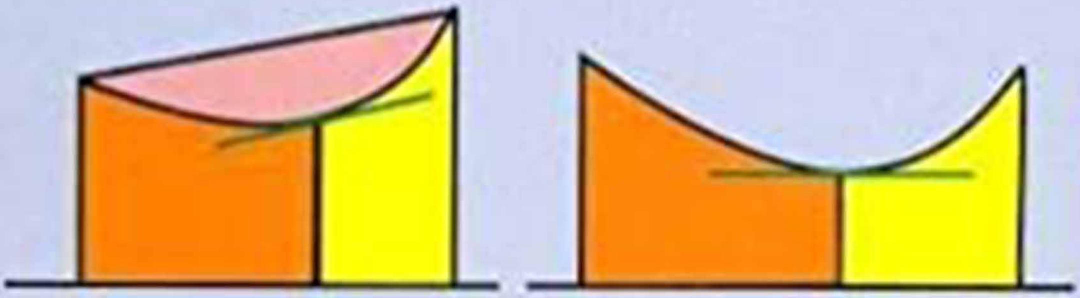
Introduction to Real Analysis

Revised 6th edition



S. K. Mapa

Introduction to
REAL ANALYSIS
(FOR DEGREE HONOURS COURSE)

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SARAT BOOK DISTRIBUTORS
18-B, Shyamacharan De Street
Kolkata-700 073

First Edition 1996
Second Edition 2000
Third Edition 2002
Fourth Edition 2004
Fifth Edition 2006
Reprint 2008
Reprint (corrected) 2009

Distribution:
Sarat Book Distributors
18-B Shyamacharan De Street
Calcutta 700 073

ISBN 81-87169-02-8

Published by Dipali Mapa, 50 DumDum Road, Calcutta 700 074 and printed at P.M.Bagchi & Co. Private Limited, 19 Gulu Ostagar Lane, Calcutta 700 006.

PREFACE

This is a thoroughly revised edition. A new chapter on 'Improper integrals' has been added. Additional materials have been incorporated in some chapters in order to stimulate the reader's interest. Some more examples have been fully worked out throughout the chapters in order to help the reader develop skill in working out exercises.

The mistakes and misprints that crept in the previous edition have been eliminated.

The author likes to appreciate suggestions given by his esteemed colleagues and by his inquisitive students for the improvement of the book.

The author conveys his sincere thanks to SARAT BOOK DISTRIBUTORS and the printer for the care and co-operation rendered by them in course of publication of the book.

CALCUTTA
July, 2006

S. K. Ma

PREFACE TO THE FIRST EDITION

This is an elementary treatise covering only a part of Real Analysis and it is designed to serve as a text book for undergraduate students of Mathematics Honours.

Keeping in mind that the volume is meant for the beginners of the subject the discussion of each topic has been supplemented by a good number of examples so that the reader can learn and understand the basic principles of Analysis.

There are many standard texts on Real Analysis. The author expresses his indebtedness to the authors of some of these texts which have been consulted during the preparation of this beginners' volume. A bibliography of such texts is given at the end.

The author likes to convey his sincere thanks to the publisher, the composer and the printer for the care and co-operation rendered by them in the process of publication of the book.

Any suggestion from the readers for the improvement of the book will be highly appreciated.

CALCUTTA
1.10.1996

S. K. MAPA

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1. SET THEORY

1.1. Introduction.

The concept of a 'set' is basic in all branches of Mathematics. In vie\ of the importance of set theory we present here a brief account of it. W take an entirely naive and intuitive view of a set.

For our purpose, a set is a well defined *collection* (or aggregate) c distinct objects (which are also called elements or points).

A set is usually denoted by capital letters $A, B, X, \ldots$, and an elemer of a set is denoted by small letters $a, b, x, \ldots$ When x is an element of set of A, it is expressed by the symbol $x \in A$ (read as x belongs to A and when x is not an element of a set A, it is expressed by the symb $x \notin A$ (read as x does not belong to A).

A set may be described by listing all its elements, usually betwee braces $\{\cdots\}$. Thus the set of all natural numbers less that 5 is $\{1, 2, 3, 4$ There is no significance in the order in which the elements are liste Therefore $\{2, 3, 1, 4\}$ and $\{1, 2, 3, 4\}$ describe the same set.

Another way of describing a set is $\{x : P(x)\}$ where $P(x)$ is tl proposition about x and $\{x : P(x)\}$ is the collection of those elements for which $P(x)$ is true. For example, the set $\{x : x$ is an even positi integer$\}$ is the set $\{2, 4, 6, \ldots \ldots\}$.

Throughout this text we use accepted notations for some familiar se of numbers.

$\mathbb{N} = \{1, 2, 3, \ldots \ldots\}$, the set of all natural numbers,
$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \cdots \cdots\}$, the set of all integers,
$\mathbb{Q} = \{\frac{p}{q} : p$ and q are integers, $q \neq 0\}$, the set of all rational numbe
$\mathbb{R} =$ the set of all real numbers.

1.2. Subsets.

Let A and B be two sets. If $x \in A \Rightarrow x \in B$ then A is said to l a *subset* of B. This means that each element of A is an element of This is expressed by $A \subset B$ or by $B \supset A$. In this case B is said to be *superset* of A.

We conceive of the existence of a set which contains no element. This is called the *null set* or *empty set* and is denoted by ϕ. For logical consistency, $\phi \subset A$ for every set A.

For every set A, $A \subset A$. Also $\phi \subset A$ for every set A. A and ϕ are said to be the *improper subsets* of A. Any other subset of A is called a *proper subset* of A.

For example, the set $\mathbb{Z}$ is a proper subset of the set $\mathbb{Q}$.

Two sets A and B are said to be *equal* if $A \subset B$ and $B \subset A$.

Definition. A set S is said to be a *finite set* if either it is empty or it contains a finite number of elements ; otherwise it is said be an *infinite set*.

1.3. Algebraic operations on sets.

In this section we shall discuss several ways of combining different sets. For this purpose we shall consider several sets, in a particular discussion, as subsets of a single fixed set, called the *universal set* in relation to its subsets. A universal set is generally denoted by U.

Let U be the universal set and $A, B, C, \ldots$ be subsets of U. We define the following operations on the class of all subsets of U.

(a) Union. The *union (or join)* of the subsets A and B is a subset of U, denoted by $A \cup B$ and is defined by

$$A \cup B = \{x : x \in A \text{ or } x \in B\} .$$

Therefore $A \cup B$ is the set of all those elements which belong to A or to B or to both. It follows that $A \subset A \cup B, B \subset A \cup B$.

(b) Intersection. The *intersection (or meet)* of two subsets A and B is a subset of U, denoted by $A \cap B$ and is defined by

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Therefore $A \cap B$ is the set of all those elements which belong to both A and B. It follows that $A \cap B \subset A, A \cap B \subset B$.

Definition. If two subsets A and B have no common element then $A \cap B = \phi$ and A and B are said to be *disjoint*.

Example.

1. Let $A = \{1, 2, 3, 4, 5\}, B = \{3, 5, 7, 9, 11\}, C = \{2, 4, 6, 8, 10\}, D = \{2, 6, 10\}$ be subsets of the set $\mathbb{N}$.

Then $A \cup B = \{1, 2, 3, 4, 5, 7, 9, 11\}, A \cap B = \{3, 5, \}, A \cap C = \{2, 4\}, B \cap C = \phi$, $C \cup D = C$, $C \cap D = D$.

B and C are disjoint subsets of $\mathbb{N}$. D is a proper subset of C.

Properties.

1. Consistency property. The three relations $A \subset B, A \cup B = B$ and $A \cap B = A$ are equivalent, i.e., one of these implies the other two.

1a. $A \cup \phi = A$, $A \cap \phi = \phi$. This follows from 1 by taking $\phi \subset A$.

1b. $A \cup U = U$, $A \cap U = A$. This follows from 1 by taking $A \subset U$.

1c. $A \cup A = A$, $A \cap A = A$. *(Idempotent property).* This follows from 1 by taking $A \subset A$.

1d. $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$. *(Absorptive property).* This follows from 1 by taking $A \cap B \subset A \subset A \cup B$.

2. $A \cup B = B \cup A$, $A \cap B = B \cap A$. *(Commutative property).*

3. $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$. *(Associative property).*

4. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. *(Distributive property).*

(c) Complementation. The *complement* of a subset A is a subset of U, denoted by A' (or A^c) and is defined by

$$A' = \{x \in U : x \notin A\}.$$

A' contains all those elements of U which are not in A.

Properties.

5. $A \cup A' = U, A \cap A' = \phi$ for any subset A.

6. $(A')' = A$ for any subset A.

7. De Morgan's Laws. $(A \cup B)' = A' \cap B', (A \cap B)' = A' \cup B'$.

(d) Difference. The *difference* of two subsets A and B is a subset of U, denoted by $A - B$ and is defined by

$$A - B = \{x \in A : x \notin B\}.$$

$A - B$ is a subset of A and is the set of those elements of A which are not in B. $A - B$ is the relative complement of B in A. The difference $A - B$ can be expressed as $A - B = A \cap B'$.

Examples (continued).

2. Let $U = \{1, 2, 3, \ldots, 10\}$ be the universal set and $A = \{1, 3, 5, 7, 9\}$. Then $A' = \{2, 4, 6, 8, 10\}$.

3. Let $A = \{1, 2, 3, 4, 5, 6\}, B = \{2, 4, 6, 8, 10\}$ be subsets of the set $\mathbb{N}$. Then $A - B = \{1, 3, 5\}, B - A = \{8, 10\}$.

(e) Symmetric Difference. The *symmetric difference* of two subsets A and B is a subset of U, denoted by $A \triangle B$ and is defined by

$$A \triangle B = (A - B) \cup (B - A).$$

$A \triangle B$ is the set of all those elements which belong either to A or to B but not to both.

Example (continued).

4. Let $A = \{1, 2, 3, 4\}, B = \{2, 4, 6, 8\}$ be subsets of the set $\mathbb{N}$. Then $A \triangle B = \{1, 3, 6, 8\}$.

Properties.

8. $A \triangle A = \phi$ for all subsets $A \subset U$.

9. $A \triangle B = B \triangle A$.

10. $A \triangle (B \triangle C) = (A \triangle B) \triangle C$.

1.4. Family of sets.

We have defined a set as a collection of its elements. If the elements be the subsets of a universal set then we have a *set of sets* or a *family of sets.*

Examples.

1. Let X be a non-empty set. The collection of all subsets of X is a family of sets. This family is called the *power set* of X and is denoted by $P(X)$.

If $X = \{1, 2, 3\}$ then $P(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}$.

If X contains n elements then $P(x)$ contains 2^n elements.

2. Let I be the finite set $\{1, 2, \ldots, n\}$ and $\mathcal{F}$ be the family of n sets $A_1, A_2, \ldots, A_n$. $\mathcal{F}$ is expressed as $\{A_\alpha : \alpha \in I\}$. I is called the *index set.* The elements of $\mathcal{F}$ are said to be indexed by the index set I.

3. Let $I = \mathbb{N}$ and $\mathcal{F}$ be the family of sets $A_1, A_2, \ldots \ldots$. $\mathcal{F}$ is expressed as $\{A_n : n \in \mathbb{N}\}$. Here $\mathbb{N}$ is the index set.

4. Let $\wedge$ be an arbitrary set. The family of sets $\mathcal{F} = \{A_\alpha : \alpha \in \wedge\}$ is the collection of sets A_α, for each $\alpha \in \wedge$. Here $\wedge$ is the index set.

5. For each $n \in \mathbb{N}$, let $I_n = \{x \in \mathbb{R} : 0 < x < \frac{1}{n}\}$. Then we have a family of sets $\{I_n : n \in \mathbb{N}\}$. The union of the family is denoted by $\bigcup_{n \in \mathbb{N}} I_n$ and the intersection of the family is denoted by $\bigcap_{n \in \mathbb{N}} I_n$.

Here $\bigcup_{n \in \mathbb{N}} I_n = \{x \in \mathbb{R} : 0 < x < 1$ and $\bigcap_{n \in \mathbb{N}} I_n = \phi$.

1.5. Cartesian product of sets.

Let A and B be non-empty sets. The *Cartesian product* of A and l denoted by $A \times B$, is the set defined by

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

$A \times B$ is the set of all ordered pairs (a, b), the first element of the pa being an element of A and the second being an element of b.

Let $A_1, A_2, \ldots, A_n$ be a finite collection of non-empty sets. The Cart sian product of the collection, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the se defined by

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_i \in A_i, i = 1, 2, \ldots, n\}.$$

In particular if $A_1 = A_2 = \cdots = A_n = A$, then the Cartesian proc uct of the collection, denoted by A^n, is the set of all ordered n tupl $\{(a_1, a_2, \ldots, a_n) : a_i \in A, i = 1, 2, \ldots, n\}$.

Examples.

1. Let $A = \{1, 2, 3\}, B = \{2, 4\}$.

Then $A \times B = \{(1,2), (1,4), (2,2), (2,4), (3,2), (3,4)\}$;

$B \times A = \{(2,1), (2,2), (2,3), (4,1), (4,2), (4,3)\}$.

2. Let $A = \mathbb{Z}, B = \mathbb{Z}$. Then $A \times B$ is the set of all ordered pairs integers.

3. Let $A = \mathbb{R}, B = \mathbb{R}, C = \mathbb{R}$. Then $A \times B \times C$ is denoted by $\mathbb{R}^3$ and is the set of all ordered triplets $\{(x, y, z) : x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}\}$.

4. $\mathbb{R}^n$ is the set of all ordered n-tuples $\{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$.

1.6. Relation on a set.

Let A and B be two non-empty sets. Intuitively, a relation ρ *betwee* A and B is a *rule* that associates some or all the elements of A with som element or elements of B.

Definition. Let A and B be two non-empty sets. A relation ρ betwee A and B is a subset of $A \times B$. If the ordered pair $(a, b) \in \rho$ then th element a of the set A is said to be related to the element b in B by th relation ρ. If $(a, b) \in (A \times B) - \rho$, then a is said to be *not related* to b t the relation ρ.

Example.

1. Let $A = \{2, 3, 4, 5\}, B = \{4, 6, 8, 9\}$. A relation ρ between A and B defined by specifying that an element a in A is related to an element in B if a is a divisor of b.

Then $\rho = \{(2,4),(2,6),(2,8),(3,6),(3,9),(4,4),(4,8)\}$.$(2,5) \in A \times B$ but $(2,5) \notin \rho$ since 2 is not a divisor of 5. 2 in A is related to three elements in B, 3 in A is related to two elements in B, 4 in A is related to two elements in B, 5 in A is related to no element in B.

Definition. Let A be a non-empty set. A relation ρ on A is a subset of $A \times A$.

If (a,b) be an element of $A \times A$ and $(a,b) \in \rho$, then a is said to be related to b by the relation ρ. This is expressed by $a\ \rho\ b$.

If (p,q) be an element of $A \times A$ and $(p,q) \notin \rho$, then p is said to be not related to q by the relation ρ. This is expressed by $p\ \bar{\rho}\ q$.

Let X be a non-empty set. A relation ρ on X is said to be *reflexive* if $a\ \rho\ a$ holds for all $a \in X$.

ρ is said to be *symmetric* if $a\ \rho\ b \Rightarrow b\ \rho\ a$ for $a,\ b \in X$.

ρ is said to be *anti-symmetric* if $a\ \rho\ b$ and $b\ \rho\ a \Rightarrow a = b$ for $a,\ b \in X$.

ρ is said to be *transitive* if $a\ \rho\ b$ and $b\ \rho\ c \Rightarrow a\ \rho\ c$ for $a,\ b,\ c \in X$.

Examples.

1. Let a relation ρ be defined on $\mathbb{Z}$ by "$a\ \rho\ b$ if and only if $a - b$ is even" for $a, b \in \mathbb{Z}$. Then ρ is reflexive, symmetric and transitive, but not anti-symmetric.

2. Let a relation ρ be defined on $\mathbb{N}$ by "$a\ \rho\ b$ if and only if a is a divisor of b". Then ρ is reflexive, anti-symmetric and transitive, but not symmetric.

1.7. Order relation on a set.

Definition. Partially ordered set.

A relation ρ on a non-empty set X is said to be a *partial order relation* if ρ is reflexive, anti-symmetric and transitive.

A set X equipped with a partial order relation ρ is said to be a *partially ordered set* (or a *poset*) and it is denoted by (X, ρ).

Note. A partial order relation is commonly denoted by the symbol $\leq$ or $\geq$ and read in usual manner. Thus $a \leq b$ is read as "a is less than or equal to b". A partially ordered set X with a partial order $\leq$ is denoted by $(X, \leq)$.

Examples (continued).

3. $(\mathbb{N}, \leq)$ is a poset where "$a \leq b$ means a is a divisor of b", for $a, b \in \mathbb{N}$.

4. Let $\mathcal{F}$ be a family of sets. Then the set inclusion $\subset$ is a partial order on $\mathcal{F}$ and $(\mathcal{F}, \subset)$ is a poset.

Definition.

A relation ρ on a set X is said to be a *strict order relation* if it is anti-symmetric and transitive and for which $(a, a) \notin \rho$ for all $a \in X$.

If a partial order be denoted by $\leq$, then the corresponding strict order is denoted by $<$.

Two elements a and b in a partially ordered set are said to be *comparable* if one of them is related to the other, i.e., one of the relations $a \leq b$, $b \leq a$ must hold. In a partially ordered set there may exist elements a and b which are not comparable. For example, in Ex.3, the integers 4 and 6 are not comparable, because neither is a divisor of the other.

If a partial order relation satisfies a fourth condition that 'any two elements are comparable' then it is called a *total order relation.*

Definition. Ordered set.

A partial order $\leq$ on a set X is said to be a *linear order* (or a *total order*) if any two elements of X be comparable, i.e., for $a, b \in X$, either $a \leq b$ or $b \leq a$.

This property is known as the *law of dichotomy* and is expressed by saying that the elements of the set X are comparable under $\leq$.

Translating in terms of strict order $<$, it says that for all $a, b \in X$, either $a < b$, or $a = b$, or $a > b$. This property is known as the *law of trichotomy.*

A set X together with a linear order $\leq$ defined on it is said to be a *linearly ordered set*, or a *totally ordered set.*

In a poset $(X, \leq)$ a subset C which is linearly ordered under the given order relation $\leq$ on X is called a *chain.*

A linearly ordered set $(X, \leq)$ is said to be a *well ordered set* if every non-empty subset S of X has a *least* element. A least element in the subset S is an element a in S such that $a \leq s$ for each element s in S.

Examples (continued).

5. $(\mathbb{N}, \leq)$ is a linearly ordered set where $a \leq b$ has its usual meaning. It is also a well-ordered set.

6. In $(\mathbb{N}, \leq)$ of Ex.3, the subset $\{1, 2, 4, 8, \ldots\}$ is a chain.

Definition.

Let $(X, \leq)$ be an ordered set. Let $S \subset X$.

An element $u \in X$ is said to be an *upper bound* for S if $s \leq u$ for each $s \in S$. An element $l \in X$ is said to be a *lower bound* for S if $l \leq s$ for each $s \in S$.

A subset S of X is said to be *bounded* above if S has an upper bound.

A subset S of X is said to be *bounded* below if S has a lower bound.

In an ordered set $(X, \leq)$ the empty set ϕ is bounded above and bounded below. Every element of X is an upper bound of ϕ and also every element of X is a lower bound of ϕ.

Let $(X, \leq)$ be an ordered set and S be a subset of X bounded above. S is said to have a *least upper bound* (or a *supremum*) (in X) if there exists an upper bound x^* of S such that $x^* \leq u$ for every upper bound u of S.

Let $(X, \leq)$ be an ordered set and S be a subset of X bounded below. S is said to have a *greatest lower bound* (or an *infimum*) (in X) if there exists a lower bound x_* of S such that $l \leq x_*$ for every lower bound l of S.

Theorem 1.7.1. In an ordered set $(X, \leq)$ if a subset S has a supremum x^*, then x^* is unique.

Proof. If possible, let x^*, x_1^* be two suprema of S.

As x^* is a supremum and x_1^* is an upper bound of S, $x^* \leq x_1^*$.

As x_1^* is a supremum and x^* is an upper bound of S, $x_1^* \leq x^*$.

It follows that $x_1^* = x^*$.

Theorem 1.7.2. In an ordered set $(X, \leq)$ if a subset S has an infimum x_*, then x_* is unique.

Similar proof.

Theorem 1.7.3. In an ordered set $(X, \leq)$ the following statements are equivalent:

(a) Every non-empty subset S that is bounded above, has a supremum.

(b) Every non-empty subset S that is bounded below, has an infimum.

Proof. We prove that (a) implies (b).

Let S be a non-empty subset of X, bounded below. Let l_0 be a lower bound of S. Let $T = \{l : l \in X \text{ and } l \text{ is a lower bound of } S\}$. Then T is a non-empty subset of X because $l_0 \in T$. Moreover $x \in T$ and $s \in S \Rightarrow x \leq s$. This shows that T is bounded above. Thus T is a non-empty subset of X, bounded above.

By (a), T has a supremum. Let $\sup T = L$.

Then (i) $t \leq L$ for every $t \in T$, since L is an upper bound of T;

and (ii) since every $s \in S$ is an upper bound of T and $L = \sup T, L \leq s$ for every $s \in S$.

(ii) shows that L is a lower bound of S and (i) shows that $L \geq$ any lower bound of S. Consequently, $L = \inf S$.

We prove that (b) implies (a).

Let S be a non-empty subset of X, bounded above.

Let u_0 be an upper bound of S. Let $T = \{u : u \in X$ and u is an upper bound of $S\}$. Then T is a non-empty subset of X because $u_0 \in T$

Moreover $x \in T$ and $s \in S \Rightarrow x \geq s$. This shows that T is bounded below. Thus T is a non-empty subset of X bounded below.

By (b) T has an infimum.

Let $\inf T = U$. Then (i) $U \leq t$ for every $t \in T$, since V is a lower bound of T; and (ii) since every $s \in S$ is a lower bound of T and $U = \inf T, U \geq s$ for every $s \in S$.

(ii) shows that U is an upper bound of S and (i) shows that $U \leq$ any upper bound of S.

Consequently, $U = \sup S$.

Definition. An ordered set $(X, \leq)$ is said to be *order complete* if every *non-empty* subset of X which has an upper bound, has a least upper bound (a supremum), or equivalently, every *non-empty* subset of X which has a lower bound, has a greatest lower bound (an infimum).

Note. In a later chapter we shall see that the set $\mathbb{R}$ is order complete but the set $\mathbb{Q}$ is not.

1.8. Function.

Let A and B be two non-empty sets. A *function* f from A to B is a *rule of correspondence* that assigns to each element x in A, *a unique* y in B.

A is said to be the *domain* of f and B, the *co-domain* of f. We say that f is a function or a mapping from A to B and we write $f : A \longrightarrow B$

The unique element y in B that corresponds to x in A is said to be *the image* of x under f and is denoted by $f(x)$. x is said to be a *pre-image* (or an inverse image) of $f(x)$.

The set $\{f(x) : x \in A\}$ is said to be the *range* of f and is denoted by $f(A)$.

The set $\{x : f(x) = y\}$ is said to be the *pre-image set* (or the *inverse image set*) of y and is denoted by $f^{-1}(y)$.

This is to emphasize that the inverse image set of an element y in B may be a void set, or a singleton set, or a set containing more than one elements.

Definitions.

A function $f : A \to B$ is said to be *injective* (or one-to-one) if $x_1 \neq x_2$ in $A \Rightarrow f(x_1) \neq f(x_2)$ in B.

A function $f : A \to B$ is said to be *surjective* if $f(A) = B$.

A function $f : A \to B$ is said to be *bijective* if f is injective as well as surjective.

If $f : A \to B$ is injective then each element of B has at most one pre-image.

If $f : A \to B$ is surjective then each element of B has at least one pre-image.

Therefore if $f : A \to B$ is bijective then each element of B has exactly one pre-image. In this case the pre-image set of each element y in B, i.e., $f^{-1}(y)$ is a singleton set.

If $f : A \to B$ is bijective, each element in A has a definite image in B and each element in B has a definte pre-image in A. Thus f sets up a one-to-one correspondence between the elements of A ana B.

Examples.

1. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x, x \in \mathbb{Z}$. f is injective but not surjective. The range set is $2\mathbb{Z}$ (the set of all even integers) and it is a proper subset of the co-domain set $\mathbb{Z}$.

Here $f(0) = 0$, $f(1) = 2$, $f(2) = 4, \ldots$

2. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = |x|, x \in \mathbb{Z}$. f is not injective, since $f(1) = f(-1) = 1$. f is not surjective, since -1 in the co-domain set has no pre-image.

Here $f^{-1}(1) = \{1, -1\}$, $f^{-1}(-1) = \phi, \ldots$

3. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x + 1, x \in \mathbb{Z}$. f is injective as well as surjective. Therefore f is bijective.

Equality of functions.

Two functions f and g are said to be *equal* if

(i) f and g have the same domain D, and (ii) $f(x) = g(x)$ for all $x \in D$.

Example (continued).

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|, x \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x, x \geq 0$ Then $f = g$.

$$= -x, x < 0.$$

1.9. Equipotent sets. Enumerable set.

Let A and B be subsets of a universal set $P(X)$, the power set of a non-empty set X. A is said to be *equipotent* with B if there exists a bijective mapping f from A to B. We write $A \sim B$.

The relation of equipotence on the set $P(X)$, is an equivalence relation, because

(i) for any subset $A \in P(X)$, the identity mapping $i : A \to A$ is a bijective mapping and therefore $A \sim A$ for all $A \in P(X)$.

(ii) If $A \sim B$ for $A, B \in P(X)$, then there exists a bijective mapping $f : A \to B$ and this ensures the existence of the bijective mapping $f^{-1} : B \to A$, proving that $B \sim A$.

(iii) If $A \sim B$ and $B \sim C$ for $A, B, C \in P(X)$, then there exist bijective mappings $f : A \to B$ and $g : B \to C$ and this ensures the existence of the bijective mapping $g.f : A \to C$, proving that $A \sim C$.

In consequence of this equivalence relation on $P(X)$, the set $P(X)$ is partitioned into classes of equipotent sets.

The sets belonging to the same equipotence class are said to have the same *potency* or the same *cardinal number*.

The cardinal number assigned to the equipotence class of finite sets each with n elements is n. The cardinal number assigned to the null set ϕ is 0.

The cardinal number of an infinite set is said to be a *transfinite cardinal number*. The cardinal number of the set $\mathbb{N}$ is denoted by d.

Definition. A set A is said to be *enumerable* (or denumerable) if A is equipotent with $\mathbb{N}$. A set which is either finite or enumerable is said to be a *countable* set. Enumerable sets are also called *countably infinite* sets.

When a set A is enumerable, there is a bijective mapping $f : \mathbb{N} \to A$. Corresponding to each $n \in \mathbb{N}$ there is a unique element $f(n)$ in A as the image of n. Thus the elements of A can be described as $f(1), f(2), \ldots, f(n), \ldots$, or as $a_1, a_2, \ldots, a_n, \ldots$, showing that the elements of A are indexed by the set $\mathbb{N}$.

Examples.

1. The cardinal number of the set $\{1, 2, 3, \ldots, 10\}$ is 10.

2. The cardinal number of the set $S = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ is d, because S is equipotent with $\mathbb{N}$. S is an enumerable set.

That there exist sets larger than the enumerable sets (i.e., the sets with cadinal number greater than d) has been established by George Cantor, a German mathematician, in his remakable theorem.

Theorem 1.9.1. Cantor's theorem.

If A be a non-empty set, there is no surjection $\phi : A \to P(A)$, where $P(A)$ is the power set of A.

Proof. Let $a \in A$. Let $f : A \to P(A)$ be a surjection. Then $f(a)$ is an element of $P(A)$, i.e., $f(a)$ is a subset of A. The element a may or may not belong to the subset $f(a)$.

Let $S = \{a \in A : a \notin f(a)\}$. Since S is a subset of A, $S \in P(A)$ and therefore there exists an element $a_0 \in A$ such that $f(a_0) = S$.

We must have either $a_0 \in S$ or $a_0 \notin S$.

$a_0 \in S \Rightarrow a_0 \notin f(a_0)$ (by definition of S), i.e., $a_0 \notin S$, a contradiction.
$a_0 \notin S \Rightarrow a_0 \in f(a_0)$ (by definition of S), i.e., $a_0 \in S$, a contradiction.
Therefore f does not exist and the proof is complete.

The theorem asserts the existence of larger and still larger sets, i.e., the sets with greater and still greater cardinal numbers.

We shall prove in a subsequent article that the set $\mathbb{R}$ is non enumerable.

Exercises 1

1. If A, B, C be subsets of $\mathbb{R}$, prove that

(i) $(A \cap B) \cup (B \cap C) \cup (C \cap A) = (A \cup B) \cap (B \cup C) \cap (C \cup A)$;

(ii) $(A \cap B \cap C) \cup (A \cap B \cap C') \cup (A \cap B' \cap C) \cup (A \cap B' \cap C') = A$;

(iii) $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$.

2. Find $\bigcup\limits_{n=1}^{\infty} I_n$ and $\bigcap\limits_{n=1}^{\infty} I_n$, where for each $n \in \mathbb{N}$,

(i) $I_n = \left\{x \in \mathbb{R} : -\frac{1}{n} < x < \frac{1}{n}\right\}$, (ii) $I_n = \left\{x \in \mathbb{R} : -1 + \frac{1}{n} \le x \le 2 - \frac{1}{n}\right\}$.

3. Let S be the set of all positive divisors of 30. Prove that $(S, \le)$ is a poset where $a \le b$ means a is a divisor of b, for $a, b \in S$.

4. Prove that the functions $f : \mathbb{R} \to \mathbb{R}$ are neither injective nor surjective.

(i) $f(x) = \frac{x}{x^2+1}, x \in \mathbb{R}$, (ii) $f(x) = \frac{|x|}{|x|+1}, x \in \mathbb{R}$, (iii) $f(x) = [x], x \in \mathbb{R}$.

5. Are the two functions f and g equal? Give reasons.

(i) $f : D \to \mathbb{R}$ defined by $f(x) = \sin x - \cos x, x \in D$,
$g : D \to \mathbb{R}$ defined by $g(x) = \sqrt{1 - \sin 2x}, x \in D$;
and $D = \{x \in \mathbb{R} : 0 \le x \le \frac{\pi}{2}\}$.

(ii) $f : D \to \mathbb{R}$ defined by $f(x) = 2\tan^{-1} x, x \in D$,
$g : D \to \mathbb{R}$ defined by $g(x) = \tan^{-1} \frac{2x}{1-x^2}, x \in D$;
and $D = \{x \in \mathbb{R} : x > 1\}$.

2. REAL NUMBER:

2.1. Natural numbers.

The natural numbers are $1, 2, 3, \ldots$ The set of all natural numbers denoted by $\mathbb{N}$.

We assume familiarity with the algebraic operations of addition a multiplication on the set $\mathbb{N}$ and also with the linear order relation $<$ $\mathbb{N}$ defined by "$a < b$ if $a, b \in \mathbb{N}$ and a is less than b".

We discuss the following fundamental properties of $\mathbb{N}$.

1. Well ordering property.

2. Principle of induction.

2.1.1. Well ordering property. Every non-empty subset of $\mathbb{N}$ ha least element.

This means that if S is a non-empty subset of $\mathbb{N}$ then there is element m in S such that $m \leq s$ for all $s \in S$. In particular, $\mathbb{N}$ itself the least element 1.

Proof. Let S be a non-empty subset of $\mathbb{N}$. Let k be an element of Then k is a natural number.

We define a subset T by $T = \{x \in S : x \leq k\}$. Then T is a non-em subset of $\{1, 2, \ldots, k\}$.

Clearly, T is a finite subset of $\mathbb{N}$ and therefore it has a least elem say m. Then $1 \leq m \leq k$.

We now show that m is the least element of S. Let s be any elen of S.

If $s > k$ then the inequality $m \leq k$ implies $m < s$.

If $s \leq k$ then $s \in T$; and m being the least element of T, we l $m \leq s$.

Thus m is the least element of S. This completes the proof.

2.1.2. Principle of induction. Let S be a subset of $\mathbb{N}$ such that

(i) $1 \in S$, and

(ii) if $k \in S$ then $k + 1 \in S$.

Then $S = \mathbb{N}$.

Proof. Let $T = \mathbb{N} - S$. We prove that $T = \phi$.

Let T be non-empty. Then by the well ordering property of $\mathbb{N}$, the non-empty subset T has a least element, say m.

Since $1 \in S$ and 1 is the least element of $\mathbb{N}, m > 1$.

Hence $m - 1$ is a natural number and $m - 1 \notin T$. So $m - 1 \in S$.

But by (ii) $m - 1 \in S \Rightarrow (m - 1) + 1 \in S$, i.e., $m \in S$.

This contradicts that m is the least element in T. Therefore our assumption is wrong and $T = \phi$.

Therefore $S = \mathbb{N}$. This completes the proof.

Theorem 2.1.3. Let $P(n)$ be a statement involving a natural number n.

If (i) $P(1)$ is true, and

(ii) $P(k + 1)$ is true whenever $P(k)$ is true,

then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. Let S be the set of those natural numbers for which the statement $P(n)$ is true.

Then S has the properties (a) $1 \in S$ by (i)

(b) $k \in S \Rightarrow k + 1 \in S$ by (ii).

By the principle of induction $S = \mathbb{N}$.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$. This completes the proof.

Note. Let a statement $P(n)$ satisfies the conditions

(i) for some $m \in \mathbb{N}, P(m)$ is true (m being the least possible)

and (ii) $P(k)$ is true $\Rightarrow P(k + 1)$ is true for all $k \geq m$.

Then $P(n)$ is true for all natural numbers $\geq m$.

Worked Examples.

1. Prove that for each $n \in \mathbb{N}, 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

The statement is true for $n = 1$, because $1 = \frac{1(1+1)}{2}$.

Let the statement be true for some natural number k.

Then $1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$ and therefore

$(1 + 2 + 3 + \cdots + k) + (k + 1) = \frac{k(k+1)}{2} + (k + 1)$

or, $1 + 2 + 3 + \cdots + (k + 1) = \frac{(k+1)(k+2)}{2}$.

This shows that the statement is true for the natural number $k + 1$ if it is true for k. By the principle of induction, the statement is true for all natural numbers.

2. Prove that for each $n \geq 2, (n + 1)! > 2^n$.

The inequality holds for $n = 2$ since $(2 + 1)! > 2^2$.

Let the inequality hold for some natural number $k \geq 2$.
Then $(k+1)! > 2^k$

$$\begin{aligned} \text{and } (k+2)! &= (k+2)(k+1)! \\ &> 2.2^k \text{ since } k+2 > 2 \\ \text{or, } (k+2)! &> 2^{k+1}. \end{aligned}$$

This shows that if the inequality holds for $k(\geq 2)$ then it also holds for $k+1$.

By the principle of induction, the inequality holds for all natural numbers ≥ 2.

[Note that the inequality does not hold for $n = 1$.]

2.1.4. Second principle of induction.

Let S be a subset of $\mathbb{N}$ such that

(i) $1 \in S$, and (ii) if $\{1, 2, 3, \ldots, k\} \subset S$ then $k+1 \in S$.
Then $S = \mathbb{N}$.

Proof. Let $T = \mathbb{N} - S$. We prove that $T = \phi$.

Let T be non-empty. Then T will have a least element, say m, by the well ordering property of $\mathbb{N}$. Since $1 \in S, 1 \notin T$.

As m is the least element in T and $1 \notin T, m > 1$.
By the choice of m, all natural numbers less than m belong to S.
That is, $1, 2, \ldots, m-1$ all belong to S.

Then by (ii) $m \in S$ and consequently, $m \notin T$, a contradiction.
It follows that $T = \phi$ and therefore $S = \mathbb{N}$.

This completes the proof.

Worked Example (continued).

3. Prove that for all $n \in \mathbb{N}, (3+\sqrt{5})^n + (3-\sqrt{5})^n$ is an even integer.

Let $P(n)$ be the statement "$(3+\sqrt{5})^n + (3-\sqrt{5})^n$ is an even integer ".
$P(1)$ is true since $(3+\sqrt{5})^1 + (3-\sqrt{5})^1 = 6$, an even integer.

Let us assume that $P(n)$ is true for $n = 1, 2, \ldots, k$.

$(3+\sqrt{5})^{(k+1)} + (3-\sqrt{5})^{(k+1)}$
$= a^{k+1} + b^{k+1}$ where $a = 3+\sqrt{5}, b = 3-\sqrt{5}$
$= (a^k + b^k)(a+b) - (a^{k-1} + b^{k-1})ab$
$= 6(a^k + b^k) - 4(a^{k-1} + b^{k-1})$.

It is an even integer, since $a^k + b^k$ and $a^{k-1} + b^{k-1}$ are even integers.
Hence $P(k+1)$ is true whenever $P(n)$ is true for $n = 1, 2, \ldots, k$.

By the second principle of induction, $P(n)$ is true for all natural numbers.

2.2. Integers.

Addition and multiplication are defined on the set $\mathbb{N}$. Subtraction is not defined on the set $\mathbb{N}$ in the sense that if $a \in \mathbb{N}$ and $b \in \mathbb{N}$ that $a - b$ is not always an element of $\mathbb{N}$. The set $\mathbb{N}$ is enlarged by the inclusion of 0 and the numbers of the form $-n$ (called *negative* of n) for all $n \in \mathbb{N}$. The new set is called the set of *all integers* and is denoted by $\mathbb{Z}$.

$$\mathbb{Z} = \{0, 1, 2, 3, \ldots, -1, -2, -3, \ldots\}.$$

On $\mathbb{Z}$, subtraction is defined as the inverse of additon. For each $a \in \mathbb{Z}, -a \in \mathbb{Z}$. If $a \in \mathbb{Z}, b \in \mathbb{Z}$ then $a - b$ is defined by $a + (-b)$ and $a - b \in \mathbb{Z}$.

Multiplication is defined on $\mathbb{Z}$. But division, the inverse operation of multiplication, is not defined on $\mathbb{Z}$ in the sense that if $a \in \mathbb{Z}, b \in \mathbb{Z}$ then $\frac{a}{b}$ is not always an element of $\mathbb{Z}$. If the set $\mathbb{Z}$ could be enlarged by the inclusion of all numbers of the form $\frac{a}{b}$ where $a \in \mathbb{Z}, b \in \mathbb{Z}$ then the new enlarged set might be rich anough to allow division as the inverse operation of multiplication. But if $a \in \mathbb{Z}$ and $b = 0$ then there is no number of the form $\frac{a}{0}$ and therefore the enlargement of the set $\mathbb{Z}$ by inclusion of all numbers of the form $\frac{a}{b}, a \in \mathbb{Z}, b \in \mathbb{Z}$ cannot be completed.

2.3. Rational numbers.

A *rational number* is of the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$. The set of all rational numbers is denoted by $\mathbb{Q}$.

Evidently every integer is a rational number. The set $\mathbb{Z}$ is a proper subset of $\mathbb{Q}$.

We now describe some fundamental properties of the set $\mathbb{Q}$.

1. Algebraic properties of $\mathbb{Q}$.

2. Order properties of $\mathbb{Q}$.

3. Density property of $\mathbb{Q}$.

2.3.1. Algebraic properties of $\mathbb{Q}$.

Addition and multiplication are defined on the set $\mathbb{Q}$ satisfying the following properties:

A1. $a + b \in \mathbb{Q}$ for all $a, b \in \mathbb{Q}$,

A2. $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{Q}$,

A3. there exists an element 0 in $\mathbb{Q}$ (called the zero element) such that $a + 0 = a$ for all $a \in \mathbb{Q}$,

A4. for each a in $\mathbb{Q}$ there exists an element $-a$ in $\mathbb{Q}$ such that $a + (-a) = 0$,

A5. $a + b = b + a$ for all $a, b \in \mathbb{Q}$, M1. $a.b \in \mathbb{Q}$ for all $a, b \in \mathbb{Q}$,

M2. $(a.b).c = a.(b.c)$ for all $a, b, c \in \mathbb{Q}$,

M3. there exists an element 1 in $\mathbb{Q}$ (called the unity) such that $a.1 =$ for all $a \in \mathbb{Q}$.

M4. for each element $a \neq 0$ in $\mathbb{Q}$ there exists an element $\frac{1}{a}$ in $\mathbb{Q}$ su that $a.\frac{1}{a} = 1$,

M5. $a.b = b.a$ for all $a, b \in \mathbb{Q}$,

D. $a.(b + c) = a.b + a.c$ for all $a, b, c \in \mathbb{Q}$.

$-a$ is the additive inverse of a. It is called the *negative* of a. $\frac{1}{a}$ the multiplicative inverse of a. It is also called the *reciprocal* of a. T reciprocal of a exists provided $a \neq 0$. The zero element 0 and the un 1 are unique elements.

A2 states that addition is associative on $\mathbb{Q}$. A5 states that additi is commutative on $\mathbb{Q}$. M2 states that multiplication is associative $\mathbb{Q}$. M5 states that multiplication is commutative on $\mathbb{Q}$. D states distributive property. Multiplication is distributive over addition.

The set $\mathbb{Q}$ is said to form a *field* under addition and multiplicatio

2.3.2. Order properties of $\mathbb{Q}$.

On the set $\mathbb{Q}$, a linear order relation $<$ is defined by "$a < b$ if $\mathbb{Q}, b \in \mathbb{Q}$ and a is less than b "and it satisfies the following condition

O1. If $a, b \in \mathbb{Q}$ then exactly one of the following statements hold $a < b$, or $a = b$, or $b < a$; (law of trichotomy)

O2. $a < b$ and $b < c \Rightarrow a < c$ for $a, b, c \in \mathbb{Q}$; (transitivity)

O3. $a < b \Rightarrow a + c < b + c$ for $a, b, c \in \mathbb{Q}$;

O4. $a < b$ and $0 < c \Rightarrow ac < bc$ for $a, b, c \in \mathbb{Q}$.

Note 1. $a < b$ is equivalently expressed as $b > a$ (b is greater than c

The law of trichotomy states that a rational number a is one of following : $a < 0, a = 0, 0 < a$. i.e., $a < 0, a = 0, a > 0$.

A rational number a is said to be *positive* if $a > 0$ and is said t *negative* if $a < 0$.

2. If $a, b, c \in \mathbb{Q}$ and $a < c$, $c < b$ both hold, we write $a < c < b$. We that c lies between a and b.

3. The field $\mathbb{Q}$ together with the order relation defined on $\mathbb{Q}$ satisf O1-O4 becomes an *ordered field.*

2.3.3. Density property of $\mathbb{Q}$.

If x and y be any two rational numbers and $x < y$, there exists a rational number r such that $x < r < y$. That is, between any two rational numbers there exists a rational number.

$$\begin{aligned} x < y \quad &\Rightarrow \quad x + y < y + y, \text{ by } O3 \\ &\Rightarrow \quad \tfrac{1}{2}(x+y) < \tfrac{1}{2}(2y), \text{ by } O4 \\ \text{i.e.,} \quad & \quad \tfrac{1}{2}(x+y) < y. \end{aligned}$$

$$\begin{aligned} \text{Again, } x < y \quad &\Rightarrow \quad x + x < x + y, \text{ by } O3 \\ &\Rightarrow \quad \tfrac{1}{2}(2x) < \tfrac{1}{2}(x+y), \text{ by } O4 \\ \text{i.e.,} \quad & \quad x < \tfrac{1}{2}(x+y). \end{aligned}$$

Therefore we have $x < \frac{1}{2}(x+y) < y$. Then $r = \frac{1}{2}(x+y)$.

We observe that between two rational numbers x and y (where $x < y$) there exists another rational number $\frac{1}{2}(x+y)$. Again between x and $\frac{1}{2}(x+y)$ (since $x < \frac{1}{2}(x+y)$) there exists another rational number and the process can be continued indefinitely.

We say that between any two rational numbers x and y (where $x < y$) there exist infinitely many rational numbers. This is expressed by saying that the set $\mathbb{Q}$ is dense and this property of $\mathbb{Q}$ is called the *density property* of $\mathbb{Q}$.

Because of this density property of $\mathbb{Q}$, between any two rational numbers x and y we can interpolate infinitely many rational numbers.

2.3.4. Geometrical representation of rational numbers.

Rational numbers can be represented by points on a straight line. Let $X'X$ be a directed line. We take a point O on the line. O divides the line into two parts. The part to the right of O is called the positive side and the part to the left of O is called the negative side.

Let us take a point A to the right of O. Let O represent the rational number *zero* and A represent the rational number *one*. Taking the distance OA as the unit distance on some chosen scale, each rational number can be represented by a unique point on the line. First of all, the positive integers $2, 3, \cdots$ are represented by the points $A_2, A_3, \cdots$ lying to the right of O where $OA_2 = 2OA, OA_3 = 3OA, \cdots$ and the negative integers $-1, -2, \cdots$ are represented by the points $A_1', A_2', \cdots$ lying to the left O such that $OA_1' = OA, OA_2' = 2OA, \cdots$

To represent a positive rational number r of the form $\frac{p}{q}$ where p, q are positive integers, we measure p times the distance OA to the right of O and get a point B and then measure the qth part of the distance OB to the right of O to the get the point P. P represents the rational number

r. If r be a negative rational number $(-s)$ then the point P' to the left of O (where $OP' = OP$ and P represents s) represents r.

Thus every rational number can be made to correspond to a point on the line. If a point that corresponds to a rational number be called a rational point then we observe that between any two rational points there lie infinitely many rational points. If all the rational numbers be plotted as points on the line it appears that the whole line is covered by rational points i.e., the whole line is composed of only rational points.

A little further examination will show that such a view point is not tenable.

If we take a point D to the right of O such that OD is the length of the diagonal of the square on the side OA, then D is not a rational point as can be established by the following example.

Example. There does not exist a rational number r such that $r^2 = 2$.

If possible, let p and q are integers such that $(p/q)^2 = 2$. It may be assumed that p and q have no common factor other than 1.

Now $(p/q)^2 = 2 \Rightarrow p^2 = 2q^2 \Rightarrow p^2$ is even $\Rightarrow p$ is even, because if p be odd then p^2 is also odd.

Let $p = 2m$, where m is an integer.
Then $p^2 = 2q^2 \Rightarrow 2m^2 = q^2 \Rightarrow q^2$ is even $\Rightarrow q$ is even.

Thus we arrive at a contradiction to the assumption that p and q have no common factor other than 1.

Therefore there is no rational number whose square is 2.

The point D, therefore corresponds to a new type of number, called an *irrational number.*

The next theorem shows the existence of many irrational numbers.

Theorem 2.3.5. Let m be a non-square positive integer. There does not exist a rational number r such that $r^2 = m$.

Proof. Since m is a non-square positive integer, there exist two consecutive square integers λ^2 and $(\lambda + 1)^2$ such that $\lambda^2 < m < (\lambda + 1)^2$.

If possible, let $r = p/q$ (where p and q are positive integers prime to each other) be such that $r^2 = m$.

Then $\lambda^2 < (p/q)^2 < (\lambda + 1)^2$
or, $\lambda < p/q < \lambda + 1$
or, $\lambda q < p < \lambda q + q$
or, $0 < p - \lambda q < q$ ... ... (i)

$$\begin{aligned} m(p-\lambda q)^2 &= mp^2 - 2\lambda mpq + \lambda^2 mq^2 \\ &= m^2q^2 - 2\lambda mpq + \lambda^2 p^2, \text{ since } mq^2 = p^2 \\ &= (mq-\lambda p)^2. \end{aligned}$$

So $m = (\frac{mq-\lambda p}{p-\lambda q})^2$.

Thus $m = (p/q)^2 \Rightarrow m = (\frac{mq-\lambda p}{p-\lambda q})^2$.

Since p and q are prime to each other and $(p/q)^2, (\frac{mq-\lambda p}{p-\lambda q})^2$ are two representations of m, we must have $p - \lambda q > q$ and this contradicts (i).

So our assumption that $r^2 = m$ is wrong and the theorem is done.

2.4. Real numbers.

The set containing all rational as well as irrational numbers is called the set of all *real numbers*. The set of all real numbers is denoted by $\mathbb{R}$.

We now describe some fundamental properties of the set $\mathbb{R}$.

1. Algebraic properties of $\mathbb{R}$.

2. Order properties of $\mathbb{R}$.

3. Completeness property of $\mathbb{R}$.

4. Archimedean property of $\mathbb{R}$.

5. Density property of $\mathbb{R}$.

2.4.1. Algebraic properties of $\mathbb{R}$.

Addition and multiplication are defined on the set $\mathbb{R}$ satisfying the following properties :

A1. $a + b \in \mathbb{R}$ for all a, b in $\mathbb{R}$;

A2. $(a + b) + c = a + (b + c)$ for all a, b, c in $\mathbb{R}$;

A3. there exists an element 0 in $\mathbb{R}$ (called the zero element) such that $a + 0 = a$ for all a in $\mathbb{R}$;

A4. for each a in $\mathbb{R}$ there exists an element $-a$ in $\mathbb{R}$ such that $a + (-a) = 0$;

A5. $a + b = b + a$ for all a, b in $\mathbb{R}$;

M1. $a.b \in \mathbb{R}$ for all a, b in $\mathbb{R}$;

M2. $(a.b).c = a.(b.c)$ for all a, b, c in $\mathbb{R}$;

M3. there exists an element 1 in $\mathbb{R}$ (called the unity) such that $a.1 = a$ for alll a in $\mathbb{R}$;

M4. for each element $a \neq 0$ in $\mathbb{R}$ there exists an element $\frac{1}{a}$ in $\mathbb{R}$, such that $a.\frac{1}{a} = 1$;

M5. $a.b = b.a$ for all a, b in $\mathbb{R}$;

D. $a.(b+c) = a.b + a.c$ for all a, b, c in $\mathbb{R}$.

$-a$ is the additive inverse of a: It is also called the *negative* of a. $1/$ is the multiplicative inverse a. It is also called the *reciprocal* of a.

The reciprocal of a exists provided $a \neq 0$.

The zero element 0 and the unity 1 are unique.

$\mathbb{R}$ is said to form a *field* under the operations- addition and multipl cation.

Addition and multiplication are both commutative and associative c the set $\mathbb{R}$. Multiplication is distributive over addition.

Theorem 2.4.2. Let $a, b, c \in \mathbb{R}$. Then

(i) $a + b = a + c$ implies $b = c$ (cancellation law for addition);

(ii) $a \neq 0$ and $a.b = a.c$ implies $b = c$ (cancellation law for multip cation).

Proof. (i) $a + b = a + c$.

$-a \in \mathbb{R}$, since $a \in \mathbb{R}$. Therefore $-a + (a + b) = -a + (a + c)$
or, $(-a + a) + b = (-a + a) + c$, by A2
or, $0 + b = 0 + c$, by A4
or, $b = c$.

(ii) $a.b = a.c$.

$\frac{1}{a} \in \mathbb{R}$, since $a \neq 0$. Therefore $(\frac{1}{a}).(a.b) = (\frac{1}{a}).(a.c)$
or, $(\frac{1}{a}.a).b = (\frac{1}{a}.a).c$, by M2
or, $1.b = 1.c$, by M4
or, $b = c$.

Theorem 2.4.3. Let $a \in \mathbb{R}$. Then

(i) $a.0 = 0$,

(ii) $(-1).a = -a$,

(iii) $-(-a) = a$,

(iv) $1/(1/a) = a$, provided $a \neq 0$.

Proof. (i) We have $0 + 0 = 0$ in $\mathbb{R}$.

Then $a.(0 + 0) = a.0$
or, $a.0 + a.0 = a.0$, by D
$-(a.0) \in \mathbb{R}$. Therefore $-(a.0) + [a.0 + a.0] = (-a.0) + a.0$
or, $[-(a.0) + a.0] + a.0 = 0$, by A2 and A4
or, $0 + a.0 = 0$, by A4
or, $a.0 = 0$, by A3.

(ii) We have $1 + (-1) = 0$ in $\mathbb{R}$.

Then $[1 + (-1)].a = 0$

or, $a + (-1).a = 0$
$-a \in \mathbb{R}$. Therefore $-a + [a + (-1).a] = -a + 0$
or, $(-a + a) + (-1).a = -a$, by A2 and A3
or, $0 + (-1).a = -a$, by A4
or, $(-1).a = -a$, by A3.

(iii) We have $a + (-a) = 0$, by A4.
Since $-a \in \mathbb{R}, -a + \{-(-a)\} = 0$, by A4.
Therefore $-a + a = -a + \{-(-a)\}$.
or, $a = -(-a)$, by cancellation law for addition.

(iv) Since $a \neq 0, \frac{1}{a} \in \mathbb{R}$ and $a.(\frac{1}{a}) = 1$.
$a.\frac{1}{a} = 1 \Rightarrow \frac{1}{a} \neq 0$, because $\frac{1}{a} = 0 \Rightarrow 1 = 0$.
Since $\frac{1}{a} \neq 0, 1/(1/a) \in \mathbb{R}$ and $\frac{1}{a}.\{1/(1/a)\} = 1$.
Therefore $\frac{1}{a}.a = \frac{1}{a}.\{1/(1/a)\}$.
Since $\frac{1}{a} \neq 0$, $a = 1/(1/a)$, by cancellation law for multiplication.

Theorem 2.4.4. Let $a, b, c \in \mathbb{R}$. Then $a.b = 0$ implies $a = 0$, or $b = 0$.

Proof. Let $a \neq 0$. Then $\frac{1}{a} \in \mathbb{R}$ and $\frac{1}{a}.a = 1$.
$a.b = 0 \Rightarrow \frac{1}{a}.(ab) = \frac{1}{a}.0 \Rightarrow (\frac{1}{a}.a).b = 0 \Rightarrow b = 0$.
Therefore $a \neq 0 \Rightarrow b = 0$.
Contrapositively, $b \neq 0 \Rightarrow a = 0$.
Therefore either $a = 0$ or $b = 0$.

Theorem 2.4.5. Let $a, b \in \mathbb{R}$. Then
(i) $a.(-b) = (-a).b = -(a.b)$,
(ii) $(-a).(-b) = a.b$.

Proof. We have $b + (-b) = 0$ in $\mathbb{R}$.
Therefore $a.[b + (-b)] = a.0$.
or, $a.b + a.(-b) = 0$, by D and theorem 2.4.3 (i)
$-(a.b) \in \mathbb{R}$. Therefore $-(a.b) + [a.b + a.(-b)] = -(a.b)$.
or, $[-(a.b) + a.b] + a.(-b) = -(a.b)$, by A2
or, $0 + a.(-b) = -(a.b)$, by A4
or, $a.(-b) = -(a.b)$, by A3.

Again $-a + a = 0$.
Therefore $[-a + a].b = 0.b$.
Proceeding similarly, we can prove $(-a).b = -(a.b)$.

Therefore $a.(-b) = (-a).b = -(a.b)$.

(ii) Let $p = -a$. Then $p \in \mathbb{R}$.
$(-a).(-b) = p.(-b) = -(p.b)$, by (i)
$= -[(-a).b] = -(-(a.b)) = a.b$, by theorem 2.4.3 (iii).

2.4.6. Order properties of $\mathbb{R}$.

On the set $\mathbb{R}$, a linear order relation $<$ is defined by "$a < b$ if $a \in \mathbb{R}, b \in \mathbb{R}$ and a is less than b" and it satisfies the following conditions :

O1. If $a, b \in \mathbb{R}$, then exactly one of the following statements holds – $a < b$, or $a = b$, or $b < a$ (law of trichotomy);

O2. $a < b$ and $b < c \Rightarrow a < c$ for $a, b, c \in \mathbb{R}$ (transitivity);

O3. $a < b \Rightarrow a + c < b + c$ for $a, b, c \in \mathbb{R}$;

O4. $a < b$ and $0 < c \Rightarrow ac < bc$ for $a, b, c \in \mathbb{R}$.

Note. $a < b$ is equivalently expressed as $b > a$ (b is greater than a).

The law of trichotomy states that a real number a is one of the following : $a < 0$, $a = 0$, $0 < a$. i.e., $a < 0$, $a = 0$, $a > 0$.

a is said to be a *positive* real number if $a > 0$.

a is said to be a *negative* real number if $a < 0$.

We use the symbol $a \geq 0$ to mean that a is either positive or zero; $a \leq 0$ to mean that a is either negative or zero.

If $a, b, c \in \mathbb{R}$ and $a < c$, $c < b$ both hold, we write $a < c < b$ and say that c lies between a and b.

Note. The field $\mathbb{R}$ together with the order relation defined on $\mathbb{R}$ satisfying O1-O4 becomes an *ordered field.*

Remark. On a field $(F, +, .)$, in general, an order relation is defined with the help of a *positive set* in F. A subset P of F is called a positive set if

(1) $a \in P$, $b \in P \Rightarrow a + b \in P$ and $a.b \in P$,

(2) if $c \in F$ then exactly one of the following statements holds-

$$c \in P,\ c = 0,\ -c \in P.$$

The positive set P is used to define an order $<$ in F.

Definition. If $a, b \in F$, then $a < b$ (a is less than b) if and only if $b - a \in P$.

$a < b$ is same as $b > a$ (b is greater than a).

From definition it follows that $a > 0$ if and only if $a - 0 \in P$, i.e., $a \in P$.

The order properties O1-O4 can be deduced from the above definition.

O1. Let $a, b \in F$. Then $a - b \in F$.

Therefore by (2) exactly one of the following statements holds-

$a - b \in P$, $a - b = 0$, $-(a - b) \in P$

i.e., $a - b > 0$, $a - b = 0$, $-(a - b) > 0$

i.e., $b < a$, $b = a$, $a < b$.

O2. Let $a < b$ and $b < c$.
Then $b - a \in P$ and $c - b \in P$ and by (1), $(b - a) + (c - b) \in P$
or, $c - a \in P$, i.e., $a < c$.

O3. Let $a, b, c \in F$ and $a < b$. Then $b - a \in P$.
Therefore $(b + c) - (a + c) \in P$, i.e., $a + c < b + c$.

O4. Let $a < b$ and $c > 0$. Then $b - a \in P, c \in P$.
By (1), $(b - a)c \in P$
or, $bc - ac \in P$, i.e., $ac < bc$.

The field $\mathbb{R}$ is an ordered field. The positive set in $\mathbb{R}$ is called the set of all positive real numbers and is denoted by $\mathbb{R}^+$.

Theorem 2.4.7. Let $a \in \mathbb{R}$. Then

(i) $a > 0 \Rightarrow -a < 0$;

(ii) $a < 0 \Rightarrow -a > 0$.

Proof. (i) $a \in \mathbb{R}$ and $a + (-a) = 0$, by A4.
By the law of trichotomy, either $-a < 0$ or $-a = 0$, or $-a >$
Let $-a > 0$.
$-a > 0, a \in \mathbb{R} \Rightarrow -a + a > a$, by O3
$\Rightarrow 0 > a$, a contradiction.
Let $-a = 0$. Then $a + (-a) = a + 0 = a$,
and also $a + (-a) = 0$, by A4.
Therefore $a = 0$, a contradiction.

We conclude that $-a < 0$.

(ii) Similar proof.

Theorem 2.4.8. Let $a, b \in \mathbb{R}$. Then

(i) $a > 0, b > 0 \Rightarrow a + b > 0$,

(ii) $a < 0, b < 0 \Rightarrow a + b < 0$,

(iii) $a > 0, b > 0 \Rightarrow ab > 0$,

(iv) $a < 0, b < 0 \Rightarrow ab > 0$,

(v) $a > 0, b < 0 \Rightarrow ab < 0$.

Proof. (i) $a > 0$ and $b \in \mathbb{R} \Rightarrow a + b > b$, by O
$a + b > b$ and $b > 0 \Rightarrow a + b > 0$, by O2

(ii) Similar proof.

(iii) $a > 0, b > 0 \Rightarrow a.b > 0.b$, by O4
i.e., $ab > 0$.

$$\begin{aligned}
\text{(iv)} a < 0, b < 0 \quad &\Rightarrow \quad a < 0, -b > 0 \\
&\Rightarrow \quad a.(-b) < 0(-b), \text{ by O4} \\
&\Rightarrow \quad -ab < 0 \\
&\Rightarrow \quad -(-ab) > 0, \text{ by Theorem 2.4.7 (ii)} \\
&\Rightarrow \quad ab > 0, \text{ by Theorem 2.4.3 (iii).}
\end{aligned}$$

(v) Similar proof.

Theorem 2.4.9. Let $a, b, c, d \in \mathbb{R}$ and $a > b$, $c > d$. Then $a + c > b + d$

Proof. $a > b$ and $c \in \mathbb{R} \Rightarrow a + c > b + c$, by O3
$c > d$ and $b \in \mathbb{R} \Rightarrow b + c > b + d$, by O3
$a + c > b + c$ and $b + c > b + d \Rightarrow a + c > b + d$, by O2.

Corollary. Let $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n \in \mathbb{R}$ and $a_i > b_i$ for $i = 1, 2, \ldots, n$.
Then $a_1 + a_2 + \cdots + a_n > b_1 + b_2 + \cdots + b_n$.

Theorem 2.4.10. Let $a, b, c, d \in \mathbb{R}$ and $a > 0, b > 0, c > 0, d > 0$. The $a > b, c > d \Rightarrow ac > bd$.

Proof. $a > b$ and $c > 0 \Rightarrow ac > bc$, by O4
$c > d$ and $b > 0 \Rightarrow bc > bd$, by O4
$ac > bc$ and $bc > bd \Rightarrow ac > bd$, by O2.

Corollary 1. Let $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n \in \mathbb{R}$ and $a_i > 0, b_i > 0$ f $i = 1, 2, \ldots, n$.
Then $a_i > b_i \Rightarrow a_1 a_2 \ldots a_n > b_1 b_2 \ldots b_n$.

Corollary 2. Let $a, b \in \mathbb{R}$ and $a > b > 0$. Then $a^n > b^n$ for all $n \in \mathbb{N}$.

Theorem 2.4.11. If $a \in \mathbb{R}$ and $a \neq 0$, then $a^2 > 0$.

Proof. Since $a \neq 0$, either $a < 0$ or $a > 0$, by O1.

Case I. Let $a < 0$. Then $-a > 0$, by Theorem 2.4.7 (ii)
By O4, $a. - a < 0. - a$. Therefore $-a^2 < 0$.
This implies $a^2 > 0$, by Theorem 2.4.7 (ii)

Case II. Let $a > 0$.
By O4, $a.a > a.0$. Therefore $a^2 > 0$.

Combining the cases, we have $a^2 > 0$ for all $a \neq 0$.

Corollary. $1 > 0$, since $1 = 1.1 = 1^2$.

Theorem 2.4.12. Let $a \in \mathbb{R}$. Then
(i) $a > 0 \Rightarrow \frac{1}{a} > 0$, (ii) $a < 0 \Rightarrow \frac{1}{a} < 0$.

Proof left to the reader.

Theorem 2.4.13. $n > 0$ for all $n \in \mathbb{N}$.

Proof. The statement holds for $n = 1$, since $1 > 0$.

Let us assume that the statement holds for $n = k$, where $k \in \mathbb{N}$. Then $k > 0$. $k > 0$ and $1 > 0 \Rightarrow k + 1 > 0$, by Theorem 2.4.8.

This shows that the statement holds for $k + 1$ if it holds for k. By the principle of induction, the statement holds for all $n \in \mathbb{N}$.

Deduction. For all $n \in \mathbb{N}$, $\frac{1}{n} > 0$.

Theorem 2.4.14. Let $a, b \in \mathbb{R}$. Then $a < b \Rightarrow a < \frac{a+b}{2} < b$.

Proof.
$$\begin{aligned} a < b &\Rightarrow a + a < a + b \\ &\Rightarrow 2a < a + b \\ &\Rightarrow \tfrac{1}{2}.2a < \tfrac{1}{2}(a + b), \text{ since } \tfrac{1}{2} \in \mathbb{R} \text{ and } \tfrac{1}{2} > 0 \\ &\Rightarrow a < \tfrac{a+b}{2}. \end{aligned}$$
Also
$$\begin{aligned} a < b &\Rightarrow a + b < b + b \\ &\Rightarrow a + b < 2b \\ &\Rightarrow \tfrac{1}{2}(a + b) < \tfrac{1}{2}.2b, \text{ since } \tfrac{1}{2} \in \mathbb{R} \text{ and } \tfrac{1}{2} > 0 \\ &\Rightarrow \tfrac{a+b}{2} < b. \end{aligned}$$
Therefore $a < \frac{1}{2}(a + b) < b$.

Corollary. There is no least positive real number.

If possible, let a be the least positive real number. Then $a > 0$. $0 < a \Rightarrow 0 < \frac{1}{2}a < a$ by the theorem.

This shows that $\frac{1}{2}a$ is a positive real number and $\frac{1}{2}a < a$ indicates that a is not the least positive real number.

It follows that there is no least positive real number.

2.4.15. Absolute value.

Let $a \in \mathbb{R}$. The absolute value of a, denoted by $|a|$, is defined by
$$\begin{aligned} |a| &= a, \text{ if } a > 0 \\ &= 0, \text{ if } a = 0 \\ &= -a, \text{ if } a < 0. \end{aligned}$$

For example, $|3| = 3$, $|-2| = 2$, $|0| = 0$.

It follows from definition that $|a|$ is a non-negative real number. $|a| = 0$ if and only if $a = 0$.

Theorem 2.4.16.

(i) $|-a| = |a|$ for all $a \in \mathbb{R}$;

(ii) $|ab| = |a||b|$ for all $a, b \in \mathbb{R}$;

(iii) if $a, c \in \mathbb{R}$ and $c > 0$, then $|a| < c \Leftrightarrow -c < a < c$;

(iv)$-|a| \leq a \leq |a|$ for all $a \in \mathbb{R}$.

Proof.(**i**)
Case I. Let $a > 0$. Then $-a < 0$ and $|-a| = -(-a) = a = |a|$.

Case II. Let $a < 0$. Then $-a > 0$ and $|-a| = -a = |a|$.

Case III. Let $a = 0$. Then $-a = 0$ and $|-a| = 0 = |a|$.

It follows that $|-a| = |a|$.

(**ii**) Case I. Let one or both of a, b be 0. Then $ab = 0$.
In this case $|ab| = 0$ and $|a||b| = 0$. Therefore $|ab| = |a||b|$.

Case II. Let $a > 0, b > 0$. Then $ab > 0$ and $|ab| = ab$, $|a| = a$, $|b| = b$. Therefore $|ab| = |a||b|$.

Case III. Let $a < 0, b > 0$. Then $ab < 0$ and $|ab| = -ab$, $|a| = -a$, $|b| = b$. Therefore $|ab| = |a||b|$.

Case IV. Let $a > 0, b < 0$.
Similar proof.

Case V. Let $a < 0, b < 0$. Then $ab > 0$ and $|ab| = ab$, $|a| = -a$, $|b| = -b$. Therefore $|ab = |a||b|$.

Combining the cases, we have $|ab| = |a||b|$.

Deduction. $|a^2| = |a|^2$ for all $a \in \mathbb{R}$.

(**iii**) Let $|a| < c$. Then if $a \geq 0$, $a < c$ and if $a < 0$, $-a < c$ and this implies $-c < a$. Therefore $|a| < c \Rightarrow -c < a < c$.

Conversely, let $c > 0$ and $-c < a < c$.
Then we have $a < c$, $0 < c$ and $-a < c$.
Combining, we have $|a| < c$.

Corollary. If $c \in \mathbb{R}$ and $c > 0$ then $|a| \leq c \Leftrightarrow -c \leq a \leq c$.

(**iv**) Let $a > 0$. Then $-|a| < 0$ and $a = |a|$.
Therefore $-|a| < a = |a|$.
Let $a = 0$. Then $-|a| = a = |a|$.
Let $a < 0$. Then $a = -|a|$ and $a < |a|$.
Therefore $-|a| = a < |a|$.
Combining the cases, we have $-|a| \leq a \leq |a|$.

Theorem 2.4.17. (Triangle inequality)

For all $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.

Proof. We have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$.

Then $-(|a| + |b|) \leq a + b \leq |a| + |b|$
This implies $|a+b| \leq |a| + |b|$, since $-c \leq a \leq c \Rightarrow |a| \leq c$.

Corollary 1. $|a - b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$.

Proof. Replacing b by $-b$ in the triangle inequality we get the inequality.

Corollary 2. $||a| - |b|| \leq |a - b|$.

Proof. $|a| = |a - b + b| \leq |a - b| + |b|$.
or, $|a| - |b| \leq |a - b|$.
Again $|b| = |b - a + a| \leq |b - a| + |a|$
or, $|b| - |a| \leq |b - a| = |a - b|$.
So we have $-|a - b| \leq |a| - |b| \leq |a - b|$.
This implies $||a| - |b|| \leq |a - b|$, since $-c \leq a \leq c \Rightarrow |a| \leq c$.

Corollary 3. Let $a_1, a_2, \ldots, a_n \in \mathbb{R}$.
Then $|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$.

Theorem 2.4.18. If $a, b \in \mathbb{R}$,

$$\max\{a, b\} = \tfrac{1}{2}\{a + b + |a - b|\},$$
$$\min\{a, b\} = \tfrac{1}{2}\{a + b - |a - b|\}.$$

Proof. $\max\{a, b\} = a$ if $a > b$
$= b$ if $a < b$
$= \frac{1}{2}(a + b)$ if $a = b$.

$\min\{a, b\} = b$ if $a > b$
$= a$ if $a < b$
$= \frac{1}{2}(a + b)$ if $a = b$.

It follows that

if $a > b$ $\quad \max\{a, b\} - \min\{a, b\} = a - b = |a - b|$,

if $a < b$ $\quad \max\{a, b\} - \min\{a, b\} = b - a = |a - b|$,

if $a = b$ $\quad \max\{a, b\} - \min\{a, b\} = 0 = |a - b|$.

Also if $a > b$ $\quad \max\{a, b\} + \min\{a, b\} = a + b$,

if $a < b$ $\quad \max\{a, b\} + \min\{a, b\} = b + a$,

if $a = b$ $\quad \max\{a, b\} + \min\{a, b\} = \frac{1}{2}(a + b) + \frac{1}{2}(a - b) = a + b$.

Therefore we have $\max\{a, b\} + \min\{a, b\} = a + b$ for $a, b \in \mathbb{R}$;

$\max\{a, b\} - \min\{a, b\} = |a - b|$ for $a, b \in \mathbb{R}$.

Consequently, $\max\{a, b\} = \frac{1}{2}\{a + b + |a - b|\}$;

$\min\{a, b\} = \frac{1}{2}\{a + b - |a - b|\}$ for all $a, b \in \mathbb{R}$.

Worked Examples.

1. Solve the equation $\left|\frac{x+2}{2x-1}\right| = 3$.

$\left|\frac{x+2}{2x-1}\right| = 3 \Rightarrow \frac{x+2}{2x-1} = \pm 3.$
$\frac{x+2}{2x-1} = 3 \Rightarrow x+2 = 6x-3 \Rightarrow x = 1$
$\frac{x+2}{2x-1} = -3 \Rightarrow x+2 = -6x+3 \Rightarrow x = \frac{1}{7}.$
Therefore $x = 1, \frac{1}{7}$.

2. Find the solution set of the inequality $\left|\frac{x+3}{2x-6}\right| \leq 1$.

The solution set is the union of two sets S_1 and S_2 where
$S_1 = \{x : 2x-6 > 0 \text{ and } -1 \leq \frac{x+3}{2x-6} \leq 1\}$
$S_2 = \{x : 2x-6 < 0 \text{ and } -1 \leq \frac{x+3}{2x-6} \leq 1\}$.
If $2x - 6 > 0$, then
$-1 \leq \frac{x+3}{2x-6} \leq 1 \Leftrightarrow -2x+6 \leq x+3 \leq 2x-6 \;\ldots\;\ldots$ (i)
If $2x - 6 < 0$, then
$-1 \leq \frac{x+3}{2x-6} \leq 1 \Leftrightarrow 2x-6 \leq x+3 \leq -2x+6 \;\ldots\;\ldots$ (ii)

From (i) $x > 3$ and $x \geq 1$ and $x \geq 9$ simultaneously.
From (ii) $x < 3$ and $x \leq 9$ and $x \leq 1$ simultaneously.

Therefore $S_1 = \{x \in \mathbb{R} : x \geq 9\}$ and $S_2 = \{x \in \mathbb{R} : x \leq 1\}$.
So the solution set is $\{x \in \mathbb{R} : x \geq 9\} \cup \{x \in \mathbb{R} : x \leq 1\}$.

2.4.19. Completeness property of $\mathbb{R}$.

Definition. Let S be a subset of $\mathbb{R}$. A real number u is said to be a *upper bound* of S if $x \in S \Rightarrow x \leq u$. A real number l is said to be a *lower bound* of S if $x \in S \Rightarrow x \geq l$.

Let S be a subset of $\mathbb{R}$. S is said to be *bounded above* if S has a upper bound. S is said to be *bounded below* if S has a lower bound.

In other words, a set $S \subset \mathbb{R}$ is said to be bounded above if there exis a real number u such that $x \in S \Rightarrow x \leq u$; S is said to be bounded bel if there exists a real number l such that $x \in S \Rightarrow x \geq l$.

S is said to be a *bounded set* if S be bounded above as well as bound below.

Examples.

1. Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$. S is bounded above, 1 being an upper boun S is bounded below, 0 being a lower bound.

2. Let $S = \{x \in \mathbb{R} : 1 < x < 2\}$. S is bounded above, 2 being an upp bound. S is bounded below, 1 being a lower bound.

3. Let $S = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$. S is bounded above, 2 being an upper bound. S is bounded below, 1 being a lower bound.

4. Let $S = \phi$. Every real number x is an upper bound of the set S. Every real number x is a lower bound of the set S. S is a bounded set.

Definition. Let S be a subset of $\mathbb{R}$. If S be bounded above, then an upper bound of S is said to be the *supremum* of S (or *the least upper bound* of S) if it is less than every other upper bound of S. If S be bounded below then a lower bound of S is said to be the *infimum* of S (or *the greatest lower bound* of S) if it is greater than every other lower bound of S.

If a set $S \subset \mathbb{R}$ be bounded above then S has an upper bound. If u be an upper bound of S then obviously each of $u+1, u+2, \cdots$ is an upper bound of S. Therefore for a set S bounded above, there exist infinitely many upper bounds.

It is not possible to ascertain if S has a least upper bound. It is a deeper property of the set $\mathbb{R}$ that if S be a *non-empty* subset of $\mathbb{R}$, bounded above, then the set of all upper bounds of S has a least element. We shall take this property of $\mathbb{R}$ as an axiom, called "the least upper bound axiom". This property is also called *the supremum property of* $\mathbb{R}$.

Statement of the property.

Every *non-empty* subset of $\mathbb{R}$ that is bounded above has a *least upper bound* (or a supremum).

A similar approach can be made in respect of a non-empty subset of $\mathbb{R}$ that is bounded below and we can obtain the greatest lower bound property of $\mathbb{R}$, or *the infimum property of* $\mathbb{R}$ in the following form.

Every *non-empty* subset of $\mathbb{R}$ that is bounded below has a *greatest lower bound* (or an infimum).

We can establish that these two properties are equivalent in the sense that one of these implies the other. However, we assume the supremum property of $\mathbb{R}$ as an axiom and call it *the completeness property* of $\mathbb{R}$ and treat the other property (the infimum property) as a theorem.

For a non-empty set S, bounded above, the supremum of S is denoted by $\sup S$. $\sup S$ may or may not belong to S. For a non-empty set S, bounded below, the infimum of S is denoted by $\inf S$. $\inf S$ may or may not belong to S.

If S happens to be a non-empty finite set, then $\sup S$ and $\inf S$ both exist and belong to S. They are said to be the *maximum* and the *minimum* of S respectively and are denoted by $\max S$ and $\min S$.

Theorem 2.4.20. Let S be a non-empty subset of $\mathbb{R}$, bounded below.

Then S has an infimum.

Proof. Let l_o be a lower bound of S. Let $T = \{l \in \mathbb{R} : l$ is a lower bound of $S\}$. Then T is a non-empty subset of $\mathbb{R}$ because $l_o \in T$.

Moreover, $x \in T$ and $s \in S \Rightarrow x \leq s$. This shows that T is bounded above.

Thus T is a non-empty subset of $\mathbb{R}$, bounded above. By the supremum property of $\mathbb{R}, T$ has a supremum. Let $\sup T = L$.

Then (i) $t \leq L$ for every $t \in T$, since L is an upper bound of T.
and (ii) since every $s \in S$ is an upper bound of T and $L = \sup T$,
$L \leq s$ for every $s \in S$.

(ii) shows that L is a lower bound of S and (i) shows that $L \geq$ any lower bound of S. Consequently, $L = \inf S$.

Therefore S has an infimum and the proof is complete.

An ordered field is said to be a *complete ordered field* if the completeness property (i.e., the supremum property, or the infimum property) holds in it. Thus $\mathbb{R}$ is a complete ordered field.

The ordered field $\mathbb{Q}$ of all rational numbers does not have the supremum property. For example, the set $S = \{1, 1 + \frac{1}{1!}, 1 + \frac{1}{1!} + \frac{1}{2!}, \cdots\}$ which is a subset of $\mathbb{Q}$, is bounded above, because each element of the set is less than 3. But there is no rational number which is the supremum of the set S. (The supremum of the set is e, an irrational number.)

It is this completeness property that distinguishes $\mathbb{R}$ from $\mathbb{Q}$ and that transforms $\mathbb{R}$ from an algebraic system into a structure rich in abundant materials of analysis.

Theorem 2.4.21. Let S be a non-empty subset of $\mathbb{R}$, bounded above. An upper bound u of S is the supremum of S if and only if for each positive ϵ there exists an element s in S such that $u - \epsilon < s \leq u$.

Proof. Let $u = \sup S$. Let us choose $\epsilon > 0$. Then $u - \epsilon$ is not an upper bound of S. Therefore there exists at least one element of S, say s, such that $s > u - \epsilon$.
Since $u = \sup S$ and $s \in S$, we have $s \leq u$. Consequently, $u - \epsilon < s \leq u$.

Conversely, let u be an upper bound of S such that for each chosen $\epsilon > 0$, there is an element, say s, of S such that $u - \epsilon < s < u$.

We prove that u is the least upper bound of S, i.e., no upper bound of S is less than u.

If possible, let u_0 be an upper bound of S such that $u_0 < u$.

Let $\epsilon = \frac{1}{2}(u - u_0)$. Then $\epsilon > 0$ and $u - \epsilon = u_0 + \epsilon$.

By the stated condition, there exists an element in S, say s', such that $u - \epsilon < s' \leq u$.

or, $s' > u_0 + \epsilon$ and this shows that u_0 can not be an upper bound of S. Hence u is the least upper bound of S.

2.4.22. Properties of the supremum and the infimum.

Let S be a non-empty subset of $\mathbb{R}$, bounded above. Then $\sup S$ exists. Let $M = \sup S$. Then $M \in \mathbb{R}$ and M satisfies the following conditions :

(i) $x \in S \Rightarrow x \leq M$, and

(ii) for each $\epsilon > 0$, there exists an element $y(\epsilon)$ in S such that $M - \epsilon < y \leq M$.

Let S be a non-empty subset of $\mathbb{R}$, bounded below. Then $\inf S$ exists. Let $m = \inf S$. Then $m \in \mathbb{R}$ and m satisfies the following conditions :

(i) $x \in S \Rightarrow x \geq m$, and

(ii) for each $\epsilon > 0$, there exists an element $y(\epsilon)$ in S such that $m \leq y < m + \epsilon$.

Note. The symbol $y(\epsilon)$ indicates dependence of y on the choice of ϵ.

Worked Examples (continued).

3. Prove that the set $\mathbb{N}$ is not bounded above.

The set $\mathbb{N}$ is a non-empty subset of $\mathbb{R}$, since $1 \in \mathbb{N}$.

Let $\mathbb{N}$ be bounded above. Then $\mathbb{N}$ being a non-empty subset of $\mathbb{R}$ bounded above, $\sup \mathbb{N}$ exists by the supremum property of $\mathbb{R}$. Let $u = \sup \mathbb{N}$. Then (i) $x \in \mathbb{N} \Rightarrow x \leq u$, and

(ii) for each $\epsilon > 0$ there exists an element, say y in $\mathbb{N}$ such that $u - \epsilon < y \leq u$.

Let us choose $\epsilon = 1$. Then there exists an element k in $\mathbb{N}$ such that $u - 1 < k \leq u$. $u - 1 < k \Rightarrow k + 1 > u$.

Since k is a natural number, $k + 1$ is also a natural number. $k + 1 > u$ implies that u is not an upper bound of $\mathbb{N}$.

Thus we arrive at a contradiction. So our assumption that $\mathbb{N}$ is bounded above is wrong. Hence the set $\mathbb{N}$ is not bounded above.

4. Let S be a non-empty subset of $\mathbb{R}$, bounded above and $T = \{-x : x \in S\}$. Prove that the set T is bounded below and $\inf T = -\sup S$.

$\sup S$ exists. Let $u = \sup S$. Then $x \in S \Rightarrow x \leq u$.

Let $y \in T$. Then $-y \in S$ and therefore $-y \leq u$, i.e., $y \geq -u$. This implies that $-u$ is a lower bound of T. Therefore the set T is bounded below.

Let us choose $\epsilon > 0$. Since $u = \sup S$, there exists an element p in S such that $u - \epsilon < p \leq u$. Therefore $-u \leq -p < -u + \epsilon$. ... (i)

Let $q = -p$. then $q \in T$.

(i) shows that for a pre-assigned positive ϵ there exists an element q in T such that $-u \leq q < -u + \epsilon$.

This proves that $-u = \inf T$. Therefore $\inf T = -\sup S$.

5. Let S be a non-empty bounded subset of $\mathbb{R}$ with $\sup S = M$ and $\inf S = m$. Prove that the set $T = \{\mid x - y \mid : x \in S, y \in S\}$ is bounded above and $\sup T = M - m$.

$x \in S \Rightarrow m \leq x \leq M$, $y \in S \Rightarrow m \leq y \leq M$.

Therefore $m - M \leq x - y \leq M - m$, i.e., $\mid x - y \mid \leq M - m$.

This shows that the set T is bounded above, $M - m$ being an upper bound.

Let $a \in S$. Then $\mid a - a \mid \in T$ showing that T is non-empty. By the supremum property of $\mathbb{R}$, $\sup T$ exists.

We now prove that no real number less than $M - m$ is an upper bound of T.

If possible, let $p < M - m$ be an upper bound of T.

Let $(M - m) - p = 2\epsilon$. Then $\epsilon > 0$ and $p + \epsilon = M - m - \epsilon$.

Since $\sup S = M$, there exists an element $x \in S$ such that

$$M - \tfrac{\epsilon}{2} < x \leq M.$$

Since $\inf S = m$, there exists an element $y \in S$ such that

$$m \leq y < m + \tfrac{\epsilon}{2}.$$

Now $x - y > M - m - \epsilon$, i.e., $x - y > p + \epsilon$.

