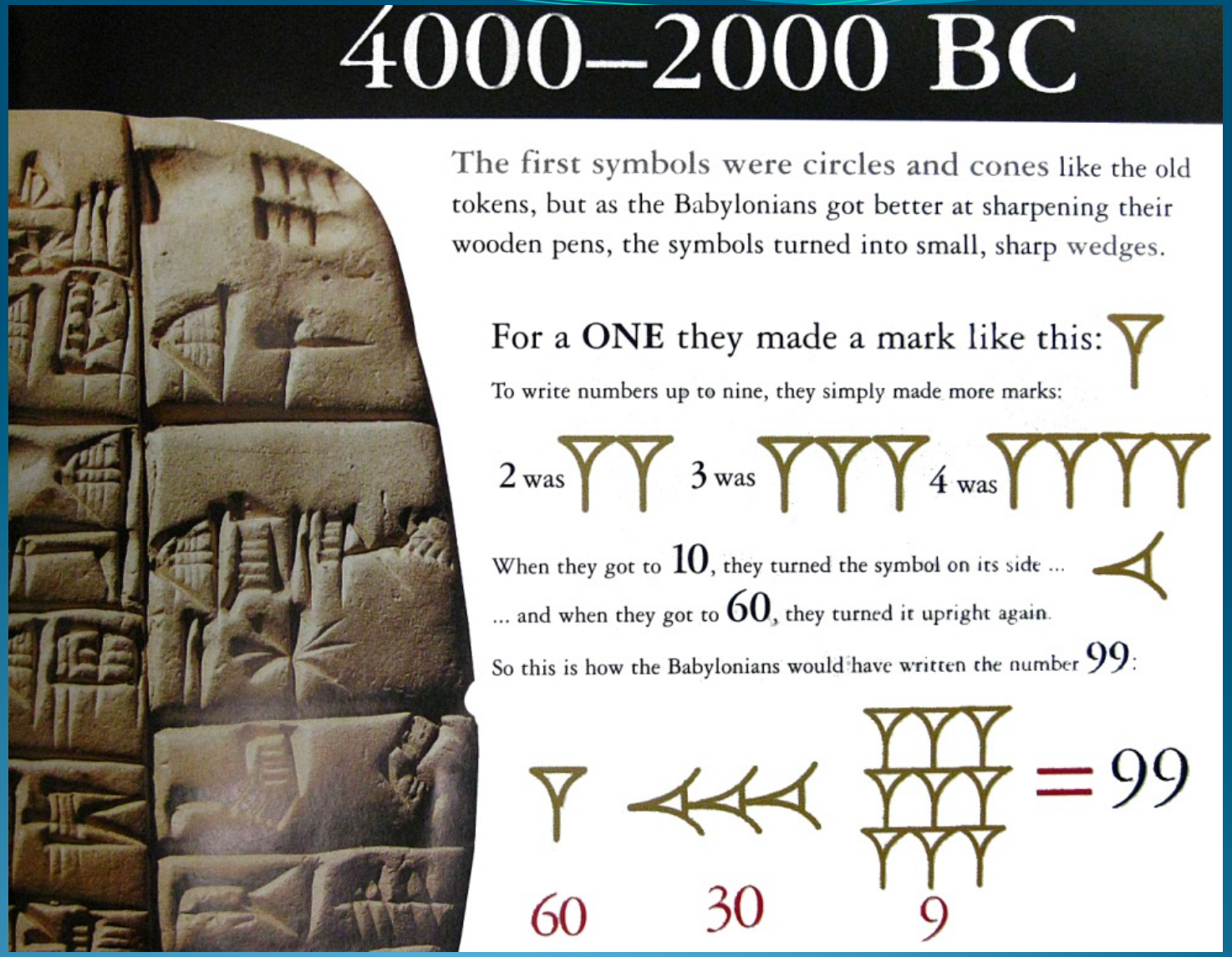


Babylonians

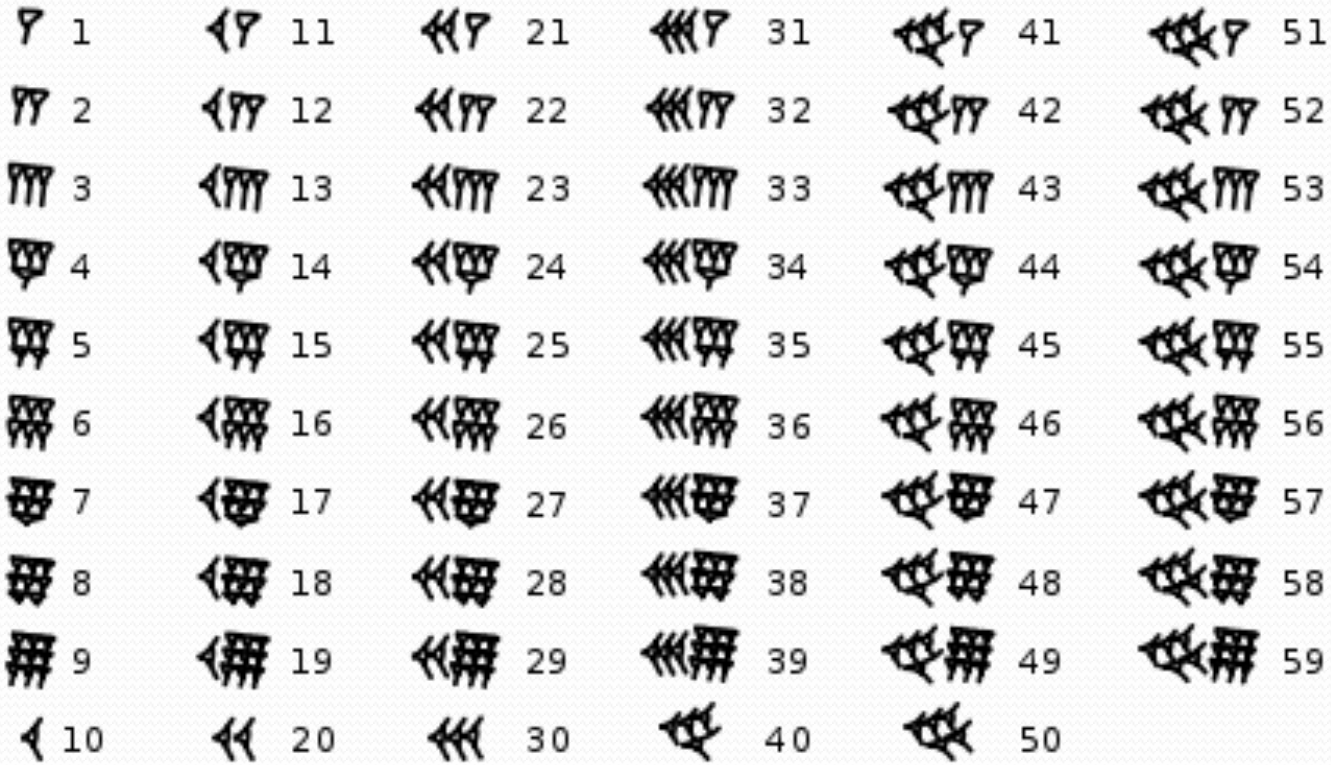
* First they used tokens to count (oval = sack of wheat, circle = oil jar)

