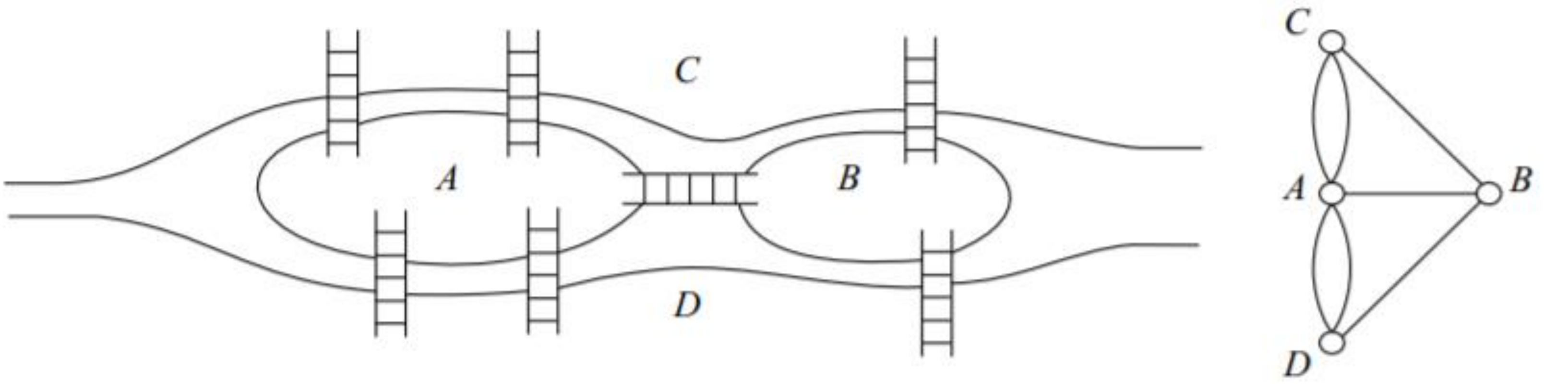
CHAPTER FOUR

ELEMENTS OF GRAPH THEORY

* Graph theory is a branch of mathematics that deals with arrangements of certain objects and relationships between these objects.



Konigsberg Bridges

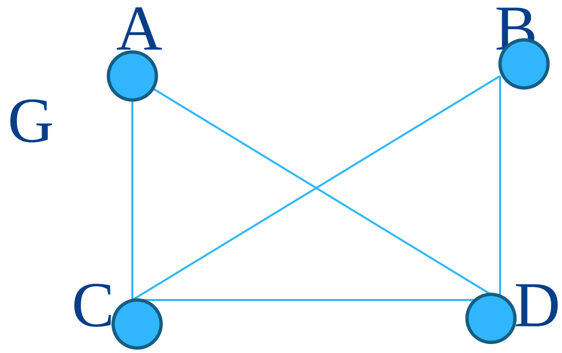
Underlying Graph

* Two islands A and B formed by the Pregal river (now Pregolya) in Konigsberg (then the capital of east Prussia, but now renamed Kaliningrad and in west Soviet Russia) were connected to each other and to the banks C and D with seven bridges

* Is it possible to walk along a route that cross **each bridge exactly once?**

Definition and examples

- **Definition:** A *graph G* = (*V*, *E*) consists of a nonempty set *V* of *vertices* (or *nodes*) and a set *E* of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to *connect* its endpoints.
- Example:-



G is a graph with G(V,E) where

i) V={A,B,C,D}

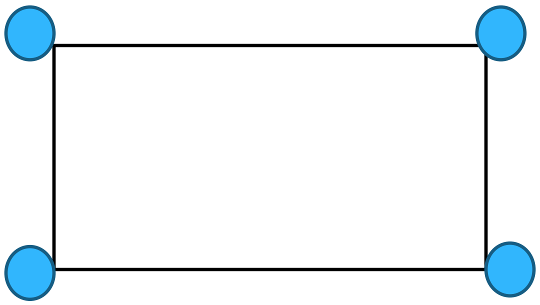
ii) E={AC,AD,BC,BD}

➢ **Note:-** An edge should not pass through more than two vertices.

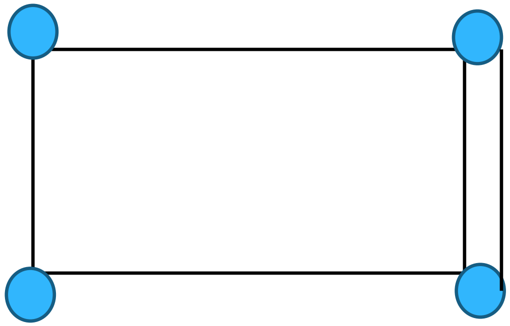
Terminology

* In a simple graph each edge connects two different vertices and no two edges connect the same pair of vertices.
* Multi graphs may have multiple edges connecting the same two vertices. When m different edges connect the vertices u and v, we say that $\{u, v\}$ is an edge of multiplicity m.
* An edge that connects a vertex to itself is called a loop.
* A pseudo graph may include loops, as well as multiple edges Connecting the same pair of vertices.