This shows that p is not an upper bound of T.

Therefore no real number less than $M - m$ is an upper bound of T That is, $\sup T = M - m$.

6. Let A, B be bounded subsets of $\mathbb{R}$ such that $x \in A$, $y \in B \Rightarrow x \leq y$ Prove that $\sup A \leq \inf B$.

Since A, B are non-empty bounded subsets of $\mathbb{R}$, $\sup A$, $\inf B$ exist Let $\sup A = a^*$, $\inf B = b_*$.

Let $b \in B$. Then $x \in A \Rightarrow x \leq b$. This shows that b is an upper bound of A. Since $\sup A = a^*$ and b is an upper bound of A it follows that $a^* \leq b$.

Now $a^* \leq b$ for all $b \in B$. Therefore a^* is a lower bound of B. Since $\inf B = b_*$ and a^* is a lower bound of B it follows that $a^* \leq b_*$, i.e. $\sup A \leq \inf B$.

7. Let S be the subset of $\mathbb{Q}$ defined by $S = \{x \in \mathbb{Q} : x > 0 \text{ and } x^2 < 2\}$. Show that S is a non-empty subset of $\mathbb{Q}$ bounded above but $\sup S$ does not belong to $\mathbb{Q}$.

S is non-empty, since $1 \in S$. S is bounded above, since 2 is an upper bound of the set.

If possible, let $\sup S \in \mathbb{Q}$ and $\sup S = u$. Then $u > 0$ and $u \in \mathbb{Q}$.

By the law of trichotomy, exactly one of the following holds :

$u^2 > 2, u^2 = 2, u^2 < 2$.

Case 1. Let $u^2 > 2$. Then $u^2 - 2 > 0$.

Let us take another rational number $r = \frac{4+3u}{3+2u}$. Then $r > 0$.

$u - r = u - \frac{4+3u}{3+2u} = \frac{2(u^2-2)}{3+2u} > 0$. Therefore $0 < r < u$ ... ... (i)

$r^2 - 2 = \frac{(4+3u)^2 - 2(3+2u)^2}{(3+2u)^2} = \frac{u^2-2}{(3+2u)^2} > 0$.

Therefore $r > 0$ and $r^2 > 2$ ... ...(ii)

(ii) shows that r is an upper bound of S and (i) shows that u is not the supremum of S. This is a contradiction to the assumption that $u = \sup S$. Therefore $u^2 \not> 2$.

Case II. $u^2 = 2$. We have seen that there exists no rational number r such that $r^2 = 2$. Therefore $u^2 \neq 2$.

Case III. $u^2 < 2$.

Let us take again the rational number $r = \frac{4+3u}{3+2u}$. Then $r > 0$ and $r - u = \frac{2(2-u^2)}{3+2u} > 0$. Therefore $0 < u < r$ ... ... (iii)

$2 - r^2 = \frac{2-u^2}{(3+2u)^2} > 0$. Therefore $r > 0$ and $r^2 < 2$ ... ... (iv)

(iv) shows that $r \in S$.

From (iii) it follows that u belongs to S and u is less than an element r of S. Therefore u is not the supremum of S, a contradiction. Therefore $u^2 \not< 2$.

None of the three possibilities provided by the law of trichotomy can hold. Hence our assumption that $\sup S$ is a rational number is wrong. Therefore no rational number can be the supremum of S.

Note 1. This example shows that the supremum property which is an important property of $\mathbb{R}$ is not satisfied in the subset $\mathbb{Q}$ of $\mathbb{R}$.

2. If we regard this set S as a subset of $\mathbb{R}$, then by the supremum property of $\mathbb{R}$, $\sup S$ exists as a real number.

2.4.23. Archimedean property of $\mathbb{R}$.

If $x, y \in \mathbb{R}$ and $x > 0, y > 0$, then there exists a natural number n such that $ny > x$.

Proof. If possible, let there exist no natural number n for which $ny > x$. Then for every natural number k, $ky \le x$.

Thus the set $S = \{ky : k \in \mathbb{N}\}$ is bounded above, x being an upper bound. S is non-empty because $y \in S$.

By the supremum property of $\mathbb{R}$, $\sup S$ exists. Let $\sup S = b$.
Then $ky \le b$ for all $k \in \mathbb{N}$.

$b - y < b$ since $y > 0$. This shows that $b - y$ is not an upper bound of S and therefore there exists a natural number p such that
$b - y < py \le b$. This implies $(p + 1)y > b$ ... ... (i)

But $p \in \mathbb{N} \Rightarrow p + 1 \in \mathbb{N}$ and therefore $(p + 1)y \in S$.
(i) shows that b is not the supremum of S, a contradiction.

Therefore our assumption is wrong and the existence of a natural number n satisfying $ny > x$ is proved.

Important deductions.

(i) If $x \in \mathbb{R}$, then there exists a natural number n such that $n > x$.

Case 1. $x > 0$.
Taking $y = 1$, by Archimedean property of $\mathbb{R}$ there exists a natural number n such that $n.1 > x$ and hence the existence is proved.

Case 2. $x \le 0$. Then $n = 1$.

(ii) If $x \in \mathbb{R}$ and $x > 0$, then there exists a natural number n such that $0 < \frac{1}{n} < x$.

Taking $y = 1$, by Archimedean property of $\mathbb{R}$ there exists a natural number n such that $nx > 1$.
Since n is a natural number, $n > 0$ and therefore $\frac{1}{n} > 0$ and also $\frac{1}{n} < x$. Therefore we have $0 < \frac{1}{n} < x$.

(iii) If $x \in \mathbb{R}$ and $x > 0$, there exists a natural number m such that $m - 1 \le x < m$.

Taking $y = 1$ and $x > 0$, by Archimedean property of $\mathbb{R}$ there exists a natural number n such that $n.1 > x$, i.e., $n > x$.

Let $S = \{k \in \mathbb{N} : k > x\}$. Then S is non-empty subset of $\mathbb{N}$, since $n \in S$. By the well ordering property of the set $\mathbb{N}$, S has a least element, say m. Since $m \in S, m > x$.

As m is the least element in S, $m - 1 \not> x$, i.e., $m - 1 \leq x$.
Hence $m - 1 \leq x < m$.

(iv) If $x \in \mathbb{R}$, then there exists an integer m such that $m - 1 \leq x < m$.

Case 1. $x > 0$.
This is (iii)

Case 2. $x = 0$.
In this case $m = 1$.

Case 3. $x < 0$.
First we assume that x is not a negative integer.
Then $-x > 0$. By case 1, there exists a natural number m' such that $m' - 1 < -x < m'$.
$-x < m' \Rightarrow -m' < x$ and $m' - 1 < -x \Rightarrow x < -m' + 1$.
Therefore $-m' < x < -m' + 1$.

Let $m = -m' + 1$. Since m' is a natural number, m is an integer ≤ 0. So we have $m - 1 < x < m$.
If however, x is a negative integer, then $x = m - 1$.
Combining, we have $m - 1 \leq x < m$.

Note. An ordered field is called an *Archimedean ordered field* if the Archimedean property holds in it. Thus $\mathbb{R}$ is an Archimedean ordered field. $\mathbb{Q}$ is also an Archimedean ordered field. But $\mathbb{Q}$ is not a complete Archimedean ordered field, while $\mathbb{R}$ is so.

Worked Examples (continued).

8. Show that there exists a unique positive real number x such that $x^2 = 2$.

Let $S = \{s \in \mathbb{R} : s \geq 0 \text{ and } s^2 < 2\}$. S is a non-empty subset of $\mathbb{R}$, since $0 \in S$. S is bounded above, 2 being an upper bound.
By the supremum property of $\mathbb{R}$, $\sup S$ exists. Let $x = \sup S$. Clearly, $x > 0$. $1 \in S$ and 1 is not an upper bound of S and therefore $x > 1$ also. We shall prove that $x^2 = 2$.

If not, let $x^2 > 2$. Then $\frac{x^2-2}{2x} > 0$.
By Archimedean property of $\mathbb{R}$, there exists a natural number m such that $0 < \frac{1}{m} < \frac{x^2-2}{2x}$. Therefore $\frac{2x}{m} < x^2 - 2$.

$$\begin{aligned} (x - \tfrac{1}{m})^2 &= x^2 - \tfrac{2x}{m} + \tfrac{1}{m^2} \\ &> x^2 - \tfrac{2x}{m} > 2. \end{aligned}$$

$x - \frac{1}{m} > 0$, since $x > 1$. $(x - \frac{1}{m})^2 > 2$ shows that $x - \frac{1}{m}$ is an upper bound of S which contradicts that $\sup S = x$.

Therefore $x^2 \not> 2$ ... ... (i)

Let $x^2 < 2$. Then $2 - x^2 > 0$ and $\frac{2-x^2}{2x+1} > 0$

By Archimedean property of $\mathbb{R}$, there exists a natural number k such that $0 < \frac{1}{k} < \frac{2-x^2}{2x+1}$, i.e., $\frac{1}{k}(2x+1) < 2 - x^2$.

$$\begin{aligned}(x + \tfrac{1}{k})^2 &= x^2 + \tfrac{1}{k}(2x + \tfrac{1}{k}) \\ &\leq x^2 + \tfrac{1}{k}(2x+1) < 2.\end{aligned}$$

This shows that $x + \frac{1}{k} \in S$ and as $x + \frac{1}{k} > x$, x fails to be the supremum of S.

Therefore $x^2 \not< 2$ ... ... (ii)

From (i) and (ii) $x \in \mathbb{R}$ and $x^2 = 2$.

We prove that x is unique.

Let us assume that there exists another real number y such that $y > 0$ and $y^2 = 2$.

Now $x > 0$ and $x^2 = 2$; $y > 0$ and $y^2 = 2$.

This implies $x^2 = y^2$.

Let $x > y$. Then $x > 0, x > y \Rightarrow x^2 > xy$
and $y > 0, x > y \Rightarrow xy > y^2$.

It follows that $x^2 > y^2$, a contradiction.

Let $x < y$. Then $x > 0, x < y \Rightarrow x^2 < xy$
and $y > 0, x < y \Rightarrow xy < y^2$.

It follows that $x^2 < y^2$, a contradiction. Consequently, $x = y$.

This proves that x is a unique positive real number such that $x^2 = 2$.

Note. x is denoted by $\sqrt{2}$. $\sqrt{2}$ is therefore an irrational real number.

9. If n be a positive integer ≥ 2 and a be a positive real number, show that there exists a unique positive real number x such that $x^n = a$.

Let $S = \{s \in \mathbb{R} : s > 0 \text{ and } s^n < a\}$.

Let $t = \frac{a}{1+a}$. Then $0 < t < 1$ and also $0 < t < a$.

This implies $t^n < t < a$.

$t > 0$ and $t^n < a \Rightarrow t \in S$, proving that S is non-empty.

Let $u = 1 + a$. Then $u > 1$ and $u > a$.

This implies $u^n > u > a$.

Since $u^n > a$ and $u > 0$, u is an upper bound of S.

Thus S is a non-empty subset of $\mathbb{R}$, bounded above and hence $\sup S$ exists.

Let $x = \sup S$. Clearly, $x > 0$. We prove that $x^n = a$.

If not, either $x^n > a$ or $x^n < a$. (by the law of trichotomy)

Case 1. Let $x^n > a$. Then $\frac{x^n - a}{(1+x)^n - x^n} > 0$.

By Archimedean property of $\mathbb{R}$, there exists a natural number m such that $0 < \frac{1}{m} < \frac{x^n - a}{(1+x)^n - x^n}$

or, $x^n - a > \frac{1}{m}[\binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2} + \cdots + \binom{n}{n}]$.

$$\begin{aligned}(x - \tfrac{1}{m})^n &= x^n - \tbinom{n}{1}x^{n-1}.\tfrac{1}{m} + \cdots + (-1)^n\tbinom{n}{n}.\tfrac{1}{m^n}\\ &> x^n - \tfrac{1}{m}[\tbinom{n}{1}x^{n-1} + \tbinom{n}{2}x^{n-2} + \cdots + \tbinom{n}{n}]\\ &> x^n - (x^n - a) = a.\end{aligned}$$

This shows that $x - \frac{1}{m}$ is an upper bound of S and this contradicts that $x = \sup S$.

Case 2. Let $x^n < a$. Then $\frac{a - x^n}{(1+x)^n - x^n} > 0$.

By Archimedean property of $\mathbb{R}$, there exists a natural number k such that $0 < \frac{1}{k} < \frac{a - x^n}{(1+x)^n - x^n}$

or, $a - x^n > \frac{1}{k}[\binom{n}{1}x^{n-1}.\frac{1}{k} + \binom{n}{2}.x^{n-2} + \cdots + \binom{n}{n}]$

$$\begin{aligned}(x + \tfrac{1}{k})^n &= x^n + \tbinom{n}{1}x^{n-1}.\tfrac{1}{k} + \tbinom{n}{2}.x^{n-2}\tfrac{1}{k^2} + \cdots + \tbinom{n}{n}.\tfrac{1}{k^n}\\ &< x^n + \tfrac{1}{k}[\tbinom{n}{1}x^{n-1} + \tbinom{n}{2}x^{n-2} + \cdots + \tbinom{n}{n}]\\ &< x^n + (a - x^n) = a.\end{aligned}$$

This shows that $x + \frac{1}{k} \in S$ and this contradicts that $x = \sup S$.

In view of the cases 1 and 2, we have $x^n = a$.

We prove that x is unique.

If possible, let $y \neq x$ and $y^n = a$.

Now $y > 0, x > 0$ and $y \neq x \Rightarrow y^n \neq x^n$.

Therefore $y^n \neq a$ and x is unique.

Note. This unique positive real number is denoted by $\sqrt[n]{a}$.

2.4.24. Density property of $\mathbb{R}$.

1. If x, y are real numbers with $x < y$ then there exists a rational number r such that $x < r < y$.

2. If x, y are real numbers with $x < y$ then there exists an irrational number s such that $x < s < y$.

Proof. **1.** $y - x > 0$. By Archimedean property of $\mathbb{R}$, there exists a natural number n such that $0 < \frac{1}{n} < y - x$

or, $ny - nx > 1$

or, $nx + 1 < ny$. ... ... (i)

$nx \in \mathbb{R}$. By deduction (iv) of Archimedean property, there exists an integer m such that $m - 1 \leq nx < m$. ... ... (ii)

$m - 1 \leq nx \Rightarrow nx + 1 \geq m$.

Therefore $m \leq nx + 1 < ny$ from (i). Also $nx < m$ from (ii).

Therefore $nx < m < ny$

or, $x < \frac{m}{n} < y$.

Since m is an integer and n is a natural number, $\frac{m}{n}$ is a rationa number.

Let $r = \frac{m}{n}$. Then the rational number r is such that $x < r < y$.

2. $\sqrt{2}x, \sqrt{2}y$ are real numbers and $\sqrt{2}x < \sqrt{2}y$.

By Density property 1, there exists a rational number r such that $\sqrt{2}x < r < \sqrt{2}y$. Without loss of generality, we assume $r \neq 0$.

Then $x < \frac{r}{\sqrt{2}} < y$.

Let $s = \frac{r}{\sqrt{2}}$. Then s is an irrational number satisfying $x < s < y$.

2.4.25. Geometrical representation of real numbers.

The real numbers can be represented by points on a straight line. Let $X'X$ be a directed line. We take a point O on the line. O divides the line into two parts. The part to the right of O is called the positive side, the part to the left of O is called the negative side. Let us take a point A to the right of O.

Let O represent the real number *zero* and A represent the real number *one.* Taking the distance OA as the unit distance on some chosen scale, each real number can be represented by a unique point on the line; a positive real number by a point lying to the right of O and a negative real number by a point lying to the left of O. A point that represents a rational number is called a rational point and a point that represents an irrational number is called an irrational point. By the density property of $\mathbb{R}$, between any two points on the line there lie infinitely many rational points as well as infinitely many irrational points.

Having a complete representation of the set $\mathbb{R}$ as points on the line, the question comes -"Does there exist any other point on the line that does not correspond to a real number?" The answer to the question is provided by Cantor-Dedekind axiom which states that there is a one-to-one correspondence between the set of all points on a line and the set of all real numbers.

Therefore each point on the line corresponds to only one real number and conversely, each real number is represented by only one point on the line.

Note. It will be convenient for us to suppose that a straight line is composed of points which correspond to all the numbers in the set $\mathbb{R}$. The points on the line can be looked upon as images of the numbers in

$\mathbb{R}$. In view of the one-to-one correspondence between the two sets (the set of points on the line and the set of numbers in $\mathbb{R}$) we shall use the word "a point" for "a real number" and vice versa.

Definition. The aggregate of all real numbers is called the *arithmetical continuum* and the aggregate of all points on a straight line is called the *linear continuum.*

2.4.26. Extended set of real numbers.

It is often convenient to extend the set $\mathbb{R}$ by the addition of two elements ∞ and $-\infty$. This enlarged set is called the *extended set of real numbers* and is often denoted by $\mathbb{R}^*$.

In the extended set $\mathbb{R}^*$ we define –

$$\begin{aligned} \text{for all } x \in \mathbb{R}, x + \infty &= \infty + x = \infty \\ x + (-\infty) &= (-\infty) + x = -\infty; \end{aligned}$$

$$\begin{aligned} \text{for all } x > 0, x.\infty &= \infty.x = \infty \quad \text{and} \\ x.(-\infty) &= (-\infty).x = -\infty; \end{aligned}$$

$$\begin{aligned} \text{for all } x < 0, x.\infty &= \infty.x = -\infty \quad \text{and} \\ x.(-\infty) &= -\infty.x = \infty; \end{aligned}$$

$$\infty + \infty = \infty, (-\infty) + (-\infty) = -\infty$$
$$\infty.\infty = \infty, \quad (-\infty).\infty = \infty.(-\infty) = -\infty, \quad (-\infty).(-\infty) = \infty.$$

$\infty + (-\infty)$, $(-\infty) + \infty$, $0.\infty$, $\infty.0$, $0. - \infty$, $-\infty.0$ are not defined.

$\mathbb{R}^*$ is not a field. It is not even an algebraic system since addition and multiplication are not fully defined on the set.

The order relation in $\mathbb{R}^*$ satisfies the inequality $-\infty < x < \infty$ for all $x \in \mathbb{R}$.

If x be a positive real number, then $0 < x < \infty$.

If x be a negative real number, then $-\infty < x < 0$.

If S be a non-empty subset of $\mathbb{R}$ having no upper bound, we define $\sup S = \infty$. If S be a non-empty subset of $\mathbb{R}$ having no lower bound, we define $\inf S = -\infty$.

Therefore for every non-empty subset S in $\mathbb{R}, \sup S$ and $\inf S$ both exist in $\mathbb{R}^*$ and $\inf S \leq \sup S$.

If S be the empty set, we have $\sup S = -\infty$ and $\inf S = \infty$.

The advantage of the extended set of real numbers is that we can speak of $\sup S$ and $\inf S$ of any type of subset S of $\mathbb{R}$.

Exercises 2

1. Use the principle of induction to prove that

(i) if $x > -1, (1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$;

(ii) $3^{2^n} - 1$ divisible by 2^{n+2} for all $n \in \mathbb{N}$;

(iii) if $u_1 = \sqrt{2}$ and $u_{n+1} = \sqrt{2 + u_n}$ for all $n \geq 1$, then $u_n < 2$ for al $n \in \mathbb{N}$;

(iv) $u_{n+2} + u_n = 4u_{n+1}$ for all $n \in \mathbb{N}$, where $u_n = (2+\sqrt{3})^n + (2-\sqrt{3})^n$

2. Show that there does not exist a rational number r such that $r^2 = 5$.

3. Show that the following numbers are irrational numbers.

(i) $1+\sqrt{2}+\sqrt{3}$, (ii) $1-\sqrt{2}+\sqrt{3}$, (iii) $1+\sqrt{2}-\sqrt{3}$, (iv) $1-\sqrt{2}-\sqrt{3}$.

4. Show that $\log_{10} n$ is not a rational number if n is any integer not a powe of 10.

5. Let $a, b \in \mathbb{R}$ and $ab > 0$. Prove that either $a > 0$ and $b > 0$, or $a < 0$ anc $b < 0$.

6. If $a, b \in \mathbb{R}$ and $0 \leq a - b < \epsilon$ holds for every positive ϵ, prove that $a = b$.

7. If $a \in \mathbb{R}$ and $0 \leq a < \frac{1}{n}$ for every natural number n, prove that $a = 0$.

8. Find the solution set of the inequality

(i) $\frac{3x}{2x-1} < 3$, (ii) $\frac{x+2}{x-1} < 4$, (iii) $\frac{4x}{2x-3} > \frac{1}{2} + \frac{3x}{2x+3}$,

(iv) $\left| \frac{x+3}{6-5x} \right| \leq 2$, (v) $\left| \frac{2x-5}{x-6} \right| < 3$.

9. Prove that $|x| + |y| + |z| \leq |x+y-z| + |y+z-x| + |z+x-y|$ for all $x, y, z \in \mathbb{R}$.

10. Find $\sup A$ and $\inf A$, where

(i) $A = \{x \in \mathbb{R} : x^2 < 1\}$, (ii) $A = \{x \in \mathbb{R} : 3x^2 + 8x - 3 < 0\}$,

(iii) $A = \{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\}$, (iv) $A = \{\frac{n+(-1)^n}{n} : n \in \mathbb{N}\}$.

11. If y be a positive real number show that there exists a natural number m such that $0 < 1/2^m < y$.

12. Let S be a bounded subset of $\mathbb{R}$ and T be a non-empty subset of S. Prove that $\inf S \leq \inf T \leq \sup T \leq \sup S$.

13. Let S and T be two non-empty bounded subsets of $\mathbb{R}$ and $U = \{x + y : x \in S, y \in T\}$. Prove that $\sup U = \sup S + \sup T$, $\inf U = \inf S + \inf T$.

14. Let S be a non-empty subset of $\mathbb{R}$, bounded below and $T = \{-x : x \in S\}$. Prove that the set T is bounded above and $\sup T = -\inf S$.

15. Let S be a bounded subset of $\mathbb{R}$ with $\sup S = M$ and $\inf S = m$. Prove that the set $T = \{x - y : x \in S, y \in S\}$ is a bounded set and $\sup T = M - m$, $\inf T = m - M$.

16. A and B are non-empty bounded subsets of $\mathbb{R}$. Prove that

(i) $\sup A \cup B = \max\{\sup A, \sup B\}$, (ii) $\inf A \cup B = \min\{\inf A, \inf B\}$.

17. Let S be a non-empty subset of $\mathbb{R}$ bounded below. A lower bound l of S is such that for each natural number n there exists an element s_n in S satisfying $s_n < l + \frac{1}{n}$. Prove that $l = \inf S$.

18. Show that the subset $S = \{x \in \mathbb{Q} : x > 0 \text{ and } x^2 > 2\}$ is a non-empty subset of $\mathbb{Q}$, bounded below; but $\inf S$ does not belong to $\mathbb{Q}$.

Hint. Assume $\inf S = l \in \mathbb{Q}$. Take $r = \frac{4-3l}{3+2l} \in \mathbb{Q}$ and show that either $r < l$ if $l^2 > 2$, or $0 < l < r$ if $l^2 < 2$.]

how that there exists a unique positive real number x such that $x^2 = 5$

3.1. Intervals.

Let $a, b \in \mathbb{R}$ and $a < b$.

The subset $\{x \in \mathbb{R} : a < x < b\}$ is said to be an *open interval.* The points a and b are called the *end points* of the interval. a and b are not points in the open interval. This open interval is denoted by (a, b).

The subset $\{x \in \mathbb{R} : a \leq x \leq b\}$ is said to be a *closed interval.* The end points a and b are points in the closed interval. This closed interval is denoted by $[a, b]$.

The subsets $\{x \in \mathbb{R} : a < x \leq b\}$ and $\{x \in \mathbb{R} : a \leq x < b\}$ are said to be *half open* (or *half closed*) intervals. One of the end points is a point in the interval. These half open intervals are denoted by $(a, b]$ and $[a, b)$ respectively.

The subset $\{x \in \mathbb{R} : x > a\}$ is an *infinite open interval.* This is denoted by (a, ∞).

The subset $\{x \in \mathbb{R} : x \geq a\}$ is an *infinite closed interval.* This is denoted by $[a, \infty)$

The subset $\{x \in \mathbb{R} : x < a\}$ is an *infinite open interval.* This is denoted by $(-\infty, a)$.

The subset $\{x \in \mathbb{R} : x \leq a\}$ is an *infinite closed interval.* This is denoted by $(-\infty, a]$.

When both the end points of an interval belong to $\mathbb{R}$, the interval is said to be a *bounded interval.*

Therefore the intervals $(a, b), [a, b], (a, b], [a, b)$ are all bounded intervals.

The intervals $(a, \infty), [a, \infty), (-\infty, a), (-\infty, a]$ are *unbounded intervals.*

If $a = b$, the closed interval $[a, a]$ is the singleton set $\{a\}$.

The set $\mathbb{R}$ is also denoted by $(-\infty, \infty)$. This is an unbounded interval without end points.

3.2. Neighbourhood.

Let $c \in \mathbb{R}$. A subset $S \subset \mathbb{R}$ is said to be a *neighbourhood* of c if there exists an open interval (a, b) such that $c \in (a, b) \subset S$.

Clearly, an open bounded interval containing the point c is a neighbourhood of c. Such a neighbourhood of c is denoted by $N(c)$.

A closed bounded interval containing the point c may not be a neighbourhood of c. For example, $1 \in [1, 3]$ but $[1, 3]$ is not a neighbourhood of 1.

Let $c \in \mathbb{R}$ and $\delta > 0$. The open interval $\boxed{(c - \delta, c + \delta)}$ is said to the δ-neighbourhood of c and is denoted by $\boxed{N(c, \delta)}$. Clearly, the δ-neighbourhood of c is an open interval symmetric about c.

Theorem 3.2.1. Let $c \in \mathbb{R}$. The union of two neighbourhoods of c is a neighbourhood of c.

Proof. Let $S_1 \subset \mathbb{R}, S_2 \subset \mathbb{R}$ be two neighbourhoods of c. Then there exist open intervals $(a_1, b_1), (a_2, b_2)$ such that $c \in (a_1, b_1) \subset S_1$ and $c \in (a_2, b_2) \subset S_2$.

Then $a_1 < b_1, a_2 < b_1; a_1 < b_2, a_2 < b_2$. Let $a_3 = \min\{a_1, a_2\}$, $b_3 = \max\{b_1, b_2\}$. Then $(a_1, b_1) \cup (a_2, b_2) = (a_3, b_3)$ and $c \in (a_3, b_3)$.

Now $(a_1, b_1) \subset S_1 \cup S_2$ and $(a_2, b_2) \subset S_1 \cup S_2$

$\Rightarrow (a_3, b_3) = (a_1, b_1) \cup (a_2, b_2) \subset S_1 \cup S_2$.

Thus $c \in (a_3, b_3) \subset S_1 \cup S_2$.

This proves that $S_1 \cup S_2$ is a neighbourhood of c.

Hold for nbd

Note. The union of a finite number of neighbourhoods of c is a neighbourhood of c.

ed, Arbitrarily Union

Theorem 3.2.2. Let $c \in \mathbb{R}$. The intersection of two neighbourhoods of c is a neighbourhood of c.

Proof. Let $S_1 \subset \mathbb{R}, S_2 \subset \mathbb{R}$ be two neighbourhoods of c. Then there exist open intervals $(a_1, b_1), (a_2, b_2)$ such that $c \in (a_1, b_1) \subset S_1$ and $c \in (a_2, b_2) \subset S_2$.

Then $a_1 < b_1, a_2 < b_1; a_1 < b_2, a_2 < b_2$.

Let $a_3 = \max\{a_1, a_2\}$, $b_3 = \min\{b_1, b_2\}$. Then $(a_1, b_1) \cap (a_2, b_2) = (a_3, b_3)$ and $c \in (a_3, b_3)$.

$$\begin{aligned} \text{Now } (a_3, b_3) &= (a_1, b_1) \cap (a_2, b_2) \subset (a_1, b_1) \subset S_1 \\ \text{and } (a_3, b_3) &= (a_1, b_1) \cap (a_2, b_2) \subset (a_2, b_2) \subset S_2 \\ &\Rightarrow (a_3, b_3) \subset S_1 \cap S_2. \end{aligned}$$

Thus $c \in (a_3, b_3) \subset S_1 \cap S_2$.

This proves that $S_1 \cap S_2$ is a neighbourhood of c.

Note. The intersection of a finite number of neighbourhoods of a point

c is a neighbourhood of c.

The intersection of an infinite number of neighbourhoods of a point c may not be a neighbourhood of c.

For example, for every $n \in \mathbb{N}$, $(-\frac{1}{n}, \frac{1}{n})$ is a neighbouhood of 0.

$\bigcap\limits_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$. This is not a neighbourhood of 0.

3.3. Interior point.

Let S be a subset of $\mathbb{R}$. A point x in S is said to be an *interior point* of S if there exists a neighbourhood $N(x)$ of x such that $N(x) \subset S$.

The set of all interior points of S is said to be the *interior* of S and is denoted by int S (or by S^o).

Examples.

1. Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\cdots\}$. $\Rightarrow S^o = \phi$

Let $x \in S$. Every neighbourhood of x contains points not belonging to S. So x can not be an interior point of S. Therefore int $S = \phi$.

2. Let $S = \mathbb{N}$.

Let $x \in S$. Every neighbourhood of x contains points not belonging to S. So x can not be an interior point of S. Therefore int $S = \phi$.

3. Let $S = \mathbb{Q}$.

Let $x \in \mathbb{Q}$. Every neighbourhood of x contains rational as well as irrational points. So x can not be an interior point of $\mathbb{Q}$. Therefore int $S = \phi$.

4. Let $S = \{x \in \mathbb{R} : 1 < x < 3\}$. Every point of S is an interior point of S. Therefore int $S = S$.

5. Let $S = \mathbb{R}$. Every point of S is an interior point of S. Therefore int $S = S$.

6. Let $S = \phi$. S has no interior point. Therefore int $S = \phi$.

3.4. Open set.

Let $S \subset \mathbb{R}$. S is said to be an *open set* if each point of S is an interior point of S.

Examples.

1. Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\cdots\}$. No point of S is an interior point of S. S is not an open set.

2. Let $S = \mathbb{Z}$. No point of S is an interior point of S. S is not an open set.

3. Let $S = \mathbb{Q}$. No point of S is an interior point of S. S is not an open set.

4. Let $S = \{x \in \mathbb{R} : 1 < x < 3\}$. Each point of S is an interior point of S. S is an open set.

5. Let $S = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$. 1 and 3 belong to S but they are not interior points of S. S is not an open set.

6. Let $S = \mathbb{R}$. Each point of S is an interior point of S. S is an open set.

7. Let $S = \phi$. S contains no point. Therefore the requirement in the definition is vacuously satisfied. S is an open set.

Theorem 3.4.1. Let $S \subset \mathbb{R}$. Then S is an open set if and only if $S =$ int S.

Proof. We prove the theorem for a non-empty set S because if $S = \phi$ then $\phi =$ int ϕ holds and also ϕ is an open set.

Let S be a non-empty open set and let $x \in S$. Then x is an interior point of S.

Thus $x \in S \Rightarrow x \in$ int S. Therefore $S \subset$ int S ... ... (i)

Let $y \in$ int S. Then $y \in S$ by the definition of an interior point.

Thus $y \in$ int $S \Rightarrow y \in S$. Therefore int $S \subset S$ ... ... (ii)

From (i) and (ii) we have $S =$ int S.

Conversely, let S be a non-empty set and $S =$ int S.

Let $x \in S$. Then $x \in$ int S, since $S =$ int S.

Thus every point of S is an interior point of S and therefore S is an open set.

This completes the proof.

Theorem 3.4.2. The union of two open sets in $\mathbb{R}$ is an open set.

Proof. Let G_1 and G_2 be two open sets in $\mathbb{R}$.

Let $x \in G_1 \cup G_2$. Then $x \in G_1$ or $x \in G_2$.

Let $x \in G_1$. Since G_1 is open set and $x \in G_1, x$ is an interior point of G_1. Therefore there exists a neighbourhood $N(x)$ of x such that $N(x) \subset G_1$.

$N(x) \subset G_1 \Rightarrow N(x) \subset G_1 \cup G_2$.

This shows that x is an interior point of $G_1 \cup G_2$.

Since x is arbitrary, every point of $G_1 \cup G_2$ is an interior point of $G_1 \cup G_2$. Therefore $G_1 \cup G_2$ is an open set.

If however, $x \in G_2$, we can prove in a similar manner that $G_1 \cup G_2$ is an open set. This completes the proof.

Theorem 3.4.3. The intersection of two open sets in $\mathbb{R}$ is an open set.

Proof. Let G_1 and G_2 be two open sets in $\mathbb{R}$.

Case 1. $G_1 \cap G_2 = \phi$. Since ϕ is an open set, $G_1 \cap G_2$ is an open set.

Case 2. $G_1 \cap G_2 \neq \phi$. Let $x \in G_1 \cap G_2$. Then $x \in G_1$ and $x \in G_2$.

Since G_1 is an open set and $x \in G_1, x$ is an interior point of G_1.

Hence there exists a positive δ_1 such that the neighbourhood $N(x, \delta_1) \subset G_1$.

Since G_2 is an open set and $x \in G_2, x$ is an interior point of G_2.

Hence there exists a positive δ_2 such that the neighbourhood $N(x, \delta_2) \subset G_2$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$.

$N(x, \delta) \subset N(x, \delta_1) \subset G_1$ and $N(x, \delta) \subset N(x, \delta_2) \subset G_2$.

Consequently, $N(x, \delta) \subset G_1 \cap G_2$.

This shows that x is an interior point of $G_1 \cap G_2$. Since x is arbitrary, $G_1 \cap G_2$ is an open set and this completes the proof.

Theorem 3.4.4. The union of a finite number of open sets in $\mathbb{R}$ is an open set.

Proof. Let $G_1, G_2, \ldots, G_m$ be m open sets in $\mathbb{R}$.

Let $G = G_1 \cup G_2 \cup \cdots \cup G_m$.

Let $x \in G$. Then x belongs to at least one of the sets, say G_k. Since G_k is an open set and $x \in G_k, x$ is an interior point of G_k. Hence there exists a neighbourhood $N(x)$ of x such that $N(x) \subset G_k$.

$N(x) \subset G_k \Rightarrow N(x) \subset G$.

This shows that x is an interior point of G. Since x is arbitrary, G is an open set. This completes the proof.

Theorem 3.4.5. The intersection of a finite number of open sets in $\mathbb{R}$ is an open set.

Proof. Let $G_1, G_2, \ldots, G_m$ be m open sets in $\mathbb{R}$.

Let $G = G_1 \cap G_2 \cap \cdots \cap G_m$.

Case 1. $G = \phi$. Then G is an open set, since ϕ is an open set.

Case 2. $G \neq \phi$. Let $x \in G$. Then $x \in G_i$ for each $i = 1, 2, \ldots, m$.

Since G_1 is an open set and $x \in G_1$, there exists a positive δ_1 such that $N(x, \delta_1) \subset G_1$.

Since G_2 is an open set and $x \in G_2$, there exists a positive δ_2 such that $N(x, \delta_2) \subset G_2$.

... ...

Since G_m is an open set and $x \in G_m$, there exists a positive δ_m such that $N(x, \delta_m) \subset G_m$.

Let $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_m\}$. Then $\delta > 0$.

$$N(x,\delta) \subset N(x,\delta_1) \subset G_1$$
$$N(x,\delta) \subset N(x,\delta_2) \subset G_2$$
$$\ldots \quad \ldots \quad \ldots$$
$$N(x,\delta) \subset N(x,\delta_m) \subset G_m.$$

Consequently, $N(x,\delta) \subset G_1 \cap G_2 \cap \cdots \cap G_m = G$.

This shows that x is an interior point of G. Since x is arbitrary, G is an open set. This completes the proof.

Theorem 3.4.6. The union of an arbitrary collection of open sets in $\mathbb{R}$ is an open set.

Proof. Let $\{G_\alpha : \alpha \in \Lambda\}$, Λ being the index set, be an arbitrary collection of open sets in $\mathbb{R}$. Let $G = \bigcup\limits_{\alpha \in \Lambda} G_\alpha$.

Let $x \in G$. Then x belongs to at least one open set of the collection, say G_λ, $(\lambda \in \Lambda)$.

Since G_λ is an open set and $x \in G_\lambda$, x is an interior point of G_λ.

Therefore there exists a neighbourhood $N(x)$ of x such that $N(x) \subset G_\lambda$. $N(x) \subset G_\lambda \Rightarrow N(x) \subset G$.

This shows that x is an interior point of G. Since x is arbitrary, G is an open set and the proof is complete.

Note. The intersection of an infinite number of open sets in $\mathbb{R}$ is not necessarily an open set.

Let us consider the sets G_i where

$$G_1 = \{x \in \mathbb{R} : -1 < x < 1\}$$
$$G_2 = \{x \in \mathbb{R} : -\tfrac{1}{2} < x < \tfrac{1}{2}\}$$
$$\ldots \quad \ldots \quad \ldots$$
$$G_n = \{x \in \mathbb{R} : -\tfrac{1}{n} < x < \tfrac{1}{n}\}$$
$$\ldots \quad \ldots \quad \ldots$$

Each G_i is an open set. $\bigcap\limits_{i=1}^{\infty} G_i = \{0\}$. This is not an open set.

Let us consider the sets G_i where

$$G_1 = \{x \in \mathbb{R} : -1 < x < 1\}$$
$$G_2 = \{x \in \mathbb{R} : -2 < x < 2\}$$
$$\ldots \quad \ldots \quad \ldots$$
$$G_n = \{x \in \mathbb{R} : -n < x < n\}$$
$$\ldots \quad \ldots \quad \ldots$$

Each G_i is an open set. $\bigcap\limits_{i=1}^{\infty} G_i = G_1$. This is an open set.

From these two examples we conclude that the intersection of an infinite number of open sets in $\mathbb{R}$ is not necessarily an open set.

Theorem 3.4.7. Let S be a subset of $\mathbb{R}$. Then int S is an open set.

Proof. **Case 1.** int $S = \phi$. Since ϕ is an open set, int S is an open set.

Case 2. int $S \neq \phi$. Let $x \in$ int S. Then x is an interior point of S. Therefore there exists a neighbourhood $N(x)$ of x such that $N(x) \subset S$.

Let $y \in N(x)$. Then $N(x)$ is a neighbourhood of y also and since $N(x) \subset S$, y is an interior point of S.

Thus $y \in N(x) \Rightarrow y \in$ int S. Therefore $N(x) \subset$ int S.

This shows that x is an interior point of int S.

Thus $x \in$ int $S \Rightarrow x$ is an interior point of int S.

Therefore int S is an open set. This completes the proof.

Theorem 3.4.8. Let $S \subset \mathbb{R}$. Then int S is the largest open set contained in S.

Proof. By the previous theorem, int S is an open set and int $S \subset S$, by definition.

Let P be any open set contained in S.

Let $x \in P$. Since P is an open set, x is an interior point of P.

Therefore there exists a neighbourhood $N(x)$ of x such that $N(x) \subset P$. But $N(x) \subset P \Rightarrow N(x) \subset S$, since $P \subset S$.

This shows that x is an interior point of S, i.e., $x \in$ int S.

Thus $x \in P \Rightarrow x \in$ int S. Therefore $P \subset$ int S.

Since P is arbitrary, int S is the largest open set contained in S.

Note. int S is the union of all open sets contained in S.

Worked Examples.

1. Prove that an open interval is an open set.

Let I be an open interval. Four cases arise.

Case 1. $I = (a, b)$ for some $a, b \in \mathbb{R}$, with $a < b$.

Let $c \in I$. Then I itself is a neighbourhood of c, say $N(c)$ and $N(c) \subset I$. This shows that c is an interior point of I. Thus every point of I is an interior point of I and therefore I is an open set.

Case 2. $I = (a, \infty)$ for some $a \in \mathbb{R}$.

Let $c \in I$. Then $a < c < \infty$. Let $d \in (c, \infty)$. Then $a < c < d$.

The open interval (a, d) is a neighbourhood of c, say $N(c)$ and $N(c) \subset I$. This shows that c is an interior point of I. Thus every point of I is an interior point of I and therefore I is an open set.

Case 3. $I = (-\infty, a)$ for some $a \in \mathbb{R}$.
Similar proof.

Case 4. $I = (-\infty, \infty)$.
Similar proof.

2. Let $S = (0, 1]$ and $T = \{\frac{1}{n} : n = 1, 2, 3, \ldots\}$. Show that $S - T$ is an open set.

$S - T = (\frac{1}{2}, 1) \cup (\frac{1}{3}, \frac{1}{2}) \cup (\frac{1}{4}, \frac{1}{3}) \cup \cdots$

$S - T$ is the union of an infinite number of open intervals. Since an open interval is an open set, $S - T$ being the union of an infinite number of open sets is an open set.

We have seen that an open interval is an open set in $\mathbb{R}$ and the union of any collection of open sets is an open set in $\mathbb{R}$. Therefore the union of an arbitrary collection of open intervals is an open set in $\mathbb{R}$.

The following theorem deals with the converse problem and it depicts the structural composition of a bounded open set in $\mathbb{R}$.

Theorem 3.4.9. A non-empty bounded open set in $\mathbb{R}$ is the union of a countable collection of disjoint open intervals.

Proof. Let G be a non-empty bounded open set in $\mathbb{R}$. Let $x \in G$. Since G is an open set, there is an element $y_o < x$ and an element $z_o > x$ such that $(y_o, x) \subset G$ and $(x, z_o) \subset G$.

Let $A = \{y : (y, x) \subset G\}$, $B = \{z : (x, z) \subset G\}$.

Then A is a non-empty set, since $y_o \in A$; A is bounded below, since G is bounded below. Let $a = \inf A$.

Similarly, B is a non-empty set bounded above. Let $b = \sup B$.

Then $a < x < b$ and $I_x = (a, b)$ is an open interval containing x. We prove $I_x \subset G$.

Let $w \in I_x$ and $a < x < w < b$.

Since $b = \sup B$, there exists an element $z' \in B$ such that $w < z' \leq b$.

Therefore $(x, z') \subset G$, since $z' \in B$. Therefore $w \in G$.

If however, $w \in I_x$ and $a < w < x < b$, then also $w \in G$.

Thus $w \in I_x \Rightarrow w \in G$ and therefore $I_x \subset G$.

We prove $a \notin G, b \notin G$.

If $b \in G$, then for some positive $\epsilon, (b - \epsilon, b + \epsilon) \subset G$, since G is an open set. Let $\delta < \epsilon$. Then $b < b + \delta < b + \epsilon$ and $b + \delta \in G$ contradicting the definition of b. Therefore $b \notin G$. Similarly, $a \notin G$.

Let $\mathcal{G}$ be the collection of open intervals $\{I_x : x \in G\}$. Let $H = \bigcup\limits_{x \in G} I_x$.

Let $x \in G$. Then $x \in I_x$ and $I_x \subset H$.
Thus $x \in G \Rightarrow x \in H$. Therefore $G \subset H$.
Let $y \in H$. Then $y \in I_y$ and $I_y \subset G$.
Thus $y \in H \Rightarrow y \in G$. Therefore $H \subset G$.
Consequently, $G = H = \bigcup\limits_{x \in G} x$.

We prove that two distinct intervals in the collection $\mathcal{G}$ are disjoint.

Let $(a, b), (c, d)$ be two intervals in this collection with a point p in common.

Then $c < b$ and $a < d$.

Since $c \notin G$, c does not belong to (a, b) and therefore $c \leq a$.

Since $a \notin G$, a does not belong to (c, b) and therefore $a \leq c$.

$c \leq a$ and $a \leq c \Rightarrow a = c$. Similarly $b = d$.

Therefore two distinct intervals of the collection are disjoint.

Thus G is the union of disjoint collection of open intervals $\{I_x : x \in G\}$.

We show that the collection is countable.

Let $\mathcal{G}'$ be the collection $\{I_\alpha : \alpha \in \Lambda\}$ where I_α is an open interval and Λ is the index set.

Let $\lambda \in \Lambda$. Then I_λ is an open interval of the collection $\mathcal{G}'$.

Let $x \in I_\lambda$. Then there exists a positive δ such that $(x-\delta, x+\delta) \subset I_\lambda$. There exists a rational number r_λ such that $x-\delta < r_\lambda < x+\delta$. Therefore $r_\lambda \in \mathbb{Q} \cap I_\lambda$.

Let us define a function $f : \Lambda \to \mathbb{Q}$ that assigns $\lambda(\in \Lambda)$ to $r_\lambda(\in \mathbb{Q})$.

Since I_α's are disjoint, the function f is injective.

Since $\mathbb{Q}$ is an enumerable set and f is an injective function, Λ is at most enumerable. Hence $\mathcal{G}'$ is a countable collection.

This completes the proof.

3.5. Limit point.

Let S be a subset of $\mathbb{R}$. A point p in $\mathbb{R}$ is said to be a *limit point* (or an *accumulation point*, or a *cluster point*) of S if every neighbourhood of p contains a point of S other than p.

Therefore p is a limit point of S if for each positive ϵ,

$$[N(p, \epsilon) - \{p\}] \cap S \neq \phi.$$

$N(p, \epsilon) - \{p\}$ is called the *deleted* ϵ-neighbourhood of p and is denoted by $N'(p, \epsilon)$. $N(p) - \{p\}$ is called the *deleted neighbourhood* of p and is denoted by $N'(p)$.

Therefore p is a limit point of S if every deleted neighbourhood of p contains a point of S.

Note. A limit point of S may or may not belong to S. When we say that a set $S \subset \mathbb{R}$ has a limit point we mean that some real number p is a limit point of S and no assertion is made as to whether p belongs to S or not.

3.6. Isolated point.

Let S be a subset of $\mathbb{R}$. A point x in S is said to be an *isolated point* of S if x is not a limit point of S.

Since x is not a limit point of S, there exists a neighbourhood $N(x)$ of x such that $N'(x) \cap S = \phi$. Since $x \in S, N(x) \cap S = \{x\}$.

Therefore x is an isolated point of S if for some positive $\epsilon, N(x, \epsilon)$ contains *no point* of S other than x.

Examples.

1. Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\cdots\}$.

Every point of S is an isolated point of S.

We prove that 0 is a limit point of S. Let $\epsilon > 0$. By Archimedean property of $\mathbb{R}$ there exists a natural number m such that $0 < \frac{1}{m} < \epsilon$. $\frac{1}{m} \in S$ and $\frac{1}{m} \in N'(0, \epsilon)$. Thus the deleted ϵ- neighbourhood of 0 contains a point of S and this holds for each positive ϵ.

Therefore 0 is a limit point of S.

2. Let $S = \mathbb{Z}$.

Every point of $\mathbb{Z}$ is an isolated point of $\mathbb{Z}$. Therefore no point of $\mathbb{Z}$ is a limit point of $\mathbb{Z}$.

Let $x \in \mathbb{R} - \mathbb{Z}$. Then there exists an integer m such that $m - 1 < x < m$. Let $\epsilon = \min\{|x - m|, |x - \overline{m - 1}|\}$. Then the neighbourhood $N(x, \epsilon)$ of x contains no point of $\mathbb{Z}$ and therefore x can not be a limit point of $\mathbb{Z}$.

3. Let $S = \mathbb{Q}$.

No point of S is an isolated point of S. Every point $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$, since each deleted neighbourhood of x contains a point of $\mathbb{Q}$.

4. Let $S = \mathbb{R}$.

No point of S is an isolated point of S. Every point x of $\mathbb{R}$ is a limit point of $\mathbb{R}$, since each deleted neighbourhood of x contains a point of $\mathbb{R}$.

Theorem 3.6.1. Let $S \subset \mathbb{R}$ and p be a limit point of S. Then every neighbourhood of p contains infinitely many elements of S.

Proof. Let $\epsilon > 0$. Since p is a limit point of S, the deleted neighbourhood $N'(p, \epsilon)$ contains a point of S. That is, $N'(p, \epsilon) \cap S \neq \phi$.

Let $B = N'(p, \epsilon) \cap S$. We prove that B is an infinite set.

If not, let B contain only a finite number of elements of S, say $a_1, a_2, \ldots, a_m$.

Let $\delta_1 = | p - a_1 |, \delta_2 = | p - a_2 |, \cdots, \delta_m = | p - a_m |$.

Let $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_m\}$. Then $\delta > 0$ and $a_i \notin N(p, \delta), i = 1, 2, \ldots, m$.

It follows that $N'(p, \delta) \cap S = \phi$ and this disallows p to be a limit point of S.

Thus B is an infinite set. In other words, $N(p, \epsilon)$ contains infinitely many elements of S. This proves the theorem.

Worked Examples.

1. Show that a finite set has no limit point.

Let S be a finite set and $S = \{x_1, x_2, \ldots, x_m\}$. Let $p \in \mathbb{R}$. p can not be a limit point of S because if p be a limit point of S then every neighbourhood of p must contain infinitely many elements of S, which is an impossibility since S contains only a finite number of elements.

Therefore S has no limit point.

2. Show that the set $\mathbb{N}$ has no limit point.

Let $p \in \mathbb{R}$. Let $\epsilon = \frac{1}{2}$. Then the ϵ-neighbourhood $N(p, \frac{1}{2})$ of p contains at most one natural number and p cannot be a limit point of $\mathbb{N}$, because, in order that p may be a limit point of $\mathbb{N}$, each neighbourhood of p must contain infinitely many elements of $\mathbb{N}$.

It follows that $\mathbb{N}$ has no limit point.

Note. By similar arguments it can be established that the set $\mathbb{Z}$ has no limit point.

3. Let S be a subset of $\mathbb{R}$. Prove that an interior point of S is a limit point of S.

Let x be an interior point of S. Then there exists a positive δ such that the neighbourhood $N(x, \delta)$ of x is entirely contained in S.

Let us choose $\epsilon > 0$.

Case 1. $0 < \epsilon < \delta$.

Then $N(x, \epsilon) \subset N(x, \delta) \subset S$ and therefore $N'(x, \epsilon) \cap S \neq \phi$.

Case 2. $\epsilon \geq \delta$. Then $N(x, \delta) \subset N(x, \epsilon)$.

$N(x, \delta) \subset S$ and $N(x, \delta) \subset N(x, \epsilon) \Rightarrow N(x, \delta) \subset N(x, \epsilon) \cap S$.

Then clearly, $N'(x, \epsilon) \cap S \neq \phi$.

In both the cases $N'(x, \epsilon) \cap S \neq \phi$ and this proves that x is a limit point of S.

Theorem 3.6.2. Bolzano-Weierstrass theorem.

Every bounded infinite subset of $\mathbb{R}$ has at least one limit point (in $\mathbb{R}$).

Proof. Let S be a bounded infinite subset of $\mathbb{R}$. Since S is a non-empty bounded subset of $\mathbb{R}$, $\sup S$ and $\inf S$ both exist. Let $s^* = \sup S$ and $s_* = \inf S$. Then $x \in S \Rightarrow s_* \leq x \leq s^*$.

Let H be a subset of $\mathbb{R}$ defined by $H = \{x \in \mathbb{R} : x$ is greater than infinitely many elements of $S\}$.

Then $s^* \in H$ and so H is a non-empty subset of $\mathbb{R}$.

Let $h \in H$. Then h is greater than infinitely many elements of S and therefore $h > s_*$, because no element $\leq s_*$ exceeds infinitely many elements of S.

Thus H is a non-empty subset of $\mathbb{R}$, bounded below, s_* being a lower bound. So $\inf H$ exists.

Let $\inf H = \xi$. We now show that ξ is a limit point of S.

Let us choose $\epsilon > 0$.

Since $\inf H = \xi$, there exists an element y in H such that $\xi \leq y < \xi + \epsilon$. Since $y \in H$, y exceeds infinitely many elements of S and consequently $\xi + \epsilon$ exceeds infinitely many element of S.

Since ξ is the infimum of H, $\xi - \epsilon$ does not belong to H and so $\xi - \epsilon$ can exceed at most a finite number of elements of S. Thus the neighbourhood $(\xi - \epsilon, \xi + \epsilon)$ contains infinitely many elements of S.

This holds for each $\epsilon > 0$. Therefore ξ is a limit point of S.

This completes the proof.

3.7. Derived set.

Let $S \subset \mathbb{R}$. The set of all limit points of S is said to be the *derived set* of S and is denoted by S'.

Examples.

1. Let S be a finite set. Then $S' = \phi$.
2. Let $S = \mathbb{N}$. Then $S' = \phi$.
3. Let $S = \mathbb{Z}$. Then $S' = \phi$.
4. Let $S = \mathbb{Q}$. Then $S' = \mathbb{R}$.
5. Let $S = \mathbb{R}$. Then $S' = \mathbb{R}$.
6. Let $S = \phi$. Then $S' = \phi$.

Theorem 3.7.1. Let A, B be subsets of $\mathbb{R}$ and $A \subset B$. Then $A' \subset B'$.

Proof. **Case 1.** $A' = \phi$. Then $A' \subset B'$.

Case 2. $A' \neq \phi$. Let $p \in A'$. Then p is a limit point point of A.
Let $\epsilon > 0$. Then $N(p, \epsilon)$ contains a point of A, say q, other than p.
$q \in A \Rightarrow q \in B$. Therefore $N'(p, \epsilon)$ contains a point q of B.

Since ϵ is arbitrary, p is a limit point of B. Therefore $p \in B'$. Thus $p \in A' \Rightarrow p \in B'$ and therefore $A' \subset B'$.

This completes the proof.

Theorem 3.7.2. Let $A \subset \mathbb{R}$. Then $(A')' \subset A'$.

Proof. **Case 1.** $(A')' = \phi$. Then $(A')' \subset A'$.

Case 2 . $(A')' \neq \phi$. Let $p \in (A')'$. Then p is a limit point of A'.
Let $\epsilon > 0$. Then $N(p, \epsilon)$ contains a point of A', say q, other than p.

Since $q \in A'$, q is a limit point of A. Therefore $N(p, \epsilon)$ being a neighbourhood of q also, contains infinitely many points of A.

Since $N(p, \epsilon)$ contains infinitely many points of A, p is a limit point of A. That is, $p \in A'$.
Thus $p \in (A')' \Rightarrow p \in A'$ and therefore $(A')' \subset A'$.

This completes the proof.

Theorem 3.7.3. Let $A, B \subset \mathbb{R}$. Then $(A \cap B)' \subset A' \cap B'$.

Proof. $A \cap B \subset A \Rightarrow (A \cap B)' \subset A'$, since $A \subset B \Rightarrow A' \subset B'$.
$A \cap B \subset B \Rightarrow (A \cap B)' \subset B'$, since $A \subset B \Rightarrow A' \subset B'$.

It follows that $(A \cap B)' \subset A' \cap B'$.

Note. $(A \cap B)' \neq A' \cap B'$, in general.
For example, let $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\}$, $B = \{0, -1, -\frac{1}{2}, -\frac{1}{3}, \cdots\}$. Then $A' = \{0\}$, $B' = \{0\}$. $A \cap B = \{0\}$, $A' \cap B' = \{0\}$, but $(A \cap B)' = \phi$.

Corollary. Let $A_1, A_2, \ldots, A_m$ be subsets of $\mathbb{R}$. Then $(A_1 \cap A_2 \cap \cdots \cap A_m)' \subset A'_1 \cap A'_2 \cap \cdots \cap A'_m$.

Theorem 3.7.4. Let A and B be subsets of $\mathbb{R}$. Then $(A \cup B)' = A' \cup B'$.

Proof. $A \subset A \cup B \Rightarrow A' \subset (A \cup B)'$, since $A \subset B \Rightarrow A' \subset B'$
$B \subset A \cup B \Rightarrow B' \subset (A \cup B)'$, since $A \subset B \Rightarrow A' \subset B'$.

It follows that $A' \cup B' \subset (A \cup B)'$ ... ... (i)

We now prove that $(A \cup B)' \subset A' \cup B'$.
Let $p \notin A' \cup B'$. Then $p \notin A'$ and $p \notin B'$.

So there exists a positive ϵ_1 such that $N'(p, \epsilon_1) \cap A = \phi$ and there exists a positive ϵ_2 such that $N'(p, \epsilon_2) \cap B = \phi$.

Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then $\epsilon > 0$ and $N'(p, \epsilon) \cap A = \phi$, $N'(p, \epsilon) \cap B = \phi$.
Therefore $N'(p, \epsilon) \cap (A \cup B) = [N'(p, \epsilon) \cap A] \cup [N'(p, \epsilon) \cap B] = \phi$.
This disallows p to be a limit point of $A \cup B$. So $p \notin (A \cup B)'$.
Thus $p \notin A' \cup B' \Rightarrow p \notin (A \cup B)'$.

Contrapositively, $p \in (A \cup B)' \Rightarrow p \in A' \cup B'$.
Consequently, $(A \cup B)' \subset A' \cup B'$ ... ... (ii)

From (i) and (ii) it follows that $(A \cup B)' = A' \cup B'$.

This completes the proof.

Corollary. Let $A_1, A_2, \ldots, A_m$ be subsets of $\mathbb{R}$. Then $(A_1 \cup A_2 \cup \cdots \cup A_m)' = A_1' \cup A_2' \cup \cdots \cup A_m'$.

Worked Examples.

1. Let S be a bounded subset of $\mathbb{R}$. Prove that S' (the derived set of S) is bounded.

Case 1. Let S be a finite subset of $\mathbb{R}$. Then $S' = \phi$ and it is bounded.

Case 2. Let S be an infinite subset of $\mathbb{R}$. By Bolzano-Weierstrass theorem, S' is a non-empty subset of $\mathbb{R}$.

Let $\sup S = m^*$. Then $x \in S \Rightarrow x \leq m^*$. Let $c > m^*$. Let us choose $\epsilon = \frac{c-m^*}{2}$. Then $m^* + \epsilon = c - \epsilon$ and the ϵ−neighbourhood $(c-\epsilon, c+\epsilon)$ of c contains no point of S. Therefore c cannot be a limit point of S, i.e., $c \notin S'$.

Thus $c > m^* \Rightarrow c \notin S'$. Contrapositively, $c \in S' \Rightarrow c \leq m^*$. This shows that m^* is an upper bound of S', i.e., S' is bounded above.

Let $\inf S = m_*$ and let $d < m_*$.
By similar arguments, d cannot be a limit point of S, i.e., $d \notin S'$.

Thus $d < m_* \Rightarrow d \notin S'$. Contrapositively, $d \in S' \Rightarrow d \geq m_*$. This shows that m_* is a lower bound of S', i.e., S' is bounded below.
Therefore S' is a bounded subset of $\mathbb{R}$.

2. Let S be a non-empty subset of $\mathbb{R}$ bounded above and $s^* = \sup S$. If s^* does not belong to S prove that s^* is a limit point of S and s^* is the greatest element of S'.

Let $\epsilon > 0$. Since $s^* = \sup S$,
(i) $x \in S \Rightarrow x < s^*$ (since $s^* \notin S$) and
(ii) there is an element y in S such that $s^* - \epsilon < y < s^*$.

We have $s^* - \epsilon < y < s^* < s^* + \epsilon$.

Thus the ϵ-neighbourhood $(s^* - \epsilon, s^* + \epsilon)$ of s^* contains a point y of S other than s^*. Since ϵ is arbitrary, s^* is a limit point of S.

Let $t > s^*$ and $\epsilon = \frac{t-s^*}{2}$. Then $\epsilon > 0$ and $s^* + \epsilon = t - \epsilon$. Since $s^* = \sup S$, no point of S is greater than S^*. Therefore the neighbourhood $(t-\epsilon, t+\epsilon)$ of t contains no point of S and so t is not a limit point of S. Consequently, S^* is the greatest element of S'.

3. Let $S = (a, b)$ an open bounded interval. Prove that $S' = [a, b]$.

Case 1. Let $x \in (a, b)$. Then x is an interior point of S.

By worked Example 4, page 53, x is a limit point of S.

Case 2. Let $x = a$.

Let us choose $\epsilon > 0$.

Let $\delta = \min\{\epsilon, b - a\}$. Then $\delta > 0$ and $a < a + \frac{\delta}{2} < a + \delta \leq a + \epsilon$,

$a < a + \frac{\delta}{2} < a + \delta \leq b$.

$a < a + \frac{\delta}{2} < a + \epsilon \Rightarrow a + \frac{\delta}{2} \in N'(a, \epsilon)$.

$a < a + \frac{\delta}{2} < b \Rightarrow a + \frac{\delta}{2} \in S$.

Therefore $a + \frac{\delta}{2} \in N'(a, \epsilon) \cap S$. As $N'(a, \epsilon) \cap S \neq \phi$, a is a limit point of S.

Case 3. Let $x = b$.

In a similar manner we can prove that b is a limit point of S.

Case 4. Let $x > b$.

Let us choose $\epsilon = \frac{x-b}{2}$. Then $\epsilon > 0$ and $b + \epsilon = x - \epsilon$.

The neighbourhood $(x-\epsilon, x+\epsilon)$ contains no point of S and this proves that x is not a limit point of S.

Case 5. Let $x < a$.

Let us choose $\epsilon = \frac{a-x}{2}$. Then $\epsilon > 0$ and $x + \epsilon = a - \epsilon$.

The neighbourhood $(x-\epsilon, x+\epsilon)$ contains no point of S and this proves that x is not a limit point of S.

From the above cases, we conclude that $S' = [a, b]$.

4. Let $S = [a, b]$, a closed bounded interval, then $S' = S$.

The proof is similar.

5. Find the derived set of the set $S = \{\frac{1}{n} : n \in \mathbb{N}\}$.

Here $S \subset (0, 1] = B$, say. Therefore $S' \subset B' = [0, 1]$.

First we prove that 0 is a limit point of S.

Let $\epsilon > 0$. By Archimedean property of $\mathbb{R}$, there exists a natural number p such that $0 < \frac{1}{p} < \epsilon$. Then $-\epsilon < \frac{1}{p} < \epsilon$.

This shows that the ϵ-neighbourhod of 0 contains a point $\frac{1}{p}$ of S, other than 0. So 0 is a limit point of S.

Let $c \in (0, 1]$. Then $c > 0$. Let us choose a positive ϵ such that $c - \epsilon > 0$.

By Archimedean property of $\mathbb{R}$, there exists a natural number k such that $0 < \frac{1}{k} < c - \epsilon$. Then $\frac{1}{n} < c - \epsilon$ for all $n \geq k$.

It follows that at most a finite number of elements of S lie in the neighbourhood $(c - \epsilon, c + \epsilon)$ of c. So c cannot be a limit point of S.

Thus 0 is the only limit point of S. That is, $S' = \{0\}$.

6. Let $S = \{\frac{1}{m} + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\}$.

(i) Show that 0 is a limit point of S.

(ii) If $k \in \mathbb{N}$, show that $\frac{1}{k}$ is a limit point of S.

(iii) Find S' (the derived set of S).

(i) Let $\epsilon > 0$. By Archimedean property of $\mathbb{R}$, there exist natural numbers p, q such that $0 < \frac{1}{p} < \frac{\epsilon}{2}$, $0 < \frac{1}{q} < \frac{\epsilon}{3}$.

Then $0 < \frac{1}{p} + \frac{1}{q} < \epsilon$.

This shows that the ϵ-neighbourhood $(-\epsilon, \epsilon)$ of 0 contains a point $\frac{1}{p} + \frac{1}{q}$ of S, other than 0. So 0 is limit point of S.

(ii) Let $\epsilon > 0$. By Archimedean property of $\mathbb{R}$, there exists a natural number s such that $0 < \frac{1}{s} < \epsilon$.

Then $-\epsilon < \frac{1}{s} < \epsilon$

or, $\frac{1}{k} - \epsilon < \frac{1}{k} + \frac{1}{s} < \frac{1}{k} + \epsilon$.

This shows that the ϵ-neighbourhood of $\frac{1}{k}$ contains a point $\frac{1}{k} + \frac{1}{s}$ of S, other than $\frac{1}{k}$. So $\frac{1}{k}$ is a limit point of S.

(iii) Let $T = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $T \subset S'$.

Now $S \subset (0, 2]$. This implies $S' \subset \{(0, 2]\}' = [0, 2]$.

We prove that $T = S'$, i.e., no point of $[0, 2] - T$ is a limit point of S.

$[0, 2] - T = \{2\} \cup I_1 \cup I_2 \cup \cdots$ where $I_1 = (1, 2)$, $I_n = (\frac{1}{n}, \frac{1}{n-1})$, for $n = 2, 3, \ldots$

2 cannot be a limit point of S, because the neighbourhood $(2 - \frac{1}{2}, 2 + \frac{1}{2})$ of 2 contains no point of S, other than 2.

Let $c \in I_1$. Then $1 < c < 2$. Let us choose $\epsilon > 0$ such that $1 < c - \epsilon < c + \epsilon < 2$.

$c - 1 - \epsilon > 0$. By Archimedean property of $\mathbb{R}$, there exists a natural number p such that $0 < \frac{1}{p} < c - 1 - \epsilon$.

Then $1 + \frac{1}{p} < c - \epsilon < c + \epsilon < 2$.

If $m \geq p$, $\frac{1}{m} + \frac{1}{n} \leq 1 + \frac{1}{p}$ for all $n \geq 1$.
If $n \geq p$, $\frac{1}{m} + \frac{1}{n} \leq 1 + \frac{1}{p}$ for all $m \geq 1$.

It follows that only a finite number of elements of S lie in the neighbourhod $(1 - \epsilon, 1 + \epsilon)$ of c. So c cannot be a limit point of S.

Let $c \in I_k, k = 2, 3, \ldots$. Then $\frac{1}{k} < c < \frac{1}{k-1}$.
Let us choose $\epsilon > 0$ such that $\frac{1}{k} < c - \epsilon < c + \epsilon < \frac{1}{k-1}$.
$c - \frac{1}{k} - \epsilon > 0$. By Archimedean property of $\mathbb{R}$, there exists a natural number p such that $0 < \frac{1}{p} < c - \frac{1}{k} - \epsilon$.

Then $\frac{1}{k} + \frac{1}{p} < c - \epsilon < c + \epsilon < \frac{1}{k-1}$ and $\frac{1}{p} < \frac{1}{k-1} - \frac{1}{k}$, i.e., $p > k$, since $k \geq 2$

If $m \leq k - 1$, $\frac{1}{m} + \frac{1}{n} > \frac{1}{k-1}$ for all $n \geq 1$.
If $m \geq k$, $\frac{1}{m} + \frac{1}{n} \leq \frac{1}{k} + \frac{1}{p}$ for all $n \geq p$.

If $n \leq k - 1$, $\frac{1}{m} + \frac{1}{n} > \frac{1}{k-1}$ for all $m \geq 1$.
If $n \geq k$, $\frac{1}{m} + \frac{1}{n} \leq \frac{1}{k} + \frac{1}{p}$ for all $m \geq p$.

It follows that at most a finite number of elements of S lie in the neighbourhood $(c - \epsilon, c + \epsilon)$ of c. So c cannot be a limit point of S.

Thus $c \in [0, 2] - T \Rightarrow c$ is not a limit point of S.

Hence $S' = T$, i.e., $S' = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$.

7. Let $S = \{(-1)^m + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\}$.
(i) Show that 1 and -1 are limit points of S.
(ii) Find S' (the derived set of S).

(i) Let $A = \{(-1)^{2m} + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\}$, $B = \{(-1)^{2m-1} + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\}$. Then $S = A \cup B$ and therefore $S' = A' \cup B'$.

$A = \{1 + \frac{1}{n} : n \in \mathbb{N}\}$. We prove that 1 is the only limit point of A.
Let $\epsilon > 0$. By Archimedean property of $\mathbb{R}$, there exists a natural number p such that $0 < \frac{1}{p} < \epsilon$. Therefore $-\epsilon < \frac{1}{p} < \epsilon$.
or, $1 - \epsilon < 1 + \frac{1}{p} < 1 + \epsilon$.

This shows that the ϵ-neighbourhood $(1 - \epsilon, 1 + \epsilon)$ of 1 contains a point $1 + \frac{1}{p}$ of A, other than 1. So 1 is a limit point of A.

$A \subset (1, 2]$. This implies $A' \subset [1, 2]$.

2 cannot be a limit point of A, because the neighbourhood $(2 - \frac{1}{2}, 2 + \frac{1}{2})$ of 2 contains no point of A other than 2.

Let $c \in (1, 2)$. Let us choose a positive ϵ such that $1 < c - \epsilon < c + \epsilon < 2$. Then $c - 1 - \epsilon > 0$.

By Archimedean property of $\mathbb{R}$, there exists a natural number p such that $0 < \frac{1}{p} < c - 1 - \epsilon$. Then $1 < 1 + \frac{1}{p} < c - \epsilon$.

Therefore for all $n \geq p$, $1 + \frac{1}{n} \leq 1 + \frac{1}{p} < c - \epsilon$. So the neighbourhood $(c - \epsilon, c + \epsilon)$ of c contains at most a finite number of elements of A and hence c cannot be a limit point of A. Therefore $A' = \{1\}$.

$B = \{-1 + \frac{1}{n} : n \in \mathbb{N}\}$. We prove that -1 is the only limit point of B.

Let $\epsilon > 0$. There exists a natural number q such that $0 < \frac{1}{q} < \epsilon$.

Therefore $-\epsilon < \frac{1}{q} < \epsilon$, or, $-1 - \epsilon < -1 + \frac{1}{q} < -1 + \epsilon$.

This shows that the ϵ-neighbourhood $(-1 - \epsilon, -1 + \epsilon)$ of -1 contains a point $-1 + \frac{1}{q}$ of B, other than -1. So -1 is a limit point of B.

$B \subset (-1, 0]$. This implies $B' \subset [-1, 0]$.

0 cannot be a limit point of B, because the neighbourhood $(-\frac{1}{2}, \frac{1}{2})$ of 0 contains no point of B other than 0.

Let $c \in (-1, 0)$. Let us choose a positive ϵ such that $-1 < c - \epsilon < c + \epsilon < 0$. Then $c + 1 - \epsilon > 0$.

By Archimedean property of $\mathbb{R}$, there exists a natural number q such that $0 < \frac{1}{q} < c + 1 - \epsilon$. Then $-1 + \frac{1}{q} < c - \epsilon$.

Therefore for all $n \geq q$, $-1 + \frac{1}{n} \leq -1 + \frac{1}{q} < c - \epsilon$.

This shows that the neighbourhood $(c - \epsilon, c + \epsilon)$ of c contains at most a finite number of elements of B and hence c cannot be a limit point of B. Therefore $B' = \{-1\}$.

Therefore $S' = A' \cup B' = \{1, -1\}$.

3.8. Closed set.

Let S be a subset of $\mathbb{R}$. S is said to be a *closed set* if $S' \subset S$. (i.e., if S contains all its limit points.)

Examples.

1. Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$. 0 is a limit point of S. As $0 \notin S$, S is not a closed set.

2. Let $S = \{x \in \mathbb{R} : 1 < x < 3\}$. Each point of S is a limit point of S. 1 and 3 are also limit points of S but $1 \notin S, 3 \notin S$. Therefore S is not a closed set.

3. Let $S = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$. Each point of S is a limit point of S. Here $S' = S$. As $S' \subset S$, S is a closed set.

4. Let $S = \mathbb{Z}$. Then $S' = \phi$. So $S' \subset S$ and S is a closed set.

5. Let $S = \mathbb{N}$. Then $S' = \phi$. So $S' \subset S$ and S is a closed set.

6. Let S be a finite set, say $S = \{a_1, a_2, \ldots, a_m\}$. Then $S' = \phi$. So $S' \subset S$ and S is a closed set.

7. Let $S = \mathbb{Q}$. Let $x \in \mathbb{R}$. Every neighbourhood of x contains infinitely many elements of $\mathbb{Q}$. Therefore $x \in \mathbb{Q}'$. Hence $S' = \mathbb{R}$. Here S' is not a subset of S. S is not a closed set.

Note that the set $\mathbb{Q}$ is neither an open set nor a closed set in $\mathbb{R}$.

8. Let $S = \mathbb{R}$. Then $S' = \mathbb{R}$. So $S' \subset S$ and S is a closed set.

9. Let $S = \phi$. Then $S' = \phi$. So $S' \subset S$ and S is a closed set.

10. Let S be a subset of $\mathbb{R}$. Then $(S')' \subset S'$, by theorem 3.7.2. Therefore S' is a closed set. It follows that the derived set of S is a closed set.

Theorem 3.8.1. The union of a finite number of closed sets in $\mathbb{R}$ is a closed set.

Proof. Let $F_1, F_2, \ldots, F_m$ be m closed sets in $\mathbb{R}$. Let $F = F_1 \cup F_2 \cup \cdots \cup F_m$.

Since F_i is a closd set, $F_i' \subset F_i$ for $i = 1, 2, \ldots, m$.

$F' = (F_1 \cup F_2 \cup \cdots \cup F_m)' = F_1' \cup F_2' \cup \cdots \cup F_m'$.
$F_1' \subset F_1 \Rightarrow F_1' \subset F, F_2' \subset F_2 \Rightarrow F_2' \subset F, \cdots, F_m' \subset F_m \Rightarrow F_m' \subset F$.
Therefore $F_1' \cup F_2' \cup \cdots \cup F_m' \subset F$, i.e., $F' \subset F$.
As $F' \subset F$, F is a closed set and the theorem is done.

Theorem 3.8.2. The intersection of a finite number of closed sets in $\mathbb{R}$ is a closed set.

Proof. Let $F_1, F_2, \ldots, F_m$ be m closed sets in $\mathbb{R}$ and let $F = F_1 \cap F_2 \cap \cdots \cap F_m$.

Since F_i is a closd set, $F_i' \subset F_i$ for $i = 1, 2, \ldots, m$.

$F' = (F_1 \cap F_2 \cap \cdots \cap F_m)' \subset F_1' \cap F_2' \cap \cdots \cap F_m'$.

$$\begin{aligned}
F_1' \cap F_2' \cap \cdots \cap F_m' &\subset F_1' \subset F_1.\\
F_1' \cap F_2' \cap \cdots \cap F_m' &\subset F_2' \subset F_2.\\
\cdots \qquad & \cdots \qquad \cdots\\
F_1' \cap F_2' \cap \cdots \cap F_m' &\subset F_m' \subset F_m.
\end{aligned}$$

Therefore $F_1' \cap F_2' \cap \cdots \cap F_m' \subset F_1 \cap F_2 \cap \cdots \cap F_m = F$.
It follows that $F' \subset F_1' \cap F_2' \cap \cdots \cap F_m' \subset F$.
As $F' \subset F$, F is a closed set and the theorem is done.

Theorem 3.8.3. The intersection of an arbitrary collection of closed sets in $\mathbb{R}$ is a closed set.

Proof. Let $\{F_\alpha : \alpha \in \Lambda\}$, Λ being the index set, be a collection of closed sets in $\mathbb{R}$. Then $(F_\alpha)' \subset F_\alpha$ for each $\alpha \in \Lambda$. Let $F = \bigcap_{\alpha\in\Lambda} F_\alpha$.

Case 1. $F' = \phi$. Then obviously $F' \subset F$.

Case 2. $F' \neq \phi$. Let $p \in F'$. Then p is a limit point of F.

Let us choose $\epsilon > 0$. Then $N'(p, \epsilon)$ contains a point, say q, of F. $q \in N'(p,\epsilon) \cap F \Rightarrow q \in N'(p,\epsilon) \cap F_\alpha$ for each $\alpha \in \Lambda$.

This implies p is a limit point of F_α for each $\alpha \in \Lambda$.

Since each F_α is a closed set, $p \in F_\alpha$ for each $\alpha \in \Lambda$.

Hence $p \in \bigcap_{\alpha\in\Lambda} F_\alpha$, i.e., $p \in F$.

Thus $p \in F' \Rightarrow p \in F$ and therefore $F' \subset F$.

This proves that F is a closed set and the theorem is done.

Note. The union of an infinite number of closed sets in $\mathbb{R}$ is not necessarily a closed set.

Let us consider the sets F_i, where

$$\begin{aligned} F_1 &= \{x \in \mathbb{R} : -1 \le x \le 1\}, \\ F_2 &= \{x \in \mathbb{R} : -\tfrac{1}{2} \le x \le \tfrac{1}{2}\}, \\ &\dots \quad \dots \quad \dots \\ F_n &= \{x \in \mathbb{R} : -\tfrac{1}{n} \le x \le \tfrac{1}{n}\}, \\ &\dots \quad \dots \quad \dots \end{aligned}$$

Each F_i is a closed set. $\bigcup_{i=1}^{\infty} F_i = F_1$ and this is a closed set.

Let us consider the sets F_i, where

$$\begin{aligned} F_1 &= \{x \in \mathbb{R} : 1 \le x \le 2\}, \\ F_2 &= \{x \in \mathbb{R} : \tfrac{1}{2} \le x \le 3 - \tfrac{1}{2}\}, \\ &\dots \quad \dots \quad \dots \\ F_n &= \{x \in \mathbb{R} : \tfrac{1}{n} \le x \le 3 - \tfrac{1}{n}\}, \\ &\dots \quad \dots \quad \dots \end{aligned}$$

Each F_i is a closed set. $\bigcup_{i=1}^{\infty} F_i = \{x \in \mathbb{R} : 0 < x < 3\}$. This is not a closed set.

These two examples establish that the union of an infinite number of closed sets in $\mathbb{R}$ is not necessarily a closed set.

Theorem 3.8.4. Let G be an open set in $\mathbb{R}$. Then the complement of G (in $\mathbb{R}$) is a closed set in $\mathbb{R}$.

Proof. **Case 1.** $G = \phi$ (an open set in $\mathbb{R}$). The complement of ϕ in $\mathbb{R}$ is $\mathbb{R}$ and $\mathbb{R}$ is a closed set.

Case 2. $G \neq \phi$. Let $x \in G$. Since G is an open set, x is an interior point of G. Therefore there exists a neighbourhood $N(x)$ of x such that $N(x) \subset G$.

That is, $N(x) \cap G^c = \phi$ where G^c is the complement of G.

This implies that x is not a limit point of G^c. That is, $x \notin (G^c)'$.

Thus $x \in G \Rightarrow x \notin (G^c)'$.

Contrapositively, $x \in (G^c)' \Rightarrow x \notin G$. i.e., $x \in G^c$.

Therefore $(G^c)' \subset G^c$ and this proves that G^c is a closed set.

This completes the proof.

Theorem 3.8.5. Let F be a closed set in $\mathbb{R}$. Then the complement of F (in $\mathbb{R}$) is an open set in $\mathbb{R}$.

Proof. **Case 1.** $F = \mathbb{R}$ (a closed set). Then the complement of F in $\mathbb{R}$ is ϕ and ϕ is an open set.

Case 2. F is a proper subset of $\mathbb{R}$. Then $F^c \neq \phi$ where F^c is the complement of F.

Let $x \in F^c$. Since F is a closed set and $x \notin F, x$ is not a limit point of F. Therefore there exists a neighbourhood $N(x)$ of x such that $N'(x) \cap F = \phi$.

Since $x \notin F$, $N(x) \cap F = \phi$. That is $N(x) \subset F^c$.

Thus $x \in F^c \Rightarrow N(x) \subset F^c$.

This shows that x is an interior point of F^c. Since x is arbitrary, F^c is an open set and the theorem is done.

Worked Examples.

1. If S be a non-empty closed and bounded subset of $\mathbb{R}$. then prove that sup S and inf S belong to S.

Since S is a non-empty bounded set, by the supremum property of $\mathbb{R}$, sup S and inf S both exist. Let sup $S = m^*$, inf $S = m_*$.

Let us assume that $m^* \notin S$.

Since $\sup S = m^*$, $x \in S \Rightarrow x < m^*$ and for each pre-assigned positive ϵ, there is an element y in S such that $m^* - \epsilon < y < m^*$. Therefore $m^* - \epsilon < y < m^* + \epsilon$.

This shows that $N'(m^*, \epsilon) \cap S \neq \phi$. It follows that m^* is a limit point of S. $m^* \notin S$ and $m^* \in S' \Rightarrow S$ is not a closed set, a contradiction.

Hence $m^* \in S$. In a similar manner it can be proved that $m_* \in S$.

2. Prove that a finite subset of $\mathbb{R}$ is a closed set.

Let $S = \{a_1, a_2, \ldots, a_m\}$ where $a_1 < a_2 < a_3 < \cdots < a_m$. Then S can be expressed as the complement of the union of $m+1$ open intervals $(-\infty, a_1), (a_1, a_2), \cdots, (a_{m-1}, a_m), (a_m, \infty)$.

Since each open interval is an open set and the union of a finite number of open sets is an open set, S is the complement of an open set. Therefore S is a closed set.

3. Prove that a closed and bounded interval is a closed set.

Let $I = [a, b]$ be a closed and bounded interval. Then I can be expressed as $\mathbb{R} - [(-\infty, a) \cup (b, \infty)]$.

$(-\infty, a)$ is an open interval and so it is an open set.

(b, ∞) is an open interval and so it is an open set.

The union $(-\infty, a) \cup (b, \infty)$ is an open set. Therefore I being the complement of an open set, is a closed set.

Note. The closed intervals $(-\infty, a], [a, \infty)$ are closed sets.

3.9. Adherent point.

Let S be a subset of $\mathbb{R}$. A point $x \in \mathbb{R}$ is said to be an *adherent point* of S if every neighbourhood of x contains a point of S.

It follows that x is an adherent point of S if $N(x, \epsilon) \cap S \neq \phi$ for every $\epsilon > 0$.

The set of all adherent points of S is said to be the *closure* of S and is denoted by $\bar{S}$.

From definition it follows that $S \subset \bar{S}$ for any set $S \subset \mathbb{R}$.

Theorem 3.9.1. Let S be a subset of $\mathbb{R}$. Then $\bar{S} = S \cup S', S'$ being the derived set of S.

Proof. Let $x \in S$. Then every neighbourhood of x contains x, a point of S. Therefore x is an adherent point of S.

Thus $x \in S \Rightarrow x \in \bar{S}$ and therefore $S \subset \bar{S}$ ... ... (i)

Let $x \in S'$. Then x is a limit point of S. Hence every neighbourhood $N(x)$ of x contains a point of S other than x, i.e., $N(x) \cap S \neq \phi$.

Therefore x is an adherent point of S, i.e., $x \in \bar{S}$.

Thus $x \in S' \Rightarrow x \in \bar{S}$ and therefore $S' \subset \bar{S}$ ... ... (ii)

From (i) and (ii) $S \cup S' \subset \bar{S}$ ... ... (iii)

Let $y \notin S \cup S'$. Then $y \notin S$ and $y \notin S'$

Since $y \notin S'$, there exists a neighbourhood $N(y)$ of y such that $N'(y) \cap S = \phi$ and also since $y \notin S$, $N'(y) \cap S = \phi \Rightarrow N(y) \cap S = \phi$.

This shows that y is not an adherent point of S.
Therefore $y \notin S \cup S' \Rightarrow y \notin \bar{S}$.

