Discrete Mathematics (Math191)

By: Department of Mathematics
College of Natural and Social Sciences
Addis Ababa Science and Technology University

Chapter 1

- **Basic Counting Principles**

Basic Counting Principles

- **Counting problems** are of the following kind:
- “**How many** different 8-letter passwords are there?”
- “**How many** possible ways are there to pick 11 soccer players out of a 20-player team?”
- Most importantly, counting is the basis for computing **probabilities of discrete events**.
- (“What is the probability of winning the lottery?”)

Basic Counting Principles

- **The sum rule:**
- If a task can be done in n_1 ways and a second task in n_2 ways, and if these two tasks cannot be done at the same time, then there are $n_1 + n_2$ ways to do either task.

- **Example:**
- The department will award a free computer to either a CS student or a CS professor.
- How many different choices are there, if there are 530 students and 15 professors?

- There are $530 + 15 = 545$ choices.

Basic Counting Principles

- **Generalized sum rule:**
- If we have tasks $T_1, T_2, \ldots, T_m$ that can be done in $n_1, n_2, \ldots, n_m$ ways, respectively, and no two of these tasks can be done at the same time, then there are $n_1 + n_2 + \ldots + n_m$ ways to do one of these tasks.

Basic Counting Principles

- **The product rule:**
- Suppose that a procedure can be broken down into two successive tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done, then there are $n_1 n_2$ ways to do the procedure.

Basic Counting Principles

- **Example:**
- How many different license plates are there that containing exactly three English letters ?
- **Solution:**
- There are 26 possibilities to pick the first letter, then 26 possibilities for the second one, and 26 for the last one.
- So there are $26 \cdot 26 \cdot 26 = 17576$ different license plates.

Basic Counting Principles

- **Generalized product rule:**
- If we have a procedure consisting of sequential tasks T_1, T_2, …, T_m that can be done in n_1, n_2, …, n_m ways, respectively, then there are $n_1 \cdot n_2 \cdot \ldots \cdot n_m$ ways to carry out the procedure.

Basic Counting Principles

- The sum and product rules can also be phrased in terms of **set theory**.

- **Sum rule:** Let $A_1, A_2, \ldots, A_m$ be disjoint sets. Then the number of ways to choose any element from one of these sets is $|A_1 \cup A_2 \cup \ldots \cup A_m| =$
$|A_1| + |A_2| + \ldots + |A_m|$.

- **Product rule:** Let $A_1, A_2, \ldots, A_m$ be finite sets. Then the number of ways to choose one element from each set in the order $A_1, A_2, \ldots, A_m$ is
$|A_1 \times A_2 \times \ldots \times A_m| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_m|$.

Inclusion-Exclusion

- How many bit strings of length 8 either start with a 1 or end with 00?
- **Task 1:** Construct a string of length 8 that starts with a 1.
- There is one way to pick the first bit (1),
- two ways to pick the second bit (0 or 1),
- two ways to pick the third bit (0 or 1),
- .
- .
- .
- two ways to pick the eighth bit (0 or 1).
- **Product rule:** Task 1 can be done in $1 \cdot 2^7 = 128$ ways.

Inclusion-Exclusion

- **Task 2:** Construct a string of length 8 that ends with 00.
- There are two ways to pick the first bit (0 or 1),
- two ways to pick the second bit (0 or 1),
- .
- .
- .
- two ways to pick the sixth bit (0 or 1),
- one way to pick the seventh bit (0), and
- one way to pick the eighth bit (0).
- **Product rule:** Task 2 can be done in $2^6 = 64$ ways.

Inclusion-Exclusion

- Since there are 128 ways to do Task 1 and 64 ways to do Task 2, does this mean that there are 192 bit strings either starting with 1 or ending with 00 ?
- No, because here Task 1 and Task 2 can be done **at the same time**.
- When we carry out Task 1 and create strings starting with 1, some of these strings end with 00.
- Therefore, we sometimes do Tasks 1 and 2 at the same time, so **the sum rule does not apply**.

Inclusion-Exclusion

- If we want to use the sum rule in such a case, we have to subtract the cases when Tasks 1 and 2 are done at the same time.
- How many cases are there, that is, how many strings start with 1 **and** end with 00?
- There is one way to pick the first bit (1),
- two ways for the second, …, sixth bit (0 or 1),
- one way for the seventh, eighth bit (0).
- **Product rule:** In $2^5 = 32$ cases, Tasks 1 and 2 are carried out at the same time.