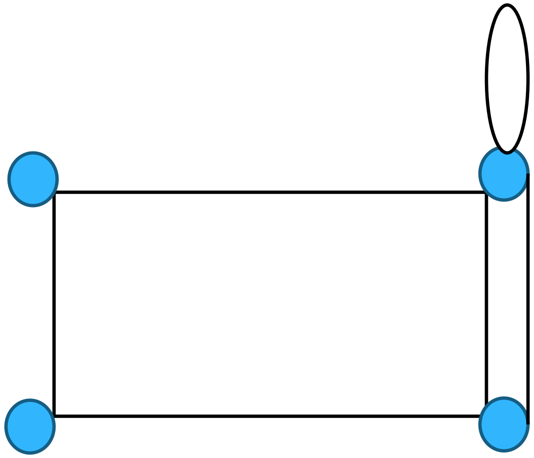
Example:-



Simple graph



multi graph



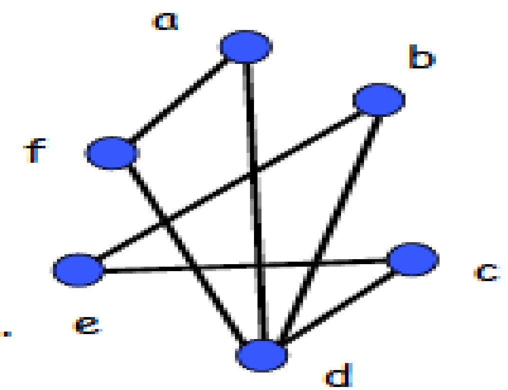
pseudo graph

Cont.

Definition:- The degree of a vertex in a undirected graph is the number of edges incident with it, except that a loop at a vertex contributes two to the degree of that vertex. The degree of the vertex v is denoted by deg(v).

Example:-

An edge uv is *incident* on the vertex u and the vertex v.



The *neighbour set* N(v) of a vertex v is the set of vertices adjacent to it.

e.g. N(a) = {d,f}, N(d) = {a,b,c,f}, N(e) = {b,c}.

degree of a vertex = # of *incident* edges

e.g. deg(d) = 4, deg(a)=deg(b)=deg(c)=deg(e)=deg(f)=2.

the degree of a vertex v = the number of neighbours of v?

For multigraphs, NO. For simple graphs, YES.

Cont.

Definition: -

i) A vertex is said to be even if its degree is even number and odd if its degree is odd number.

ii) A vertex of degree zero is called **isolated** vertex.

iii) A vertex of degree one is called **pendant** vertex.

Theorem (Handshaking Theorem)

If $G = (V, E)$ is an undirected graph with m edges, then

$$2m = \sum_{v \in V} \deg(v)$$

Proof: Each edge contributes twice to the degree count of all vertices. Hence, both the left-hand and right-hand sides of this equation equal twice the number of edges.

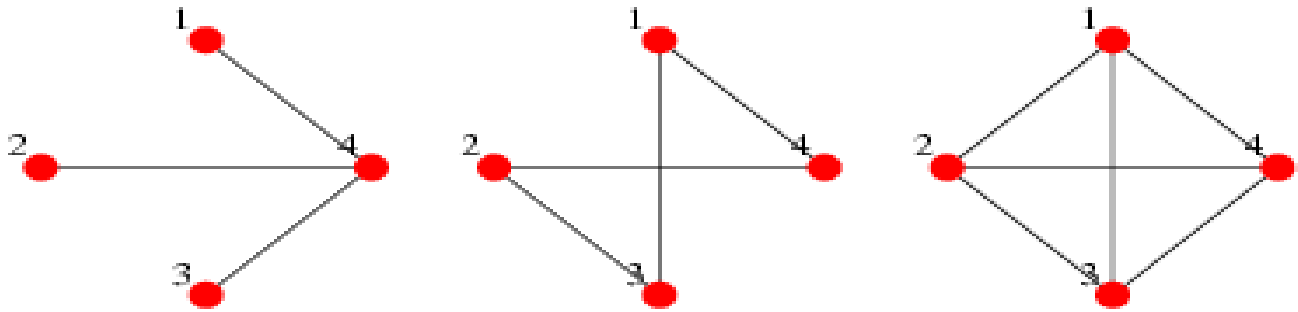
4.2 matrix representation of graph

Adjacency matrix

Definition:- suppose G is a graph with m vertices and suppose the vertices have been ordered, say $v_1, v_2, \dots v_m$. Then the adjacency matrix $A = [a_{ij}]$ of the graph G is the $m \times n$ matrix define by