Contrapositively, $y \in \bar{S} \Rightarrow y \in S \cup S'$ and therefore $\bar{S} \subset S \cup S'$ ... (iv)

From (iii) and (iv) $\bar{S} = S \cup S'$ and this completes the proof.

Theorem 3.9.2. Let S be a subset of $\mathbb{R}$. Then S is a closed set if and only if $S = \bar{S}$.

Proof. Let S be a closed set. Then $S' \subset S$.
$S \cup S' = S$, i.e., $\bar{S} = S$.

Conversely, let S be a subset of $\mathbb{R}$ such that $\bar{S} = S$. Then $S = S \cup S'$.
This implies $S' \subset S$ and therefore S is a closed set.
This completes the proof.

Theorem 3.9.3. Let S be a subset of $\mathbb{R}$. Then $\bar{S}$ is a closed set and it is the smallest closed set containing S.

Proof. $(\bar{S})' = (S \cup S')' = S' \cup (S')'$, since $(A \cup B)' = A' \cup B'$
$= S'$, since $(S')' \subset S'$
$\subset S \cup S' = \bar{S}$.

This proves that $\bar{S}$ is a closed set.

Again, $\bar{S} = S \cup S' \supset S$. Therefore $\bar{S}$ is a closed set containing S.

Let P be any closed set containing S.
$S \subset P$ implies $S' \subset P' \subset P$, since P is a closed set.
$S \subset P$ and $S' \subset P \Rightarrow S \cup S' \subset P$, i.e., $\bar{S} \subset P$.

Since P is arbitrary, $\bar{S}$ is the smallest closed set containing $\bar{S}$.
This completes the proof.

Note. $\bar{S}$ is the intersection of all closed supersets of S.

Worked Example.

1. Let A be a non-empty subset of $\mathbb{R}$ and $d(x, A) = \inf\{|x - y| : y \in A\}$. Prove that $d(x, A) = 0$, if and only if $x \in \bar{A}$.

Let $d(x, A) = 0$. Then $\inf\{| x - y |: y \in A\} = 0$.

Therefore $| x - y | \geq 0$ for all $y \in A$ and for a pre-assigned positive ϵ, there exists a point a in A such that $0 \leq | x - a | < \epsilon$.

This shows that the ϵ-neighbourhood of x contains a point a of A and this holds for each $\epsilon > 0$. Therefore $x \in \bar{A}$.

Conversely, let $x \in \bar{A}$. Then $x \in A \cup A'$.
If $x \in A$ then $d(x, A) = \inf\{|x - y| : y \in A\} = 0$.

If $x \in A'$, then for a chosen positive ϵ, there exists a point $y \in A$ such that $y \in N(x, \epsilon)$. Therefore $0 <| x - y |< \epsilon$ and this holds for each $\epsilon > 0$.
So $\inf\{| x - y |: y \in A\} = 0$.

Theorem 3.9.4. Let A, B be subsets of $\mathbb{R}$ and $A \subset B$. Then $\bar{A} \subset \bar{B}$.

Proof. Since $B \subset \bar{B}$, $A \subset B \Rightarrow A \subset \bar{B}$.
Thus $\bar{B}$ is a closed set containing A. But $\bar{A}$ is the smallest closed set containing A. Therefore $\bar{A} \subset \bar{B}$ and the proof is complete.

Theorem 3.9.5. Let A, B be subsets of $\mathbb{R}$. Then $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

Proof. $A \subset A \cup B \Rightarrow \bar{A} \subset \overline{A \cup B}$ and $B \subset A \cup B \Rightarrow \bar{B} \subset \overline{A \cup B}$, by Theorem 3.9.4. Therefore $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$... (i)

$A \subset \bar{A} \subset \bar{A} \cup \bar{B}$ and $B \subset \bar{B} \subset \bar{A} \cup \bar{B}$. Therefore $A \cup B \subset \bar{A} \cup \bar{B}$.
Because $\bar{A}$ and $\bar{B}$ are closed sets, $\bar{A} \cup \bar{B}$ is a closed set containing the set $A \cup B$. But $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$. Therefore $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$... (ii)

From (i) and (ii) it follows that $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
This completes the proof.

Corllary. $\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \bar{A}_1 \cup \bar{A}_2 \cup \cdots \cup A_n$.

Theorem 3.9.6. Let A, B be subsets of $\mathbb{R}$. Then $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$.

Proof. $A \cap B \subset A \Rightarrow \overline{A \cap B} \subset \bar{A}$. $A \cap B \subset B \Rightarrow \overline{A \cap B} \subset \bar{B}$.
Therefore $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$.

Note. The equality $\overline{A \cap B} = \bar{A} \cap \bar{B}$ may not hold.
For example, let $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, $B = \{-1, -\frac{1}{2}, -\frac{1}{3}, \ldots\}$. Then $\overline{A \cap B} = \phi$, $\bar{A} \cap \bar{B} = \{0\}$. $\overline{A \cap B} \neq \bar{A} \cap \bar{B}$.

Definition. Closed set. A set $S \subset \mathbb{R}$ is said to be a closed set if $\bar{S} \subset S$.

With this definition of a closed set, the theorems 3.8.1 and 3.8.2 will have the following alternative proofs.

Theorem 3.9.7. The union of a finite number of closed sets in $\mathbb{R}$ is a closed set.

Proof. Let $F_1, F_2, \ldots, F_m$ be m closed sets in $\mathbb{R}$ and $F = \bigcup\limits_{i=1}^{m} F_i$.

Since F_i is a closed set, $\bar{F}_i \subset F_i$ for $i = 1, 2, \ldots, m$.
$\bar{F} = \bar{F}_1 \cup \bar{F}_2 \cup \cdots \cup \bar{F}_m$.

$\bar{F}_i \subset F_i \Rightarrow \bar{F}_i \subset F$ for $i = 1, 2, \ldots, m$.

Therefore $\bar{F}_1 \cup \bar{F}_2 \cup \cdots \cup \bar{F}_m \subset F$, i.e., $\bar{F} \subset F$, showing that F is a closed set. This completes the proof.

Theorem 3.9.8. The intersection of a finite number of closed sets in $\mathbb{R}$ is a closed set.

Proof. Let $F_1, F_2, \ldots, F_m$ be m closed sets in $\mathbb{R}$ and $F = \bigcap_{i=1}^{m} F_i$.

Since F_i is a closed set, $\bar{F}_i \subset F_i$ for $i = 1, 2, \ldots, m$.
$\bar{F} = \overline{F_1 \cap F_2 \cap \cdots \cap F_m} \subset \bar{F}_1 \cap \bar{F}_2 \cap \cdots \cap \bar{F}_m$.

$\bar{F}_1 \cap \bar{F}_2 \cap \cdots \cap \bar{F}_m \subset \bar{F}_i \subset F_i$ for each $i = 1, 2, \ldots, m$ and therefore $\bar{F}_1 \cap \bar{F}_2 \cap \cdots \cap \bar{F}_m \subset F$, i.e., $\bar{F} \subset F$.
Therefore F is a closed set. This completes the proof.

Another definition of a closed set.

A set $S \subset \mathbb{R}$ is said to be a closed set if $\mathbb{R} - S$ is an open set.

Theorem 3.9.9. A set $S \subset \mathbb{R}$ is closed if and only if $S' \subset S$.

Proof. Let S be a closed set in $\mathbb{R}$. Then $\mathbb{R} - S$ is an open set, by definition.

Let $x \in \mathbb{R} - S$. Then x is an interior point of $\mathbb{R} - S$. Therefore there exists a neighbourhood $N(x)$ of x such that $N(x) \subset \mathbb{R} - S$, i.e., $N(x) \cap S = \phi$.

It follows that x is not a limit point of S, i.e., $x \notin S'$.
Thus $x \in \mathbb{R} - S \Rightarrow x \notin S'$.

Contrapositively, $x \in S' \Rightarrow x \notin \mathbb{R} - S$, i.e., $x \in S$.
This proves that $S' \subset S$.

Conversely, let S be a subset of $\mathbb{R}$ such that $S' \subset S$.

Let $x \in \mathbb{R} - S$. Then $x \notin S$ and therefore $x \notin S'$ since $S' \subset S$.

Thus there exists a neighbourhood $N(x)$ of x such that $N(x) \cap S = \phi$ and therefore $N(x) \subset \mathbb{R} - S$.

Thus $x \in \mathbb{R} - S \Rightarrow N(x) \subset \mathbb{R} - S$.

So x is an interior point of $\mathbb{R} - S$, proving that $\mathbb{R} - S$ is an open set. Therefore S is a closed set.

This completes the proof.

Remark. Since $\mathbb{R}$ is an open set , ϕ being the complement of $\mathbb{R}$, is a closed set. Since ϕ is an open set, $\mathbb{R}$ being the complement of ϕ, is a closed set.

Therefore the set $\mathbb{R}$ is both open and closed; the set ϕ is both open and closed in $\mathbb{R}$.

We are now in search of subsets of $\mathbb{R}$ which are both open and closed in $\mathbb{R}$. The next theorem throws light on our search.

Theorem 3.9.10. No non-empty proper subset of $\mathbb{R}$ is both open and closed in $\mathbb{R}$.

Proof. If possible, let S be a non-empty proper subset of $\mathbb{R}$ which is both open and closed. Since S is a proper subset of $\mathbb{R}$, there exists an element c in $\mathbb{R} - S$. Therefore $S \subset (-\infty, c) \cup (c, \infty)$.

Since S is non-empty, at least one of $S \cap (-\infty, c)$ and $S \cap (c, \infty)$ is non-empty.

Let $A = S \cap (-\infty, c) \neq \phi$. A is bounded above, c being an upper bound. Therefore $\sup A$ exists. Let $a = \sup A$. Then $a \leq c$.

For each $\epsilon > 0$, there is an element b in A such that $a - \epsilon < b \leq a$.
$b \in A \Rightarrow b \in S$.
Therefore each ϵ-neighbourhood of a contains a point of S.
So $a \in \bar{S}$ and since S is closed, $a \in S$. Therefore $a < c$.

Since S is open and $a \in S$, for some positive δ, $(a - \delta, a + \delta) \subset S$.

Let $d < \min\{\delta, c - a\}$. Then $a + d < a + \delta$ and $a + d < c$.
Therefore $a + d \in S$ and $a + d \in (-\infty, c)$.
Therefore $a + d \in A$ and this contradicts the definition of a.

Hence our assumption is wrong and the theorem is established.

3.10. Dense set. Perfect set.

Definition. Let S be a subset of $\mathbb{R}$. A subset $T \subset S$ is said to be *dense in* S if $S \subset T'$.

In particular, S is said to be *dense in* $\mathbb{R}$ (or *dense*, or *everywhere dense*) if every point of $\mathbb{R}$ is a limit point of S, or equivalently $S' = \mathbb{R}$.

Definition. Let S be a subset of $\mathbb{R}$. S is said to be *dense-in-itself* if $S \subset S'$.

Definition. Let S be a subset of $\mathbb{R}$. S is said to be a *perfect* set if S is dense-in-itself and closed, i.e., if $S = S'$.

Examples.

1. The set $\mathbb{Q}$ is dense in $\mathbb{R}$, since $\mathbb{Q}' = \mathbb{R}$.

2. Let $S = \{x \in \mathbb{R} : 1 < x < 2\}$. Then $S \subset S'$. S is dense-in-itself.

3. The set $\mathbb{Q}$ is dense-in-itself, since $\mathbb{Q} \subset \mathbb{Q}'$. The set $\mathbb{R}$ is dense-in-itself.

4. Let $S = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$. Then S is a perfect set.

Exercises 3

1. Give an example of an infinite set $S \subset \mathbb{R}$ such that
   (i) S has no limit point, (ii) S has only one limit point,
   (iii) S has three limit points, (iv) S has four limit points,
   (v) S is a proper subset of the derived set S'.
2. Let $S = \{m + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\}$. Find the derived set of S.
3. Let $S = \{(-1)^n(1 + \frac{1}{n}) : n \in \mathbb{N}\}$.
   (i) Show that -1 and 1 are limit points of S.
   (ii) Find the derived set of S.
4. Let $S = \{\frac{(-1)^m}{m} + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\}$.
   (i) Show that 0 is a limit point of S.
   (ii) If $k \in \mathbb{N}$, show that $\frac{1}{k}$ is a limit point of S.
   (iii) If $k \in \mathbb{N}$, show that $\frac{-1}{2k-1}$ is a limit point of S.
5. Verify Bolzano-Weierstrass theorem for the set $S \subset \mathbb{R}$.
   (i) $S = \{(-1)^n(1 + \frac{1}{n}) : n \in \mathbb{N}\}$, (ii) $S = \{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\}$,
   (iii) $S = \{\frac{n}{n+1} : n \in \mathbb{N}\}$, (iv) $S = \{\frac{n-1}{n+1} : n \in \mathbb{N}\}$.
6. Let $S = \{\frac{1}{2^m} + \frac{1}{2^n} : m \in \mathbb{N}, n \in \mathbb{N}\}$.
   (i) Show that 0 is a limit point of S.
   (ii) If $k \in \mathbb{N}$, show that $\frac{1}{2^k}$ is a limit point of S.
   (iii) Find S'.(the derived set of S).
7. Give an example of a set $S \subset \mathbb{R}$ such that
   (i) S is neither open nor closed in $\mathbb{R}$,
   (ii) S is both open and closed in $\mathbb{R}$,
   (iii) S is a proper subset of the derived set S', (iv) $S = S'$.
8. Prove that the set S is an open set, where
   (i) $S = \{x \in \mathbb{R} : 2x^2 - 5x + 2 < 0\}$, (ii) $S = \{x \in \mathbb{R} : 2x^2 - 5x + 2 > 0\}$,
   (iii) $S = A - B$ where $A = (0, 1)$, $B = \{\frac{1}{2^n} : n \in \mathbb{N}\}$;
   (iv) $S = \{x \in \mathbb{R} : \sin x \neq 0\}$.
9. Examine if the set S is closed in $\mathbb{R}$.
   (i) $S = \{x \in \mathbb{R} : \sin x = 0\}$, (ii) $S = \{x \in \mathbb{R} : \sin \frac{1}{x} = 0\}$,
   (iii) $S = \{\frac{1}{m} + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\}$, (iv) $S = \{(-1)^m + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\}$,
   (v) $S = \bigcup_{n=1}^{\infty} I_n$ where $I_n = \{x \in \mathbb{R} : (\frac{1}{3})^n \leq x \leq 1\}$.

10. Let $G \subset \mathbb{R}$ be an open set and $F \subset \mathbb{R}$ be a closed set. Prove that $G - F$ is an open set and $F - G$ is a closed set.

11. Let G be an open set in $\mathbb{R}$ and S be a non-empty finite subset of G. Prove that $G - S$ is an open set.

12. Let G be an open set in $\mathbb{R}$ and S be a subset of $\mathbb{R}$ such that $G \cap S = \phi$. Prove that $G \cap S' = \phi$.

[**Hint.** $G \cap S = \phi \Rightarrow S \subset G^c \Rightarrow S' \subset (G^c)' \Rightarrow S' \subset G^c$(since G^c is closed in $\mathbb{R}$)$\Rightarrow G \cap S' = \phi$.]

13. Let S be a bounded subset of $\mathbb{R}$ and $\sup S = b, \inf S = a$ and $a \neq b$. Prove that $[a, b]$ is the smallest closed interval containing the set S.

14. (i) Let S be a non-empty subset of $\mathbb{R}$ bounded above and $s^* = \sup S$. If $s^* \notin S$, prove that $s^* \in S'$ and $s^* = \sup S'$, S' being the derived set of S.

(ii) Let S be a non-empty subset of $\mathbb{R}$ bounded below and $s_* = \inf S$. If $s_* \notin S$, prove that $s_* \in S'$ and $s_* = \inf S'$, S' being the derived set of S.

15. If S be a non-empty bounded subset of $\mathbb{R}$ prove that $\sup S \in \bar{S}$ and $\inf S \in \bar{S}$.

16. Let $S \subset \mathbb{R}$. Prove that

(i) $(\bar{S})^c = (S^c)^o$, i.e., the complement of the closure of S is the interior of the complement of S;

(ii) $(S^o)^c = (\bar{S^c})$, i.e., the complement of the interior of S is the closure of the complement of S.

17. A set $S \subset \mathbb{R}$ is said to be a *discrete* set if $S' = \phi$ (i.e., if S has no limit point).

A set $S \subset \mathbb{R}$ is said to be an *isolated* set if $S \cap S' = \phi$ (i.e., if each point of S is an isolated point).

(i) Prove that every discrete set is an isolated set, but not conversely.

(ii) Give an example of an infinite discrete set $S \subset \mathbb{R}$.

(iii) Give an example of a bounded discrete set $S \subset \mathbb{R}$.

(iv) Can there be an infinite bounded discrete set $S \subset \mathbb{R}$?

18. Let $S \subset \mathbb{R}$. A point $x \in \mathbb{R}$ is said to be a *boundary point* of S if every neighbourhood $N(x)$ of x contains a point of S and also a point of $\mathbb{R} - S$.

If a boundary point of S is not a point of S prove that it is a limit point of S. Prove that a set $S \subset \mathbb{R}$ is closed if and only if S contains all it boundary points.

3.11. Nested intervals.

If $\{I_n : n \in \mathbb{N}\}$ be a family of intervals such that $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$, then the family $\{I_n\}$ is said to be a family of *nested intervals*.

Examples.

1. Let $I_n = \{x \in \mathbb{R} : 0 < x < \frac{1}{n}\}$.
Then $I_1 = (0, 1)$, $I_2 = (0, \frac{1}{2})$, $I_3 = (0, \frac{1}{3}), \cdots \cdots$
$I_1 \supset I_2 \supset I_3 \supset \cdots \cdots$
$\{I_n : n \in \mathbb{N}\}$ is a family of nested open and bounded intervals.

2. Let $I_n = \{x \in \mathbb{R} : x > n\}$.
Then $I_1 \supset I_2 \supset I_3 \supset \cdots \cdots$
$\{I_n : n \in \mathbb{N}\}$ is a family of nested open infinite intervals.

3. Let $I_n = \{x \in \mathbb{R} : -\frac{1}{n} \leq x \leq \frac{1}{n}\}$.
Then $I_1 \supset I_2 \supset I_3 \supset \cdots \cdots$
$\{I_n : n \in \mathbb{N}\}$ is a family of nested closed and bounded intervals.

4. Let $I_n = \{x \in \mathbb{R} : x \leq \frac{1}{n}\}$.
Then $I_1 \supset I_2 \supset I_3 \supset \cdots \cdots$
$\{I_n : n \in \mathbb{N}\}$ is a family of nested closed infinite intervals.

Theorem 3.11.1. (Theorem on nested intervals)

If $\{[a_n, b_n] : n \in \mathbb{N}\}$ be a family of nested closed and bounded intervals then $\bigcap_{n=1}^{\infty} [a_n, b_n]$ is non-empty.

Furthermore, if $\inf\{(b_n - a_n) : n \in \mathbb{N}\} = 0$, then there is *one and only one point* x such that $x \in \bigcap_{n=1}^{\infty} [a_n, b_n]$.

Proof. $[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots \cdots$

Then $a_1 \leq a_2 \leq a_3 \cdots \leq a_n \leq \cdots \leq b_n \leq \cdots \leq b_3 \leq b_2 \leq b_1$.

The set $A = \{a_i : i \in \mathbb{N}\}$ is a non-empty subset of $\mathbb{R}$ bounded above, b_1 being an upper bound. By the supremum property of $\mathbb{R}$, $\sup A$ exists. Let $\sup A = x$. Then $a_n \leq x$ for all $n \in \mathbb{N}$.

We now establish that $b_n \geq x$ for all $n \in \mathbb{N}$.

If not, let $b_m < x$ for some $m \in \mathbb{N}$.

Since x is the lub of the set $\{a_1, a_2, a_3, \ldots\}$ and $b_m < x$, there is an element a_k such that $b_m < a_k < x$.

Let $q = \max\{m, k\}$. Then $b_q \leq b_m$ and $a_k \leq a_q$.

Consequently, $b_q \leq b_m < a_k \leq a_q$.

This shows that $b_q < a_q$, a contradiction, since $[a_q, b_q]$ is an interval of the family.

Hence $b_n \geq x$ for all $n \in \mathbb{N}$ and therefore $a_n \leq x \leq b_n$ for all $n \in \mathbb{N}$. That is, $x \in \bigcap_{n=1}^{\infty} [a_n, b_n]$.

This proves that $\bigcap_{n=1}^{\infty} [a_n, b_n]$ is non-empty.

Second part. If possible, let $x' \in \bigcap\limits_{n=1}^{\infty}[a_n, b_n]$.

Then $a_n \leq x \leq b_n$, $a_n \leq x' \leq b_n$ for all $n \in \mathbb{N}$.

Therefore $a_n - b_n \leq x - x' \leq b_n - a_n$.

or, $0 \leq | x - x' | \leq b_n - a_n$ for all $n \in \mathbb{N}$.

Let $\epsilon > 0$. Since $\inf\{(b_n - a_n) : n \in \mathbb{N}\} = 0, b_n - a_n \geq 0$ for all $n \in \mathbb{N}$ and there exists an element $b_m - a_m$ of the set (corresponding to some natural number m) such that $0 \leq b_m - a_m < \epsilon$.

Therefore $0 \leq | x - x' | < \epsilon$. Since ϵ is arbitrary, $x = x'$.

This proves that x is unique and the proof is complete.

Note 1. The set $B = \{b_i : i \in \mathbb{N}\}$ is a non-empty subset of $\mathbb{R}$ bounded below. If $\inf B = y$, then $y \in \bigcap\limits_{n=1}^{\infty}[a_n, b_n]$.

Note 2. If $\{I_n : n \in \mathbb{N}\}$ be a family of nested open bounded intervals then $\bigcap\limits_{n=1}^{\infty} I_n$ may not be non-empty.

For example, if $I_n = (0, \frac{1}{n})$, then $\bigcap\limits_{n=1}^{\infty} I_n = \phi$.

Note 3. If $\{I_n : n \in \mathbb{N}\}$ be a family of nested closed unbounded intervals then $\bigcap\limits_{n=1}^{\infty} I_n$ may not be non-empty.

For example, if $I_n = [n, \infty)$, then $\bigcap\limits_{n=1}^{\infty} I_n = \phi$.

Utilising Nested intervals theorem we now give an alternative proof of Bolzano-Weierstrass theorem (Theorem 3.6.2).

Another proof of Bolzano-Weierstrass theorem.

Every bounded infinite subset of $\mathbb{R}$ has at least one limit point (in $\mathbb{R}$).

Proof. Let S be a bounded subset of $\mathbb{R}$ containing infinite number of elements. Since S is a non-empty bounded subset of $\mathbb{R}$, $\sup S$ and $\inf S$ exist. Let $a_1 = \inf S$, $b_1 = \sup S$.

Then $x \in S \Rightarrow a_1 \leq x \leq b_1$, i.e., $x \in [a_1, b_1]$. Thus S is contained in the closed and bounded interval $I_1 = [a_1, b_1]$.

Let $c_1 = \frac{a_1+b_1}{2}$. Then at least one of the closed intervals $[a_1, c_1], [c_1, b_1]$ must contain infinitely many elements of S. Because, otherwise, S would be a finite set. We take one such subinterval containing infinitely many elements of S and call it $I_2 = [a_2, b_2]$.

$I_2 \subset I_1$ and $| I_2 | = \frac{b_1-a_1}{2}$.

Let $c_2 = \frac{a_2+b_2}{2}$. Then at least one of the closed intervals $[a_2, c_2], [c_2, b_2]$ must contain infinitely many elements of S. We take one such subinterval containing infinitely many elements of S and call it $I_3 = [a_3, b_3]$.

$I_3 \subset I_2 \subset I_1$ and $| I_3 |= \frac{b_1 - a_1}{2^2}$.

Let $c_3 = \frac{a_3 + b_3}{2}$. Continuing in a similar manner we obtain a family of closed and bounded intervals $\{I_n\}$ such that

(i) $I_1 \supset I_2 \supset I_3 \supset \cdots \cdots$

(ii) $| I_n |= \frac{1}{2^{n-1}}(b_1 - a_1)$, for each $n \in \mathbb{N}$

(iii) I_n contains infinitely many elements of S, for each $n \in \mathbb{N}$.

So $\{I_n : n \in \mathbb{N}\}$ is a family of nested closed and bounded intervals and $\inf\{(b_n - a_n) : n \in \mathbb{N}\} = 0$.

By the nested intervals theorem, there exists precisely one point x such that $\{x\} = \bigcap\limits_{n=1}^{\infty} [a_n, b_n]$.

We now prove that x is a limit point of S.

Let $\epsilon > 0$. Since $\inf\{(b_n - a_n) : n \in \mathbb{N}\} = 0$, there exists a natural number m such that $0 \leq b_m - a_m < \epsilon$.

Since $x \in I_m$ and $b_m - a_m < \epsilon, I_m \subset N(x, \epsilon)$.

Since I_m contains infinitely many elements of $S, N(x, \epsilon)$ contains infinitely many elements of S and this happens for each $\epsilon > 0$.

Therefore x is a limit point of S.

Thus S has a limit point and the theorem is done.

Theorem 3.11.2. (Cantor's intersection theorem)

Let $F_1, F_2, F_3, \ldots \ldots$ be a countable collection of non-empty closed and bounded subsets of $\mathbb{R}$ such that $F_1 \supset F_2 \supset F_3 \supset \cdots \cdots$

Then the intersection $\bigcap\limits_{i=1}^{\infty} F_i$ is non-empty.

Proof. **Case 1.** Let the collection be a finite collection containing m non-empty closed and bounded subsets $F_1, F_2, \ldots, F_m$ such that $F_1 \supset F_2 \supset \cdots \supset F_m$.

Then obviously, $\bigcap\limits_{i=1}^{m} F_i = F_m$ and this is non-empty by hypothesis.

Case 2. Let the collection be countably infinite. Without loss of generality, we we assume that no two sets of the collection are equal sets, because if there exists some block (or blocks) of equal sets, the inetersection of the whole collection remains same if we replace a block of equal sets by any one set of the block, and re-index the distinct elements of the collection as $F_1, F_2, F_3, \ldots$

Then $F_k - F_{k+1}$ is non-empty for each $k \in \mathbb{N}$.

Since the collection contains infinitely many non-empty closed sets, each F_k contains infinite number of elements , because, if we suppose, on the contrary, that F_j is a finite set containing p elements for some $j \in \mathbb{N}$ then F_{j+p} must be ϕ, a contradiction to hypothesis.

Let us take a point x_1 in $F_1 - F_2, x_2$ in $F_2 - F_3, \ldots, x_k$ in $F_k - F_{k+1}, \ldots \ldots$

Then we obtain an infinite set of points $S = \{x_1, x_2, x_3, \ldots \ldots\}$. Clearly, $S \subset F_1$.

Since F_1 is bounded, S is an infinite bounded subset of $\mathbb{R}$ and by Bolzano-Weierstrass theorem S has a limit point $x \in \mathbb{R}$.

We prove that x is a limit point of each $F_k, k = 1, 2, 3, \ldots \ldots$

Let x be not a limit point of F_m for some $m \in \mathbb{N}$. Then there exists a positive ϵ such that the neighbourhood $N(x, \epsilon)$ of x contains at most a finite number of points of F_m.

That is, $N(x, \epsilon) \cap F_m$ is a finite set.

Since $\{x_m, x_{m+1}, x_{m+2}, \ldots \ldots\} \subset F_m$, $N(x, \epsilon) \cap \{x_m, x_{m+1}, x_{m+2}, \ldots\}$ is a finite set.

Consequently, $N(x, \epsilon) \cap S$ is a finite set.

This disallows x to be a limit point of S. So our assumption that x is not a limit point of F_m is wrong.

Thus x is a limit point of F_m for every $m \in \mathbb{N}$ and since F_m is a closed set, $x \in F_m$ for every $m \in \mathbb{N}$.

That is, $x \in \bigcap_{i=1}^{\infty} F_i$ and this proves that the intersection $\bigcap_{i=1}^{\infty} F_i$ is non-empty. This completes the proof.

Corollary. If I_n be the closed and bounded interval $[a_n, b_n]$ and $[a_1, b_1] \supset [a_2, b_2] \supset \cdots \cdots$ then the intersection $\bigcap_{n=1}^{\infty} [a_n, b_n]$ is non-empty.

This is "Nested intervals theorem".

3.12. Decimal representation of a real number.

Let $x \in [0, 1]$. If we divide $[0, 1]$ into 10 equal subintervals, then x lies in at least one of the subintervals $[\frac{a}{10}, \frac{a+1}{10}]$ where a is one of integers $0, 1, 2, \ldots, 9$.

If x be a point of division, then two values of a are possible. We choose one of then and call it a_1. Then

$$\frac{a_1}{10} \leq x \leq \frac{a_1+1}{10}, \text{ where } 0 \leq a_1 \leq 9.$$

The chosen interval $[\frac{a_1}{10}, \frac{a_1+1}{10}]$ is again divided into 10 equal subintervals. Then x lies in at least one of them and

$$\frac{a_1}{10} + \frac{a_2}{10^2} \leq x \leq \frac{a_1}{10} + \frac{a_2+1}{10^2} \text{ where } 0 \leq a_i \leq 9, i = 1, 2.$$

The process is continued and we obtain integers $a_1, a_2, a_3, \cdots$ with $0 \leq a_n \leq 9$ for all $n \in \mathbb{N}$ such that

$$\frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} \leq x \leq \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n+1}{10^n} \text{ for all } n \in \mathbb{N}.$$

We write $x = .a_1a_2a_3 \cdots \cdots$ and call it a *decimal representation* of x.

Conversely, we now show that every decimal of the form $.a_1a_2a_3 \cdots \cdots$ is the decimal representation of some real number in $[0, 1]$.

Let $I_1 = [0, 1], I_2 = [\frac{a_1}{10}, \frac{a_1+1}{10}], I_3 = [\frac{a_1}{10} + \frac{a_2}{10^2}, \frac{a_1}{10} + \frac{a_2+1}{10^2}], \cdots,$
$I_{n+1} = [\frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n}, \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n+1}{10^n}], \cdots$

We obtain a family of closed and bounded intervals $\{I_n\}$ satisfying the conditions (i) $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$ and

(ii) $| I_n | = \frac{1}{10^{n-1}}$.

By the theorem on nested intervals, there exists a unique real number x such that $x \in I_n$ for $n \in \mathbb{N}$. Then

$$\frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} \leq x \leq \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n+1}{10^n} \text{ for all } n \in \mathbb{N} \ldots \text{ (i)}$$

Therefore $x \in [0, 1]$ and the inequality (i) shows that $0.a_1a_2a_3 \cdots$ is the decimal representation of the real number x.

The decimal representation of $x \in (0, 1)$ is unique except when x is a point of subdivision at some stage.

Let n be the least positive integer for which x is a point of subdivision at the nth stage, i.e., x is an end point of the subinterval I_{n+1}.

Then $x = \frac{m}{10^n}$ for some positive integer $m < 10^n$ not divisible by 10.

If we choose a_n such that x is the left end point of the subinterval I_{n+1}, then $\frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} = x$ and $a_k = 0$ for all $k \geq n + 1$.

In this case $x = .a_1a_2 \cdots a_n000 \cdots$

If we choose a_n such that x is the right end point of the subinterval I_{n+1}, then $x = \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n}$ and $a_k = 9$ for all $k \geq n + 1$.

In this case $x = .a_1a_2 \cdots a_n999 \cdots$

[For example, if $x = \frac{437}{1000}, x = .437000 \cdots$, or $x = .436999 \cdots$]

The decimal representation of 0 is $0.000 \cdots$

The decimal representation of 1 is $0.999 \cdots$

If $x \geq 1$, then there exists a natural number p such that $p \leq x < p+1$.

Therefore $x - p \in [0, 1]$. Then $x = p.a_1a_2a_3 \cdots$ where $.a_1a_2a_3 \cdots$ is the decimal representation of $x - p$.

A decimal $p.a_1a_2a_3 \cdots$ is said to be a **periodic decimal** (or a **recurring decimal**) if there exist natural numbers m and k such that $a_n = a_{n+m}$ for all $n \geq k$. The smallest natural number m with this property is called the **period** of the decimal. In this case the block of digits $a_k a_{k+1} \cdots a_{k+m-1}$ is repeated once the kth digit is reached.

[For example, the decimal $0.235636363\cdots$ is periodic with repeating block 63.]

A decimal $p.a_1a_2a_3\cdots$ is said to be a *terminating decimal* if there exists a natural number k such that $a_n = 0$ for all $n \geq k$. A terminating decimal can be regarded as a periodic decimal with the repeating block 0.

Theorem 3.12.1. A positive real number is rational if and only if its decimal representation is periodic.

Proof. Let x be a positive rational number. Let $x = p/q$ where p, q are natural numbers relatively prime. In the process of long division of p by q, the quotient gives the decimal representation of p/q. Each step in the division process gives a remainder which is an integer r satisfying $0 \leq r \leq q-1$. Therefore after at most q steps, some remainder will occur a second time or the remainder will be zero.

If some remainder recurs then the digits in the quotient will begin to repeat in blocks and we obtain a periodic decimal representation.

If the remainder be zero at some step, then the quotient gives a terminating decimal representation.

Conversely, let $x = p.a_1a_2a_3\cdots$ be a periodic decimal with period m and the period starts from the kth stage.
Then $x = p.a_1a_2\cdots a_n\cdots$ where $a_n = a_{n+m}$ for all $n \geq k$.
$10^{k-1}x = pa_1a_2\cdots a_{k-1}.a_ka_{k+1}\cdots$ and $10^{k+m-1}x =$

$pa_1a_2\cdots a_{k+m-1}.a_{k+m}a_{k+m+1}\cdots = pa_1a_2\cdots a_{k+m-1}.a_ka_{k+1}\cdots$

We have $(10^{k+m-1} - 10^{k-1})x = pa_1a_2\cdots a_{k+m-1} - pa_1a_2\cdots a_{k-1}$.

As $pa_1a_2\cdots a_{k+m-1} - pa_1a_2\cdots a_{k-1}$ is an integer and $10^{k+m-1} - 10^{k-1}$ is an integer, x is a rational number, and the theorem is done.

Corollary. A non-terminating non recurring decimal represents a positive irrational number.

3.13. Enumerable set.

Let S be a subset of $\mathbb{R}$. S is said to be *enumerable* (or *denumerable*) if there exists a bijective mapping $f : \mathbb{N} \to S$, i.e., if S and $\mathbb{N}$ are equipotent sets.

A set which is either finite or enumerable is said to be a *countable* (or, an *at most enumerable*) set.

An enumerable set is also called a *countably infinite* set.

If a set S is finite and contains n elements, its elements can be described as $a_1, a_2, \ldots, a_n$, the elements being indexed by the finite set $\{1, 2, \ldots, n\}$.

If S is enumerable, there exists a bijective mapping $f : \mathbb{N} \to S$ and f assigns to each element $n \in \mathbb{N}$ an element $f(n)$ in S. The elements of S can be described as $f(1), f(2), \ldots, f(n), \ldots$, or as $a_1, a_2, \ldots, a_n, \ldots$ showing that the elements are indexed by the set $\mathbb{N}$.

Note. Since an enumerable set is equipotent with the set $\mathbb{N}$, the cardinal number of an enumerable set is d.

Examples.

1. The set $\mathbb{N}$ is enumerable, because the mapping $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n, n \in \mathbb{N}$ is a bijection.

2. The set $S = \{2, 4, 6, \ldots\}$ is enumerable, because the mapping $f : \mathbb{N} \to S$ defined by $f(n) = 2n, n \in \mathbb{N}$ is a bijection.

3. The set $S = \{1^2, 2^2, 3^2, \ldots\}$ is enumerable because the mapping $f : \mathbb{N} \to S$ defined by $f(n) = n^2, n \in \mathbb{N}$ is a bijection .

4. The set $\mathbb{Z}$ is enumerable, because the mapping $f : \mathbb{N} \to \mathbb{Z}$ by

$$\begin{aligned} f(n) &= \tfrac{1}{2}n, \text{ if } n \text{ be even} \\ &= \tfrac{1}{2}(1-n), \text{ if } n \text{ be odd,} \end{aligned}$$

is a bijection.

Theorem 3.13.1. An infinite subset of an enumerable set is enumerable.

Proof. Let S be an enumerable set and T be an infinite subset of S. Since S is an enumerable set, its element can be described as $a_1, a_2, a_3, \ldots \ldots$

Since T is an infinite subset of S, T contains infinite number of a's and the suffixes of the elements of T form an infinite subset P of $\mathbb{N}$. By the well ordering property of $\mathbb{N}, P$ contains a least element, say μ_1.

$a_{\mu_1} \in T$. Let $T_1 = T - \{a_{\mu_1}\}$. Then T_1 is an infinite subset of S and the suffixes of the elements of T_1 form an infinite subset P_1 of $\mathbb{N}$. Therefore P_1 contains a least element say μ_2. $a_{\mu_2} \in T_1$. Let $T_2 = T - \{a_{\mu_1}, a_{\mu_2}\}$. Then T_2 is an infinite subset of S. Proceeding with similar arguments, we obtain the elements $a_{\mu_3}, a_{\mu_4}, \ldots$

Let us define a mapping $f : \mathbb{N} \to T$ by $f(n) = a_{\mu_n}, n \in \mathbb{N}$. To prove that f is injective, let $p, q \in \mathbb{N}$ and $p < q$.

Then $f(p) \in \{a_{\mu_1}, a_{\mu_2}, \ldots, a_{\mu_{q-1}}\}$ and $f(q) \in T - \{a_{\mu_1}, a_{\mu_2}, \ldots, a_{\mu_{q-1}}\}$.

Therefore $f(p) \neq f(q)$ and this proves that f is injective.

To prove that f is surjective, let $a_r \in T$ for some natural number r.

There are at most $r-1$ elements in T whose suffixes are less than r. So a_r is one of $f(1), f(2), \ldots, f(r)$. Therefore f is surjecive.

Therefore f is a bijection and T is enumerable.

Corollary. A subset of an enumerable set is either finite or enumerable.

Theorem 3.13.2. The union of a finite set and an enumerable set is enumerable.

Proof. Let S be an enumerable set with elements $a_1, a_2, a_3, \ldots \ldots$ and T be a finite set with elements $b_1, b_2, \ldots, b_m$.

Case 1. $S \cap T = \phi$.

Let us define a mapping $f : \mathbb{N} \to S \cup T$ by

$$f(i) = b_i, \quad i = 1, 2, \ldots, m$$
$$f(m+i) = a_i, \quad i = 1, 2, 3, \ldots$$

Then f is a bijective mapping. This proves that $S \cup T$ is enumerable.

Case 2. $S \cap T \neq \phi$.

Let $S_1 = S - T$. Then $S_1 \cup T = S \cup T$ and $S_1 \cap T = \phi$.

S_1 is an infinite subset of the enumerable set S and therefore S_1 is enumerable. By case 1, $S_1 \cup T$ is enumerable.

That is, $S \cup T$ is enumerable and the theorem is done.

Theorem 3.13.3. The union of two enumerable sets is enumerable.

Proof. Let S_1, S_2 be two enumerable sets and let
$S_1 = \{a_1, a_2, a_3, \ldots \ldots\}, S_2 = \{b_1, b_2, b_3, \ldots \ldots\}$.

Case 1. $S_1 \cap S_2 = \phi$.

Let us define a mapping $f : \mathbb{N} \to S_1 \cup S_2$ by

$$f(n) = a_{(n+1)/2}, \quad \text{if } n \text{ be odd}$$
$$= b_{n/2}, \quad \text{if } n \text{ be even.}$$

Then f is a bijection and therefore $S_1 \cup S_2$ is enumerable.

Case 2. $S_1 \cap S_2 \neq \phi$

Let $A_1 = S_1, A_2 = S_2 - S_1$. Then $A_1 \cup A_2 = S_1 \cup S_2$ and $A_1 \cap A_2 = \phi$.

A_2 is a subset of S_2. So A_2 is either finite or enumerable. If A_2 is finite then $A_1 \cup A_2$ is the union of an enumerable set and a finite set and therefore it is enumerable.

If A_2 is enumerable, then $A_1 \cup A_2$ is enumerable by Case 1.

Therefore $S_1 \cup S_2$ is enumerable and the proof is complete.

Theorem 3.13.4. The union of an enumerable number of enumerable sets is enumerable.

Proof. Let $S_1, S_2, S_3, \ldots$ be an enumerable family of enumerable sets.

Let $S_1 = \{a_{11}, a_{12}, a_{13}, \ldots, a_{1n}, \ldots\}$
$S_2 = \{a_{21}, a_{22}, a_{23}, \ldots, a_{2n}, \ldots\}$
$\ldots \quad \ldots \quad \ldots$
$S_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots, a_{nn}, \ldots\}$
$\ldots \quad \ldots \quad \ldots$

Case 1. Let $S_i \cap S_j = \phi$ for all i, j.

Let $A = \bigcup\limits_{i=1}^{\infty} S_i$. Each element of A is of the form a_{mn}, where $m, n \in \mathbb{N}$.

Let us define a mapping $f : A \to \mathbb{N}$ by $f(a_{mn}) = 2^m.3^n$.

f is injective because for two distinct elements $a_{mn}, a_{pq} \in A, (m, n) \neq (p, q) \Rightarrow 2^m 3^n \neq 2^p 3^q$.

$f(A)$ is a proper subset of $\mathbb{N}$, because there are elements in $\mathbb{N}$ (for example 5, 7, 11) which have no pre-image in A.

Let $f(A) = N_1$. Then $f : A \to N_1$ is a bijection.

Since N_1 is an infinite subset of $\mathbb{N}$, N_1 is enumerable and since A is equipotent with N_1, A is enumerable.

Case 2. Let the sets $\{S_i\}$ be not pairwise disjoint.

Let us define sets A_i by

$A_1 = S_1, A_2 = S_2 - S_1, A_3 = S_3 - (S_1 \cup S_2), \cdots \cdots,$
$A_k = S_k - (S_1 \cup S_2 \cup \cdots \cup S_{k-1}), \cdots$

Then $A_k \subset S_k$ for all k, $\bigcup\limits_{i=1}^{\infty} A_i = \bigcup\limits_{i=1}^{\infty} S_i$ and $A_i \cap A_j = \phi$ for all i, j.

Since $A_k \subset S_k, A_k$ is either finite or enumerable

Therefore $\bigcup\limits_{i=1}^{\infty} A_i$ is enumerable.

This completes the proof.

Examples.

1. The set $\mathbb{Q}$ is enumerable.

Let P be the set of all positive rational numbers, P' be the set of all negative rational numbers. Then $\mathbb{Q} = P \cup P' \cup \{0\}$.

The sets P and P' are equipotent since the mapping $f : P \to P'$ defined by $f(x) = -x, x \in P$ is a bijection.

P can be described as the union $\bigcup\limits_{k=1}^{\infty} A_k$, where

$A_1 = \{\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \cdots, \frac{n}{1}, \cdots\}$
$A_2 = \{\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \cdots, \frac{n}{2}, \cdots\}$
$\ldots \quad \ldots$
$A_k = \{\frac{1}{k}, \frac{2}{k}, \frac{3}{k}, \cdots, \frac{n}{k}, \cdots\}$
$\ldots \quad \ldots$

Each A_i is enumerable. P being the union of an enumerable number of enumerable sets is enumerable.

Since P is equipotent with P', P' is also enumerable.

Therefore $P \cup P'$ is enumerable.

$\mathbb{Q}$ being the union of an enumerable set and the finite set $\{0\}$, $\mathbb{Q}$ is enumerable.

2. The set of all algebraic numbers is enumerable.

[A real number is called an **algebraic number** if it is a root of a polynomial of the form $a_o x^n + a_1 x^{n-1} + \cdots + a_n$ where $a_o, a_1, \ldots, a_n$ are all integers and $a_o \neq 0$.

For example, $\sqrt{2}$ is an algebraic number, since $\sqrt{2}$ is a root of the polynomial $x^2 - 2$. Every rational number $\frac{p}{q}$ is an algebraic number since it is a root of the polynomial $qx - p$.

Not every irrational number is algebraic. e is not an algebraic number, π is not an algebraic number.]

Let $f(x) = a_o x^n + a_1 x^{n-1} + \cdots + a_n$ be a polynomial with integral coefficients. Then every real root of $f(x)$ is an algebraic number.

Let us define the *height* h of the polynomial $f(x)$ by

$h = n + \mid a_o \mid + \mid a_1 \mid + \cdots + \mid a_n \mid$. Then h is a positive integer ≥ 1.

Corresponding to every positive integer h, there exists a finite number of polynomials with integral coefficients with height h. Since each polynomial of degree n has at most n real roots, the number of algebraic numbers corresponding to a positive integer h (as height of a polynomial) is finite.

Therefore for every positive integer h, there corresponds a finite number of algebraic numbers.

The set of all algebraic numbers is the union of an enumerable number of finite sets and therefore it is enumerable.

Theorem 3.13.5. The set $\mathbb{R}$ is not enumerable.

Proof. Let $I = [a, b]$ be a closed and bounded interval.

First we prove that I is non-enumerable.

Let I be enumerable. Then the elements of I can be expressed as $x_1, x_2, x_3, \ldots \ldots$

We divide I into three subintervals $[a, c], [c, d], [d, b]$ by the points $c = \frac{2a+b}{3}, d = \frac{a+2b}{3}$. At least one of these subintervals does not contain x_1. We choose it and call it $I_1 = [a, b]$.

I_1 does not contain x_1 and $\mid I_1 \mid = \frac{b-a}{3}$.

We divide I_1 into three subintervals $[a_1, c_1], [c_1, d_1], [d_1, b_1]$ by the points $c_1 = \frac{2a_1+b_1}{3}, d_1 = \frac{a_1+2b_1}{3}$. At least one of these subintervals does not contain x_2. We choose it and call it $I_2 = [a_2, b_2]$.

I_2 contains none of x_1, x_2; $I_2 \subset I_1 \subset I$ and $| I_2 |= \frac{b-a}{3^2}$.

We divide I_2 into three subintervals in a similar manner. Continuing this process we obtain a family of closed and bounded intervals $\{I_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$,

(i) I_n contains none of $x_1, x_2, \ldots, x_n$;

(ii) $I_{n+1} \subset I_n$, and

(iii) $| I_n |= \frac{(b-a)}{3^n}$ and consequently, $\inf \{| I_n | : n \in \mathbb{N}\} = 0$.

By the theorem on nested intervals (Theorem 3.11.1) there exists one and only one point α such that $\{\alpha\} = \bigcap_{n=1}^{\infty} I_n$. Therefore $\alpha \in [a, b]$.

But by the construction of I_n, $\bigcap_{n=1}^{\infty} I_n$ contains none of $x_1, x_2, x_3, \ldots \ldots$, i.e., $\alpha \notin \{x_1, x_2, x_3, \ldots\}$

Thus α being an element of $[a, b]$ escapes enumeration. So our assumption that $[a, b]$ is enumerable is not tenable and so $[a, b]$ is not enumerable.

Now $[a, b]$ is an infinite subset of $\mathbb{R}$. Therefore $\mathbb{R}$ is not enumerable because every infinite subset of an enumerable set is enumerable.

This completes the proof.

Another proof. Let $I = \{x \in \mathbb{R} : 0 < x < 1\}$.

First we prove that I is non-enumerable. If not, let I be enumerable. Then the elements of the set can be described as $x_1, x_2, x_3, \ldots \ldots$

We consider the decimal representation of each number in I. The integral part of each of them is 0.

Each irrational number in the set has a unique representation as an infinite decimal. Each rational number has either a finite decimal representation, or a recurring decimal representation.

A finite decimal can be expressed as an infinite decimal in two ways - either by using 9's or by using 0's. For example, .23 can be expressed as .22999 $\cdots$ or as .23000 $\cdots$

If we stick to the recurring decimal representation by using 9's only and ignore the second type (by using 0's) then every real number in I can have a unique non-terminating decimal representation.

$$\begin{array}{lcl} \text{Let } x_1 & = & 0.x_{11}x_{12}x_{13}\cdots\cdots \\ x_2 & = & 0.x_{21}x_{22}x_{23}\cdots\cdots \\ x_3 & = & 0.x_{31}x_{32}x_{33}\cdots\cdots \\ \cdots & & \cdots \quad \cdots \text{ where } 0 \le x_{ij} \le 9. \end{array}$$

By our assumption every real number in I must have a place in the enumeration $x_1, x_2, x_3, \ldots \ldots$

Let us consider the real number $a = 0.a_1a_2a_3 \cdots \cdots$

where $a_k = 1$ if $x_{kk} \neq 1$
$= 2$ if $x_{kk} = 1$.

Now a is different from each x_i because a differs from x_i in the ith decimal place.

Therefore a does not belong to $\{x_1, x_2, x_3, \ldots \ldots\}$. But $0 < a < 1$.

Thus a belongs to I but does not belong to $\{x_1, x_2, x_3, \ldots \ldots\}$, a contradiction. Therefore I is not enumerable.

Now I is an infinite subset of $\mathbb{R}$. Therefore $\mathbb{R}$ must be non-enumerable, because every infinite subset of an enumerable set is enumerable.

This completes the proof.

Corollary. The set S of all irrational numbers is non-enumerable.

Note. The set $\mathbb{R}$ has a cardinal number different from that of the set $\mathbb{N}$. The cardinal number of the set $\mathbb{R}$ is denoted by c. It is also called the *power* (or, the *potency*) of the continuum.

Worked Examples.

1. Prove that the set of all open intervals having rational end points is enumerable.

Let the set of all rational numbers be enumerated as

$$\{x_1, x_2, x_3, \ldots \ldots\}.$$

The set of all open intervals having x_1 as the left end point is the set of open intervals of the form (x_1, x_r) such that $x_r > x_1$.

The set $A_1 = \{x_r \in \mathbb{Q} : x_r > x_1\}$ is a proper subset of $\mathbb{Q}$.

Since $\mathbb{Q}$ is enumerable, the set A_1 is at most enumerable. But A_1 is clearly an infinite set so that A_1 is enumerable.

Thus the set of all open intervals having x_1 as the left end point is an enumerable set, say I_1.

The set of all open intervals in question is the set $I_1 \cup I_2 \cup I_3 \cup \cdots$

This being the union of an enumerable collection of enumerable sets, is enumerable.

Thus the set of all open intervals having rational end points is enumerable.

2. Let S be a subset of $\mathbb{R}$ such that no point of S is a cluster point of S. Prove that S is a countable set.

Let $x \in S$. Since x is not a limit point of S, there exists an open interval $I_x = (a_x, b_x)$ containing x such that I_x contains a finite number of points of S.

Let us choose rational numbers r_x, s_x in I_x such that $a_x < r_x < x < s_x < b_x$. Then $J_x = (r_x, s_x)$ is an open interval containing x and having rational end points. Also $J_x \cap S$ being a subset of $I_x \cap S$ contains a finite number of points of S.

The set of *all* open intervals having rational end points being an enumerable set, we can enumerate them as $J_1, J_2, J_3, \cdots \cdots$

Each point of S is contained in some $J_k (k \in \mathbb{N})$ and $J_k \cap S$ is a finite set. Also $S \subset \bigcup_{k=1}^{\infty} (J_k \cap S)$.

Thus S is contained in the countable union of finite sets and therefore S is a countable set.

Exercises 4

1. Give an example of a family $\{I_n : n \in \mathbb{N}\}$ of non-empty closed intervals such that $I_1 \supset I_2 \supset I_3 \supset \cdots$ and $\bigcap_{n=1}^{\infty} I_n = \phi$.

2. Give an example of a family $\{I_n : n \in \mathbb{N}\}$ of bounded open intervals such that $I_1 \supset I_2 \supset I_3 \supset \cdots$ and $\bigcap_{n=1}^{\infty} I_n = \phi$.

3. Let $I_n = [a_n, b_n]$ and $I_1 \supset I_2 \supset I_3 \supset \cdots$
If $\alpha = \sup\{a_n : n \in \mathbb{N}\}, \beta = \inf\{b_n : n \in \mathbb{N}\}$ prove that
(i) $[\alpha, \beta] = \bigcap_{n=1}^{\infty} I_n$ if $\alpha \neq \beta$, (ii) $\{\alpha\} = \bigcap_{n=1}^{\infty} I_n$ if $\alpha = \beta$.

4. Let S be an enumerable subset and T be a non-enumerable infinite subset of $\mathbb{R}$. Prove that
(i) $S \cup T$ is non-enumerable, (ii) $S \cap T$ is at most enumerable,
(iii) $S - T$ is at most enumerable, (iv) $T - S$ is non- enumerable.

6. Prove that $\mathbb{N} \times \mathbb{N}$ is an enumerable set. Deduce that if S be an enumerable set, then $S \times S$ is enumerable.

7. Prove that the set of all circles in the plane having rational radii and centres with rational co-ordinates is enumerable.

8. Prove that the set of all transcendental numbers is a non-enumerable set.
[A real number is said to be *transcendental* if it is not algebraic.]

9. Show that each of the following sets has the cardinal number c.
(i) $A = \{x \in \mathbb{R} : 0 < x < \infty\}$, (ii) $B = \{x \in \mathbb{R} : 0 < x < 1\}$.

Hint. (i) The mapping $f : \mathbb{R} \to A$ defined by $f(x) = e^x, x \in \mathbb{R}$ is a bijection.
(ii) The mapping $f : \mathbb{R} \to B$ defined by $f(x) = \frac{e^x}{e^x+1}, x \in \mathbb{R}$ is a bijection.

3.14. Point of condensation.

Let S be a subset of $\mathbb{R}$. A point $x \in \mathbb{R}$ is said to be a *point of condensation* of S if every neighbourhood of x contains uncountably many points of S.

It follows that every point of condensation of a set S is a limit point of S, but not conversely.

It also follows from the definition that a countable set cannot have a point of condensation.

The set of all points of condensation of a set S is denoted by S_c.

Examples.

1. Let $S = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$. Then every point of S is a point of condensation of S. Here $S_c = S$.

2. Let $S = \{x \in \mathbb{R} : 1 < x < 3\}$. Then $S_c = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$. Here $S \subset S_c$.

3. Let $S = \mathbb{Q}$. The set S being an enumerable set, $S_c = \phi$.

Theorem 3.14.1. Every uncountable subset S of $\mathbb{R}$ has at least one point of condensation in S.

Proof. Suppose S has no point of condensation. Let $x \in S$. Then x is not a point of condensation of S and there exists an open interval $I_x = (a_x, b_x)$ containing x such that I_x contains a countable number of points of S.

Let us choose rational numbers r_x, s_x in I_x such that $a_x < r_x < x < s_x < b_x$. Then $J_x = (r_x, s_x)$ is an open interval containing x and having rational end points. Also $J_x \cap S$ being a subset of $I_x \cap S$ is a countable set.

The set of *all* open intervals having rational end points being an enumerable set, we can enumerate them as $J_1, J_2, J_3, \ldots \ldots$

Each point of S is contained in some $J_k (k \in \mathbb{N})$. We conclude that $S \subset \bigcup_{k=1}^{\infty} (J_k \cap S)$. Further, each $J_k \cap S$ is countable.

Thus S is a subset of a countable union of countable sets and therefore S is a countable set, a contradiction to the hypothesis. So our assumption is wrong and there is at least one point of condensation in S.

This completes the proof.

Corollary. If no point of S is a condensation point of S then S is a countable set.

Note. This theorem is analogous to Bolzano-Weierstrass theorem but stronger in the sense that the condition of boundedness is not required for the existence of a point of condensation. Moreover, Bolzano-Weierstrass theorem assures the existence of a limit point which may not belong to the set but this theorem assures that some point of condensation is actually contained in the set.

Theorem 3.14.2. For every set $S \subset \mathbb{R}$, $S - S_c$ is a countable set.

Proof. **Case 1.** Let $S - S_c = \phi$. Then $S - S_c$ is a countable set.

Case 2. Let $S - S_c \neq \phi$. Let $x \in S - S_c$. Then x is not a point of condensation of S. So there exists a neighbourhood $N(x)$ of x such that $N(x) \cap S$ is a countable set. Consequently, $N(x) \cap S$ contains only countably many points of $S - S_c$ and as such x cannot be a point of condensation of $S - S_c$.

Thus no point of $S - S_c$ is a point of condensation of $S - S_c$.

By Theorem 3.14.1, $S - S_c$ must be a countable set.

This completes the proof.

Corollary. As $S = (S - S_c) \cup (S \cap S_c)$ it follows from the theorem that if S be an uncountable set then $S \cap S_c$ is uncountable.

That is, if S be an uncountable set, S *contains* uncountably many points of condensation of S.

Theorem 3.14.3. For every set $S \subset \mathbb{R}$, S_c is a closed set.

Proof. **Case 1.** Let $S_c = \phi$. Then S_c is a closed set.

Case 2. Let $S_c \neq \phi$. Let $x \in \bar{S}_c$ and $N(x)$ be a neighbourhood of x. Then $N(x)$ contains at least a point, say y of S_c.

Since $y \in S_c$ and $N(x)$ can be considered as a neighbourhood of y, $N(x)$ contains uncountably many points of S and as such x happens to be a point of condensation of S.

Thus $x \in \bar{S}_c \Rightarrow x \in S_c$ and therefore $\bar{S}_c \subset S_c$.

But by definition, $S_c \subset \bar{S}_c$ and therefore $\bar{S}_c = S_c$. This proves that S_c is a closed set.

3.15. Borel set.

We have seen that the union of an infinite collection of closed sets in $\mathbb{R}$ may not be a closed set in $\mathbb{R}$; the intersection of an infinite collection of open set may not be an open set in $\mathbb{R}$.

If however, the infinite collection be an enumerable collection, then we have a special type of sets.

Definition. The union of an enumerable collection of closed sets in $\mathbb{R}$ is said to be an F_σ **set.**

The intersection of an enumerable collection of open sets in $\mathbb{R}$ is said to be a G_δ **set.**

Examples.

1. $\mathbb{Q}$ is an F_σ set.

Since the set of all rational numbers is enumerable, $\mathbb{Q}$ can be described as the set $\{x_1, x_2, x_3, \ldots \ldots\}$.

Therefore $\mathbb{Q}$ can be expressed as the union of an enumerable collection of closed sets $\{x_1\}, \{x_2\}, \{x_3\}, \cdots \cdots$

So $\mathbb{Q}$ is an F_σ set.

2. A closed and bounded interval $[a, b]$ is a G_δ set.

The interval $[a, b]$ can be expressed as the intersection $\bigcap_{n=1}^{\infty}(a-\frac{1}{n}, b+\frac{1}{n})$.

For each $n \in \mathbb{N}$, $(a - \frac{1}{n}, b + \frac{1}{n})$ is an open set.

Thus $[a, b]$ is the intersection of an enumerable collection of open sets in $\mathbb{R}$. So $[a, b]$ is a G_δ set.

3. An open bounded interval (a, b) is an F_σ set.

The interval (a, b) can be expressed as the union $\bigcup_{n=1}^{\infty}[a + \frac{1}{n}, b - \frac{1}{n}]$.

For each $n \in \mathbb{N}$, $[a + \frac{1}{n}, b - \frac{1}{n}]$ is a closed set.

Thus (a, b) is the union of an enumerable collection of closed sets in $\mathbb{R}$. So (a, b) is an F_σ set.

Theorem 3.15.1. (i) The union of a countable collection of F_σ sets is an F_σ set.
(ii) The intersection of a countable collection of G_δ sets is a G_δ set.

Proof. (i) Let $\{F_1, F_2, F_3, \ldots\}$ be a countable collection of F_σ sets in $\mathbb{R}$. Let $F = \bigcup_{i=1}^{\infty} F_i$.

Since F_i is an F_σ set, $F_i = F_{i1} \cup F_{i2} \cup F_{i3} \cup \cdots \cdots$ where each F_{ij} is a closed set in $\mathbb{R}$.

Thus F is the union of a countable collection of enumerable number of closed sets.

It follows that F is the union of an enumerable collection of closed sets in $\mathbb{R}$ and therefore F is an F_σ set.

(ii) Let $\{G_1, G_2, G_3, \ldots \ldots\}$ be a countable collection of G_δ sets in $\mathbb{R}$. Let $G = \bigcup_{i=1}^{\infty} G_i$.

Since each G_i is a G_δ set, $G_i = G_{i1} \cap G_{i2} \cap G_{i3} \cap \cdots \cdots$ where each G_{ij} is an open set in $\mathbb{R}$.

Thus G is the intersection of a countable collection of enumerable number of open sets.

It follows that G is the intersection of an enumerable collection of open sets in $\mathbb{R}$ and therefore G is a G_δ set.

Note. Every open set in $\mathbb{R}$ can be expressed as the union of an enumerable number of closed sets in $\mathbb{R}$.

Every closed set in $\mathbb{R}$ can be expressed as the intersection of an enumerable number of open sets in $\mathbb{R}$.

Theorem 3.15.2. (i) The complement of an F_σ set in $\mathbb{R}$ is a G_δ set.

(ii) The complement of a G_δ set in $\mathbb{R}$ is an F_σ set.

Proof. (i) Let F be an F_σ set and $F = F_1 \cup F_2 \cup F_3 \cup \cdots \cdots$ where each F_i is a closed set in $\mathbb{R}$.

$$\begin{aligned}\mathbb{R} - F &= \mathbb{R} - (F_1 \cup F_2 \cup F_3 \cup \cdots) \\ &= (\mathbb{R} - F_1) \cap (\mathbb{R} - F_2) \cap (\mathbb{R} - F_3) \cap \cdots\end{aligned}$$

Each $\mathbb{R} - F_i$ is an open set in $\mathbb{R}$. Therefore $\mathbb{R} - F$ is the intersection of an enumerable collection of open sets.

So $\mathbb{R} - F$ is a G_δ set and the theorem is done.

(ii) similar proof.

Definition. A set that can be obtained as the union and intersection of an enumerable collection of closed sets and open sets in $\mathbb{R}$ is said to be a **Borel set**.

Examples.

1. A G_δ set is a Borel set. An F_σ set is a Borel set.

2. The union of a countable collection of G_δ sets is a Borel set. This is denoted by $G_{\delta\sigma}$.

3. The intersection of a countable collection of F_σ sets is a Borel set. This is denoted by $F_{\sigma\delta}$.

3.16. Cover, open cover.

Let S be a subset of $\mathbb{R}$. A collection C of sets $\{A_\alpha : \alpha \in \Lambda\}$, Λ being the index set, is said to be a *cover* (or a covering) of S if $S \subset \bigcup_{\alpha \in \Lambda} A_\alpha$.

If $\mathcal{G}$ be a collection of open sets $\{G_\alpha : \alpha \in \Lambda\}$, Λ being the index set, such that $S \subset \bigcup_{\alpha \in \Lambda} G_\alpha$ then $\mathcal{G}$ is said to be an *open cover* of S.

A family $\mathcal{G}$ of open intervals $\{I_\alpha : \alpha \in \Lambda\}, \Lambda$ being the index set, is said to be an open cover of S if $S \subset \bigcup_{\alpha \in \Lambda} I_\alpha$.

Worked Examples.

1. Let C be the collection of closed intervals $I_n = \{x \in \mathbb{R} : 0 \leq x \leq n\}$, where $n \in \mathbb{N}$. Show that C is a cover of the set $S = \{x \in \mathbb{R} : x \geq 0\}$.

Let $c \in S$. Then $c \geq 0$. There exists a natural number m such that $m - 1 \leq c < m$. [Archimedean property of $\mathbb{R}$, deduction 3].

$m - 1 \leq c < m \Rightarrow c \in I_m$. Therefore $c \in S \Rightarrow c \in I_m \Rightarrow c \in \bigcup_{n=1}^{\infty} I_n$ and therefore $S \subset \bigcup_{n=1}^{\infty} I_n$.

Hence the collection C is a cover of the set S.

2. Let $\mathcal{G}$ be the collection of open intervals $I_n = \{x \in \mathbb{R} : -n < x < n\}$, where $n \in \mathbb{N}$. Show that the collection $\mathcal{G}$ is an open cover of the set $\mathbb{Q}$.

Let $x \in \mathbb{Q}$. Then $|x| \geq 0$. There exists a natural number k such that $|x| < k$, i.e., $x \in I_k$ and therefore $x \in \bigcup_{n=1}^{\infty} I_n$.

$x \in \mathbb{Q} \Rightarrow x \in \bigcup_{n=1}^{\infty} I_n$ and therefore $\mathbb{Q} \subset \bigcup_{n=1}^{\infty} I_n$.

This proves that the collection $\mathcal{G}$ is an open cover of the set $\mathbb{Q}$.

3. Let $\mathcal{G}$ be the family of open intervals $I_n = \{x \in \mathbb{R} : \frac{1}{2^n} < x < 2\}$, $n = 1, 2, 3, \ldots$. Show that $\mathcal{G}$ is an open cover of the set $S = \{x \in \mathbb{R} : 0 < x < 1\}$.

Let $c \in S$. Then $0 < c < 1$. By Archimedean property of $\mathbb{R}$, there exists a natural number k such that $0 < \frac{1}{k} < c$.

Since the sequence $\{2^n\}$ is a strictly increasing sequence of natural numbers, there exists a natural number m such that $2^m > k$. Then $0 < \frac{1}{2^m} < \frac{1}{k} < c$.

$0 < c < 1 \Rightarrow 0 < \frac{1}{2^m} < \frac{1}{k} < c < 2$.

$c \in S \Rightarrow 0 < c < 1 \Rightarrow \frac{1}{2^m} < c < 2 \Rightarrow c \in I_m$.

$c \in S \Rightarrow c \in \bigcup_{n=1}^{\infty} I_n$ and therefore $S \subset \bigcup_{n=1}^{\infty} I_n$.

This proves that the collection $\mathcal{G}$ is an open cover of the set S.

4. Let $\mathcal{G}$ be the family of open intervals $\{I_n : n \in \mathbb{N}\}$, where $I_n = \{x \in \mathbb{R} : -\frac{1}{n} < x < n\}$. Show that the family $\mathcal{G}$ is an open cover of the set $S = \{x \in \mathbb{R} : x \geq 0\}$.

Let $c \in S$. Then $c \geq 0$. There exists a natural number m such that $m - 1 \leq c < m$. [Archimedean property of $\mathbb{R}$, deduction 3].

$m-1 \leq c < m \Rightarrow c \in I_m$. Therefore $c \in S \Rightarrow c \in I_m \Rightarrow c \in \bigcup_{n=1}^{\infty} I_n$ and therefore $S \subset \bigcup_{n=1}^{\infty} I_n$.

Hence the collection $\mathcal{G}$ is a cover of the set S.

Sub cover.

Let S be subset of $\mathbb{R}$. Let $\mathcal{C}$ be a collection of sets in $\mathbb{R}$ that covers S. If $\mathcal{C}'$ be a subcollection of $\mathcal{C}$ such that $\mathcal{C}'$ also covers S then $\mathcal{C}'$ is said to be a *sub cover* of the cover $\mathcal{C}$.

If the subcollection $\mathcal{C}'$ contains a finite number of sets of $\mathcal{C}$ and $\mathcal{C}'$ covers S then $\mathcal{C}'$ is said to be a *finite sub cover* of the cover $\mathcal{C}$.

Worked Examples (continued).

5. Show that there is a subcollection of the family $\mathcal{G}$ of Ex.2 that can cover the set $\mathbb{Q}$.

Let $\mathcal{G}' = \{I_{2n} : n \in \mathbb{N}\}$. Then $\mathcal{G}'$ is a sub collection of the family $\mathcal{G}$.

Let $x \in \mathbb{Q}$. Then $|x| \geq 0$. There exists a natural number k such that $|x| < 2k$, i.e., $x \in I_{2k}$ and therefore $x \in \bigcup_{n=1}^{\infty} I_{2n}$.

$x \in \mathbb{Q} \Rightarrow x \in \bigcup_{n=1}^{\infty} I_{2n}$ and therefore $\mathbb{Q} \subset \bigcup_{n=1}^{\infty} I_{2n}$.

This proves that the collection $\mathcal{G}'$ is an open cover of the set $\mathbb{Q}$. So $\mathcal{G}'$ is a subcover of $\mathcal{G}$.

6. Show that there is no finite subcollection of the family $\mathcal{G}$ of Ex.2 that can cover the set $\mathbb{Q}$.

Let us assume that there is a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers the set S.

Let $\mathcal{G}' = \{I_{r_1}, I_{r_2}, \ldots, I_{r_m}\}$. Then $r_1, r_2, \ldots, r_m$ are natural numbers and $\mathbb{Q} \subset I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m}$.

Let $p = \max\{r_1, r_2, \ldots, r_m\}$. Then p is a natural number and $I_p = \{x \in \mathbb{R} : -p < x < p\}$.

Now $I_{r_1} \subset I_p, I_{r_2} \subset I_p, \ldots, I_{r_m} \subset I_p$ and therefore $I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m} \subset I_P$.

consequently, $\mathbb{Q} \subset I_p$ ... ...(i)

But $p \in \mathbb{Q}$ and $p \notin I_p$. This contradicts (i).

Therefore $\mathcal{G}$ has no finite subcover.

7. Show that there is no finite subcollection of the family $\mathcal{G}$ of Ex.3 that can cover the interval $I = (0, 1)$.

Let us assume that there is a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers I.

Let $\mathcal{G}' = \{I_{r_1}, I_{r_2}, \ldots, I_{r_m}\}$. Then $r_1, r_2, \ldots, r_m$ are natural numbers and $I \subset I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m}$.

Let $p = \max\{r_1, r_2, \ldots, r_m\}$. Then p is a natural number and $I_p = \{x \in \mathbb{R} : -p < x < p\}$.

Now $I_{r_1} \subset I_p, I_{r_2} \subset I_p, \ldots, I_{r_m} \subset I_p$ and therefore $I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m} \subset I_P$.

consequently, $I \subset I_p$ ... ...(i)

But $\frac{1}{2^p} \in I$ and $\frac{1}{2^p} \notin I_p$. This contradicts (i).

Therefore there is no finite subcollection of $\mathcal{G}$ that can cover I.

8. Show that there is no finite subcollection of the family $\mathcal{G}$ of Ex.4 that can cover the set $S = x \in \mathbb{R} : x \geq 0$.

Let us assume that there is a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers the set S.

Let $\mathcal{G}' = \{I_{r_1}, I_{r_2}, \ldots, I_{r_m}\}$. Then $r_1, r_2, \ldots, r_m$ are natural numbers and $S \subset I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m}$.

Let $p = \max\{r_1, r_2, \ldots, r_m\}$ and $q = \min\{r_1, r_2, \ldots, r_m\}$.

Then $I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m} = \{x \in \mathbb{R} : -\frac{1}{q} < x < p\}$ and $S \subset \{x \in \mathbb{R} : -\frac{1}{q} < x < p\}$.

But $p \in S$ and $p \notin \{x \in \mathbb{R} : -\frac{1}{q} < x < p\}$. This shows that $\mathcal{G}'$ cannot cover the set S.

Therefore no finite subcollection of $\mathcal{G}$ can cover S.

Theorem 3.16.1. Heine-Borel theorem.

Let S be a closed and bounded subset of $\mathbb{R}$. Then every open cover of S has a finite sub cover.

Proof. Let $\mathcal{G}$ be a collection of open sets $\{G_\alpha : \alpha \in \Lambda$, Λ being the index set$\}$ in $\mathbb{R}$ such that $\mathcal{G}$ is an open cover of S.

Let us assume that $\mathcal{G}$ contains no finite sub cover. Therefore S is not contained in the union of a finite number of open sets in $\mathcal{G}$.

Since S is bounded there exist two real numbers a_1, b_1 such that $x \in S \Rightarrow a_1 \leq x \leq b_1$. Therefore $S \subset [a_1, b_1]$.

Let $I_1 = [a_1, b_1]$ and let $c_1 = \frac{a_1+b_1}{2}$. Let $I_1' = [a_1, c_1], I_1'' = [c_1, b_1]$. At least one of the two subsets $S \cap I_1'$ and $S \cap I_1''$ has the property that it must be non-empty and it is not contained in the union of a finite number of open sets in $\mathcal{G}$. For if both of the sets $S \cap I_1'$ and $S \cap I_1''$ be contained in the union of a finite number of open sets in $\mathcal{G}$ then S would be contained in the union of a finite number of open sets in $\mathcal{G}$, a contradiction to the assumption.

If $S \cap I_1'$ be not contained in the union of a finite number of open sets in $\mathcal{G}$, we call $I_2 = I_1'$. If not, we call $I_2 = I_1''$.

We now bisect I_2 into closed subintervals I_2' and I_2'' and at least one of the sets $S \cap I_2'$ and $S \cap I_2''$ has the property and that it is non-empty and it not contained in the union of a finite number of open sets in $\mathcal{G}$. If $S \cap I_2'$ is not contained in the union of a finite number of open sets in $\mathcal{G}$ we call $I_3 = I_2'$. If not, we call $I_3 = I_2''$.

Continuing this process of bisection, we obtain a family of closed and bounded intervals $\{I_n\}$ such that

(i) $I_n \supset I_{n+1}$ for all $n \in \mathbb{N}$;

(ii) for all $n \in \mathbb{N}, S \cap I_n$ is non-empty and is not contained in the union of a finite number of open sets in $\mathcal{G}$;

(iii) $| I_n |= \frac{b_1-a_1}{2^{n-1}}$ and therefore $\lim | I_n |= 0$.

By the nested intervals theorem there exists one and only one point α such that $\alpha \in \bigcap_{n=1}^{\infty} I_n$.

Let us choose $\delta > 0$. There exists a natural number k such that $0 < \frac{b_1-a_1}{2^{k-1}} < \delta$.

As $| I_k |= \frac{b_1-a_1}{2^{k-1}}$, we have $| I_k |< \delta$.

Since $\alpha \in I_k$ and $| I_k |< \delta, I_k \subset (\alpha - \delta, \alpha + \delta)$. Since $I_k \cap S$ is not contained in the union of a finite number of open sets in $\mathcal{G}, I_k$ contains infinite number of elements of S. Consequently, α is a limit point of S. Since S is a closed set, $\alpha \in S$.

Since $\mathcal{G}$ covers S and $\alpha \in S$, there exists an open set $G_\lambda \in \mathcal{G}$ such that $\alpha \in G_\lambda$. Since G_λ is an open set, the neighbourhood $(\alpha - \epsilon, \alpha + \epsilon) \subset G_\lambda$ for some $\epsilon > 0$.

As $0 < \epsilon$, there exists a natural number m such that $0 < \frac{b_1-a_1}{2^{m-1}} < \epsilon$.

As $| I_m |= \frac{b_1-a_1}{2^{m-1}}$, we have $| I_m |< \epsilon$.

Since $\alpha \in I_m$ and $| I_m |< \epsilon, I_m \subset (\alpha - \epsilon, \alpha + \epsilon)$. That is, $I_m \subset G_\lambda$.

This shows that $S \cap I_m$ is contained in the single open set G_λ of the collection $\mathcal{G}$ but this is contrary to our construction of $\{I_n\}$.

Therefore our assumption that S is not contained in the union of a finite number of open sets in $\mathcal{G}$ is not tenable.

Hence there exists a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ covers S. This completes the proof.

Note. In Heine-Borel theorem the hypothesis that the set S is a closed and bounded subset of $\mathbb{R}$ is crucial. The theorem does not hold if S be not closed, or if S be unbounded.

Definition. Let S be a subset of $\mathbb{R}$. S is said to be a **compact set** i every open cover $\mathcal{G}$ of S has a finite subcover. That is, if $\mathcal{G}$ be a family of open sets that covers S then there exists a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers S.

To be explicit, if $\{G_\alpha : \alpha \in \Lambda\}$ be an open cover of $S \subset \mathbb{R}$ then S will be compact if there exists a finite number of indices $\alpha_1, \alpha_2, \ldots, \alpha_m \in \Lambda$ such that $S \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_m}$.

Heine-Borel theorem states that a closed and bounded subset of $\mathbb{R}$ is compact.

Theorem 3.16.2. (Converse of Heine-Borel theorem)

A compact subset of $\mathbb{R}$ is closed and bounded in $\mathbb{R}$.

Proof. Let K be a compact subset of $\mathbb{R}$. We first prove that K is bounded.

Let $I_n = \{x \in \mathbb{R} : -n < x < n\}$, where $n \in \mathbb{N}$ and $\mathcal{G} = \{I_n : n \in \mathbb{N}\}$. Then $\mathcal{G}$ is a collection of open sets in $\mathbb{R}$.

Clearly, $\mathbb{R} \subset \bigcup\limits_{n=1}^{\infty} I_n$, and therefore $K \subset \bigcup\limits_{n=1}^{\infty} I_n$.

This shows that $\mathcal{G}$ is an open cover of K.

Since K is compact, there exists a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers K.

Let $\mathcal{G}' = \{I_{r_1}, I_{r_2}, \ldots, I_{r_m}\}$. Then $K \subset I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m}$.

Let $p = \max\{r_1, r_2, \ldots, r_m\}$.

Then $I_{r_1} \subset I_p, I_{r_2} \subset I_p, \cdots, I_{r_m} \subset I_p$ and $K \subset I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m} \subset I_p = (-p, p)$.

This shows that K is a bounded subset of $\mathbb{R}$.

We now prove that K is a closed set.

Let $y \in \mathbb{R} - K$. Let us consider the collection of open sets $\{I_n\}$ where

$$I_1 = \{x \in \mathbb{R} : |\, y - x\,| > 1\}$$
$$I_2 = \{x \in \mathbb{R} : |\, y - x\,| > \tfrac{1}{2}\}$$
$$I_3 = \{x \in \mathbb{R} : |\, y - x\,| > \tfrac{1}{3}\}$$
$$\ldots \quad \ldots$$

Clearly, $\bigcup\limits_{n=1}^{\infty} I_n = \mathbb{R} - \{y\}$ and $K \subset \bigcup\limits_{n=1}^{\infty} I_n$.

Let $\mathcal{G} = \{I_n : n \in \mathbb{N}\}$. Then $\mathcal{G}$ is an open cover of K. Since K is compact, there exists a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers K.

Let $\mathcal{G}' = \{I_{r_1}, I_{r_2}, \ldots, I_{r_m}\}$. Then $K \subset I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m}$.

Let $p = \max\{r_1, r_2, \ldots, r_m\}$. Then $I_p \supset I_{r_1}, I_p \supset I_{r_2}, \cdots, I_p \supset I_{r_m}$ and $K \subset I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m} \subset I_p$.

But $I_p = \{x \in \mathbb{R} : |\, y - x\,| > \frac{1}{p}\}$. Let $G = \{x \in \mathbb{R} : |\, y - x\,| < \frac{1}{p}\}$.

Then G is a neighbourhood of y and $G \cap K = \phi$, since $K \subset I_p$.

It follows that y is not a limit point of K.

Thus $y \in \mathbb{R} - K \Rightarrow y \notin K'$ (the derived set of K).
Contrapositively, $y \in K' \Rightarrow y \notin \mathbb{R} - K$, i.e., $y \in K$.
Therefore $K' \subset K$ and this proves that K is a closed set.
This completes the proof.

Theorem 3.16.3. If K be a compact set in $\mathbb{R}$, every infinite subset of K has a limit point in K.

Proof. Let T be an infinite subset of K. Let us suppose that T has no limit point in K.

Let $x \in K$. Then x is not a limit point of T.

Therefore there exists a positive δ_x such that $N'(x, \delta_x) \cap T = \phi$, where $N'(x, \delta_x) = N(x, \delta_x) - \{x\}$.

Let us consider the family $\mathcal{G}$ of neighbourhoods $\{N(x, \delta_x) : x \in K \;\&\; N'(x, \delta_x) \cap T = \phi\}$. Clearly $\mathcal{G}$ is an open cover of K. Since K is compact, there is a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers K.

Let $\mathcal{G}' = \{N(x_1, \delta_{x_1}), N(x_2, \delta_{x_2}), \ldots, N(x_m, \delta_{x_m})\}$.
Then $K \subset N(x_1, \delta_{x_1}) \cup N(x_2, \delta_{x_2}) \cup \cdots \cup N(x_m, \delta_{x_m})$.

As $T \subset K, T \subset N(x_1, \delta_{x_1}) \cup N(x_2, \delta_{x_2}) \cup \cdots \cup N(x_m, \delta_{x_m})$
$= N'(x_1, \delta_{x_1}) \cup N'(x_2, \delta_{x_2}) \cup \cdots \cup N'(x_m, \delta_{x_m}) \cup \{x_1, x_2, \ldots, x_m\}$.

It follows that $T \subset \{x_1, x_2, \ldots, x_m\}$, since $N'(x_i, \delta_{x_i}) \cap T = \phi$, for $i = 1, 2, \ldots, m$.

This shows that T is a finite set, a contradiction.
Thus T has limit point in K and the proof is complete.

Corollary. The set $\mathbb{R}$ is not compact, since the set $\mathbb{Z}$ is an infinite subset of $\mathbb{R}$ having no limit point in $\mathbb{R}$.

Theorem 3.16.4. If K be a subset of $\mathbb{R}$ such that every infinite subset of K has a limit point in K then K is compact.

Proof. First we prove that K is closed.

Let p be a limit point of K. Then for each positive ϵ, $N'(p, \epsilon)$ contains infinitely many elements of K.

Let $\epsilon = 1$. Then $N'(p, 1)$ contains infinitely many elements of K.
Let us choose one such element and call it x_1.

Let $\epsilon = \frac{1}{2}$. Then $N'(p, \frac{1}{2})$ contains infinitely many elements of K.
Let $x_2 (\neq x_1)$ be one such. Then $x_2 \in N'(p, \frac{1}{2})$.
Let $\epsilon = \frac{1}{3}$.

Proceeding in a similar manner, we obtain an infinite subset $S = \{x_1, x_2, x_3, \ldots\}$ of K such that $x_i \in N'(p, \frac{1}{i})$. We now prove that p is the only limit point of S.

Let $N(p, \delta)$ be a neighbourhood of p. Since $0 < \delta$, there exists a natural number m such that $0 < \frac{1}{m} < \delta$.

$N(p, \delta) \supset N(p, \frac{1}{m})$ and $N(p, \frac{1}{m})$ contains each of $x_m, x_{m+1}, \ldots$

Thus $N(p, \delta)$ contains infinitely many elements of S, for each $\delta > 0$. So p is a limit point of S.

Let $q \neq p$. Let $\delta_0 = \frac{1}{2} \mid p - q \mid > 0$.

Then $N(p, \delta_0)$ and $N(q, \delta_0)$ are disjoint neighbourhoods.

There exists a natural number k such that $0 < \frac{1}{k} < \delta_0$.

$N(p, \delta_0) \supset N(p, \frac{1}{k})$ and as $N(p, \frac{1}{k})$ contains each of $x_k, x_{k+1}, \cdots$, $N(q, \delta_0)$ contains at most a finite number of elements of S and therefore q is not a limit point of S.

Hence p is the only limit point of S and by hypothesis, $p \in K$.

Thus $p \in K' \Rightarrow p \in K$. Therefore $K' \subset K$ and K is a closed set.