Inclusion-Exclusion

- Since there are 128 ways to complete Task 1 and 64 ways to complete Task 2, and in 32 of these cases Tasks 1 and 2 are completed at the same time, there are
- 128 + 64 – 32 = 160 ways to do either task.

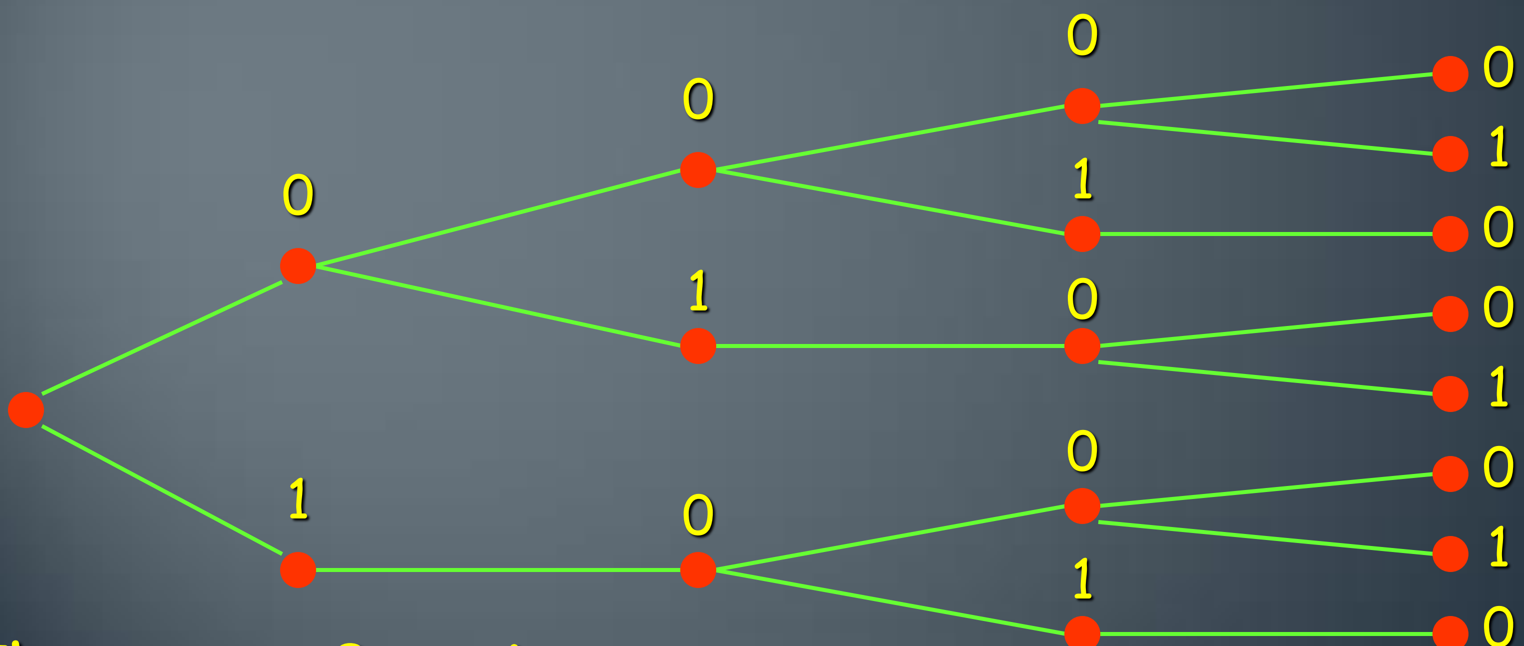
- In set theory, this corresponds to sets A_1 and A_2 that are **not** disjoint. Then we have:
- $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

- This is called the **principle of inclusion-exclusion.**

Tree Diagrams

- How many bit strings of length four do not have two consecutive 1s?
- Task 1 Task 2 Task 3 Task 4

Task 1	Task 2	Task 3	Task 4
(1st bit)	(2nd bit)	(3rd bit)	(4th bit)



There are 8 strings.

The Pigeonhole Principle

- **The pigeonhole principle:** If $(k + 1)$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.
- **Example 1:** If there are 11 players in a soccer team that wins 12-0, there must be at least one player in the team who scored at least twice.
- **Example 2:** If you have 6 classes from Monday to Friday, there must be at least one day on which you have at least two classes.

The Pigeonhole Principle

- **The generalized pigeonhole principle:** If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ of the objects.

- **Example 1:** In our 60-student class, at least 12 students will get the same letter grade (A, B, C, D, or F).