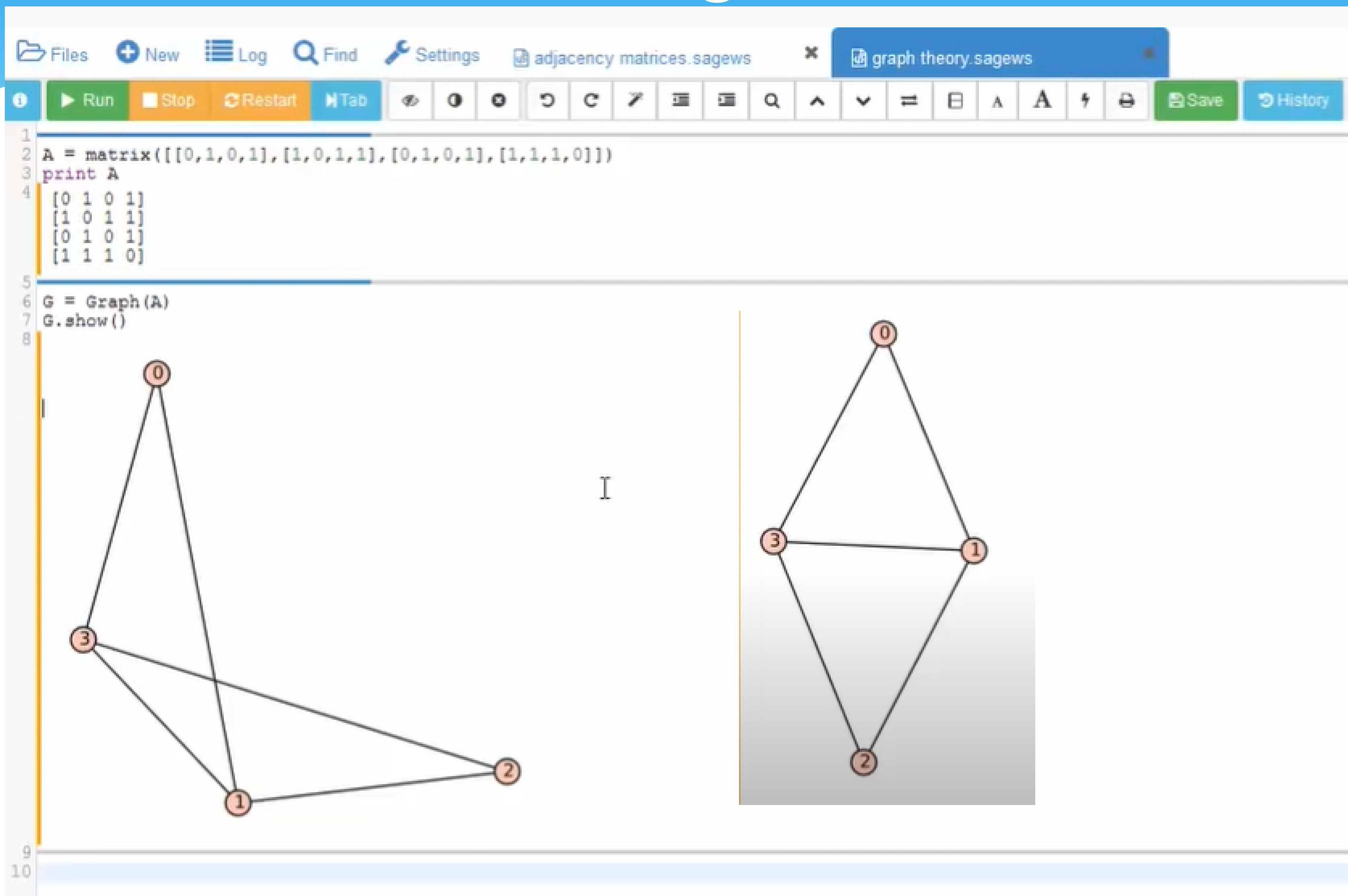
$$a_{ij} = \begin{cases} 1, & if\ ij\ is\ an\ edge\ joining\ v_i\ and\ v_j \\ & 0, otherwise \end{cases}$$

Example :-



$$\begin{array}{cc} & \text{Vertex} \\ \text{Vertex} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array} \qquad \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

4.2 Matrix representation of graph using Sage



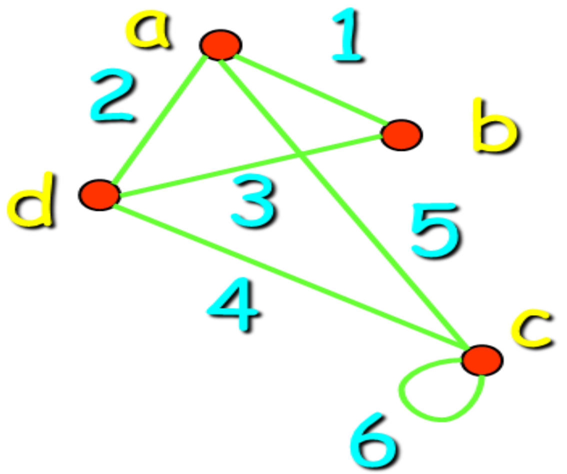
Cont.