We prove that K is bounded.

Let us assume that K is unbounded above.

Let us choose an element $x_1 \in K$. Let us choose x_2 in K such that $x_2 > x_1 + 1$. Let us choose x_3 in K such that $x_3 > x_2 + 1, \cdots \cdots$

Thus we obtain an infinite set $\{x_1, x_2, x_3, \ldots \ldots\}$ in K.

But this set has no limit point by the construction of the elements. This contradicts the hypothesis that every infinite subset of K has a limit point in K.

Therefore K is not unbounded above.

Similarly, K is not unbounded below. Therefore K is bounded.

Since K is closed and bounded, K is compact and the proof is complete.

Let K be a subset of $\mathbb{R}$. Then the following statements are equivalent.

(i) K is closed and bounded.

(ii) Every open cover of K has a finite subcover.

(iii) Every infinite subset of K has a limit point in K.

Proof. (i) $\Rightarrow$ (ii) by Heine-Borel theorem. (ii) $\Rightarrow$ (iii) by theorem 3.16.3. (iii) $\Rightarrow$ (i) by theorem 3.16.4.

Therefore the three statements are such that one of them implies the other two. Hence they are equivalent.

Worked Examples (continued).

9. Using the definition of a compact set, prove that a finite subset of $\mathbb{R}$ is a compact set in $\mathbb{R}$.

Let $S = \{a_1, a_2, \ldots, a_m\}$ be a finite subset of $\mathbb{R}$. Let $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$, Λ being the index set, be a collection of open sets in $\mathbb{R}$ such that $\mathcal{G}$ covers S.

Since $a_1 \in S$, $a_1 \in G_{\alpha_1}$ of $\mathcal{G}$ for some $\alpha_1 \in \Lambda$.
Since $a_2 \in S$, $a_2 \in G_{\alpha_2}$ of $\mathcal{G}$ for some $\alpha_2 \in \Lambda$.

... ...

Since $a_m \in S$, $a_m \in G_{\alpha_m}$ of $\mathcal{G}$ for some $\alpha_m \in \Lambda$.

Therefore $S \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_m}$.

Let $\mathcal{G}' = \{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_m}\}$. Then $\mathcal{G}'$ is a finite subcollection of $\mathcal{G}$ and $\mathcal{G}'$ covers S.

Since $\mathcal{G}$ is an arbitrary open cover of S, it follows that every open cver of S has a finite subcover. Therefore S is a compact set in $\mathbb{R}$.

10. Let K be a compact subset of $\mathbb{R}$ and $F \subset K$ be a closed subset in $\mathbb{R}$. Prove that F is compact in $\mathbb{R}$.

Since F is closed, $\mathbb{R} - F$ is open.

Let $\mathcal{G}$ be a collection of open sets $\{G_\alpha : \alpha \in \Lambda\}$, Λ being the index set, such that $\mathcal{G}$ is an open cover of F. Let us suppose that $\mathcal{G}$ is not an open cover of K.

Let $\mathcal{G}'$ be the collection of open sets $\{G_\alpha : \alpha \in \Lambda\}$ together with $\mathbb{R} - F$. Clearly, $\mathbb{R} \subset (\underset{\alpha\in\Lambda}{\cup} G_\alpha) \cup (\mathbb{R} - F)$.

Therefore $K \subset \{\underset{\alpha\in\Lambda}{\cup} G_\alpha\} \cup (\mathbb{R} - F)$, i.e., $\mathcal{G}'$ is an open cover of K.

Since K is compact there exists a finite subcollection $\mathcal{G}''$ of $\mathcal{G}'$ such that $\mathcal{G}''$ also covers K.

Let $\mathcal{G}'' = \{G_{r_1}, G_{r_2}, \ldots, G_{r_m}, \mathbb{R} - F\}$, $r_i \in \Lambda$.

$\mathcal{G}''$ must contain $\mathbb{R} - F$ because K cannot be contained in the union $G_{r_1} \cup G_{r_2} \cup \cdots \cup G_{r_m}$.

Therefore $K \subset G_{r_1} \cup G_{r_2} \cup \cdots \cup G_{r_m} \cup (\mathbb{R} - F)$ and as $F \subset K, F \subset G_{r_1} \cup G_{r_2} \cup \cdots \cup G_{r_m}$.

Let $\mathcal{G}''' = \{G_{r_1}, G_{r_2}, \ldots, G_{r_m}\}$. Then $\mathcal{G}'''$ is a finite subcollection of $\mathcal{G}$ such that $\mathcal{G}'''$ also covers F. So F is compact.

11. Let K be a non-empty compact set in $\mathbb{R}$. Show that K has a least element.

Let us assume that K has no least element. For each $a \in K$, let $G_a = \{x \in \mathbb{R} : x > a\}$. Then G_a is an open interval.

Let us consider the family of open intervals $\mathcal{G} = \{G_a : a \in K\}$.

Let $b \in K$. Since K has no least element, there is an element c in K such that $c < b$ and therefore $b \in G_c$.

Thus $b \in K \Rightarrow b \in G_c$ for some $c \in K$. This shows that the family $\mathcal{G}$ is an open cover of K.

Since K is compact, there is a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers K.

Let $\mathcal{G}' = \{G_{a_1}, G_{a_2}, \ldots, G_{a_m}\}$. Then each $a_i \in K$ and $K \subset G_{a_1} \cup G_{a_2} \cup \cdots \cup G_{a_m}$.

Let $a_0 = \min\{a_1, a_2, \ldots, a_m\}$. Then $a_0 \in K$. But $a_0 \notin \bigcup_{i=1}^{m} G_{a_i}$.

We arrive at a contradiction and therefore K has a least element.

Note. In a similar manner we can prove that K has a greatest element.

Theorem 3.16.5. Lindelof's theorem.

If S be a subset of $\mathbb{R}$ every open cover of S has a countable subcover. That is, if $\mathcal{G}$ be a collection of open sets in $\mathbb{R}$ that covers S then there is a countable subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers S.

Proof. Let $\mathcal{G}$ be a collection of open sets $\{G_\alpha : \alpha \in \Lambda$, Λ being the index set$\}$ in $\mathbb{R}$ such that $\mathcal{G}$ covers S.

Let $x \in S$. Then x belongs to at least one open set, say G_λ, of the collection. Therefore there exists an open interval I_x such that $x \in I_x \subset G_\lambda$.

Let us take an open interval $J(x)$ with rational end points such that $x \in J(x) \subset I_x$.

Let $\mathcal{G}'$ be the collection of all distinct open intervals $\{J(x) : x \in S\}$. Obviously $\mathcal{G}'$ covers S.

The set of all open intervals in $\mathbb{R}$ with rational end points is enumerable. The collection $\{J(x) : x \in S\}$ being a subset of this is countable. Therefore the collection $\mathcal{G}'$ can be enumerated as $\mathcal{G}' = \{J_1, J_2, J_3, \ldots\}$.

Now corresponding to each $J_m \in \mathcal{G}'$ let us choose a point $x_m \in S$ such that $x_m \in J_m \subset I_{x_m} \subset G_{\lambda_m}$, say. Then $J_m \subset G_{\lambda_m} \in \mathcal{G}$.

Thus for each $J_m \in \mathcal{G}'$ there corresponds one $G_{\lambda_m} \in \mathcal{G}$.

Let $\mathcal{G}''$ be the family of open sets $\{G_{\lambda_1}, G_{\lambda_2}, G_{\lambda_3}, \ldots\}$.

Then $\mathcal{G}''$ is a countable subcollection of $\mathcal{G}$ and $\mathcal{G}''$ covers S.

This completes the proof.

Another proof of Heine-Borel theorem.

If S be closed and bounded subset of $\mathbb{R}$ then every open cover of S has a finite subcover.

Proof. Let $\mathcal{G}$ be a collection of open sets in $\mathbb{R}$ that covers S. By Lindelof's theorem, there exists a countable subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers S. Let $\mathcal{G}' = \{J_1, J_2, \ldots, J_n, \ldots\}$. Then $S \subset \bigcup_{i=1}^{\infty} J_i$.

Let $C_1 = S - J_1$,
$C_2 = S - (J_1 \cup J_2)$,
$C_3 = S - (J_1 \cup J_2 \cup J_3)$,
$\ldots \quad \ldots \quad \ldots$

$C_k \subset S$ for all $k \in \mathbb{N}$. Also $C_1 \supset C_2 \supset C_3 \supset \cdots$
Since S is a bounded set each C_k is a bounded set.

Since each J_i is open, $C_k = S - (J_1 \cup J_2 \cup \cdots \cup J_k)$ is a closed set.

Therefore the collection $\{C_1, C_2, C_3, \ldots\}$ is a countable collection of closed and bounded sets in $\mathbb{R}$ and $C_1 \supset C_2 \supset C_3 \supset \cdots$

We shall prove that $c_m = \phi$ for some $m \in \mathbb{N}$.

If none of the sets of the collection $\{C_1, C_2, C_3, \ldots \ \ldots\}$ be empty then the collection is an enumerable collection of non-empty closed and bounded sets with $C_1 \supset C_2 \supset C_3 \supset \cdots$

By Cantor's intersection theorem, there exists a point x in $\mathbb{R}$ such that $x \in \bigcap_{i=1}^{\infty} C_i$. [Theorem 3.11.2]

But $C_k \subset S$ for all $k \in \mathbb{N}$. Therefore $x \in S$ ... ... (A)

Again $x \in \bigcap_{i=1}^{\infty} C_i \Rightarrow x \notin \bigcup_{i=1}^{\infty} J_i$ ... ... (B)

(A) and (B) together imply that $\{J_1, J_2, J_3, \ldots\}$ is not a cover of S, a contradiction.

So our assumption that none of the sets of the collection $\{C_1, C_2, C_3, \ldots\}$ is empty is wrong.

Therefore at least one of the sets, say C_m, is empty. Consequently, $S \subset J_1 \cup J_2 \cup \cdots \cup J_m$. That is, a finite subcollection of $\mathcal{G}$ also covers S.

This proves the theorem.

Exercises 5

1. Define a compact set. Use your definition to prove that

(i) the set $\mathbb{R}$ is not compact; (ii) the set $\mathbb{Z}$ is not compact;

(iii) the set $\mathbb{N}$ is not compact.

[**Hint.** Let $I_n = \{x \in \mathbb{R} : -n < x < n\}$. Then the family $\mathcal{F}$ of open intervals $\{I_n : n \in \mathbb{N}\}$ is an open cover of the set.]

2. Let $\mathcal{F}$ be the family of open intervals $\{I_n : n \in \mathbb{N}\}$, where $I_n = \{x \in \mathbb{R} : \frac{1}{n+2} < x < 1 - \frac{1}{n+2}\}$. Show that the family $\mathcal{F}$ is an open cover of the interval $I = \{x \in \mathbb{R} : 0 < x < 1\}$. Does there exist a finite subfamily of $\mathcal{F}$ that can cover I? Justify your answer.

3. For each $x \in (0,2)$, let $I_x = (\frac{x}{2}, \frac{x+2}{2})$. Show that the family $\mathcal{G} = \{I_x : x \in (0,2)\}$ is an open cover of the set $S = \{x \in \mathbb{R} : 0 < x < 2\}$. Show that no finite subcollection of $\mathcal{G}$ can cover S.

Hint. Let $c \in S$. Then $0 < c < 2$. This implies $0 < \frac{c}{2} < c < \frac{c+2}{2} < 2 \Rightarrow c \in I_c$.

4. Give an example of an open cover of the set $(0,1]$ which does not have a finite sub cover.

5. Give an example of an open cover of the set $[0,\infty)$ which does not have a finite sub cover.

6. Use the definition of a compact set to prove that the union of two compact sets in $\mathbb{R}$ is a compact set.

Give an example to show that the union of an infinite number of compact sets in $\mathbb{R}$ is not necessarily a compact set.

7. Use the definition of a compact set to prove that

(i) the intersection of two compact sets in $\mathbb{R}$ is compact,

(ii) the intersection of an infinite collection of compact sets in $\mathbb{R}$ is compact.

8. Let A and B be subsets of $\mathbb{R}$ of which A is closed and B is compact. Prove that $A \cap B$ is compact.

9. Give an alternative proof of the theorem- A compact set K in $\mathbb{R}$ is closed.

[**Hint.** Let $y \in \mathbb{R} - K$. For every $x \in K$, there exists $\epsilon_x > 0$ such that $N(x, \epsilon_x) \cap N(y, \epsilon_x) = \phi$. The family of neighbourhoods $\{N(x, \epsilon_x) : x \in K\}$ is an open cover of K. Extract a finite sub cover. Show that $y \notin K'$.]

10. If p be a limit point of a set $S \subset \mathbb{R}$ prove that there exists a countably infinite subset of S having p as its only limit point.

11. Let S be a subset of $\mathbb{R}$ such that every infinite subset of S has at least one limit point in S. Prove that S is a closed set.

4. REAL FUNCTIONS

4.1. Real function.

Let X be a non-empty set. A function $f : X \longrightarrow \mathbb{R}$ is called a *real valued function* on X. For each $x \in X$, the f-image, which is also called the value of f at x, denoted by $f(x)$, is a real number.

For example, the function $f : \mathbb{C} \longrightarrow \mathbb{R}$ defined by $f(z) = |z|, z \in \mathbb{C}$ is a real valued function of complex numbers.

Let D be a non-empty subset of $\mathbb{R}$. A function $f : D \longrightarrow \mathbb{R}$ is said to be a *real valued function of real numbers*. Such a function is also called a *real function*.

D is said to be the *domain* of f. The set $f(D) = \{f(x) : x \in D\}$ is a subset of $\mathbb{R}$ and it is called the *range* of f.

Examples.

1. Let $c \in \mathbb{R}$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = c, x \in \mathbb{R}$. The range of the function f is the singleton set $\{c\}$. f is called a **constant function**. f is also expressed as $f(x) = c, x \in \mathbb{R}$.

2. Let $D = \{x \in \mathbb{R} : x \neq 0\}$ and $f : D \longrightarrow \mathbb{R}$ is defined by $f(x) = \frac{1}{x}, x \neq 0$. The range of f is $\{x \in \mathbb{R} : x \neq 0\}$.

3. Let $D = \{x \in \mathbb{R} : x \geq 0\}$ and $f : D \longrightarrow \mathbb{R}$ is defined by $f(x) = \sqrt{x}, x \in D$. The range of f is $\{x \in \mathbb{R} : x \geq 0\}$. f is also expressed as $f(x) = \sqrt{x}, x \geq 0$. f is called the **square root function**.

4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined $f(x) = \sin x, x \in \mathbb{R}$. The range of f is $\{x \in \mathbb{R} : -1 \leq x \leq 1\}$. f is also expressed as $f(x) = \sin x, x \in \mathbb{R}$. f is called the real **sine function**.

5. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = |x|, x \in \mathbb{R}$. The range of the function is $\{x \in \mathbb{R} : x \geq 0\}$. f is equivalently expressed as
$$\begin{aligned} f(x) &= x, x > 0 \\ &= 0, x = 0 \\ &= -x, x < 0. \end{aligned}$$
f is called the **absolute value function.**

6. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = \text{ sgn } x, x \in \mathbb{R}$.

$$\begin{aligned} \operatorname{sgn} x &= \tfrac{|x|}{x}, x \neq 0 \\ &= 0, x = 0. \end{aligned}$$

The range of f is the finite set $\{-1, 0, 1\}$. f is equivalently expressed as

$$\begin{aligned} f(x) &= 1, x > 0 \\ &= 0, x = 0 \\ &= -1, x < 0. \end{aligned}$$

f is called the **signum function.**

7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = [x], x \in \mathbb{R}$. $[x]$ is the greatest integer not greater than x. The range of the function is $\mathbb{Z}$. f is equivalently expressed as

$$\begin{aligned} f(x) &= 0, 0 \leq x < 1 \\ &= 1, 1 \leq x < 2 \\ &= 2, 2 \leq x < 3 \\ &\cdots \quad \cdots \\ &= -1, -1 \leq x < 0 \\ &= -2, -2 \leq x < -1 \\ &\cdots \quad \cdots \end{aligned}$$

f is called the **greatest integer function.**

For every $x \in \mathbb{R}$, $x \geq [x]$. The difference between x and its integral part $[x]$ is called the *fractional part* of x and is denoted by $\{x\}$.

Therefore $\{x\} = x - [x]$ for all real x. It also follows that $0 \leq \{x\} < 1$ for all real x.

For example, $\{.3\} = .3$, $\{2.3\} = .3$, $\{2\} = 0$, $\{-.3\} = .7$.

Definition.

A function f defined on $I = [a, b]$ is said to be a **piecewise constant function** on I (or a **step function** on I) if there exist finite number of points $x_0, x_1, \ldots, x_n (a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b)$ such that f is a constant on each open subinterval (x_{k-1}, x_k) of $[a, b]$. That is, for each $k = 1, 2, \ldots, n$ there is a real number s_k such that $f(x) = s_k$ for all $x \in (x_{k-1}, x_k)$. $f(x_{k-1}), f(x_k)$ need not be same as s_k, $k = 1, 2, \ldots, n$.

4.2. Injective function, Surjective function.

Let $D \subset \mathbb{R}$. A function $f : D \to \mathbb{R}$ is said to be *injective* (or one-one) if for two distinct elements x_1, x_2 in D the functional values $f(x_1)$ and $f(x_2)$ are distinct.

Let $D \subset \mathbb{R}, E \subset \mathbb{R}$. A function $f : D \to E$ is said to be surjective (or onto) if $f(D) = E$.

For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin\ x, x \in \mathbb{R}$ is not injective, because two distinct points π and 2π in the domain $\mathbb{R}$ have the same functional value. f is not surjective, because the range of $f = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$, a proper subset of the codomain set $\mathbb{R}$.

4.3. Equal functions.

Let $D \subset \mathbb{R}$. The functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ having the same domain D are said to be *equal* if $f(x) = g(x)$ for all $x \in D$.

Examples.

1. Let $f(x) = |x|, x > 0; g(x) = x, x > 0$
Then f and g have the same domain $\{x \in \mathbb{R} : x > 0$ and $f(x) = g(x)$ for all x in the domain. Therefore $f = g$.

2. Let $f(x) = \sqrt{\frac{2x}{x-1}}, x \in A \subset \mathbb{R}; g(x) = \frac{\sqrt{2x}}{\sqrt{x-1}}, x \in B \subset \mathbb{R}$.
Here $A = \{x \in \mathbb{R} : x > 1\} \cup \{x \in \mathbb{R} : x \leq 0\}$, $B = \{x \in \mathbb{R} : x > 1\}$.
f and g have different domains. Therefore $f \neq g$.

4.4. Restriction function.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let D_o be a non-empty subset of D. The function $g : D_o \to \mathbb{R}$ defined by $g(x) = f(x), x \in D_o$ is said to be the *restriction* of f to D_o and g is denoted by f/D_o.

Examples.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \text{ sgn } x, x \in \mathbb{R}$.
Let $D_o = \{x \in \mathbb{R} : x > 0\}$. Then the restriction function f/D_o is defined by $f/D_o\ (x) = 1, x > 0$.
Let $D_1 = \{x \in \mathbb{R} : x < 0\}$. Then the restriction function f/D_1 is defined by $f/D_1\ (x) = -1, x < 0$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = [x], x \in \mathbb{R}$.
Let $D_o = \{x \in \mathbb{R} : 0 \leq x < 1\}$. Then the restriction function f/D_o is defined by $f/D_o\ (x) = 0, 0 \leq x < 1$.
Let $D_1 = \{x \in \mathbb{R} : 1 \leq x < 2\}$. Then the restriction function f/D_1 is defined by $f/D_1\ (x) = 1, 1 \leq x < 2$.

3. Let $D = \{x \in \mathbb{R} : 0 \leq x \leq \frac{\pi}{2}\}$ and $f : D \to \mathbb{R}$ is defined by
$f(x) = \sqrt{1 - \sin 2x}, x \in D$.
Let $D_o = \{x \in \mathbb{R} : 0 \leq x \leq \frac{\pi}{4}\}$. Then the restriction function f/D_o is defined by $f/D_o(x) = \cos x - \sin x, 0 \leq x \leq \pi/4$.
Let $D_1 = \{x \in \mathbb{R} : \frac{\pi}{4} \leq x \leq \frac{\pi}{2}\}$. Then the restriction function f/D_1 is defined by $f/D_1(x) = \sin x - \cos x, \frac{\pi}{4} \leq x \leq \frac{\pi}{2}$.

4.5. Composite function.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $g : E \to \mathbb{R}$ be a function on E where $f(D) \subset E \subset \mathbb{R}$. Then for each $x \in D, f(x) \in E$ and therefore $g(f(x)) \in \mathbb{R}$. We can conceive of a real function $h : D \to \mathbb{R}$ such that $h(x) = g(f(x)), x \in D$. Then h is said to be the *composite function* of f and g and the function h is expressed as gf or as $g \circ f$.

Examples.

1. Let $D = \{x \in \mathbb{R} : x \geq 0\}$ and $f : D \to \mathbb{R}$ is defined by $f(x) = \sqrt{x}, x \in D$, and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = e^x, x \in \mathbb{R}$.

$f(D) = \{x \in \mathbb{R} : x \geq 0\}$. $f(D)$ is a subset of the domain of g. The composite function $g \circ f : D \to \mathbb{R}$ is defined by $g \circ f(x) = e^{\sqrt{x}}, x \in D$, i.e., $g \circ f(x) = e^{\sqrt{x}}, x \geq 0$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + 1, x \in \mathbb{R}$. Let $D = \{x \in \mathbb{R} : x \geq 0\}$ and $g : D \to \mathbb{R}$ be defined by $g(x) = \sqrt{x}, x \in D$. The range of f is $\{x \in \mathbb{R} : x \geq 1\}$ and this is a subset of the domain of g.

The composite function $gf : \mathbb{R} \to \mathbb{R}$ is defined by $gf(x) = \sqrt{x^2 + 1}, x \in \mathbb{R}$.

4.6. Inverse function.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be an injective function. Let $f(D) = E \subset \mathbb{R}$. Then $f : D \to E$ is injective as well as surjective.

Let $x \in D$. Then $f(x) = y \in E$. Each y in E has exactly one pre-image x in D. We can define a function $g : E \to D$ by $g(y) = x, y \in E$ where $f(x) = y$.

Therefore $gf(x) = x$ for all $x \in D$ and $fg(y) = y$ for all $y \in E$.

g is said to be the *inverse* of f and is denoted by f^{-1}.

The domain of the inverse function f^{-1} is the range of f and the range of f^{-1} is the domain of f.

Also $f^{-1}f(x) = x$ for all $x \in D$ and $ff^{-1}(y) = y$ for all $y \in E$.

Examples.

1. Let $D = \{x \in \mathbb{R} : x \geq 0\}$ and $f(x) = x^2, x \in D$. $f(D) = \{x \in \mathbb{R} : x \geq 0\} = E$,say. Then $f : D \to E$ is injective as well as surjective.
The inverse function $f^{-1} : E \to D$ is defined by $f^{-1}(y) = \sqrt{y}, y \in E$.

Also $f^{-1}f(x) = x$ for all $x \geq 0$ and $ff^{-1}(y) = y$ for all $y \geq 0$;

i.e., $\sqrt{x^2} = x$ for all $x \geq 0$ and $(\sqrt{y})^2 = y$ for all $y \geq 0$.

This inverse function is called **the square root function**.

2. Let $D = \{x \in \mathbb{R} : x \leq 0\}$ and $f(x) = x^2, x \leq 0$. The range of is $\{x \in \mathbb{R} : x \geq 0\} = E$,say. Then $f : D \to E$ is injective as well a surjective.

The inverse function $f^{-1} : E \to D$ is defined by $f^{-1}(y) = -\sqrt{y}, y \in E$.

Also $f^{-1}f(x) = x$ for all $x \leq 0$ and $ff^{-1}(y) = y$ for all $y \geq 0$;

i.e., $-\sqrt{x^2} = x$ for all $x \leq 0$ and $(-\sqrt{y})^2 = y$ for all $y \geq 0$.

This inverse function is called the **negative square root** function

Note. The function $f(x) = x^2, x \in \mathbb{R}$ admits of two inverse function the *principal* inverse function is the function described in Example 1.

3. The real sine function defined on $\mathbb{R}$ is not injective on $\mathbb{R}$. The ran of the function is $E = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$.

Let us consider the subset $D = \{x \in \mathbb{R} : -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\}$. The $f : D \to E$ defined by $f(x) = \sin x, x \in D$ is injective as well as surjectiv The inverse function $f^{-1} : E \to D$ is defined by $f^{-1}(y) = \sin^{-1} y, y \in$

Also $f^{-1}f(x) = x$ for all $x \in D$ and $ff^{-1}(y) = y$ for all $y \in E$;

i.e., $\sin^{-1}(\sin x) = x$, for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and $\sin(\sin^{-1} y) = y$, f $-1 \leq y \leq 1$.

This inverse function is called the **principal inverse sine functio** The domain of the inverse function is $\{y \in \mathbb{R} : -1 \leq y \leq 1\}$ and t range is $\{x \in \mathbb{R} : \frac{\pi}{2} \leq x \leq \frac{\pi}{2}\}$

Therefore $-\frac{\pi}{2} \leq \sin^{-1} y \leq \frac{\pi}{2}$ for $-1 \leq y \leq 1$.

Note. If instead of D, we choose $D_1 = \{x \in \mathbb{R} : 3\pi/2 \leq x \leq 5\pi/2\}$ the domain then the function $f(x) = \sin x, x \in D_1$, is injective as well surjective and therefore it admits of an inverse function $f^{-1} : E \to$ satisfying the conditions

$f^{-1}f(x) = x$ for all $x \in D_1$ and $ff^{-1}(y) = y$ for all $y \in E$.

But this inverse function differs from the principal inverse sine fun tion as they have different ranges.

Equivalently, we can define *many* inverse sine functions on the san domain E with their respective ranges different. This is expressed saying that inverse of real sine function is a *many-valued function* a this is denoted by Sin^{-1} (or Arc sin). The *principal* inverse function denoted by $\sin^{-1}$ (or arc sin).

Thus sin (Arc $\sin y) = y$, for $-1 \leq y \leq 1$ but Arc sin $(\sin x) \neq x$, general.

4. The real cosine function $f(x) = \cos x, x \in \mathbb{R}$ is not injective. T range of the function is $E = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$.

Let us consider the subset $D = \{x \in \mathbb{R} : 0 \leq x \leq \pi\}$. Then t function $f : D \to E$ defined by $f(x) = \cos x, x \in D$ is injective as well surjective.

The inverse function $f^{-1} : E \to D$ is defined by $f^{-1}(y) = \cos^{-1} y, y \in E$.

Also $f^{-1}f(x) = x$ for all $x \in D$ and $ff^{-1}(y) = y$ for all $y \in E$, i.e., $\cos^{-1}(\cos x) = x$, for $0 \leq x \leq \pi$ and $\cos(\cos^{-1} y) = y$, for $-1 \leq y \leq 1$.

This inverse function is called the *principal inverse cosine function.* The domain of this inverse function is $\{y \in \mathbb{R} : -1 \leq y \leq 1\}$ and the range is $\{x \in \mathbb{R} : 0 \leq x \leq \pi\}$.

Therefore $0 \leq \cos^{-1} y \leq \pi$ for $-1 \leq y \leq 1$.

Note. If instead of D we choose $D_1 = \{x \in \mathbb{R} : 2\pi \leq x \leq 3\pi\}$ as the domain, then the function $f : D_1 \to E$ defined by $f(x) = \cos x, x \in D_1$ is injective as well as surjective and therefore it admits of an inverse function $f^{-1} : E \to D_1$.

Thus as in the case of sine function, we can define *many* inverse cosine functions on the same domain E with their respective ranges different. Therefore inverse cosine function is also a many valued function and this is denoted by Cos^{-1} or (Arc cos). The *principal* inverse function is denoted by $\cos^{-1}$ (or arc cos).

5. Let n be an integer and $I_n = \{x \in \mathbb{R} : (2n-1)\frac{\pi}{2} < x < (2n+1)\frac{\pi}{2}\}$. Let $I = \bigcup_{n \in z} I_n$. The real tangent function is defined on the domain I and the range of the function is $\mathbb{R}$. Let us consider the subset $D = \{x \in \mathbb{R} : -\frac{\pi}{2} < x < \frac{\pi}{2}\}$. Then $f : D \to \mathbb{R}$ defined by $f(x) = \tan x, x \in D$ is injective as well as surjective.

The inverse function $f^{-1} : \mathbb{R} \to D$ is defined by $f^{-1}(y) = \tan^{-1} y, y \in \mathbb{R}$.

Also $f^{-1}f(x) = x$ for all $x \in D$ and $ff^{-1}(y) = y$ for all $y \in \mathbb{R}$, i.e., $\tan^{-1}(\tan x) = x$, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $\tan(\tan^{-1} y) = y$, for $y \in \mathbb{R}$.

This inverse function is called the **principal inverse tangent function.** The domain of the inverse function is $\mathbb{R}$ and the range is $\{x \in \mathbb{R} : -\frac{\pi}{2} < x < \frac{\pi}{2}\}$

Therefore $-\frac{\pi}{2} < \tan^{-1} y < \frac{\pi}{2}$ for all $y \in \mathbb{R}$.

Note. If instead of D, we choose $D_1 = \{x \in \mathbb{R} : \frac{3\pi}{2} < x < 5\frac{\pi}{2}\}$ as the domain then the function $f : D_1 \to \mathbb{R}$ defined by $f(x) = \tan x, x \in D_1$, is injective as well as surjective and therefore it admits of an inverse function $f^{-1} : \mathbb{R} \to D_1$.

Thus we can define many inverse tangent functions on $\mathbb{R}$ with their respective ranges different. Therefore inverse tangent function is also a many valued function and this is denoted by Tan^{-1} (or Arc tan).

The principal inverse function is denoted by $\tan^{-1}$ (or arc tan).

6. Let n be an integer and $I_n = \{x \in \mathbb{R} : n\pi < x < (n+1)\pi\}$. Let $I = \bigcup\limits_{n \in z} I_n$. The real cotangent function is defined on the domain I and the range of the function is $\mathbb{R}$. Let $D = \{x \in \mathbb{R} : 0 < x < \pi\}$ then $f(x) = \cot x, x \in D$ is injective as well as surjective.
The inverse function $f^{-1} : \mathbb{R} \to D$ is defined by $f^{-1}(y) = \cot^{-1} y, y \in \mathbb{R}$.

This inverse function is called the **principal inverse cotangent function.** The domain of the inverse function is $\mathbb{R}$ and the range is $\{x \in \mathbb{R} : 0 < x < \pi\}$.

Therefore $0 < \cot^{-1} y < \pi$ for all $y \in \mathbb{R}$.

4.7. Algebraic operations on functions.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}, g : D \to \mathbb{R}$ be functions on D.

(i) The sum function $f + g$ is defined on D by
$$(f+g)(x) = f(x) + g(x), x \in D.$$

(ii) The product function $f.g$ is defined on D by
$$f.g(x) = f(x)g(x), x \in D.$$

(iii) Let $k \in \mathbb{R}$. The function kf is defined on D by
$$kf(x) = k.f(x), x \in D.$$

(iv) If $g(x) \neq 0, x \in D$, the quotient f/g is defined on D by
$$f/g(x) = f(x)/g(x), x \in D.$$

Note. If f and g be two functions on $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ respectively then the function $f + g$ and $f.g$ are defined on $A \cap B$. If $g(x) \neq 0, x \in D \subset A \cap B$, then f/g is defined on D.

Examples.

1. Let $f(x) = \sqrt{x}, x \geq 0; g(x) = x, x \in \mathbb{R}$.
Then $(f+g)(x) = \sqrt{x} + x, x \geq 0. fg(x) = x\sqrt{x}, x \geq 0$.
$g(x) \neq 0$ on $\{x \in \mathbb{R} : x \neq 0\}$. $\therefore$ $f/g(x) = \frac{\sqrt{x}}{x}, x > 0$.

2. Let $f(x) = \sqrt{x}, x \geq 0; g(x) = \sqrt{x-1}, x \geq 1$.
Then $(f+g)(x) = \sqrt{x} + \sqrt{x-1}, x \geq 1$
$f.g(x) = \sqrt{x}.\sqrt{x-1}, x \geq 1$
$f/g(x) = \frac{\sqrt{x}}{\sqrt{x-1}}, x > 1$.

Note. If $h(x) = \sqrt{x(x-1)}$, then the domain of h is $\{x \in \mathbb{R} : x < 0\} \cup \{x \in \mathbb{R} : x > 1\}$. Therefore $f.g \neq h$.

If $h(x) = \sqrt{\frac{x}{x-1}}$, then the domain of h is $\{x \in \mathbb{R} : x < 0\} \cup \{x \in \mathbb{R} : x > 1\}$. Therefore $f/g \neq h$.

4.8. Monotone functions.

Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be **monotone increasing** on I if $x_1, x_2 \in I$ and $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$.

$f : I \to \mathbb{R}$ is said to be **monotone decreasing** on I if $x_1, x_2 \in I$ and $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$.

A function $f : I \to \mathbb{R}$ is said to be *monotone* on I if f is either monotone increasing or monotone decreasing on I.

A function $f : I \to \mathbb{R}$ is said to be *strictly increasing* on I if $x_1, x_2 \in I$ and $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

$f : I \to \mathbb{R}$ is said to be *strictly decreasing* on I if $x_1, x_2 \in I$ and $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

A function $f : I \to \mathbb{R}$ is said to be *strictly monotone* on I if f is either strictly increasing or strictly decreasing on I.

Let $I = [a, b]$ be a closed and bounded interval.
A function $f : I \to \mathbb{R}$ is said to be monotone increasing on I if $x_1, x_2 \in I$ and $a \leq x_1 < x_2 \leq b \Rightarrow f(x_1) \leq f(x_2)$.

Similar definitons for a monotone decreasing function.

Let I be an interval and $f : I \to \mathbb{R}, g : I \to \mathbb{R}$ are both monotone increasing (decreasing) on I. Then

(i) $f + g$ is monotone increasing (decreasing) on I.

(ii) if $k \in \mathbb{R}$ and $k > 0, kf$ is monotone increasing (decreasing) on I.

(iii) if $k \in \mathbb{R}$ and $k < 0, kf$ is monotone decreasing (increasing) on I.

Examples.

1. Let $f(x) = 1 - x, x \in \mathbb{R}$.
$x_1, x_2 \in \mathbb{R}$ and $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.
Therefore f is strictly decreasing on $\mathbb{R}$.

2. Let $f(x) = x^2, x \in \mathbb{R}$.
$x_1, x_2 \in \mathbb{R}$ and $0 \leq x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
$x_1, x_2 \in \mathbb{R}$ and $x_1 < x_2 \leq 0 \Rightarrow f(x_1) > f(x_2)$.
Therefore f is strictly increasing on $[0, \infty)$ and strictly decreasing on $(-\infty, 0]$.

3. Let $f(x) = \operatorname{sgn} x, x \in [-1, 1]$.
$x_1 < 0, x_2 < 0$ and $x_1 < x_2 \Rightarrow f(x_1) = f(x_2)$.
$x_1 < 0, x_2 > 0$ and $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
$x_1 > 0, x_2 > 0$ and $x_1 < x_2 \Rightarrow f(x_1) = f(x_2)$.
Therefore f is monotone increasing on $[-1, 1]$.

4.9. Even function, odd function.

For $a \in \mathbb{R}^*$, let D be the symmetric interval $(-a, a)$.

A function $f : D \to \mathbb{R}$ is said to be an *even* function if $f(-x) = f(x)$ for all $x \in D$.

A function $f : D \to \mathbb{R}$ is said to be an *odd* function if $f(-x) = -f(x)$ for all $x \in D$.

For example,the functions $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2, f(x) = \cos x$ are even functions on $\mathbb{R}$ and defined by $f(x) = x, f(x) = \operatorname{sgn} x, f(x) = \sin x$ are odd functions on $\mathbb{R}$.

If f be an odd function on $(-a, a)$ then $f(0) = 0$.

Let f be an odd function on $(-a, a)$, for some $a \in \mathbb{R}^*$. If $(x, f(x))$ be a point on the graph of f then $(-x, -f(x))$ is also a point on the graph. It follows that the graph of f is symmetrical about the origin.

Let f be an even function on $(-a, a)$, for some $a \in \mathbb{R}^*$. If (x, y) be a point on the graph of f then $(-x, y)$ is also a point on the graph. It follows that the graph of f is symmetrical about the y axis.

4.10. Power functions.

A. Positive Integral powers.

Case 1. Let n be an even positive integer.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^n, x \in \mathbb{R}$. The range of f is $[0, \infty)$.

f is not injective on $\mathbb{R}$ since $f(c) = f(-c)$ for all $c \in \mathbb{R}$.

Let $x_1, x_2 \in [0, \infty)$ and $0 \le x_1 < x_2$. Then $f(x_1) < f(x_2)$. f is a strictly increasing function on $[0, \infty)$.

Let $x_1, x_2 \in (-\infty, 0]$ and $x_1 < x_2 \le 0$. Then $f(x_1) > f(x_2)$. f is a strictly decreasing function on $(-\infty, 0]$.

If we restrict the domain of f to $[0, \infty)$, then the function $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = x^n, x \in [0, \infty)$ is a strictly increasing function on $[0, \infty)$ and therefore f is injective on $[0, \infty)$.

For each $y \in (0, \infty)$ there exists a unique $x \in (0, \infty)$ such that $x^n = y$ [2.4.23, worked Ex 8]. This together with $f(0) = 0$ shows that f is surjective.

Therefore f is a bijective function and the inverse function f^{-1} is defined by $f^{-1}(x) = x^{\frac{1}{n}}, x \in [0, \infty)$.

This inverse function is called the *nth rootfunction* (n even positive integer) and the domain of this function is $[0, \infty)$.

Case 2. Let n be an odd positive integer.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^n, x \in \mathbb{R}$. The range of f is $\mathbb{R}$.

Let $x_1, x_2 \in \mathbb{R}$ and $x_1 < x_2$. Then $f(x_1) < f(x_2)$.

Therefore f is a strictly increasing function on $\mathbb{R}$. Consequently f is injective.

For each $y \in (0, \infty)$ there exists a unique $x > 0$ such that $x^n = y$ [2.4.23, worked Ex 8].

f is an odd function. Hence for each $y \in (-\infty, 0)$ there exists a unique $x < 0$ such that $x^n = y$. Also $f(0) = 0$.

Thus for each $y \in \mathbb{R}$ there is a unique $x \in \mathbb{R}$ such that $x^n = y$. Consequently, f is surjective.

Since f is a bijective function, the inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$ is defined by $f^{-1}(x) = x^{\frac{1}{n}}, x \in \mathbb{R}$.

This inverse function is called the *nth rootfunction* (n odd positive integer) and the domain of this function is $\mathbb{R}$.

B. Negative integral powers.

Let n be a positive integer. We define $x^{-n} = (\frac{1}{x})^n$ for all $x \neq 0$.

C. Rational powers.

In A, we have defined the nth root function $x^{\frac{1}{n}}$ (n even positive integer) for all $x \geq 0$ and the nth root function $x^{\frac{1}{n}}$ (n odd positive integer) for all $x \in \mathbb{R}$.

Therefore for all positive integers n, the nth root function $x^{\frac{1}{n}}$ is defined for all real $x \geq 0$ and $x^{\frac{1}{n}} \geq 0$ for all real $x \geq 0$.

Let r be a positive rational number, say $\frac{p}{q}$, where $p \in \mathbb{N}, q \in \mathbb{N}$.

We define $x^r = x^{\frac{p}{q}} = (x^{\frac{1}{q}})^p$ for all $x \geq 0$; and $x^{-r} = (\frac{1}{x})^r$ for all $x > 0$. Also we define $x^0 = 1$ for all $x > 0$.

Thus for *all rational* n, the power function $f(x) = x^n$ is defined for all real $x > 0$.

D. Irrational powers.

The power function for irrational powers is defined in terms of exponential functions discussed in 4.11.

Theorem 4.10.1. If $x \in \mathbb{R}, x > 0$ and r be a positive rational number p/q then $x^r = (x^p)^{1/q}$.

By definition, $x^r = (x^{1/q})^p$.

$x^{1/q} > 0$, since $x > 0$. Let $y = x^r$. Then $y = (x^{1/q})^p$.

$$\begin{aligned} y^q = \{(x^{1/q})^p\}^q &= (x^{1/q})^{pq}, \text{ since } p, q \text{ are positive integers} \\ &= (x^{1/q})^{qp} \\ &= \{(x^{1/q})^q\}^p = x^p. \end{aligned}$$

Since $x^p > 0$, $y = (x^p)^{1/q}$. That is, $(x^{1/q})^p = (x^p)^{1/q}$.

Note. The theorem says that it is immeterial whether we define $x^{p/q}$ by $(x^{1/q})^p$ or by $(x^p)^{1/q}$.

Theorem 4.10.2. If $x \in \mathbb{R}, x > 0$ and r be a positive rational number and $r = p/q = m/n$ where p, q, m, n are natural numbers then
$x^r = (x^{1/q})^p = (x^{1/n})^m$.

Here $np = qm$. Let $y = (x^{1/q})^p$.

Then $y^n = \{(x^{1/q})^p\}^n = (x^{1/q})^{pn} = (x^{1/q})^{qm} = \{(x^{1/q})^q\}^m = x^m$.
Therefore $y = (x^m)^{1/n} = (x^{1/n})^m$, by the previous theorem.
That is, $(x^{1/q})^p = (x^{1/n})^m$.

Note. The theorem says that although r can be written in many ways, the definition of x^r is unambiguous.

Theorem 4.10.3. If $x \in \mathbb{R}, x > 0$ and r, s are rational numbers then
(i) $x^r.x^s = x^{r+s}$; (ii) $(x^r)^s = x^{rs}$.

Proof left to the reader.

Theorem 4.10.4. If $x, y \in \mathbb{R}$ and $x > 0, y > 0$ and r is a rational number then
(i) $(xy)^r = x^r y^r$; (ii) $(\frac{x}{y})^r = \frac{x^r}{y^r}$.

Proof left to the reader.

Theorem 4.10.5. If $x \in \mathbb{R}, x > 0$ and r, s are rational numbers, then
(i) $x^r > 0$;
(ii) for $x > 1$, $x^r < x^s$ if $r < s$; and for $0 < x < 1$, $x^r > x^s$ if $r < s$.

Proof. (i) Let $r = \frac{p}{q}$ where $q \in \mathbb{N}, p \in \mathbb{Z}$. Then $x^r = (x^{\frac{1}{q}})^p = y^p$, say.
$x > 0 \Rightarrow x^{\frac{1}{q}} > 0$, i.e., $y > 0$ and $y > 0 \Rightarrow y^p > 0$, i.e., $x^r > 0$.
Therefore $x^r > 0$.

(ii) $x > 1 \Rightarrow \cdots < x^{-2} < x^{-1} < 1 < x < x^2 < \cdots$
It follows that $x^m < x^n$ if m, n are integers and $m < n$.

Let $r = \frac{u}{q}, s = \frac{v}{q}$ where u, v, q are integers and $q > 1$. $r < s \Rightarrow u < v$.

$x > 1 \Rightarrow x^{\frac{1}{q}} > 1 \Rightarrow (x^{\frac{1}{q}})^u < (x^{\frac{1}{q}})^v$, since $u < v$.
That is, $x^r < x^s$ if $r < s$.

Second part. $0 < x < 1 \Rightarrow \frac{1}{x} > 1$.

$$\begin{aligned} \tfrac{1}{x} > 1 \text{ and } r < s &\Rightarrow (\tfrac{1}{x})^r < (\tfrac{1}{x})^s, \text{ by the first part} \\ &\Rightarrow x^r > x^s. \end{aligned}$$

This completes the proof.

4.11. Exponential function.

We define a^x where a and x are real numbers and $a > 0$.

Case 1. Let $a > 1, x \in \mathbb{R}$.

If x be a rational number, a^x is already defined. We define a^x when x is irrational.

There exist rational numbers r and s such that $r < x < s$.

Since $a > 1$ and $r < s$, we have $a^r < a^s$.

Let $S = \{a^r : r \in \mathbb{Q} \text{ and } r < x\}$.

S is a non-empty subset of $\mathbb{R}$ having a^s as an upper bound. By the supremum property of $\mathbb{R}, \sup S$ exists.

We define $a^x = \sup S$.

Case 2. Let $0 < a < 1$, $x \in \mathbb{R}$.

We define $a^x = b^{-x}$ where $b = \frac{1}{a} > 1$.

Case 3. Let $a = 1, x \in \mathbb{R}$.

In this case $a^x = 1$.

Theorem 4.11.1. If $a \in \mathbb{R}, a > 0$ and $x \in \mathbb{R}$ then $a^x > 0$.

Proof. When $x \in \mathbb{Q}$, this reduces to the Theorem 4.10.5 (i).

We prove the theorem when x is irrational.

Case 1. $a > 1$.

$a^x = \sup S$ where $S = \{a^r : r \in \mathbb{Q} \text{ and } r < x\}$.

Each element of S is positive by Theorem 4.10.5 (i) and a^x being the supremum of S must be positive.

Thus $a^x > 0$ for all irrational x.

Case 2. $0 < a < 1$.

$a^x = b^{-x}$ where $b = \frac{1}{a} > 1$.

Since $b > 1$ and $-x$ is irrational, $b^{-x} > 0$ by case 1.

Thus $a^x > 0$ for all irrational x.

Case 3. $a = 1$.

Then $a^x = 1 > 0$.

Combining all cases, the theorem is done.

Theorem 4.11.2. If $a \in \mathbb{R}, a > 0, x \in \mathbb{R}$ and r, s are rational numbers such that $r < x < s$, then

(i) for $a > 1, a^r < a^x < a^s$; and (ii) for $0 < a < 1, a^r > a^x > a^s$.

Proof. (i) **Case 1.** $x \in \mathbb{Q}$.

$r < x < s \Rightarrow a^r < a^x < a^s$, by Theorem 4.10.5 (ii).

Case 2. x is irrational.

We have $a^x = \sup S$ where $S = \{a^r : r \in \mathbb{Q} \text{ and } r < x\}$.

First we prove that $a^x \notin S$. If not, let $a^x \in S$. Then $a^x = a^{r_1}$ for some rational number $r_1 < x$. By Density property of $\mathbb{R}$, there exists $r_2 \in \mathbb{Q}$ such that $r_1 < r_2 < x$.

Then $a^{r_2} \in S$, by the definition of S and $a^{r_1} < a^{r_2}$, by Theorem 4.10.5 (ii). That is, $a^x < a^{r_2}$ and this contradicts that $a^x = \sup S$.

Therefore $a^x \notin S$.

$$\begin{aligned} r < x \quad &\Rightarrow \quad a^r \in S \\ &\Rightarrow \quad a^r < a^x \text{ since } a^x = \sup S \text{ and } a^x \notin S \ldots \qquad \ldots\text{(i)} \end{aligned}$$

$$\begin{aligned} x < s \quad &\Rightarrow \quad s > r \text{ for all rational } r < x \\ &\Rightarrow \quad a^s > a^r \text{ for all rational } r < x \\ &\Rightarrow \quad a^s \text{ is an upper bound of } S \ldots \qquad \ldots\text{(ii)} \\ &\Rightarrow \quad a^s \geq a^x, \text{ since } a^x = \sup S. \end{aligned}$$

If possible, let $a^s = a^x$. By Density property of $\mathbb{R}$, there exists $s_1 \in \mathbb{Q}$ such that $x < s_1 < s$.

$$\begin{aligned} s_1 < s \quad &\Rightarrow \quad a^{s_1} < a^s \text{ by Theorem 4.10.5 (ii)} \\ &\Rightarrow \quad a^{s_1} < \sup S, \text{ since } a^s = a^x = \sup S. \end{aligned}$$

Again $x < s_1 \quad \Rightarrow \quad a^{s_1}$ is an upper bound of S, by (ii).

We arrive at a contradiction. Therefore $a^x < a^s \quad \ldots \qquad \ldots$(iii)

From (i) and (iii) $\quad a^r < a^x < a^s$.

(ii) $0 < a < 1$.

$0 < a < 1 \Rightarrow \frac{1}{a} > 1$. By (i) $(\frac{1}{a})^r < (\frac{1}{a})^x < (\frac{1}{a})^s$.

Therefore $a^s < a^x < a^r$.

This completes the proof.

Theorem 4.11.3. If $a \in \mathbb{R}, a > 0$ and $x_1, x_2 \in \mathbb{R}$, then

(i) for $a > 1, a^{x_1} < a^{x_2}$ if $x_1 < x_2$;

(ii) for $0 < a < 1, a^{x_1} > a^{x_2}$ if $x_1 < x_2$.

Proof. (i) Let r_1, r_2 be rational numbers such that $r_1 < x_1 < r_2 < x_2$. Also $a^{r_2} < a^{x_2}$, by Theorem 4.11.2.

Then $a^{r_1} < a^{x_1} < a^{r_2}$, by Theorem 4.11.2. Therefore $a^{x_1} < a^{x_2}$.

(ii) Similar proof.

Theorem 4.11.4. If $a \in \mathbb{R}, a > 0$ and $x, y \in \mathbb{R}$ then $a^{x+y} = a^x \cdot a^y$.

Proof. **Case 1.** $x, y \in \mathbb{Q}$.

In this case the theorem reduces to Theorem 4.10.3.

Case 2. x, y are both irrational.

Subcase (i) $a > 1$.

$a^x = \sup\{a^r : r \in \mathbb{Q} \text{ and } r < x\}$, $a^y = \sup\{a^r : r \in \mathbb{Q} \text{ and } r < y\}$.
Since $x, y \in \mathbb{R}$, $a^x > 0$ and $a^y > 0$.

Let us choose ϵ such that $0 < \epsilon < \min\{a^x(a^x + a^y), a^y(a^x + a^y)\}$.

There exist rational numbers p, q such that $p < x, q < y$ and $0 < a^x - \frac{\epsilon}{A} < a^p < a^x$ and $0 < a^y - \frac{\epsilon}{A} < a^q < a^y$, where $A = a^x + a^y > 0$.

Therefore $(a^x - \frac{\epsilon}{A})(a^y - \frac{\epsilon}{A}) < a^p.a^q < a^x.a^y$
or, $a^x.a^y - \epsilon < a^{p+q} < a^x.a^y$.
$p + q < x + y \Rightarrow a^{p+q} < a^{x+y}$, by Theorem 4.11.3.
Therefore $a^x.a^y - \epsilon < a^{x+y}$ ... ... (i)

$a^{x+y} = \sup\{a^r : r \in \mathbb{Q} \text{ and } r < x + y\}$.
Let $\epsilon > 0$. There exists a rational number s such that $s < x + y$ and $a^{x+y} - \epsilon < a^s < a^{x+y}$.

Let $x+y-s = 2k$. Let $u, v \in \mathbb{Q}$ such that $x-k < u < x, y-k < v < y$. Then $s < u + v < x + y$.

$u < x, v < y \Rightarrow a^u < a^x, a^v < a^y \Rightarrow a^{u+v} < a^x.a^y$
$s < u + v \Rightarrow a^{x+y} - \epsilon < a^s < a^{u+v}$.
Therefore $a^{x+y} - \epsilon < a^x.a^y$ ... ... (ii)

From (i) and (ii) $a^{x+y} - \epsilon < a^x.a^y < a^{x+y} + \epsilon$.

As ϵ is arbitrary, it follows that $a^{x+y} = a^x.a^y$.

Subcase (ii) $0 < a < 1$.

In this case $\frac{1}{a} > 1$ and $(\frac{1}{a})^{x+y} = (\frac{1}{a})^x.(\frac{1}{a})^y$, by subcase (i)
or, $a^{x+y} = a^x.a^y$.

Subcase (iii) $a = 1$. In this case $a^{x+y} = 1$, $a^x = 1, a^y = 1$ and hence $a^{x+y} = a^x.a^y$.

Case 3. One of x, y is rational and the other is irrational. Let $x \in \mathbb{Q}, y \in \mathbb{R} - \mathbb{Q}$. Then $x + y$ is irrational.

Subcase (i) $a > 1$.

$a^{x+y} = \sup\{a^r : r \in \mathbb{Q} \text{ and } r < x + y\}$.

Let $\epsilon > 0$. There exists a rational number s such that $s < x + y$ and $a^{x+y} - \epsilon < a^s < a^{x+y}$.

Let $x + y - s = k$. Let $u \in \mathbb{Q}$ such that $y - k < u < y$.
Then $s < x + u < x + y$.
$u < y \Rightarrow a^u < a^y$.

$$\begin{aligned} s < x+u \quad &\Rightarrow \quad a^s < a^{x+u} \\ &\Rightarrow \quad a^{x+y} - \epsilon < a^{x+u} \\ &\Rightarrow \quad a^{x+y} - \epsilon < a^x.a^u \\ &\Rightarrow \quad a^{x+y} - \epsilon < a^x.a^y \ldots \quad \ldots \text{(i)} \end{aligned}$$

$a^y = \sup\{a^r : r \in \mathbb{Q} \text{ and } r < y\}$.

There exists a rational number p such that $p < y$ and $a^y - \frac{\epsilon}{a^x} < a^p < a^y$.

Therefore $a^x(a^y - \frac{\epsilon}{a^x}) < a^x.a^p < a^x.a^y$

or, $a^x.a^y - \epsilon < a^{x+p} < a^x.a^y$

$x + p < x + y \Rightarrow a^{x+p} < a^{x+y}$.

Therefore $a^x.a^y - \epsilon < a^{x+y}$.

or, $a^x.a^y < a^{x+y} + \epsilon \ldots \quad \ldots$ (ii)

From (i) and (ii) $a^{x+y} - \epsilon < a^x.a^y < a^{x+y} + \epsilon$.

Since ϵ is arbitrary, $a^{x+y} = a^x.a^y$.

Subcase (ii) $0 < a < 1$.

Similar proof.

Subcase (iii) $a = 1$.

Similar proof.

Combining all cases, the proof is complete.

Definition. If $a \in \mathbb{R}$ and $a > 0$ the function $f(x) = a^x, x \in \mathbb{R}$ is called the **exponential function**. The domain of the exponential function is $\mathbb{R}$.

When $a = 1$, $f(x) = 1$ for all $x \in \mathbb{R}$.

When $a > 1$, the exponential function is a strictly increasing function on $\mathbb{R}$. The range of the function is $(0, \infty)$.

When $0 < a < 1$, the exponential function is a strictly decreasing function on $\mathbb{R}$. The range of the function is $(0, \infty)$.

In particular, the exponential function $f(x) = e^x, x \in \mathbb{R}$ is a strictly increasing function on $\mathbb{R}$, since $e > 1$. The range of the function is $(0, \infty)$.

4.12. Logarithmic function.

Let $a \in \mathbb{R}, a > 1$.

In this case the function $f(x) = a^x, x \in \mathbb{R}$ is strictly increasing on $\mathbb{R}$. The range of the function is $(0, \infty)$.

Therefore the function is bijective and the inverse function $f^{-1} : (0, \infty) \to \mathbb{R}$ exists.

Let $a \in \mathbb{R}, 0 < a < 1$.

In this case, the function $f(x) = a^x, x \in \mathbb{R}$ is strictly decreasing on $\mathbb{R}$. The range of the function is $(0, \infty)$.

Therefore the function is bijective and the inverse function $f^{-1} : (0, \infty) \to \mathbb{R}$ exists.

In both the cases the inverse function is called the *logarithmic function* and it is denoted by $\log_a x$. a is called the *base* of the logarithmic function. The logarithmic function is a monotone function on $(0, \infty)$ (monotone increasing if $a > 1$, monotone decreasing if $0 < a < 1$) and the range of this function is $\mathbb{R}$.

Also we have $\log_a(a^x) = x$ for all $x \in \mathbb{R}$ and $a^{\log_a x} = x$ for $x > 0$.

In particular, the inverse function $\log_e x$ is called the **natural logarithmic** function. The domain of the function is $(0, \infty)$ and the range of the function is $\mathbb{R}$.

Also we have $\log_e(e^x) = x$ for all $x \in \mathbb{R}$ and $e^{\log_e x} = x$ for all $x > 0$.

[The base e in the natural logarithmic function is often dropped and it is expressed as $\log x$.]

Remark. The exponential function a^x is defined on $\mathbb{R}$ for all real $a > 0$.

For $a = 1$, the function a^x is a constant function.

For $a > 1$, the function a^x is a strictly increasing function on $\mathbb{R}$. The range of the function is $(0, \infty)$.

For $0 < a < 1$, the function a^x is a strictly decreasing function on $\mathbb{R}$. The range of the function is $(0, \infty)$.

For $a > 0 (\neq 1)$ the exponential function admits of an inverse function (called the logarithmic function) $\log_a x$ on $(0, \infty)$. The range of the logarithmic function is $\mathbb{R}$.

For $a > 1$, the logarithmic function $\log_a x$ is a strictly increasing function on $(0, \infty)$.

For $0 < a < 1$, the logarithmic function $\log_a x$ is a strictly decreasing function on $(0, \infty)$.

In particular, when $a = e$, the logarithmic function $\log_e x$ is called the natural logarithmic function.

Theorem 4.12.1. If $a \in \mathbb{R}, a > 0 (\neq 1)$ and $x, y \in \mathbb{R}, x > 0, y > 0$, then $\log_a x + \log_a y = \log_a xy$.

Proof. Since $x > 0, y > 0$ and $a > 0 (\neq 1), \log_a x \in \mathbb{R}, \log_a y \in \mathbb{R}$.

$$\begin{aligned} a^{\log_a x + \log_a y} &= a^{\log_a x} . a^{\log_a y}, \text{ by Theorem 4.11.4} \\ &= xy. \end{aligned}$$

Since $xy > 0$ and $a^{\log_a x + \log_a y} = xy$ it follows from the property of the inverse function that

$\log_a xy = \log_a x + \log_a y$.

In particular, $\log_e xy = \log_e x + \log_e y$.

Corollary. $\log_a \frac{1}{x} = -\log_a x$.

Theorem 4.12.2. If $a \in \mathbb{R}, a > 0 (\neq 1)$ and $x \in \mathbb{R}, x > 0$ then $\log_a x^n = n \log_a x$ if n be a rational number.

Proof left to the reader.

In particular, $\log_e x^n = n \log_e x$.

Definition. If $x \in \mathbb{R}, x > 0$ and $\alpha \in \mathbb{R}$ we define the **power function** x^α by $x^\alpha = e^{\alpha \log x}, x > 0$.

This definition is consistent with the definition of the power function for rational α. Because, if $\alpha = \frac{m}{n}$ where $m \in \mathbb{Z}$, $n \in \mathbb{N}$ and $x > 0$ then $\log_e x^{\frac{m}{n}} = \frac{m}{n} \log_e x$.

By the property of the inverse function, $x^\alpha = x^{\frac{m}{n}} = e^{\log_e x^{\frac{m}{n}}} = e^{\frac{m}{n} \log_e x} = e^{\alpha \log_e x}$.

Theorem 4.12.3. If $x > 0$ and $\alpha, \beta \in \mathbb{R}$ then

(i) $x^{\alpha+\beta} = x^\alpha . x^\beta$;

(ii) $(x^\alpha)^\beta = x^{\alpha\beta}$;

(iii) $x^\alpha > 0$;

(iv) for $x > 1, x^\alpha < x^\beta$ if $\alpha < \beta$; and for $0 < x < 1, x^\alpha > x^\beta$ if $\alpha < \beta$

(v) $\log_e x^\alpha = \alpha \log_e x$.

Proof. (v) $\log_e x^\alpha = \log_e(e^{\alpha \log_e x})$, by definition
$= \alpha \log_e x$, since $\log_e e^x = x$ for all $x \in \mathbb{R}$.

Proofs for other parts left to the reader.

The general exponential function a^x can be expressed in terms of the exponential function e^x by $a^x = e^{x \log_e a}$.

For $a \in \mathbb{R}, x \in \mathbb{R}$ and $a > 0$, we have $a^x > 0$.

Therefore for all $x \in \mathbb{R}$, $a^x = e^{\log_e (a^x)}$, since $e^{\log_e x} = x$ for all $x > 0$.

Since $a \in \mathbb{R}, a > 0$ and $x \in \mathbb{R}, \log_e(a^x) = x \log_e a$, by Theorem 4.12.3 (v).

Consequently, $a^x = e^{x \log_e a}$.

4.13. Hyperbolic functions.

The hyperbolic functions $\sinh x$ and $\cosh x$ are defined by
$\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$.

The other hyperbolic functions $\tanh x$, $\coth x$, $\operatorname{cosech} x$ and $\operatorname{sech} x$ are defined by $\tanh x = \frac{\sinh x}{\cosh x}$, $\coth x = \frac{\cosh x}{\sinh x}$, $\operatorname{cosech} x = \frac{1}{\sinh x}$, $\operatorname{sech} x = \frac{1}{\cosh x}$.

Domain and Range:

sinh x **:** Since the domains of e^x and e^{-x} are both $\mathbb{R}$, the domain of $\sinh x$ is $\mathbb{R}$.

Let $y \in \mathbb{R}$, the co-domain set and let x be a pre-image of y.
Then $e^x - e^{-x} = 2y$
or, $e^{2x} - 2ye^x - 1 = 0$.
Therefore $e^x = y \pm \sqrt{y^2 + 1}$.
Since $e^x > 0$ for all real $x, x = \log(y + \sqrt{y^2 + 1}) \in \mathbb{R}$.
So the range of the function $\sinh x$ is $\mathbb{R}$.

cosh x **:** Since the domains of e^x and e^{-x} are both $\mathbb{R}$, the domain of $\cosh x$ is $\mathbb{R}$.

Let $y \in \mathbb{R}$, the co-domain set and let x be a pre-image of y.
Then $e^x + e^{-x} = 2y$
or, $e^{2x} - 2ye^x + 1 = 0$.
Therefore $e^x = y \pm \sqrt{y^2 - 1}$.
Since $e^x > 0$ for all real x, $y \geq 1$ and $x = \log(y \pm \sqrt{y^2 - 1})$ shows that, there are two pre-images of y.
So the range of the function $\cosh x$ is $\{y \in \mathbb{R} : y \geq 1\}$.

tanh x **:** Since $\cosh x \geq 1$, the domain of $\tanh x$ is $\mathbb{R}$.

Let $y \in \mathbb{R}$, the co-domain set and let x be a pre-image of y.
Then $\frac{e^x - e^{-x}}{e^x + e^{-x}} = y$
or, $e^{2x} - 1 = (e^{2x} + 1)y$.
Therefore $e^{2x} = \frac{y+1}{1-y}, y \neq 1$.
But $e^{2x} > 0$ for all real x. $-1 < y < 1$.
So the range of the function $\tanh x$ is $\{y \in \mathbb{R} : -1 < y < 1\}$.

coth x **:** $\sinh x = 0$ gives $x = 0$.

So the domain of $\coth x$ is $\{x \in \mathbb{R} : x \neq 0\}$.
Since $-1 < \tanh x < 1$, therefore $|\coth x| > 1$.
So the range of the function $\coth x$ is $\{y \in \mathbb{R} : |y| > 1\}$.

sech x **:** Since $\cosh x \geq 1$ for all real x, the domain of $\operatorname{sech} x$ is $\mathbb{R}$.

Let $y \in \mathbb{R}$, the co-domain set and let x be a pre-image of y.

Then $\frac{2}{e^x+e^{-x}} = y$
or, $ye^{2x} - 2e^x + y = 0$.
Therefore $e^x = \frac{1 \pm \sqrt{1-y^2}}{y}, y \neq 0$.
Since $e^{2x} > 0$ for all real $x, 0 < y \leq 1$.
So the range of the function sech x is $\{y \in \mathbb{R} : 0 < y \leq 1\}$.

cosech x : sinh $x = 0$ gives $x = 0$.
So the domain of cosech x is $\{x \in \mathbb{R} : x \neq 0\}$.
Let $y \in \mathbb{R}$, the co-domain set and let x be a pre-image of y.
Then $\frac{2}{e^x-e^{-x}} = y$
or, $ye^{2x} - 2e^x - y = 0$.
Therefore $e^x = \frac{1 \pm \sqrt{y^2+1}}{y}, y \neq 0$.

$$\text{Since } e^x > 0 \text{ for all real } x, x = \log \frac{1+\sqrt{y^2+1}}{y}, \text{ for } y > 0$$
$$= \log \frac{1-\sqrt{y^2+1}}{y}, \text{ for } y < 0.$$

So the range of the function cosech x is $\{y \in \mathbb{R} : y \neq 0\}$.

Properties.

1. $\cosh^2 x - \sinh^2 x = 1, x \in \mathbb{R}$
$\text{sech}^2 x = 1 - \tanh^2 x, x \in \mathbb{R}$
$\text{cosech}^2 x = \coth^2 x - 1, x \neq 0$.

2. $\sinh(x+y) = \sinh x \cosh x + \cosh x \sinh y$
$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$
$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$
$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$
$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$
$\tanh(x-y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$.

4.14. Bounded function.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. f is said to be *bounded above* on D if there exists a real number B such that $f(x) \leq B$ for all $x \in D$. B is said to be *an upper bound* of f on D. f is said to be *bounded below* on D if there exists a real number b such that $f(x) \geq b$ for all $x \in D$. b is said to be a *lower bound* of f on D.

f is said to be **bounded** on D if f is bounded above as well as bounded below on D.

In other words, f is bounded on D if the range set $f(D)$ be a bounded set in $\mathbb{R}$.

f is said to be *unbounded* on D if f is either *unbounded* above or *unbounded* below or both.

Let f be bounded above on D. Then the range set $f(D) = \{f(x) : x \in D\}$ is a non-empty subset of $\mathbb{R}$ bounded above. Therefore by the supremum property of $\mathbb{R}$, the subset $f(D)$ has a least upper bound M. M is called the *supremum* of f on D and is expressed as $M = \sup\limits_{x \in D} f(x)$.

Similarly, if f be bounded below on D there exists a real number m which is called the *infimum* of f on D and is expressed as $m = \inf\limits_{x \in D} f(x)$.

Let f be a bounded function on D. Then M, the supremum of f on D, satisfies the following conditions :

(i) $f(x) \leq M$ for all $x \in D$,

(ii) for each pre-assigned positive ϵ there exists an element y in D such that $M - \epsilon < f(y) \leq M$.

Also m, the infimum of f on D, satisfies the following conditions:

(i) $f(x) \geq m$ for all $x \in D$,

(ii) for each pre-assigned positive ϵ there exists an element y in D such that $m \leq f(y) < m + \epsilon$.

Examples.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{1}{1+x^2}, x \in \mathbb{R}$. Then f is bounded above as well as bounded below and $\sup\limits_{x \in \mathbb{R}} f(x) = 1$, $\inf\limits_{x \in \mathbb{R}} f(x) = 0$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^x, x \in \mathbb{R}$. Then f is unbounded above but bounded below and $\inf\limits_{x \in \mathbb{R}} f(x) = 0$.

Definition. Let $f : D \to \mathbb{R}$ be bounded on D. Then $\sup\limits_{x \in D} f(x) - \inf\limits_{x \in D} f(x)$ is said to be the *oscillation* of f on D.

Definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Let $c \in D$. f is said to be *bounded at c* if there exists a neighbourhood $N(c)$ of c such that f is bounded on $N(c) \cap D$.

If f be bounded on D then it follows from the definition that f is bounded at each point of D. But f can be bounded at each point of D without being bounded on D.

For example, the function $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is bounded at each point of the interval $(0, 1)$, but is not bounded on the interval $(0, 1)$.

The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is bounded at each point of $\mathbb{R}$, but is not bounded on $\mathbb{R}$.

The following theorem gives a condition for which boundedness of a function f at each point of a set implies boudedness on the whole set.

Theorem 4.14.1. If a function $f : D \to \mathbb{R}$ be bounded at each point of D and D is a closed and bounded set in $\mathbb{R}$, then f is bounded on D.

Proof. Let $x \in D$. Since f is bounded at each point of D, there is a neighbourhood $N(x)$ of x such that f is bounded on $N(x) \cap D$.

Let us consider the collection of neighbourhoods $\mathcal{G} = \{N(x) : x \in D\}$ such that f is bounded on $N(x) \cap D$. Clearly, $\mathcal{G}$ is an open cover of D. Since D is a closed and bounded set in $\mathbb{R}$, by Heine-Borel theorem, there exists a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers D.

Let $\mathcal{G}' = \{N(x_1), N(x_2), \ldots, N(x_m)\}$. Then $D \subset N(x_1) \cup N(x_2) \cup \ldots \cup N(x_m)$ and f is bounded on $D \cap N(x_i)$ for $i = 1, 2, \ldots, m$.

So there exists a positive M_i such that $|f(x)| \leq M_i$ for all $x \in D \cap N(x_i)$ and this holds for $i = 1, 2, \ldots, m$.

Let $M = \max\{M_1, M_2, M_3, \ldots, M_m\}$.

Let $x \in D$. Then $x \in D \cap N(x_k)$ for some $k \in \{1, 2, \ldots, m\}$ and therefore $|f(x)| \leq M_k \leq M$.

This proves that f is bounded on D and this completes the proof.

Note. In particular, if f be bounded at each point of a closed and bounded interval $[a, b]$, then f is bounded on $[a, b]$.

Exercises 6

1. Determine the domain of the real function f.

 (i) $f(x) = \log \frac{2+x}{2-x}$, (ii) $f(x) = \cos^{-1} \frac{2x}{1+x}$, (iii) $f(x) = \sqrt{2 + x - x^2}$,

 (iv) $f(x) = \sqrt{-x} + \frac{1}{\sqrt{2+x}}$, (v) $f(x) = \log \sin x$, (vi) $f(x) = \frac{x}{[x+1]}$.

2. Show that the pair of functions f and g are not equal by specifying their domains.

 (i) $f(x) = \sqrt{x(x-3)}$; $g(x) = \sqrt{x}.\sqrt{x-3}$

 (ii) $f(x) = \log x^2$; $g(x) = 2 \log x$

 (iii) $f(x) = \sqrt{\frac{x}{x-3}}$; $g(x) = \frac{\sqrt{x}}{\sqrt{x-3}}$

 (iv) $f(x) = \log \frac{x+3}{x}$; $g(x) = \log(x+3) - \log x$.

 Show that in each of the cases, g is a restriction of f.

3. Determine which of the following functions are even and which are odd.

 (i) $f(x) = \log \frac{1+x}{1-x}$, $x \in (-1, 1)$ (ii) $f(x) = \log(x + \sqrt{1 + x^2})$, $x \in \mathbb{R}$

(iii) $f(x) = \sqrt[3]{(x+1)^2} + \sqrt[3]{(x-1)^2}$, $x \in \mathbb{R}$

(iv) $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$, $x \in \mathbb{R}$.

4. Prove that every function $f : D \to \mathbb{R}$, where D is a symmetric interval (i.e., $x \in D \Rightarrow -x \in D$) can be expressed as the sum of an even and an odd function.

Express f as the sum of an even and an odd function, where

(i) $f(x) = \sqrt{1+x}$, $-1 \leq x \leq 1$, (ii) $f(x) = x + \sqrt{1+x^2}$, $x \in \mathbb{R}$.

5. A function $f : D \to \mathbb{R}$ is said to be a *periodic function* if there exists a positive real number p such that for all $n \in \mathbb{Z}$, $f(x+np) = f(x)$ holds in D.

The least positive p is said to be the *period* of f.

For example , (i) let $f(x) = \sin x, x \in \mathbb{R}$. Then $p = 2\pi$ since $\sin(x+2n\pi) = \sin x$ for all $n \in \mathbb{Z}$;

(ii) let $f(x) = x - [x], x \in \mathbb{R}$. Then $f(x+n) = f(x)$ for all $n \in \mathbb{Z}$, since for all real x, $[x+n] = [x] + n$, if n be an integer. Therefore $p = 1$.

Find the period of the periodic function f, where

(i) $f(x) = a \sin 3x + b \cos 3x, a, b \in \mathbb{R}$, (ii) $f(x) = \sqrt{\tan x}$,

(iii) $f(x) = \sin^2 x$.

6. Verify the following.

(i) The domain of the inverse function $\sinh^{-1} x$ is $\mathbb{R}$ and $\sinh^{-1} x = \log(x + \sqrt{x^2+1}), x \in \mathbb{R}$.

(ii) The domain of the inverse function $\cosh^{-1} x$ is $\{x \in \mathbb{R} : x \geq 1\}$ and $\cosh^{-1} x$ has two values $\pm \log(x + \sqrt{x^2-1}), x \geq 1$.

Note : $\cosh^{-1} x$ has two branches, the principle branch is given by $\cosh^{-1} x = \log(x + \sqrt{x^2-1}), x \geq 1$.

(iii) The domain of the inverse function $\tanh^{-1} x$ is $\{x \in \mathbb{R} : |x| < 1\}$ and $\tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}, -1 < x < 1$.

(iv) The domain of the inverse function $\coth^{-1} x$ is $\{x \in \mathbb{R} : |x| > 1\}$ and $\coth^{-1} x = \frac{1}{2} \log \frac{x+1}{x-1}, |x| > 1$.

(v) The domain of the inverse function $\text{sech}^{-1} x$ is $\{x \in \mathbb{R} : 0 < x \leq 1\}$ and $\text{sech}^{-1} x = \log \frac{1+\sqrt{1-x^2}}{x}, 0 < x \leq 1$.

(vi) The domain of the inverse function $\text{cosech}^{-1} x$ is $\{x \in \mathbb{R} : x \neq 0\}$ and

$$\begin{aligned} \text{cosech}^{-1} x &= \log \frac{1+\sqrt{x^2+1}}{x}, x > 0 \\ &= \log \frac{1-\sqrt{x^2+1}}{x}, x < 0. \end{aligned}$$

5. SEQUENCE

5.1. Real Sequence.

A mapping $f : \mathbb{N} \to \mathbb{R}$ is said to be a *sequence in* $\mathbb{R}$, or a *real sequence.*

The f-images $f(1), f(2), f(3), \ldots \ldots$ are real numbers. The image of the nth element, $f(n)$, is said to be the nth *element* (or the nth term) of the real sequence.

We shall be mainly concerned with real sequences and we shall use the term 'sequence' to mean a 'real sequence'.

A sequence f is generally denoted by the symbol $\{f(n)\}$. Also the symbol $\{f(1), f(2), f(3), \ldots \ldots\}$ is used to denote the sequence f.

The *range* of the real sequence $\{f(n)\}$ is a subset of $\mathbb{R}$, denoted by the symbol $\{f(n) : n \in \mathbb{N}\}$.

The symbols like $\{u_n\}, \{v_n\}, \{x_n\}$, etc. shall also be used to denote a sequence.

Examples.

1. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = n, n \in \mathbb{N}$. Then $f(1) = 1, f(2) = 2, \ldots \ldots$ The sequence is denoted by $\{n\}$. It is also denoted by $\{1, 2, 3, \ldots \ldots\}$.

2. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = n^2, n \in \mathbb{N}$. The sequence is $\{n^2\}$. It is also denoted by $\{1^2, 2^2, 3^2, \ldots \ldots\}$.

3. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = \frac{n}{n+1}, n \in \mathbb{N}$. The sequence is $\{\frac{n}{n+1}\}$. It is also denoted by $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \ldots\}$.

4. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = (-1)^n, n \in \mathbb{N}$. The sequence is $\{(-1)^n\}$. It is also denoted by $\{-1, 1, -1, \ldots \ldots\}$. The range of the sequence is $\{-1, 1\}$.

5. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = \sin \frac{n\pi}{2}, n \in \mathbb{N}$. The sequence is $\{1, 0, -1, 0, 1, 0, \ldots \ldots\}$. The range of the sequence is $\{-1, 0, 1\}$.

6. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = 2$ for all $n \in \mathbb{N}$. The sequence is $\{2, 2, 2, \ldots \ldots\}$. It is called a *constant sequence* .

Sometimes it is convenient to specify $f(1)$ and describe $f(n+1)$ in terms of $f(n)$ for all $n \geq 1$.

For example, $f(1) = \sqrt{2}$ and $f(n+1) = \sqrt{2f(n)}$ for $n \geq 1$ defines the sequence $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2}\sqrt{2}}, \ldots \ldots\}$.

5.2. Bounded Sequence.

A real sequence $\{f(n)\}$ is said to be *bounded above* if there exists a real number G such that $f(n) \leq G$ for all $n \in \mathbb{N}$. G is said to be *an upper bound* of the sequence.

A real sequence $\{f(n)\}$ is said to be *bounded below* if there exists a real number g such that $f(n) \geq g$ for all $n \in \mathbb{N}$. g is said to be *a lower bound* of the sequence.

A real sequence $\{f(n)\}$ is said to be a *bounded sequence* if there exist real numbers G, g such that $g \leq f(n) \leq G$ for all $n \in \mathbb{N}$.

Therefore a real sequence is bounded if and only if it is bounded above as well as bounded below. In this case, the range of the sequence is a bounded set.

For a real sequence $\{f(n)\}$ bounded above, the range of the sequence is a set bounded above and by the supremum property of $\mathbb{R}$, the range set has the least upper bound, which is also called the *least upper bound* of the sequence $\{f(n)\}$ and is denoted by $\sup\{f(n)\}$.

The least upper bound of a real sequence $\{f(n)\}$ is a real number M satisfying the following conditions :

(i) $f(n) \leq M$ for all $n \in \mathbb{N}$,

(ii) for each pre-assiged positive ϵ, there exists a *natural number* k such that $f(k) > M - \epsilon$.

By similar arguments, for a real sequence $\{f(n)\}$ bounded below, there exists a *greatest lower bound* and it is denoted by $\inf\{f(n)\}$.

The greatest lower bound of a real sequence $\{f(n)\}$ is a real number m satisfying the following conditions :

(i) $f(n) \geq m$ for all $n \in \mathbb{N}$,

(ii) for each pre-assigned positive ϵ, there exists a *natural number* k such that $f(k) < m + \epsilon$.

For a real sequence $\{f(n)\}$ unbounded above, we define $\sup\{f(n)\} = \infty$

For a real sequence $\{f(n)\}$ unbounded below, we define $\inf\{f(n)\} = -\infty$.

Examples.

1. The sequence $\{\frac{1}{n}\}$ is a bounded sequence. 0 is the greatest lower bound and 1 is the least upper bound of the sequence.

2. The sequence $\{n^2\}$ is bounded below and unbounded above. Here

$\sup\{f(n)\} = \infty, \inf\{f(n)\} = 1$.

3. The sequence $\{-2n\}$ is bounded above and unbounded below. Here $\sup\{f(n)\} = -2, \inf\{f(n)\} = -\infty$.

4. Let $f(n) = (-1)^n n, n \in \mathbb{N}$. The sequence $\{f(n)\}$ is unbounded above and unbounded below. The sequence is $\{-1, 2, -3, 4, \ldots \ldots\}$.

Here $\sup\{f(n)\} = \infty, \inf\{f(n)\} = -\infty$.

5.3. Limit of a sequence.

Let $\{f(n)\}$ be a real sequence. A *real number* l is said to be a *limit* of the sequence $\{f(n)\}$ if corresponding to a pre-assigned positive ϵ there exists a *natural number* k (depending on ϵ) such that

$$| f(n) - l | < \epsilon \text{ for all } n \geq k$$
$$\text{i.e., } l - \epsilon < f(n) < l + \epsilon \text{ for all } n \geq k.$$

To be explicit, a real number l is said to be a limit of the sequence $\{f(n)\}$, if for a pre-assigned positive ϵ there exists a natural number k such that *all elements* of the sequence, excepting the first $k-1$ at most, lie in the ϵ- neighbourhood of l.

Theorem 5.3.1. A sequence can have at most one limit.

Proof. If possible, let a sequence $\{f(n)\}$ have two distinct limits l_1 and l_2 where $l_1 < l_2$.

Let $\epsilon = \frac{1}{2}(l_2 - l_1)$. Then $\epsilon > 0$ and $l_1 + \epsilon = l_2 - \epsilon$. Therefore the neighbourhoods $(l_1 - \epsilon, l_1 + \epsilon)$ and $(l_2 - \epsilon, l_2 + \epsilon)$ are disjoint.

Since l_1 is a limit of the sequence, for the chosen ϵ, there exists a natural number k_1 such that

$$l_1 - \epsilon < f(n) < l_1 + \epsilon \text{ for all } n \geq k_1.$$

Since l_2 is a limit of the sequence, for the same chosen ϵ, there exists a natural number k_2 such that

$$l_2 - \epsilon < f(n) < l_2 + \epsilon \text{ for all } n \geq k_2.$$

Let $k = \max\{k_1, k_2\}$.

Then $l_1 - \epsilon < f(n) < l_1 + \epsilon$ and $l_2 - \epsilon < f(n) < l_2 + \epsilon$ for all $n \geq k$.

This cannot happen since the neighbourhoods $N(l_1, \epsilon)$ and $N(l_2, \epsilon)$ are disjoint. Therefore our assumption that $l_1 \neq l_2$ is wrong.

Hence $l_1 = l_2$ and this proves the theorem.

5.4. Convergent sequence.

A real sequence $\{f(n)\}$ is said to be a *convergent sequence* if it has a limit $l \in \mathbb{R}$. In this case the sequence is said to converge to l.

We write $\lim\limits_{n \to \infty} f(n) = l$, or $\lim f(n) = l$.

A sequence is said to be a *divergent sequence* if it is not convergent.

Examples.

1. The sequence $\{\frac{1}{n}\}$ converges to 0.

Let us choose a positive ϵ.

By Archimedean property of $\mathbb{R}$, there exists a natural number k such that $0 < \frac{1}{k} < \epsilon$. This implies $0 < \frac{1}{n} < \epsilon$ for all $n \geq k$.

It follows that $|\frac{1}{n} - 0| < \epsilon$ for all $n \geq k$.

This proves $\lim \frac{1}{n} = 0$.

2. The sequence $\{\frac{n^2+1}{n^2}\}$ converges to 1.

Let us choose a positive ϵ.

Now $|\frac{n^2+1}{n^2} - 1| < \epsilon$ will hold if $\frac{1}{n^2} < \epsilon$, i.e., if $n > \frac{1}{\sqrt{\epsilon}}$.

Let $k = [\frac{1}{\sqrt{\epsilon}}] + 1$. [For example, if $\epsilon = .01$ then $k = 11$; if $\epsilon = .001$ then $k = 32$.] Then k is a natural number and $|\frac{n^2+1}{n^2} - 1| < \epsilon$ for all $n \geq k$.

This proves $\lim \frac{n^2+1}{n^2} = 1$.

3. Let $f(n) = 2$ for all $n \in \mathbb{N}$. The sequence is $\{2, 2, 2, \ldots \ldots\}$. We prove that the sequence converges to 2.

Let us choose a positive ϵ.

Now $|f(n) - 2| < \epsilon$ holds for all $n \geq 1$.

Therefore $\lim f(n) = 2$.

Note. A constant sequence is a convergent sequence.

Theorem 5.4.1. A convergent sequence is bounded.