*Several tokens were wrapped together in a clay envelope, marked on the outside with a stick

* Then clay tables marked with wooden pens were used, not bothering with tokens anymore

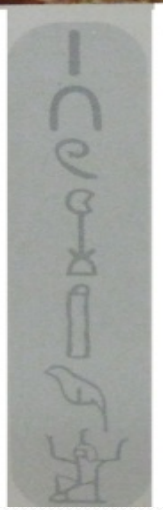


Babylonian Numerals: Base 60



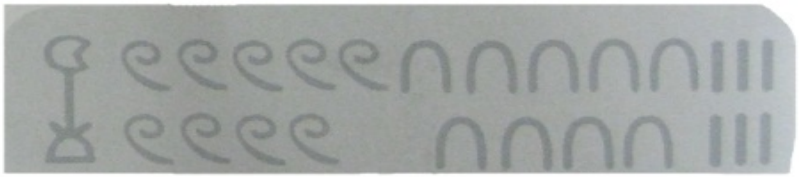
EGYPTIAN *numbers*

Egyptians counted in base 10 and wrote numbers as little pictures, or "hieroglyphs." Simple lines stood for 1, 10, and 100. For 1,000 they drew a lotus flower; 10,000 was a finger; 100,000 was a frog; and a million was a god.



1
10
100
1000
10,000
100,000
1,000,000

The hieroglyphics were stacked up in piles to create bigger numbers. This is how the Egyptians wrote 1,996:



While hieroglyphics were carved in stone, a different system was used for writing on paper.

MAYAN *numbers*

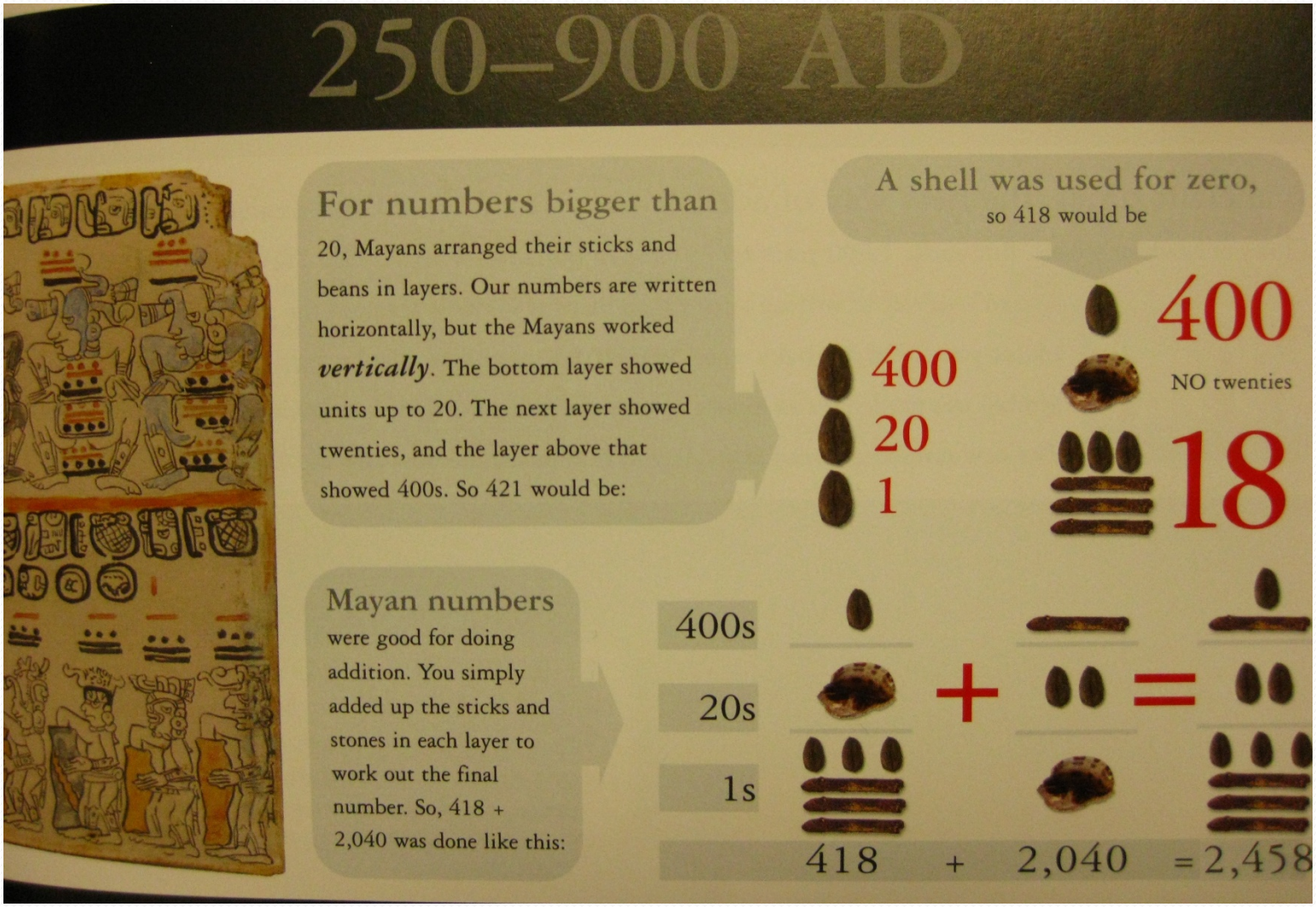
Native Americans also discovered farming and invented ways of writing numbers. The Mayans had a number system even better than that of the Egyptians. They kept perfect track of the date and calculated that a year is 365.242 days long. They counted in twenties, perhaps using toes as well as fingers. Their numbers look like beans, sticks, and shells—objects they may once have used like an abacus.

1
2
3
4
5

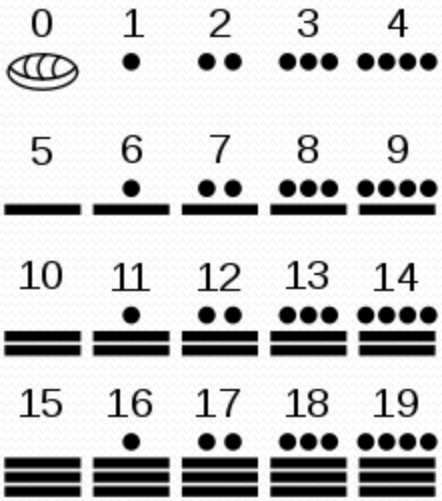
The symbols for 1–4 looked like cocoa beans or pebbles. The symbol for 5 looked like a stick.

The sticks and beans were piled up in groups to make numbers up to 20, so 18 would be:

Mayan Numbers



Mayan numerals: Base 20



Roman numbers (500BC – 1,500 AD)

ROMAN *numbers*

Roman numbers spread across Europe during the Roman empire. The Romans counted in tens and used letters as numerals. For Europeans, this was the main way of writing numbers for 2,000 years. We still see Roman numbers today in clocks, the names of royalty (like Queen Elizabeth II), and books with paragraphs numbered (i), (ii), and (iii).