Incidence matrix

Definition:-Suppose G is a graph with vertices $v_1, v_2, \dots v_n$. The incidence matrix $I = [b_{ij}]$of the graph G is given by:

$$b_{ij} = \begin{cases} 1, if\ e_j\ is\ incident\ on\ v_i \\ \qquad 0, otherwise \end{cases}$$

Example:-



Edges

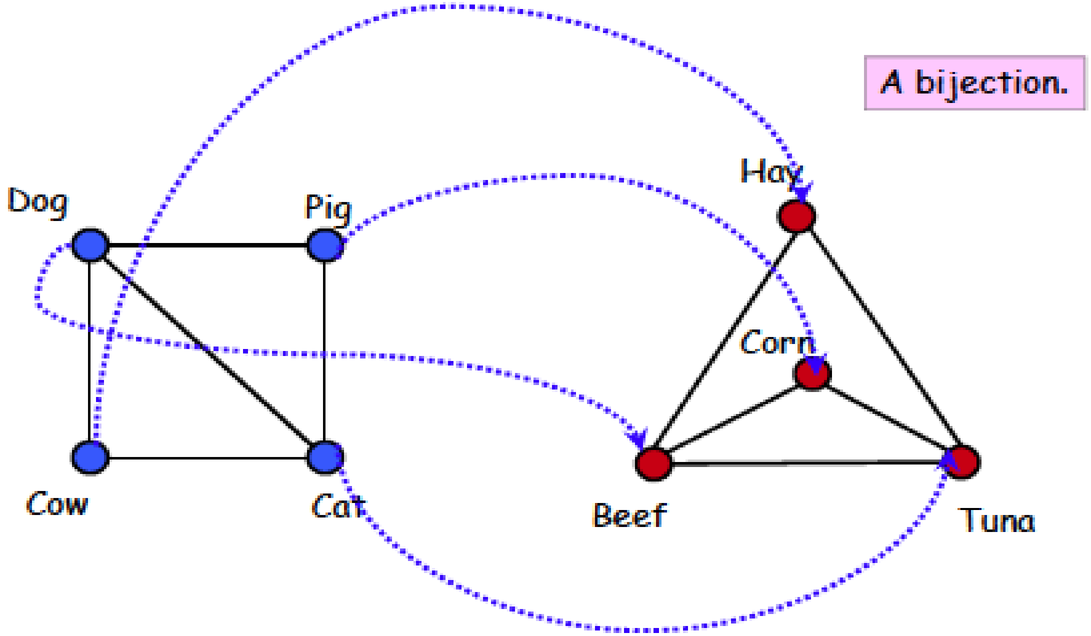
$$M = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Vertex

4.3 isomorphic of graphs

Definition :- The simple graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ are isomorphic if there is an injective (one-to-one) and surjective (onto) function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if f (a) and f (b) are adjacent in G_2, for all a and b in V_1. Such a function f is called an isomorphism.

Example :-



f(Dog) = Beef

f(Cat) = Tuna

f(Cow) = Hay

f(Pig) = Corn

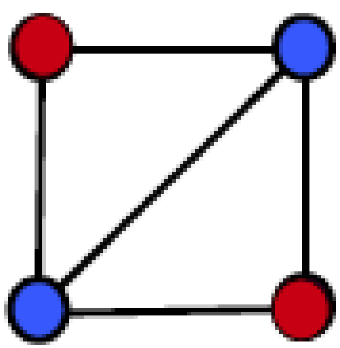
Cont.

How to show two graphs are isomorphic?

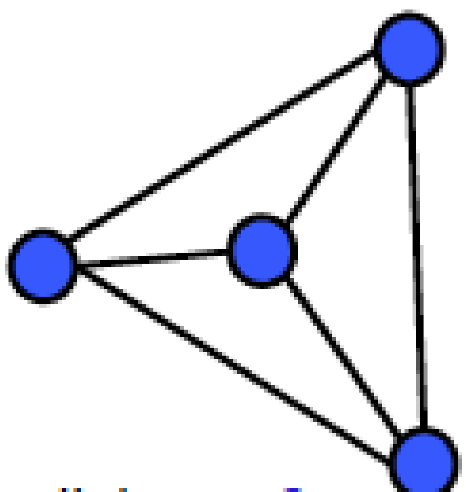
Find a mapping and show that it is edge-preserving.

How to show two graphs are non-isomorphic?

Find some isomorphic-preserving properties which is satisfied in one graph but not the other.