The Pigeonhole Principle

- **Example 2:** Assume you have a drawer containing a random distribution of a dozen brown socks and a dozen black socks. It is dark, so how many socks do you have to pick to be sure that among them there is a matching pair?

- There are two types of socks, so if you pick at least 3 socks, there must be either at least two brown socks or at least two black socks.

- Generalized pigeonhole principle: $\lceil 3/2 \rceil = 2$.

Permutations and Combinations

- How many ways are there to pick a set of 3 people from a group of 6?
- There are 6 choices for the first person, 5 for the second one, and 4 for the third one, so there are
$6 \cdot 5 \cdot 4 = 120$ ways to do this.
- This is not the correct result!
- For example, picking person C, then person A, and then person E leads to the **same group** as first picking E, then C, and then A.
- However, these cases are counted **separately** in the above equation.

Permutations and Combinations

- So how can we compute how many different subsets of people can be picked (that is, we want to disregard the order of picking) ?

- To find out about this, we need to look at **permutations.**

- A **permutation** of a set of distinct objects is an ordered arrangement of these objects.

- An ordered arrangement of r elements of a set is called an **r-permutation**.

Permutations and Combinations

- **Example:** Let $S = \{1, 2, 3\}$.
- The arrangement 3, 1, 2 is a permutation of S.
- The arrangement 3, 2 is a 2-permutation of S.

- The number of r-permutations of a set with n distinct elements is denoted by **$P(n, r)$.**

- We can calculate $P(n, r)$ with the product rule:
- $P(n, r) = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-r+1)$.
- (n choices for the first element, $(n-1)$ for the second one, $(n-2)$ for the third one…)

Permutations and Combinations

- **Example:**
- $P(8, 3) = 8 \cdot 7 \cdot 6 = 336$
- $= (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)/(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$
- **General formula:**
- $P(n, r) = n!/(n - r)!$
- Knowing this, we can return to our initial question:
- How many ways are there to pick a set of 3 people from a group of 6 (disregarding the order of picking)?

Permutations and Combinations

- An **r-combination** of elements of a set is an unordered selection of r elements from the set.
- Thus, an r-combination is simply a subset of the set with r elements.

- **Example:** Let $S = \{1, 2, 3, 4\}$.
- Then $\{1, 3, 4\}$ is a 3-combination from S.

- The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$.

- **Example:** $C(4, 2) = 6$, since, for example, the 2-combinations of a set $\{1, 2, 3, 4\}$ are $\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}$.

Permutations and Combinations

- How can we calculate $C(n, r)$?
- Consider that we can obtain the r-permutation of a set in the following way:
- **First,** we form all the r-combinations of the set (there are $C(n, r)$ such r-combinations).
- **Then,** we generate all possible orderings in each of these r-combinations (there are $P(r, r)$ such orderings in each case).
- Therefore, we have:
- $P(n, r) = C(n, r) \cdot P(r, r)$

Permutations and Combinations

- $C(n, r) = P(n, r)/P(r, r)$
- $\quad = n!/(n - r)!/(r!/(r - r)!)$
- $\quad = n!/(r!(n - r)!)$

- Now we can answer our initial question:
- How many ways are there to pick a set of 3 people from a group of 6 (disregarding the order of picking)?
- $C(6, 3) = 6!/(3!\cdot 3!) = 720/(6\cdot 6) = 720/36 = 20$

- There are 20 different ways, that is, 20 different groups to be picked.

Permutations and Combinations

- **Corollary:**
- Let n and r be nonnegative integers with $r \leq n$.
- Then $C(n, r) = C(n, n - r)$.

- Note that **"picking a group of r people from a group of n people"** is the same as **"splitting a group of n people into a group of r people and another group of (n – r) people".**

- Please try to proof.

Permutations and Combinations

- **Example:**
- A soccer club has 8 female and 7 male members. For today's match, the coach wants to have 6 female and 5 male players on the grass. How many possible configurations are there?

- $C(8, 6) \cdot C(7, 5) = 8!/(6!\cdot 2!) \cdot 7!/(5!\cdot 2!)$
- $= 28\cdot 21$
- $= 588$

Combinations

- We also saw the following:

$$C(n,n-r)=\frac{n!}{(n-r)![n-(n-r)]!}=\frac{n!}{(n-r)!r!}=C(n,r)$$

This symmetry is intuitively plausible. For example, let us consider a set containing six elements (n = 6).