Like most counting systems, Roman numbers start off as a tally:

1 is I 2 is II 3 is III

Different letters are then used for bigger numerals:

V	X	L	C	D	M
5	10	50	100	500	1000

* Indian numbers : 200 BC to now
* They invented the place system, - a way of writing the numbers so that the symbols matched the rows on the abacus
* A symbol was needed for the empty row, so the Indians invented zero
* The numbers spread to Asia and became the numbers we use today

Other bases

- Most languages with both numerals and counting use bases 8, 10, 12, or 20.
- Base 10 (*decimal*) --comes from counting one's fingers
- Base 20 (*vigesimal*) comes from the fingers and toes
- Base 8 (*octal*) comes from counting the spaces between the fingers
- Base 12 (*duodecimal*) comes from counting the knuckles (3 each for the four fingers)
- Base 60 (*sexagesimal*) appears to come from a combination of base 10 and base 12 – origin of modern degrees, minutes and seconds
- No base (use body parts to count) Example: 1-4 fingers, , 5 'thumb', 6 'wrist', 7 'elbow', 8 'shoulder', etc., across the body and down the other arm, opposite pinkie is 17

Base *b* Representations

- We can use positive integer *b* greater than 1 as a base, because of this theorem:

  Theorem 1: Let *b* be a positive integer greater than 1. Then if *n* is a positive integer, it can be expressed uniquely in the form:

  $$n = a_k b^k + a_{k-1} b^{k-1} + \ldots + a_1 b + a_0$$

  where *k* is a nonnegative integer, $a_0, a_1, \ldots a_k$ are nonnegative integers less than *b*, and $a_k \neq 0$. The a_j, $j = 0,\ldots,k$ are called the base-*b* digits of the representation.

  (We can prove this using mathematical induction in Section 5.1.)

- The representation of *n* given in Theorem 1 is called the *base b expansion of n* and is denoted by $(a_k a_{k-1} \ldots a_1 a_0)_b$.
- We usually omit the subscript 10 for base 10 expansions.

Binary Expansions

Most computers represent integers and do arithmetic with binary (base 2) expansions of integers. In these expansions, the only digits used are 0 and 1.

Example: What is the decimal expansion of the integer that has $(1\ 0101\ 1111)_2$ as its binary expansion?

Example: What is the decimal expansion of the integer that has $(11011)_2$ as its binary expansion?

Binary Expansions

Most computers represent integers and do arithmetic with binary (base 2) expansions of integers. In these expansions, the only digits used are 0 and 1.

Example: What is the decimal expansion of the integer that has $(1\ 0101\ 1111)_2$ as its binary expansion?

Solution:

$(1\ 0101\ 1111)_2 = 1\cdot 2^8 + 0\cdot 2^7 + 1\cdot 2^6 + 0\cdot 2^5 + 1\cdot 2^4 + 1\cdot 2^3 + 1\cdot 2^2 + 1\cdot 2^1 + 1\cdot 2^0 = 351.$

Example: What is the decimal expansion of the integer that has $(11011)_2$ as its binary expansion?

Solution: $(11011)_2 = 1\cdot 2^4 + 1\cdot 2^3 + 0\cdot 2^2 + 1\cdot 2^1 + 1\cdot 2^0 = 27.$

Octal Expansions

The octal expansion (base 8) uses the digits {0,1,2,3,4,5,6,7}.

Example: What is the decimal expansion of the number with octal expansion $(7016)_8$?

Example: What is the decimal expansion of the number with octal expansion $(111)_8$?

Octal Expansions

The octal expansion (base 8) uses the digits {0,1,2,3,4,5,6,7}.

Example: What is the decimal expansion of the number with octal expansion $(7016)_8$?

Solution: $7 \cdot 8^3 + 0 \cdot 8^2 + 1 \cdot 8^1 + 6 \cdot 8^0 = 3598$

Example: What is the decimal expansion of the number with octal expansion $(111)_8$?

Solution: $1 \cdot 8^2 + 1 \cdot 8^1 + 1 \cdot 8^0 = 64 + 8 + 1 = 73$

Hexadecimal Expansions

The hexadecimal expansion needs 16 digits, but our decimal system provides only 10. So letters are used for the additional symbols. The hexadecimal system uses the digits {0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F}. The letters A through F represent the decimal numbers 10 through 15.

Example: What is the decimal expansion of the number with hexadecimal expansion $(2AE0B)_{16}$?

Example: What is the decimal expansion of the number with hexadecimal expansion $(1E5)_{16}$?

Hexadecimal Expansions

The hexadecimal expansion needs 16 digits, but our decimal system provides only 10. So letters are used for the additional symbols. The hexadecimal system uses the digits {0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F}. The letters A through F represent the decimal numbers 10 through 15.

Example: What is the decimal expansion of the number with hexadecimal expansion $(2AE0B)_{16}$?

Solution:

$2 \cdot 16^4 + 10 \cdot 16^3 + 14 \cdot 16^2 + 0 \cdot 16^1 + 11 \cdot 16^0 = 175627$

Example: What is the decimal expansion of the number with hexadecimal expansion $(1E5)_{16}$?

Solution: $1 \cdot 16^2 + 14 \cdot 16^1 + 5 \cdot 16^0 = 256 + 224 + 5 = 485$

Base Conversion

To construct the base b expansion of an integer n (in base 10):

- Divide n by b to obtain a quotient and remainder.

  $n = bq_0 + a_0 \quad 0 \leq a_0 \leq b$
- The remainder, a_0, is the rightmost digit in the base b expansion of n. Next, divide q_0 by b.

  $q_0 = bq_1 + a_1 \quad 0 \leq a_1 \leq b$
- The remainder, a_1, is the second digit from the right in the base b expansion of n.
- Continue by successively dividing the quotients by b, obtaining the additional base b digits as the remainder. The process terminates when the quotient is 0.

continued →

Algorithm: Constructing Base *b* Expansions

procedure *base b expansion*(n, b: positive integers with $b > 1$)
$q := n$
$k := 0$
while ($q \neq 0$)
    $a_k := q$ **mod** b
    $q := q$ **div** b
    $k := k + 1$
return$(a_{k-1}, \ldots, a_1, a_0)$ {$(a_{k-1} \ldots a_1 a_0)_b$ is base b expansion of n}

- q represents the quotient obtained by successive divisions by b, starting with $q = n$.
- The digits in the base b expansion are the remainders of the division given by q **mod** b.
- The algorithm terminates when $q = 0$ is reached.

Base Conversion

Example: Find the octal expansion of $(12345)_{10}$

Solution: Successively dividing by 8 gives:

- $12345 = 8 \cdot 1543 + 1$
- $1543 = 8 \cdot 192 + 7$
- $192 = 8 \cdot 24 + 0$
- $24 = 8 \cdot 3 + 0$
- $3 = 8 \cdot 0 + 3$

The remainders are the digits from right to left yielding $(30071)_8$.

Comparison of Hexadecimal, Octal, and Binary Representations

TABLE 1 Hexadecimal, Octal, and Binary Representation of the Integers 0 through 15.

Decimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Hexadecimal	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Octal	0	1	2	3	4	5	6	7	10	11	12	13	14	15	16	17
Binary	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	1110	1111

Initial 0s are not shown

Each octal digit corresponds to a block of 3 binary digits.
Each hexadecimal digit corresponds to a block of 4 binary digits.
So, conversion between binary, octal, and hexadecimal is easy.

Conversion Between Binary, Octal, and Hexadecimal Expansions

Example: Find the octal and hexadecimal expansions of $(11\ 1110\ 1011\ 1100)_2$.