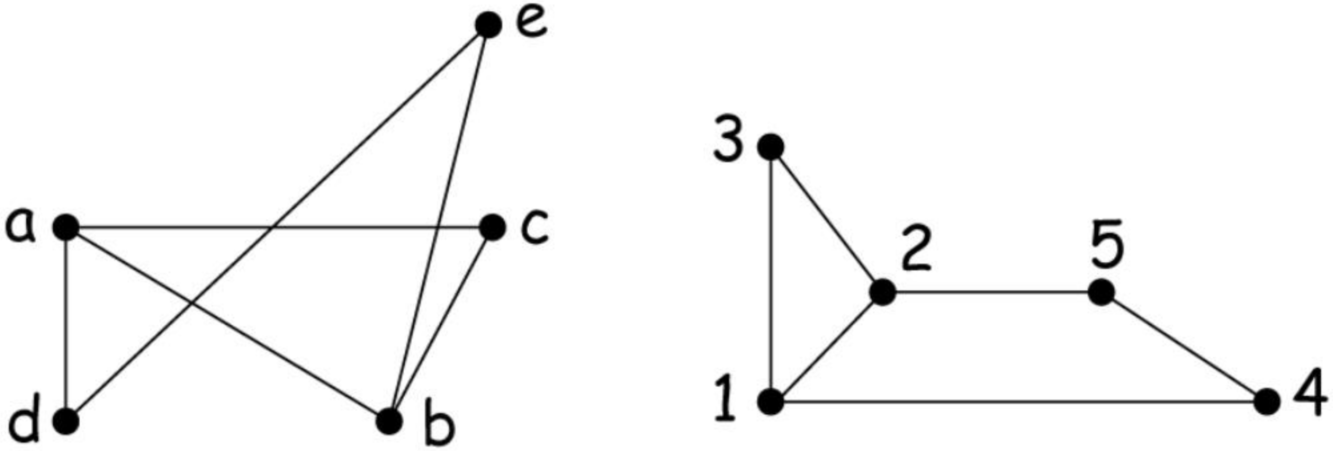
degree 2



all degree 3

Non-isomorphic

Cont.



$$a = f(1)$$
$$b = f(2)$$
$$c = f(3)$$
$$d = f(4)$$
$$e = f(5)$$

Example of Isomorphic Graphs

4.4 path and connectivity

Definition :-

A walk in a graph G is a sequence of edges of the form

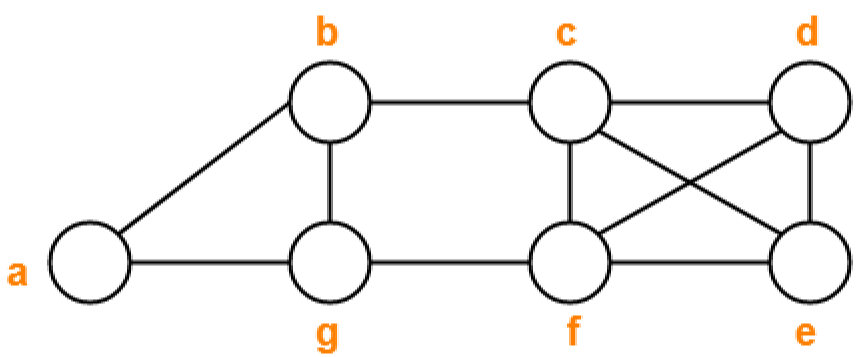
$$v_0v_1, v_1v_2, v_2v_3 \dots v_{n-1}v_n$$

The number of edges is called the length of the walk .

A walk in which all the edges are distinct is called a trial. A trail in which all vertices are distinct (except possibly $v_n = v_0$) is called a path.

A path $v_0 \to \cdots \to v_n$ with $v_n = v_0$ is called a cycle.

Example :-



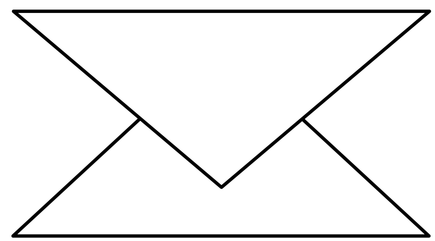
f-d-e-c-b is path and a-b-g-f-c-b is trial

Cont.

Definition :- a graph is connected if for each pair v, u of vertices, there is a path from v to u. A graph which is not connected is made up of a number of connected pieces, called components.

Example :- is G connected?

G



Remark:

1. A disconnected graph is made up of two or more disjoint connected subgraphs.

2. A graph is connected iff it has only one component.
3. The component of a graph is denoted by C(G).

Cont.

Special classes of graphs

Cycle graph:

A cycle graph of order n is a connected graph whose edges form a cycle of length n. Cycle graphs are denoted by C_n.

Path graph:

A path graph of length n is a graph obtained by removing an edge from a cycle graph C_n. Path graph of order n is denoted by P_n.

Wheel Graph:

A wheel of order n is a graph obtained by joining a single vertex (the 'hub') to each vertex of a cycle graph. Wheel graph is denoted by W_n.

Null Graph:

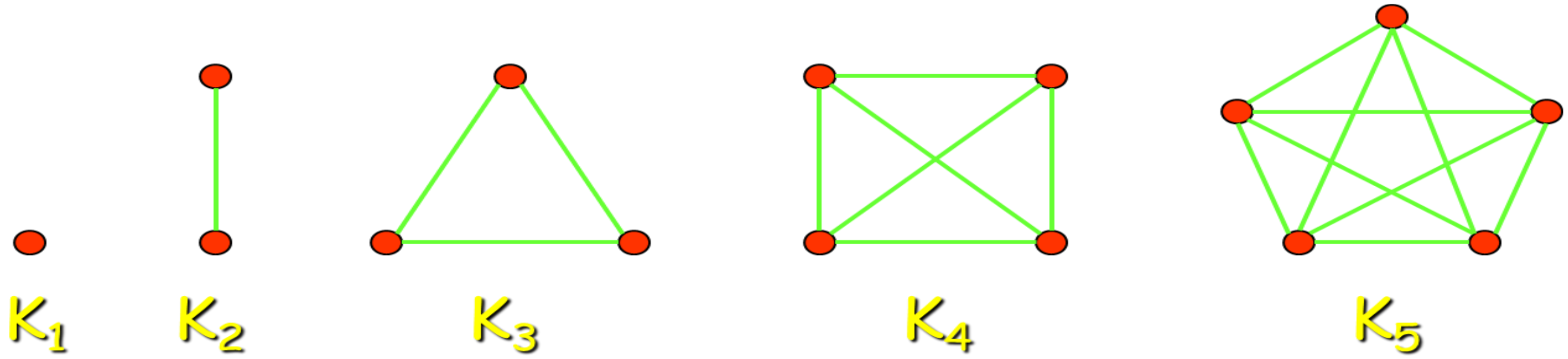
A null graph of order n is a graph with n vertices and no edges and is denoted byN_n.