Proof. Let $\{f(n)\}$ be a convergent sequence and let l be its limit. Let us choose $\epsilon = 1$. For this chosen ϵ there exists a natural number k such that $\quad l - 1 < f(n) < l + 1$ for all $n \geq k$.

Let $B = \max\{f(1), f(2), \ldots, f(k-1), l+1\}$;
$\quad b = \min\{f(1), f(2), \ldots, f(k-1), l-1\}$.

Then $b \leq f(n) \leq B$ for all $n \in \mathbb{N}$.

This proves that the sequence $\{f(n)\}$ is a bounded sequence.

Corollary. An unbounded sequence is not convergent.

Note. A bounded sequence may not be a convergent sequence.

For example, the sequence $\{(-1)^n\}$ is a bounded sequence but the sequence does not converge to a limit.

5.5. Limit theorems.

Theorem 5.5.1. Let $\{u_n\}$ and $\{v_n\}$ be two convergent sequences that converge to u and v respectively.

Then (i) $\lim(u_n + v_n) = u + v$;

(ii) if $c \in \mathbb{R}, \lim(cu_n) = cu$;

(iii) $\lim u_n v_n = uv$;

(iv) $\lim \frac{u_n}{v_n} = \frac{u}{v}$, provided $\{v_n\}$ is a sequence of non zero real numbers and $v \neq 0$.

Proof. (i) To show that $\lim(u_n + v_n) = u + v$, we need to establish that for a pre-assigned positive ϵ there exists a natural number k such that $|(u_n + v_n) - (u + v)| < \epsilon$ for all $n \geq k$.

Using triangle inequality, we have

$|(u_n + v_n) - (u + v)| = |(u_n - u) + (v_n - v)| < |u_n - u| + |v_n - v|$.

Let $\epsilon > 0$. Since $\lim u_n = u$, there exists a natural number k_1 such that $|u_n - u| < \frac{\epsilon}{2}$ for all $n \geq k_1$.

Since $\lim v_n = v$, there exists a natural number k_2 such that

$|v_n - v| < \frac{\epsilon}{2}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $|u_n - u| < \frac{\epsilon}{2}$ and $|v_n - v| < \frac{\epsilon}{2}$ for all $n \geq k$. It follows that $|(u_n + v_n) - (u + v)| < \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, $\lim(u_n + v_n) = u + v$.

(ii) Let us assume $c \neq 0$. When $c = 0$ the theorem is obvious.

To show that $\lim cu_n = cu$, we need to establish that for a pre-assigned positive ϵ there exists a natural number k such that

$|cu_n - cu| < \epsilon$ for all $n \geq k$.

We have $|cu_n - cu| = |c||u_n - u|$.

Let $\epsilon > 0$. Since $\lim u_n = u$, there exists a natural number k such that $|u_n - u| < \frac{\epsilon}{|c|}$ for all $n \geq k$.

It follows that $|cu_n - cu| < \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, $\lim cu_n = cu$.

(iii) To show that $\lim u_n v_n = uv$, we need to establish that for a pre-assigned positive ϵ there exists a natural number k such that

$|u_n v_n - uv| < \epsilon$ for all $n \geq k$.

We have $|u_n v_n - uv| = |u_n(v_n - v) + v(u_n - u)|$

$\leq |u_n||v_n - v| + |v||u_n - u|$.

Since $\{u_n\}$ is a convergent sequence, it is bounded. Therefore there exists a positive number B_1 such that $|u_n| < B_1$ for all $n \in \mathbb{N}$.

Let $B = \max\{B_1, |v|\}$.

Then $|u_n v_n - uv| < B|v_n - v| + B|u_n - u|$.

Let $\epsilon > 0$. Since $\lim u_n = u$ and $\lim v_n = v$, there exist natural numbers k_1 and k_2 such that

$| u_n - u | < \frac{\epsilon}{2B}$ for all $n \geq k_1$ and $| v_n - v | < \frac{\epsilon}{2B}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $| u_n - u | < \frac{\epsilon}{2B}$ and $| v_n - v | < \frac{\epsilon}{2B}$ for all $n \geq k$.

It follows that $| u_n v_n - uv | < B.\frac{\epsilon}{2B} + B.\frac{\epsilon}{2B}$ for all $n \geq k$
or, $| u_n v_n - uv | < \epsilon$ for all $n \geq k$.
Since ϵ is arbitrary, $\lim u_n v_n = uv$.

(iv) First we prove that if $\lim v_n = v$ where $\{v_n\}$ is a sequence of non-zero real numbers and $v \neq 0, \lim 1/v_n = 1/v$.

Let $\alpha = \frac{1}{2} | v |$. Then $\alpha > 0$. Since $\lim v_n = v$, there exists a natural number k_1 such that

$| v_n - v | < \alpha$ for all $n \geq k_1$.
We have $| \, | v_n | - | v | \, | \leq v_n - v | < \alpha$ for all $n \geq k_1$
or, $| v | - \alpha < | v_n | < | v | + \alpha$ for all $n \geq k_1$.
Therefore $| v_n | > \frac{1}{2} | v |$ for all $n \geq k_1$.
Now $| \frac{1}{v_n} - \frac{1}{v} | = \frac{|v_n - v|}{|v||v_n|} < \frac{2}{|v|^2} | v_n - v |$ for all $n \geq k_1$.

Let $\epsilon > 0$. Since $\lim v_n = v$, there exists a natural number k_2 such that $| v_n - v | < \frac{|v|^2}{2}\epsilon$ for all $n \geq k_2$.
Let $k = \max\{k_1, k_2\}$. Then $| \frac{1}{v_n} - \frac{1}{v} | < \epsilon$ for all $n \geq k$.
Since ϵ is arbitrary, $\lim \frac{1}{v_n} = \frac{1}{v}$.

The proof of the theorem is now completed by considering the convergence of the product of two sequences $\{u_n\}$ and $\{\frac{1}{v_n}\}$.

Therefore $\lim \frac{u_n}{v_n} = \lim(u_n.\frac{1}{v_n}) = u.\frac{1}{v} = \frac{u}{v}$.

Note. If $\{u_n\}, \{v_n\}, \{w_n\}$ be three convergent sequences of real numbers that converge to u, v, w respectively, then

(i) $\lim(u_n + v_n + w_n) = u + v + w$ and
(ii) $\lim(u_n v_n w_n) = uvw$.

The theorem can be generalised to the sum and the product of a *finite* number of convergent sequences.

Theorem 5.5.2. Let $\{u_n\}$ be a convergent sequence of real numbers converging to u. Then the sequence $\{| u_n |\}$ converges to $| u |$.

Proof. We have $| \, | u_n | - | u | \, | \leq | u_n - u |$.

Let $\epsilon > 0$. Since $\lim u_n = u$, there exists a natural number k such that $| u_n - u | < \epsilon$ for all $n \geq k$.
It follows that $| \, | u_n | - | u | \, | < \epsilon$ for all $n \geq k$.
Since ϵ is arbitrary, $\lim | u_n | = | u |$.

Note 1. The converse of the theorem is not true. That is, if $\{|u_n|\}$ is a convergent sequence it does not necessarily imply that $\{u_n\}$ is a convergent sequence.

For example, let $u_n = (-1)^n$. Then the sequence $\{|u_n|\}$ converges to 1 but the sequence $\{u_n\}$ is a divergent sequence.

Note 2. The theorem states that $\lim |u_n| = |\lim u_n|$, provided the limit in R.H. S. exists.

Theorem 5.5.3. Let $\{u_n\}$ be a convergent sequence of real numbers and there exists a natural number m such that $u_n > 0$ for all $n \geq m$. Then $\lim u_n \geq 0$.

Proof. Let $\lim u_n = u$ and if possible let $u < 0$.

Let us choose a positive ϵ such that $u + \epsilon < 0$.

Since $\lim u_n = u$, there exists a natural number k_1 such that

$$u - \epsilon < u_n < u + \epsilon \text{ for all } n \geq k_1.$$

Let $k = \max\{k_1, m\}$.

Then by hypothesis, $u_n > 0$ for all $n \geq k$ and we have from above $u_n < u + \epsilon < 0$ for all $n \geq k$.

This is a contradiction. Therefore $\lim u_n \geq 0$.

Note 1. The theorem also says that a convergent sequence of positive numbers may converge to 0. For example, for the sequence $\{u_n\}$ where $u_n = \frac{1}{n}$; $u_n > 0$ for all $n \in \mathbb{N}$ but $\lim u_n = 0$.

Note 2. If $\{u_n\}$ be a convergent sequence and $u_n \geq 0$ for all $n \geq m$ (m being a natural number) then $\lim u_n \geq 0$.

Theorem 5.5.4. Let $\{u_n\}$ and $\{v_n\}$ be two convergent sequences and there exists a natural number m such that $u_n > v_n$ for all $n \geq m$.

Then $\lim u_n \geq \lim v_n$.

Proof. Let $\lim u_n = u, \lim v_n = v$ and $w_n = u_n - v_n$.

Then $\{w_n\}$ is a convergent sequence such that $w_n > 0$ for all $n \geq m$ and $\lim w_n = u - v$.

By the previous theorem, $u - v \geq 0$.

Consequently, $\lim u_n \geq \lim v_n$.

Note. If $\{u_n\}$ and $\{v_n\}$ be two convergent sequences and $u_n \geq v_n$ for all $n \geq m$ then $\lim u_n \geq \lim v_n$.

If $w_n = u_n - v_n$ then $\{w_n\}$ is a convergent sequence such that $w_n \geq 0$ for all $n \geq m$ and $\lim w_n = u - v$.

So $u - v \geq 0$ and therefore $\lim u_n \geq \lim v_n$.

Corollary 1. If $\{x_n\}$ is a convergent sequence of points in $[a, b]$ and $\lim x_n = c$, then $c \in [a, b]$.

Corollary 2. If $\{x_n\}$ is a convergent sequence of points in (a, b) and $\lim x_n = c$, then $c \in [a, b]$. [Here c may not be in (a, b)].

Theorem 5.5.5. (Sandwich theorem)

Let $\{u_n\}, \{v_n\}, \{w_n\}$ be three sequences of real numbers and there is a natural number m such that

$$u_n < v_n < w_n \text{ for all } n \geq m.$$

If $\lim u_n = \lim w_n = l$ then $\{v_n\}$ is convergent and $\lim v_n = l$.

Proof. Let $\epsilon > 0$. It follows from the convergence of the sequences $\{u_n\}$ and $\{w_n\}$ that there exist natural numbers k_1 and k_2 such that

$| u_n - l |< \epsilon$ for all $n \geq k_1$ and $| w_n - l |< \epsilon$ for all $n \geq k_2$.

Let $k_3 = \max\{k_1, k_2\}$.

Then $l - \epsilon < u_n < l + \epsilon$ and $l - \epsilon < w_n < l + \epsilon$ for all $n \geq k_3$.

Let $k = \max\{k_3, m\}$.

Then $l - \epsilon < u_n < v_n < w_n < l + \epsilon$ for all $n \geq k$.

Consequently, $| v_n - l |< \epsilon$ for all $n \geq k$.

This shows that the sequence $\{v_n\}$ is convergent and $\lim v_n = l$.

Note. If $u_n \leq v_n \leq w_n$ for all $n \geq m$ and $\lim u_n = \lim w_n = l$ then $\lim v_n = l$.

Worked Examples.

1. Prove that $\lim\limits_{n\to\infty} \frac{3n^2+2n+1}{n^2+1} = 3$.

$\lim\limits_{n\to\infty} \frac{3n^2+2n+1}{n^2+1} = \lim\limits_{n\to\infty} \frac{u_n}{v_n}$ where $u_n = 3 + \frac{2}{n} + \frac{1}{n^2}$ and $v_n = 1 + \frac{1}{n^2}$.

But $\lim u_n = 3$ and $\lim v_n = 1$.

Therefore $\lim\limits_{n\to\infty} \frac{3n^2+2n+1}{n^2+1} = \lim \frac{u_n}{v_n} = 3$.

2. Prove that $\lim\limits_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0$.

$$\begin{aligned} \lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) &= \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} \\ &= \lim_{n\to\infty} u_n v_n \text{ where } u_n = \tfrac{1}{\sqrt{n}}, v_n = \tfrac{1}{\sqrt{1+\frac{1}{n}}+1} \\ &= 0, \text{ since } \lim u_n = 0 \text{ and } \lim v_n = \tfrac{1}{2}. \end{aligned}$$

3. Prove that

$\lim\limits_{n\to\infty} (\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}) = 1$.

Let $u_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}$.

We have $\frac{1}{\sqrt{n^2+2}} < \frac{1}{\sqrt{n^2+1}}$

$\frac{1}{\sqrt{n^2+3}} < \frac{1}{\sqrt{n^2+1}}$

$\cdots \quad \cdots$

$\frac{1}{\sqrt{n^2+n}} < \frac{1}{\sqrt{n^2+1}}$.

Therefore $u_n < \frac{n}{\sqrt{n^2+1}}$ for all $n \geq 2$.

Again, $\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} > \frac{2}{\sqrt{n^2+2}}$

$\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \frac{1}{\sqrt{n^2+3}} > \frac{3}{\sqrt{n^2+3}}$

$\cdots \quad \cdots \quad \cdots \quad \cdots$

Therefore $u_n > \frac{n}{\sqrt{n^2+n}}$ for all $n \geq 2$.

Thus $\frac{n}{\sqrt{n^2+n}} < u_n < \frac{n}{\sqrt{n^2+1}}$ for all $n \geq 2$.

But $\lim\limits_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = 1$ and $\lim\limits_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1$.

By Sandwich theorem, $\lim u_n = 1$.

5.6. Null sequence.

A sequence $\{u_n\}$ is said to be a *null sequence* if $\lim u_n = 0$.

Theorem 5.6.1. If $\{u_n\}$ be null sequence then $\{|u_n|\}$ is a null sequence and conversely.

Proof. Let $\epsilon > 0$. Since $\lim u_n = 0$, there exists a natural number k such that $|u_n| < \epsilon$ for all $n \geq k$.

As $||u_n| - 0| = |u_n|$, it follows that $||u_n| - 0| < \epsilon$ for all $n \geq k$.

This proves $\lim |u_n| = 0$.

Conversely, let $\lim |u_n| = 0$.

Let $\epsilon > 0$. There exists a natural number k such that $||u_n| - 0| < \epsilon$ for all $n \geq k$. That is, $|u_n| < \epsilon$ for all $n \geq k$.

This proves $\lim u_n = 0$.

5.7. Divergent sequence.

A real sequence $\{f(n)\}$ is said to *diverge* to ∞ if corresponding to a pre-assigned positive number G, however large, there exists a natural number k such that

$$f(n) > G \text{ for all } n \geq k.$$

In this case we write $\lim f(n) = \infty$ and also say that the sequence $\{f(n)\}$ tends to ∞.

A real sequence $\{f(n)\}$ is said to diverge to $-\infty$ if corresponding to a pre-assigned positive number G, however large, there exists a natural number k such that

$$f(n) < -G \text{ for all } n \geq k.$$

In this case we write $\lim f(n) = -\infty$ and also say that the sequence $\{f(n)\}$ tends to $-\infty$.

A real sequence $\{f(n)\}$ is said to be a *properly divergent sequence* if it either diverges to ∞, or diverges to $-\infty$.

Theorem 5.7.1. A sequence diverging to ∞ is unbounded above but bounded below.

Proof. Let a sequence $\{f(n)\}$ diverge to ∞. Then for each pre-assigned positive number G there exists a natural number k such that $f(k) > G$.

Therefore there does not exist a real number B such that $f(n) \leq B$ holds for all $n \in \mathbb{N}$. In other words, $\{f(n)\}$ is unbounded above.

Let $G > 0$. Then there exists a natural number k such that
$$f(n) > G \text{ for all } n \geq k.$$

Let $b = \min\{f(1), f(2), \ldots, f(k-1), G\}$. Then $f(n) \geq b$ for all $n \in \mathbb{N}$. This proves that the sequence $\{f(n)\}$ is bounded below.

Note. A sequence unbounded above but bounded below may not diverge to ∞.

For example, let us consider the sequence $\{f(n)\}$ where $f(n) = n^{(-1)^n}$. The sequence is $\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, \ldots \ldots\}$.

The sequence is unbounded above and bounded below, 0 being a lower bound. The sequence does not diverge to ∞, because for a pre-assigned positive number G there does not exist a natural number k such that $f(n) > G$ holds for all $n \geq k$.

Theorem 5.7.2. A sequence diverging to $-\infty$ is unbounded below but bounded above.

Proof left to the reader.

Note. A sequence unbounded below but bounded above may not diverge to $-\infty$.

Definitions. A bounded sequence that is not convergent is said to be an *oscillatory sequence of finite oscillation.*

An unbounded sequence that is not properly divergent is said to be an *oscillatory sequence of infinite oscillation.*

An oscillatory sequence is therefore neither convergent nor properly divergent. It is called an *improperly divergent sequence.*

Examples.

1. The sequence $\{2^n\}$ diverges to ∞.
2. The sequence $\{-n^2\}$ diverges to $-\infty$.

3. The sequence $\{(-1)^n\}$ is a bounded sequence, but not convergent. It is an oscillatory sequence of finite oscillation.

4. The sequence $\{(-1)^n n\}$ is an unbounded sequence, and it is not properly divergent. It is an oscillatory sequence of infinite oscillation.

5.8. Some important limits.

1. $\lim r^n = 0$ if $|r| < 1$.

Case 1. $r = 0$. In this case the sequence is $\{0, 0, 0, \ldots \ldots\}$.
The sequence converges to 0.
That is, $\lim r^n = 0$ when $r = 0$.

Case 2. $r \neq 0$ and $|r| < 1$.
$\frac{1}{|r|} > 1$, since $|r| < 1$. Let $\frac{1}{|r|} = a + 1$ where $a > 0$.
$|r^n - 0| = |r^n| = |r|^n = \frac{1}{(a+1)^n}$.
We have $(1 + a)^n > na$ for all $n \in \mathbb{N}$.
So $|r^n - 0| < \frac{1}{na}$ for all $n \in \mathbb{N}$.

Let $\epsilon > 0$. Then $|r^n - 0| < \epsilon$ holds if $n > \frac{1}{a\epsilon}$.
Let $k = [\frac{1}{a\epsilon}] + 1$. Then k is a natural number and $|r^n - 0| < \epsilon$ for all $n \geq k$.
Since ϵ is arbitrary, $\lim r^n = 0$.

Combining the cases, $\lim r^n = 0$ if $|r| < 1$.

2. $\lim a^{1/n} = 1$ if $a > 0$.

Case 1. $a = 1$. In this case the sequence converges to 1.

Case 2. $a > 1$. Then $a^{1/n} > 1$. Let $a^{1/n} = 1 + x_n$ where $x_n > 0$.
Then $a = (1 + x_n)^n$
$> 1 + nx_n$ for $n > 1$.
Let $\epsilon > 0$. Then $|a^{1/n} - 1| < \epsilon$ holds if $\frac{a-1}{n} < \epsilon$ i.e., if $n > \frac{a-1}{\epsilon}$.
Let $k = [\frac{a-1}{\epsilon}] + 1$. Then k is a natural number and $|a^{1/n} - 1| < \epsilon$ for all $n \geq k$.
Since ϵ is arbitrary, $\lim a^{1/n} = 1$.

Case 3. $0 < a < 1$. Let $b = 1/a$. Then $b > 1$ and
$\lim a^{1/n} = \lim \frac{1}{b^{1/n}} = 1$, by case 2.
Combining the cases, $\lim a^{1/n} = 1$ if $a > 0$.

3. If $\lim x_n = 0$ and $a > 0$, then $\lim a^{x_n} = 1$.

We have $\lim a^{1/n} = 1$ and $\lim a^{-1/n} = 1$.
Let us choose $\epsilon > 0$. There exist natural numbers k_1, k_2 such that

$1-\epsilon < a^{1/n} < 1+\epsilon$ for all $n \geq k_1$ and $1-\epsilon < a^{-1/n} < 1+\epsilon$ for all $n \geq k_2$

Let $k = \max\{k_1, k_2\}$.
Then $1-\epsilon < a^{1/k} < 1+\epsilon$ and $1-\epsilon < a^{-1/k} < 1+\epsilon$.
Since $\lim x_n = 0$, there exists a natural number p such that
$-\frac{1}{k} < x_n < \frac{1}{k}$ for all $n \geq p$.

Let $a > 1$. Then $a^{-1/k} < a^{x_n} < a^{1/k}$ for all $n \geq p$
or, $1-\epsilon < a^{-1/k} < a^{x_n} < a^{1/k} < 1+\epsilon$ for all $n \geq p$.

Let $0 < a < 1$. Then $a^{1/k} < a^{x_n} < a^{-1/k}$ for all $n \geq p$
or, $1-\epsilon < a^{1/k} < a^{x_n} < a^{-1/k} < 1+\epsilon$ for all $n \geq p$.
Therefore if $a > 0, 1-\epsilon < a^{x_n} < 1+\epsilon$ for all $n \geq p$.
This implies $\lim a^{x_n} = 1$.

Corollary 1. If $\lim x_n = l$ and $a > 0$, then $\lim a^{x_n} = a^l$.

Corollary 2. If $\lim x_n = l$, then $\lim e^{x_n} = e^l$.

4. If $\lim x_n = 0$, then $\lim \log(1+x_n) = 0$.

Let $\epsilon > 0$. $-\epsilon < \log(1+x_n) < \epsilon$ will hold if $e^{-\epsilon} - 1 < x_n < e^{\epsilon} - 1$
Since $\epsilon > 0, e^{\epsilon} - 1 > 0$ and $e^{-\epsilon} - 1 < 0$.

By Archimedean property of $\mathbb{R}$, there exists a natural number m_1 such that $0 < \frac{1}{m_1} < e^{\epsilon} - 1$ and also there exists a natural number m_2 such that $0 < \frac{1}{m_2} < 1 - e^{-\epsilon}$.

Let $m = \max\{m_1, m_2\}$.
Then $0 < \frac{1}{m} < e^{\epsilon} - 1$ and $0 < \frac{1}{m} < 1 - e^{-\epsilon}$.
Combining, $e^{-\epsilon} - 1 < -\frac{1}{m} < \frac{1}{m} < e^{\epsilon} - 1$.
Since $\lim x_n = 0$, there exists a natural number k such that
$| x_n - 0 | < \frac{1}{m}$ for all $n \geq k$
or, $-\frac{1}{m} < x_n < \frac{1}{m}$ for all $n \geq k$.
Consequently, $e^{-\epsilon} - 1 < x_n < e^{\epsilon} - 1$ for all $n \geq k$
or, $-\epsilon < \log(1+x_n) < \epsilon$ for all $n \geq k$.
This proves $\lim \log(1+x_n) = 0$.

Corollary. Let $\{x_n\}$ be a sequence such that $x_n > 0$ for all $n \in \mathbb{N}$ and $\lim x_n = c > 0$. Then $\lim(\log x_n - \log c) = \lim \log \frac{x_n}{c} = \lim \log(1 + \frac{x_n - c}{c}) = 0$, since $\lim \frac{x_n - c}{c} = 0$.

Therefore $\lim \log x_n = \log c$.

5. If $u_n > 0$ and $\lim u_n = u > 0$ for all $n \in \mathbb{N}$ and $\lim v_n = v$, then $\lim (u_n)^{v_n} = u^v$.

By definition, $(u_n)^{v_n} = e^{v_n \log u_n}$.
As $\lim u_n = u, \lim \log u_n = \log u$. So $\lim(v_n \log u_n) = v \log u$.

By corollary 2 of Ex. 3, $\lim e^{v_n \log u_n} = e^{v \log u} = u^v$
or, $\lim (u_n)^{v_n} = u^v$.

6. $\lim n^{1/n} = 1$.

$n^{1/n} > 1$ for all $n > 1$.
Let $n^{1/n} = 1 + x_n$ where $x_n > 0$.
Then $n = (1 + x_n)^n$

$$= 1 + nx_n + \tfrac{n(n-1)}{2} x_n^2 + \cdots + x_n^n$$
$$> \tfrac{1}{2} n(n-1) x_n^2.$$

Clearly, $x_n^2 < \frac{2}{n-1}$ for all $n > 1$
or, $| x_n | < \frac{\sqrt{2}}{\sqrt{n-1}}$.
Let $\epsilon > 0$. Then $| n^{1/n} - 1 | = | x_n | < \epsilon$ holds if $n > 1 + \frac{2}{\epsilon^2}$.

Let $k = [1 + \frac{2}{\epsilon^2}] + 1$. Then k is a natural number and $| n^{1/n} - 1 | < \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, $\lim n^{1/n} = 1$.

7. Behaviour of the sequence $\{r^n\}$ for different real values of r.

Case 1. $r > 1$. Let $r = 1 + a$ where $a > 0$.
Then $r^n = (1 + a)^n > 1 + na$ for $n > 1$.
Let $G > 0$. Then $1 + na > G$ holds if $n > \frac{G-1}{a}$.

Let $k = [\frac{G-1}{a}] + 1$. Then k is a natural number and $r^n > G$ for all $n \geq k$.

Since G is an arbitrary positive number, $\lim r^n = \infty$.
Therefore in this case the sequence diverges to ∞.

Case 2. $r = 1$. In this case the sequence is $\{1, 1, 1, \ldots \ldots\}$ and the sequence converges to 1.

Case 3. $| r | < 1$. In this case the sequence converges to 0, by Example 1.

Case 4. $r = -1$. In this case the sequence is $\{-1, 1, -1, \ldots \ldots\}$. The sequence is bounded but not convergent. The sequence is an oscillatory sequence of finite oscillation.

Case 5. $r < -1$. let $r = -s$. Then $s > 1$.

The sequence is $\{(-1)^n s^n\}$. It is an unbounded sequence. It neither diverges to ∞ nor diverges to $-\infty$. It is an oscillatory sequence of infinite oscillation.

Theorem 5.8.1. Let $\{u_n\}$ be a sequence of positive real numbers such that $\lim \frac{u_{n+1}}{u_n} = l$.

(i) If $0 \leq l < 1$ then $\lim u_n = 0$,

(ii) if $l > 1$ then $\lim u_n = \infty$.

Proof. (i) Let us choose a positive ϵ such that $l + \epsilon < 1$.

Since $\lim \frac{u_{n+1}}{u_n} = l$, there exists a natural number k such that
$$l - \epsilon < \frac{u_{n+1}}{u_n} < l + \epsilon \text{ for all } n \geq k.$$
Let $l + \epsilon = r$. Then $0 < r < 1$.

Therefore $\frac{u_{n+1}}{u_n} < r$ for all $n \geq k$.

Hence we have $\frac{u_{k+1}}{u_k} < r, \frac{u_{k+2}}{u_{k+1}} < r, \cdots, \frac{u_n}{u_{n-1}} < r$ for $n \geq k+1$.

Multiplying, $\frac{u_n}{u_k} < r^{n-k}$ for $n \geq k+1$

or, $u_n < \frac{u_k}{r^k}.r^n$ for $n \geq k+1$.

Now $\lim r^n = 0$ since $0 < r < 1$; and $\frac{u_k}{r^k}$ is a fixed positive number.

Therefore $\lim u_n = 0$.

(ii) Let us choose a positive ϵ such that $l - \epsilon > 1$.

Since $\lim \frac{u_{n+1}}{u_n} = l$, there exists a natural number m such that
$$l - \epsilon < \frac{u_{n+1}}{u_n} < l + \epsilon \text{ for all } n \geq m.$$
Let $l - \epsilon = s$. Then $s > 1$.

Therefore $\frac{u_{n+1}}{u_n} > s$ for all $n \geq m$.

Hence we have $\frac{u_{m+1}}{u_m} > s, \frac{u_{m+2}}{u_{m+1}} > s, \cdots, \frac{u_n}{u_{n-1}} > s$ for $n \geq m+1$.

Multiplying, $\frac{u_n}{u_m} > s^{n-m}$ for $n \geq m+1$

or, $u_n > \frac{u_m}{s^m}.s^n$ for $n \geq m+1$.

Now $\lim s^n = \infty$ since $s > 1$; and $\frac{u_m}{s^m}$ is a fixed positive number.

Therefore $\lim u_n = \infty$.

Note. If $\lim \frac{u_{n+1}}{u_n} = 1$, no definite conclusion can be made about the nature of the sequence. For example, (i) if $u_n = \frac{n+1}{n}$ then $\lim \frac{u_{n+1}}{u_n} = 1$ and $\lim u_n = 1$; (ii) if $u_n = \frac{1}{n}$ then $\lim \frac{u_{n+1}}{u_n} = 1$ and $\lim u_n = 0$.

Theorem 5.8.2. Let $\{u_n\}$ be a sequence of positive real numbers such that $\lim \sqrt[n]{u_n} = l$.

(i) If $0 \leq l < 1$ then $\lim u_n = 0$.

(ii) If $l > 1$ then $\lim u_n = \infty$.

Proof. (i) Let us choose a positive ϵ such that $l + \epsilon < 1$.

Since $\lim \sqrt[n]{u_n} = l$, there exists a natural number k such that
$$l - \epsilon < \sqrt[n]{u_n} < l + \epsilon \text{ for all } n \geq k.$$
Let $l + \epsilon = r$. Then $0 < r < 1$ and $\sqrt[n]{u_n} < r$ for all $n \geq k$.

So we have $0 < u_n < r^n$ for all $n \geq k$.

Since $\lim r^n = 0$, $\lim u_n = 0$, by Sandwich theorem.

(ii) Let us choose a positive ϵ such that $l - \epsilon > 1$.

Since $\lim \sqrt[n]{u_n} = l$, there exists a natural number m such that

$$l - \epsilon < \sqrt[n]{u_n} < l + \epsilon \text{ for all } n \geq m.$$

Let $l - \epsilon = s$. Then $s > 1$ and $\sqrt[n]{u_n} > s$ for all $n \geq m$.
So we have $u_n > s^n$ for all $n \geq m$.
Since $s > 1, \lim s^n = \infty$ and therefore $\lim u_n = \infty$.

Note. If $\lim \sqrt[n]{u_n} = 1$, no definite conclusion can be made about the nature of the sequence $\{u_n\}$.

For example, (i) if $u_n = \frac{n+1}{n}$ then $\lim \sqrt[n]{u_n} = 1$ and $\lim u_n = 1$; (ii) if $u_n = \frac{n+1}{2n}$ then $\lim \sqrt[n]{u_n} = 1$ and $\lim u_n = \frac{1}{2}$.

Worked Examples.

1. A sequence $\{u_n\}$ is defined by $u_{n+2} = \frac{1}{2}(u_{n+1} + u_n)$ for $n \geq 1$ and $0 < u_1 < u_2$. Prove that the sequence $\{u_n\}$ converges to $\frac{u_1+2u_2}{3}$.

$u_2 - u_1 > 0$
$u_3 - u_2 = \frac{1}{2}(u_2 + u_1) - u_2 = -\frac{1}{2}(u_2 - u_1)$
$u_4 - u_3 = \frac{1}{2}(u_3 + u_2) - u_3 = \frac{1}{2}(u_2 - u_3) = (-\frac{1}{2})^2(u_2 - u_1)$
$\ldots \quad \ldots \quad \ldots$
$u_n - u_{n-1} = (-\frac{1}{2})^{n-2}(u_2 - u_1)$.
Therefore $u_n - u_1 = (u_2 - u_1)[1 + (-\frac{1}{2}) + (-\frac{1}{2})^2 + \cdots + (-\frac{1}{2})^{n-2}]$
$= \frac{2(u_2-u_1)}{3}[1 - (-\frac{1}{2})^{n-1}]$.
Now $\lim(u_n - u_1) = \frac{2}{3}(u_2 - u_1)$ since $\lim(-\frac{1}{2})^{n-1} = 0$.

Therefore $\lim u_n = u_1 + \frac{2}{3}(u_2 - u_1) = \frac{u_1+2u_2}{3}$.

2. If $x_n = (a^n + b^n)^{1/n}$ for all $n \in \mathbb{N}$ and $0 < a < b$, show that $\lim x_n = b$.

$x_n = b[(\frac{a}{b})^n + 1]^{1/n}$
$> b$ for all $n \in \mathbb{N}$, since $(\frac{a}{b})^n + 1 > 1$ for all $n \in \mathbb{N}$.
Again, $0 < a < b \Rightarrow a^n < b^n$ for all $n \in \mathbb{N}$.
Therefore $a^n + b^n < 2b^n$
or, $x_n < 2^{1/n}.b$ for all $n \in \mathbb{N}$.

Let $u_n = b$ for all $n \in \mathbb{N}, v_n = 2^{1/n}b$ for all $n \in \mathbb{N}$.
Then $\lim u_n = b$ and $\lim v_n = b$ since $\lim 2^{1/n} = 1$.
Now $u_n < x_n < v_n$ for all $n \in \mathbb{N}$.
Since $\lim u_n = \lim v_n = b, \lim x_n = b$ by Sandwich theorem.

3. A sequence $\{u_n\}$ is defined by $u = \sqrt{2}$ and $u_{n+1} = \sqrt{2u_n}$ for $n \geq 1$. Prove that $\lim u_n = 2$.

$$\begin{aligned} u_1 = 2^{1/2}, u_2 = \sqrt{2\sqrt{2}} &= 2^{1/2+1/2^2} = 2^{1-\frac{1}{2^2}}, \\ u_3 &= 2^{1/2+1/2^2+1/2^3} = 2^{1-\frac{1}{2^3}}, \\ &\ldots \quad \ldots \\ u_n &= 2^{1/2+1/2^2+\cdots+1/2^n} = 2^{1-\frac{1}{2^n}}. \end{aligned}$$

$\lim u_n = \lim 2^{1-1/2^n} = \lim 2^{x_n}$ where $x_n = 1 - \frac{1}{2^n}$.

As $\lim x_n = 1$, we have $\lim u_n = \lim 2^{x_n} = 2$, since $\lim x_n = l$ and $a > 0 \Rightarrow \lim a^{x_n} = a^l$.

4. If $u_n > 0$ for all n and $\lim \sqrt[n]{u_n} = \mu > 0$ prove that $\lim \sqrt[n]{(n+1)u_{n+1}} = \mu$.

$\lim \sqrt[n]{n+1} = \lim\{(n+1)^{\frac{1}{n+1}}\}^{\frac{n+1}{n}}$.

Since $\lim n^{\frac{1}{n}} = 1$, it follows that $\lim(n+1)^{\frac{1}{n+1}} = 1$.

Since $\lim \frac{n+1}{n} = 1$ and $\lim(n+1)^{\frac{1}{n+1}} = 1$, we have $\lim \sqrt[n]{n+1} = 1$, by Ex 5 of 5.8.

$\lim \sqrt[n]{u_{n+1}} = \lim\{(u_{n+1})^{\frac{1}{n+1}}\}^{\frac{n+1}{n}}$.

Since $\lim \sqrt[n]{u_n} = \mu$, it follows that $\lim(u_{n+1})^{\frac{1}{n+1}} = \mu$.

Since $\lim \frac{n+1}{n} = 1$ and $\lim(u_{n+1})^{\frac{1}{n+1}} = \mu > 0$, we have $\lim \sqrt[n]{u_{n+1}} = \mu$.

Therefore $\lim \sqrt[n]{(n+1)u_{n+1}} = \lim(\sqrt[n]{n+1}.\sqrt[n]{u_{n+1}}) = \mu$.

5.9. Monotone sequence.

A real sequence $\{f(n)\}$ is said to be a *monotone increasing sequence* if $f(n+1) \geq f(n)$ for all $n \in \mathbb{N}$.

A real sequence $\{f(n)\}$ is said to be a *monotone decreasing sequence* if $f(n+1) \leq f(n)$ for all $n \in \mathbb{N}$.

A real sequence $\{f(n)\}$ is said to be a *monotone sequence* if it is either a monotone increasing sequence or a monotone decreasing sequence.

Note. If $f(n+1) > f(n)$ for all $n \in \mathbb{N}$, the sequence $\{f(n)\}$ is said to be a *strictly monotone increasing sequence.*

If $f(n+1) < f(n)$ for all $n \in \mathbb{N}$, the sequence $\{f(n)\}$ is said to be a *strictly monotone decreasing sequence.*

If for some natural number $m, f(n+1) \geq f(n)$ for all $n \geq m$ the sequence $\{f(n)\}$ is said to be an 'ultimately' monotone increasing sequence.

If for some natural number $m, f(n+1) \leq f(n)$ for all $n \geq m$ the sequence $\{f(n)\}$ is said to be an 'ultimately' monotone decreasing sequence.

Examples.

1. Let $f(n) = 2^n, n \geq 1$.

Then $f(n+1) > f(n)$ for all $n \in \mathbb{N}$.

Therefore the sequence $\{f(n)\}$ is a monotone increasing sequence. It is also strictly monotone.

2. Let $f(n) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}, n \geq 1$.

$$\begin{aligned} \text{Then } f(n+1) - f(n) &= \tfrac{1}{2n+1} + \tfrac{1}{2n+2} - \tfrac{1}{n+1} \\ &= \tfrac{1}{(2n+1)(2n+2)} > 0 \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Therefore the sequence $\{f(n)\}$ is a monotone increasing sequence. It is also strictly monotone.

3. Let $f(n) = \frac{1}{n}, n \geq 1$.

Then $f(n+1) - f(n) < 0$ for all $n \in \mathbb{N}$.

Therefore the sequence $\{f(n)\}$ is a monotone decreasing sequence. It is also strictly monotone.

4. The sequence $\{(-2)^n\}$ is neither a monotone increasing sequence, nor a monotone decreasing sequence. Therefore it is not a monotone sequence.

Theorem 5.9.1. A monotone increasing sequence, if bounded above, is convergent and it converges to the least upper bound.

Proof. Let $\{f(n)\}$ be a monotone increasing sequence bounded above and let M be its its least upper bound.

Then (i) $f(n) \leq M$ for all $n \in \mathbb{N}$ and

(ii) for a pre-assigned positive ϵ, there exists a natural number k such that $f(k) > M - \epsilon$.

Since $\{f(n)\}$ is a monotone increasing sequence,

$M - \epsilon < f(k) \leq f(k+1) \leq f(k+2) \leq \cdots \cdots \leq M$.

That is, $M - \epsilon < f(n) < M + \epsilon$ for all $n \geq k$.

This shows that the sequence $\{f(n)\}$ is convergent and $\lim f(n) = M$.

Theorem 5.9.2. A monotone decreasing sequence, if bounded below, is convergent and it converges to the greatest lower bound.

Similar proof.

Theorem 5.9.3. A monotone increasing sequence that is unbounded above diverges to ∞.

Proof. Let $\{f(n)\}$ be a monotone increasing sequence, not bounded above. Since the sequence is unbounded above, for a pre-assigned positive number G, however large, there exists a natural number k such that $f(k) > G$.

Since the sequence $\{f(n)\}$ is monotone increasing,

$$G < f(k) \leq f(k+1) \leq f(k+2) \leq \cdots \cdots$$

That is, $f(n) > G$ for all $n \geq k$.

This proves that the sequence $\{f(n)\}$ diverges to ∞.

Theorem 5.9.4. A monotone decreasing sequence that is unbounded below diverges to $-\infty$.

Similar proof.

Note. A monotone sequence has a definite behaviour. It is either convergent, or properly divergent.

The theorems on monotone sequences are important and useful in the sense that the convergence of the sequence can be established without prior knowledge of the limit. The limit of the sequence, however, can be determined if the l.u.b. of the increasing sequence (or the g.l.b. of the decreasing sequence) be evaluated.

Theorem 5.9.5. (Cantor's theorem on nested intervals)

Let $\{[a_n, b_n]\}$ be a sequence of *closed and bounded* intevals such that

(i) $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all $n \in \mathbb{N}$, and

(ii) $\lim \delta_n = 0$ where $\delta_n = b_n - a_n =$ length of $[a_n, b_n]$.

Then $\bigcap\limits_{n=1}^{\infty} [a_n, b_n]$ contains precisely one point.

Proof. $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all $n \in \mathbb{N}$.

So we have $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$ for all $n \in \mathbb{N}$.

Also $a_n \leq b_n$ for $n \in \mathbb{N}$.

Therefore $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1$.

This shows that the sequence $\{a_n\}$ is a monotone increasing sequence, bounded above and the sequence $\{b_n\}$ is a monotone decreasing sequence, bounded below.

Hence both the sequences are convergent. Let $\lim a_n = l, \lim b_n = m$.

Since $\lim(b_n - a_n) = 0$, $l = m = \alpha$, say,

Therefore α is the least upper bound of the sequence $\{a_n\}$ and the greatest lower bound of the sequence $\{b_n\}$.

Hence $a_n \leq \alpha$ and $\alpha \leq b_n$ for all $n \in \mathbb{N}$.'

This implies $\alpha \in [a_n, b_n]$ for all $n \in \mathbb{N}$. That is, $\alpha \in \bigcap\limits_{n=1}^{\infty} [a_n, b_n]$.

We now prove that α is the only point in $\bigcap\limits_{n=1}^{\infty} [a_n, b_n]$.

If possible, let $\beta \in \bigcap\limits_{n=1}^{\infty} [a_n, b_n]$. Then $a_n \leq \beta \leq b_n$ for all $n \in \mathbb{N}$.

Let us define a sequence $\{u_n\}$ by $u_n = \beta$ for all $n \in \mathbb{N}$. Then $\lim u_n = \beta$. Now $a_n \leq u_n \leq b_n$ for $n \geq 1$ and $\lim a_n = \lim b_n = \alpha$.

By Sandwich theorem, $\lim u_n = \alpha$ and therefore $\beta = \alpha$.
This proves that α is unique.

Note. The theorem says that a nested sequence of closed and bounded intervals has a non-empty intersecion.

A nested sequence of open and bounded intervals $\{I_n\}$ may not have a non-empty intersection.

For example, let $I_n = \{x \in \mathbb{R} : 0 < x < \frac{1}{n}\}$ for all $n \in \mathbb{N}$. Then $\{I_n\}$ is a nested sequence of open bounded intervals since $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$. Here $\bigcap\limits_{n=1}^{\infty} I_n = \phi$.

A nested sequence of closed and unbounded intervals $\{I_n\}$ may not have a non-empty intersection.

For example, let $I_n = \{x \in \mathbb{R} : x \geq n\}$ for all $n \in \mathbb{N}$. Then $\{I_n\}$ is a nested sequence of closed and unbounded intervals since $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$. Here $\bigcap\limits_{n=1}^{\infty} I_n = \phi$.

5.10. Some important sequences.

1. The sequence $\{(1 + \frac{1}{n})^n\}$ is a monotone increasing sequence, bounded above.

Let $u_n = (1 + \frac{1}{n})^n$. Then $u_{n+1} = (1 + \frac{1}{n+1})^{n+1}$.

Let us consider $n + 1$ positive numbers $1 + \frac{1}{n}, 1 + \frac{1}{n}, \cdots, 1 + \frac{1}{n}$ (n times) and 1.

Applying A.M.> G.M., we have $\frac{n(1+\frac{1}{n})+1}{n+1} > (1 + \frac{1}{n})^{\frac{n}{n+1}}$

or, $(1 + \frac{1}{n+1})^{n+1} > (1 + \frac{1}{n})^n$

i.e., $u_{n+1} > u_n$ for all $n \in \mathbb{N}$.

This shows that the sequence $\{u_n\}$ is a monotone increasing sequence.

$$\begin{aligned} \text{Now } u_n &= 1 + 1 + \frac{n(n-1)}{2!}\frac{1}{n^2} + \cdots + \frac{n(n-1)\cdots 2.1}{n!}\frac{1}{n^n} \\ &= 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \cdots + \frac{1}{n!}(1 - \frac{1}{n})(1 - \frac{2}{n}) \cdots \frac{2}{n}.\frac{1}{n} \\ &< 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} \text{ for all } n \geq 2. \end{aligned}$$

We have $n! > 2^{n-1}$ for all $n > 2$. Utilising this

$1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$, for $n > 2$.

Also $1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 1 + 2[1 - (\frac{1}{2})^n] < 3$ for all $n \in \mathbb{N}$.

It follows that $u_n < 3$ for all $n \in \mathbb{N}$, proving that the sequence $\{u_n\}$ is bounded above.

Thus the sequence $\{u_n\}$ being a monotone increasing sequence bounded above, is convergent. The limit of the sequence is denoted by e.

Since $u_1 = 2$, it follows that $2 < u_n < 3$ for all $n \geq 2$.

2. The sequence $\{x_n\}$ where $x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$ is a monotone increasing sequence, bounded above. And $\lim x_n = e.$

$x_{n+1} - x_n = [1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{(n+1)!}]-[1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}] = \frac{1}{(n+1)!} > 0$ for all $n \geq 1$.

So $x_{n+1} > x_n$ for all $n \geq 1$.

This shows that the sequence $\{x_n\}$ is a monotone increasing sequence.

$x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$

$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$, for $n \geq 3$, since $n! > 2^{n-1}$ for all $n \geq 3$.

Again $1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 1 + 2[1 - (\frac{1}{2})^n] < 3$ for all $n \in \mathbb{N}$

It follows that $x_n < 3$ for all $n \in \mathbb{N}$, proving that the sequence $\{x_n\}$ is bounded above.

Thus the sequence $\{x_n\}$ being a monotone increasing sequence bounded above, is convergent.

Let $u_n = (1 + \frac{1}{n})^n$.

$$\begin{aligned}
\text{Then } u_n &= 1 + 1 + \frac{n(n-1)}{2!}.\frac{1}{n^2} + \cdots + \frac{n(n-1)\cdots 2.1}{n!}.\frac{1}{n^n} \\
&= 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \cdots + \frac{1}{n!}(1 - \frac{1}{n})(1 - \frac{2}{n})\cdots\frac{2}{n}.\frac{1}{n}. \\
&< 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} \text{ for all } n \geq 2.
\end{aligned}$$

Therefore $\lim u_n \leq \lim x_n$ (since both the limits exist).

or, $e \leq \lim x_n$ ... ... (A)

Let us choose a natural number m. Then for each $n > m$,

$u_n = 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \cdots + \frac{1}{m!}(1 - \frac{1}{n})(1 - \frac{2}{n})\cdots(1 - \frac{m-1}{n}) + \cdots + \frac{1}{n!}(1 - \frac{1}{n})(1 - \frac{2}{n})\cdots\frac{2}{n}.\frac{1}{n}$

$> 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \cdots + \frac{1}{m!}(1 - \frac{1}{n})(1 - \frac{2}{n})\cdots(1 - \frac{m-1}{n})$.

Keeping m fixed, let $n \to \infty$. Then

$\lim\limits_{n\to\infty} u_n \geq 1 + 1 + \dfrac{1}{2!} + \cdots + \dfrac{1}{m!}$

or, $e \geq 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!}$

or, $x_m \leq e$.

The inequality holds for all natural numbers m.

Proceeding to limit as $m \to \infty$, $\lim\limits_{m\to\infty} x_m \leq e$ ... ... (B)

From (A) and (B), $\lim x_n = e$.

3. The sequence $\{(1 + \frac{1}{n})^{n+1}\}$ is a monotone decreasing sequence with limit e.

Let $v_n = (1 + \frac{1}{n})^{n+1}$.

Let us consider $n + 2$ positive numbers $1 - \frac{1}{n+1}, 1 - \frac{1}{n+1}, \cdots, 1 - \frac{1}{n+1}$ [$(n + 1)$ times] and 1.

Applying A.M.> G.M., we have $\frac{(n+1)(1-\frac{1}{n+1})+1}{n+2} > (1 - \frac{1}{n+1})^{\frac{n+1}{n+2}}$

or, $(\frac{n+1}{n+2})^{n+2} > (\frac{n}{n+1})^{n+1}$
or, $(\frac{n+1}{n})^{n+1} > (\frac{n+2}{n+1})^{n+2}$
or, $(1+\frac{1}{n})^{n+1} > (1+\frac{1}{n+1})^{n+2}$
i.e., $v_n > v_{n+1}$ for all $n \in \mathbb{N}$.

This shows that the sequence $\{v_n\}$ is a monotone decreasing sequence.
Again $v_n = 1 + \frac{n+1}{n} + \frac{(n+1)n}{2!}.\frac{1}{n^2} + \cdots + \frac{1}{n^{n+1}} > 1$ for all $n \in \mathbb{N}$.
This shows that the sequence $\{v_n\}$ is bounded below.
Hence the sequence $\{v_n\}$ is convergent.

Let $u_n = (1+\frac{1}{n})^n$. Then $v_n - u_n = (1+\frac{1}{n})^n.\frac{1}{n}$
and $\lim(v_n - u_n) = \lim(1+\frac{1}{n})^n.\frac{1}{n} = 0$.
This implies $\lim v_n = \lim u_n$, since both the limits exist.
As $\lim u_n = e$, it follows that $\lim v_n = e$.

Note. $v_n - u_n = (1+\frac{1}{n})^n.\frac{1}{n} > 0$ for all $n \in \mathbb{N}$.

Since $u_n < u_{n+1}$ and $v_{n+1} < v_n$ for all $n \in \mathbb{N}$, we have $u_n < u_{n+1} < v_{n+1} < v_n$.

Let $I_n = [u_n, v_n]$ be an interval. Then $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$.
The sequence of intervals $\{I_n\}$ is such that

(i) $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$; and (ii) $\lim |I_n| = 0$.

By Cantor's theorem on nested intervals, $\bigcap_{n=1}^{\infty} I_n$ is a singleton set and the set is $\{e\}$.

Worked Examples.

1. Prove that the sequence $\{u_n\}$ defined by
$u_1 = \sqrt{2}$ and $u_{n+1} = \sqrt{2u_n}$ for all $n \geq 1$ converges to 2.

The sequence is $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \cdots \cdots\}$
$u_{n+1}^2 - u_n^2 = 2(u_n - u_{n-1})$
or, $(u_{n+1}+u_n)(u_{n+1}-u_n) = 2(u_n - u_{n-1})$.
Since $u_n > 0$ for all n, $u_{n+1} >$ or $< u_n$ according as $u_n >$ or $< u_{n-1}$.

But $u_2 > u_1$. Consequently, $u_3 > u_2, u_4 > u_3, \cdots \cdots$ and therefore $\{u_n\}$ is a monotone increasing sequence.

Again $2u_n = u_{n+1}^2 > u_n^2$ for all $n \in \mathbb{N}$.
That is, $u_n^2 - 2u_n < 0$ for all $n \in \mathbb{N}$
or, $u_n(u_n - 2) < 0$ for all $n \in \mathbb{N}$.
But $u_n > 0$. Therefore $u_n < 2$ for all $n \in \mathbb{N}$.

This shows that the sequence $\{u_n\}$ is bounded above and therefore it is convergent.

Let $\lim u_n = l$.

By definition, $u_{n+1}^2 = 2u_n$ for all $n \in \mathbb{N}$.

Taking limit as $n \to \infty$, we have $l^2 = 2l$. Therefore l is either 0 or 2. But l cannot be 0 since the sequence $\{u_n\}$ is monotone increasing and $u_1 = \sqrt{2} > 1$.

Therefore $l = 2$. That is, the sequence converges to 2.

2. Prove that the sequence $\{u_n\}$ defined by
$u_1 = \sqrt{7}$ and $u_{n+1} = \sqrt{7 + u_n}$ for all $n \geq 1$ converges to the positive root of the equation $x^2 - x - 7 = 0$.

The sequence is $\{\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}, \cdots \cdots\}$

$u_{n+1}^2 - u_n^2 = u_n - u_{n-1}$.

or, $(u_{n+1} + u_n)(u_{n+1} - u_n) = u_n - u_{n-1}$.

Since $u_n > 0$ for all n, $u_{n+1} >$ or $< u_n$ according as $u_n >$ or $< u_{n-1}$.

But $u_2 > u_1$. Consequently, $u_3 > u_2, u_4 > u_3, \cdots \cdots$ and therefore $\{u_n\}$ is a monotone increasing sequence.

Again $u_n^2 < u_{n+1}^2 = 7 + u_n$ for all $n \in \mathbb{N}$

or, $u_n^2 - u_n - 7 < 0$

or, $(u_n - \alpha)(u_n - \beta) < 0$ where α, β are the roots of the equation $x^2 - x - 7 = 0$. One of the roots is negative and the other is positive. Let $\alpha < 0$. & $\beta \angle 0$.

Since $u_n > 0$ for all $n \in \mathbb{N}, u_n - \alpha > 0$. Consequently, $u_n < \beta$ for all $n \in \mathbb{N}$.

This proves that the sequence $\{u_n\}$ is bounded above and therefore the sequence $\{u_n\}$ is convergent.

Let $\lim u_n = l$.

By definition, $u_{n+1}^2 = 7 + u_n$ for all $n \in \mathbb{N}$.

Taking limit as $n \to \infty$, we have $l^2 = 7 + l$.

Therefore $(l - \alpha)(l - \beta) = 0$.

But $l \neq \alpha$, since each element of the sequence is positive and $\alpha < 0$. Therefore $l = \beta$. That is, the sequence converges to the positive root of the equation $x^2 - x - 7 = 0$.

3. Let $u_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n; v_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \log n, n \geq 2$.

Show that $\{u_n\}$ is a monotone decreasing sequence and $\{v_n\}_{n=2}^{\infty}$ is a monotone increasing one and they converge to the same limit.

$u_{n+1} - u_n = \frac{1}{n+1} - \log(n+1) + \log n = \frac{1}{n+1} - \log(1 + \frac{1}{n})$.

$v_{n+1} - v_n = \frac{1}{n} - \log(n+1) + \log n = \frac{1}{n} - \log(1 + \frac{1}{n})$.

As the sequence $\{(1+\frac{1}{n})^{n+1}\}$ is a strictly monotone decreasing sequence converging to e, $(1+\frac{1}{n})^{n+1} > e$ for all $n \in \mathbb{N}$.

Therefore $\log(1+\frac{1}{n}) > \frac{1}{n+1}$ for all $n \in \mathbb{N}$
or, $u_{n+1} < u_n$ for all $n \in \mathbb{N}$.

This shows that the sequence $\{u_n\}$ is a strictly monotone decreasing sequence.

As the sequence $\{(1+\frac{1}{n})^n\}$ is a strictly monotone increasing sequence converging to e, $(1+\frac{1}{n})^n < e$ for all $n \in \mathbb{N}$.

Therefore $\log(1+\frac{1}{n}) < \frac{1}{n}$ for all $n \in \mathbb{N}$
or, $v_{n+1} > v_n$ for all $n \geq 2$.

Therefore the sequence $\{v_n\}$ is a strictly monotone increasing sequence.

Again $\frac{1}{n} > \log\frac{n+1}{n} = \log(n+1) - \log n$.
Therefore $1 > \log 2 - \log 1$, $\frac{1}{2} > \log 3 - \log 2$, $\cdots$, $\frac{1}{n} > \log(n+1) - \log n$.

So we have $1 + \frac{1}{2} + \cdots + \frac{1}{n} > \log(n+1) > \log n$.
Hence $u_n > 0$ for all $n \in \mathbb{N}$.
Therefore $\{u_n\}$ is a monotone decreasing sequence bounded below.
Hence the sequence $\{u_n\}$ is convergent.

Let $\lim u_n = \gamma$.
Now $u_n - v_n = \frac{1}{n}$ for $n \geq 2$. Therefore $\lim v_n = \gamma$.

Thus the sequences $\{u_n\}$ and $\{v_n\}$ converge to the same limit γ.

Note 1. This limit γ is called **Euler's constant.**

Since $u_1 = 1$ and $\{u_n\}$ is a strictly monotone decreasing sequence, $\gamma < 1$. Since $v_2 = 1 - \log 2 = 1 - .69315 > .3$ and $\{v_n\}$ is a monotone increasing sequence, $\gamma > .3$. Therefore $.3 < \gamma < 1$.

The approximation of γ upto six places of decimal is given by $\gamma = 0.577215$.

Note 2. $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$ is denoted by γ_n. Then the sequence $\{\gamma_n\}$ converges to γ and $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \gamma_n + \log n$.

Evaluation of the limit of some sequences can be done by the help of Euler's constant.

For example, if $s_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$, then

$$\begin{aligned}\lim s_n &= \lim[(1 + \tfrac{1}{2} + \cdots + \tfrac{1}{2n}) - (1 + \tfrac{1}{2} + \cdots + \tfrac{1}{n})]\\ &= \lim[(\gamma_{2n} + \log 2n) - (\gamma_n + \log n)]\\ &= \lim[\gamma_{2n} - \gamma_n + \log 2]\\ &= \log 2,\ \text{since}\ \lim \gamma_{2n} = \lim \gamma_n = \gamma.\end{aligned}$$

4. Two sequences $\{x_n\}, \{y_n\}$ are defined by
$x_{n+1} = \frac{1}{2}(x_n + y_n); y_{n+1} = \sqrt{x_n y_n}$ for $n \geq 1$ and $x_1 > 0, y_1 > 0$.
Prove that both the sequences converge to a common limit.

Case 1. Let $x_1 \neq y_1$.
$x_2 = \frac{1}{2}(x_1 + y_1) > \sqrt{x_1 y_1} = y_2$.
Let us assume that $x_k > y_k$.
Then $x_{k+1} = \frac{1}{2}(x_k + y_k) > \sqrt{x_k y_k} = y_{k+1}$.
$x_k > y_k$ implies $x_{k+1} > y_{k+1}$ and $x_2 > y_2$.
By the principle of induction, $x_n > y_n$ for all $n \geq 2$.

$x_{n+1} = \frac{1}{2}(x_n + y_n) < \frac{1}{2}(x_n + x_n) = x_n$ for all $n \geq 2$
$y_{n+1} = \sqrt{x_n y_n} > \sqrt{y_n . y_n} = y_n$ for all $n \geq 2$.
So we have $y_2 < y_3 < y_4 < \cdots \cdots < x_4 < x_3 < x_2$.

Therefore the sequence $\{x_n\}_{n=2}^{\infty}$ is a monotone decreasing sequence bounded below and the sequence $\{y_n\}_{n=2}^{\infty}$ is a monotone increasing sequence bounded above. Hence both the sequences are convergent.

Let $\lim x_n = l$, $\lim y_n = m$.
$x_{n+1} = \frac{1}{2}(x_n + y_n)$ for all $n \in \mathbb{N}$.
Proceeding to limit as $n \to \infty$, we have $l = \frac{1}{2}(l + m)$. i.e., $l = m$.

Therefore the sequences $\{x_n\}$ and $\{y_n\}$ converge to a common limit.

Case 2. Let $x_1 = y_1$.
In this case $x_n = y_n = x_1$ for all $n \in \mathbb{N}$.
Therefore $\{x_n\}$ and $\{y_n\}$ both converge to the same limit x_1.

5. If $u_1 > 0$ and $u_{n+1} = \frac{1}{2}(u_n + \frac{9}{u_n})$ for $n \geq 1$, prove that the sequence $\{u_n\}$ converges to 3.

$u_n^2 - 2u_n u_{n+1} + 9 = 0$. This is a quadratic equation in u_n having real roots. Therefore $4u_{n+1}^2 - 36 \geq 0$.
This implies $u_{n+1} \geq 3$ for all $n \geq 1$, since $u_{n+1} > 0$ for all $n \geq 1$.

$$u_n - u_{n+1} = u_n - \frac{1}{2}(u_n + \frac{9}{u_n})$$
$$= \frac{1}{2}(u_n - \frac{9}{u_n}) = \frac{1}{2}(\frac{u_n^2 - 9}{u_n}) \geq 0 \text{ for all } n \geq 2.$$

Therefore $u_{n+1} \leq u_n$ for all $n \geq 2$.
This shows that the sequence $\{u_n\}_{n=2}^{\infty}$ is a monotone decreasing sequence bounded below and hence the sequence $\{u_n\}$ is convergent.

Let $\lim u_n = l$.
$u_{n+1} = \frac{1}{2}(u_n + \frac{9}{u_n})$ for $n \geq 1$. Proceeding to limit as $n \to \infty$, we have $l = \frac{1}{2}(l + \frac{9}{l})$. This gives $l = 3$, since $l > 0$.

$(1+x/n)^{n}$ $< e^{x/n}$

Exercises 7

1. (i) Give an example of a sequence of rational numbers that converges to an irrational number.

(ii) Give an example of a sequence of irrational numbers that converges to a rational number.

(iii) Give an example of divergent sequences $\{u_n\}$ and $\{v_n\}$ such that the sequence $\{u_n + v_n\}$ is convergent.

(iv) Give an example of divergent sequences $\{u_n\}$ and $\{v_n\}$ such that the sequence $\{u_n v_n\}$ is convergent.

2. Find $\sup\{u_n\}$ and $\inf\{u_n\}$ where

(i) $u_n = (-1)^n + \cos \frac{n\pi}{4}$, (ii) $u_n = \frac{(-1)^n}{n} + \sin \frac{n\pi}{2}$.

3. Let $\{u_n\}$ be a bounded sequence and $\lim v_n = 0$. Prove that $\lim u_n v_n = 0$. Utilise this to prove that

(i) $\lim \frac{\sin n}{n} = 0$, (ii) $\lim \frac{(-1)^n n}{n^2+1} = 0$, (iii) $\lim(-1)^n a_n = 0$ if $\lim a_n = 0$.

4. Let $\{u_n\}, \{v_n\}$ be two real sequences with $\lim u_n = l, \lim v_n = m$.
If $x_n = \max\{u_n, v_n\}, y_n = \min\{u_n, v_n\}$ prove that the sequence $\{x_n\}$ converges to $\max\{l, m\}$ and the sequence $\{y_n\}$ converges to $\min\{l, m\}$.

[**Hint.** $\max\{a, b\} = \frac{1}{2}\{a + b + \mid a - b \mid\}, \min\{a, b\} = \frac{1}{2}\{a + b - \mid a - b \mid\}$ for all $a, b \in \mathbb{R}$.]

5. If $\{u_n\}$ be a bounded sequence and $x_r = \min\{u_r, u_{r+1}, u_{r+2}, \cdots\}$, $y_r = \max\{u_r, u_{r+1}, u_{r+2}, \cdots\}$, for $r \geq 1$, prove that $\{x_n\}$ and $\{y_n\}$ are both monotone convergent sequences.
If $\lim x_n = \lim y_n = l$ prove that the sequence $\{u_n\}$ converges to l.

[**Hint.** $x_n \leq x_{n+1}$ and $y_n \geq y_{n+1}$ for all n. $x_n \leq u_n \leq y_n$ for all n.]

6. Prove that the sequence $\{u_n\}$ is a null sequence.

(i) $u_n = \frac{n!}{n^n}$, (ii) $u_n = \frac{4^{3n}}{3^{4n}}$, (iii) $u_n = \frac{b^n}{n!}, b > 1$.

7. Use Sandwich theorem to prove that

(i) $\lim(\sqrt{n+1} - \sqrt{n}) = 0$, (ii) $\lim(2^n + 3^n)^{1/n} = 3$,

(iii) $\lim[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}] = 0$, (iv) $\lim \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n} = 0$.

[**Hint.** (iv) Let $u_n = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}$. Then $u_n < \frac{2.4\ldots 2n}{3.5\ldots(2n+1)}$, since $\frac{n}{n+1} < \frac{n+1}{n+2}$ for all $n \geq 1$. Therefore $u_n^2 = u_n.u_n < \frac{1}{2n+1}$ for all $n \geq 1$.]

8. If $0 < u_1 < 1$ and $u_{n+1} = 1 - \sqrt{1 - u_n}$ for $n \geq 1$, prove that

(i) the sequence $\{u_n\}$ converges to 0 and (ii) $\lim \frac{u_{n+1}}{u_n} = \frac{1}{2}$.

[**Hint.** $1 - u_{n+1} = (1 - u_n)^{1/2} = (1 - u_{n-1})^{1/2^2} = \cdots = (1 - u_1)^{1/2^n}$.]

9. Prove that (i) $\lim \sqrt[n]{n+1} = 1$, (ii) $\lim \sqrt[n+1]{n} = 1$,

(iii) $\lim \frac{(n+1)^{2n}}{(n^2+1)^n} = e^2$, (iv) $\lim\{(1+\frac{1}{n^2})(1+\frac{2}{n^2})(1+\frac{3}{n^2})\}^{n^2} = e^6$.

[**Hint.** $\frac{(n+1)^{2n}}{(n^2+1)^n} = \{(1+\frac{1}{n})^n\}^2/\{(1+\frac{1}{n^2})^{n^2}\}^{\frac{n}{n^2}}$.]

10. Prove that the sequence $\{u_n\}$ defined by

(i) $u_1 = \sqrt{3}$ and $u_{n+1} = \sqrt{3u_n}$ for $n \geq 1$, converges to 3;

(ii) $u_1 = \sqrt{6}$ and $u_{n+1} = \sqrt{6+u_n}$ for $n \geq 1$, converges to 3.

11. A sequence $\{u_n\}$ is defined by $u_1 > 0$ and $u_{n+1} = \sqrt{6+u_n}$ for $n \geq 1$. Show that

(i) the sequence $\{u_n\}$ is monotone increasing if $0 < u_1 < 3$;

(ii) the sequence $\{u_n\}$ is monotone decreasing if $u_1 > 3$.

Find $\lim u_n$.

12. A sequence $\{u_n\}$ is defined by $u_1 > 0$ and $u_{n+1} = \frac{3(1+u_n)}{5+u_n}$ for $n \geq 1$. Prove that

(i) the sequence $\{u_n\}$ is a decreasing sequence if $u_1 > 1$;

(ii) the sequence $\{u_n\}$ is an increasing sequence if $0 < u_1 < 1$.

(iii) $\lim u_n = 1$ in both cases.

13. Prove that the sequence $\{u_n\}$ defined by

(i) $0 < u_1 < u_2$ and $u_{n+2} = \frac{2u_{n+1}+u_n}{3}$ for $n \geq 1$, converges to $\frac{u_1+3u_2}{4}$,

(ii) $0 < u_1 < u_2$ and $u_{n+2} = \frac{u_{n+1}+2u_n}{3}$ for $n \geq 1$, converges to $\frac{2u_1+3u_2}{5}$,

(iii) $0 < u_1 < u_2$ and $u_{n+2} = \sqrt{u_{n+1}u_n}$ for $n \geq 1$, converges to the limit $\sqrt[3]{u_1u_2^2}$,

(iv) $0 < u_1 < u_2$ and $\frac{2}{u_{n+2}} = \frac{1}{u_{n+1}} + \frac{1}{u_n}$ for $n \geq 1$, converges to the limit $3/(\frac{1}{u_1} + \frac{2}{u_2})$.

14. If $s_1 > 0$ and $s_{n+1} = \frac{1}{2}(s_n + \frac{4}{s_n})$ for $n \geq 1$, prove that the sequence $\{s_n\}$ is a monotone decreasing sequence bounded below and $\lim s_n = 2$.

15. Prove that the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$x_{n+1} = \sqrt{x_ny_n}$ and $\frac{2}{y_{n+1}} = \frac{1}{x_n} + \frac{1}{y_n}$ for $n \geq 1, x_1 > 0, y_1 > 0$ converge to a common limit.

16. Prove that the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$x_{n+1} = \frac{1}{2}(x_n + y_n)$, $\frac{2}{y_{n+1}} = \frac{1}{x_n} + \frac{1}{y_n}$ for $n \geq 1, x_1 > 0, y_1 > 0$ converge to a common limit l where $l^2 = x_1y_1$.

17. Prove that the sequence $\{\gamma_n\}$ where $\gamma_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$ is convergent. Hence find

(i) $\lim[1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{2n}]$, (ii) $\lim[\frac{1}{1.3} + \frac{1}{2.5} + \cdots + \frac{1}{n(2n+1)}]$.

18. Let S be a non-empty subset of $\mathbb{R}$ having a limit point l. Show that there exists a sequence $\{u_n\}$ of distinct elements of S such that $\lim u_n = l$.

19. Let S be an infinite subset of $\mathbb{R}$ that is bounded above and let $\sup S \notin S$. Show that there exists a monotone increasing sequence $\{u_n\}$ with $u_n \in S$, such that $\lim u_n = \sup S$. $S = \{\{1-\frac{1}{n}\} \mid n \in N\}$

5.11. Subsequence.

Let $\{u_n\}$ be a real sequence and $\{r_n\}$ be a strictly increasing sequence of natural numbers, i.e., $r_1 < r_2 < r_3 < \cdots < r_n < \cdots$ Then the sequence $\{u_{r_n}\}$ is said to be a *subsequence* of the sequence $\{u_n\}$. The elements of the subsequence $\{u_{r_n}\}$ are $u_{r_1}, u_{r_2}, \ldots, u_{r_n}, \ldots \ldots$

Let $r : \mathbb{N} \to \mathbb{N}$ be a sequence of natural numbers such that $r_1 < r_2 < \cdots < r_n < \cdots \cdots$ and $u : \mathbb{N} \to \mathbb{R}$ be a real sequence. Then the composite mapping $u \circ r : \mathbb{N} \to \mathbb{R}$ is said to be a *subsequence* of the real sequence u. The elements of the subsequence $u \circ r$ are $u_{r_1}, u_{r_2}, \ldots, u_{r_n}, \ldots \ldots$

Examples.

1. Let $u_n = \frac{1}{n}$ and $r_n = 2n$ for all $n \in \mathbb{N}$.
   Then $\{u_{r_n}\} = \{u_2, u_4, u_6, \ldots \ldots\}$
   $= \{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \cdots \cdots\}$ is a subsequence of $\{\frac{1}{n}\}$.

2. Let $u_n = \frac{1}{n}$ and $r_n = 2n - 1$ for all $n \in \mathbb{N}$.
   Then $\{u_{r_n}\} = \{u_1, u_3, u_5, \ldots \ldots\}$
   $= \{1, \frac{1}{3}, \frac{1}{5}, \cdots \cdots\}$ is a subsequence of $\{\frac{1}{n}\}$.

3. Let $u_n = (-1)^n$ and $r_n = 2n$ for all $n \in \mathbb{N}$.
   Then $\{u_{r_n}\} = \{u_2, u_4, u_6, \ldots \ldots\}$
   $= \{1, 1, 1, \ldots \ldots\}$ is a subsequence of $\{(-1)^n\}$.

4. Let $u_n = 1 + 1/n$ and $r_n = n^2$ for all $n \in \mathbb{N}$.
   Then $\{u_{r_n}\} = \{1+1, 1+\frac{1}{2^2}, 1+\frac{1}{3^2}, \cdots \cdots\}$ is a subsequence of $\{1+\frac{1}{n}\}$.

Theorem 5.11.1. If a sequence $\{u_n\}$ converges to l then every subsequence of $\{u_n\}$ also converges to l.

Proof. Let $\{r_n\}$ be a strictly increasing sequence of natural numbers. Then $\{u_{r_n}\}$ is subsequence of the sequence $\{u_n\}$.

Let $\epsilon > 0$. Since $\lim u_n = l$, there exists a natural number k such that $l - \epsilon < u_n < l + \epsilon$ for all $n \geq k$.

Since $\{r_n\}$ is a strictly increasing sequence of natural numbers, there exists a natural number k_0 such that $r_n > k$ for all $n \geq k_0$.

Therefore $l - \epsilon < u_{r_n} < l + \epsilon$ for all $n \geq k_0$.

Since ϵ is arbitrary, $\lim\limits_{n\to\infty} u_{r_n} = l$.

Note. If there exist two different subsequences $\{u_{r_n}\}$ and $\{u_{k_n}\}$ of a sequence $\{u_n\}$ such that $\{u_{r_n}\}$ and $\{u_{k_n}\}$ converge to two different limits, then the sequence $\{u_n\}$ is not convergent.

If a sequence $\{u_n\}$ has a divergent subsequence then $\{u_n\}$ is divergent.

Worked Examples.

1. Prove that $\lim(1 + \frac{1}{2n})^n = \sqrt{e}$.

Let $u_n = (1+\frac{1}{n})^n, v_n = (1+\frac{1}{2n})^{2n}$ and $w_n = (1+\frac{1}{2n})^n$ for all $n \in \mathbb{N}$.

$\{u_n\}$ is a convergent sequence and $\lim u_n = e$.

Since $v_n = u_{2n}$ for all $n \in \mathbb{N}$, $\{v_n\}$ is a subsequence of $\{u_n\}$ and therefore $\lim v_n = e$.

Now $w_n = \sqrt{v}_n$ for all $n \in \mathbb{N}$. Therefore $\lim w_n = \lim \sqrt{v_n} = \sqrt{e}$.

2. Prove that the sequence $\{(-1)^n\}$ is divergent.

Let $u_n = (-1)^n, v_n = u_{2n}, w_n = u_{2n-1}$.

Then $\{v_n\}$ is the subsequence $\{1, 1, 1, \ldots \ldots\}$ and $\lim v_n = 1$,

$\{w_n\}$ is the subsequence $\{-1, -1, -1, \ldots \ldots\}$ and $\lim w_n = -1$.

Since two different subsequences converge to two different limits, the sequence $\{u_n\}$ is divergent.

Theorem 5.11.2. If the subsequences $\{u_{2n}\}$ and $\{u_{2n-1}\}$ of a sequence $\{u_n\}$ converge to the same limit l then the sequence $\{u_n\}$ is convergent and $\lim u_n = l$.

Proof. Let us choose $\epsilon > 0$. Since $\lim u_{2n} = l$, there exists a natural number k_1 such that $| u_{2n} - l | < \epsilon$ for all $n \geq k_1$.

Since $\lim u_{2n-1} = l$, there exists a natural number k_2 such that $| u_{2n-1} - l | < \epsilon$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then k is a natural number and for all $n \geq k$, $l - \epsilon < u_{2n} < l + \epsilon$ and $l - \epsilon < u_{2n-1} < l + \epsilon$.

That is, $l - \epsilon < u_n < l + \epsilon$ for all $n \geq 2k - 1$.

As $2k - 1$ is a natural number, it follows that $\lim u_n = l$.

Note 1. If two subsequences of a sequence converge to the same limit l, the sequence $\{u_n\}$ may not be convergent.

For example, let $u_n = \sin \frac{n\pi}{4}$.

Then the subsequence $\{u_{8n-7}\}$ is $\{\sin \frac{\pi}{4}, \sin \frac{9\pi}{4}, \sin \frac{17\pi}{4}, \cdots\}$ and this converges to $\frac{1}{\sqrt{2}}$.

The subsequence $\{u_{8n-5}\}$ is $\{\sin \frac{3\pi}{4}, \sin \frac{11\pi}{4}, \sin \frac{19\pi}{5}, \cdots\}$ and this converges to $\frac{1}{\sqrt{2}}$.

But the sequence $\{u_n\}$ is not convergent.

2. If $k \in \mathbb{N}$ and k subsequences $\{u_{kn}\}$, $\{u_{kn-1}\}, \{u_{kn-2}\}, \cdots \{u_{kn-k+1}\}$ converge to the same limit l then the sequence $\{u_n\}$ is convergent and $\lim u_n = l$.

Worked Examples (continued.)

4. Prove that the sequence $\{u_n\}$ defined by $0 < u_1 < u_2$ and $u_{n+2} = \frac{1}{2}(u_n + u_{n+1})$, is convergent.

$u_3 - u_1 = \frac{u_1+u_2}{2} - u_1 = \frac{(u_2-u_1)}{2} > 0$, i.e., $u_1 < u_3$.

$\frac{u_1 + 2u_2}{3}.$

$u_3 - u_2 = \frac{u_1+u_2}{2} - u_2 = \frac{(u_1-u_2)}{2} < 0$, i.e., $u_3 < u_2$.

So $u_1 < u_3 < u_2$.

Similarly, $u_3 < u_4 < u_2, u_3 < u_5 < u_4, u_5 < u_6 < u_4, \cdots \cdots$

The inequalities give

$u_1 < u_3 < u_5 < \cdots < u_6 < u_4 < u_2$.

This shows that the sequence $\{u_{2n-1}\}$ is a monotone increasing sequence bounded above, u_2 being an upper bound; and the sequence $\{u_{2n}\}$ is a monotone decreasing sequence bounded below, u_1 being a lower bound .

So both the sequences $\{u_{2n}\}$ and $\{u_{2n-1}\}$ are convergent.

Let $\lim u_{2n} = l, \lim u_{2n-1} = m$.

Now $2u_{2n+2} = u_{2n} + u_{2n+1}$ for all $n \in \mathbb{N}$.

Proceeding to limit as $n \to \infty$, we have $2l = l + m$, i.e., $l = m$.

Thus the subsequences $\{u_{2n}\}$ and $\{u_{2n-1}\}$ converge to the same limit l and therefore the sequence $\{u_n\}$ is convergent.

5. A sequence $\{u_n\}$ is defined by $u_n > 0$ and $u_{n+1} = \frac{6}{1+u_n}$ for all $n \in \mathbb{N}$.

(i) Prove that the subsequences $\{u_{2n-1}\}$ and $\{u_{2n}\}$ converge to a common limit.

(ii) Find $\lim u_n$.

$u_{n+1} - u_n = \frac{6}{1+u_n} - u_n = \frac{6-u_n-u_n^2}{1+u_n} = \frac{(2-u_n)(3+u_n)}{1+u_n}$.

Therefore $\underline{u_n < 2} \Rightarrow u_n < u_{n+1}$; $\underline{u_n > 2} \Rightarrow u_n > u_{n+1}$.

Again $u_n < 2 \Rightarrow u_{n+1} = \frac{6}{1+u_n} > 2; u_n > 2 \Rightarrow u_{n+1} = \frac{6}{1+u_n} < 2$.

It follows that

$u_n < 2 \Rightarrow u_n < 2 < u_{n+1}$; $u_n > 2 \Rightarrow u_{n+1} < 2 < u_n$ ... (i)

$u_{n+2} - u_n = \frac{6(1+u_n)}{7+u_n} - u_n = \frac{6-u_n-u_n^2}{7+u_n} = \frac{(2-u_n)(3+u_n)}{7+u_n}$.

$u_n < 2 \Rightarrow u_n < u_{n+2}$; $u_n > 2 \Rightarrow u_n > u_{n+2}$ ... (ii)

Case 1. Let $u_1 < 2$. Then $u_2 > 2$.

From (i) $u_1 < 2 \Rightarrow u_1 < 2 < u_2$; $u_2 > 2 \Rightarrow u_3 < 2 < u_2$; $u_3 < 2 \Rightarrow u_3 < 2 < u_4$; $u_4 > 2 \Rightarrow u_5 < 2 < u_4; \cdots$

From (ii) $u_1 < 2 \Rightarrow u_1 < u_3; u_3 < 2 \Rightarrow u_3 < u_5; \cdots$
$u_2 > 2 \Rightarrow u_2 > u_4;\ u_4 > 2 \Rightarrow u_4 > u_6;\ \cdots$
Therefore $u_1 < u_3 < u_5 < \cdots < u_6 < u_4 < u_2$.

This shows that the subsequence $\{u_{2n-1}\}$ is a monotone increasing sequence, bounded above and the subsequence $\{u_{2n}\}$ is a monotone decreasing sequence, bounded below. Hence both the subsequences are convegent.

Let $\lim u_{2n-1} = l$, $\lim u_{2n} = m$.

We have $u_{2n} = \frac{6}{1+u_{2n-1}}$, $u_{2n+1} = \frac{6}{1+u_{2n}}$ for all $n \in \mathbb{N}$.

Taking limit as $n \to \infty$, we have $m = \frac{6}{1+l}, l = \frac{6}{1+m}$.

Therefore $l = m$ and the subsequences $\{u_{2n-1}\}$ and $\{u_{2n}\}$ converge to a common limit.

Case 2. $u_1 > 2$.

From (i) and (ii) it follows that $u_2 < u_4 < u_6 < \cdots < u_5 < u_3 < u_1$.

The subsequence $\{u_{2n}\}$ is a monotone increasing sequence, bounded above and the subsequence $\{u_{2n-1}\}$ is a monotone decreasing sequence, bounded below.

Hence both the sequences are convergent.

Proceeding as in case 1, it can be shown that they converge to a common limit.

(ii) Let the limit be l. We have $u_{n+1} = \frac{6}{1+u_n}$ for all $n \in \mathbb{N}$.

Taking limit as $n \to \infty$, we have $l = \frac{6}{1+l}$. This gives $l = 2$, or $l = -3$.
As $u_n > 0$ for all $n \in \mathbb{N}, l \neq -3$. Therefore $\lim u_n = 2$.

Theorem 5.11.3. Every subsequence of a monotone increasing (decreasing) sequence of real numbers is monotone increasing (decreasing).

Proof. (i) Let $\{u_n\}$ be a monotone increasing sequence. Then for any two natural numbers p, q with $p > q, u_p \geq u_q$.

Let $\{u_{r_n}\}$ be a subsequence of $\{u_n\}$. Then $\{r_n\}$ is a strictly increasing sequence of natural numbers. This implies $r_{n+1} > r_n$ for all $n \in \mathbb{N}$.

$r_{n+1} > r_n \Rightarrow u_{r_{n+1}} \geq u_{r_n}$ for all n.

This proves that $\{u_{r_n}\}$ is a monotone increasing subseqeunce.

(ii) Similar proof for a monotone decreasing sequence $\{u_n\}$.

Theorem 5.11.4. A monotone sequence of real numbers having a convergent subsequence with limit l, is convergent with limit l.

Proof. Let $\{u_n\}$ be a monotone increasing sequence and $\{u_{r_n}\}$ be a subsequence of $\{u_n\}$ such that $\lim u_{r_n} = l$.

Since $\{u_n\}$ is a monotone increasing sequence, the subsequence $\{u_{r_n}\}$ is also monotone increasing, by Theorem 5.11.3.

Since $\{u_{r_n}\}$ is a convergent sequence, it is bounded above.

We assert that the sequence $\{u_n\}$ is bounded above. If not, let $\{u_n\}$ be unbounded above. Then being a monotone increasing sequence it must diverge to ∞ and therefore for a pre-assigned positive number G, however large, there must exist a natural number k such that $u_n > G$ for all $n \geq k$. Since $\{r_n\}$ is a strictly increasing sequence of natural numbers, there exists a natural number k_0 such that $r_n > k$ for all $n \geq k_0$. Consequently, $u_{r_n} > G$ holds for all $n \geq k_0$.

Since G is arbitrary, the sequence $\{u_{r_n}\}$ must diverge to ∞, a contradiction. So our assertion is established and the sequence $\{u_n\}$ is bounded above.

Thus the sequence $\{u_n\}$ being a monotone increasing sequence, bounded above, is convergent.

Let $\lim u_n = m$. Then $\{u_{r_n}\}$ being subsequence of $\{u_n\}$ converges to m, by Theorem 5.11.1. Therefore $l = m$. This completes the proof.

Theorem 5.11.5. A monotone sequence of real numbers having a divergent subsequence is properly divergent.

Proof. Let $\{u_n\}$ be a montone increasing sequence having a divergent subsequence $\{u_{r_n}\}$. Since the sequence $\{u_n\}$ is monotone increasing, the subsequence $\{u_{r_n}\}$ is also monotone increasing and therefore it is a properly divergent subsequence. Consequently, the subsequence $\{u_{r_n}\}$ is unbounded above. Hence the sequence $\{u_n\}$ must be unbounded above and therefore it is properly divergent.