Picking two **elements and leaving four** is essentially the same as **picking four** elements and **leaving two.**

In either case, our number of choices is the number of possibilities to **divide** the set into one set containing two elements and another set containing four elements.

Combinations

- **Pascal's Identity:**
- Let n and k be positive integers with $n \geq k$.
Then $C(n + 1, k) = C(n, k - 1) + C(n, k)$.
- How can this be explained?
- What is it good for?

Combinations

- Imagine a set S containing n elements and a set T containing (n + 1) elements, namely all elements in S plus a new element **a**.
- Calculating C(n + 1, k) is equivalent to answering the question: How many subsets of T containing k items are there?
- Case I: The subset contains (k – 1) elements of S plus the element **a**: C(n, k – 1) choices.
- Case II: The subset contains k elements of S and does not contain **a**: C(n, k) choices.
- **Sum Rule:** C(n + 1, k) = C(n, k – 1) + C(n, k).

Pascal's Triangle

- In Pascal's triangle, each number is the sum of the numbers to its upper left and upper right:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
\ldots & & \ldots & & \ldots & & \ldots & & \ldots & & \ldots
\end{array}
$$

Pascal's Triangle

- Since we have $C(n + 1, k) = C(n, k - 1) + C(n, k)$ and $C(0, 0) = 1$, we can use Pascal's triangle to simplify the computation of $C(n, k)$:

k

n

$C(0, 0) = 1$

$C(1, 0) = 1 \quad C(1, 1) = 1$

$C(2, 0) = 1 \quad C(2, 1) = 2 \quad C(2, 2) = 1$

$C(3, 0) = 1 \quad C(3, 1) = 3 \quad C(3, 2) = 3 \quad C(3, 3) = 1$

$C(4, 0) = 1 \quad C(4, 1) = 4 \quad C(4, 2) = 6 \quad C(4, 3) = 4 \quad C(4, 4) = 1$

Binomial Coefficients

- Expressions of the form C(n, k) are also called **binomial coefficients.**
- How come?
- A **binomial expression** is the sum of two terms, such as (a + b).
- Now consider $(a + b)^2 = (a + b)(a + b)$.
- When expanding such expressions, we have to form all possible products of a term in the first factor and a term in the second factor:
- $(a + b)^2 = a\cdot a + a\cdot b + b\cdot a + b\cdot b$
- Then we can sum identical terms:
- $(a + b)^2 = a^2 + 2ab + b^2$

Binomial Coefficients

- For $(a + b)^3 = (a + b)(a + b)(a + b)$ we have
- $(a + b)^3 = aaa + aab + aba + abb + baa + bab + bba + bbb$
- $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- There is only one term a^3, because there is only one possibility to form it: Choose **a** from all three factors: $C(3, 3) = 1$.
- There is three times the term a^2b, because there are three possibilities to choose **a** from two out of the three factors: $C(3, 2) = 3$.
- Similarly, there is three times the term ab^2 ($C(3, 1) = 3$) and once the term b^3 ($C(3, 0) = 1$).

Binomial Coefficients

- This leads us to the following formula:

$$(a+b)^n = \sum_{j=0}^{n} C(n, j) \cdot a^{n-j} b^j$$ (Binomial Theorem)

With the help of Pascal's triangle, this formula can considerably simplify the process of expanding powers of binomial expressions.

For example, the fifth row of Pascal's triangle (1 – 4 – 6 – 4 – 1) helps us to compute $(a + b)^4$:

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Binomial Coefficients

Q. What is the coefficient of the term $a^8\, b^{12}$ in the expansion of $(a+b)^{20}$

Solution:

By the binomial theorem, we have that

$$(a+b)^{20} = \sum_{j=0}^{20} C(20,j)\, a^{20-j} b^{j}$$

When j=12, we get the term $a^8\, b^{12}$

Thus, the coefficient of the term $a^8\, b^{12}$ is C(20,12)

which is C(20,12) $= \frac{20!}{12!8!} = 125{,}970$

Binomial Coefficients

Q. What is the coefficient of the term $a^8\ b^{12}$ in the expansion of $(3a+4b)^{20}$

Solution:

By the binomial theorem, we have that

$$(3a+4b)^{20} = \sum_{j=0}^{20} C(20,j)\,(3a)^{20-j}(4b)^{j}$$

When j=12, we have $C(20,12)(3a)^8\ (4b)^{12}$

Thus, the coefficient of the term $a^8\ b^{12}$ is

$C(20,12)3^8 4^{12}$=13,866,187,326,750,720

End of Chapter 1
Thank you!