Cont.

Complete graph: A graph G is said to be complete if every vertex in G is connected to every other vertex in G. Thus a complete graph G must be connected.

Notation: The complete graph with n-vertices is denoted by K_n.

Example :-



Remark: In a complete graph;

There are $\frac{n(n-1)}{2}$ edges.

- All vertices are mutually adjacent

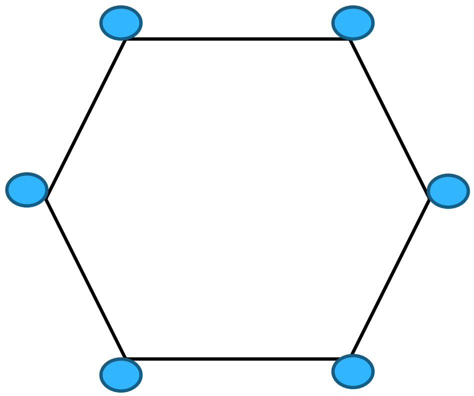
Cont.

Definition:-

A simple graph G is bipartite if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2. In other words, there are no edges which connect two vertices in V_1 or in V_2.

Note: An equivalent definition of a bipartite graph is a graph where it is possible to color the vertices red or blue so that no two adjacent vertices are the same color.

Example :- show that C_6 is bipartite graph

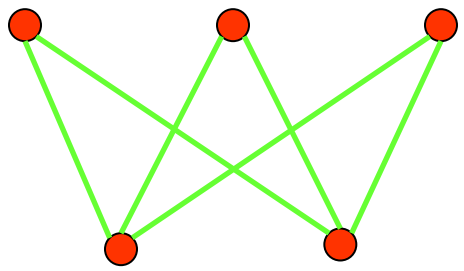


Cont.

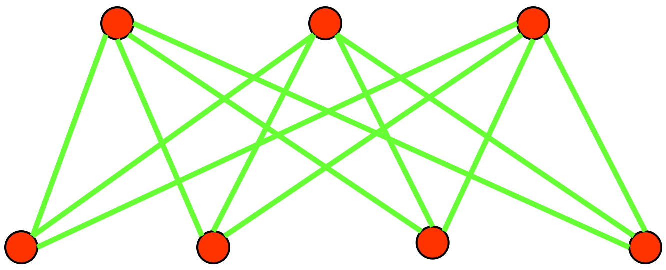
Complete bipartite graphs

Definition: A complete bipartite graph $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets V1of size m and V2 of size n such that there is an edge from every vertex in V1 to every vertex in V2.

Example :-



$K_{3,2}$



$K_{3,4}$

Theorem: In a complete bipartite graph $K_{m,n}$; the number of edges is given by $E = |mn|$.

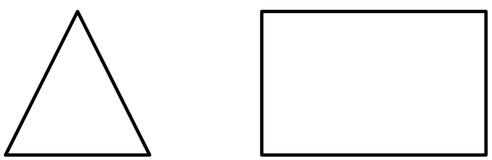
Cont.

Regular graph: A graph G is said to be regular of degree k or K-regular if every vertex has degree k. In other words, a graph is regular if every vertex has the same degree.

Example: Here are some examples of regular graphs.

0-regular

1- regular



2-regular

Theorem: A complete graph with n-vertices K_n is a regular graph of degree n-1.

Euler and Hamilton Graphs

Eulerian Path:

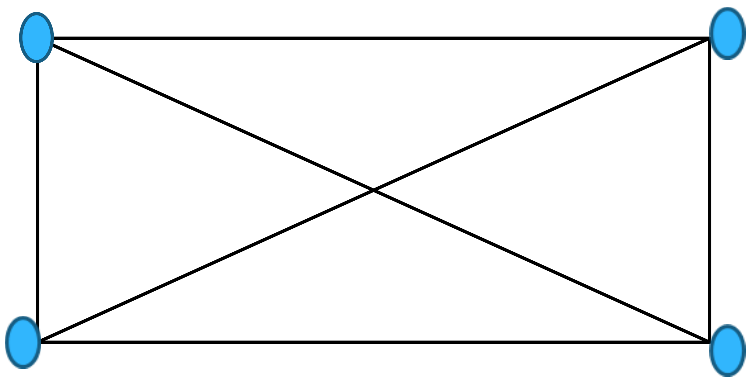
Definition: An Eulerian path in a graph G(V,E) is a path which uses each edge in E exactly once.