Solution:

- To convert to octal, we group the digits into blocks of three $(011\ 111\ 010\ 111\ 100)_2$, adding initial 0s as needed. The blocks from left to right correspond to the digits 3,7,2,7, and 4. Hence, the solution is $(37274)_8$.
- To convert to hexadecimal, we group the digits into blocks of four $(0011\ 1110\ 1011\ 1100)_2$, adding initial 0s as needed. The blocks from left to right correspond to the digits 3,E,B, and C. Hence, the solution is $(3EBC)_{16}$.

Binary Addition of Integers

- Algorithms for performing operations with integers using their binary expansions are important as computer chips work with binary numbers. Each digit is called a *bit*.

procedure *add*(a, b: positive integers)

{the binary expansions of a and b are $(a_{n-1},a_{n-2},\ldots,a_0)_2$ and $(b_{n-1},b_{n-2},\ldots,b_0)_2$, respectively}

$c := 0$

for $j := 0$ to $n - 1$

$d := \lfloor (a_j + b_j + c)/2 \rfloor$

$s_j := a_j + b_j + c - 2d$

$c := d$

$s_n := c$

return$(s_0,s_1,\ldots, s_n)$ {the binary expansion of the sum is $(s_n,s_{n-1},\ldots,s_0)_2$}

- The number of additions of bits used by the algorithm to add two n-bit integers is $O(n)$.

Binary Multiplication of Integers

- Algorithm for computing the product of two n bit integers.

procedure *multiply*(a, b: positive integers)
{the binary expansions of a and b are $(a_{n-1},a_{n-2},\ldots,a_0)_2$ and $(b_{n-1},b_{n-2},\ldots,b_0)_2$, respectively}
for $j := 0$ to $n - 1$
    if $b_j = 1$ **then** $c_j = a$ shifted j places
    else $c_j := 0$
{$c_0,c_1,\ldots, c_{n-1}$ are the partial products}
$p := 0$
for $j := 0$ to $n - 1$
  $p := p + c_j$
return p {p is the value of ab}

- The number of additions of bits used by the algorithm to multiply two n-bit integers is $O(n^2)$.

Primes and Greatest Common Divisors

Section 4.3

Section Summary

- Prime Numbers and their Properties
- Conjectures and Open Problems About Primes
- Greatest Common Divisors and Least Common Multiples
- The Euclidean Algorithm
- gcd as Linear Combination

Primes

Definition: A positive integer *p* greater than 1 is called *prime* if the only positive factors of *p* are 1 and *p*. A positive integer that is greater than 1 and is not prime is called *composite*.

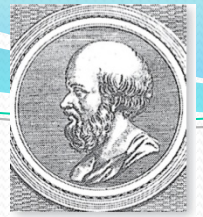
Example: The integer 7 is prime because its only positive factors are 1 and 7, but 9 is composite because it is divisible by 3.

The Fundamental Theorem of Arithmetic

Theorem: Every positive integer greater than 1 can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of nondecreasing size.

Examples:

- $100 = 2 \cdot 2 \cdot 5 \cdot 5 = 2^2 \cdot 5^2$
- $641 = 641$
- $999 = 3 \cdot 3 \cdot 3 \cdot 37 = 3^3 \cdot 37$
- $1024 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^{10}$



Erastothenes
(276-194 B.C.)

The Sieve of Erastosthenes

- The *Sieve of Erastosthenes* can be used to find all primes not exceeding a specified positive integer. For example, begin with the list of integers between 1 and 100.
  a. Delete all the integers, other than 2, divisible by 2.
  b. Delete all the integers, other than 3, divisible by 3.
  c. Next, delete all the integers, other than 5, divisible by 5.
  d. Next, delete all the integers, other than 7, divisible by 7.
  e. Since all the remaining integers are not divisible by any of the previous integers, other than 1, the primes are:

{2,3,7,11,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89, 97}

continued →

The Sieve of Erastosthenes

TABLE 1 The Sieve of Eratosthenes.

Integers divisible by 2 other than 2 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 3 other than 3 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 5 other than 5 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 7 other than 7 receive an underline; integers in color are prime.

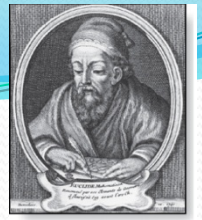
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

If an integer n is a composite integer, then it has a prime divisor less than or equal to $\sqrt{n}$.

To see this, note that if $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Trial division, a very inefficient method of determining if a number n is prime, is to try every integer $i \leq \sqrt{n}$ and see if n is divisible by i.

Infinitude of Primes



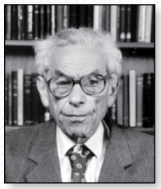
Euclid
(325 B.C.E. – 265 B.C.E.)

Theorem: There are infinitely many primes. (Euclid)

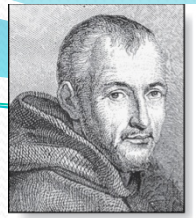
Proof: By contradiction. Assume there are finitely many primes: $p_1, p_2, \ldots, p_n$

- Let $q = p_1 p_2 \cdots p_n + 1$
- Either q is prime or by the fundamental theorem of arithmetic it is a product of primes.
  - But none of the primes p_j divides q since if $p_j \mid q$, then p_j divides $q - p_1 p_2 \cdots p_n = 1$.
  - Hence, there is a prime not on the list $p_1, p_2, \ldots, p_n$. It is either q, or if q is composite, it is a prime factor of q. This contradicts the assumption that $p_1, p_2, \ldots, p_n$ are all the primes.
- Consequently, there are infinitely many primes.

This proof was given by Euclid *The Elements*. The proof is considered to be one of the most beautiful in all mathematics. It is the first proof in *The Book*, inspired by the famous mathematician Paul Erdős' imagined collection of perfect proofs maintained by God.



Paul Erdős
(1913-1996)



Marin Mersenne
(1588-1648)

Mersenne Primes (optional)

Definition: Prime numbers of the form $2^p - 1$, where p is prime, are called *Mersenne primes*.

- $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 31$, and $2^7 - 1 = 127$ are Mersenne primes.
- $2^{11} - 1 = 2047$ is not a Mersenne prime since $2047 = 23 \cdot 89$.
- There is an efficient test for determining if $2^p - 1$ is prime.
- The largest known prime numbers are Mersenne primes.
- As of mid 2011, 47 Mersenne primes were known, the largest is $2^{43,112,609} - 1$, which has nearly 13 million decimal digits.
- The *Great Internet Mersenne Prime Search* (*GIMPS*) is a distributed computing project to search for new Mersenne Primes.

http://www.mersenne.org/

Generating Primes (optional)

- The problem of generating large primes is of both theoretical and practical interest.
- Finding large primes with hundreds of digits is important in cryptography.
- So far, no useful closed formula that always produces primes has been found. There is no simple function $f(n)$ such that $f(n)$ is prime for all positive integers n.
- $f(n) = n^2 - n + 41$ is prime for all integers 1,2,..., 40. Because of this, we might conjecture that $f(n)$ is prime for all positive integers n. But $f(41) = 41^2$ is not prime.
- More generally, there is no polynomial with integer coefficients such that $f(n)$ is prime for all positive integers n.
- Fortunately, we can generate large integers which are almost certainly primes.