Similar proof if $\{u_n\}$ be a monotone decreasing sequence.

Worked Example.

1. Prove that the sequence $\{u_n\}$ is divergent where $u_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$.

$u_{n+1} - u_n = \frac{1}{n+1} > 0$ for all n. Therefore the sequence $\{u_n\}$ is a monotone increasing sequence.

Let $r_n = 2^n$. Then $\{r_n\}$ is a strictly increasing sequence of natural numbers and so the sequence $\{u_{r_n}\}$ is a subsequence of $\{u_n\}$.

Now $u_{r_n} = u_{2^n}$

$$
\begin{aligned}
&= 1 + \tfrac{1}{2} + \cdots + \tfrac{1}{2^n} \\
&= 1 + \tfrac{1}{2} + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \cdots + \tfrac{1}{8}) + \cdots + (\tfrac{1}{2^{n-1}+1} + \cdots + \tfrac{1}{2^n}) \\
&> 1 + \tfrac{1}{2} + (\tfrac{1}{4} + \tfrac{1}{4}) + (\tfrac{1}{8} + \cdots + \tfrac{1}{8}) + \cdots + (\tfrac{1}{2^n} + \cdots + \tfrac{1}{2^n}) \\
&= 1 + \tfrac{1}{2} + 2\tfrac{1}{2^2} + 2^2.\tfrac{1}{2^3} + \cdots + 2^{n-1}.\tfrac{1}{2^n} \\
&= 1 + \tfrac{n}{2}.
\end{aligned}
$$

Let $v_n = 1 + \frac{n}{2}$. Then $u_{r_n} > v_n$ for all $n > 2$ and $\lim v_n = \infty$.
Therefore $\lim u_{r_n} = \infty$.

Thus the sequence $\{u_n\}$ is a monotone increasing sequence having a properly divergent subsequence $\{u_{r_n}\}$ and therefore the sequence $\{u_n\}$ is properly divergent.

Theorem 5.11.6. Every sequence of real numbers has a monotone subsequence.

Proof. Let $\{u_n\}$ be a sequence of real numbers. An elerment u_k is said to be a *peak* of the sequence $\{u_n\}$ if $u_k \geq u_n$ for all $n > k$, i.e., u_k is greater than or equal to all subsequent elements beyond u_k. A sequence may or may not have a peak or else it may have a finite or an infinite number of peaks.

We consider the following cases.

Case 1. Let the sequence $\{u_n\}$ have infinitely many peaks.

Let the peaks be $u_{r_1}, u_{r_2}, u_{r_3}, \ldots$ (u_{r_1} being the first peak, u_{r_2} being the second,...). Then $u_{r_1} \geq u_{r_2} \geq u_{r_3} \geq \cdots$

The subsequence $\{u_{r_1}, u_{r_2}, u_{r_3}, \ldots\}$ is a monotone decreasing subsequence.

Case 2. Let the sequence have either no peak or a finite number of peaks.

Let the peaks be arranged in ascending order of the subscripts as $u_{r_1}, u_{r_2}, \ldots, u_{r_m}$. Let $s_1 = r_m + 1$. Then u_{s_1} is not a peak and there is no peak beyond the element u_{s_1}.

Since u_{s_1} is not a peak, there is an $s_2 \in \mathbb{N}$ with $s_2 > s_1$ such that $u_{s_2} > u_{s_1}$.

Since u_{s_2} is not a peak, there is an $s_3 \in \mathbb{N}$ with $s_3 > s_2$ such that $u_{s_3} > u_{s_2}$.

Proceeding thus we obtain natural numbers s_i such that $s_1 < s_2 < s_3 < \cdots$ and $u_{s_1} < u_{s_2} < u_{s_3} < \cdots$

Clearly, the subsequence $\{u_{s_n}\}$ is a monotone increasing subsequence of the sequence $\{u_n\}$.

This completes the proof.

Examples

1. Let $u_n = \sin \frac{n\pi}{2}, n \in \mathbb{N}$. The sequence is $\{1, 0, -1, 0, 1, 0, -1, 0, \cdots\}$
   Here $u_1 \geq u_n$ for all $n > 1$. Therefore u_1 is a peak.
   $u_5 \geq u_n$ for all $n > 5$. Therefore u_5 is a peak.
   Here $u_9 \geq u_n$ for all $n > 9$. Therefore u_9 is a peak.
   ... ...

The infinitely many peaks are $u_1, u_5, u_9, \ldots$

The subsequence $\{u_1, u_5, u_9, u_{13}, \ldots\}$ is a monotone subsequence of the sequence $\{u_n\}$.

2. Let $u_n = n^{(-1)^n}$. The sequence is $\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \frac{1}{7}, \cdots\}$.

Here the sequence $\{u_n\}$ has no peak.

u_1 is not a peak. Let $s_1 = 1$. Since u_{s_1} is not a peak, there is a natural number $s_2 > s_1$ such that $u_{s_2} > u_{s_1}$. Here $s_2 = 2$.

Since u_{s_2} is not a peak, there is a natural number $s_3 > s_2$ such that $u_{s_3} > u_{s_2}$. Here $s_3 = 4$.

By similar arguments, $s_4 = 6, s_5 = 8, \ldots$

Thus $\{u_1, u_2, u_4, u_6, u_8, \ldots\}$ is a monotone increasing subsequence of the sequence $\{u_n\}$.

5.12. Subsequential limit.

Let $\{u_n\}$ be a real sequence. A real number l is said to be a *subsequential limit* of the sequence $\{u_n\}$ if there exists a subsequence of $\{u_n\}$ that converges to l.

Theorem 5.12.1. A real number l is a subsequential limit of a sequence $\{u_n\}$ if and only if every neighbourhood of l contains infinitely many elements of the sequence $\{u_n\}$.

Proof. Let l be a subsequential limit of the sequence $\{u_n\}$. Then there exists a subsequence $\{u_{r_n}\}$ such that $\lim\limits_{n\to\infty} u_{r_n} = l$.

Let us choose a positive ϵ. Then there exists a natural number k such that $l - \epsilon < u_{r_n} < l + \epsilon$ for all $n \geq k$.

Therefore $l - \epsilon < u_n < l + \epsilon$ for infinitely many values of n.

Since ϵ is arbitrary, every neighbourhood of l contains infinite number of elements of the sequence $\{u_n\}$.

Conversely, let the sequence $\{u_n\}$ be such that for each pre-assigned positive ϵ the ϵ-neighbourhood of l contains infinitely many elements of the sequence.

Let $\epsilon = 1$. Then $l - 1 < u_n < l + 1$ for infinitely many values of n. Therefore the set $S_1 = \{n : l - 1 < u_n < l + 1\}$ is an infinite subset of the set $\mathbb{N}$. By the well ordering property of the set $\mathbb{N}$, S_1 has a least element, say r_1.

Therefore $l - 1 < u_{r_1} < l + 1$.

Let $\epsilon = \frac{1}{2}$. Then $l - \frac{1}{2} < u_n < l + \frac{1}{2}$ for infinitely many values of n. Therefore the set $S_2 = \{n : l - \frac{1}{2} < u_n < l + \frac{1}{2}\}$ is an infinite subset of $\mathbb{N}$ and hence there exists a natural number $r_2 (> r_1)$ in S_2 such that $l - \frac{1}{2} < u_{r_2} < l + \frac{1}{2}$.

Continuing thus, we obtain a strictly increasing sequence of natural numbers $\{r_1, r_2, r_3, \ldots \ldots\}$ such that $l - \frac{1}{n} < u_{r_n} < l + \frac{1}{n}$ for all $n \in \mathbb{N}$.

By Sandwich theorem, $\lim u_{r_n} = l$.

In other words the subsequence $\{u_{r_n}\}$ converges to l.

That is, l is a subsequential limit of the sequence $\{u_n\}$.

Note. The limit of a sequence, if it exists, is also a subsequential limit of the sequence.

Theorem 5.12.2. (Bolzano-Weierstrass theorem)

Every bounded sequence of real numbers has a convergent subsequence.

Proof. Let $\{u_n\}$ be a bounded sequence. Then there is a closed and bounded interval, say $I = [a, b]$, such that $u_n \in I$ for every $n \in \mathbb{N}$.

Let $c = \frac{a+b}{2}$ and $I' = [a, c], I'' = [c, b]$. Then at least one of the intervals I' and I'' contains infinitely many elements of $\{u_n\}$.

Let $I_1 = [a_1, b_1]$ be such an interval. Then $I_1 \subset I$ and $\mid I_1 \mid =$ the length of the interval $= \frac{1}{2}(b - a)$.

Let $c_1 = \frac{a_1+b_1}{2}$ and $I_1' = [a_1, c_1], I_1'' = [c_1, b_1]$. Then at least one of the intervals I_1' and I_1'' contains infinitely many elements of $\{u_n\}$. Let $I_2 = [a_2, b_2]$ be such an interval.

Then $I_2 \subset I_1$ and $\mid I_2 \mid = \frac{1}{2} \mid I_1 \mid$.

Continuing thus, we obtain a sequence of closed and bounded intervals $\{I_n\}$ such that

(i) $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$;

(ii) $\mid I_n \mid = \frac{1}{2^n}(b - a)$ and therefore $\lim\limits_{n \to \infty} \mid I_n \mid = 0$; and

(iii) each I_n contains infinitely many elements of $\{u_n\}$.

By Cantor's theorem on nested intervals, there exists a unique point α such that $\alpha \in \bigcap\limits_{n=1}^{\infty} I_n$.

We prove that α is a subsequential limit of the sequence $\{u_n\}$.

Let us choose $\epsilon > 0$. There exists a natural number k such that

$0 < \frac{b-a}{2^k} < \epsilon$. That is, $\mid I_k \mid < \epsilon$.

Since $\alpha \in I_k$ and $\mid I_k \mid < \epsilon, I_k$ is entirely contained in the neighbourhood $(\alpha - \epsilon, \alpha + \epsilon)$ and consequently, the ϵ-neighbourhood of α contains infinitely many elements of $\{u_n\}$.

Since ϵ is arbitrary, each neighbourhood of α contains infinitely many elements of $\{u_n\}$. Therefore α is a subsequential limit of $\{u_n\}$.

Therefore there exists a subsequence of $\{u_n\}$ that converges to α. In other words, $\{u_n\}$ has a convergent subsequence.

This completes the proof.

Note. Another version of the theorem is -- Every bounded sequence of real numbers has a subsequential limit.

Examples.

1. The sequence $\{u_n\}$ where $u_n = \sin \frac{n\pi}{2}, n \geq 1$ is a bounded sequence since $|u_n| \leq 1$ for all $n \geq 1$.

(i) The subsequence $\{u_1, u_5, u_9, \ldots \ldots\}$, i.e., $\{u_{4n-3}\}$ is a convergent subsequence that converges to 1.

(ii) The subsequence $\{u_2, u_4, u_6, \ldots \ldots\}$, i.e., $\{u_{2n}\}$ is a convergent subsequence that converges to 0.

(iii) The subsequence $\{u_1, u_3, u_5, \ldots \ldots\}$, i.e., $\{u_{2n-1}\}$ is a $\boxed{\text{divergent}}$ subsequence. $S_n = \sin\frac{n\pi}{2}$, inwhich S.S IS C

Note. The example 1(iii) shows that a bounded sequence may have a ~~divergent~~ Not Congernt subsequence.

2. The sequence $\{u_n\}$ where $u_n = n^{(-1)^n}$ is an unbounded sequence.

The sequence is $\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \ldots \ldots\}$.

(i) The sequence $\{u_2, u_4, u_6, \ldots \ldots\}$, i.e., $\{u_{2n}\}$ is a properly divergent subsequence.

(ii) The sequence $\{u_1, u_3, u_5, \ldots \ldots\}$, i.e., $\{u_{2n-1}\}$ is a convergent subsequence.

Note. The example 2(ii) shows that an unbounded sequence may have a convergent subsequence.

5.13. Characterisation of a compact set.

Theorem 5.13.1. Let K be a non-empty subset of $\mathbb{R}$. Then K is compact if and only if every sequence in K has a subsequence convergent to a point in K.

Proof. Let K be a compact set. Let $\{x_n\}$ be a sequence in K.

Since K is compact, K is a closed and bounded set. Since $\{x_n\}$ is a sequence in K, it is a bounded sequence and by Bolzano-Weierstrass theorem it has a convergent subsequence, say $\{x_{r_n}\}$. Let $\lim\limits_{n\to\infty} x_{r_n} = l$.

We prove that $l \in K$.

Let $l \notin K$. Then $l \in \mathbb{R} - K$. Since K is a closed set, it follows that $\mathbb{R} - K$ is an open set and l is an interior point of $\mathbb{R} - K$. So there exists a neighbourhood $N(l)$ of l such that $N(l) \subset \mathbb{R} - K$.

Hence $N(l)$ contains no element of the sequence $\{x_{r_n}\}$ and therefore l cannot be the limit of the sequence $\{x_{r_n}\}$, a contradiction.

Therefore $l \in K$.

Thus every sequence in K has a subsequence convergent to a point in K.

Conversely, suppose that K is a non-empty subset of $\mathbb{R}$ with the property that every sequence in K has a subsequence convergent to a point in K. Let T be an infinite subset of K.

Let $x_1 \in T, x_2 \in T - \{x_1\}, x_3 \in T - \{x_1, x_2\}, \ldots \ldots$

Continuing thus we obtain a sequence $\{x_n\}$ of distinct elements in K. By hypothesis there is a subsequence $\{x_{r_n}\}$ which converges to some point x in K. Therefore x is a limit point of the set T.

Thus K is such that every infinite subset of K has a limit point in K and therefore K is compact. [Theorem 3.16.4]

This completes the proof.

Worked Examples.

1. If S and T are disjoint compact subsets of $\mathbb{R}$ prove that $d(S,T) > 0$, where $d(S,T) = \inf\{| x - y |: x \in S, y \in T\}$.

Let $P = \{| x - y |: x \in S, y \in T\}$.

Since S and T are disjoint subsets, none is empty. Let $s \in S, t \in T$.

Then $| s - t | \in P$ and therefore P is a non-empty subset of $\mathbb{R}$.

Also P is a set of non-negative elements and therefore P is bounded below, 0 being a lower bound.

By the infimum property of $\mathbb{R}$, $\inf P$ exists and $\inf P \geq 0$. We prove that $\inf P > 0$.

If $\inf P = 0$, then for a pre-assigned positive ϵ there exist points $x' \in S, y' \in T$ such that $0 \leq | x' - y' | < 0 + \epsilon$.

Since S and T are disjoint $x' \neq y'$. Therefore $0 < | x' - y' | < \epsilon$.

For $\epsilon = \frac{1}{n}$, there exist points $x_n \in S, y_n \in T$ such that

$$0 < | x_n - y_n | < \tfrac{1}{n}.$$

This holds for all $n \in \mathbb{N}$.

$\{x_n\}$ is a sequence of points in S. Since S is compact, there is a subsequence $\{x_{r_n}\}$ of $\{x_n\}$ converging to a point, say x, in S.

Now $| x_{r_n} - y_{r_n} | < \frac{1}{r_n}$ for all $n \in \mathbb{N}$.

Therefore $\lim(x_{r_n} - y_{r_n}) = 0$ and since $\lim x_{r_n} = x$, we have $\lim y_{r_n} = x$.

But $\{y_{r_n}\}$ is a convergent sequence in T and since T is compact, $\lim y_{r_n} \in T$. That is, $x \in T$.

Therefore $x \in S$ and $x \in T$. This contradicts that S and T are disjoint.

Hence $d(S,T) = \inf P > 0$.

2. Let K be a non-empty compact set in $\mathbb{R}$ and $p \in \mathbb{R}$. Prove that there exists a point c in K such that $\sup\{| p - x |: x \in K\} = | p - c |$.

Since K is compact, K is bounded. Therefore the set $H = \{| p - x |: x \in K\}$ is a bounded set. Since K is non-empty, H is non-empty.

By the supremum property of $\mathbb{R}$, $\sup H$ exists. Let $\sup H = M$.

Let $\epsilon > 0$. Then there exists an element in the set, say $| p - x_0 |$, such that $M - \epsilon < | p - x_0 | \leq M$..

Let $\epsilon = 1$. Then there is an element $x_1 \in K$ such that
$M - 1 < | p - x_1 | \leq M$.

Let $\epsilon = \frac{1}{2}$. Then there exists an element $x_2 \in K$ such that
$M - \frac{1}{2} < | p - x_2 | \leq M$.

... ...

Proceeding in this way we obtain a sequence $\{x_n\}$ in K. Since K is compact, $\{x_n\}$ has a convergent subsequence $\{x_{r_n}\}$ that converges to a point c in K.

Now $M - \frac{1}{r_n} < | p - x_{r_n} | \leq M$ for all $n \in \mathbb{N}$.

$\lim x_{r_n} = c \Rightarrow \lim(p - x_{r_n}) = p - c \Rightarrow \lim | p - x_{r_n} | = | p - c |$.

Also $\lim M - \frac{1}{r_n} = M$.

By Sandwich theorem, $| p - c | = M$.

5.14. The upper limit and the lower limit.

Let $\{u_n\}$ be a bounded sequence of real numbers. Then by Bolzano-Weierstrass theorem there is a convergent subsequence of $\{u_n\}$. In other words, there is a subsequential limit of $\{u_n\}$. Since $\{u_n\}$ is bounded, the set S of all subsequential limits of $\{u_n\}$ is a bounded set.

Case 1. S is a finite set. Then S has a greatest element. → Maximum element

Case 2. S is an infinite set. Being a bounded set, S has a least upper bound. sup S

Let u^* be the lub of S. Then there is an element of S greater than $u^* - 1$. That is, $\{u_n\}$ has a subsequential limit $l_1 > u^* - 1$. Since every neighbourhood of l_1 contains an infinite number of elements of $\{u_n\}$, there is a natural number r_1 such that $u_{r_1} > u^* - 1$.

There is an element of S greater than $u^* - \frac{1}{2}$. That is, $\{u_n\}$ has a subsequential limit $l_2 > u^* - \frac{1}{2}$. Therefore there is a natural number $r_2 > r_1$ such that $u_{r_2} > u^* - \frac{1}{2}$.

Proceeding similarly, we obtain a strictly increasing sequence of natural numbers $\{r_1, r_2, r_3, \ldots \ldots\}$ such that $u_{r_n} > u^* - \frac{1}{n}$ for all $n \in \mathbb{N}$.

Let $\epsilon > 0$. There is a natural number k such that $0 < \frac{1}{n} < \epsilon$ for all $n \geq k$.

Therefore $| u_{r_n} - u^* | < \frac{1}{n} < \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, the subsequence $\{u_{r_n}\}$ converges to u^*.

Therefore u^* is a subsequential limit of the sequence $\{u_n\}$.

Since u^* is the lub of the set S, u^* is the *greatest subsequential limit* of the sequence $\{u_n\}$.

In a similar manner it can be established that a bounded sequence $\{u_n\}$ has a *least subsequential limit* u_*.

Definition. Let $\{u_n\}$ be a bounded sequence of real numbers.

The greatest subsequential limit of $\{u_n\}$ is said to be *the upper limit* or *the limit superior* of $\{u_n\}$ and this is denoted by $\overline{\lim}\, u_n$ or $\lim\sup u_n$.

The least subsequential limit of $\{u_n\}$ is said to be *the lower limit* or *the limit inferior* of $\{u_n\}$ and this is denoted by $\underline{\lim}\, u_n$ or $\lim\inf u_n$.

If $\{u_n\}$ is unbounded above then we define $\overline{\lim}\, u_n = \infty$

If $\{u_n\}$ is unbounded below then we define $\underline{\lim}\, u_n = -\infty$

If $\{u_n\}$ be unbounded above but bounded below, then $\underline{\lim}\, u_n$ is defined to be the *least subsequential limit.* If there is no subsequential limit, we define $\underline{\lim}\, u_n = \infty$.

If $\{u_n\}$ be unbounded below but bounded above, then $\overline{\lim}\, u_n$ is defined to be the *greatest subsequential limit.* If there is no subsequential limit, we define $\overline{\lim}\, u_n = -\infty$.

Examples.

1. Let $u_n = (-1)^n(1 + \frac{1}{n}), n \geq 1$. Then the sequence $\{u_n\}$ is a bounded sequence. $\overline{\lim}\, u_n = 1, \underline{\lim}\, u_n = -1$.

2. Let $u_n = \frac{1}{n}, n \geq 1$. Then the sequence $\{u_n\}$ is a bounded sequence. $\overline{\lim}\, u_n = \underline{\lim}\, u_n = 0$.

3. Let $u_n = (-1)^n n^2, n \geq 1$. Then the sequence $\{u_n\}$ is unbounded above and unbounded below. $\overline{\lim}\, u_n = \infty, \underline{\lim}\, u_n = -\infty$.

4. Let $u_n = n^{(-1)^n . n}, n \geq 1$. Then the sequence $\{u_n\}$ is unbounded above and bounded below. $\overline{\lim}\, u_n = \infty, \underline{\lim}\, u_n = 0$.

5. Let $u_n = n^2, n \geq 1$. Then the sequence $\{u_n\}$ is unbounded above and bounded below. $\overline{\lim}\, u_n = \infty, \underline{\lim}\, u_n = \infty$.

6. Let $u_n = -n^2, n \geq 1$. Then the sequence $\{u_n\}$ is unbounded below and bounded above. $\overline{\lim}\, u_n = -\infty, \underline{\lim}\, u_n = -\infty$.

Let $\{u_n\}$ be a bounded sequence and $u^* = \overline{\lim}\, u_n, u_* = \lim u$

Let B be the least upper bound and b be the greatest lower bound of the sequence $\{u_n\}$. Then $b \leq u^* \leq B$ and also $b \leq u_* \leq B$.

Since u^* is a subsequential limit of $\{u_n\}$, each ϵ-neighbourhood of u^* contains infinite number of elements of the sequence $\{u_n\}$.

Therefore for a positive $\epsilon, u^* - \epsilon < u_n < u^* + \epsilon$ for infinitely many values of n.

Also $u_n > u^* + \epsilon$ for at most a finite number of elements of$\{u_n\}$.

Because, if $u_n > u^* + \epsilon$ for infinitely many values of n, then there is a subsequence $\{u_{r_n}\}$ whose elements lie in the closed interval $[u^* + \epsilon, B]$, and $\{u_{r_n}\}$ being itself a bounded sequence, must have a subsequential limit l lying in $[u^* + \epsilon, B]$. l being also a subsequential limit of $\{u_n\}$, is greater than u^* and thereby u^* fails to be the upper limit of $\{u_n\}$.

Thus the upper limit u^* satisfies the following conditions.

For each positive ϵ,

(i) $u_n > u^* - \epsilon$ for infinitely many values of n, and

(ii) there exists a natural number k such that $u_n < u^* + \epsilon$ for all $n \geq k$.

By similar arguments, the lower limit u_* satisfies the following conditions.

For each positive ϵ,

(i) $u_n < u_* + \epsilon$ for infinitely many values of n, and

(ii) there exists a natural number k such that $u_n > u_* - \epsilon$ for all $n \geq k$.

Theorem 5.14.1. A bounded sequence $\{u_n\}$ is convergent if and only if $\overline{\lim}\, u_n = \underline{\lim}\, u_n$.

Proof. Let $\{u_n\}$ be a convergent sequence and $\lim u_n = l$.

Since $\{u_n\}$ is convergent, every subsequence of $\{u_n\}$ converges to l. Therefore l is the greatest as well as the least subsequential limit.

That is, $\overline{\lim}\, u_n = \underline{\lim}\, u_n$.

Conversely, let $\{u_n\}$ be a bounded sequence such that $\overline{\lim}\, u_n = \underline{\lim}\, u_n$.

Let $\overline{\lim}\, u_n = \underline{\lim}\, u_n = l$.

Since $\overline{\lim}\, u_n = l$, for a pre-assigned positive ϵ there exists a natural number k_1 such that $u_n < l + \epsilon$ for all $n \geq k_1$.

Since $\underline{\lim}\, u_n = l$, corresponding to the same ϵ there exists a natural number k_2 such that $u_n > l - \epsilon$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $l - \epsilon < u_n < l + \epsilon$ for all $n \geq k$.

This proves that $\lim u_n = l$.

In other words, the sequence $\{u_n\}$ is convergent.

Note. The theorem can be restated as –

A bounded sequence is convergent if and only if it has only one subsequential limit.

Theorem 5.14.2. Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences. Then

(i) $\overline{\lim}\, u_n + \overline{\lim}\, v_n \geq \overline{\lim}\, (u_n + v_n)$

(ii) $\underline{\lim}\, u_n + \underline{\lim}\, v_n \leq \underline{\lim}\, (u_n + v_n)$.

Proof. (i) Since $\{u_n\}$ and $\{v_n\}$ are both bounded sequences, the sequence $\{u_n + v_n\}$ is a bounded sequence.

Let $\overline{\lim}\, u_n = l_1, \overline{\lim}\, v_n = l_2, \overline{\lim}\, (u_n + v_n) = p$.

Let us choose $\epsilon > 0$.

Since $\overline{\lim}\, u_n = l_1$, there exists a natural number k_1 such that

$$u_n < l_1 + \tfrac{\epsilon}{2} \text{ for all } n \geq k_1.$$

Since $\overline{\lim}\, v_n = l_2$, there exists a natural number k_2 such that

$$v_n < l_2 + \tfrac{\epsilon}{2} \text{ for all } n \geq k_2.$$

Let $k = \max\{k_1, k_2\}$.

Then $u_n < l_1 + \frac{\epsilon}{2}$ and $v_n < l_2 + \frac{\epsilon}{2}$ for all $n \geq k$.

So $u_n + v_n < l_1 + l_2 + \epsilon$ for all $n \geq k$.

It follows that no subsequential limit of $\{u_n + v_n\}$ can be greater than $l_1 + l_2 + \epsilon$. Since $\epsilon (> 0)$ is arbitrary, every subsequential limit $\leq l_1 + l_2$. Hence $p \leq l_1 + l_2$.

(ii) proof left to the reader.

Note. Strict inequality in both (i) and (ii) may occur.

For example; if $u_n = \sin \frac{n\pi}{2}, n \in \mathbb{N}; v_n = \cos \frac{n\pi}{2}, n \in \mathbb{N}$ then

$\underline{\lim}\, (u_n + v_n) = -1, \underline{\lim}\, u_n = -1, \underline{\lim}\, v_n = -1$.

$\overline{\lim}\, u_n = 1, \overline{\lim}\, v_n = 1, \overline{\lim}\, (u_n + v_n) = 1$.

So in this case $\underline{\lim}\, u_n + \underline{\lim}\, v_n < \underline{\lim}\, (u_n + v_n)$ and $\overline{\lim}\, u_n + \overline{\lim}\, v_n > \overline{\lim}\, (u_n + v_n)$.

5.15. Cauchy criterion.

We discussed several methods of establishing convergence of a real sequence. In most of the methods, a prior knowledge of the limit is necessary. If however a sequence is monotone, the convergence can be established without any pre-conceived limit.

Cauchy's method of establishing convergence of a sequence does not require any knowledge of its limit, nor does it require the sequence to be monotone.

The method is very powerful as it is concerned only with the elements of the sequence.

Theorem 5.15.1. (Cauchy's general principle of convergence)

A necessary and sufficient condition for the convergence of a sequence $\{u_n\}$ is that for a pre-assigned positive ϵ there exists a natural number m such that

$| u_{n+p} - u_n |< \epsilon$ for all $n \geq m$ and for $p = 1, 2, 3, \ldots \ldots$

Proof. Let $\{u_n\}$ be convergent and $\lim u_n = l$. Then for a pre-assigned positive ϵ there exists a natural number m such that

$$| u_n - l |< \tfrac{\epsilon}{2} \text{ for all } n \geq m.$$

Therefore $| u_{n+p} - l |< \frac{\epsilon}{2}$ for all $n \geq m$ and $p = 1, 2, 3, \ldots \ldots$

$$\begin{aligned} \text{Now } | u_{n+p} - u_n | &\leq | u_{n+p} - l | + | u_n - l | \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \text{ for all } n \geq m \text{ and } p = 1, 2, 3, \ldots \end{aligned}$$

That is, $| u_{n+p} - u_n |< \epsilon$ for all $n \geq m$ and $p = 1, 2, 3, \ldots \ldots$

This proves that the condition is necessary.

We now prove that the sequence $\{u_n\}$ is convergent under the stated condition. First we prove that the sequence $\{u_n\}$ is bounded.

Let $\epsilon = 1$. Then there exists a natural number k such that

$$| u_{n+p} - u_n |< 1 \text{ for all } n \geq k \text{ and } p = 1, 2, 3, \ldots$$

Therefore $| u_{k+p} - u_k |< 1$ for $p = 1, 2, 3, \ldots \ldots$

or, $u_k - 1 < u_{k+p} < u_k + 1$ for $p = 1, 2, 3, \ldots \ldots$

Let $B = \max\{u_1, u_2, \ldots, u_k, u_k + 1\}, b = \min\{u_1, u_2, \ldots, u_k, u_k - 1\}$.

Then $b \leq u_n \leq B$ for all $n \in \mathbb{N}$.

This proves that $\{u_n\}$ is a bounded sequence.

By Bolzano-Weierstrass theorem, the sequence $\{u_n\}$ has a convergent subsequence. Let l be the limit of that subsequence. Then l is a subsequential limit of $\{u_n\}$.

Let $\epsilon > 0$. Then by the given condition, there exists a natural number m such that

$$| u_{n+p} - u_n |< \tfrac{\epsilon}{3} \text{ for all } n \geq m \text{ and } p = 1, 2, 3, \ldots$$

Taking $m = n$, it follows that

$$| u_{m+p} - u_m |< \tfrac{\epsilon}{3} \text{ for } p = 1, 2, 3, \ldots \ldots \ldots \quad \text{(i)}$$

Since l is a subsequential limit of $\{u_n\}$, each ϵ-neighbourhood of l contains infinite number of elements of $\{u_n\}$. Therefore there exists a natural number $q > m$ such that $| u_q - l |< \frac{\epsilon}{3}$.

As $q > m$, it follows from (i) that $| u_q - u_m |< \frac{\epsilon}{3}$.

$$\begin{aligned} \text{Now } | u_{m+p} - l | &\leq | u_{m+p} - u_m | + | u_m - u_q | + | u_q - l | \\ &< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} \text{ for } p = 1, 2, 3, \ldots \ldots \end{aligned}$$

Therefore $| u_n - l |< \epsilon$ for all $n \geq m + 1$.

Since ϵ is arbitrary, the sequence $\{u_n\}$ converges to l.

In other words, $\{u_n\}$ is a convergent sequence. This completes the proof.

Note. The condition stated in the theorem is called the "*Cauchy condition*" for convergence of a sequence.

Therefore a sequence $\{u_n\}$ is convergent if and only if the *Cauchy condition* is satisfied.

Worked Examples.

1. Use Cauchy's general principle of convergence to prove that the sequence $\{\frac{n}{n+1}\}$ is convergent.

Let $u_n = \frac{n}{n+1}$. Let p be a natural number.

Then $u_{n+p} = \frac{n+p}{n+p+1}$.

$$\begin{aligned} | u_{n+p} - u_n | &= \left| \frac{n+p}{n+p+1} - \frac{n}{n+1} \right| \\ &= \frac{p}{(n+p+1)(n+1)} \\ &< \frac{1}{n+1} < \frac{1}{n} \text{ for all } p, \text{ since } \frac{p}{n+p+1} < 1 \text{ for all } p. \end{aligned}$$

Let $\epsilon > 0$. Then $\frac{1}{n} < \epsilon$ holds for $n > \frac{1}{\epsilon}$.

Let $m = [\frac{1}{\epsilon}] + 1$. Then m is a natural number and $| u_{n+p} - u_n |< \epsilon$ for all $n \geq m$ and $p = 1, 2, 3, \ldots \ldots$

This proves that the sequence $\{u_n\}$ is convergent.

2. Use Cauchy's general principle of convergence to prove that the sequence $\{u_n\}$ where $u_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, is not convergent.

Let p be a natural number.

$| u_{n+p} - u_n |= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p}$.

Let us choose $n = m$ and $p = m$.

$$\begin{aligned} \text{Then } | u_{2m} - u_m | &= \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m} \\ &> \frac{1}{2m} + \frac{1}{2m} + \cdots + \frac{1}{2m} \\ &= \frac{1}{2}. \end{aligned}$$

If we choose $\epsilon = \frac{1}{2}$ then no natural number k can be found such that $| u_{n+p} - u_n |< \epsilon$ will hold for all $n \geq k$ and for every natural number p.

This shows that Cauchy condition is not satisfied by the sequence and the sequence $\{u_n\}$ is not convergent.

Cauchy sequence:

Definition. A sequence $\{u_n\}$ is said to be a *Cauchy sequence* if for a pre-assigned positive ϵ there exists a natural number k such that

$$| u_m - u_n |< \epsilon \text{ for all } m, n \geq k.$$

Replacing m by $n+p$ where $p = 1, 2, 3, \ldots \ldots$ the above condition can be equivalently stated as

$$| u_{n+p} - u_n | < \epsilon \text{ for all } n \geq k \text{ and } p = 1, 2, 3, \ldots \ldots$$

Theorem 5.15.2. A Cauchy sequence of real numbers is convergent.

Proof. Let $\{u_n\}$ be a Cauchy sequence. First we prove that the sequence $\{u_n\}$ is bounded.

Let $\epsilon = 1$. Then there exists a natural number k such that

$$| u_m - u_n | < 1 \text{ for all } m, n \geq k.$$

Therefore $| u_k - u_n | < 1$ for all $n \geq k$.
or, $u_k - 1 < u_n < u_k + 1$ for all $n \geq k$.

Let $B = \max\{u_1, u_2, \ldots, u_{k-1}, u_k + 1\}$,
$b = \min\{u_1, u_2, \ldots, u_{k-1}, u_k - 1\}$.

Then $b \leq u_n \leq B$ for all $n \in \mathbb{N}$ and this proves that the sequence $\{u_n\}$ is bounded.

By Bolzano-Weierstrass theorem, $\{u_n\}$ has a convergent subsequence.

Let l be the limit of that convergent subsequence. Then l is a subsequential limit of $\{u_n\}$.

We now prove that the sequence $\{u_n\}$ converges to l.

Let us choose $\epsilon > 0$. There exists a natural number k such that

$$| u_m - u_n | < \tfrac{\epsilon}{2} \text{ for all } m, n \geq k \ \ldots \ldots \ \text{(i)}$$

Since l is a subsequential limit of $\{u_n\}$, there exists a natural number $q > k$ such that $| u_q - l | < \frac{\epsilon}{2}$.

Since $q > k$, from (i) $| u_q - u_n | < \frac{\epsilon}{2}$ for all $n \geq k$.

$$\begin{aligned} \text{Now } | u_n - l | &= | u_n - u_q | + | u_q - l | \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \text{ for all } n \geq k. \end{aligned}$$

That is, $| u_n - l | < \epsilon$ for all $n \geq k$.

This implies $\lim u_n = l$. In other words, the sequence $\{u_n\}$ is convergent and the theorem is done.

Theorem 5.15.3. A convergent sequence is a Cauchy sequence.

Proof. Let $\{u_n\}$ be a convergent sequence and let $\lim u_n = l$.

For a pre-assigned positive ϵ there exists a natural number k such that $| u_n - l | < \frac{\epsilon}{2}$ for all $n \geq k$.

If m, n be natural numbers $\geq k$, then

$$| u_m - l | < \tfrac{\epsilon}{2} \text{ and } | u_n - l | < \tfrac{\epsilon}{2}.$$

$$\begin{aligned} \text{Now } | u_m - u_n | &\leq | u_m - l | + | l - u_n | \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \text{ for all } m, n \geq k. \end{aligned}$$

That is, $| u_m - u_n | < \epsilon$ for all $m, n \geq k$.

This proves that the sequence $\{u_n\}$ is a Cauchy sequence.

Worked Examples (continued).

3. Prove that the sequence $\{\frac{1}{n}\}$ is a Cauchy sequence.

Let $u_n = \frac{1}{n}$. Let us choose a positive ϵ. There is a natural number k such that $\frac{2}{k} < \epsilon$.

$$\text{Then } |u_m - u_n| = |\tfrac{1}{m} - \tfrac{1}{n}| \le \tfrac{1}{m} + \tfrac{1}{n} < \epsilon \text{ if } m, n \ge k.$$

This proves that the sequence $\{u_n\}$ is a Cauchy sequence.

4. Prove that the sequence $\{(-1)^n\}$ is not a Cauchy sequence.

Let $u_n = (-1)^n$. Then

$$\begin{aligned} |u_m - u_n| &= |(-1)^m - (-1)^n|. \\ |u_m - u_n| &= 0 \text{ if } m \text{ and } n \text{ are both odd or both even,} \\ |u_m - u_n| &= 2 \text{ if one of } m, n \text{ is odd and the other is even.} \end{aligned}$$

Let us choose $\epsilon = \frac{1}{2}$. Then it is not possible to find a natural number k such that $|u_m - u_n| < \epsilon$ for all $m, n \ge k$.

Hence $\{u_n\}$ is not a Cauchy sequence.

5. Prove that the sequence $\{u_n\}$ where $u_1 = 0, u_2 = 1$ and $u_{n+2} = \frac{1}{2}(u_{n+1} + u_n)$ for all $n \ge 1$ is a Cauchy sequence.

$u_{n+2} - u_{n+1} = \frac{1}{2}(u_{n+1} + u_n) - u_{n+1} = -\frac{1}{2}(u_{n+1} - u_n)$
or, $|u_{n+2} - u_{n+1}| = \frac{1}{2} |u_{n+1} - u_n|$ for all $n \in \mathbb{N}$.

Therefore $|u_{n+2} - u_{n+1}| = \frac{1}{2} |u_{n+1} - u_n| = \frac{1}{2^2} |u_n - u_{n-1}| = \cdots = \frac{1}{2^n} |u_2 - u_1| = \frac{1}{2^n}$.

Let $m > n$. Then $|u_m - u_n|$
$\le |u_m - u_{m-1}| + |u_{m-1} - u_{m-2}| + \cdots + |u_{n+1} - u_n|$
$= (\frac{1}{2})^{m-2} + (\frac{1}{2})^{m-3} + \cdots + (\frac{1}{2})^{n-1}$
$= \frac{4}{2^n}[1 - (\frac{1}{2})^{m-n}] < \frac{4}{2^n}$.

Let $\epsilon > 0$. Then there exists a natural number k such that $\frac{4}{2^n} < \epsilon$ for all $n \ge k$.

Hence $|u_m - u_n| < \epsilon$ for all $m, n \ge k$.

This proves that the sequence $\{u_n\}$ is a Cauchy sequence.

6. Prove that the sequence $\{u_n\}$ satisfying the condition $|u_{n+2} - u_{n+1}| \le c\, |u_{n+1} - u_n|$ for all $n \in \mathbb{N}$, where $0 < c < 1$, is a Cauchy sequence.

$$\begin{aligned} | u_{n+2} - u_{n+1} | &\leq c | u_{n+1} - u_n | \\ &\leq c^2 | u_n - u_{n-1} | \\ &\leq \cdots \\ &\leq c^n | u_2 - u_1 |. \end{aligned}$$

Let $m > n$.

$$\begin{aligned} \text{Then } | u_m - u_n | &\leq | u_m - u_{m-1} | + \cdots + | u_{n+1} - u_n | \\ &\leq | u_2 - u_1 | \{c^{m-2} + c^{m-3} + \cdots + c^{n-1}\} \\ &= | u_2 - u_1 | c^{n-1} . \tfrac{1-c^{m-n}}{1-c} \\ &< \tfrac{c^{n-1}}{1-c} | u_2 - u_1 |. \end{aligned}$$

Let $\epsilon > 0$. Since $0 < c < 1$, the sequence $\{c^{n-1}\}$ is a convergent sequence. Therefore there exists a natural number k such that

$$\tfrac{c^{n-1}}{1-c} | u_2 - u_1 | < \epsilon \text{ for all } n \geq k.$$

It follows that $| u_m - u_n | < \epsilon$ for all $m, n \geq k$ and this proves that the sequence $\{u_n\}$ is a Cauchy sequence.

5.16. Cauchy's theorems on limits.

Theorem 5.16.1. If $\lim u_n = l$ then $\lim \frac{u_1+u_2+\cdots+u_n}{n} = l$.

Proof. **Case 1.** $l = 0$.

Since $\{u_n\}$ is a convergent sequence, it is bounded. Therefore there exists a positive number B such that $| u_n | < B$ for all $n \in \mathbb{N}$.

Let $\epsilon > 0$. Since $\lim u_n = 0$, there exists a natural number k_1 such that $| u_n | < \frac{\epsilon}{2}$ for all $n \geq k_1$.

$$\begin{aligned} \text{Now } \left|\frac{u_1+u_2+\cdots+u_n}{n}\right| &\leq \left|\frac{u_1+u_2+\cdots+u_{k_1-1}}{n}\right| + \left|\frac{u_{k_1}+u_{k_1+1}+\cdots+u_n}{n}\right| \\ &\leq \left|\frac{|u_1|+|u_2|+\cdots+|u_{k_1-1}|}{n}\right| + \left|\frac{|u_{k_1}|+|u_{k_1+1}|+\cdots+|u_n|}{n}\right| \\ &< \frac{B(k_1-1)}{n} + \frac{n-k_1+1}{n} . \frac{\epsilon}{2} \text{ for all } n \geq k_1 \\ &< \frac{Bk_1}{n} + \frac{\epsilon}{2} \text{ for all } n \geq k_1. \end{aligned}$$

Since $\lim \frac{1}{n} = 0$, there exists a natural number k_2 such that $\frac{Bk_1}{n} < \frac{\epsilon}{2}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $\left|\frac{u_1+u_2+\cdots+u_n}{n}\right| < \epsilon$ for all $n \geq k$.

This proves that $\lim \frac{u_1+u_2+\cdots+u_n}{n} = 0$.

Case 2. $l \neq 0$.

Let $v_n = u_n - l$. Then $\lim v_n = 0$.

Now $\frac{u_1+u_2+\cdots+u_n}{n} - l = \frac{v_1+v_2+\cdots+v_n}{n}$.

By case 1, $\lim \frac{v_1+v_2+\cdots+v_n}{n} = 0$. Therefore $\lim \frac{u_1+u_2+\cdots+u_n}{n} = l$.

This completes the proof.

Note. The converse of the theorem is not true.

Let us consider the sequence $\{u_n\}$ where $u_n = (-1)^n$.

Then $\lim \frac{u_1+u_2+\cdots+u_n}{n} = 0$ but the sequence $\{u_n\}$ is not convergent.

Corollary. If $\lim u_n = l$ where $u_n > 0$ for all n and $l \neq 0$, then $\lim \sqrt[n]{u_1 u_2 \ldots u_n} = l$.

Since each u_n is positive and $\lim u_n = l > 0$, the sequence $\{\log u_n\}$ converges to $\log l$, by the Corollary of 4.(Art 5.8).

Therefore $\lim \frac{\log u_1+\log u_2+\cdots+\log u_n}{n} = \log l$.

or, $\lim \log \sqrt[n]{(u_1 u_2 \ldots u_n)} = \log l$.

It follows that, $\lim \sqrt[n]{(u_1 u_2 \ldots u_n)} = l$.

Worked Examples.

1. Prove that $\lim \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n} = 0$.

Let $u_n = \frac{1}{n}$. Then $\lim u_n = 0$.

By Cauchy's theorem, $\lim \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n} = 0$.

2. Prove that $\lim \frac{1+\sqrt{2}+\sqrt[3]{3}+\cdots+\sqrt[n]{n}}{n} = 1$.

Let $u_n = \sqrt[n]{n}$. Then $\lim u_n = 1$.

By Cauchy's theorem, $\lim \frac{1+\sqrt{2}+\sqrt[3]{3}+\cdots+\sqrt[n]{n}}{n} = 1$.

Theorem 5.16.2. If $u_n > 0$ for all $n \in \mathbb{N}$ and $\lim \frac{u_{n+1}}{u_n} = l(\neq 0)$ then $\lim \sqrt[n]{u_n} = l$.

Proof. Let $v_1 = u_1, v_2 = \frac{u_2}{u_1}, v_3 = \frac{u_3}{u_2}, \cdots, v_n = \frac{u_n}{u_{n-1}}, \cdots$

Then $v_n > 0$ for all $n \in \mathbb{N}$ and $\lim v_n = l > 0$

This implies $\lim \log v_n = \log l$.

By the first theorem, $\lim \frac{\log v_1+\log v_2+\cdots+\log v_n}{n} = \log l$.

or, $\lim \log \sqrt[n]{(v_1 v_2 \ldots v_n)} = \log l$.

It follows that $\lim \sqrt[n]{(v_1 v_2 \ldots v_n)} = l$. That is, $\lim \sqrt[n]{u_n} = l$.

Theorem 5.16.3. Let $u_n > 0$ for all $n \in \mathbb{N}$ and $\lim \frac{u_{n+1}}{u_n} = l$ (finite or infinite). Then $\lim \sqrt[n]{u_n} = l$.

Proof. **Case 1.** $0 < l < \infty$.

Let us choose $\epsilon > 0$ such that $l - \epsilon > 0$. Since $\lim \frac{u_{n+1}}{u_n} = l$, there exists a natural number k such that $l - \frac{\epsilon}{2} < \frac{u_{n+1}}{u_n} < l + \frac{\epsilon}{2}$ for all $n \geq k$.

Then $l - \frac{\epsilon}{2} < \frac{u_{k+1}}{u_k} < l + \frac{\epsilon}{2}$.

$$l - \frac{\epsilon}{2} < \frac{u_{k+2}}{u_k} < l + \frac{\epsilon}{2}$$

$$\cdots \qquad \cdots$$

$$l - \frac{\epsilon}{2} < \frac{u_n}{u_{n+1}} < l + \frac{\epsilon}{2}.$$

We have $(l-\frac{\epsilon}{2})^{n-k} < \frac{u_n}{u_k} < (l+\frac{\epsilon}{2})^{n-k}$ for all $n > k$
or, $(l-\frac{\epsilon}{2})^n.B < u_n < A.(l+\frac{\epsilon}{2})^n$, where $A = \frac{u_k}{(l+\frac{\epsilon}{2})^k} > 0, B = \frac{u_k}{(l-\frac{\epsilon}{2})^k} > 0$.
or, $(l-\frac{\epsilon}{2})B^{\frac{1}{n}} < u_n^{\frac{1}{n}} < A^{\frac{1}{n}}(l+\frac{\epsilon}{2})$.

Since $A > 0, \lim A^{\frac{1}{n}} = 1$. Since $B > 0, \lim B^{\frac{1}{n}} = 1$.

Since $\lim A^{\frac{1}{n}}(l+\frac{\epsilon}{2}) = l+\frac{\epsilon}{2}$, there exists a natural number k_2 such that $A^{\frac{1}{n}}(l+\frac{\epsilon}{2}) < l+\epsilon$ for all $n \geq k_2$.

Since $\lim B^{\frac{1}{n}}(l-\frac{\epsilon}{2}) = l-\frac{\epsilon}{2}$, there exists a natural number k_3 such that $B^{\frac{1}{n}}(l-\frac{\epsilon}{2}) > l-\epsilon$ for all $n \geq k_3$.

Let $k_0 = \max\{k_1, k_2, k_3\}$. Then $l-\epsilon < u_n^{\frac{1}{n}} < l+\epsilon$ for all $n > k_0$.

Therefore $\lim u_n^{\frac{1}{n}} = l$.

Case 2. $l = 0$.

Let $\epsilon > 0$. There exists a natural number k such that

$$0 < \frac{u_{n+1}}{u_n} < \frac{\epsilon}{2} \text{ for all } n \geq k.$$

Therefore $0 < \frac{u_{k+1}}{u_k} < \frac{\epsilon}{2}, 0 < \frac{u_{k+2}}{u_{k+1}} < \frac{\epsilon}{2}, \cdots\cdots, 0 < \frac{u_n}{u_{n-1}} < \frac{\epsilon}{2}$.

We have $0 < \frac{u_n}{u_k} < (\frac{\epsilon}{2})^{n-k}$ for all $n > k$
or, $0 < u_n < \frac{u_k}{(\frac{\epsilon}{2})^k}.(\frac{\epsilon}{2})^n$
or, $0 < u_n < \Lambda(\frac{\epsilon}{2})^n$ where $\Lambda = u_k(\frac{2}{\epsilon})^k > 0$
or, $0 < u_n^{\frac{1}{n}} < \Lambda^{\frac{1}{n}}\frac{\epsilon}{2}$.

Since $\Lambda > 0, \lim \Lambda^{\frac{1}{n}} = 1$.

Since $\lim \Lambda^{\frac{1}{n}}\frac{\epsilon}{2} = \frac{\epsilon}{2}$, there exists a natural number k_1 such that $\Lambda^{\frac{1}{n}}\frac{\epsilon}{2} < \epsilon$ for all $n \geq k_1$.

Let $k_0 = \max\{k, k_1\}$. Then $0 < u_n^{\frac{1}{n}} < \epsilon$ for all $n > k_0$.

Therefore $\lim u_n^{\frac{1}{n}} = 0$.

Case 3. $\lim \frac{u_{n+1}}{u_n} = \infty$.

Let us choose $G > 0$. There exists a natural number k such that $\frac{u_{n+1}}{u_n} > G+1$ for all $n \geq k$.

Therefore $\frac{u_{k+1}}{u_k} > G+1, \frac{u_{k+2}}{u_{k+1}} > G+1, \cdots, \frac{u_n}{u_{n-1}} > G+1$.

We have $\frac{u_n}{u_k} > (G+1)^{n-k}$ for all $n > k$
or, $u_n > \mu(G+1)^n$ where $\mu = \frac{u_k}{(G+1)^k} > 0$
or, $u_n^{\frac{1}{n}} > \mu^{\frac{1}{n}}(G+1)$.

Since $\mu > 0, \lim \mu^{\frac{1}{n}} = 1$.

Since $\lim \mu^{\frac{1}{n}}(G+1) = G+1$, there exists a natural number k_1 such that $\mu^{\frac{1}{n}}(G+1) > G$ for all $n \geq k_1$.

Let $k_0 = \max\{k, k_1\}$. Then $u_n^{\frac{1}{n}} > G$ for all $n > k_0$.

Therefore $\lim u_n^{\frac{1}{n}} = \infty$. This completes the proof.

Note. The converse of the theorem is not true.

For example, let us consider the sequence $\{u_n\}$ where $u_n = \frac{3+(-1)^n}{2}$.
The sequence is $\{1, 2, 1, 2, 1, 2, \ldots \ldots\}$.
Here $\lim \sqrt[n]{u_n} = 1$, since $\lim \sqrt[2n]{u_{2n}} = \lim 2^{\frac{1}{2n}} = 1$ and $\lim (u_{2n-1})^{\frac{1}{2(n-1)}} = 1$. But $\lim \frac{u_{n+1}}{u_n}$ does not exist.

Theorem 5.16.4. If $u_n > 0$ for all $n \in \mathbb{N}$ then
$\underline{\lim} \frac{u_{n+1}}{u_n} \leq \underline{\lim} \sqrt[n]{u_n} \leq \overline{\lim} \sqrt[n]{u_n} \leq \overline{\lim} \frac{u_{n+1}}{u_n}$.

Proof. Let $\underline{\lim} \frac{u_{n+1}}{u_n} = \lambda_*, \overline{\lim} \frac{u_{n+1}}{u_n} = \lambda^*, \underline{\lim} \sqrt[n]{u_n} = \mu_*, \overline{\lim} \sqrt[n]{u_n} = \mu^*$.
We first prove $\mu^* \leq \lambda^*$

Case 1. Let $\lambda^* = \infty$. Then $\mu^* \leq \lambda^*$ trivially.

Case 2. Let λ^* be finite.

Let us choose $\epsilon > 0$. Since $\overline{\lim} \frac{u_{n+1}}{u_n} = \lambda^*$, there exists a natural number k such that $\frac{u_{n+1}}{u_n} < \lambda^* + \epsilon$ for all $n \geq k$.

Then $\frac{u_{k+1}}{u_k} < \lambda^* + \epsilon, \frac{u_{k+2}}{u_{k+1}} < \lambda^* + \epsilon, \cdots, \frac{u_n}{u_{n-1}} < \lambda^* + \epsilon$

So $u_n < u_k(\lambda^* + \epsilon)^{n-k}$ for all $n > k$.

Hence for all $n > k, u_n < A(\lambda^* + \epsilon)^n$ where $A = \frac{u_k}{(\lambda^*+\epsilon)^k} > 0$.

Therefore $\sqrt[n]{u_n} < A^{1/n}(\lambda^* + \epsilon)$ for all $n > k$.

Consequently, $\overline{\lim} \sqrt[n]{u_n} \leq \overline{\lim} A^{1/n}(\lambda^* + \epsilon)$
$= \lambda^* + \epsilon$ since $\lim A^{1/n} = 1$.

Since ϵ is arbitrary, $\overline{\lim} \sqrt[n]{u_n} \leq \lambda^*$, i.e., $\mu^* \leq \lambda^*$.

In a similar manner we can prove $\lambda_* \leq \mu_*$.

Also the inequality $\mu_* \leq \mu^*$ follows from the property of the limit inferior and the limit superior of a sequence.

This completes the proof.

Note. If $u_n > 0$ for all n and $\lim \frac{u_{n+1}}{u_n}$ exists, then it follows from the theorem that $\lim \sqrt[n]{u_n}$ also exists.

Worked Examples (continued).

3. Prove that $\lim \sqrt[n]{n} = 1$.

Let $u_n = n$. Then $u_n > 0$ for all $n \in \mathbb{N}$ and $\lim \frac{u_{n+1}}{u_n} = 1 > 0$.
It follows that from the theorem that $\lim \sqrt[n]{n} = 1$.

4. Prove that $\lim \frac{(n!)^{1/n}}{n} = \frac{1}{e}$.

Let $u_n = \frac{n!}{n^n}$. Then $u_n > 0$ for all $n \in \mathbb{N}$ and $\lim \frac{u_{n+1}}{u_n} = \frac{1}{e} > 0$.

It follows from the theorem that $\lim \sqrt[n]{u_n} = \frac{1}{e}$, i.e., $\lim \frac{(n!)^{\frac{1}{n}}}{n} = \frac{1}{e}$.

5. Prove that $\lim \frac{\{(n+1)(n+2)\cdots(2n)\}^{1/n}}{n} = \frac{4}{e}$.

Let $u_n = \frac{(n+1)(n+2)\cdots 2n}{n^n}$. Then $u_n > 0$ for all $n \in \mathbb{N}$ and $\lim \frac{u_{n+1}}{u_n} = \lim \frac{2(2n+1)}{n+1} \cdot \frac{1}{(1+\frac{1}{n})^n} = \frac{4}{e} > 0$.

It follows from the theorem that $\lim \sqrt[n]{u_n} = \frac{4}{e}$.

Exercises 8

1. Establish the convergence and find the limit of the sequence $\{u_n\}$, where u_n is

(i) $(1 + \frac{1}{3n})^n$, (ii) $(1 + \frac{1}{n^2})^{n^2}$, (iii) $(1 + \frac{1}{3n+1})^n$, (iv) $(1 + \frac{1}{n^2+2})^{n^2}$.

2. Prove that the sequence $\{u_n\}$ is convergent by showing that the subsequences $\{u_{2n}\}$ and $\{u_{2n-1}\}$ converge to the same limit.

(i) $0 < u_1 < u_2$ and $u_{n+2} = \frac{1}{3}(u_{n+1} + 2u_n)$ for $n \geq 1$;

(ii) $0 < u_1 < u_2$ and $u_{n+2} = \sqrt{u_{n+1}u_n}$ for $n \geq 1$;

(iii) $0 < u_1 < u_2$ and $\frac{2}{u_{n+2}} = \frac{1}{u_{n+1}} + \frac{1}{u_n}$ for $n \geq 1$.

3. Prove that the sequence

(i) $\{2, \frac{2}{1+2}, \frac{2}{1+\frac{2}{1+2}}, \cdots\cdots\}$ converges to 1;

(ii) $\{6, \frac{6}{1+6}, \frac{6}{1+\frac{6}{1+6}}, \cdots\cdots\}$ converges to 2.

4. $\{x_n\}$ and $\{y_n\}$ are bounded sequences and a sequence $\{z_n\}$ is defined by $z_1 = x_1, z_2 = y_1; z_3 = x_2, z_4 = y_2; z_5 = x_3, z_6 = y_3; \ldots \ldots$. Prove that the sequence $\{z_n\}$ is convergent if and only if both the sequences $\{x_n\}$ and $\{y_n\}$ are convergent with the same limit.

5. (a) If $\overline{\lim}\, u_n = \infty$ prove that there exists a properly divergent subsequence $\{u_{r_n}\}$ of the sequence $\{u_n\}$ such that $\lim u_{r_n} = \infty$.

(b) If $\underline{\lim}\, u_n = -\infty$ prove that there exists a properly divergent subsequence $\{u_{r_n}\}$ of the sequence $\{u_n\}$ such that $\lim u_{r_n} = -\infty$.

6. A sequence $\{u_n\}$ is such that every subsequence of $\{u_n\}$ has a subsequence that converges to 0. Prove that $\lim u_n = 0$.

[**Hint.** Prove that $\overline{\lim}\, u_n$ and $\underline{\lim}\, u_n$ are both finite. Assume $\overline{\lim}\, u_n = l, \underline{\lim}\, u_n = m$. Prove that $l = 0, m = 0$.]

7. Find $\overline{\lim}\, u_n$ and $\underline{\lim}\, u_n$ where $u_n =$

(i) $(-1)^n(1 + \frac{1}{n})$, (ii) $n + \frac{(-1)^n}{n}$, (iii) $n^{(-1)^n}$, (iv) $(\cos \frac{n\pi}{4})^{(-1)^n}$.

8. Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequences and $u_n > 0, v_n > 0$ for all $n \in \mathbb{N}$. Prove that

(i) $\overline{\lim}\, u_n . \overline{\lim}\, v_n \geq \overline{\lim}\, u_n v_n$; (ii) $\underline{\lim}\, u_n . \underline{\lim}\, v_n \leq \underline{\lim}\, u_n v_n$.

9. Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequences such that the sequence $\{v_n\}$ is convergent. Prove that

(i) $\overline{\lim}\ (u_n + v_n) = \overline{\lim}\ u_n + \lim v_n$; (ii) $\underline{\lim}\ (u_n + v_n) = \underline{\lim}\ u_n + \lim v_n$.

10. Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequences with $u_n > 0, v_n > 0$ for all $n \in \mathbb{N}$ such that the sequence $\{v_n\}$ is convergent. Prove that

(i) $\overline{\lim}\ (u_n v_n) = \overline{\lim}\ u_n . \lim v_n$; (ii) $\underline{\lim}\ (u_n v_n) = \underline{\lim}\ u_n . \lim v_n$.

11. Let $\{u_n\}$ be a bounded sequence of real numbers and E be the set of all subsequential limits of $\{u_n\}$. Prove that E is a non-empty closed and bounded set and $\sup E \in E$, $\inf E \in E$.

12. $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. Prove directly that

(i) $\{u_n + v_n\}$ is a Cauchy sequence, (ii) $\{u_n v_n\}$ is a Cauchy sequence.

13. Establish from definition that $\{u_n\}$ is a Cauchy sequence, where

(i) $u_n = \frac{n}{n+1}$, (ii) $u_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$,

(iii) $| u_{n+2} - u_{n+1} | \leq \frac{1}{2} | u_{n+1} - u_n |$ for all $n \in \mathbb{N}$.

[**Hint.** (ii) $(n+1)! \geq 2^n$ for $n \geq 2$. (iii) $|u_{n+2} - u_{n+1}| \leq (\frac{1}{2})^n |u_2 - u_1|$.]

14. Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}$ and $\{y_n\}$ is a sequence in $\mathbb{R}$ such that $| x_n - y_n | < \frac{1}{n}$ for all $n \geq 1$. Prove that $\{y_n\}$ is a Cauchy sequence and $\lim x_n = \lim y_n$.

15. If $\{u_n\}$ be a Cauchy sequence in $\mathbb{R}$ having a subsequence converging to a real number l, prove that $\lim u_n = l$.

16.(i) Let $u_1 = 2$ and $u_{n+1} = 2 + \frac{1}{u_n}$ for $n \geq 1$. Prove that the sequence $\{u_n\}$ converges to the limit $\sqrt{2} + 1$.

(ii) Let $u_1 > 0$ and $u_{n+1} = \frac{1}{2+u_n}$ for $n \geq 1$. Prove that the sequence $\{u_n\}$ converges to the limit $\sqrt{2} - 1$.

[**Hint.** (i) $|u_{n+2} - u_{n+1}| < \frac{1}{4}|u_{n+1} - u_n|$. (ii) $|u_{n+2} - u_{n+1}| < \frac{1}{4}|u_{n+1} - u_n|$.]

17. If $u_n = \frac{1}{1.n} + \frac{1}{2(n-1)} + \frac{1}{3(n-2)} + \cdots + \frac{1}{n.1}$, prove that $\lim u_n = 0$.

[**Hint.** $(n+1)u_n = (1 + \frac{1}{n}) + (\frac{1}{2} + \frac{1}{n-1}) + \cdots + (\frac{1}{n} + 1)$.]

18. Prove that $\lim\limits_{n\to\infty} \dfrac{1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1}}{2n+1} = 0$.

19. Prove that (i) $\lim\limits_{n\to\infty} \frac{1}{n}\{(2n+1)(2n+2)\cdots(2n+n)\}^{\frac{1}{n}} = \frac{27}{4e}$,

(ii) $\lim\limits_{n\to\infty} \frac{1}{n}\{(a+1)(a+2)\cdots(a+n)\}^{\frac{1}{n}} = \frac{1}{e}$ if $a > 0$.

[**Hint.** (i) Let $u_n = \frac{(2n+1)(2n+2)\cdots(2n+n)}{n^n}$. Then $\lim \frac{u_{n+1}}{u_n} = \frac{27}{4e}$.]

6. SERIES

6.1. Infinite Series.

Let $\{u_n\}$ be a sequence. Then the sequence $\{s_n\}$ defined by

$$s_1 = u_1, s_2 = u_1 + u_2, s_3 = u_1 + u_2 + u_3, \cdots \cdots$$

is represented by the symbol $u_1 + u_2 + u_3 + \cdots \cdots$, which is said to be an *infinite series* (or a *series*) generated by the sequence $\{u_n\}$. The series is denoted by $\sum\limits_{n=1}^{\infty} u_n$ or by Σu_n. u_n is said to be the *nth term* of the series. The elements s_k of the sequence $\{s_n\}$ are called the partial sums of the series Σu_n.

The sequence $\{s_n\}$ is called the *sequence of partial sums* of the series Σu_n.

If $\{u_n\}$ be a real sequence then Σu_n is a series of real numbers.

We shall be mainly concerned with the series of real numbers.

The infinite series Σu_n is said to be *convergent* or *divergent* according as the sequence $\{s_n\}$ is convergent or divergent.

In case of convergence, if $\lim s_n = s$ then s is said to be the *sum* of the series Σu_n.

If, however, $\lim s_n = \infty$ (or $-\infty$) the series Σu_n is said to *diverge* to ∞ (or $-\infty$).

Examples.

1. Let us consider the series $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots \cdots$

Let the series be $\sum\limits_{n=1}^{\infty} u_n$. Then $u_n = \frac{1}{n(n+1)}$.

Let $s_n = u_1 + u_2 + \cdots + u_n$.

$$\begin{aligned} \text{Then } s_n &= \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{n(n+1)} \\ &= (1 - \tfrac{1}{2}) + (\tfrac{1}{2} - \tfrac{1}{3}) + \cdots + (\tfrac{1}{n} - \tfrac{1}{n+1}) \\ &= 1 - \frac{1}{n+1}, \end{aligned}$$

and $\lim s_n = 1$. Hence the series Σu_n is convergent and the sum of the series is 1.

2. Let us consider the series $1 + 2 + 3 + \cdots \cdots$

Let $s_n = 1+2+3+\cdots+n$. Then $s_n = \frac{n(n+1)}{2}$ and $\lim s_n = \infty$.
Hence the series is divergent.

3. Let us consider the series $1+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots$

Let $s_n = 1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}$.
Then $s_n = 2(1-\frac{1}{2^n}) = 2-\frac{1}{2^{n-1}}$ and $\lim s_n = 2$ since $\lim(\frac{1}{2})^{n-1} = 0$.
Therefore the series is convergent and the sum of the series is 2.

4. Let us consider the series $1-1+1-1+\cdots\cdots$

Let $s_n = 1-1+1-1+\cdots+(-1)^{n-1}$.
Then $s_n = 0$ if n be even,
$= 1$ if n be odd.
The sequence $\{s_n\}$ is divergent. Therefore the series is divergent.

5. Geometric series.

A. Let us consider the series $1+a+a^2+\cdots$ where $|a|<1$.

Let $s_n = 1+a+a^2+\cdots+a^{n-1}$. Then $s_n = \frac{1-a^n}{1-a} = \frac{1}{1-a} - \frac{a^n}{1-a}$.
$\lim s_n = \frac{1}{1-a}$ since $\lim a^n = 0$.
Therefore the series is convergent and the sum of the series is $\frac{1}{1-a}$.

B. Let us consider the series $1+a+a^2+\cdots$ where $|a|\geq 1$.
Let $s_n = 1+a+a^2+\cdots+a^{n-1}$.

Case 1. $a=1$. In this case $s_n = n$ and $\lim s_n = \infty$.
Therefore the series is divergent.

Case 2. $a>1$. In this case $s_n = \frac{a^n-1}{a-1}$ and $\lim s_n = \infty$ since $\lim a^n = \infty$ in this case.
Therefore the series is divergent.

Case 3. $a=-1$. In this case $s_n = 1$ if n be odd,
$= 0$ if n be even.
The sequence $\{s_n\}$ is divergent. Therefore the series is divergent.

Case 4. $a<-1$. In this case the sequence $\{s_n\}$ is divergent and therefore the series is divergent.

From (A) and (B), the geometric series $1+a+a^2+\cdots\cdots$ is convergent if $|a|<1$, and divergent if $|a|\geq 1$.

6. Harmonic series.

$1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots$

Let $\sum_{n=1}^{\infty} u_n$ be the series. Then $u_n = \frac{1}{n}$.

$$\begin{aligned}
\text{Let } s_n &= u_1 + u_2 + \cdots + u_n.\\
\text{Then } s_2 &= 1 + \tfrac{1}{2}\\
s_4 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} > 1 + \tfrac{1}{2} + (\tfrac{1}{4} + \tfrac{1}{4}) = 1 + 2.\tfrac{1}{2}\\
s_8 &= 1 + \tfrac{1}{2} + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8})\\
&> 1 + \tfrac{1}{2} + (\tfrac{1}{4} + \tfrac{1}{4}) + (\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}) = 1 + 3.\tfrac{1}{2}\\
s_{16} &> 1 + 4.\tfrac{1}{2}\\
&\cdots \quad \cdots\\
s_{2^n} &> 1 + n.\tfrac{1}{2}.
\end{aligned}$$

Therefore $\lim\limits_{n \to \infty} s_{2^n} = \infty$.

The sequence $\{s_n\}$ is a monotone increasing sequence, since $s_{n+1} - s_n = a_{n+1} > 0$ for all $n \in \mathbb{N}$. Since the subsequence $\{s_{2^n}\}$ diverges to ∞, the sequence $\{s_n\}$ is unbounded above and therefore the series $\sum\limits_{n=1}^{\infty} u_n$ is divergent.

Theorem 6.1.1. Let m be a natural number. Then the two series $u_1 + u_2 + u_3 + \cdots \;\cdots$ and $u_{m+1} + u_{m+2} + u_{m+3} + \cdots \;\cdots$ converge or diverge together.

Proof. Let $s_n = u_1 + u_2 + \cdots + u_n, t_n = u_{m+1} + u_{m+2} + \cdots + u_{m+n}$.

Then $t_n = s_{m+n} - s_m$, where s_m is a fixed number.

If $\{s_n\}$ converges then $\{t_n\}$ converges and conversely.

If $\{s_n\}$ diverges then $\{t_n\}$ diverges and conversely.

Therefore both the sequences $\{s_n\}$ and $\{t_n\}$ and consequently the series Σu_n and Σu_{m+n} converge or diverge together.

Note. The theorem states that we can remove from the beginning a finite number of terms from a given series, or add to the beginning a finite number of terms to a given series without changing its behaviour regarding convergence or divergence.

Theorem 6.1.2. If Σu_n and Σv_n be two convergent series having the sums s and t respectively then

(i) the series $\Sigma(u_n + v_n)$ converges to the sum $s + t$;

(ii) the series $\Sigma k u_n$, where k is a real number, converges to the sum ks.

The proof is immediate.

Theorem 6.1.3. (Cauchy's principle of convergence)

A necessary and sufficient condition for the convergence of a series Σu_n is that corresponding to a pre-assigned positive ϵ there exists a natural number m such that

$| u_{n+1} + u_{n+2} + \cdots + u_{n+p} | < \epsilon$ for all $n \geq m$ and for every natural number p.

Proof. Let $s_n = u_1 + u_2 + \cdots + u_n$.

Let Σu_n be convergent. Then the sequence $\{s_n\}$ is convergent. Therefore by Cauchy's principle of convergence for the sequence, corresponding to a pre-assigned positive ϵ there exists a natural number m such that

$| s_{n+p} - s_n | < \epsilon$ for all $n \geq m$ and for every natural number p.

or, $| u_{n+1} + u_{n+2} + \cdots + u_{n+p} | < \epsilon$ for all $n \geq m$ and for every natural number p.

Conversely, let us assume that for a pre-assigned positive ϵ there exists a natural number m such that

$| u_{n+1} + u_{n+2} + \cdots + u_{n+p} | < \epsilon$ for all $n \geq m$ and for every natural number p.

Then $| s_{n+p} - s_n | < \epsilon$ for all $n \geq m$ and for every natural number p.

This implies that the sequence $\{s_n\}$ is convergent by Cauchy's principle of convergence. Therefore Σu_n is convergent.

This completes the proof.

Theorem 6.1.4. A necessary condition for the convergence of a series Σu_n is $\lim u_n = 0$.

Proof. Let Σu_n be convergent. Then for a pre-assigned positive ϵ there exists a natural number m such that

$| u_{n+1} + u_{n+2} + \cdots + u_{n+p} | < \epsilon$ for all $n \geq m$ and for every natural number p.

Taking $p = 1$, $| u_{n+1} | < \epsilon$ for all $n \geq m$.

This implies $\lim u_n = 0$.

Note. The converse of the theorem is not true.

That is, $\lim u_n = 0$ does not necessarily imply convergence of the series Σu_n. Because the sufficient condition for the convergence of Σu_n states that for a chosen positive ϵ there must exist a natural number m such that

$| u_{n+1} + u_{n+2} + \cdots + u_{n+p} | < \epsilon$ for all $n \geq m$ and for $p = 1, 2, 3, \cdots$.

Therfore the sum of p consecutive terms of the series must be less than ϵ whatever natural number p may be. The condition must be satisfied for all p and not for only a particular p.

Let us consider the series Σu_n where $u_n = \frac{1}{n}$.

Here $\lim u_n = 0$. But Σu_n is a divergent series.

Here $| s_{n+p} - s_n | = | \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p} |$.

If we take $p = n$, $|\ s_{n+p} - s_n\ | \ = \ \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$
$$> \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{1}{2}.$$

Therefore $|\ s_{n+p} - s_n\ |$ cannot be made less than a chosen positive $\epsilon < \frac{1}{2}$ for every natural number p.

Worked Examples.

1. Prove that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \cdots$ is convergent.

Let the series be $\sum\limits_{n=1}^{\infty} u_n$. Then $u_n = (-1)^{n+1}\frac{1}{n}$.

Let $s_n = u_1 + u_2 + \cdots + u_n$. Then

$$\begin{aligned} |\ s_{n+p} - s_n\ | &= \left|\ \frac{1}{n+1} - \frac{1}{n+2} + \cdots + (-1)^{p-1}\frac{1}{n+p}\ \right| \\ &= \left|\ \frac{1}{n+1} - \left(\frac{1}{n+2} - \frac{1}{n+3}\right) - \cdots\ \right| \\ &< \frac{1}{n+1}. \end{aligned}$$

Let $\epsilon > 0$. Then $|\ s_{n+p} - s_n\ | < \epsilon$ holds if $n > \frac{1}{\epsilon} - 1$.

Let $m = [\frac{1}{\epsilon} - 1] + 2$. Then m is a natural number and $|\ s_{n+p} - s_n\ | < \epsilon$ for all $n \geq m$ and for $p = 1, 2, 3, \cdots$

This proves that the sequence $\{s_n\}$ is convergent and consequently the series Σu_n is convergent.

2. Prove that the series $\sum\limits_{n=1}^{\infty} u_n$ where $u_n = \frac{n}{n+1}$, is divergent.

Here $\lim u_n = 1$. Since $\lim u_n$ is not $0, \Sigma u_n$ is divergent because a necessary condition for convergence of the series Σu_n is $\lim u_n = 0$.

6.2. Series of positive terms.

A series Σu_n is said to be a *series of positive terms* if u_n is a positive real number for all $n \in \mathbb{N}$.

Theorem 6.2.1. A series of positive real numbers Σu_n is convergent if and only if the sequence $\{s_n\}$ of partial sums is bounded above.

Proof. $s_n = u_1 + u_2 + \cdots + u_n$. Then $s_{n+1} - s_n = u_{n+1} > 0$ for all $n \in \mathbb{N}$.

Hence the sequence $\{s_n\}$ is a monotone increasing sqeuence. Therefore $\{s_n\}$ is convergent if and only if it is bounded above.

Consequently, the series Σu_n is convergent if and only if the sequence $\{s_n\}$ is bounded above.

Note. If not bounded above, the sequence $\{s_n\}$ being a monotone increasing sequence, diverges to ∞. In this case the series diverges to ∞.

Therefore a series of positive real numbers either converges to a real number, or diverges to ∞.

Introduction and removal of brackets.

Let Σu_n be a series of positive real numbers. Let the terms of the series be arranged in groups without changing the order of the terms. Let us denote the nth group by v_n. Then a new series Σv_n is obtained.

Example.

Let $\Sigma u_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots\cdots$

Let us introduce brackets and the series takes the form
$1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + (\frac{1}{9} + \cdots + \frac{1}{16}) + \cdots$

If the new series be Σv_n then
$v_1 = 1, v_2 = \frac{1}{2}, v_3 = \frac{1}{3} + \frac{1}{4}, v_4 = \frac{1}{5} + \cdots + \frac{1}{8}, v_5 = \frac{1}{9} + \cdots + \frac{1}{16}, \cdots$

Σv_n is obtained from Σu_n by introduction of brackets and Σu_n is obtained from Σv_n by removal of brackets.

Theorem 6.2.2. Let Σu_n be a series of positive real numbers and Σv_n is obtained from Σu_n by grouping its terms. Then

(i) if Σu_n converges to the sum s, so does Σv_n ;

(ii) if Σv_n converges to the sum t, so does Σu_n.

Proof. Let $v_1 = u_1 + u_2 + \cdots + u_{r_1}, v_2 = u_{r_1+1} + \cdots + u_{r_2}, v_3 = u_{r_2+1} + \cdots + u_{r_3}, \cdots \cdots$

Then $\{r_n\}$ is a strictly increasing sequence of natural numbers.

Let $s_n = u_1 + u_2 + \cdots + u_n$, $t_n = v_1 + v_2 + \cdots + v_n$.

Then $t_n = u_1 + u_2 + \cdots + u_{r_n} = s_{r_n}$.

Let Σu_n be convergent and the sum of the series be s. Then $\lim s_n = s$. The sequence $\{s_{r_n}\}$ being a subsequence of the convergent sequence $\{s_n\}$, is convergent and $\lim s_{r_n} = s$. That is, $\lim t_n = s$.

This proves that the series Σv_n is convergent and the sum of the series is also s.

Let Σv_n be convergent and the sum of the series be t. Then $\lim t_n = t$. That is, $\lim s_{r_n} = t$.

$\{s_{r_n}\}$ is a convergent subsequence of the monotone increasing sequence $\{s_n\}$. By Theorem 5.11.4, the sequence $\{s_n\}$ is convergent and $\lim s_n = t$.

This proves that the series Σu_n is convergent and the sum of the series is also t. This completes the proof.

Note. The theorem does not hold if Σu_n be a series of arbitrary terms.

Let us consider the series
$1 - 1 + 1 - 1 + \cdots \cdots \ldots \quad \ldots$(A)

Introducing brackets we get the series

$(1-1)+(1-1)+\cdots\cdots\ldots\quad\ldots(B)$

Introducing brackets in another manner we get the series

$1-(1-1)-(1-1)-\cdots\cdots\ldots\quad\ldots(C)$

The series (A) is divergent but the series (B) converges to 0 and the series (C) converges to 1.

Re-arrangement of terms.

Let Σu_n be a given series. If a new series Σv_n is obtained by using each term of Σu_n exactly once, the order of the terms being disturbed, then Σv_n is called a re-arrangement of Σu_n.

If $f:\mathbb{N}\to\mathbb{N}$ be a bijective mapping, $\Sigma u_{f(n)}$ is a re-arrangement of Σu_n and conversely if Σv_n be a re-arrangement of the series Σu_n then $v_n=u_{f(n)}$ for some bijection $f:\mathbb{N}\to\mathbb{N}$.

For example, let
$$\begin{aligned} f(n) &= n+1 \text{ if } n \text{ be odd },\\ &= n-1 \text{ if } n \text{ be even.}\end{aligned}$$

$f(1)=2, f(2)=1, f(3)=4, f(4)=3,\cdots\cdots$

$\Sigma u_{f(n)}=u_2+u_1+u_4+u_3+\cdots\cdots$ is a re-arrangement of Σu_n.

Theorem 6.2.3. Let Σu_n be a convergent series of positive real numbers.Then any re-arrangement of Σu_n is convergent and the sum remains unaltered.

Proof. Let Σu_n converge to s and Σv_n be a re-arrangement of Σu_n. Then $v_n=u_{f(n)}$ for some bijection $f:\mathbb{N}\to\mathbb{N}$.

Let $s_n=u_1+u_2+\cdots+u_n,\ \ t_n=v_1+v_2+\cdots+v_n$.

Since $u_n>0$, the sequence $\{s_n\}$ is a monotone increasing sequence. As Σu_n converges to s, $\lim s_n=s$. Therefore the sequence $\{s_n\}$ is bounded above and $s_n\le s$ for all $n\in\mathbb{N}$.

$$\begin{aligned} t_n &= v_1+v_2+\cdots+v_n\\ &= u_{f(1)}+u_{f(2)}+\cdots+u_{f(n)}\\ &\le u_1+u_2+\cdots+u_{m(n)}, \text{ where } m(n)=\max\{f(1),\ldots,f(n)\}.\end{aligned}$$

But $u_1+u_2+\cdots+u_{m(n)}=s_{m(n)}\le s$.

Thus the sequence $\{t_n\}$ is bounded above and being a monotone increasing sequence, it is convergent. Let $\lim t_n=t$. Then $t\le s$.

$$\begin{aligned} s_n &= u_1+u_2+\cdots+u_n\\ &= v_{f^{-1}(1)}+v_{f^{-1}(2)}+\cdots+v_{f^{-1}(n)}\\ &\le v_1+v_2+\cdots+v_{k(n)}, \text{ where } k(n)=\max\{f^{-1}(1),\ldots,f^{-1}(n)\}.\end{aligned}$$

But $v_1+v_2+\cdots+v_{k(n)}=t_{k(n)}\le t$.

$s_n\le t\Rightarrow \lim s_n\le t$, i.e., $s\le t$

It follows that $s=t$.

This proves that the series Σv_n is convergent and the sum of the series is also s.

6.3. Tests for convergence of a series of positive terms.

The convergence or divergence of a particular series is decided by examining the sequence of partial sums of the series. In most cases the expression for s_n (the nth partial sum) becomes not so nice as can be easily handled to determine its nature in a straightforward manner. Some other elegent methods will be applied to the series that will decide the convergence of the series without prior knowledge of the nature of the sequence $\{s_n\}$. These methods, called 'tests for convergence', will be discussed here.

Theorem 6.3.1.(Comparison test [First type]).

A. Let Σu_n and Σv_n be two series of positive real numbers and there is a natural number m such that $u_n \leq kv_n$ for all $n \geq m, k$ being a fixed positive number.

Then (i) Σu_n is convergent if Σv_n is convergent

(ii) Σv_n is divergent if Σu_n is divergent.

Proof. Let $s_n = u_1 + u_2 + \cdots + u_n$, $t_n = v_1 + v_2 + \cdots + v_n$.

$$\begin{aligned} \text{Then } s_n - s_m &= u_{m+1} + u_{m+2} + \cdots + u_n \\ &\leq k(v_{m+1} + v_{m+2} + \cdots + v_n) \\ &= k(t_n - t_m) \end{aligned}$$

or, $s_n \leq kt_n + h$ where $h = s_m - kt_m$, a finite number.

(i) Let Σv_n be convergent. Then the sequence $\{t_n\}$ is bounded.

Let B be an upper bound. Then $t_n < B$ for all $n \in \mathbb{N}$.

Therefore $s_n < kB + h$ for all $n \geq m$.

This shows that the sequence $\{s_n\}$ is bounded above. $\{s_n\}$ being a monotone increasing sequence bounded above, is convergent.

Therefore Σu_n is convergent.

(ii) Let Σu_n is divergent. Then the sequence $\{s_n\}$ is not bounded above.

Since $s_n \leq kt_n + h$, the sequence $\{t_n\}$ is not bounded above. Therefore the series Σv_n is divergent.

B. Limit form.

Let Σu_n and Σv_n be two series of positive real numbers and $\lim \frac{u_n}{v_n} = l$ where l is a *non-zero* finite number.

Then the two series Σu_n and Σv_n converge or diverge together.

Proof. $l > 0$. Let us choose a positive ϵ such that $l - \epsilon > 0$. There a natural number m such that $l - \epsilon < \frac{u_n}{v_n} < l + \epsilon$ for all $n \geq m$.

Therefore $u_n < kv_n$ for all $n \geq m$ where $k = l + \epsilon > 0$ ... ... (i) and $v_n < k'u_n$ for all $n \geq m$ where $k' = \frac{1}{l-\epsilon} > 0$ ... ... (ii)

By comparison test A, it follows from (i) that Σu_n is convergent if Σv_n is convergent and Σv_n is divergent if Σu_n is divergent.

By comparison test A, it follows from (ii) that Σv_n is convergent if Σu_n is convergent and Σu_n is divergent if Σv_n is divergent.

Therefore the two series Σu_n and Σv_n converge or diverge together.

Note. If $\lim \frac{u_n}{v_n} = 0$, then for a pre-asigned positive number ϵ there exists a natural number m such that $0 < \frac{u_n}{v_n} < \epsilon$, for all $n \geq m$.

Therefore Σu_n is convergent if Σv_n is convergent.