An Euler path that begins and ends at the same vertex is called Eulerian trial.

A graph that contains an Eulerian (closed) trial is called an Eulerian graph.

Theorem: A connected graph is Eulerian if and only if all of its vertices have even degree.

Example :-



Cont.

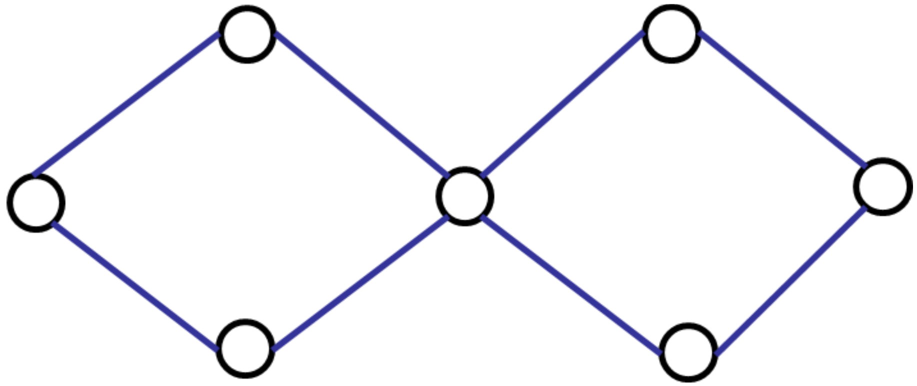
Hamilton Paths:

Definition :- a path that visits every vertex in a graph G exactly once is called a Hamilton path.

A closed Hamilton path is called a Hamilton cycle.

G is called a Hamiltonian graph if it admits a Hamiltonian cycle.

Example:-



Theorem: If G is a simple graph with vertices $n \geq 3$ and if $\deg(u) + \deg(v) \geq 3$ for all pairs of non-adjacent vertices u and v, then G is Hamiltonian. (The converse is not always true)

Trees and forest

Trees

Definition:

A tree is a connected acyclic (cannot contain multiple edges or loops) graph.

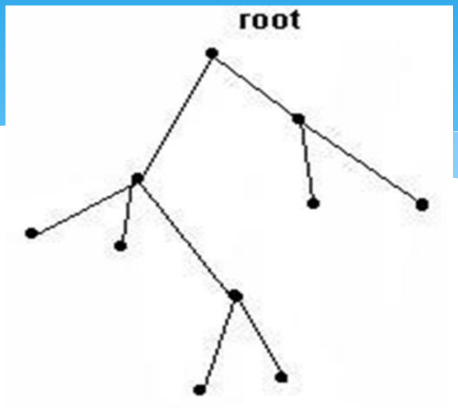
- Therefore, any tree must be a simple graph.

Theorem: An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

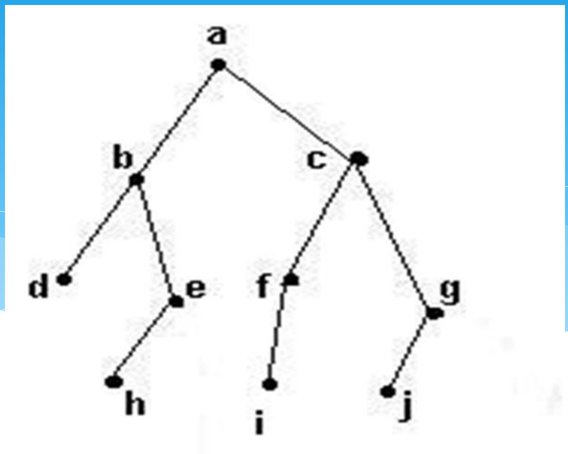
In general, we use trees to represent hierarchical structures.

Cont.

Example :-



Rooted tree



binary tree

Theorem: Let G be a graph with $n > 1$ vertices. Then the following are equivalent.

i) G is a tree

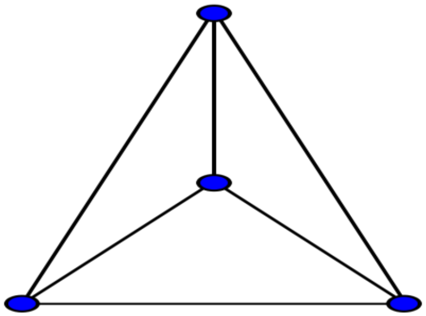
ii) G is cycle-free and has n-1 edges

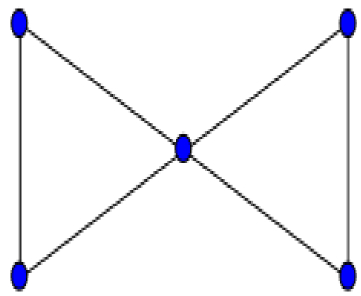
iii) G is connected and has n-1 edges.