Conjectures about Primes (optional)

- Even though primes have been studied extensively for centuries, many conjectures about them are unresolved, including:
- *Goldbach's Conjecture*: Every even integer n, $n > 2$, is the sum of two primes. It has been verified by computer for all positive even integers up to $1.6 \cdot 10^{18}$. The conjecture is believed to be true by most mathematicians.
- There are infinitely many primes of the form $n^2 + 1$, where n is a positive integer. So far, it has been shown that there are infinitely many numbers of the form $n^2 + 1$ which are either a prime or a product of two primes (where n is a positive integer).
- *The Twin Prime Conjecture*: There are infinitely many pairs of twin primes. Twin primes are pairs of primes that differ by 2. Examples are 3 and 5, 5 and 7, 11 and 13, etc. The current world's record for twin primes (as of mid 2011) consists of numbers $65{,}516{,}468{,}355 \cdot 23^{33{,}333} \pm 1$, which have 100,355 decimal digits.

Greatest Common Divisor

Definition: Let a and b be integers, not both zero. The largest integer d such that $d \mid a$ and also $d \mid b$ is called the greatest common divisor of a and b. The greatest common divisor of a and b is denoted by $\gcd(a,b)$.

One can find greatest common divisors of small numbers by inspection.

Example:What is the greatest common divisor of 24 and 36?

Solution: gcd(24,36) = 12

Example:What is the greatest common divisor of 17 and 22?

Solution: gcd(17,22) = 1

Greatest Common Divisor

Definition: The integers a and b are *relatively prime* if their greatest common divisor is 1.

Example: 17 and 22

Definition: The integers $a_1, a_2, \ldots, a_n$ are *pairwise relatively prime* if $\gcd(a_i, a_j) = 1$ whenever $1 \leq i < j \leq n$.

Example: Determine whether the integers 10, 17 and 21 are pairwise relatively prime.

Example: Determine whether the integers 10, 19, and 24 are pairwise relatively prime.

Greatest Common Divisor

Definition: The integers a and b are *relatively prime* if their greatest common divisor is 1.

Example: 17 and 22

Definition: The integers $a_1, a_2, \ldots, a_n$ are *pairwise relatively prime* if $\gcd(a_i, a_j) = 1$ whenever $1 \leq i < j \leq n$.

Example: Determine whether the integers 10, 17 and 21 are pairwise relatively prime.

Solution: Because $\gcd(10,17) = 1$, $\gcd(10,21) = 1$, and $\gcd(17,21) = 1$, 10, 17, and 21 are pairwise relatively prime.

Example: Determine whether the integers 10, 19, and 24 are pairwise relatively prime.

Solution: Because $\gcd(10,24) = 2$, 10, 19, and 24 are not pairwise relatively prime.

Finding the Greatest Common Divisor Using Prime Factorizations

- Suppose the prime factorizations of a and b are:

$$a = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n} , \qquad b = p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n} ,$$

where each exponent is a nonnegative integer, and where all primes occurring in either prime factorization are included in both. Then:

$$\gcd(a, b) = p_1^{\min(a_1, b_1)} p_2^{\min(a_2, b_2)} \cdots p_n^{\min(a_n, b_n)} .$$

- This formula is valid since the integer on the right (of the equals sign) divides both a and b. No larger integer can divide both a and b.

Example: $120 = 2^3 \cdot 3 \cdot 5$ $\quad 500 = 2^2 \cdot 5^3$

$\gcd(120,500) = 2^{\min(3,2)} \cdot 3^{\min(1,0)} \cdot 5^{\min(1,3)} = 2^2 \cdot 3^0 \cdot 5^1 = 20$

- Finding the gcd of two positive integers using their prime factorizations is not efficient because there is no efficient algorithm for finding the prime factorization of a positive integer.

Least Common Multiple

Definition: The least common multiple of the positive integers a and b is the smallest positive integer that is divisible by both a and b. It is denoted by $\text{lcm}(a,b)$.

- The least common multiple can also be computed from the prime factorizations.

$$\text{lcm}(a,b) = p_1^{\max(a_1,b_1)} p_2^{\max(a_2,b_2)} \cdots p_n^{\max(a_n,b_n)}$$

This number is divided by both a and b and no smaller number is divided by a and b.

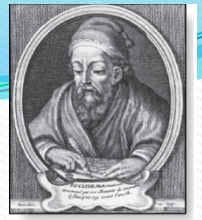
Example: $\text{lcm}(2^3 3^5 7^2, 2^4 3^3) = 2^{\max(3,4)}\, 3^{\max(5,3)}\, 7^{\max(2,0)} = 2^4\, 3^5\, 7^2$

- The greatest common divisor and the least common multiple of two integers are related by:

Theorem 5: Let a and b be positive integers. Then

$$ab = \gcd(a,b) \cdot \text{lcm}(a,b)$$

Euclidean Algorithm



Euclid
(325 B.C.E. – 265 B.C.E.)

- The Euclidean algorithm is an efficient method for computing the greatest common divisor of two integers. It is based on the idea that, if $a > b$ and r is the remainder when a is divided by b, then $\gcd(a,b)$ is equal to $\gcd(b,r)$.

Example: Find $\gcd(91, 287)$:

- $287 = 91 \cdot 3 + 14$ — Divide 287 by 91
- $91 = 14 \cdot 6 + 7$ — Divide 91 by 14
- $14 = 7 \cdot 2 + 0$ — Divide 14 by 7

Stopping condition

$$\gcd(287, 91) = \gcd(91, 14) = \gcd(14, 7) = 7$$

continued →

Euclidean Algorithm

- The Euclidean algorithm expressed in pseudocode is:

```
procedure gcd(a, b: positive integers)
x := a
y := b
while   y ≠ 0
      r := x mod y
      x := y
      y := r
return x {gcd(a,b) is x}
```

- Note: The time complexity of the algorithm is $O(\log b)$, where $a > b$.

Correctness of Euclidean Algorithm

Lemma 1: Let $a = bq + r$, where a, b, q, and r are integers. Then $\gcd(a,b) = \gcd(b,r)$.

Proof:

- Suppose that d divides both a and b. Then d also divides $a - bq = r$ (by Theorem 1 of Section 4.1). Hence, any common divisor of a and b must also be any common divisor of b and r.
- Suppose that d divides both b and r. Then d also divides $bq + r = a$. Hence, any common divisor of a and b must also be a common divisor of b and r.
- Therefore, $\gcd(a,b) = \gcd(b,r)$. ◀

Correctness of Euclidean Algorithm

- Suppose that a and b are positive integers with $a \geq b$.
  Let $r_0 = a$ and $r_1 = b$.
  Successive applications of the division algorithm yields:

$$
\begin{aligned}
r_0 &= r_1 q_1 + r_2 && 0 \leq r_2 < r_1, \\
r_1 &= r_2 q_2 + r_3 && 0 \leq r_3 < r_2, \\
&\vdots \\
r_{n-2} &= r_{n-1} q_{n-1} + r_2 && 0 \leq r_n < r_{n-1}, \\
r_{n-1} &= r_n q_n . &&
\end{aligned}
$$

- Eventually, a remainder of zero occurs in the sequence of terms: $a = r_0 > r_1 > r_2 > \cdots \geq 0$. The sequence can't contain more than a terms.
- By Lemma 1 $\gcd(a,b) = \gcd(r_0,r_1) = \cdots = \gcd(r_{n-1},r_n) = \gcd(r_n, 0) = r_n$.
- Hence the greatest common divisor is the last nonzero remainder in the sequence of divisions.