If $\lim \frac{u_n}{v_n} = \infty$, then for a pre-asigned positive number G there exists a natural number m such that $\frac{u_n}{v_n} > G$, for all $n \geq m$.

Therefore Σu_n is divergent if Σv_n is divergent.

For example, if Σu_n be a convergent series of positive real numbers, then the series $\Sigma \frac{1}{n^p} u_n$ is convergent for all $p > 0$ and if Σu_n be a divergent series of positive real numbers, then the series $\Sigma n^p u_n$ is divergent for all $p > 0$.

In order to make use of the Comparison test we need to have a collection of series whose nature are known. The series $\Sigma \frac{1}{n^p}$ discussed in the following theorem will be an addition to the collection.

Theorem 6.3.2. The series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots\cdots$ converges for $p > 1$ and diverges for $p \leq 1$.

Proof. **Case 1.** $p > 1$. Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series. Then $u_n = \frac{1}{n^p}$.

Let Σv_n be obtained from Σu_n by grouping the terms as

$1 + (\frac{1}{2^p} + \frac{1}{3^p}) + (\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}) + (\frac{1}{8^p} + \cdots + \frac{1}{15^p}) + \cdots$

Then $v_1 = 1, v_2 = \frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{1}{2^{p-1}}$,

$v_3 = \frac{1}{4^p} + \cdots + \frac{1}{7^p} < \frac{4}{4^p} = \frac{1}{2^{2(p-1)}}$,

$v_4 = \frac{1}{8^p} + \cdots + \frac{1}{15^p} < \frac{8}{8^p} = \frac{1}{2^{3(p-1)}}$,

$\cdots \quad \cdots \quad \cdots$

Let $w_n = \{\frac{1}{2^{p-1}}\}^{n-1}$. Then $v_n < w_n$ for all $n \geq 2$.

But Σw_n is a geometric series of common ratio $\frac{1}{2^{p-1}}$.

Since $p > 1, 0 < \frac{1}{2^{p-1}} < 1$ and hence Σw_n is convergent.

Therefore Σv_n is convergent by Comparison test.

Since Σv_n is obtained from Σu_n by introduction of brackets, Σu_n is convergent.

Case 2. $p = 1$. In this case the series is $1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots$

Let $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

Then $s_{2n} - s_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$
$> \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{1}{2}$.

This shows that the sequence $\{s_n\}$ is not a Cauchy sequence and therefore is not convergent. Hence the series $\Sigma\frac{1}{n}$ is not convergent.

Case 3. $0 < p < 1$. Then $\frac{1}{2^p} > \frac{1}{2}, \frac{1}{3^p} > \frac{1}{3}, \cdots \cdots$

Therefore $\frac{1}{n^p} > \frac{1}{n}$ for all $n \geq 2$.

But $\Sigma\frac{1}{n}$ is divergent. Therefore $\Sigma\frac{1}{n^p}$ is divergent by Comparison test.

Case 4. $p \leq 0$. Then $\lim \frac{1}{n^p} \neq 0$ and therefore $\Sigma\frac{1}{n^p}$ is not convergent.

This completes the proof.

Worked Examples.

1. Test the convergence of the series $\frac{1+2}{2^3} + \frac{1+2+3}{3^3} + \frac{1+2+3+4}{4^3} + \cdots \cdots$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series. Then $u_n = \frac{(n+1)(n+2)}{2(n+1)^3} = \frac{n+2}{2(n+1)^2}$.

Let $v_n = \frac{1}{n}$. Then $\lim \frac{u_n}{v_n} = \lim \frac{n(n+2)}{2(n+1)^2} = \frac{1}{2}$.

Since Σv_n is divergent, Σu_n is divergent by Comparison test.

2. Test the convergence of the series $\frac{1}{1.2^2} + \frac{1}{2.3^2} + \frac{1}{3.4^2} + \cdots \cdots$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series. Then $u_n = \frac{1}{n(n+1)^2}$.

Let $v_n = \frac{1}{n^3}$. Then $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = 1$.

But Σv_n is convergent. Therefore Σu_n is convergent by Comparison test.

3. Test the convergence of the series Σu_n where $u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$

$u_n = \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$. Let $v_n = \frac{1}{n^2}$.

Then $\lim \frac{u_n}{v_n} = \lim \frac{2n^2}{n^2\{\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}\}} = 1$.

Since Σv_n is convergent, Σu_n is convergent by Comparison test.

Theorem 6.3.3. (Comparison test [Second type])

Let Σu_n and Σv_n be two series of positive real numbers and there is natural number m such that

$$\frac{u_{n+1}}{u_n} \leq \frac{v_{n+1}}{v_n} \text{ for all } n \geq m.$$

Then (i) Σu_n is convergent if Σv_n is convergent,

(ii) Σv_n is divergent if Σu_n is divergent.

Proof. $\frac{u_{m+1}}{u_m} \leq \frac{v_{m+1}}{v_m}, \frac{u_{m+2}}{u_{m+1}} \leq \frac{v_{m+2}}{v_{m+1}}, \cdots, \frac{u_n}{u_{n-1}} \leq \frac{v_n}{v_{n-1}}$, where $n > m$.

Therefore $\frac{u_n}{u_m} \leq \frac{v_n}{v_m}$ for all $n > m$

or, $u_n \le \frac{u_m}{v_m} v_n$ for all $n > m$.
or, $u_n \le k v_n$ for all $n > m$ and $k(= \frac{u_m}{v_m})$ is a positive number.

By Comparison test (first type), Σu_n is convergent if Σv_n is convergent and Σv_n is divergent if Σu_n is divergent.

Theorem 6.3.4. (D'Alembert's ratio test)

Let Σu_n be a series of positive real numbers and let $\lim \frac{u_{n+1}}{u_n} = l$.

Then Σu_n is convergent if $l < 1$, Σu_n is divergent if $l > 1$.

Proof. **Case 1.** $l < 1$.
Let us choose a positive ϵ such that $l + \epsilon < 1$.
Since $\lim \frac{u_{n+1}}{u_n} = l$, there exists a natural number m such that

$$l - \epsilon < \frac{u_{n+1}}{u_n} < l + \epsilon \text{ for all } n \ge m.$$

Let $l + \epsilon = r$. Then $0 < r < 1$.
We have $\frac{u_{m+1}}{u_m} < r, \frac{u_{m+2}}{u_{m+1}} < r, \cdots \cdots, \frac{u_n}{u_{n-1}} < r$ where $n > m$.
Consequently, $\frac{u_n}{u_m} < r^{n-m}$ for all $n > m$
or, $u_n < \frac{u_m}{r^m}.r^n$ for all $n > m$.

$\frac{u_m}{r^m}$ is a positive number and Σr^n is a geometric series of common ratio r where $0 < r < 1$ and therefore Σr^n is convergent.

Therefore Σu_n is convergent by Comparison test.

Case 2. $l > 1$.
Let us choose a positive ϵ such that $l - \epsilon > 1$.
Since $\lim \frac{u_{n+1}}{u_n} = l$, there exists a natural number k such that

$$l - \epsilon < \frac{u_{n+1}}{u_n} < l + \epsilon \text{ for all } n \ge k.$$

Let $l - \epsilon = p$. Then $p > 1$.
We have $\frac{u_{k+1}}{u_k} > p, \frac{u_{k+2}}{u_{k+1}} > p, \cdots, \frac{u_n}{u_{n-1}} > p$ where $n > k$.
Consequently, $\frac{u_n}{u_k} > p^{n-k}$ for all $n > k$ or, $u_n > \frac{u_k}{p^k}.p^n$ for all $n > k$.

$\frac{u_k}{p^k}$ is a positive number and Σp^n is a geometric series of common ratio $p > 1$ and therefore Σp^n is divergent.

Therefore Σu_n is divergent by Comparison test.

Note. When $l = 1$, the test fails to give a decision.
Let $u_n = \frac{1}{n}$. Then Σu_n is a divergent series and $\lim \frac{u_{n+1}}{u_n} = 1$
Let $u_n = \frac{1}{n^2}$. Then Σu_n is a convergent series and $\lim \frac{u_{n+1}}{u_n} = 1$.

Although for both the series $\lim \frac{u_{n+1}}{u_n} = 1$, one is a convergent series and the other is a divergent series.

Therefore if $\lim \frac{u_{n+1}}{u_n} = 1$, nothing can be said about the convergence or divergence of the series Σu_n.

Theorem 6.3.5. (Cauchy's root test)

Let Σu_n be a series of positive real numbers and let $\lim u_n^{1/n} = 1$.

Then Σu_n is convergent if $l < 1$, Σu_n is divergent if $l > 1$.

Proof. **Case 1.** $l < 1$.

Let us choose a positive ϵ such that $l + \epsilon < 1$.

Since $\lim u_n^{1/n} = l$, there exists a natural number m such that

$$l - \epsilon < u_n^{1/n} < l + \epsilon \text{ for all } n \geq m$$

or, $(l - \epsilon)^n < u_n < (l + \epsilon)^n$ for all $n \geq m$.

Let $l + \epsilon = r$. Then $o < r < 1$ and $u_n < r^n$ for all $n \geq m$.

But Σr^n is a geometric series of common ratio r where $0 < r < 1$. So Σr^n is convergent.

Therefore Σu_n is convergent by Comparison test.

Case 2. $l > 1$.

Let us choose a positive ϵ such that $l - \epsilon > 1$.

Since $\lim u_n^{1/n} = l$, there exists a natural number k such that

$$l - \epsilon < u_n^{1/n} < l + \epsilon \text{ for all } n \geq k$$

or, $(l - \epsilon)^n < u_n < (l + \epsilon)^n$ for all $n \geq k$.

Let $l - \epsilon = p$. Then $p > 1$ and $u_n > p^n$ for all $n \geq k$.

But Σp^n is a geometric series of common ratio $p > 1$. So Σp^n is divergent.

Therefore Σu_n is divergent by Comparison test.

Note. When $l = 1$, the test fails to give a decision.

Let $u_n = 1/n$. Then $\lim u_n^{1/n} = 1$ and Σu_n is a divergent series.

Let $u_n = 1/n^2$. Then $\lim u_n^{1/n} = 1$ and Σu_n is a convergent series.

Although for both the series $\lim u_n^{1/n} = 1$, one is a convergent series and the other is a divergent series.

Thus if $\lim u_n^{1/n} = 1$, nothing can be said about the convergence or divergence of the series Σu_n.

Worked Examples (continued).

4. Test the convergence of the series $1 + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \cdots \cdots$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series. Then $u_n = \frac{2n-1}{n!}$.

$\frac{u_{n+1}}{u_n} = \frac{2n+1}{(n+1)(2n-1)}$ and $\lim \frac{u_{n+1}}{u_n} = 0 < 1$.
By D'Alembert's ratio test, Σu_n is convergent.

5. Examine the convergence of the series $x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \cdots, x > 0$.

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series. Since $x > 0, \Sigma u_n$ is a series of positive terms. $\frac{u_{n+1}}{u_n} = \frac{nx}{n+1}$ and $\lim \frac{u_{n+1}}{u_n} = x$.
By D'Alembert's ratio test,
Σu_n is convergent if $x < 1$, Σu_n is divergent if $x > 1$.

When $x = 1$, the series becomes $1 + \frac{1}{2} + \frac{1}{3} + \cdots \cdots$ and this is divergent.

6. Test the convergence of the series $1 + \frac{1}{1!} + \frac{2^2}{2!} + \frac{3^3}{3!} + \cdots \cdots$

Ignoring the first term, let $\sum\limits_{n=1}^{\infty} u_n$ be the series. Then $u_n = \frac{n^n}{n!}$.
$\frac{u_{n+1}}{u_n} = (\frac{n+1}{n})^n$ and $\lim \frac{u_{n+1}}{u_n} = e > 1$.
Σu_n is divergent by D'Alembert's ratio test.
Therefore the given series is divergent.

7. Test the convergence of the series
$1 + \frac{1}{2^3} + \frac{1}{2^2} + \frac{1}{2^5} + \frac{1}{2^4} + \cdots \cdots$

Here $u_n = \{2^{n+(-1)^n}\}^{-1}$ and $\lim u_n^{1/n} = \lim\{2^{1+\frac{(-1)^n}{n}}\}^{-1} = \frac{1}{2}$.
Therefore the series is convergent by Cauchy's root test.

Note. Here $\frac{u_{n+1}}{u_n} = \frac{1}{8}$ if n be odd,
$= 2$ if n be even.

$\lim \frac{u_{n+1}}{u_n}$ does not exist and therefore the convergence of the series cannot be decided by D'Alembert's ratio test.

It follows that the root test is more powerful than the ratio test in deciding convergence of a series of positive real numbers.

The fact is explained by the theorem 5.16.4 which states that if $u_n > 0$ then
$\underline{\lim} \frac{u_{n+1}}{u_n} \leq \underline{\lim}\, u_n^{1/n} \leq \overline{\lim}\, u_n^{1/n} \leq \overline{\lim} \frac{u_{n+1}}{u_n}$.

If for some series Σu_n of positive terms $\lim \frac{u_{n+1}}{u_n}$ exists and equals l, then $\lim u_n^{1/n}$ also exists and equals l. Therefore if the ratio test decides the convergence of the series Σu_n then the root test also does.

But if for some series Σu_n of positive terms $\lim u_n^{1/n}$ exists and equals l, then $\lim \frac{u_{n+1}}{u_n}$ does not necessarily exist. Therefore if the root test decides the convergence of the series Σu_n, the ratio test may not do so.

Theorem 6.3.6. (General form of ratio test)

Let Σu_n be a series of positive real numbers and let

$$\overline{\lim}\ \frac{u_{n+1}}{u_n} = R, \quad \underline{\lim}\ \frac{u_{n+1}}{u_n} = r.$$

Then Σu_n is convergent if $R < 1$, Σu_n is divergent if $r > 1$.

Proof. **Case 1.** $R < 1$.

Let us choose a positive ϵ such that $R + \epsilon < 1$.

Since $\overline{\lim}\ \frac{u_{n+1}}{u_n} = R$, there exists a natural number m such that

$$\frac{u_{n+1}}{u_n} < R + \epsilon \text{ for all } n \geq m.$$

Let $R + \epsilon = p$. Then $0 < p < 1$.

We have $\frac{u_{m+1}}{u_m} < p, \frac{u_{m+2}}{u_{m+1}} < p, \cdots, \frac{u_n}{u_{n-1}} < p$ where $n > m$.

Consequently, $\frac{u_n}{u_m} < p^{n-m}$ for all $n > m$

or, $u_n < \frac{u_m}{p^m} p^n$ for all $n > m$.

$\frac{u_m}{p^m}$ is a positive number and Σp^n is convergent since $0 < p < 1$.

Therefore Σu_n is convergent by Comparison test.

Case 2. $r > 1$

Let us choose a positive ϵ such that $r - \epsilon > 1$.

Since $\underline{\lim}\ \frac{u_{n+1}}{u_n} = r$, there exists a natural number k such that

$$\frac{u_{n+1}}{u_n} > r - \epsilon \text{ for all } n \geq k.$$

Let $r - \epsilon = q$. Then $q > 1$.

We have $\frac{u_{k+1}}{u_k} > q, \frac{u_{k+2}}{u_{k+1}} > q, \cdots, \frac{u_n}{u_{n-1}} > q$.

Consequently, $\frac{u_n}{u_k} > q^{n-k}$ for all $n > k$

or, $u_n > \frac{u_k}{q^k} q^n$ for all $n > k$.

$\frac{u_k}{q^k}$ is a positive number and Σq^n is divergent since $q > 1$.

Therefore Σu_n is divergent by Comparison test.

Theorem 6.3.7. (General form of root test)

Let Σu_n be a series of positive real numbers and let $\overline{\lim}\ u_n^{1/n} = r$.

Then Σu_n is convergent if $r < 1$, Σu_n is divergent if $r > 1$.

Proof. **Case 1.** $r < 1$.

Let us choose a positive ϵ such that $r + \epsilon < 1$.

Since $\overline{\lim}\ u_n^{1/n} = r$, there exists a natural number m such that

$$u_n^{1/n} < r + \epsilon \text{ for all } n \geq m.$$

Let $r + \epsilon = p$. Then $0 < p < 1$ and $u_n < p^n$ for all $n \geq m$.

But Σp^n is a convergent series since $0 < p < 1$.

Therefore Σu_n is convergent by Comparison test.

Case 2. $r > 1$.

Let us choose a positive ϵ such that $r - \epsilon > 1$.

Since $\overline{\lim}\, u_n^{1/n} = r, u_n^{1/n} > r - \epsilon$ for infinite number of values of n.

That is, infinite number of elements of the sequence $\{u_n\}$ are greater than 1 and therefore $\lim u_n$ cannot be 0.

Therefore Σu_n is divergent, since a necessary condition for convergence of the series Σu_n is $\lim u_n = 0$.

Worked Examples (continued).

8. Test the convergence of the series.

$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots\cdots$

Let $\sum\limits_1^\infty u_n$ be the given series.

Then $u_{2n} = \frac{1}{3^n}, u_{2n+1} = \frac{1}{2^{n+1}}, u_{2n-1} = \frac{1}{2^n}$.

$\lim \frac{u_{2n}}{u_{2n-1}} = \lim(\frac{2}{3})^n = 0, \lim \frac{u_{2n+1}}{u_{2n}} = \lim \frac{1}{3}(\frac{3}{2})^{n+1} = \infty.$

It follows that $\limsup \frac{u_{n+1}}{u_n} = \infty, \liminf \frac{u_{n+1}}{u_n} = 0$.

Clearly, the ratio test gives no decision.

$\lim(u_{2n})^{1/2n} = \frac{1}{\sqrt{3}}, \lim(u_{2n+1})^{1/(2n+1)} = \frac{1}{\sqrt{2}}$.

It follows that $\limsup(u_n)^{1/n} = \frac{1}{\sqrt{2}} < 1$.

Therefore Σu_n is convergent by the root test.

9. Test the series

$a + b + a^2 + b^2 + a^3 + b^3 + \cdots\cdots$ where $0 < a < b < 1$.

Let $\sum\limits_1^\infty u_n$ be the given series.

Here $\frac{u_{2n}}{u_{2n-1}} = (\frac{b}{a})^n, \frac{u_{2n+1}}{u_{2n}} = a(\frac{a}{b})^n$. $\lim \frac{u_{2n}}{u_{2n-1}} = \infty, \lim \frac{u_{2n+1}}{u_{2n}} = 0$.

It follows that $\overline{\lim}\, \frac{u_{n+1}}{u_n} = \infty, \underline{\lim}\, \frac{u_{n+1}}{u_n} = 0$.

The ratio test gives no decision.

$\lim u_{2n}^{1/2n} = \lim(b^n)^{1/2n} = \sqrt{b}$,

$\lim(u_{2n+1})^{1/(2n+1)} = \lim(a^{n+1})^{\frac{1}{2n+1}} = \sqrt{a}$.

It follows that $\limsup(u_n)^{1/n} = \sqrt{b} < 1$.

Therefore Σu_n is convergent by the root test.

Note. Here the ratio test does not decide convergence of the series but the root test does. The root test is more powerful than the ratio test for deciding convergence of a series positive real numbers.

Theorem 6.3.8. (Cauchy's condensation test)

Let $\{f(n)\}$ be a monotone decreasing sequence of positive real numbers and a be a positive integer > 1.

Then the series $\sum_1^\infty f(n)$ and $\sum_1^\infty a^n f(a^n)$ converge or diverge together.

Proof. Grouping the terms of $\Sigma f(n)$ as

$\{f(1)\}+\{f(2)+\cdots+f(a)\}+\{f(a+1)+\cdots+f(a^2)\}+\cdots$ and ignoring the first term, let Σv_n be the new series.

Then $v_n = f(a^{n-1}+1) + f(a^{n-1}+2) + \cdots + f(a^n)$ for all $n \geq 1$.

The number of terms in v_n is $a^n - a^{n-1}$. Since $\{f(n)\}$ is a monotone decreasing sequence, each term of $v_n \leq f(a^{n-1}+1)$ and $\geq f(a^n)$.

Therefore $(a^n - a^{n-1})f(a^n) \leq v_n$ for all $n \geq 1$

or, $\frac{a-1}{a} a^n f(a^n) \leq v_n$ for all $n \geq 1$.

Let $w_n = a^n f(a^n)$. Then $w_n \leq \frac{a}{a-1} v_n$ for all $n \geq 1$.

$\frac{a}{a-1}$ is positive. By Comparison test,

Σw_n is convergent if Σv_n is convergent

and Σv_n is divergent if Σw_n is divergent. ... ... (A)

$$\begin{aligned} \text{Again, } v_n &\leq (a^n - a^{n-1})f(a^{n-1}+1) \\ &\leq (a^n - a^{n-1})f(a^{n-1}) \text{ for all } n \geq 2. \end{aligned}$$

That is, $v_n \leq (a-1)w_{n-1}$ for all $n \geq 2$.

$a - 1$ is positive. By Comparison test,

Σv_n is convergent if Σw_n is convergent

and Σw_n is divergent if Σv_n is divergent. ... ... (B)

From (A) and (B), Σv_n and Σw_n converge or diverge together.

But Σv_n and $\Sigma f(n)$ converge or diverge together.

Therefore $\Sigma f(n)$ and Σw_n, i.e., $\Sigma f(n)$ and $\Sigma a^n f(a^n)$ converge or diverge together.

Worked Examples (continued).

10. Test the convergence of the series $\sum_1^\infty \frac{1}{n}$.

Let $f(n) = \frac{1}{n}$. Then $\{f(n)\}$ is a monotone decreasing sequence of positive real numbers.

By Cauchy's condensation test, the two series $\Sigma f(n)$ and $\Sigma 2^n f(2^n)$ converge or diverge together.

$2^n f(2^n) = 1$ and therefore $\Sigma 2^n f(2^n)$ is divergent.

It follows that $\Sigma f(n)$ is divergent, i.e., $\Sigma 1/n$ is divergent.

11. Discuss the convergence of the series $\sum_1^\infty 1/n^p$, $p > 0$.

Let $f(n) = 1/n^p$. As $p > 0$, the sequence $\{f(n)\}$ is a monotone decreasing sequence of positive real numbers.

By Cauchy's condensation test, the two series $\Sigma f(n)$ and $\Sigma 2^n f(2^n)$ converge or diverge together.

$2^n f(2^n) = 2^n . \frac{1}{2^{np}} = \frac{1}{2^{n(p-1)}}$.

But $\Sigma(\frac{1}{2^{p-1}})^n$ is a geometric series and it converges if $p > 1$ and diverges if $p \leq 1$.

Therefore $\Sigma 1/n^p$ is convergent when $p > 1$ and is divergent when $0 < p \leq 1$.

12. Discuss the convergence of the series $\sum\limits_{2}^{\infty} \dfrac{1}{n(\log n)^p}$, $p > 0$.

Let $f(n) = \frac{1}{n(\log n)^p}, n \geq 2$. As $\{\log n\}$ is an increasing sequence and $p > 0$, $\{\log(n+1)\}^p > \{\log n\}^p$ and therefore $(n+1)\{\log(n+1)\}^p > n(\log n)^p$.

Therefore $\{f(n)\}_2^\infty$ is a monotone decreasing sequence of positive real numbers.

By Cauchy's condensation test, the two series $\Sigma f(n)$ and $\Sigma 2^n f(2^n)$ converge or diverge together.

$\Sigma 2^n f(2^n) = \Sigma \frac{1}{(n \log 2)^p}$ and this converges when $p > 1$ and diverges when $p \leq 1$.

Therefore $\sum\limits_{n=2}^{\infty} f(n)$ is convergent when $p > 1$ and divergent when $0 < p \leq 1$.

If the limits $\lim \frac{u_{n+1}}{u_n}$ or $\lim \sqrt[n]{u_n}$ be equal to 1, D'Alembert's ratio test and Cauchy's root test fail to decide convergence of the series Σu_n. In such cases it is often helpful to use a more delicate test due to Raabe.

Theorem 6.3.9. (Raabe's test)

Let Σu_n be a series of positive real numbers and let $\lim n(\frac{u_n}{u_{n+1}} - 1) = l$.

Then Σu_n is convergent if $l > 1$, Σu_n is divergent if $l < 1$.

Proof. **Case 1.** $l > 1$.

Let us choose a positive ϵ such that $l - \epsilon > 1$.

Since $\lim n(\frac{u_n}{u_{n+1}} - 1) = l$, there exists a natural number m such that

$$l - \epsilon < n(\frac{u_n}{u_{n+1}} - 1) < l + \epsilon \text{ for all } n \geq m.$$

Let $l - \epsilon = r$. Then $r > 1$.

We have $nu_n - nu_{n+1} > ru_{n+1}$ for all $n \geq m$

or, $nu_n - (n+1)u_{n+1} > (r-1)u_{n+1}$ for all $n \geq m$.

We have $mu_m - (m+1)u_{m+1} > (r-1)u_{m+1}$
$(m+1)u_{m+1} - (m+2)u_{m+2} > (r-1)u_{m+2}$
$\cdots \quad \cdots$
$(n-1)u_{n-1} - nu_n > (r-1)u_n$ where $n > m$.

Consequently, $mu_m - nu_n > (r-1)(u_{m+1} + u_{m+2} + \cdots + u_n)$ for all $n > m$

or, $u_{m+1} + u_{m+2} + \cdots + u_n < \frac{1}{r-1}(mu_m - nu_n) < \frac{1}{r-1}mu_m$

or, $s_n - s_m < \frac{1}{r-1}mu_m$ where $s_n = u_1 + u_2 + \cdots + u_n$

or, $s_n < \frac{1}{r-1}mu_m + s_m$ for all $n > m$.

This shows that the sequence $\{s_n\}$ is bounded above and therefore the series Σu_n is convergent.

Case 2. $l < 1$.

Let us choose a positive ϵ such that $l + \epsilon < 1$. There exists a natural number k such that

$$l - \epsilon < n(\tfrac{u_n}{u_{n+1}} - 1) < l + \epsilon \text{ for all } n \geq k.$$

Let $l + \epsilon = p$. Then $p < 1$.

We have $n(\frac{u_n}{u_{n+1}} - 1) < p < 1$ for all $n \geq k$.

Therefore $n(u_n - u_{n+1}) < pu_{n+1} < u_{n+1}$ for all $n \geq k$

or, $nu_n < (n+1)u_{n+1}$ for all $n \geq k$.

We have $ku_k < (k+1)u_{k+1}$
$(k+1)u_{k+1} < (k+2)u_{k+2}$
$\cdots \quad \cdots$
$(n-1)u_{n-1} < nu_n$ where $n > k$.

Consequently, $nu_n > ku_k$ for all $n > k$

or, $u_n > ku_k.\frac{1}{n}$.

ku_k is a positive number and $\Sigma\frac{1}{n}$ is a divergent series.

Therefore Σu_n is divergent by Comparison test.

Note. If $l = 1$, the test is inconclusive. This can be established by taking the series Σu_n where $u_n > 0$ for all $n \in \mathbb{N}$ and $\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{2}{n \log n}, n \geq 2$ and the series Σv_n where $v_n = \frac{1}{n}$, for all $n \in \mathbb{N}$.

Theorem 6.3.10. (General form of Raabe's test)

Let Σu_n be a series of positive real numbers and let $\overline{\lim}\, n(\frac{u_n}{u_{n+1}} - 1) = R$ and $\underline{\lim}\, n(\frac{u_n}{u_{n+1}} - 1) = r$.
Then Σu_n is convergent if $r > 1$, Σu_n is divergent if $R < 1$.

Proof. **Case 1.** $r > 1$.

Let us choose a positive ϵ such that $r - \epsilon > 1$.

Since $\underline{\lim}\, n(\frac{u_n}{u_{n+1}} - 1) = r$, there exists a natural number m such that $n(\frac{u_n}{u_{n+1}} - 1) > r - \epsilon$ for all $n \geq m$.

Let $r - \epsilon = k$. Then $k > 1$.
We have $nu_n - nu_{n+1} > ku_{n+1}$ for all $n \geq m$
or, $nu_n - (n+1)u_{n+1} > (k-1)u_{n+1}$ for all $n \geq m$
We have $mu_m - (m+1)u_{m+1} > (k-1)u_{m+1}$

$$(m+1)u_{m+1} - (m+2)u_{m+2} > (k-1)u_{m+2}$$
$$\cdots \quad \cdots$$
$$(n-1)u_{n-1} - nu_n > (k-1)u_n \quad \text{where } n > m.$$

Consequently, $mu_m - nu_n > (k-1)(u_{m+1} + u_{m+2} + \cdots + u_n)$ for all $n > m$

$$\begin{aligned} \text{or, } u_{m+1} + u_{m+2} + \cdots + u_n &< \tfrac{1}{k-1}(mu_m - nu_n) \\ &< \tfrac{1}{k-1}mu_m. \end{aligned}$$

Let $s_n = u_1 + u_2 + \cdots + u_n$.
Then $s_n < s_m + \frac{1}{k-1}mu_m$ for all $n > m$.
This shows that the sequence $\{s_n\}$ is bounded above and therefore the series Σu_n is convergent.

Case 2. $R < 1$.

Let us choose a positive ϵ such that $R + \epsilon < 1$.
Since $\overline{\lim}\, n(\frac{u_n}{u_{n+1}} - 1) = R$, there exists a natural number k such that $n(\frac{u_n}{u_{n+1}} - 1) < R + \epsilon$ for all $n \geq k$.

Let $R + \epsilon = p$. Then $p < 1$.
We have $n(u_n - u_{n+1}) < pu_{n+1}$ for all $n \geq k$
i.e., $n(u_n - u_{n+1}) < u_{n+1}$ for all $n > k$
or, $nu_n < (n+1)u_{n+1}$ for all $n \geq k$.

We have $ku_k < (k+1)u_{k+1}$

$$(k+1)u_{k+1} < (k+2)u_{k+2}$$
$$\cdots \quad \cdots$$
$$(n-1)u_{n-1} < nu_n \quad \text{where } n > k.$$

Therefore $ku_k < nu_n$ for all $n > k$
or, $u_n > ku_k.\frac{1}{n}$ for all $n > k$.
ku_k is positive and $\Sigma 1/n$ is a divergent series.
Therefore Σu_n is divergent by Comparison test.

Worked Example (continued).

13. Test the convergence of the series
$1 + \frac{1}{2}.\frac{1}{3} + \frac{1.3}{2.4}.\frac{1}{5} + \frac{1.3.5}{2.4.6}.\frac{1}{7} + \cdots \cdots$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series.

Then $u_n = \frac{1.3.5\cdots(2n-3)}{2.4.6\cdots(2n-2)} \cdot \frac{1}{2n-1}$ for all $n \geq 2$.

Therefore $\frac{u_{n+1}}{u_n} = \frac{(2n-1)^2}{2n(2n+1)}$ and $\lim \frac{u_{n+1}}{u_n} = 1$.

D'Alembert's ratio test gives no decision.

Let us apply Raabe's test.

$\lim\limits_{n\to\infty} n(\frac{u_n}{u_{n+1}} - 1) = \lim\limits_{n\to\infty} \frac{6n^2-n}{(2n-1)^2} = \frac{3}{2} > 1$.

Therefore Σu_n is convergent by Raabe's test.

Theorem 6.3.11. (Logarithmic test)

Let Σu_n be a series of positive real numbers and $\lim n \log \frac{u_n}{u_{n+1}} = l$. Then Σu_n is convergent if $l > 1$, Σu_n is divergent if $l < 1$.

Proof. **Case 1.** $l > 1$.

Let us choose a positive ϵ such that $l - \epsilon > 1$.

Since $\lim n \log \frac{u_n}{u_{n+1}} = l$, there exists a natural number m such that

$$l - \epsilon < n \log \frac{u_n}{u_{n+1}} < l + \epsilon \text{ for all } n \geq m.$$

Let $l - \epsilon = r$. Then $r > 1$.

We have $n \log \frac{u_n}{u_{n+1}} > r > 1$ for all $n \geq m$

or, $\frac{u_n}{u_{n+1}} > e^{r/n}$ for all $n \geq m$.

Since $\{(1 + \frac{1}{n})^n\}$ is a monotonic increasing sequence converging to e and e is irrational, $(1 + \frac{1}{n})^n < e$ for all $n \in \mathbb{N}$.

$$\text{It follows that } \frac{u_n}{u_{n+1}} > (1 + \tfrac{1}{n})^r \text{ for all } n \geq m$$
$$= \frac{(n+1)^r}{n^r}.$$

Let $v_n = \frac{1}{n^r}$. Then $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$ for all $n \geq m$.

That is, $\frac{u_{n+1}}{u_n} < \frac{v_{n+1}}{v_n}$ for all $n \geq m$.

By Comparison test, Σu_n is convergent since Σv_n is convergent.

Case 2. $0 \leq l < 1$.

Let us choose a positive ϵ such that $l + \epsilon < 1$.

Since $\lim n \log \frac{u_n}{u_{n+1}} = l$, there exists a natural number p such that

$$l - \epsilon < n \log \frac{u_n}{u_{n+1}} < l + \epsilon \text{ for all } n \geq p.$$

Let $l + \epsilon = k$. Then $0 < k < 1$.

$n \log \frac{u_n}{u_{n+1}} < k$ for all $n \geq p$

or, $\frac{u_n}{u_{n+1}} < e^{k/n}$ for all $n \geq p$.

Since $\{(1 + \frac{1}{n-1})^n\}_{n=2}^{\infty}$ is a monotone decreasing sequence converging to e and e is irrational, $(1 + \frac{1}{n-1})^n > e$ for all $n \geq 2$.

Therefore $\frac{u_n}{u_{n+1}} < (\frac{n}{n-1})^k$ for all $n \geq p > 1$.

Let $w_n = \frac{1}{(n-1)^k}$ for $n \geq 2$. Then $\sum\limits_2^\infty w_n$ is divergent and $\frac{u_{n+1}}{u_n} > \frac{w_{n+1}}{w_n}$ for all $n \geq p > 1$.

By Comparison test, Σu_n is divergent.

Case 3. $l < 0$.

Let us choose a positive ϵ such that $l + \epsilon < 0$.

Since $\lim n \log \frac{u_n}{u_{n+1}} = l$, there exists a natural number q such that

$$n \log \frac{u_n}{u_{n+1}} < l + \epsilon \text{ for all } n \geq q.$$

Let $l + \epsilon = s$. Then $s < 0$ and $n \log \frac{u_n}{u_{n+1}} < s < 0$ for all $n \geq q$

or, $n \log \frac{u_{n+1}}{u_n} > -s > 0$

or, $n \log \frac{u_{n+1}}{u_n} > p' > 0$ (where $p' = -s$) for all $n \geq q$

or, $\frac{u_{n+1}}{u_n} > e^{p/n}$ for all $n \geq q$.

Since $e > (1 + \frac{1}{n})^n$ for all $n \in \mathbb{N}$, it follows that $\frac{u_{n+1}}{u_n} > (1 + \frac{1}{n})^{p'}$ for all $n \geq q$.

Let $w_n = n^{p'}$. Then $\frac{u_{n+1}}{u_n} > \frac{w_{n+1}}{w_n}$ for all $n \geq q$.

As Σw_n is a divergent series, Σu_n is divergent by Compasiron test.

Worked Example (continued).

14. Test the convergence of the series

$1 + \frac{x}{1!} + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \cdots \quad \cdots, x > 0$

Ignoring the first term, let $\sum\limits_{n=1}^\infty u_n$ be the given series.

Then $u_n = \frac{n^n x^n}{n!}$. $\frac{u_{n+1}}{u_n} = (1 + \frac{1}{n})^n x$ and $\lim \frac{u_{n+1}}{u_n} = ex$.

By D'Alembert's ratio test,

Σu_n is convergent if $0 < x < 1/e$, Σu_n is divergent if $x > 1/e$.

When $x = 1/e$, let us apply Logarithmic test.

$\lim n \log \frac{u_n}{u_{n+1}} = \lim n[1 + n \log \frac{n}{n+1}] = \underline{\lim[n + n^2 \log \frac{n}{n+1}] = \frac{1}{2}}$.

By Logarithmic test, Σu_n is divergent when $x = 1/e$.

So the given series is convergent if $0 < x < \frac{1}{e}$ and divergent if $x \geq \frac{1}{e}$.

Theorem 6.3.12. (Kummer's test)

Let Σu_n and $\Sigma 1/d_n$ be two series of positive real numbers and let $w_n = \frac{u_n}{u_{n+1}} d_n - d_{n+1}$.

If $\lim w_n = k > 0$ then Σu_n is convergent.

If $\lim w_n = k < 0$ and $\Sigma 1/d_n$ is divergent then Σu_n is divergent.

Proof. Case 1. $k > 0$.

Let us choose a positive ϵ such that $k - \epsilon > 0$.

Since $\lim w_n = k$, there exists a natural number m such that

$$k - \epsilon < w_n < k + \epsilon \text{ for all } n \geq m.$$

Let $k - \epsilon = r$. Then $r > 0$ and $\frac{u_n d_n}{u_{n+1}} - d_{n+1} > r$ for all $n \geq m$

or, $u_n d_n - u_{n+1} d_{n+1} > r u_{n+1}$ for all $n \geq m$.

Then we have $u_m d_m - u_{m+1} d_{m+1} > r u_{m+1}$

$u_{m+1} d_{m+1} - u_{m+2} d_{m+2} > r u_{m+2}$

$\ldots \quad \ldots \quad \ldots$

$u_{n-1} d_{n-1} - u_n d_n > r u_n$, where $n > m$.

So $u_m d_m - u_n d_n > r(u_{m+1} + u_{m+2} + \cdots + u_n)$ for all $n > m$

or, $u_{m+1} + u_{m+2} + \cdots + u_n < \frac{1}{r}(u_m d_m - u_n d_n)$

$$< \tfrac{1}{r} u_m d_m.$$

or, $s_n - s_m < \frac{1}{r} u_m d_m$, where $s_n = u_1 + u_2 + \cdots + u_n$

or, $s_n < s_m + \frac{1}{r} u_m d_m$ for all $n > m$.

The sequence $\{s_n\}$ is bounded above and therefore Σu_n is convergent.

Case 2. $k < 0$.

Let us choose a positive ϵ such that $k + \epsilon < 0$.

Then there exists a natural number p such that

$$k - \epsilon < w_n < k + \epsilon \text{ for all } n \geq p.$$

So $\frac{u_n d_n}{u_{n+1}} - d_{n+1} < 0$ for all $n \geq p$.

or, $u_n d_n < u_{n+1} d_{n+1}$ for all $n \geq p$.

We have $u_p d_p < u_{p+1} d_{p+1}$

$u_{p+1} d_{p+1} < u_{p+2} d_{p+2}$

$\ldots \quad \ldots \quad \ldots$

$u_{n-1} d_{n-1} < u_n d_n$ for all $n > p$.

So $u_p d_p < u_n d_n$ for all $n > p$.

or, $u_n > \frac{u_p d_p}{d_n}$ for all $n > p$.

$u_p d_p$ is positive and $\Sigma \frac{1}{d_n}$ is a divergent series.

Therefore Σu_n is divergent by Comparison test.

Corollary 1. If we take $d_n = n$ then

$w_n = n \frac{u_n}{u_{n+1}} - (n+1) = n(\frac{u_n}{u_{n+1}} - 1) - 1$.

$\lim w_n = \lim n(\frac{u_n}{u_{n+1}} - 1) - 1$.

Let $\lim n(\frac{u_n}{u_{n+1}} - 1) = l$. Then Kummer's test gives Σu_n is convergent if $l > 1$, Σu_n is divergent if $l < 1$. This is Rabbe's test.

Corollary 2. If we take $d_n = 1$ then $w_n = \frac{u_n}{u_{n+1}} - 1$ and $\lim w_n = \lim(\frac{u_n}{u_{n+1}} - 1)$.

Let $\lim \frac{u_{n+1}}{u_n} = l$. Then Kummer's test gives
Σu_n is convergent if $\frac{1}{l} > 1$, i.e., if $l < 1$;
Σu_n is divergent if $\frac{1}{l} < 1$, i.e., if $l > 1$.
This is D'Alembert's ratio test.

Theorem 6.3.13. (Gauss's test).

Let Σu_n be a series of positive real numbers and let $\frac{u_n}{u_{n+1}} = 1+\frac{a}{n}+\frac{b_n}{n^p}$, where $p > 1$ and the sequence $\{b_n\}$ is bounded.

Then Σu_n is convergent if $a > 1$, Σu_n is divergent if $a \leq 1$.

Proof. **Case 1.** $a \neq 1$.
$\lim n(\frac{u_n}{u_{n+1}} - 1) = \lim n(\frac{a}{n} + \frac{b_n}{n^p}) = a$, since $\lim \frac{b_n}{n^{p-1}} = 0$.
By Raabe's test,
Σu_n is convergent if $a > 1$ and Σu_n is divergent if $a < 1$.

Case 2. $a = 1$.
Then $\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{b_n}{n^p}$.
Let us apply Kummer's test by taking $d_n = n \log n$.

$$\begin{aligned}
\text{Then } w_n &= \frac{d_n u_n}{u_{n+1}} - d_{n+1}\\
&= n \log n(1 + \frac{1}{n} + \frac{b_n}{n^p}) - (n+1)\log(n+1)\\
&= (n+1)\log n + \frac{b_n \log n}{n^{p-1}} - (n+1)\log(n+1)\\
&= (n+1)\log \frac{n}{n+1} + \frac{\log n}{n^{p-1}} b_n.
\end{aligned}$$

$$\begin{aligned}
\lim w_n &= \lim(n+1)\log(1 - \tfrac{1}{n+1}) + \lim \tfrac{\log n}{n^{p-1}}.b_n\\
&= -1, \text{ since } \lim \log(1 - \tfrac{1}{n+1})^{n+1} = \log e^{-1} = -1 \text{ and}
\end{aligned}$$

$\lim \frac{\log n}{n^{p-1}} = 0$ and $\{b_n\}$ is a bounded sequence.
By Kummer's test, Σu_n is divergent.

Worked Examples (continued).

15. Examine the convergence of the series $1 + \frac{2^2}{3^2} + \frac{2^2.4^2}{3^2.5^2} + \frac{2^2.4^2.6^2}{3^2.5^2.7^2} + \cdots\cdots$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series.

Then $u_n = \frac{2^2.4^2\cdots(2n-2)^2}{3^2.5^2\cdots(2n-1)^2}$ for all $n \geq 2$.

$\frac{u_{n+1}}{u_n} = \frac{4n^2}{4n^2+4n+1}$ and $\lim \frac{u_{n+1}}{u_n} = 1$.
D'Alembert's ratio test gives no decision.

Let us apply Raabe's test.

$\lim n(\frac{u_n}{u_{n+1}} - 1) = \lim n(\frac{4n+1}{4n^2+4n+1}) = 1$.
Raabe's test gives no decision.

Let us apply Gauss's test.

$\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{1}{4n^2}$.

$\frac{u_n}{u_{n+1}}$ is of the form $1 + \frac{a}{n} + \frac{b_n}{n^2}$, where $a = 1$ and $b_n = \frac{1}{4}$ and so $\{b_n\}$ is a bounded sequence.

By Gauss's test, Σu_n is divergent.

16. Hypergeometric series.

$$1 + \frac{\alpha.\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3 + \cdots,$$

where $\alpha, \beta, \gamma, x > 0$.

Ignoring the first term, let $\sum\limits_{1}^{\infty} u_n$ be the series.

Then $u_n = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{1.2\cdots.n\gamma(\gamma+1)\cdots(\gamma+n-1)}x^n$ for $n \geq 1$.

$\frac{u_{n+1}}{u_n} = \frac{(\alpha+n)(\beta+n)}{(1+n)(\gamma+n)}x$ and $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = x$.

By D'Alembert's ratio test, Σu_n is convergent if $0 < x < 1$ and Σu_n is divergent if $x > 1$.

When $x = 1$,

$\frac{u_n}{u_{n+1}} = \frac{(n+1)(n+\gamma)}{(n+\alpha)(n+\beta)}$

$= 1 + (\frac{(\gamma+1-\alpha-\beta)n+(\gamma-\alpha\beta)}{n^2+(\alpha+\beta)n+\alpha\beta})$

$= 1 + (\frac{\gamma+1-\alpha-\beta}{n} + \frac{\gamma-\alpha\beta}{n^2})[1 + \frac{\alpha+\beta}{n} + \frac{\alpha\beta}{n^2}]^{-1}$

$= 1 + (\frac{\gamma+1-\alpha-\beta}{n} + \frac{\gamma-\alpha\beta}{n^2})[1 - \frac{\alpha+\beta}{n} - \frac{\alpha\beta}{n^2} + \cdots]$

$= 1 + \frac{\gamma+1-\alpha-\beta}{n} + \frac{1}{n^2}[(\gamma-\alpha\beta) - (\alpha+\beta)(\gamma+1-\alpha-\beta) +$ terms containing $\frac{1}{n}$ and higher powers of $\frac{1}{n}]$

$= 1 + \frac{\gamma+1-\alpha-\beta}{n} + \frac{\phi(n)}{n^2}$, where $\lim\limits_{n\to\infty} \phi(n)$ is finite and therefore $\{\phi(n)\}$ is bounded.

By Gauss's test, when $x = 1$,
Σu_n is convergent if $\gamma + 1 - \alpha - \beta > 1$ and
Σu_n is divergent if $\gamma + 1 - \alpha - \beta \leq 1$.

Therefore the series is convergent if $0 < x < 1$ and divergent if $x > 1$.

When $x = 1$, the series is convergent if $\gamma > \alpha + \beta$ and divergent if $\gamma \leq \alpha + \beta$.

The order symbol O.

Let f and ϕ be two functions of n defined for all $n \geq m$, m being a natural number; and ϕ be ultimately monotone with $\phi(n) > 0$ for sufficiently large n.

If there exists a natural number $m_o \geq m$ such that $|f(n)| \leq k\phi(n)$ for all $n \geq m_o, k$ being a positive constant, we write $f = O(\phi)$.

Thus $O(\phi)$ denotes a function f such that $f(n) = h(n)\phi(n)$ where h is a bounded function of n.

In particular, $f = O(1)$ means that f is a bounded function of n.

Examples.

1. Let $f(n) = 5n^2 + 3n + 1$. Then $f(n) = O(n^2)$, since $f(n) \leq 5n^2$ for all $n \geq 1$.

2. Let $f(n) = \frac{(-1)^n}{n}$. Then $f(n) = O(\frac{1}{n})$, since $\frac{|f(n)|}{\frac{1}{n}} \to 1$ as $n \to \infty$

3. Let $f(n) = \frac{2n^2-3n+1}{5n^3-3}$. Then $f(n) = O(\frac{1}{n})$, since $\frac{f(n)}{\frac{1}{n}} \to \frac{2}{5}$ as $n \to \infty$.

4. Let $f(n) = \frac{1}{\sqrt{n^2-1}+\sqrt{n^2+1}}$. Then $f(n) = O(\frac{1}{n})$, since $\frac{f(n)}{\frac{1}{n}} \to \frac{1}{2}$ as $n \to \infty$.

5. Let $f(n) = 2\sin\frac{n\pi}{4}$. Then $f(n) = O(1)$, since $| f(n) | \leq 2$ for all $n \geq 1$.

Alternative form of Gauss's test.

Let Σu_n be a series of positive real numbers and let
$\frac{u_n}{u_{n+1}} = 1 + \frac{a}{n} + O(\frac{1}{n^p})$ where $p > 1$.

Then Σu_n is convergent if $a > 1$, Σu_n is divergent if $a \leq 1$.

$O(\frac{1}{n^p})$ denotes a sequence f such that $f(n) = h(n).\frac{1}{n^p}$, where h is a bounded sequence.

Therefore $\frac{u_n}{u_{n+1}} = 1 + \frac{a}{n} + \frac{h(n)}{n^p}$, where $h(n)$ is a bounded sequence and $p > 1$.

By Gauss's test, Σu_n is convergent if $a > 1$, Σu_n is divergent if $a \geq 1$.

Worked Example (continued).

17. Test the convergence of the series
$(\frac{1}{2})^2 + (\frac{1.3}{2.4})^2 + (\frac{1.3.5}{2.4.6.})^2 + \cdots$

Let $\sum_{n=1}^{\infty} u_n$ be the given series.

Then $u_n = \{\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}\}^2$ for all $n \geq 1$.

$$\begin{aligned}\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} &= (1+\tfrac{1}{n})^2(1+\tfrac{1}{2n})^{-2}\\ &= (1+\tfrac{2}{n}+\tfrac{1}{n^2})[1-\tfrac{1}{n}+O(\tfrac{1}{n^2})]\\ &= 1+\tfrac{1}{n}+O(\tfrac{1}{n^2}).\end{aligned}$$

By Gauss's test, Σu_n is divergent.

Theorem 6.3.14. **(De Morgan and Bertrand's test)**

Let Σu_n be a series of positive real numbers and $\lim[n(\frac{u_n}{u_{n+1}} - 1) - 1]\log n = l$.

Then Σu_n is convergent if $l > 1$; and divergent if $l < 1$.

Proof. Let $b_n = [n(\frac{u_n}{u_{n+1}} - 1) - 1]\log n$.
Then $\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{b_n}{n\log n}$.
Let $w_n = \frac{u_n}{u_{n+1}}d_n - d_{n+1}$ where $d_n = n\log n$.
Then $\sum\limits_{n=2}^{\infty}\frac{1}{d_n}$ is a divergent series and

$$\begin{aligned} w_n &= \frac{u_n}{u_{n+1}}n\log n - (n+1)\log(n+1) \\ &= (1 + \frac{1}{n} + \frac{b_n}{n\log n})n\log n - (n+1)\log(n+1) \\ &= (n+1)\log\frac{n}{n+1} + b_n. \end{aligned}$$

$\lim w_n = \lim\log\frac{1}{(1+\frac{1}{n})^{n+1}} + l = -1 + l$.

By Kummer's test, Σu_n is convergent if $l - 1 > 0$, i.e., if $l > 1$ and Σu_n is divergent if $l - 1 < 0$, i.e., if $l < 1$.

Worked Example (continued).

18. Test the convergence of the series
$(\frac{1}{2})^3 + (\frac{1.4}{2.5})^3 + (\frac{1.4.7}{2.5.8})^3 + \cdots$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series. Then $u_n = \{\frac{1.4.7\ldots(3n-2)}{2.5.8\ldots(3n-1)}\}^3$.

$\frac{u_{n+1}}{u_n} = (\frac{3n+1}{3n+2})^3$.

$$\begin{aligned} \lim_{n\to\infty} n(\frac{u_n}{u_{n+1}} - 1) &= \lim n[\frac{(3n+2)^3-(3n+1)^3}{(3n+1)^3}] \\ &= \lim\frac{27n^3+27n^2+7n}{27n^3+27n^2+9n+1} = 1. \end{aligned}$$

Raabe's test gives no decision.

Let us apply De Morgan and Bertrand's test.

$$\begin{aligned} \lim_{n\to\infty}[n(\frac{u_n}{u_{n+1}} - 1) - 1]\log n &= \lim\frac{(-2n-1)\log n}{27n^3+27n^2+9n+1} \\ &= \lim\frac{-2n^3-n^2}{27n^3+27n^2+9n+1}.\frac{\log n}{n^2} \\ &= \frac{-2}{27}.0, \text{ since } \lim\frac{\log n}{n^2} = 0 \\ &= 0 < 1. \end{aligned}$$

By De Morgan and Bertrand's test, Σu_n is divergent.

Theorem 6.3.15. **(Abel's theorem or Pringsheim's theorem)**

If Σu_n be a convergent series positive real numbers and $\{u_n\}$ is a monotone decreasing sequence then $\lim nu_n = 0$.

Proof. Since Σu_n is convergent, for a pre-assigned positive ϵ there exists a natural number m such that

$| u_{n+1} + u_{n+2} + \cdots + u_{n+p} | < \frac{\epsilon}{2}$ for all $n \geq m$ and for every natural number p.

Let $n = m$.

Then $u_{m+1} + u_{m+2} + \cdots + u_{m+p} < \frac{\epsilon}{2}$ for every natural number p.

But $u_{m+1} + u_{m+2} + \cdots + u_{m+p} \geq pu_{m+p}$, since $\{u_n\}$ is a monotone decreasing sequence.

Consequently, $pu_{m+p} < \frac{\epsilon}{2}$ for every natural number p.

Let $p = m$. Then $2mu_{2m} < \epsilon$ ... ... (i)

Let $p = m + 1$. Then $(m + 1)u_{2m+1} < \frac{\epsilon}{2}$.

Therefore $(2m + 1)u_{2m+1} < (2m + 2)u_{2m+1} < \epsilon$ ... ... (ii)

From (i) and (ii) $nu_n < \epsilon$ for all $n \geq 2m$.

This shows that $\lim nu_n = 0$.

Note. If $\{u_n\}$ be a monotone decreasing sequence of positive real numbers and $\lim nu_n = 0$, then Σu_n is not necessarily convergent.

For example, let $u_n = \frac{1}{n \log n}, n > 1$. Then $u_{n+1} < u_n$ for all $n > 1$ and $\lim nu_n = 0$. But $\sum\limits_{2}^{\infty} u_n$ is a divergent series.

Worked Examples (continued).

19. Prove that the series $(\frac{1}{2})^p + (\frac{1.3}{2.4})^p + (\frac{1.3.5}{2.4.6})^p + \cdots$ is convergent for $p > 2$ and divergent for $p \leq 2$.

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series.

$$\begin{aligned} \text{Then } \frac{u_n}{u_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^p &= (1 + \tfrac{1}{n})^p(1 + \tfrac{1}{2n})^{-p} \\ &= \{1 + \tfrac{p}{n} + O(\tfrac{1}{n^2})\}\{1 - \tfrac{p}{2n} + O(\tfrac{1}{n^2})\} \\ &= 1 + \tfrac{p}{2n} + O(\tfrac{1}{n^2}). \end{aligned}$$

By Gauss's test,

the series Σu_n is convergent if $\frac{p}{2} > 1$, i.e., if $p > 2$ and divergent if $\frac{p}{2} \leq 1$, i.e., if $p \leq 2$.

20. If Σu_n be a divergent series of positive real numbers prove that the series $\Sigma \frac{u_n}{1+u_n}$ is divergent.

Let $s_n = u_1 + u_2 + \cdots + u_n$.

Since the series Σu_n is a divergent series of positive real numbers, the sequence $\{s_n\}$ is a monotone increasing sequence and $\lim s_n = \infty$.

Therefore for every natural number n we can choose a natural number p such that $s_{n+p} > 1 + 2s_n$.

Now $\frac{u_{n+1}}{1+u_{n+1}} + \frac{u_{n+2}}{1+u_{n+2}} + \cdots + \frac{u_{n+p}}{1+u_{n+p}}$

$> \frac{u_{n+1}}{1+s_{n+p}} + \frac{u_{n+2}}{1+s_{n+p}} + \cdots + \frac{u_{n+p}}{1+s_{n+p}}$, since $s_{n+p} \geq s_{n+1} > u_{n+1}$, $s_{n+p} \geq s_{n+2} > u_{n+2}, \cdots, s_{n+p} > u_{n+p}$

$= \frac{s_{n+p}-s_n}{1+s_{n+p}}$

$> \frac{\frac{1}{2}(1+s_{n+p})}{1+s_{n+p}} = \frac{1}{2}$.

This shows that Cauchy's principle of convergence is not satisfied by the series $\Sigma\frac{u_n}{1+u_n}$. Hence the series is divergent.

Exercises 9

1. If Σu_n be a convergent series of positive real numbers prove that Σu_n^2 is convergent.

[**Hint.** There exists an $m \in \mathbb{N}$ such that $u_n < 1$ for all $n \geq m$. $\therefore$ $u_n^2 < u_n$ for all $n \geq m$.]

2. If Σu_n be a convergent series of positive real numbers prove that $\Sigma\frac{u_n}{n}$ is convergent.

[**Hint.** $u_n.\frac{1}{n} < \frac{u_n^2+1/n^2}{2}$.]

3. If Σu_n be a convergent series of positive real numbers prove that the series Σu_{2n} is convergent.

Hint. Let $s_n = u_1 + u_2 + \cdots + u_n$, $t_n = u_2 + u_4 + \cdots + u_{2n}$. Then $t_n < s_{2n}$ for all $n \in \mathbb{N}$. The sequence $\{t_n\}$ is a monotone increasing sequence bounded above.

4. If Σu_n be a convergent series of positive real numbers prove that $\Sigma\frac{u_n}{1+u_n}$ is convergent.

[**Hint.** Let $v_n = \frac{u_n}{1+u_n}$. Then $\lim \frac{u_n}{v_n} = 1$.]

5. If Σu_n be a series of positive real numbers and $v_n = \frac{u_1+u_2+\cdots+u_n}{n}$, prove that Σv_n is divergent.

[**Hint.** $v_1 + v_2 + \cdots + v_n > u_1(1 + \frac{1}{2} + \cdots + \frac{1}{n})$.]

6. If Σu_n be a divergent series of positive real numbers and $s_n = u_1 + u_2 + \cdots + u_n$, prove that the series $\Sigma\frac{u_n}{s_n}$ is divergent.

[**Hint.** Since $\{s_n\}$ is a monotone increasing sequence diverging to ∞, for every natural number n, we can choose a natural number p such that $s_{n+p} > 2s_n$. Then $\frac{u_{n+1}}{s_{n+1}} + \frac{u_{n+2}}{s_{n+2}} + \cdots + \frac{u_{n+p}}{s_{n+p}} > \frac{1}{2}$.]

7. If $\{a_1, a_2, a_3, \cdots\}$ be the collection of those natural numbers that end with 1, prove that the series $\Sigma\frac{1}{a_n}$ is divergent.

Hint. Considering the collection as an increasing sequence of natural numbers, $a_n = 10n - 9$ for all $n \in \mathbb{N}$. $\frac{1}{a_n} > \frac{1}{10n}$ for all $n \in \mathbb{N}$.

8. Test the convergence of the following series

(i) $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \cdots\cdots$ (ii) $\frac{1}{1+2} + \frac{1}{1+2^2} + \frac{1}{1+2^3} + \cdots\cdots$

(iii) $\frac{1}{1+2^{-1}} + \frac{1}{1+2^{-2}} + \frac{1}{1+2^{-3}} + \cdots\cdots$ (iv) $\sin\frac{\pi}{2} + \sin\frac{\pi}{4} + \sin\frac{\pi}{6} + \cdots\cdots$

(v) $\frac{1}{1.3} + \frac{2}{3.5} + \frac{3}{5.7} + \cdots\cdots$ (vi) $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \cdots\cdots$

[**Hint.** (iv) $\frac{2x}{\pi} < \sin x$ for $0 < x < \frac{\pi}{2}$.]

9. Test the series Σu_n for convergence where $u_n =$

(i) $\frac{2^n+1}{3^n+2}$, (ii) $\sqrt{n^4+1} - n^2$, (iii) $\sqrt[3]{n^3+1} - n$, (iv) $\frac{\sqrt{n+1}-\sqrt{n-1}}{n}$,

(v) $\frac{1}{\sqrt{n}} \tan\frac{1}{n}$, (vi) $\frac{1}{n}\sin\frac{1}{n}$, (vii) $\frac{3^n}{2^n+3^n}$, (viii) $\frac{1}{n\log n}, n \geq 2$,

(ix) $\frac{1}{n\log n(\log\log n)}, n \geq 3$.

10. Test the convergence of the following series

(i) $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \cdots\cdots$

(ii) $\frac{1^2.2^2}{1!} + \frac{2^2.3^2}{2!} + \frac{3^2.4^2}{3!} + \cdots\cdots$

(iii) $\frac{1^2}{2^2} + \frac{1^2.3^2}{2^2.4^2} + \frac{1^2.3^2.5^2}{2^2.4^2.6^2} + \cdots\cdots$

(iv) $(\frac{2^2}{1^2} - \frac{2}{1})^{-1} + (\frac{3^3}{2^3} - \frac{3}{2})^{-2} + (\frac{4^4}{3^4} - \frac{4}{3})^{-3} + \cdots\cdots$

(v) $1 + \frac{1}{2} + \frac{1}{4^2} + \frac{1}{2^3} + \frac{1}{4^4} + \frac{1}{2^5} + \frac{1}{4^6} + \cdots\cdots$

(vi) $1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2^2.3} + \frac{1}{2^2.3^2} + \frac{1}{2^3.3^2} + \cdots$

(vii) $\frac{1}{3} + \frac{1}{5} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{3^3} + \frac{1}{5^3} + \cdots\cdots$

(viii) $\frac{1}{3} + \frac{1}{3^3} + \frac{1}{3^2} + \frac{1}{3^5} + \frac{1}{3^4} + \cdots\cdots$

(ix) $\frac{1}{\log 2} + \frac{1}{\log 3} + \frac{1}{\log 4} + \cdots\cdots$

[**Hint.** $\log(1+x) < x$ for $x > 0$.]

(x) $\tan\frac{\pi}{4} + \tan\frac{\pi}{8} + \tan\frac{\pi}{12} + \cdots\cdots$

[**Hint.** $x < \tan x$ for $0 < x < \frac{\pi}{2}$.]

(xi) $(\frac{1}{2})^{\log 1} + (\frac{1}{2})^{\log 2} + (\frac{1}{2})^{\log 3} + \cdots\cdots$

(xii) $\frac{1}{3} + (\frac{1}{3})^{1+\frac{1}{2}} + (\frac{1}{3})^{1+\frac{1}{2}+\frac{1}{3}} + \cdots\cdots$

(xiii) $\frac{1}{4} + (\frac{1}{4})^{1+\frac{1}{3}} + (\frac{1}{4})^{1+\frac{1}{3}+\frac{1}{5}} + \cdots\cdots$

[**Hint.** (xi), (xii), (xiii). Use Logarithmic test.]

(xiv) $1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \cdots\cdots, x > 0$

(xv) $\frac{2}{3} + \frac{2.4}{3.5}x + \frac{2.4.6}{3.5.7}x^2 + \cdots\cdots, x > 0$

(xvi) $1 + \frac{1}{2}x + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \cdots\cdots, x > 0$

(xvii) $\frac{3}{7} + \frac{3.6}{7.10}x + \frac{3.6.9}{7.10.13}x^2 + \cdots\cdots, x > 0$

(xviii) $\frac{1+x}{1!} + \frac{(1+2x)^2}{2!} + \frac{(1+3x)^3}{3!} + \cdots\cdots, x > 0$

[**Hint.** Compare with the series $\Sigma\frac{n^n x^n}{n!}, x > 0$.]

(xix) $1 + \frac{2^2}{3^2}x + \frac{2^2.4^2}{3^2.5^2}x^2 + \cdots\cdots, x > 0$

(xx) $1 + \frac{1!}{2!}x + \frac{(2!)^2}{4!}x^2 + \frac{(3!)^2}{6!}x^3 + \cdots\cdots, x > 0$

11. Prove that the series $\frac{a}{b} + \frac{a(a+c)}{b(b+c)} + \frac{a(a+c)(a+2c)}{b(b+c)(b+2c)} + \cdots\cdots, a, b, c > 0$ is convergent if $b > a + c$ and divergent if $b \leq a + c$.

12. Prove that the series $1 + \frac{\alpha^2}{1.\beta} + \frac{\alpha^2(\alpha+1)^2}{1.2.\beta(\beta+1)} + \frac{\alpha^2(\alpha+1)^2(\alpha+2)^2}{1.2.3.\beta(\beta+1)(\beta+2)} + \cdots\cdots, \alpha, \beta > 0$ is convergent if $\beta > 2\alpha$ and divergent if $\beta \leq 2\alpha$.

6.4. Series of arbitrary terms.

Let Σu_n be a series of positive and negative real numbers.

Let $u_n' = |u_n|$. Then $\Sigma u_n'$ is a series of positive real numbers.

If $\Sigma u_n'$ is convergent then Σu_n is said to be an *absolutely convergent series.*

Theorem 6.4.1. An absolutely convergent series is convergent.

Proof. Let Σu_n be a series of positive and negative real numbers and be absolutely convergent. Then $\Sigma|u_n|$ is a convergent series of positive terms.

Let us choose a positive ϵ. Then there exists a natural number m such that

$|\ |u_{n+1}| + |u_{n+2}| + \cdots + |u_{n+p}|\ | < \epsilon$ for all $n \geq m$ and for every natural number p.

That is, $|u_{n+1}| + |u_{n+2}| + \cdots + |u_{n+p}| < \epsilon$ for all $n \geq m$ and for every natural number p.

But $|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| \leq |u_{n+1}| + |u_{n+2}| + \cdots + |u_{n+p}|$.

Therefore $|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \epsilon$ for all $n \geq m$ and for every natural number p.

By Cauchy's principle of convergence, Σu_n is convergent.

Examples.

1. The series $1 - \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{6^2} + \cdots$ is convergent since it is absolutely convergent.

2. The series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots$ is convergent since it is absolutely convergent.

3. For a fixed value of x the series $\Sigma \frac{\sin nx}{n^2}$ is absolutely convergent.

$\Sigma \frac{\sin nx}{n^2}$ is a series of arbitrary terms.

$\left| \frac{\sin nx}{n^2} \right| \leq \frac{1}{n^2}$. Since $\Sigma \frac{1}{n^2}$ is a convergent series, $\Sigma \left| \frac{\sin nx}{n^2} \right|$ is a convergent series, by Comparison test.

Consequently, $\Sigma \frac{\sin nx}{n^2}$ is an absolutely convergent series.

Theorem 6.4.2. If the series Σu_n be absolutely convergent and $\{v_n\}$ be a bounded sequence, then the series $\Sigma u_n v_n$ is absolutely convergent.

Proof. There exists a positive real number k such that $|v_n| < k$ for all $n \in \mathbb{N}$.

Now $|u_{n+1}v_{n+1}| + |u_{n+2}v_{n+2}| + \cdots + |u_{n+p}v_{n+p}|$
$< k\{| u_{n+1} | + | u_{n+2} | + \cdots + | u_{n+p} |\}$.

Since Σu_n is absolutely convergent, the series $\Sigma|u_n|$ is convergent.

Therefore for a chosen positive ϵ there exists a natural number m such that $| u_{n+1} | + | u_{n+2} | + \cdots + | u_{n+p} | < \epsilon/k$ for all $n \geq m$ and for every natural number p.

Therefore $| u_{n+1}v_{n+1} | + | u_{n+2}v_{n+2} | + \cdots + | u_{n+p}v_{n+p} | < \epsilon$ for all $n \geq m$ and for every natural number p.

By Cauchy's principle of convergence, the series $\Sigma \mid u_n v_n \mid$ is convergent and consequently, the series $\Sigma u_n v_n$ is absolutely convergent.

Worked Example.

1. Test the series $\frac{2}{1^3} - \frac{3}{2^3} + \frac{4}{3^3} - \frac{5}{4^3} + \cdots\cdots$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series. Then $u_n = (-1)^{n+1}\frac{n+1}{n^3} = \frac{(-1)^{n+1}}{n^2}.(1+\frac{1}{n}) = a_n b_n$, say, where $a_n = \frac{(-1)^{n+1}}{n^2}, b_n = 1 + \frac{1}{n}$.

The series Σa_n is absolutely convergent and $\{b_n\}$ is a bounded sequence and therefore the series $\Sigma a_n b_n$, i.e., the given series is absolutely convergent.

Theorem 6.4.3. Ratio test.

Let Σu_n be a series of arbitrary terms and let $\lim \frac{|u_{n+1}|}{|u_n|} = l$.

Then (i) Σu_n is absolutely convergent if $l < 1$,

(ii) Σu_n is divergent if $l > 1$.

Proof. (i) Let us choose a positive ϵ such that $l + \epsilon < 1$.

There exists a natural number m such that

$$l - \epsilon < \frac{|u_{n+1}|}{|u_n|} < l + \epsilon \text{ for all } n \geq m.$$

Let $l + \epsilon = r$. Then $0 < r < 1$ and $\frac{|u_{n+1}|}{|u_n|} < r$ for all $n \geq m$.

Then $\frac{|u_{m+1}|}{|u_m|} < r, \frac{|u_{m+2}|}{|u_{m+1}|} < r, \cdots, \frac{|u_n|}{|u_{n-1}|} < r$.

Consequently, $\frac{|u_n|}{|u_m|} < r^{n-m}$ for all $n > m$

or, $|u_n| < \frac{|u_m|}{r^m} r^n$ for all $n > m$.

But Σr^n is a convergent series, since $0 < r < 1$.

By Comparison test, the series $\Sigma|u_n|$ is convergent. Therefore the series Σu_n is absolutely convergent.

(ii) Let us choose a positive ϵ such that $l - \epsilon > 1$.

There exists a natural number k such that

$$l - \epsilon < \frac{|u_{n+1}|}{|u_n|} < l + \epsilon \text{ for all } n \geq k.$$

Therefore $\frac{|u_{n+1}|}{|u_n|} > l - \epsilon > 1$ for all $n \geq k$.

Hence the sequence $\{|u_n|\}$ is ultimately a monotone increasing sequence of positive real numbers.

So $\lim |u_n| \neq 0$ and this implies $\lim u_n \neq 0$. Consequently, the series Σu_n is divergent.

Theorem 6.4.4. Root test.

Let Σu_n be a series of arbitrary terms and let $\lim |u_n|^{1/n} = l$.

Then (i) Σu_n is absolutely convergent if $l < 1$,

(ii) Σu_n is divergent if $l > 1$.

Proof. (i) Let us choose a positive ϵ such that $l + \epsilon < 1$.

There exists a natural number m such that

$$l - \epsilon < |u_n|^{1/n} < l + \epsilon \text{ for all } n \geq m.$$

Let $l + \epsilon = r$. Then $0 < r < 1$.

We have $|u_n|^{1/n} < r$ for all $n \geq m$.

or, $|u_n| < r^n$ for all $n \geq m$.

But Σr^n is a convergent series, since $0 < r < 1$.

By Comparison test, the series $\Sigma \mid u_n \mid$ is convergent. Therefore the series Σu_n is absolutely convergent.

(ii) Let us choose a positive ϵ such that $l - \epsilon > 1$.

There exists a natural number k such that

$$l - \epsilon < |u_n|^{1/n} < l + \epsilon \text{ for all } n \geq k.$$

Therefore $|u_n| > 1$ for all $n \geq k$.

So $\lim |u_n| \neq 0$ and this implies $\lim u_n \neq 0$. Consequently, the series Σu_n is divergent.

Worked Examples (continued).

2. Examine the convergence of the series

$1 - \frac{2^2}{2!} + \frac{3^3}{3!} - \frac{4^4}{4!} + \cdots \cdots$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series. Then $u_n = (-1)^{n+1}\frac{n^n}{n!}$.

$\frac{|u_{n+1}|}{|u_n|} = \frac{(n+1)^{n+1}}{n+1} \cdot \frac{1}{n^n} = (1 + \frac{1}{n})^n$ and $\lim \frac{|u_{n+1}|}{|u_n|} = e > 1$.

By Ratio test, the series Σu_n is divergent.

3. Examine the convergence of the series $1 - \frac{(2!)^2}{4!} + \frac{(3!)^2}{6!} - \cdots \cdots$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series. Then $u_n = (-1)^{n+1}\frac{(n!)^2}{(2n)!}$ for $n \geq 2$.

$\lim \frac{|u_{n+1}|}{|u_n|} = \lim \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4} < 1$.

By Ratio test, the series Σu_n is absolutely convergent.

Alternating series.

Definition. If $u_n > 0$ for all n, the series $\sum\limits_{1}^{\infty}(-1)^{n+1}u_n$ called an *alternating series.*

Theorem 6.4.5. (Leibnitz's test)

If $\{u_n\}$ be a monotone decreasing sequence of positive real numbers and $\lim u_n = 0$ then the alternating series

$u_1 - u_2 + u_3 - u_4 + \cdots$ is convergent.

Proof. Let $s_n = u_1 - u_2 + u_3 - \cdots + (-1)^{n+1}u_n$.

Then $s_{2n+2} - s_{2n} = u_{2n+1} - u_{2n+2} \geq 0$ for all $n \in \mathbb{N}$.

The sequence $\{s_{2n}\}$ is a monotone increasing sequence.

$s_{2n+1} - s_{2n-1} = -u_{2n} + u_{2n+1} \leq 0$ for all $n \in \mathbb{N}$.

The sequence $\{s_{2n+1}\}$ is a monotone decreasing sequence.

Again $s_{2n} = u_1 - u_2 + u_3 - u_4 + \cdots - u_{2n}$

$= u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - u_{2n} < u_1$

The sequence $\{s_{2n}\}$ is bounded above.

$s_{2n+1} = u_1 - u_2 + u_3 - u_4 + \cdots + u_{2n+1}$

$= (u_1 - u_2) + (u_3 - u_4) + \cdots + u_{2n+1} > u_1 - u_2$

The sequence $\{s_{2n+1}\}$ is bounded below.

Therefore both the sequences $\{s_{2n}\}$ and $\{s_{2n+1}\}$ are convergent.

Now $\lim(s_{2n+1} - s_{2n}) = \lim u_{2n+1} = 0$.

This shows that both the sequences $\{s_{2n+1}\}$ and $\{s_{2n}\}$ converge to the same limit.

Hence the sequence $\{s_n\}$ is convergent and consequently the series $\Sigma(-1)^{n+1}u_n$ is convergent.

Note. If s be the sum of the series and s_n be the nth partial sum then $0 < (-1)^n(s - s_n) < u_{n+1}$ for all $n \in \mathbb{N}$.

$$s - s_n = (-1)^{n+2}\{u_{n+1} - u_{n+2} + u_{n+3} - \cdots\}$$
or, $(-1)^n(s - s_n) = u_{n+1} - u_{n+2} + u_{n+3} - \cdots$
$$= u_{n+1} - (u_{n+2} - u_{n+3}) - \cdots < u_{n+1}.$$
Also $(-1)^n(s - s_n) = (u_{n+1} - u_{n+2}) + (u_{n+3} - u_{n+4}) + \cdots > 0$.
Combining, we have $0 < (-1)^n(s - s_n) < u_{n+1}$ for all $n \in \mathbb{N}$.

Examples.

1. The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent by Leibnitz's test.

2. The series $\frac{1}{1+a^2} - \frac{1}{2+a^2} + \frac{1}{3+a^2} - \cdots$ is convergent by Leibnitz's test.

3. The series $1 - \frac{1}{2} + \frac{1.3}{2.4} - \frac{1.3.5}{2.4.6} + \cdots$ is convergent by Leibnitz's test, since $\lim \frac{1.3.....(2n-1)}{2.4.....2n} = 0$ and $\frac{1.3.....(2n-1)}{2.4.....2n} > \frac{1.3.....(2n+1)}{2.4.....(2n+2}$.

Theorem 6.4.6.(Abel's test)

If the sequence $\{b_n\}$ is a monotone bounded sequence and Σa_n is a convergent series then the series $\Sigma a_n b_n$ is convergent.

Proof. Let $s_n = a_1 + a_2 + \cdots + a_n$, $t_n = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$.
Then $t_n = s_1 b_1 + (s_2 - s_1)b_2 + (s_3 - s_2)b_3 + \cdots + (s_n - s_{n-1})b_n$
$$= s_1(b_1 - b_2) + s_2(b_2 - b_3) + \cdots + s_n(b_n - b_{n+1}) + s_n b_{n+1}.$$

Since Σa_n is convergent, the sequence $\{s_n\}$ is convergent. Since the sequence $\{b_n\}$ is monotonic and bounded, $\{b_n\}$ is convergent.

Therefore $s_n b_{n+1}$ tends to a limit ... ... (i)

Let $d_n = b_n - b_{n+1}$. Then either $d_n \geq 0$ for all n, or ≤ 0 for all n; and $d_1 + d_2 + \cdots + d_n = b_1 - b_{n+1}$ tends to a definite limit since $\{b_n\}$ is convergent. Therefore Σd_n is absolutely convergent.

Since the sequence $\{s_n\}$ is bounded and the series Σd_n is absolutely convergent, the series $\Sigma s_n d_n$ is absolutely convergent, by Theorem 6.4.2.

Therefore the sequence $\{\Sigma s_n d_n\}$ is convergent ... ... (ii)

From (i) and (ii) it follows that the sequence $\{t_n\}$ is convergent and this proves that the series $\Sigma a_n b_n$ is convergent.

Examples.

1. The series $\sum\limits_{2}^{\infty} \frac{(-1)^{n+1}}{n \log n}$ is convergent by Abel's test, since $\Sigma \frac{(-1)^{n+1}}{n}$ is a convergent series and the sequence $\{\frac{1}{\log n}\}_2^\infty$ is a monotone decreasing sequence bounded below.

2. The series $\sum\limits_{1}^{\infty} \frac{(-1)^{n+1} \cdot n^n}{(n+1)^{n+1}}$ is convergent by Abel's test, since $\Sigma \frac{(-1)^{n+1}}{n+1}$ is a convergent series and the sequence $\{(1+\frac{1}{n})^{-n}\}$ is a monotone decreasing sequence bounded below.

Theorem 6.4.7.(Dirichlet's test)

If the sequence $\{b_n\}$ is a monotone sequence converging to 0 and the sequence of partial sums $\{s_n\}$ of the series Σa_n is bounded, then the series $\Sigma a_n b_n$ is convergent.

Proof. $s_n = a_1 + a_2 + \cdots + a_n$.

Let $t_n = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$.

Then $t_n = s_1 b_1 + (s_2 - s_1) b_2 + (s_3 - s_2) b_3 + \cdots + (s_n - s_{n-1}) b_n$
$= s_1(b_1 - b_2) + s_2(b_2 - b_3) + \cdots + s_n(b_n - b_{n+1}) + s_n b_{n+1}$.

Since the sequence $\{s_n\}$ is bounded and $\lim b_n = 0, \lim s_n b_{n+1} = 0$.

Let $d_n = b_n - b_{n+1}$. Then either $d_n \geq 0$ for all n, or ≤ 0 for all n; and $d_1 + d_2 + \cdots + d_n (= b_1 - b_{n+1})$ tends to a definite limit since $\lim b_n = 0$. Therefore Σd_n is absolutely convergent.

Since the sequence $\{s_n\}$ is bounded and the series Σd_n is absolutely convergent, the series $\Sigma s_n d_n$ is absolutely convergent by Theorem 6.4.2.

Hence the series $\Sigma s_n d_n$ is convergent and therefore the sequence $\{t_n\}$ is convergent.

This proves that the series $\Sigma a_n b_n$ is convergent.

Note. Leibnitz's test is a particular case of Dirichlet's test. If $\{b_n\}$ is a monotone decreasing sequence converging to 0, then the series $\sum\limits_{1}^{\infty} (-1)^{n+1} b_n$ is convergent by Dirichlet's test, since the sequence of partial sums $\{s_n\}$ of the series $\Sigma(-1)^{n+1}$ is bounded.

This is Leibnitz's test for an alternating series.

Examples.

1. The series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ is convergent by Dirichlet's test, since the sequence of partial sums $\{s_n\}$ of the series $\Sigma(-1)^{n+1}$ is bounded and the sequence $\{\frac{1}{\sqrt{n}}\}$ is a monotone decreasing sequence converging to 0.

2. The series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{\log(n+1)}$ is convergent by Dirichlet's test, since the sequence of partial sums $\{s_n\}$ of the series $\Sigma(-1)^{n+1}$ is bounded and the sequence $\{\frac{1}{\log(n+1)}\}$ is a monotone decreasing sequence converging to 0.

6.5. Conditionally convergent series.

Definition. A series Σu_n is called *conditionally convergent* if Σu_n is convergent but $\Sigma \mid u_n \mid$ is not convergent.

A conditionally convergent series is also called a *semi convergent* series or a *non-absolutely convergent* series.

Examples.

1. The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \cdots$ is convergent, but the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots$ is divergent.

Therefore the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \cdots$ is conditionally convergent.

2. Let $u_n = \frac{1}{2n-1}$. Then $\{u_n\}$ is a monotone decreasing sequence of positive real numbers and $\lim u_n = 0$.

By Leibnitz's test, $\Sigma(-1)^{n+1} u_n$ is convergent. But Σu_n is a divergent series. Therefore $\Sigma(-1)^{n+1} u_n$ is a conditionally convergent series.

3. Show that the series

$\frac{1}{(1+a)^p} - \frac{1}{(2+a)^p} + \frac{1}{(3+a)^p} - \cdots \cdots, a > 0$

is (i) absolutely convergent if $p > 1$,

(ii) conditionally convergent if $0 < p \leq 1$.

Let Σu_n be the given series and $v_n = \mid u_n \mid$.

Then Σv_n is a series of positive real numbers and $v_n = \frac{1}{(n+a)^p}$.

Let $w_n = \frac{1}{n^p}$. Then $\lim \frac{v_n}{w_n} = 1$.

By Comparison test, Σv_n is convergent if $p > 1$,

Σv_n is divergent if $0 < p \leq 1$.

Case 1. $p > 1$.

In this case Σu_n is an alternating series and $\Sigma \mid u_n \mid$ is convergent. Therefore Σu_n is absolutely convergent.

Case 2. $0 < p \leq 1$.

In this case $\{v_n\}$ is a monotone decreasing sequence of positive real numbers and $\lim v_n = 0$.

By Leibnitz's test, $\Sigma(-1)^{n+1} v_n$, i.e., Σu_n is convergent.

Since $\Sigma \mid u_n \mid$ is divergent, Σu_n is conditionally convergent.

Let Σu_n be a series of positive real numbers and let

$$p_n = u_n \text{ if } u_n > 0 \qquad q_n = 0 \text{ if } u_n \geq 0$$
$$= 0 \text{ if } u_n \leq 0; \qquad = u_n \text{ if } u_n < 0.$$

Then Σp_n is a series of positive real numbers along with some 0's and Σq_n is a series of negative real numbers along with some 0's.

For example, for the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \cdots$
$\Sigma p_n = 1 + 0 + \frac{1}{3} + 0 + \frac{1}{5} + \cdots$ and $\Sigma q_n = 0 - \frac{1}{2} + 0 - \frac{1}{4} + 0 - \cdots$

$p_n = \frac{u_n + |u_n|}{2}, q_n = \frac{u_n - |u_n|}{2}$ and $u_n = p_n + q_n$.

Theorem 6.5.1. Let Σu_n be a series of arbitrary terms and $p_n = \frac{u_n+|u_n|}{2}, q_n = \frac{u_n-|u_n|}{2}$.

(i) If Σu_n is absolutely convergent then both Σp_n and Σq_n are convergent.

(ii) If Σu_n is conditionally convergent then both Σp_n and Σq_n are divergent.

Proof. (i) Since Σu_n is absolutely convergent, both Σu_n and $\Sigma |u_n|$ are convergent.

But $p_n = \frac{u_n+|u_n|}{2}$, $q_n = \frac{u_n-|u_n|}{2}$. Hence Σp_n and Σq_n are both convergent.

(ii) Since Σu_n is conditionally convergent, Σu_n is convergent but $\Sigma |u_n|$ is divergent.

Now $|u_n| = 2p_n - u_n$ ... ... (A)

If we assume that Σp_n is convergent then if follows from (A) that $\Sigma |u_n|$ is convergent, a contradiction. Therefore Σp_n is divergent.