Planar graph

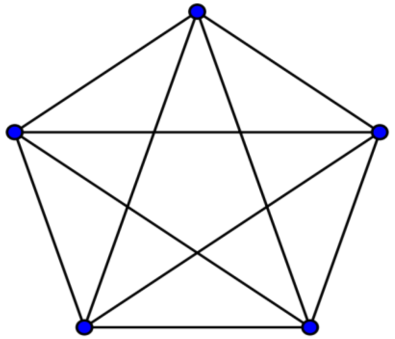
A **planar graph** is one that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges do not intersect except only at their endpoints.

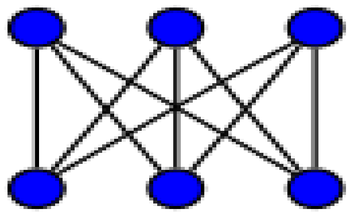
Example :-





planar





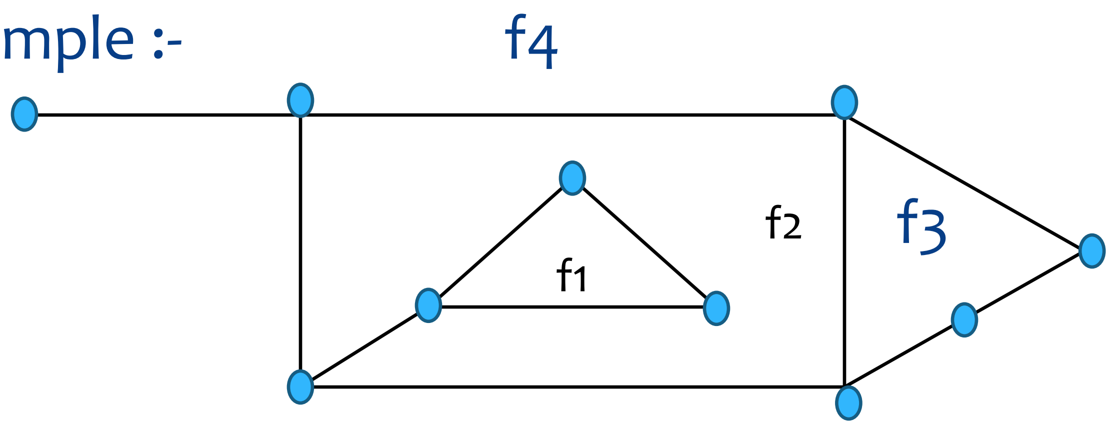
non planar

Cont.

Faces and planar graphs

Definition: If G is a planar graph, then the plane representation of G splits the plane into regions called faces of G. The unbounded region outside the graph is called the infinite face. If f is any face of a graph, then the degree of f , denoted by deg(f), is the number of edges encountered in a walk(or path)that begins and ends at the same vertex around the boundaries of the face f. If all faces have the same degree r, then G is a face-regular graph of degree r.

Example :-



Cont.

Remark: A connected planar graph has exactly one infinite face.

Theorem (Euler's Formula)

For any connected planar graph G,

$$/V/ - /E/ + /F/ = 2$$

Where /V/ denotes the number of vertices, /E/ the number of edges and /F/ the number of faces in G.

Example :- Verify Euler's formula for the planar graph K4.

Solution:-

$$/V/ - /E/ + /F/ = 2$$

$$|4| - |6| + |F| = 2$$

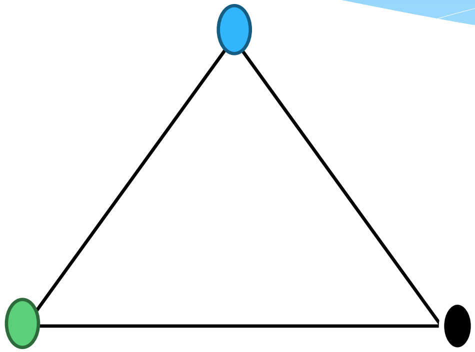
$$-2 + |F| = 2$$

$$F = 4$$

Graph Coloring

Definition: Consider a graph G. a vertex coloring or simply a coloring of G is an assignment of colors to the vertices of G such that adjacent vertices have different colors.

Example :-



Definition :- the minimum number of color needed to produce a proper coloring of G is called chromatic number of G. it is denoted by $\aleph(G)$

Example :- $\aleph(c_3) = 3$

Cont.

Theorem

The following are equivalent for a graph G

i. G is two colourable
ii. G is bipartite
iii. Every cycle of G has even length

Theorem (five colorable theorem)

Any planar graph G is 5 colorable.

Proof :- Exercise

Definition :- the number of ways to properly color G with x colors is denoted by $P_G(x)$ is called chromatic polynomial of G.

Example :- the chromatic polynomial of C_5

Solution:- $p(k) = (k)(k-1)^3(k-2)$

Cont.

Coloring application problem

Scheduling Final Exams How can the final exams at a university be scheduled so that no student has two exams at the same time?

Solution:-

This scheduling problem can be solved using a graph model, with vertices representing courses and with an edge between two vertices if there is a common student in the courses they represent. Each time slot for a final exam is represented by a different color. A scheduling of the exams corresponds to a coloring of the associated graph.