Étienne Bézout
(1730-1783)

gcds as Linear Combinations

Bézout's Theorem: If a and b are positive integers, then there exist integers s and t such that $\gcd(a,b) = sa + tb$.

Definition: If a and b are positive integers, then integers s and t such that $\gcd(a,b) = sa + tb$ are called *Bézout coefficients* of a and b. The equation $\gcd(a,b) = sa + tb$ is called *Bézout's identity*.

- By Bézout's Theorem, the gcd of integers a and b can be expressed in the form $sa + tb$ where s and t are integers. This is a *linear combination* with integer coefficients of a and b.
  - $\gcd(6,14) = (-2)\cdot 6 + 1\cdot 14$

Finding gcds as Linear Combinations

Example: Express gcd(252,198) = 18 as a linear combination of 252 and 198.

Solution: First use the Euclidean algorithm to show gcd(252,198) = 18

i. $252 = 1 \cdot 198 + 54$
ii. $198 = 3 \cdot 54 + 36$
iii. $54 = 1 \cdot 36 + 18$
iv. $36 = 2 \cdot 18$

- Now working backwards, from iii and i above
  - $18 = 54 - 1 \cdot 36$
  - $36 = 198 - 3 \cdot 54$
- Substituting the 2nd equation into the 1st yields:
  - $18 = 54 - 1 \cdot (198 - 3 \cdot 54) = 4 \cdot 54 - 1 \cdot 198$
- Substituting $54 = 252 - 1 \cdot 198$ (from i)) yields:
  - $18 = 4 \cdot (252 - 1 \cdot 198) - 1 \cdot 198 = 4 \cdot 252 - 5 \cdot 198$

- This method illustrated above is a two pass method. It first uses the Euclidean algorithm to find the gcd and then works backwards to express the gcd as a linear combination of the original two integers.

Consequence of Bézout's Theorem

Lemma 2: If a, b, and c are positive integers such that $\gcd(a, b) = 1$ and $a \mid bc$, then $a \mid c$.

Proof: Assume $\gcd(a, b) = 1$ and $a \mid bc$

- Since $\gcd(a, b) = 1$, by Bézout's Theorem there are integers s and t such that

$$sa + tb = 1.$$

- Multiplying both sides of the equation by c gives $sac + tbc = c$.
- From Theorem 1 of Section 4.1:

  $a \mid tbc$ (part ii) and a divides $sac + tbc$ since $a \mid sac$ and $a \mid tbc$ (part i)
- We conclude $a \mid c$, since $sac + tbc = c$. ◄

Dividing Congruences by an Integer

- Dividing both sides of a valid congruence by an integer does not always produce a valid congruence (Sect.4.1).
- But dividing by an integer relatively prime to the modulus does produce a valid congruence:

Theorem 7: Let m be a positive integer and let a, b, and c be integers. If $ac \equiv bc \pmod{m}$ and $\gcd(c,m) = 1$, then $a \equiv b \pmod{m}$.

Proof: Since $ac \equiv bc \pmod{m}$, $m \mid ac - bc = c(a - b)$ by Lemma 2 and the fact that $\gcd(c,m) = 1$, it follows that $m \mid a - b$. Hence, $a \equiv b \pmod{m}$. ◀

Solving Congruences

Section 4.4

Linear Congruences

Definition: A congruence of the form

$$ax \equiv b \pmod{m},$$

where m is a positive integer, a and b are integers, and x is a variable, is called a *linear congruence*.

- The solutions to a linear congruence $ax \equiv b \pmod{m}$ are all integers x that satisfy the congruence.

Definition: An integer $\bar{a}$ such that $\bar{a}a \equiv 1 \pmod{m}$ is said to be an *inverse* of a modulo m.

Example: 5 is an inverse of 3 modulo 7 since $5 \cdot 3 = 15 \equiv 1 \pmod{7}$

- One method of solving linear congruences makes use of an inverse $\bar{a}$, if it exists. Although we can not divide both sides of the congruence by a, we can multiply by $\bar{a}$ to solve for x.

Inverse of a modulo m

- The following theorem guarantees that an inverse of a modulo m exists whenever a and m are relatively prime. Two integers a and b are relatively prime when $\gcd(a,b) = 1$.

Theorem 1: If a and m are relatively prime integers and $m > 1$, then an inverse of a modulo m exists. Furthermore, this inverse is unique modulo m. (This means that there is a unique positive integer $\bar{a}$ less than m that is an inverse of a modulo m and every other inverse of a modulo m is congruent to $\bar{a}$ modulo m.)

Proof: Since $\gcd(a,m) = 1$, by Theorem 6 of Section 4.3, there are integers s and t such that $sa + tm = 1$.

- Hence, $sa + tm \equiv 1 \pmod{m}$.
- Since $tm \equiv 0 \pmod{m}$, it follows that $sa \equiv 1 \pmod{m}$
- Consequently, s is an inverse of a modulo m.
- The uniqueness of the inverse is Exercise 7.

◀

Finding Inverses

- The Euclidean algorithm and Bézout coefficients gives us a systematic approaches to finding inverses.

  Example: Find an inverse of 3 modulo 7.

  Solution: Because gcd(3,7) = 1, by Theorem 1, an inverse of 3 modulo 7 exists.

  - Using the Euclidean algorithm: $7 = 2 \cdot 3 + 1$.
  - From this equation, we get $-2 \cdot 3 + 1 \cdot 7 = 1$, and see that -2 and 1 are Bézout coefficients of 3 and 7.
  - Hence, -2 is an inverse of 3 modulo 7.
  - Also every integer congruent to -2 modulo 7 is an inverse of 3 modulo 7, i.e., 5, -9, 12, etc.

Finding Inverses

Example: Find an inverse of 101 modulo 4620.

Finding Inverses

Example: Find an inverse of 101 modulo 4620.

Solution: First use the Euclidean algorithm to show that gcd(101,4620) = 1.

$4620 = 45 \cdot 101 + 75$
$101 = 1 \cdot 75 + 26$
$75 = 2 \cdot 26 + 23$
$26 = 1 \cdot 23 + 3$
$23 = 7 \cdot 3 + 2$
$3 = 1 \cdot 2 + 1$
$2 = 2 \cdot 1$

Working Backwards:

$1 = 3 - 1 \cdot 2$
$1 = 3 - 1 \cdot (23 - 7 \cdot 3) = -1 \cdot 23 + 8 \cdot 3$
$1 = -1 \cdot 23 + 8 \cdot (26 - 1 \cdot 23) = 8 \cdot 26 - 9 \cdot 23$
$1 = 8 \cdot 26 - 9 \cdot (75 - 2 \cdot 26) = 26 \cdot 26 - 9 \cdot 75$
$1 = 26 \cdot (101 - 1 \cdot 75) - 9 \cdot 75$
$= 26 \cdot 101 - 35 \cdot 75$
$1 = 26 \cdot 101 - 35 \cdot (4620 - 45 \cdot 101)$
$= -35 \cdot 4620 + 1601 \cdot 101$

Since the last nonzero remainder is 1, gcd(101,4260) = 1

Bézout coefficients : – 35 and 1601

1601 is an inverse of 101 modulo 42620

Using Inverses to Solve Congruences

- We can solve the congruence $ax \equiv b \pmod{m}$ by multiplying both sides by $\bar{a}$.

  Example: What are the solutions of the congruence $3x \equiv 4 \pmod{7}$.

Using Inverses to Solve Congruences

- We can solve the congruence $ax \equiv b \pmod{m}$ by multiplying both sides by $\bar{a}$.

Example: What are the solutions of the congruence $3x \equiv 4 \pmod{7}$.

Solution: We found that -2 is an inverse of 3 modulo 7 (two slides back). We multiply both sides of the congruence by -2 giving

$$-2 \cdot 3x \equiv -2 \cdot 4 \pmod{7}.$$

Because $-6 \equiv 1 \pmod{7}$ and $-8 \equiv 6 \pmod{7}$, it follows that if x is a solution, then $x \equiv -8 \equiv 6 \pmod{7}$

We need to determine if every x with $x \equiv 6 \pmod{7}$ is a solution. Assume that $x \equiv 6 \pmod{7}$. By Theorem 5 of Section 4.1, it follows that $3x \equiv 3 \cdot 6 = 18 \equiv 4 \pmod{7}$ which shows that all such x satisfy the congruence.

The solutions are the integers x such that $x \equiv 6 \pmod{7}$, namely, 6,13,20 … and $-1, -8, -15, \ldots$

Applications of Congruences

Section 4.5

Section Summary

- Hashing Functions
- Pseudorandom Numbers
- Check Digits

Hashing Functions

Definition: A *hashing function h* assigns memory location $h(k)$ to the record that has k as its key.

- A common hashing function is $h(k) = k$ **mod** m, where m is the number of memory locations.
- Because this hashing function is onto, all memory locations are possible.

Example: Let $h(k) = k$ **mod** 111. This hashing function assigns the records of customers with social security numbers as keys to memory locations in the following manner:

h(064212848) = 064212848 **mod** 111 = 14

h(037149212) = 037149212 **mod** 111 = 65

h(107405723) = 107405723 **mod** 111 = 14, but since location 14 is already occupied, the record is assigned to the next available position, which is 15.

- The hashing function is not one-to-one as there are many more possible keys than memory locations. When more than one record is assigned to the same location, we say a *collision* occurs. Here a collision has been resolved by assigning the record to the first free location.
- For collision resolution, we can use a *linear probing function* to find the first free memory location: $h(k,i) = (h(k) + i)$ **mod** m, where i runs from 0 to $m - 1$.
- There are many other methods of handling with collisions. You may cover these in a later CS course.

Pseudorandom Numbers

- Randomly chosen numbers are needed for many purposes, including computer simulations.
- *Pseudorandom numbers* are not truly random since they are generated by systematic methods.
- The *linear congruential method* is one commonly used procedure for generating pseudorandom numbers.
- Four integers are needed: the *modulus* m, the *multiplier* a, the *increment* c, and *seed* x_0, with $2 \leq a < m$, $0 \leq c < m$, $0 \leq x_0 < m$.
- We generate a sequence of pseudorandom numbers $\{x_n\}$, with

$$x_{n+1} = (ax_n + c) \bmod m.$$

$0 \leq x_n < m$ for all n, by successively using the recursively defined function

Pseudorandom Numbers

- **Example**: Find the sequence of pseudorandom numbers generated by the linear congruential method with modulus $m = 9$, multiplier $a = 7$, increment $c = 4$, and seed $x_0 = 3$.
- **Solution**: Compute the terms of the sequence by successively using the congruence $x_{n+1} = (7x_n + 4) \bmod 9$, with $x_0 = 3$.

$x_1 = 7x_0 + 4 \bmod 9 = 7 \cdot 3 + 4 \bmod 9 = 25 \bmod 9 = 7,$
$x_2 = 7x_1 + 4 \bmod 9 = 7 \cdot 7 + 4 \bmod 9 = 53 \bmod 9 = 8,$
$x_3 = 7x_2 + 4 \bmod 9 = 7 \cdot 8 + 4 \bmod 9 = 60 \bmod 9 = 6,$
$x_4 = 7x_3 + 4 \bmod 9 = 7 \cdot 6 + 4 \bmod 9 = 46 \bmod 9 = 1,$
$x_5 = 7x_4 + 4 \bmod 9 = 7 \cdot 1 + 4 \bmod 9 = 11 \bmod 9 = 2,$
$x_6 = 7x_5 + 4 \bmod 9 = 7 \cdot 2 + 4 \bmod 9 = 18 \bmod 9 = 0,$
$x_7 = 7x_6 + 4 \bmod 9 = 7 \cdot 0 + 4 \bmod 9 = 4 \bmod 9 = 4,$
$x_8 = 7x_7 + 4 \bmod 9 = 7 \cdot 4 + 4 \bmod 9 = 32 \bmod 9 = 5,$
$x_9 = 7x_8 + 4 \bmod 9 = 7 \cdot 5 + 4 \bmod 9 = 39 \bmod 9 = 3.$

The sequence generated is 3,7,8,6,1,2,0,4,5,3,7,8,6,1,2,0,4,5,3,...

It repeats after generating 9 terms.

- Commonly, computers use a linear congruential generator with increment $c = 0$. This is called a *pure multiplicative generator*. Such a generator with modulus $2^{31} - 1$ and multiplier $7^5 = 16{,}807$ generates $2^{31} - 2$ numbers before repeating.

Check Digits: UPCs

- A common method of detecting errors in strings of digits is to add an extra digit at the end, which is evaluated using a function. If the final digit is not correct, then the string is assumed not to be correct.

  Example: Retail products are identified by their *Universal Product Codes* (*UPC*s). Usually these have 12 decimal digits, the last one being the check digit. The check digit is determined by the congruence

  $$3x_1 + x_2 + 3x_3 + x_4 + 3x_5 + x_6 + 3x_7 + x_8 + 3x_9 + x_{10} + 3x_{11} + x_{12} \equiv 0 \pmod{10}.$$

  a. Suppose that the first 11 digits of the UPC are 79357343104. What is the check digit?
  b. Is 041331021641 a valid UPC?

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a. Suppose that the first 11 digits of the UPC are 79357343104. What is the check digit?

b. Is 041331021641 a valid UPC?

Solution:

a. $3\cdot 7 + 9 + 3\cdot 3 + 5 + 3\cdot 7 + 3 + 3\cdot 4 + 3 + 3\cdot 1 + 0 + 3\cdot 4 + x_{12} \equiv 0 \pmod{10}$

$21 + 9 + 9 + 5 + 21 + 3 + 12 + 3 + 3 + 0 + 12 + x_{12} \equiv 0 \pmod{10}$

$98 + x_{12} \equiv 0 \pmod{10}$

$x_{12} \equiv 2 \pmod{10}$ So, the check digit is 2.

b. $3\cdot 0 + 4 + 3\cdot 1 + 3 + 3\cdot 3 + 1 + 3\cdot 0 + 2 + 3\cdot 1 + 6 + 3\cdot 4 + 1 \equiv 0 \pmod{10}$

$0 + 4 + 3 + 3 + 9 + 1 + 0 + 2 + 3 + 6 + 12 + 1 = 44 \equiv 4 \not\equiv 0 \pmod{10}$

Hence, 041331021641 is not a valid UPC.

Check Digits:ISBNs

Books are identified by an *International Standard Book Number* (ISBN-10), a 10 digit code. The first 9 digits identify the language, the publisher, and the book. The tenth digit is a check digit, which is determined by the following congruence

$$x_{10} \equiv \sum_{i=1}^{9} i x_i \pmod{11}.$$

The validity of an ISBN-10 number can be evaluated with the equivalent

$$\sum_{i=1}^{10} i x_i \equiv 0 \pmod{11}.$$

X is used for the digit 10.

a. Suppose that the first 9 digits of the ISBN-10 are 007288008. What is the check digit?

b. Is 084930149X a valid ISBN10?

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a. Suppose that the first 9 digits of the ISBN-10 are 007288008. What is the check digit?

b. Is 084930149X a valid ISBN10?

Solution:

a. $X_{10} \equiv 1\cdot 0 + 2\cdot 0 + 3\cdot 7 + 4\cdot 2 + 5\cdot 8 + 6\cdot 8 + 7\cdot 0 + 8\cdot 0 + 9\cdot 8 \pmod{11}$.
$X_{10} \equiv 0 + 0 + 21 + 8 + 40 + 48 + 0 + 0 + 72 \pmod{11}$.
$X_{10} \equiv 189 \equiv 2 \pmod{11}$. Hence, $X_{10} = 2$.

b. $1\cdot 0 + 2\cdot 8 + 3\cdot 4 + 4\cdot 9 + 5\cdot 3 + 6\cdot 0 + 7\cdot 1 + 8\cdot 4 + 9\cdot 9 + 10\cdot 10 =$
$0 + 16 + 12 + 36 + 15 + 0 + 7 + 32 + 81 + 100 = 299 \equiv 2 \not\equiv 0 \pmod{11}$
Hence, 084930149X is not a valid ISBN-10.

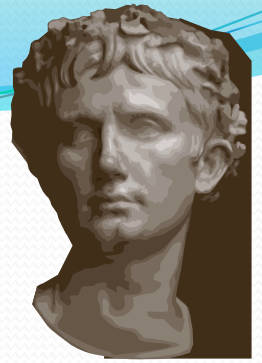
X is used for the digit 10.

- A *single error* is an error in one digit of an identification number and a *transposition error* is the accidental interchanging of two digits. Both of these kinds of errors can be detected by the check digit for ISBN-10.

Cryptography

Section 4.6

Caesar Cipher



Julius Caesar created secret messages by shifting each letter three letters forward in the alphabet (sending the last three letters to the first three letters.) For example, the letter B is replaced by E and the letter X is replaced by A. This process of making a message secret is an example of *encryption*.

Here is how this encryption process works:

- Replace each letter by an integer from $\mathbf{Z}_{26}$, that is an integer from 0 to 25 representing one less than its position in the alphabet.
- The encryption function is $f(p) = (p + 3) \bmod 26$. It replaces each integer p in the set $\{0,1,2,\ldots,25\}$ by $f(p)$ in the set $\{0,1,2,\ldots,25\}$.
- Replace each integer p by the letter with the position $p + 1$ in the alphabet.

Example: Encrypt the message "MEET YOU IN THE PARK" using the Caesar cipher.

Solution: Plaintext is 12 4 4 19 24 14 20 8 13 19 7 4 15 0 17 10.

Now replace each of these numbers p by $f(p) = (p + 3) \bmod 26$.

15 7 7 22 1 17 23 11 16 22 10 7 18 3 20 13.

Translating the numbers back to letters produces the encrypted message

"PHHW BRX LQ WKH SDUN" (called also cryptotext)

Caesar Cipher

- To recover the original message, use $f^{-1}(c) = (c-3) \bmod 26$. So, each letter in the coded message is shifted back three letters in the alphabet, with the first three letters sent to the last three letters. This process of recovering the original message from the encrypted message is called *decryption*.
- The Caesar cipher is one of a family of ciphers called *shift ciphers*. Letters can be shifted by an integer k, with 3 being just one possibility. The encryption function is

  $$f(p) = (p + k) \bmod 26$$

  and the decryption function is

  $$f^{-1}(c) = (c-k) \bmod 26$$

  The integer k is called a *key*.

Shift Cipher

Example 1: Encrypt the message "STOP GLOBAL WARMING" using the shift cipher with $k = 11$.

Shift Cipher

Example 1: Encrypt the message "STOP GLOBAL WARMING" using the shift cipher with $k = 11$.

Solution: Replace each letter with the corresponding element of $\mathbf{Z}_{26}$.

18 19 14 15 6 11 14 1 0 11 22 0 17 12 8 13 6.

Apply the shift $f(p) = (p + 11)$ **mod** 26, yielding

3 4 25 0 17 22 25 12 11 22 7 11 2 23 19 24 17.

Translating the numbers back to letters produces the ciphertext

"DEZA RWZMLW HLCXTYR."

Shift Cipher

Example 2: Decrypt the message "LEWLYPLUJL PZ H NYLHA ALHJOLY" that was encrypted using the shift cipher with $k = 7$.

Shift Cipher

Example 2: Decrypt the message "LEWLYPLUJL PZ H NYLHA ALHJOLY" that was encrypted using the shift cipher with $k = 7$.

Solution: Replace each letter with the corresponding element of $\mathbf{Z}_{26}$.

11 4 22 11 24 15 11 20 9 11 15 25 7 13 24 11 7 0 0 11 7 9 14 11 24.

Shift each of the numbers by $-k = -7$ modulo 26, yielding

4 23 15 4 17 8 4 13 2 4 8 18 0 6 17 4 0 19 19 4 0 2 7 4 17.

Translating the numbers back to letters produces the decrypted message

"EXPERIENCE IS A GREAT TEACHER."

Affine Ciphers

- We can generalize shift ciphers further to slightly enhance security by using a function of the form

$$f(p) = (ap + b) \bmod 26$$

where a and b are integers chosen so that f is a bijection.

Such a mapping is called an affine transformation, and the resulting cipher is called an *affine cipher*.

Example. What letter replaces K when the function $f(p) = (7p + 3) \bmod 26$ is used for encryption?

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Example. What letter replaces K when the function

$f(p) = (7p + 3) \bmod 26$ is used for encryption?

Solution: K represents the number 10, f(10) = (7 x 10+3) mod 26 = 21, which represents the letter V.

Affine Ciphers

To decrypt messages encrypted using an affine cipher

$c = (ap + b) \bmod 26$ (where gcd $(a, 26) = 1$,

we need to find p *(plaintext)* in terms of c *(cryptotext)*.

To do this, we solve the congruence (with unknown p)

$$c \equiv (ap + b) \bmod 26$$

i) Subtract b from both sides to obtain

$$(c - b) \equiv ap \bmod 26$$

ii) Multiply both sides by the inverse a' of a mod 26

$$p \equiv a'(c - b) \bmod 26$$

Example

- What is the decryption function for an affine cipher if the encryption function is

$$f(x) = (3x+7) \bmod 26 \ ?$$

Decrypt the message

"UTTQ CTOA"

that was encrypted using the above affine cipher.

Example (Solution)

- The decryption function is $f(x) = 9x + 15$
- The plain text is NEED HELP