Again $|u_n| = u_n - 2q_n$ ... ... (B)

If we assume that Σq_n is convergent then if follows from (B) that $\Sigma |u_n|$ is convergent, a contradiction. Therefore Σq_n is divergent.

Note. For all n, $p_n \geq 0$ and $q_n \leq 0$. From (i) it follows that in an absolutely convergent series, the series formed by the positive terms alone and the series formed by the negative terms alone are both convergent.

From (ii) it follows that in a conditionally convergent series, the series formed by the positive terms alone and the series formed by the negative terms alone are both divergent.

Introduction and removal of brackets.

Theorem 6.5.2. Let Σu_n be a series of positive and negative real numbers and Σv_n is obtained from Σu_n by grouping it terms. Then

(i) if Σu_n converges to the sum s, then Σv_n also converges to s,

(ii) if Σv_n converges, then Σu_n may not be convergent.

Proof. (i) Let $v_1 = u_1 + u_2 + \cdots + u_{r_1}, v_2 = u_{r_1+1} + u_{r_1+2} + \cdots + u_{r_2}, \cdots, v_n = u_{r_{n-1}+1} + u_{r_{n-1}+2} + \cdots + u_{r_n}, \cdots \cdots$

Then $\{r_n\}$ is a strictly increasing sequence of natural numbers.

Let $s_n = u_1 + u_2 + \cdots + u_n$, $t_n = v_1 + v_2 + \cdots + v_n$.
Then $t_n = u_1 + u_2 + \cdots + u_{r_n} = s_{r_n}$.
Since Σu_n converges to the sum s, $\lim s_n = s$.

The sequence $\{t_n\}$ is a subsequence of the sequence $\{s_n\}$ and therefore the sequence $\{t_n\}$ also converges to the sum s.

In other words, the series Σv_n converges to the sum s.

(ii) That the converse is not true can be established by the following example.

Let $u_n = (-1)^{n+1}$. Then the series Σu_n is $1 - 1 + 1 - 1 + \cdots$
This is not a convergent series.

Let Σv_n be obtained from Σu_n by grouping the terms as
$(1-1) + (1-1) + (1-1) + \cdots \cdots$
Then Σv_n is clearly a convergent series.

Examples.

1. Prove that $\log 2 = \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \cdots$

The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent, by Leibnitz's test.

We have $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ when $-1 < x \leq 1$.

So $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2$.

Grouping the terms of the series as $(1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{6}) + \cdots$ we have the series $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \cdots$

By Theorem 6.5.2, the sum of the series is $\log 2$.

2. Prove that $\frac{\pi}{8} = \frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \cdots$

The series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ is convergent by Leibnitz's test.

We have $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ when $-1 \leq x \leq 1$ (Gregory's series)

So $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$.

Grouping the terms of the series as $(1 - \frac{1}{3}) + (\frac{1}{5} - \frac{1}{7}) + \cdots$ we have the series $\frac{2}{1.3} + \frac{2}{5.7} + \frac{2}{9.11} + \cdots$

By Theorem 6.5.2, the sum of the series is $\frac{\pi}{4}$.

Hence $\frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \cdots = \frac{\pi}{8}$.

Re-arrangement of terms.

Theorem 6.5.3. If the terms of an absolutely convergent series be rearranged the series remains convergent and its sum remains unaltered.

Proof. Let Σu_n be an absolutely convergent series and let the terms be re-arranged in any manner.

Let the new series be Σv_n. Then every u is a v and every v is a u.

Let $\Sigma|u_n| = s$. Then $\Sigma|u_n| + u_n$ is a series of positive real numbers and $u_n + |u_n| leq 2|u_n|$.

By Comparison test, $\Sigma(|u_n| + u_n)$ is convergent.

Let $\Sigma(|u_n| + u_n) = s'$. Then $\Sigma u_n = s' - s$.

Since $\Sigma|u_n|$ and $\Sigma|u_n| + u_n)$ are convergent series of positive real numbers their sums are not altered by re-arrangement of terms.

Therefore $\Sigma|v_n| = s$ and $\Sigma(|v_n| + v_n) = s'$.

Consequently, $\Sigma v_n = s' - s$. This shows that Σv_n is convergent and $\Sigma v_n = \Sigma u_n$. This proves the theorem.

We state here without proof an important theorem of Riemann about the behaviour of a conditionally convergent series.

Theorem 6.5.4. (Riemann's theorem)

By appropriate re-arrangement of terms, a conditionally convergent series Σu_n can be made

(i) to converge to any number l, or (ii) to diverge to $+\infty$, or

(iii) to diverge to $-\infty$, or (iv) to oscillate finitely, or

(v) to oscillate infinitely.

Worked Example.

Prove that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges to $\log 2$, but the re-arranged series

$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots\cdots$ converges to $\frac{1}{2}\log 2$.

We have $\lim\limits_{n\to\infty}(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n) = \gamma$.

Let $1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n = \gamma_n$. Then $\lim\limits_{n\to\infty}\gamma_n = \gamma$.

Let $s_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\frac{1}{n}$.

Then $s_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$

$= (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n}) - 2(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n})$

$= \log 2n + \gamma_{2n} - (1 + \frac{1}{2} + \cdots + \frac{1}{n})$

$= \log 2n + \gamma_{2n} - (\log n + \gamma_n) = \log 2 + \gamma_{2n} - \gamma_n$.

Therefore $\lim s_{2n} = \log 2$.

$s_{2n+1} = s_{2n} + \frac{1}{2n+1}$. Therefore $\lim s_{2n+1} = \lim s_{2n} = \log 2$.

This proves $\lim s_n = \log 2$. That is, the series converges to $\log 2$.

Let $t_n = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots\cdots$ to n terms.

Then $t_{3n} = (1 - \frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{6} - \frac{1}{8}) + \cdots + (\frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n})$

$= (1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}) - \frac{1}{2}(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n})$

$= (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n}) - (\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}) - \frac{1}{2}(1 + \frac{1}{2} + \cdots + \frac{1}{2n})$

$= \log 2n + \gamma_{2n} - \frac{1}{2}(\log n + \gamma_n) - \frac{1}{2}(\log 2n + \gamma_{2n})$

$= \frac{1}{2}(\log 2n + \gamma_{2n}) - \frac{1}{2}(\log n + \gamma_n)$

$= \frac{1}{2}\log 2 + \frac{1}{2}\gamma_{2n} - \frac{1}{2}\gamma_n$.

Therefore $\lim t_{3n} = \frac{1}{2}\log 2$.

Again $t_{3n+1} = t_{3n} + \frac{1}{2n+1}$ and $t_{3n+2} = t_{3n+1} - \frac{1}{4n+2}$. Therefore $\lim t_{3n+1} = \lim t_{3n} = \frac{1}{2}\log 2$ and $\lim t_{3n+2} = \lim t_{3n+1} = \frac{1}{2}\log 2$.

This proves that $\lim t_n = \frac{1}{2}\log 2$ and hence the series
$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \cdots\cdots$ converges to $\frac{1}{2}\log 2$.

6.6. Multiplication of series.

Let $a_0 + a_1 + a_2 + a_3 + \cdots$, $b_0 + b_1 + b_2 + b_3 + \cdots$ be two series. Then the product $(a_0 + a_1 + a_2 + a_3 + \cdots)(b_0 + b_1 + b_2 + b_3 + \cdots)$ contains doubly infinite number of terms of the type $a_i b_j$ and they can be arranged in the form of a doubly infinite array

$$\begin{array}{ccccc} a_0b_0 & a_0b_1 & a_0b_2 & a_0b_3 & \cdots \\ a_1b_0 & a_1b_1 & a_1b_2 & a_1b_3 & \cdots \\ a_2b_0 & a_2b_1 & a_2b_2 & a_2b_3 & \cdots \\ a_3b_0 & a_3b_1 & a_3b_2 & a_3b_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

This array extends to the right and also downwards. We can arrange the terms of the array in the form of an infinite series in many ways.

Two particular arrangements are described below.

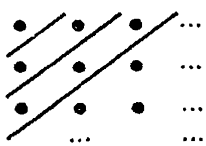

Figure 1

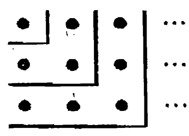

Figure 2

1. We take the first term as a_0b_0 in which the sum of the suffixes in the product a_0b_0 is 0; the second term as $a_0b_1 + a_1b_0$ in which the sum of the suffixes in each product $a_i b_j$ is 1; the third term as $a_0b_2 + a_1b_1 + a_2b_0$ in which the sum of the suffixes in each product a_ib_j is 2; and so on.

The series takes the form

$$a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots \quad \text{(i)}$$

The nth term of the series is the sum of all products lying between the $(n-1)$th and the nth lines as shown in the figure 1 where dots represent the products (the dot appearing in the ijth entry represents the product a_ib_j).

If $c_n = a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \cdots + a_nb_0$ then the series (i) is $c_0 + c_1 + c_2 + \cdots$. The series $c_0 + c_1 + c_2 + \cdots$ is said to be the *Cauchy product* of the series $a_0 + a_1 + a_2 + a_3 + \cdots$ and $b_0 + b_1 + b_2 + b_3 + \cdots$

[Note that c_n is the co-efficient of x^n in the product of the two series $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ and $b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots$]

2. We take the first term as a_0b_0 in which both the suffixes are 0; the second term as $a_1b_0 + a_1b_1 + a_0b_1$ where each term contains the suffix 1 but no higher suffix; the third term as $a_2b_0 + a_2b_1 + a_2b_2 + a_1b_2 + a_0b_2$ where each term contains the suffix 2 but no higher suffix; and so on.

The series takes the form $a_0b_0 + (a_1b_0 + a_1b_1 + a_0b_1) + (a_2b_0 + a_2b_1 + a_2b_2 + a_1b_2 + a_0b_2) + \cdots$

The nth term of the series is the sum of all products lying between the $(n-1)$th and the nth squares as shown in the figure 2 where dots represent the products (the dot appearing in the ijth entry represents the product a_ib_j).

The sum of the first $(n+1)$ terms of the series is $(a_0 + a_1 + a_2 + a_3 + \cdots + a_n)(b_0 + b_1 + b_2 + b_3 + \cdots + b_n)$.

Theorem 6.6.1. If $a_0 + a_1 + a_2 + a_3 + \cdots$ and $b_0 + b_1 + b_2 + b_3 + \cdots$ be two convergent series of positive terms with s and t as their sums, then the series $a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots$ (i.e., the Cauchy product) is convergent and has the sum st.

Proof. Let us arrange the terms of the product $(a_0 + a_1 + a_2 + a_3 + \cdots)(b_0 + b_1 + b_2 + b_3 + \cdots)$ in the form of a doubly infinite array

$$\begin{array}{ccccc}
a_0b_0 & a_0b_1 & a_0b_2 & a_0b_3 & \ldots \\
a_1b_0 & a_1b_1 & a_1b_2 & a_1b_3 & \ldots \\
a_2b_0 & a_2b_1 & a_2b_2 & a_2b_3 & \ldots \\
a_3b_0 & a_3b_1 & a_3b_2 & a_3b_3 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{array}$$

This array extends to the right and also downwards. Let us arrange the elements of the array in the form of an infinite series in two ways –

(i) $a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots$ (A)

(ii) $a_0b_0 + (a_1b_0 + a_1b_1 + a_0b_1) + (a_2b_0 + a_2b_1 + a_2b_2 + a_1b_2 + a_0b_2) + \cdots$ ... (B)

Let $s_n = a_0 + a_1 + a_2 + \cdots + a_n$, $t_n = b_0 + b_1 + b_2 + \cdots + b_n$. Then $\lim s_n = s$, $\lim t_n = t$.

Let σ_n be the sum of the first $(n+1)$ terms of the series (B).

Then $\sigma_0 = s_0t_0$, $\sigma_1 = s_1t_1$, $\sigma_2 = s_2t_2, \ldots$ $\sigma_n = s_nt_n, \ldots$

$\lim \sigma_n = \lim s_nt_n = st$. Therefore the series (B) is convergent and has the sum st.

Since the series (B) is a convergent series of positive terms, the series remains convergent with the same sum st after removal of brackets. The series (A) is obtained from the resulting series by rearrangement of terms and then by introduction of brackets. Hence the series (A) remains convergent with the same sum st.

This completes the proof.

Theorem 6.6.2. If $a_0 + a_1 + a_2 + a_3 + \cdots$ and $b_0 + b_1 + b_2 + b_3 + \cdots$ be two absolutely convergent series with s and t as their sums, then the series $a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots$ (i.e., the Cauchy product) is absolutely convergent and has the sum st.

Proof. Since an absolutely convergent series remains convergent either by rearrangement of terms or by introduction of brackets and in either case the sum remains unaltered, the theorem can be established by following the same lines of proof as discussed in the previous theorem.

The following theorem due to Mertens is a further extension of the previous one and it is stated below without proof.

Theorem 6.6.3. (Mertens)

If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be convergent series with sums s and t respectively and one of the series, say $\sum_{n=0}^{\infty} a_n$ be absolutely convergent, then the series $\sum_{n=0}^{\infty} c_n$, where $c_n = a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \cdots + a_nb_0$, is convergent and its sum is st.

Note. If both the series Σa_n and Σb_n be non-absolutely convergent then their Cauchy product may not be convergent.

For example, let us consider the series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \cdots$.

The series is non-absolutely convergent. Let the series be $\sum\limits_{n=1}^{\infty} a_n$.

Let the Cauchy product of tne series $\sum\limits_{n=1}^{\infty} a_n . \sum\limits_{n=1}^{\infty} a_n$ be $\sum\limits_{n=1}^{\infty} c_n$.

Then $c_n = (-1)^{n-1}[\frac{1}{\sqrt{1.n}} + \frac{1}{\sqrt{2(n-1}} + \frac{1}{\sqrt{3(n-2}} + \cdots + \frac{1}{\sqrt{n.1}}]$.

$r(n-r+1) = (\frac{n+1}{2})^2 - (\frac{n+1}{2} - r)^2 \leq (\frac{n+1}{2})^2$ for all r satisfying $1 \leq r \leq n$.

$|c_n| \geq \frac{2n}{n+1}$ and this implies $\lim c_n \neq 0$. The necessary condition for convergence of the series $\sum\limits_{n=1}^{\infty} c_n$ is not satisfied.

This establishes that the Cauchy product of two non-absolutely convergent series may not be convergent.

If, however, the Cauchy product of two non-absolutely convergent series be convergent, then the following theorem due to Abel establishes that the sum of the Cauchy product is the product of the sums of the series.

Theorem 6.6.4. (Abel)

If $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ be convergent series with sums A and B respectively and if their Cauchy product $\sum\limits_{n=1}^{\infty} c_n$ be convergent with the sum C, then $C = AB$.

First we prove the following lemma.

Lemma. If $\lim u_n = u, \lim v_n = v$, then $\lim \frac{u_n v_1 + u_{n-1} v_2 + \cdots + u_1 v_n}{n} = uv$.

Proof. Let $u_n = u + \alpha_n$, $v_n = v + \beta_n$. Then $\lim \alpha_n = 0, \lim \beta_n = 0$.

$$\frac{u_n v_1 + u_{n-1} v_2 + \cdots + u_1 v_n}{n}$$
$$= \frac{(u+\alpha_n)(v+\beta_1) + (v+\alpha_{n-1})(v+\beta_2) + \cdots + (u+\alpha_1)(v+\beta_n)}{n} = uv + \frac{u}{n}[\beta_1 + \beta_2 + \cdots + \beta_n] + \frac{v}{n}[\alpha_1 + \alpha_2 + \cdots + \alpha_n] + \frac{\alpha_n\beta_1 + \alpha_{n-1}\beta_2 + \cdots + \alpha_1\beta_n}{n} \quad \text{... (i)}$$

By Cauchy's theorem, $\lim \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n} = 0, \lim \frac{\beta_1 + \beta_2 + \cdots + \beta_n}{n} = 0$.

Since $\lim \alpha_n = 0$, the sequence $\{\alpha_n\}$ is bounded. So there exists a positive real number k such that $\alpha_n < k$ for all n.

Therefore $\lim \frac{\alpha_n\beta_1 + \alpha_{n-1}\beta_2 + \cdots + \alpha_1\beta_n}{n} \leq k . \lim \frac{\beta_1 + \beta_2 + \cdots + \beta_n}{n} = 0$.

From (i) it follows that $\lim \frac{u_n v_1 + u_{n-1} v_2 + \cdots + u_1 v_n}{n} = uv$.

Proof of the theorem.

Let $s_n = a_1+a_2+\cdots+a_n$, $t_n = b_1+b_2+\cdots+b_n$, $p_n = c_1+c_2+\cdots+c_n$.
Then $c_1 = a_1b_1$, $c_2 = a_1b_2 + a_2b_1$, $c_3 = a_1b_3 + a_2b_2 + a_3b_1, \cdots$

$$p_n = (a_1b_1) + (a_1b_2 + a_2b_1) + \cdots + (a_1b_n + a_2b_{n-1} + \cdots + a_nb_1)$$
$$= a_1t_n + a_2t_{n-1} + \cdots + a_nt_1.$$

$$p_1 + p_2 + \cdots + p_n = s_nt_1 + s_{n-1}t_2 + \cdots + s_1t_n.$$

Since $\lim s_n = A, \lim t_n = B$; by the lemma we have
$\lim \frac{s_nt_1+s_{n-1}t_2+\cdots+s_1t_n}{n} = AB$ ... (ii)

Again since $\lim p_n = C$, by Cauchy's theorem we have
$\lim \frac{p_1+p_2+\cdots+p_n}{n} = C$ ... (iii)

From (ii) and (iii) it follows that $AB = C$.

Examples.

1. The series $1 + x + x^2 + x^3 + \cdots$ is absolutely convergent for $|x| < 1$ and the sum of the series is $\frac{1}{1-x}$, $|x| < 1$.

(i) Let us consider the product $(1+x+x^2+\cdots+x^n+\cdots)(1+x+x^2+\cdots+x^n+\cdots)$.

Let the Cauchy product be the series $c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$. This series is absolutely convergent for $|x| < 1$ and the sum of the series is $\frac{1}{(1-x)^2}$, $|x| < 1$.

Here $c_n = n + 1$.

Therefore $1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots = \frac{1}{(1-x)^2}$, $|x| < 1$.

(ii) Let us consider the product $(1+2x+3x^2+\cdots+(n+1)x^n+\cdots)(1+x+x^2+\cdots+x^n+\cdots)$.

Let the Cauchy product be the series $c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$. This series is absolutely convergent for $|x| < 1$ and the sum of the series is $\frac{1}{(1-x)^3}$, $|x| < 1$.

Here $c_n = 1 + 2 + 3 + \cdots + (n+1) = \frac{1}{2}(n+1)(n+2)$.

Therefore $1+3x+6x^2+\cdots+\frac{1}{2}(n+1)(n+2)x^n+\cdots = \frac{1}{(1-x)^3}$, $|x| < 1$.

2. The series $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$ is a convergent series of positive terms and the sum of the series is e.

Let us consider the product $(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots)(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots)$.

Let the Cauchy product be the series $c_0 + c_1 + c_2 + c_3 + \cdots$. This series is convergent and the sum of the series is e^2.

$$c_n = \frac{1}{n!} + \frac{1}{1!(n-1)!} + \frac{1}{2!(n-2)!} + \cdots + \frac{1}{(n-1)!1!} + \frac{1}{n!}$$
$$= \frac{1}{n!}[1 + {}^n c_1 + {}^n c_2 + \cdots + 1] = \frac{2^n}{n!}.$$

Therefore $1 + \frac{2}{1!} + \frac{2^2}{2!} + \cdots + \frac{2^n}{n!} + \cdots = e^2$.

Exercises 10

1. If $\Sigma a_n{}^2$ and $\Sigma b_n{}^2$ be both convergent prove that the series $\Sigma a_n b_n$ is absolutely convergent.

2. If $\{u_n\}$ be a sequence of real numbers and Σu_n^2 is convergent, prove that $\Sigma(u_n/n)$ is absolutely convergent.

3. If $\{a_n\}$ be a monotone decreasing sequence of positive real numbers and $\lim a_n = 0$, prove that the following series are convergent.

(i) $a_1 - \frac{1}{2}(a_1 + a_2) + \frac{1}{3}(a_1 + a_2 + a_3) - \cdots\cdots$

(ii) $a_1 - \frac{1}{2}(a_1 + a_3) + \frac{1}{3}(a_1 + a_3 + a_5) - \cdots\cdots$

(iii) $a_1 - \frac{1}{3}(a_1 + a_3) + \frac{1}{5}(a_1 + a_3 + a_5) - \cdots\cdots$

Hint. (i) Let $b_n = \frac{a_1+a_2+\cdots+a_n}{n}$. Then $b_n > 0, b_{n+1} \le b_n$ for all $n \ge 1$ and $\lim b_n = 0$; (ii) The sequence $\{a_{2n-1}\}$ is a monotone decreasing null sequence.

4. Prove that the following series are convergent.

(i) $1 - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{6^2} + \cdots\cdots$

(ii) $1 - \frac{1}{3.4} + \frac{1}{5.4^2} - \frac{1}{7.4^3} + \cdots\cdots$

(iii) $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots\cdots$

(iv) $1 - \frac{1}{3.2^2} + \frac{1}{5.3^2} - \frac{1}{7.4^2} + \cdots\cdots$

(v) $1 - \frac{1}{2}(1 + \frac{1}{3}) + \frac{1}{3}(1 + \frac{1}{3} + \frac{1}{5}) - \cdots\cdots$

(vi) $\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} - \cdots\cdots$

(vii) $\frac{1}{2^2 \log 2} - \frac{1}{3^2 \log 3} + \frac{1}{4^2 \log 4} - \cdots\cdots$

(viii) $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \cdots, 0 < x < 1$.

5. Prove that the following series are conditionally convergent.

(i) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots\cdots$

(ii) $1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \cdots\cdots$

(iii) $1 - \frac{1}{2} + \frac{1.3}{2.4} - \frac{1.3.5}{2.4.6} + \cdots\cdots$

(iv) $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \cdots\cdots$

(v) $\sin\frac{\pi}{2} - \sin\frac{\pi}{4} + \sin\frac{\pi}{6} - \cdots\cdots$

(vi) $(\frac{1}{2})^2 - (\frac{1.3}{2.4})^2 + (\frac{1.3.5}{2.4.6})^2 - \cdots \cdots$

(vii) $(\frac{1}{2})^2 - (\frac{1.4}{2.5})^2 + (\frac{1.4.7}{2.5.8})^2 - \cdots \cdots$

6. Show that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \cdots$ converges to $\log 2$, but the rearranged series

(i) $1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16} + \frac{1}{5} - \cdots \cdots$ converges to 0,

(ii) $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots \cdots$ converges to $\frac{3}{2} \log 2$,

(iii) $1 + \frac{1}{3} - \frac{1}{2} - \frac{1}{4} + \frac{1}{5} + \frac{1}{7} - \frac{1}{6} - \frac{1}{8} + \cdots \cdots$ converges to $\log 2$,

(iv) $1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{4} + \cdots \cdots$ converges to $\frac{1}{2} \log 12$.

7. Prove that the series

$1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \cdots$ is divergent.

[**Hint.** Let s_n be the nth partial sum. Then $s_{3n} > (\frac{1}{3} + \frac{1}{3} - \frac{1}{3}) + (\frac{1}{6} + \frac{1}{6} - \frac{1}{6}) + \cdots + (\frac{1}{3n} + \frac{1}{3n} - \frac{1}{3n})$.]

8. If Σu_n be a convergent series, show that the following series are convergent.

(i) $\Sigma \frac{u_n}{n}$, (ii) $\Sigma n^{1/n} u_n$, (iii) $\Sigma \frac{u_n}{\log(n+1)}$.

9. Test the series for convergence. If the series be convergent, determine whether it is absolutely or conditionally convergent.

(i) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\sqrt{n}}$, (ii) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{2^n}$, (iii) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^2+1}$,

(iv) $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n!}$, (v) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n!)^2}{(2n)!} .5^n$, (vi) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n!)^2}{(2n)!} .3^n$,

(vii) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \log(n+1)}{n+1}$, (viii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)\log(n+1)}$.

7. LIMITS

7.1. Limit of a function.

Let f be a real function defined on a domain $D \subset \mathbb{R}$. In order that f may have a limit $l(\in \mathbb{R})$ at a point c, for x sufficiently close to c, $f(x)$ should be arbitrarily close to l. For this to be meaningful, it is necessary that c be a limit point of the domain D. Keeping this requirement in view, we give the formal definition.

Definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of D. A real number l is said to be a *limit of f at c* if corresponding to any neighbourhood V of l there exists a neighbourhood W of c such that $f(x) \in V$ for all $x \in [W - \{c\}] \cap D$.

This is expressed by the symbol $\lim\limits_{x \to c} f(x) = l$.

Equivalent definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of D. A real number l is said to be a *limit of f at c* if corresponding to a pre-assigned positive ϵ there exists a positive δ such that

$l - \epsilon < f(x) < l + \epsilon$ for all $x \in N'(c, \delta) \cap D$,

where $N'(c, \delta) = \{x \in \mathbb{R} : 0 <\mid x - c \mid< \delta\} = (c - \delta, c + \delta) - \{c\}$.

Note 1. In order that we may enquire if $\lim\limits_{x \to c} f(x)$ exists, c must be a *limit point* of the domain D of the function f.

Note 2. The definition demands that all values of f in some deleted δ-neighbourhood $N'(c, \delta)$ contained in D must lie in the chosen ϵ-neighbourhood of l. It does not matter whether c belongs to D or not. Even if $c \in D$, $f(c)$ need not lie in the ϵ-neighbourhood of l.

Theorem 7.1.1. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $c \in D'$. Then f can have at most one limit at c.

Proof. Suppose, on the contrary, there exist two different limits, l, m of the function f at c.

Since $l \neq m$, we assume $m > l$, without loss of generality. Let $\epsilon = \frac{m-l}{2} > 0$. Then the neighbourhoods $(l - \epsilon, l + \epsilon)$ and $(m - \epsilon, m + \epsilon)$ are disjoint.

Since l is a limit of f at c, there exists a positive δ_1 such that
$l - \epsilon < f(x) < l + \epsilon$ for all $x \in N'(c, \delta_1) \cap D$.

Since m is a limit of f at c, there exists a positive δ_2 such that
$m - \epsilon < f(x) < m + \epsilon$ for all $x \in N'(c, \delta_2) \cap D$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $l - \epsilon < f(x) < l + \epsilon$ and $m - \epsilon < f(x) < m + \epsilon$ for all $x \in N'(c, \delta) \cap D$. This is a contradiction, since the neighbourhoods $(l - \epsilon, l + \epsilon)$ and $(m - \epsilon, m + \epsilon)$ are disjoint.

Therefore $l = m$ and the theorem is done.

Worked Examples.

1. Show that $\lim\limits_{x \to 2} f(x) = 4$, where $f(x) = \frac{x^2 - 4}{x - 2}, x \neq 2$.

Here the domain D of f is $\mathbb{R} - \{2\}$. 2 is a limit point of D.

When $x \in D, \mid f(x) - 4 \mid = \mid \frac{x^2-4}{x-2} - 4 \mid = \mid x - 2 \mid$.

Let us choose $\epsilon > 0$.

$\mid f(x) - 4 \mid < \epsilon$ whenever $\mid x - 2 \mid < \epsilon$ and $x \in D$, i.e., for all $x \in D$ satisfying $0 < \mid x - 2 \mid < \epsilon$.

Therefore $\mid f(x) - 4 \mid < \epsilon$ for all $x \in N'(2, \delta) \cap D$ [taking $\delta = \epsilon$].

So we have $\lim\limits_{x \to 2} f(x) = 4$.

2. Show that $\lim\limits_{x \to 2} f(x) = 4$,

$$\text{where } f(x) = \begin{cases} \frac{x^2-4}{x-2}, & x \neq 2 \\ 10, & x = 2. \end{cases}$$

Here the domain D of f is $\mathbb{R}$. 2 is a limit point of D.

When $x \in D - \{2\}, \mid f(x) - 4 \mid = \mid \frac{x^2-4}{x-2} - 4 \mid = \mid x - 2 \mid$.

Let us choose $\epsilon > 0$.

$\mid f(x) - 4 \mid < \epsilon$ whenever $\mid x - 2 \mid < \epsilon$ and $x \neq 2$.

Therefore $\mid f(x) - 4 \mid < \epsilon$ for all $x \in N'(2, \delta) \cap D$ [taking $\delta = \epsilon$].

So we have $\lim\limits_{x \to 2} f(x) = 4$.

3. Show that $\lim\limits_{x \to 0} f(x) = 0$ where $f(x) = \sqrt{x}, x \geq 0$.

Here the domain D of f is $\{x \in \mathbb{R} : x \geq 0\}$. 0 is a limit point of D.

Let us choose $\epsilon > 0$.

When $x \geq 0, \mid f(x) - 0 \mid = \sqrt{x}$.

Therefore $\mid f(x) - 0 \mid < \epsilon$ for all x satisfying $0 < x < \epsilon^2$, i.e., for all $x \in N'(0, \delta) \cap D$ [taking $\delta = \epsilon^2$].

So we have $\lim\limits_{x \to 0} f(x) = 0$.

Note. Here $N'(0, \delta) \cap D = (0, \delta)$, since $D = \{x \in \mathbb{R} : x \geq 0\}$.

Theorem 7.1.2. (Sequential criterion)

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of D and $l \in \mathbb{R}$. Then $\lim\limits_{x \to c} f(x) = l$ if and only if for *every* sequence $\{x_n\}$ in $D - \{c\}$ converging to c, the sequence $\{f(x_n)\}$ converges to l.

Proof. Let $\lim\limits_{x \to c} f(x) = l$. Then for a pre-assigned positive ϵ, there exists a positive δ such that

$l - \epsilon < f(x) < l + \epsilon$ for all $x \in N'(c, \delta) \cap D$ ... ... (i)

Let $\{x_n\}$ be a sequence in $D - \{c\}$ converging to c.

Since $\lim x_n = c$, there exists a natural number k such that

$c - \delta < x_n < c + \delta$ for all $n \geq k$.

Therefore from (i) $l - \epsilon < f(x_n) < l + \epsilon$ for all $n \geq k$.

This proves that $\lim f(x_n) = l$.

Conversely, let for every sequence $\{x_n\}$ in $D - \{c\}$ converging to $c, \lim f(x_n) = l$. We prove that $\lim\limits_{x \to c} = l$.

If not, there exists a neighbourhood V of l such that for every neighbourhood W of c there exists at least one element $x_w \in [W - \{c\}] \cap D$ for which $f(x_w)$ does not belong to V.

Let $W_1 = N(c, 1)$. Then there exists an element $x_1 \in N'(c, 1) \cap D$ such that $f(x_1) \notin V$.

Let $W_2 = N(c, \frac{1}{2})$. Then there exists an element $x_2 \in N'(c, \frac{1}{2}) \cap D$ such that $f(x_2) \notin V$.

Proceeding in this manner, we obtain a sequence $\{x_1, x_2, x_3, \ldots\}$ in D such that $\lim x_n = c$, since $x_n \in W_n = N(c, \frac{1}{n})$ for all $n \in \mathbb{N}$; but the sequence $f(x_n)$ does not converge to l, since $f(x_n)$ does not belong to the neighbourhood V of l for all $n \in \mathbb{N}$. This is a contradiction to the hypothesis and therefore $\lim\limits_{x \to c} f(x) = l$.

This completes the proof.

Worked Examples (continued).

4. Prove that $\lim\limits_{x \to 0} f(x)$ does not exist where $f(x) = \sin \frac{1}{x}, x \neq 0$.

Here the domain D of f is $\mathbb{R} - \{0\}$. 0 is a limit point of D.

Let us consider the sequence $\{x_n\}$ in D defined by $x_n = \frac{2}{(4n-3)\pi}, n \in \mathbb{N}$. The sequence is $\{\frac{2}{\pi}, \frac{2}{5\pi}, \frac{2}{9\pi}, \cdots\}$ and this converges to 0.

The sequence $\{f(x_n)\}$ is $\{\sin \frac{\pi}{2}, \sin \frac{5\pi}{2}, \sin \frac{9\pi}{2}, \cdots\}$, i.e., $\{1, 1, 1, \ldots\}$ and this converges to 1.

Let us consider the sequence $\{y_n\}$ in D defined by $y_n = \frac{1}{n\pi}, n \in \mathbb{N}$.

The sequence is $\{\frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \cdots\}$ and this converges to 0.

The sequence $\{f(y_n)\}$ is $\{\sin \pi, \sin 2\pi, \sin 3\pi, \cdots\}$, i.e., $\{0, 0, 0, \ldots\}$ and this converges to 0.

Thus we have two sequences $\{x_n\}$ and $\{y_n\}$ in D both converging to 0 but the sequences $\{f(x_n)\}$ and $\{f(y_n)\}$ converge to two different limits.

Therefore $\lim\limits_{x \to 0} \sin \frac{1}{x}$ does not exist.

5. Show that $\lim\limits_{x \to 0} [x]$ does not exist.

Let $f(x) = [x]$. The domain of f is $\mathbb{R}$. In order to examine whether $\lim\limits_{x \to 0} f(x)$ exists or not, it is sufficient to consider the function f in an arbitrary nieghbourhood of 0, say $N(0, 1)$.

$$\begin{aligned} f(x) &= -1, \text{ if } -1 < x < 0 \\ &= 0, \quad \text{if } 0 \leq x < 1. \end{aligned}$$

Let us consider the sequence $\{x_n\}$ in $N(0, 1)$ defined by $x_n = \frac{1}{n+1}, n \in \mathbb{N}$. The sequence $\{x_n\}$ converges to 0.

The sequence $\{f(x_n)\}$ is $\{0, 0, 0, \ldots\}$. This converges to 0.

Let us consider the sequence $\{y_n\}$ in $N(0, 1)$ defined by $y_n = -\frac{1}{n+1}, n \in \mathbb{N}$. The sequence $\{y_n\}$ converges to 0.

The sequence $\{f(y_n)\}$ is $\{-1, -1, -1, \cdots\}$. This converges to -1.

Thus we have two sequences $\{x_n\}$ and $\{y_n\}$ in $N(0, 1)$ both converging to 0 but the sequences $\{f(x_n)\}$ and $\{f(y_n)\}$ converge to two different limits. Therefore $\lim\limits_{x \to 0} f(x)$, i.e., $\lim\limits_{x \to 0} [x]$ does not exist.

6. Show that $\lim\limits_{x \to 0} \operatorname{sgn} x$ does not exist.

Let $f(x) = \operatorname{sgn} x$. Then
$$\begin{aligned} f(x) &= 1 \text{ for } x > 0 \\ &= 0 \text{ for } x = 0 \\ &= -1 \text{ for } x < 0. \end{aligned}$$

Here the domain of f is $\mathbb{R}$. 0 is a limit point of the domain of f.

Let us consider the sequence $\{x_n\}$ in $\mathbb{R}$ defined by $x_n = \frac{1}{n}, n \in \mathbb{N}$. Then $\lim x_n = 0$. $f(x_n) = 1$ for all $n \in \mathbb{N}$ and therefore $\lim f(x_n) = 1$.

Let us consider the sequence $\{y_n\}$ in $\mathbb{R}$ defined by $y_n = \frac{-1}{n}, n \in \mathbb{N}$. Then $\lim y_n = 0$. $f(y_n) = -1$ for all $n \in \mathbb{N}$ and $\lim f(y_n) = -1$.

Thus we have two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{R}$ both converging to 0 but $\lim f(x_n) \neq \lim f(y_n)$. Therefore $\lim\limits_{x \to 0} f(x)$ does not exist.

Theorem 7.1.3. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $c \in D'$. If f has a limit $l \in \mathbb{R}$ at c then f is bounded on $N(c) \cap D$ for some neighbourhood $N(c)$ of c.

Proof. Let $\lim_{x \to c} f(x) = l$.

Let us choose $\epsilon = 1$. Then there exists a positive δ such that $| f(x) - l | < 1$ for all $x \in N'(c, \delta) \cap D$.

But $| f(x) - l | \geq | \, | f(x) | - | l | \, |$.

It follows that $| f(x) | < | l | + 1$ for all $x \in N'(c, \delta) \cap D$.

Therefore if $c \notin D$, $| f(x) | < | l | + 1$ for all $x \in N(c, \delta) \cap D$, showing that f is bounded on $N(c, \delta) \cap D$.

If, however, $c \in D$, let $B = \max\{| f(c) | + 1, | l | + 1\}$.

Then $| f(x) | < B$ for all $x \in N(c, \delta) \cap D$, showing that f is bounded on $N(c, \delta) \cap D$.

This completes the proof.

Corollary. If f be not bounded on $N(c, \delta) \cap D$ for some δ-neighbourhood $N(c, \delta)$ of c then $\lim_{x \to c} f(x)$ does not exist in $\mathbb{R}$.

For example, $\lim_{x \to 0} \frac{1}{x}$ does not exist in $\mathbb{R}$.

Let $f(x) = \frac{1}{x}, x \in D$. Here $D = \mathbb{R} - \{0\}$ and $0 \in D'$. f is unbounded on every neighbourhood of 0. Therefore $\lim_{x \to 0} \frac{1}{x}$ does not exist in $\mathbb{R}$.

Theorem 7.1.4. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $c \in D'$ and $\lim_{x \to c} f(x) = l$.

(i) If $l > 0$ then there exists a positive δ such that $f(x) > 0$ for all $x \in N'(c, \delta) \cap D$.

(ii) If $l < 0$ then there exists a positive δ such that $f(x) < 0$ for all $x \in N'(c, \delta) \cap D$.

Proof. (i) Let us choose a positive ϵ such that $l - \epsilon > 0$.

Since $\lim_{x \to c} f(x) = l$, there exists a $\delta > 0$ such that $l - \epsilon < f(x) < l + \epsilon$ for all $x \in N'(c, \delta) \cap D$.

Since $l - \epsilon > 0, f(x) > 0$ for all $x \in N'(c, \delta) \cap D$.

(ii) Let us choose a positive ϵ such that $l + \epsilon < 0$.

Since $\lim_{x \to c} f(x) = l$, there exists a $\delta > 0$ such that $l - \epsilon < f(x) < l + \epsilon$ for all $x \in N'(c, \delta) \cap D$.

Since $l + \epsilon < 0, f(x) < 0$ for all $x \in N'(c, \delta) \cap D$.

This completes the proof.

Theorem 7.1.5. Let $D \subset \mathbb{R}$ and f and g are functions on D to $\mathbb{R}$.

Let $c \in D'$ and $\lim_{x \to c} f(x) = l, \lim_{x \to c} g(x) = m$. Then

(i) $\lim_{x \to c} (f + g)(x) = l + m$, where $f + g : D \to \mathbb{R}$ is defined by $(f + g)(x) = f(x) + g(x), x \in D$.

(ii) if $k \in \mathbb{R}$, $\lim\limits_{x \to c}(k.f)(x) = kl$, where $kf : D \to \mathbb{R}$ is defined by $(kf)(x) = k.f(x), x \in D$.

(iii) $\lim\limits_{x \to c}(f.g)(x) = lm$, where $f.g : D \to \mathbb{R}$ is defined by $(f.g)(x) = f(x).g(x), x \in D$.

Proof. (i) $(f+g)(x) - (l+m) = \mid \overline{f(x)+g(x)} \mid - \overline{l+m} \mid$
$\leq \mid f(x) - l \mid + \mid g(x) - m \mid$.

Let us choose $\epsilon > 0$.

Since $\lim\limits_{x \to c} f(x) = l$, there exists a positive δ_1 such that
$\mid f(x) - l \mid < \frac{\epsilon}{2}$ for all $x \in N'(c, \delta_1) \cap D$.

Since $\lim\limits_{x \to c} g(x) = m$, there exists a positive δ_2 such that
$\mid g(x) - m \mid < \frac{\epsilon}{2}$ for all $x \in N'(c, \delta_2) \cap D$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $\mid f(x) - l \mid < \frac{\epsilon}{2}$ and $\mid g(x) - m \mid < \frac{\epsilon}{2}$ for all $x \in N'(c, \delta) \cap D$.

Hence $\mid (f+g)(x) - \overline{l+m} \mid \leq \mid f(x) - l \mid + \mid g(x) - m \mid$
$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$ for all $x \in N'(c, \delta) \cap D$.

That is, $\mid (f+g)(x) - \overline{l+m} \mid < \epsilon$ for all $x \in N'(c, \delta) \cap D$.

This proves that $\lim\limits_{x \to c}(f+g)(x) = l + m$.

Another Proof. Let $\{x_n\}$ be a sequence in $D - \{c\}$ converging to c. Since $\lim\limits_{x \to c} f(x) = l$ and $\lim\limits_{x \to c} g(x) = m$, we have
$\lim\limits_{n \to \infty} f(x_n) = l$ and $\lim\limits_{n \to \infty} g(x_n) = m$, by sequential criterion for limits.
Therefore $\lim\limits_{n \to \infty}[f(x_n) + g(x_n)] = l + m$.
That is, $\lim\limits_{n \to \infty}(f+g)(x_n) = l + m$.

Since $\{x_n\}$ is an arbitrary sequence in $D - \{c\}$ converging to c, it follows from the sequential criterion for limits that $\lim\limits_{x \to c}(f+g)(x) = l+m$.

(ii) Proof left to the reader.

(iii) $\mid (f.g)(x) - lm \mid = \mid f(x)g(x) - lm \mid$
$= \mid (f(x) - l)g(x) + l(g(x) - m) \mid$
$\leq \mid f(x) - l \mid \mid g(x) \mid + \mid l \mid \mid g(x) - m \mid$.

Since $\lim\limits_{x \to c} g(x)$ exists, there exists a positive number B and a positive δ_1 such that $\mid g(x) \mid < B$ for all $x \in N(c, \delta_1) \cap D$.

Let $k = \max\{B, \mid l \mid\}$. Then $k > 0$ and
$\mid (f.g)(x) - lm \mid < k\{\mid f(x) - l \mid + \mid g(x) - m \mid\}$ for all $x \in N(c, \delta_1) \cap D$.

Let us choose $\epsilon > 0$. Since $\lim\limits_{x \to c} f(x) = l$, there exists a positive δ_2 such that $\mid f(x) - l \mid < \frac{\epsilon}{2k}$ for all $x \in N'(c, \delta_2) \cap D$.

Since $\lim_{x\to c} g(x) = m$, there exists a positive δ_3 such that
$| g(x) - m | < \frac{\epsilon}{2k}$ for all $x \in N'(c, \delta_3) \cap D$.

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$.
Then $| (f.g)(x) - lm | < k.\frac{\epsilon}{2k} + k.\frac{\epsilon}{2k}$ for all $x \in N'(c, \delta) \cap D$.

That is, $| (f.g)(x) - lm | < \epsilon$ for all $x \in N'(c, \delta) \cap D$.
This proves that $\lim_{x\to c}(f.g)(x) = lm$.

Note 1. Using sequential criterion for limits as in the alternative proof of part (i), the proof of the parts (ii) and (iii) can also be done.

Note 2. Let $f_1, f_2, \ldots, f_n$ be n functions each defined on some domain $D \subset \mathbb{R}$ and let $c \in D'$. If $\lim_{x\to c} f_1(x) = l_1, \lim_{x\to c} f_2(x) = l_2, \cdots, \lim_{x\to c} f_n(x) = l_n$ then

(i) $\lim_{x\to c}(f_1+f_2+\cdots+f_n)(x) = l_1+l_2+\cdots+l_n$, where $(f_1+f_2+\cdots+f_n)(x) = f_1(x) + f_2(x) + \cdots + f_n(x), x \in D$;

(ii) $\lim_{x\to c}(f_1 f_2 \ldots f_n)(x) = l_1 l_2 \ldots l_n$, where $(f_1 f_2 \ldots f_n)(x) = f_1(x) f_2(x) \ldots f_n(x)$, $x \in D$.

In particular, if $f_1 = f_2 = \cdots = f_n = f$ and $\lim_{x\to c} f(x) = l$, then $\lim_{x\to c}[f(x)]^n = l^n$. Therefore if n be a positive integer and $\lim_{x\to c} f(x) = l$, then $\lim_{x\to c}[f(x)]^n = l^n$.

Theorem 7.1.6. Let $D \subset \mathbb{R}$ and f and g be functions on D to $\mathbb{R}$ and $g(x) \neq 0$ for all $x \in D$. Let $c \in D'$ and $\lim_{x\to c} f(x) = l, \lim_{x\to c} g(x) = m \neq 0$.

Then $\lim_{x\to c} \frac{f}{g}(x) = \frac{l}{m}$ where the function $\frac{f}{g} : D \to \mathbb{R}$ is defined by $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$.

Proof. First we prove that $\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{m}$.

$\left|\frac{1}{g(x)} - \frac{1}{m}\right| = \frac{|g(x)-m|}{|g(x)||m|}$.

Let $p = \frac{1}{2} | m |$. Then $p > 0$. Since $\lim_{x\to c} g(x) = m$, there exists a positive δ_1 such that $| g(x) - m | < p$ for all $x \in N'(c, \delta_1) \cap D$.

Since $|| g(x) | - | m || \leq | g(x) - m |$, it follows that $|| g(x) | - | m || < p$ for all $x \in N'(c, \delta_1) \cap D$.

Therefore $| m | -p < | g(x) | < | m | +p$ for all $x \in N'(c, \delta_1) \cap D$
or, $| g(x) | > \frac{1}{2} | m |$ for all $x \in N'(c, \delta_1) \cap D$.

Let us choose $\epsilon > 0$. Since $\lim_{x\to c} g(x) = m$, there exists a positive δ_2

such that

$| g(x) - m | < \frac{1}{2} | m |^2 \epsilon$ for all $x \in N'(c, \delta_2) \cap D$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\left| \frac{1}{g(x)} - \frac{1}{m} \right| < \epsilon$ for all $x \in N'(c, \delta) \cap D$.

This proves that $\lim\limits_{x \to c} \frac{1}{g(x)} = \frac{1}{m}$.

The proof of the theorem is completed by considering the limit of the product of the functions f and $1/g$.

Hence $\lim\limits_{x \to c} \frac{f}{g}(x) = \lim\limits_{x \to c} \left\{ f(x) . \frac{1}{g(x)} \right\} = l . \frac{1}{m} = \frac{l}{m}$.

Theorem 7.1.7. Let $D \subset \mathbb{R}$ and f and g be functions on D to $\mathbb{R}$. Let $c \in D'$. If f is bounded on $N'(c) \cap D$ for some deleted neighbourhood $N'(c)$ of c and $\lim\limits_{x \to c} g(x) = 0$ then $\lim\limits_{x \to c} (f.g)(x) = 0$.

Proof. Since f is bounded on $N'(c) \cap D$ for some neighbourhood $N(c)$ of c, there exists a positive number B and a positive δ_1 such that $| f(x) | < B$ for all $x \in N'(c, \delta_1) \cap D$.

Let us choose $\epsilon > 0$. Since $\lim\limits_{x \to c} g(x) = 0$, there exists a positive δ_2 such that $| g(x) - 0 | < \frac{\epsilon}{B}$ for all $x \in N'(c, \delta_2) \cap D$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $| f(x) | < B$ for all $x \in N'(c, \delta) \cap D$ and $| g(x) | < \frac{\epsilon}{B}$ for all $x \in N'(c, \delta) \cap D$.

Therefore $| f.g(x) - 0 | = | f(x) | | g(x) | < \epsilon$ for all $x \in N'(c, \delta) \cap D$.

This proves $\lim\limits_{x \to c} (f.g)(x) = 0$.

Worked Examples (continued).

7. Prove that $\lim\limits_{x \to 0} x \sin \frac{1}{x^2} = 0$.

Here $\lim\limits_{x \to 0} x = 0$ and $\sin \frac{1}{x^2}$ is bounded in some deleted neighbourhood of 0. Therefore $\lim\limits_{x \to 0} x \sin \frac{1}{x^2} = 0$.

8. Prove that $\lim\limits_{x \to 0} \sqrt{x} \sin \frac{1}{x} = 0$

Let $f(x) = \sqrt{x} \sin \frac{1}{x}, x \in D$. Then $D = \{x \in \mathbb{R} : x > 0\}$.
$0 \in D'$. Let $g(x) = \sqrt{x}, x \in D, h(x) = \sin \frac{1}{x}, x \in D$.

Here $\lim\limits_{x \to 0} g(x) = 0$ and h is bounded on $N'(0) \cap D$ for some neighbourhood $N(0)$ of 0.

Therefore $\lim\limits_{x \to 0} g(x)h(x) = 0$, i.e., $\lim\limits_{x \to 0} \sqrt{x} \sin \frac{1}{x} = 0$.

Theorem 7.1.8.(a) Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $c \in D'$. If $f(x) \leq b$ for all $x \in D - \{c\}$ and $\lim_{x \to c} f(x) = l$, then $l \leq b$.

Proof. Let $\{x_n\}$ be any sequence in $D - \{c\}$ converging to c.
Since $\lim_{x \to c} f(x) = l$, $\lim_{n \to \infty} f(x_n) = l$.
Let us define a sequence $\{u_n\}$ by $u_n = b$ for all $n \in \mathbb{N}$.
Then $f(x_n) \leq u_n$ for all $n \in \mathbb{N}$.
Since $\lim f(x_n) = l$ and $\lim u_n = b, l \leq b$, by Theorem 5.5.4.

Note. If there exists a positive δ and a real number b such that $f(x) \leq b$ for all $x \in N'(c, \delta) \cap D$ and $\lim_{x \to c} f(x) = l$, then $l \leq b$.

Theorem 7.1.8.(b) Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $c \in D'$. If $f(x) \geq a$ for all $x \in D - \{c\}$ and $\lim_{x \to c} f(x) = l$, then $l \geq a$.

Proof. Let $\{x_n\}$ be any sequence in $D - \{c\}$ convergeing to c.
Since $\lim_{x \to c} f(x) = l$, $\lim_{n \to \infty} f(x_n) = l$.
Let us define a sequence $\{v_n\}$ by $v_n = a$ for all $n \in \mathbb{N}$.
Then $f(x_n) \geq v_n$ for all $n \in \mathbb{N}$.
Since $\lim_{x \to c} f(x) = l$ and $\lim v_n = a, l \geq a$, by Theorem 5.5.4.

Note. If there exists a positive δ and a real number a such that $f(x) \geq a$ for all $x \in N'(c, \delta) \cap D$ and $\lim_{x \to c} f(x) = l$, then $l \geq a$.

Theorem 7.1.9. (Sandwich theorem)

Let $D \subset \mathbb{R}$ and f, g, h be functions on D to $\mathbb{R}$. Let $c \in D'$.
If $f(x) \leq g(x) \leq h(x)$ for all $x \in D - \{c\}$ and if $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = l$ then $\lim_{x \to c} g(x) = l$.

Proof. Let us choose $\epsilon > 0$.
Since $\lim_{x \to c} f(x) = l$, there exists a positive δ_1 such that
$l - \epsilon < f(x) < l + \epsilon$ for all $x \in N'(c, \delta_1) \cap D$.
Since $\lim_{x \to c} h(x) = l$, there exists a positive δ_2 such that
$l - \epsilon < h(x) < l + \epsilon$ for all $x \in N'(c, \delta_2) \cap D$.
Let $\delta = \min\{\delta_1, \delta_2\}$.
Then $l - \epsilon < f(x) < l + \epsilon$ and $l - \epsilon < h(x) < l + \epsilon$ for all $x \in N'(c, \delta) \cap D$.
Therefore $l - \epsilon < f(x) \leq g(x) \leq h(x) < l + \epsilon$ for all $x \in N'(c, \delta) \cap D$
or, $l - \epsilon < g(x) < l + \epsilon$ for all $x \in N'(c, \delta) \cap D$.
This proves $\lim_{x \to c} g(x) = l$.

Note 1. If $f(x) < g(x) < h(x)$ for all $x \in D - \{c\}$ and if $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} h(x) = l$, then $\lim\limits_{x \to c} g(x) = l$.

Note 2. If there exists a positive δ such that $f(x) \leq g(x) \leq h(x)$ for all $x \in N'(c, \delta) \cap D$ and $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} h(x) = l$, then $\lim\limits_{x \to c} g(x) = l$.

Worked Example (continued).

9. Show that $\lim\limits_{x \to 0} x \cos \frac{1}{x} = 0$.

Let $f(x) = \cos \frac{1}{x}, x \in D$. The domain of f is $D = \{x \in \mathbb{R} : x \neq 0\}$.
$-1 \leq f(x) \leq 1$ for all $x \in D$.

Hence $-x \leq xf(x) \leq x$ for all $x > 0$ and $x \leq xf(x) \leq -x$ for all $x < 0$.

Therefore $- \mid x \mid \leq xf(x) \leq \mid x \mid$ for all $x \neq 0$.
$\lim\limits_{x \to 0} \mid x \mid = 0$ and $\lim\limits_{x \to 0} - \mid x \mid = 0$.
By Sandwich theorem, $\lim\limits_{x \to 0} xf(x) = 0$, i.e., $\lim\limits_{x \to 0} x \cos \frac{1}{x} = 0$.

Theorem 7.1.10. (Cauchy's principle)

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $c \in D'$.

A necessary and sufficient condition for the existence of $\lim\limits_{x \to c} f(x)$ is that for a pre-assigned positive ϵ there exists a positive δ such that
$\mid f(x') - f(x'') \mid < \epsilon$ for every pair of points $x', x'' \in N'(c, \delta) \cap D$.

Proof. Let $\lim\limits_{x \to c} f(x) = l$. Then for a pre-assigned positive ϵ there exists a $\delta > 0$ such that $l - \frac{\epsilon}{2} < f(x) < l + \frac{\epsilon}{2}$ for all $x \in N'(c, \delta) \cap D$.

So for every pair of points $x', x'' \in N'(c, \delta) \cap D$, we have
$l - \frac{\epsilon}{2} < f(x') < l + \frac{\epsilon}{2}$ and $l - \frac{\epsilon}{2} < f(x'') < l + \frac{\epsilon}{2}$.

$$\begin{aligned} \text{But } \mid f(x') - f(x'') \mid &= \mid f(x') - l + l - f(x'') \mid \\ &\leq \mid f(x') - l \mid + \mid f(x'') - l \mid < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2}. \end{aligned}$$

It follows that $\mid f(x') - f(x'') \mid < \epsilon$ for all $x', x'' \in N'(c, \delta) \cap D$, proving that the condition is necessary.

Let us assume that for a given $\epsilon > 0$ we can find a $\delta > 0$ such that the given condition holds.

Let us take a sequence $\{x_n\}$ such that $x_n \in D - \{c\}$ for all $n \in \mathbb{N}$ and $\lim x_n = c$. Then there exists a natural number k such that $c - \delta < x_n < c + \delta$ for all $n \geq k$.

In other words, $x_n \in N'(c, \delta) \cap D$ for all $n \geq k$.

Hence for every natural number p and every $n \geq k$,
$x_n \in N'(c, \delta) \cap D$ and $x_{n+p} \in N'(c, \delta) \cap D$.

By the condition, $| f(x_n) - f(x_{n+p}) | < \epsilon$ for all $n \geq k$ and $p = 1, 2, 3, \cdots$

This proves that the sequence $\{f(x_n)\}$ is a Cauchy sequence and is therefore convergent.

Thus for every sequence $\{x_n\}$ in $D - \{c\}$ such that $\lim x_n = c$, the sequence $\{f(x_n)\}$ is convergent.

We now prove that all such sequences $\{f(x_n)\}$ converge to a common limit. Let $\{p_n\}$ and $\{q_n\}$ be two sequences in $D - \{c\}$ such that $\lim p_n = c, \lim q_n = c$ and $\lim f(p_n) = p, \lim f(q_n) = q$.

Let us consider the sequence $\{x_n\}$ where $x_{2n-1} = p_n, x_{2n} = q_n$ i.e., $\{x_n\} = \{p_1, q_1, p_2, q_2, \cdots\}$

Then $\lim x_n = c$ and therfore $\{f(x_n)\}$ is a convergent sequence. $\{f(x_{2n-1})\}$ and $\{f(x_{2n})\}$ are subsequences of $\{f(x_n)\}$. Since $\{f(x_n)\}$ is a convergent sequence, $\lim f(x_{2n-1}) = \lim f(x_{2n})$.

Therefore $p = q = l$, say.

Thus for every sequence $\{x_n\}$ in $D - \{c\}$ such that $\lim x_n = c$, $\{f(x_n)\}$ converges to l.

By the sequential criterion for limits, $\lim\limits_{x \to c} f(x)$ exists and this proves that the condition is sufficient. This completes the proof.

Worked Examples (continued).

10. A function $f : (0, 1) \to \mathbb{R}$ is defined by

$$\begin{aligned} f(x) &= 1, \text{ if } x \text{ is rational, } x \in (0, 1) \\ &= -1, \text{ if } x \text{ is irrational, } x \in (0, 1). \end{aligned}$$

Using Cauchy's principle prove that $\lim\limits_{x \to a} f(x)$ does not exist, where $a \in [0, 1]$.

Here the domain of f is $D = \{x \in \mathbb{R} : 0 < x < 1\}$. $a \in D'$.

Let us choose $\epsilon = 1$. In order that $\lim\limits_{x \to a} f(x)$ should exist, it is necessary that there exists a $\delta > 0$ such that $| f(x') - f(x'') | < 1$ for every pair of points $x', x'' \in N'(a, \delta) \cap (0, 1)$.

Whatever $\delta (> 0)$ may be, the set $N'(a, \delta) \cap (0, 1)$ contains rational and irrational points. Let x' be rational and x'' be irrational in $N'(a, \delta) \cap (0, 1)$.

Then $f(x') = 1, f(x'') = -1$ and $| f(x') - f(x'') | = 2$.

Therefore $| f(x') - f(x'') | \not< \epsilon$ for some pair of points $x', x'' \in N'(a, \delta) \cap (0, 1)$ for every $\delta > 0$.

Therefore Cauchy's condition for the existence of $\lim\limits_{x \to a} f(x)$ is not satisfied and $\lim\limits_{x \to a} f(x)$ does not exist.

11. Using Cauchy's principle prove that $\lim\limits_{x \to 0} \cos \frac{1}{x}$ does not exist.

Let $f(x) = \cos \frac{1}{x}, x \neq 0$. Here the domain D of f is $\{x \in \mathbb{R} : x \neq 0\}$.

Let us choose $\epsilon = \frac{1}{2}$. In order that $\lim\limits_{x \to 0} f(x)$ should exist, it is necessary that there exists a $\delta > 0$ such that $\mid f(x') - f(x'') \mid < \frac{1}{2}$ for every pair of points $x', x'' \in N'(0, \delta) \cap D$.

For a given positive δ we can find a natural number n such that $\left| \frac{1}{2n\pi} - \frac{2}{(4n+1)\pi} \right| < \delta$, because $\left| \frac{1}{2n\pi} - \frac{2}{(4n+1)\pi} \right| = \frac{1}{\pi} \frac{1}{2n(4n+1)}$ and this can be made less than δ for a suitable natural number n.

Let $x' = \frac{1}{2n\pi}$ and $x'' = \frac{2}{(4n+1)\pi}$. Then $x', x'' \in N'(0, \delta) \cap D$ and $f(x') = 1, f(x'') = 0$.

Therefore $\mid f(x') - f(x'') \mid \not< \epsilon$ for some pair of points $x', x'' \in N'(0, \delta) \cap D$ for every $\delta > 0$.

Therefore Cauchy's condition for the existence of $\lim\limits_{x \to 0} \cos \frac{1}{x}$ is not satisfied and $\lim\limits_{x \to 0} \cos \frac{1}{x}$ does not exist.

7.2. One sided limits.

There are cases where a function f does not have a limit at a limit point c of its domain D, but the restriction of the function f to an interval at one side of c (either right or left) may have a limit.

For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) =$ sgn x does not possess a limit at 0 but the restriction of f to $(0, \infty)$ does have a limit at 0 and also the restriction of f to $(-\infty, 0)$ does have a limit at 0.

In the former case we say that f has a *right hand limit* at 0 and in the latter case we say that f has a *left hand limit* at 0.

Definitions.

Right hand limit. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ is a function. Let c be a limit point of $D_1 = D \cap (c, \infty) = \{x \in D : x > c\}$.

f is said to have a *right hand limit* $l (\in \mathbb{R})$ at c if corresponding to a pre-assigned positive ϵ there exists a positive δ such that

$$\mid f(x) - l \mid < \epsilon \text{ for all } x \in N'(c, \delta) \cap D_1$$

i.e., $l - \epsilon < f(x) < l + \epsilon$ for all x in D satisfying $c < x < c + \delta$.

In this case we write $\lim\limits_{x \to c+} f(x) = l$.

Left hand limit. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ is a function. Let c be a limit point of $D_2 = D \cap (-\infty, c) = \{x \in D : x < c\}$.

f is said to have a *left hand limit* $l(\in \mathbb{R})$ at c if corresponding to a pre-assigned positive ϵ there exists a positive δ such that

$$| f(x) - l |< \epsilon \text{ for all } x \in N'(c, \delta) \cap D_2$$

i.e., $l - \epsilon < f(x) < l + \epsilon$ for all x in D satisfying $c - \delta < x < c$.

In this case we write $\lim\limits_{x \to c-} f(x) = l$.

Note. In order that we may enquire if $\lim\limits_{x \to c+} f(x)$ exists, the domain D of the function f must be such that c is a limit point of $D \cap (c, \infty)$.

Similarly, in order that we may enquire if $\lim\limits_{x \to c-} f(x)$ exists, D must be such that c is a limit point of $D \cap (-\infty, c)$.

Sequential criterion.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of $D_1 = D \cap (c, \infty)$. Then $\lim\limits_{x \to c+} f(x) = l$ if and only if for every sequence $\{x_n\}$ in D_1 converging to c, the sequence $\{f(x_n)\}$ converges to l.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of $D_2 = D \cap (-\infty, c)$. Then $\lim\limits_{x \to c-} f(x) = l$ if and only if for every sequence $\{x_n\}$ in D_2 converging to c, the sequence $\{f(x_n)\}$ converges to l.

Note. It is possible that both the right hand limit and the left hand limit may exist, or both may not exist, or one of them exists while the other does not.

Worked Examples.

1. Let $f(x) = \operatorname{sgn} x$. Examine if $\lim\limits_{x \to 0+} f(x)$ and $\lim\limits_{x \to 0-} f(x)$ exist.

Here the domain D of f is $\mathbb{R}$.

Let $D_1 = D \cap (0, \infty)$. Then $D_1 = \{x \in \mathbb{R} : x > 0\}$. 0 is a limit point of D_1. $f(x) = 1$ for all $x \in D_1$. Therefore $\lim\limits_{x \to 0+} f(x) = 1$.

Let $D_2 = D \cap (-\infty, 0)$. Then $D_2 = \{x \in \mathbb{R} : x < 0\}$. 0 is a limit point of D_2. $f(x) = -1$ for all $x \in D_2$. Therefore $\lim\limits_{x \to 0-} f(x) = -1$.

Note. Here both the right hand limit and the left hand limit of f at 0 exist. f is defined at 0 but $f(0) \neq \lim\limits_{x \to 0+} f(x)$ and also $f(0) \neq \lim\limits_{x \to 0-} f(x)$.

2. Let $f(x) = e^{1/x}$. Examine if $\lim\limits_{x \to 0+} f(x)$ and $\lim\limits_{x \to 0-} f(x)$ exist.

Here the domain D of f is $\mathbb{R} - \{0\}$.

Let $D_1 = D \cap (0, \infty)$. Then $D_1 = \{x \in \mathbb{R} : x > 0\}$. 0 is a limit point of D_1.

f is unbounded on $N(0) \cap D_1$ for any neighbourhood $N(0)$ of 0.

Therefore $\lim\limits_{x \to 0+} f(x)$ does not exist in $\mathbb{R}$.

Let $D_2 = D \cap (-\infty, 0)$. Then $D_2 = \{x \in \mathbb{R} : x < 0\}$. 0 is a limit point of D_2.

We have $e^t > t > 0$ for all $t > 0$. Taking $t = -\frac{1}{x}$, we have $e^{-\frac{1}{x}} > -\frac{1}{x} > 0$ for all $x < 0$ and this implies $0 < e^{\frac{1}{x}} < -x$ for all $x < 0$.

By Sandwich theorem, $\lim\limits_{x \to 0-} f(x) = 0$.

3. Let $f(x) = \sin \frac{1}{x}$. Using sequential criterion for limits examine if $\lim\limits_{x \to 0+} f(x)$ and $\lim\limits_{x \to 0-} f(x)$ exist.

Here the domain D of f is $\mathbb{R} - \{0\}$.

Let $D_1 = D \cap (0, \infty)$. Then $D_1 = \{x \in \mathbb{R} : x > 0\}$. 0 is a limit point of D_1.

Let us consider the sequence $\{x_n\}$ where $x_n = \frac{1}{n\pi}, n \in \mathbb{N}$.

Then $x_n \in D_1$ for all $n \in \mathbb{N}$ and $\lim x_n = 0$.

$f(x_n) = 0$ for all $n \in \mathbb{N}$ and therefore $\lim f(x_n) = 0$.

Let us consider the sequence $\{y_n\}$ where $y_n = \frac{2}{(4n+1)\pi}, n \in \mathbb{N}$.

$y_n \in D_1$ for all $n \in \mathbb{N}$ and $\lim y_n = 0$.

$f(y_n) = 1$ for all $n \in \mathbb{N}$ and therefore $\lim f(y_n) = 1$.

Therefore by sequential criterion $\lim\limits_{x \to 0+} f(x)$ does not exist.

Let $D_2 = D \cap (-\infty, 0)$. Then $D_2 = \{x \in \mathbb{R} : x < 0\}$. 0 is a limit point of D_2.

Considering two sequences $\{u_n\}$ and $\{v_n\}$ where $u_n = -\frac{1}{n\pi}$ and $v_n = -\frac{2}{(4n+1)\pi}$ we can establish that $\lim\limits_{x \to 0-} f(x)$ does not exist.

Theorem 7.2.1. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of both the sets $D_1 (= D \cap (c, \infty))$ and $D_2 (= D \cap (-\infty, c))$. Then $\lim\limits_{x \to c} f(x) = l \in \mathbb{R}$ if and only if $\lim\limits_{x \to c+} f(x) = l = \lim\limits_{x \to c-} f(x)$.

Proof. Let $\lim\limits_{x \to c} f(x) = l$. Then for a pre-assigned positive ϵ, there exists a positive δ such that

$$| f(x) - l | < \epsilon \text{ for all } x \in N'(c, \delta) \cap D.$$

That is, for all x in D satisfying $0 < | x - c | < \delta, | f(x) - l | < \epsilon$.

Therefore for all x in D satisfying $c < x < c + \delta$, $|f(x) - l| < \epsilon$; and also for all x in D satisfying $c - \delta < x < c$, $|f(x) - l| < \epsilon$.

It follows that $\lim\limits_{x \to c+} f(x) = l$ and $\lim\limits_{x \to c-} f(x) = l$.

Conversely, let $\lim\limits_{x \to c+} f(x) = l$ and $\lim\limits_{x \to c-} f(x) = l$.

Let $\epsilon > 0$. Then there exists a $\delta_1 > 0$ such that
$|f(x) - l| < \epsilon$ for all x in D satisfying $c < x < c + \delta_1$.
Also there exists a $\delta_2 > 0$ such that
$|f(x) - l| < \epsilon$ for all x in D satisfying $c - \delta_2 < x < c$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $|f(x) - l| < \epsilon$ for all x in D satisfying $c - \delta < x < c$ and $c < x < c + \delta$, i.e., for all $x \in N'(c, \delta) \cap D$.
Therefore $\lim\limits_{x \to c} f(x) = l$. This proves the theorem.

This completes the proof.

Worked Examples (continued).

4. Find $\lim\limits_{x \to 0+} \sin x$, $\lim\limits_{x \to 0-} \sin x$, $\lim\limits_{x \to 0} \sin x$.

In $0 < x < \frac{\pi}{2}$, $0 < \sin x < x$.
Let $\phi(x) = 0$ in $0 < x < \frac{\pi}{2}$. Then $\lim\limits_{x \to 0+} \phi(x) = 0$. Also $\lim\limits_{x \to 0+} x = 0$.
By Sandwich theorem, $\lim\limits_{x \to 0+} \sin x = 0$.

In $-\frac{\pi}{2} < x < 0$, $x < \sin x < 0$.
Let $\psi(x) = 0$ in $-\frac{\pi}{2} < x < 0$. Then $\lim\limits_{x \to 0-} \psi(x) = 0$. Also $\lim\limits_{x \to 0-} x = 0$.
By Sandwich theorem, $\lim\limits_{x \to 0-} \sin x = 0$.

Since $\lim\limits_{x \to 0+} \sin x = \lim\limits_{x \to 0-} \sin x = 0$, we have $\lim\limits_{x \to 0} \sin x = 0$.

5. Prove that $\lim\limits_{x \to 0} \cos x = 1$. Deduce that $\lim\limits_{x \to 0+} \cos x = \lim\limits_{x \to 0-} \cos x = 1$.

$|\cos x - 1| = |2 \sin^2 \frac{x}{2}| < 2\frac{x^2}{4}$, since $|\sin x| \le |x|$ for all $x \in \mathbb{R}$.
Let us choose $\epsilon > 0$.
Then $|\cos x - 1| < \epsilon$ for all x satisfying $\frac{x^2}{2} < \epsilon$,
i.e., for all x satisfying $-\sqrt{2\epsilon} < x < \sqrt{2\epsilon}$.
Therefore $\lim\limits_{x \to 0} \cos x = 1$.

Let $f(x) = \cos x$. The domain D of f is $\mathbb{R}$.
Let $D_1 = D \cap (0, \infty)$ and $D_2 = D \cap (-\infty, 0)$.

Since $\lim\limits_{x \to 0} \cos x = 1$ and 0 is a limit point of both D_1 and D_2, each c
$\lim\limits_{x \to 0-} \cos x$ and $\lim\limits_{x \to 0-} \cos x$ exists and equals 1.

6. Prove that $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$.

In $0 < x < \frac{\pi}{2}$, $\quad \sin x < x < \tan x$.
Therefore $1 < \frac{x}{\sin x} < \frac{1}{\cos x}$ in $0 < x < \frac{\pi}{2}$.

As $\lim\limits_{x\to 0+} \frac{1}{\cos x} = 1$, we have $\lim\limits_{x\to 0+} \frac{x}{\sin x} = 1$, by Sandwich theorem.
Therefore $\lim\limits_{x\to 0+} \frac{\sin x}{x} = 1$.

In $-\frac{\pi}{2} < x < 0$, $\tan x < x < \sin x$.
Therefore $1 < \frac{x}{\sin x} < \frac{1}{\cos x}$ in $-\frac{\pi}{2} < x < 0$.
As $\lim\limits_{x\to 0-} \frac{1}{\cos x} = 1$, we have $\lim\limits_{x\to 0-} \frac{x}{\sin x} = 1$, by Sandwich theorem.
Therefore $\lim\limits_{x\to 0-} \frac{\sin x}{x} = 1$.

Since $\lim\limits_{x\to 0+} \frac{\sin x}{x} = \lim\limits_{x\to 0-} \frac{\sin x}{x} = 1$, it follows that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$.

7. Evaluate the one sided limits – (i) $\lim\limits_{x\to 0-} [x]$, (ii) $\lim\limits_{x\to 0+} [x]$.
Show that $\lim\limits_{x\to 0} [x]$ does not exist.

Let $f(x) = [x]$. The domain of f is $\mathbb{R}$.
Since we like to evaluate $\lim\limits_{x\to 0-} f(x)$ and $\lim\limits_{x\to 0+} f(x)$, we are interested in the nature of the function f in the neighbourhood of 0.

$$\begin{aligned} f(x) &= -1, \text{ if } -1 < x < 0 \\ &= 0, \text{ if } 0 \le x < 1. \end{aligned}$$

$\lim\limits_{x\to 0-} f(x) = -1$, $\lim\limits_{x\to 0+} f(x) = 0$.

Since $\lim\limits_{x\to 0-} f(x) \neq \lim\limits_{x\to 0+} f(x)$, $\lim\limits_{x\to 0} f(x)$ does not exist.

8. Evaluate $\lim\limits_{x\to 3}([x] - [\frac{x}{3}])$.

Let $f(x) = [x] - [\frac{x}{3}]$. The domain of f is $\mathbb{R}$.
Since we like to evaluate $\lim\limits_{x\to 3} f(x)$, we are interested in the nature of the function f in the neighbourhood of 3.

$$\begin{aligned} [x] &= 2, \text{ if } 2 < x < 3 \\ &= 3, \text{ if } 3 \le x < 4. \end{aligned} \qquad \begin{aligned} [\tfrac{x}{3}] &= 0, \text{ if } 0 < x < 3 \\ &= 1, \text{ if } 3 \le x < 6. \end{aligned}$$

$$\text{Therefore } \begin{aligned} f(x) &= 2, \text{ if } 2 < x < 3 \\ &= 2, \text{ if } 3 \le x < 4. \end{aligned}$$

$\lim\limits_{x\to 3-} f(x) = 2$ and $\lim\limits_{x\to 3+} f(x) = 2$. Therefore $\lim\limits_{x\to 3} f(x) = 2$.

7.3. Infinite limits.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of D. We have seen that if f is not bounded on $N(c) \cap D$ for some neighbourhood $N(c)$ of c, f does not approach a finite limit l.

Definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of D. If corresponding to a pre-assigned positive number G, there exists a positive δ such that

$f(x) > G$ for all $x \in N'(c,\delta) \cap D$,

then we say that f tends to ∞ as $x \to c$ and we write $\lim_{x \to c} f(x) = \infty$.

Definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of D. If corresponding to a pre-assigned positive number G, there exists a positive δ such that

$f(x) < -G$ for all $x \in N'(c,\delta) \cap D$,

then we say that f tends to $-\infty$ as $x \to c$ and we write $\lim_{x \to c} f(x) = -\infty$.

Note. In both these cases we say that limit of the function f at c *exists in* $\mathbb{R}^*$. When $\lim_{x \to c} f(x) = l, (l \in \mathbb{R})$ we say that limit of the function f at c *exists* and it is expressed by saying that the limit of the function f at c *exists in* $\mathbb{R}$.

As in the case of finite limits, the sequential criteria can be formulated in the case of infinite limits.

Sequential criterion.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of D. Then $\lim_{x \to c} f(x) = \infty$ if and only if for every sequence $\{x_n\}$ in $D - \{c\}$ converging to c, the sequence $\{f(x_n)\}$ diverges to ∞.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of D. Then $\lim_{x \to c} f(x) = -\infty$ if and only if for every sequence $\{x_n\}$ in $D - \{c\}$ converging to c, the sequence $\{f(x_n)\}$ diverges to $-\infty$.

Worked Examples.

1. Show that $\lim_{x \to 0} f(x) = \infty$, where $f(x) = \frac{1}{x^2}$.

In every neighbourhood of 0, f is unbounded above.

Let us choose $G > 0$. Then $f(x) > G$ for all x satisfying $-\frac{1}{\sqrt{G}} < x < \frac{1}{\sqrt{G}}, x \neq 0$.

That is, $f(x) > G$ for all $x \in N'(0,\delta)$ where $\delta = \frac{1}{\sqrt{G}}$.

Therefore $\lim_{x \to 0} f(x) = \infty$.

2. Show that $\lim_{x \to 0} f(x)$ does not exist in $\mathbb{R}^*$, where $f(x) = \frac{1}{x}$.

Here the domain f is $D = \mathbb{R} - \{0\}$. 0 is a limit point of D.

Let us consider the sequence $\{x_n\}$ where $x_n = \frac{1}{n}, n \in \mathbb{N}$.

Then $x_n \in D$ and $\lim x_n = 0, f(x_n) = n$, $\lim f(x_n) = \infty$.

Let us consider the sequence $\{y_n\}$ where $y_n = -\frac{1}{n}, n \in \mathbb{N}$.

Then $y_n \in D$ and $\lim y_n = 0, f(y_n) = -n$, $\lim f(y_n) = -\infty$.

We have two sequences $\{x_n\}$ and $\{y_n\}$ in D both converging to 0 but the sequences $\{f(x_n)\}$ and $\{f(y_n)\}$ approach to two different limits in $\mathbb{R}^*$. Therefore $\lim\limits_{x\to 0} \frac{1}{x}$ does not exist in $\mathbb{R}^*$.

Theorem 7.3.1. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of both the sets $D_1 = D\cap(c,\infty)$ and $D_2 = D\cap(-\infty, c)$. Then $\lim\limits_{x\to c} f(x) = \infty(\infty)$ if and only if $\lim\limits_{x\to c+} f(x) = \lim\limits_{x\to c-} f(x) = \infty(-\infty)$.

Proof left to the reader.

Worked Example (continued).

3. Examine if $\lim\limits_{x\to\frac{\pi}{2}} \tan x$ exists.

Let $f(x) = \tan x$. The domain of f is $D = \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n$ is an integer $\}$. $D_1 = D \cap (\frac{\pi}{2}, \infty) \neq \phi, D_2 = D \cap (-\infty, \frac{\pi}{2}) \neq \phi$. Also $\frac{\pi}{2}$ is a limit point of both D_1 and D_2.

In $\frac{\pi}{2} < x < \pi$, f is a monotone increasing function unbounded below. Therefore $\lim\limits_{x\to\frac{\pi}{2}+} f(x) = -\infty$.

In $0 < x < \frac{\pi}{2}$, f is a monotone increasing function unbounded above. Therefore $\lim\limits_{x\to\frac{\pi}{2}-} f(x) = \infty$. [Theorem 7.6.1]

Since $\lim\limits_{x\to\frac{\pi}{2}+} f(x)$ and $\lim\limits_{x\to\frac{\pi}{2}-} f(x)$ both exist in $\mathbb{R}^*$ and $\lim\limits_{x\to\frac{\pi}{2}+} f(x) \neq \lim\limits_{x\to\frac{\pi}{2}-} f(x)$, $\lim\limits_{x\to\frac{\pi}{2}} f(x)$ does not exist.

7.4. Limits at infinity.

Definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $(c,\infty) \subset D$ for some $c \in \mathbb{R}$. We say that f tends to $l(\in \mathbb{R})$ as $x \to \infty$ if corresponding to a pre-assigned positive ϵ there exists a real number $G > c$ such that $| f(x) - l | < \epsilon$ for all $x > G$.

In this case we write $\lim\limits_{x\to\infty} f(x) = l$.

Definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $(-\infty, c) \subset D$ for some $c \in \mathbb{R}$. We say that f tends to $l(\in \mathbb{R})$ as $x \to -\infty$ if corresponding to a pre-assigned positive ϵ there exists a real number $G < c$ such that $| f(x) - l | < \epsilon$ for all $x < G$.

In this case we write $\lim\limits_{x\to-\infty} f(x) = l$.

Note. In order that we may enquire if $\lim\limits_{x\to\infty} f(x)$ exists, the domain D of f must be such that $(c, \infty) \subset D$ for some $c \in \mathbb{R}$.

In order that we may enquire if $\lim\limits_{x \to -\infty} f(x)$ exists, the domain D of f must be such that $(-\infty, c) \subset D$ for some $c \in \mathbb{R}$.

Sequential criterion.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $(c, \infty) \subset D$ for some $c \in \mathbb{R}$. Then $\lim\limits_{x \to \infty} f(x) = l(\in \mathbb{R})$ if and only if for every sequence $\{x_n\}$ in (c, ∞) diverging to ∞, the sequence $\{f(x_n)\}$ converges to l.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $(-\infty, c) \subset D$ for some $c \in \mathbb{R}$. Then $\lim\limits_{x \to -\infty} f(x) = l(\in \mathbb{R})$ if and only if for every sequence $\{x_n\}$ in $(-\infty, c)$ diverging to $-\infty$, the sequence $\{f(x_n)\}$ converges to l.

Worked Examples.

1. Prove that $\lim\limits_{x \to \infty} \frac{1}{x} = 0$.

Let $f(x) = \frac{1}{x}$. The domain of f is $D = \mathbb{R} - \{0\}.(0, \infty) \subset D$.
Let us choose $\epsilon > 0$.
When $x \neq 0, \mid f(x) - 0 \mid = \frac{1}{x}$
Therefore $\mid f(x) - 0 \mid < \epsilon$ for all $x > G$ where $G = 1/\epsilon > 0$.

Hence $\lim\limits_{x \to \infty} f(x) = 0$.

2. Show that $\lim\limits_{x \to \infty} x \sin x$ does not exist in $\mathbb{R}^*$.

Let $f(x) = x \sin x$.

Let us consider the sequence $\{x_n\}$ where $x_n = \frac{\pi}{2} + 2n\pi, n \in \mathbb{N}$. Then $\lim x_n = \infty$.

$f(x_n) = f(\frac{\pi}{2} + 2n\pi) = \frac{\pi}{2} + 2n\pi$ for all $n \in \mathbb{N}$ and therefore $\lim f(x_n) = \infty$.

Let us consider the sequence $\{y_n\}$ where $y_n = -\frac{\pi}{2} + 2n\pi, n \in \mathbb{N}$. Then $\lim y_n = \infty$.

$f(y_n) = f(-\frac{\pi}{2} + 2n\pi) = \frac{\pi}{2} - 2n\pi$ for all $n \in \mathbb{N}$ and therefore $\lim f(y_n) = -\infty$.

We have two sequences $\{x_n\}$ and $\{y_n\}$ both diverging to ∞ but the sequences $\{f(x_n)\}$ and $\{f(y_n)\}$ approach to two different limits in $\mathbb{R}^*$.

By sequential criterion, $\lim\limits_{x \to \infty} f(x)$ does not exist in $\mathbb{R}^*$.

Theorem 7.4.1. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function.

Let $(c, \infty) \subset D$ for some $c \in \mathbb{R}$. If $f(x) > 0$ for all $x \in (c, \infty)$ and $\lim\limits_{x \to \infty} f(x) = l$, then $l \geq 0$.

Proof. If possible, let $l < 0$.

Let us choose a positive ϵ such that $l + \epsilon < 0$.

Since $\lim_{x\to\infty} f(x) = l$, there exists a real number $G > c$ such that $| f(x) - l |< \epsilon$ for all $x > G$

or, $l - \epsilon < f(x) < l + \epsilon$ for all $x > G$.

$l + \epsilon < 0 \Rightarrow f(x) < 0$ for all $x > G$.

But by hypothesis $f(x) > 0$ for all $x > c$. Thus we arrive at a contradiction. Therefore $l \geq 0$.

Theorem 7.4.2. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$ be functions.

Let $(c, \infty) \subset D$ for some $c \in \mathbb{R}$. Let $f(x) < g(x)$ for all $x \in (c, \infty)$.

If $\lim_{x\to\infty} f(x) = l$ and $\lim_{x\to\infty} g(x) = m$ then $l \leq m$.

Proof. Let $h(x) = g(x) - f(x), x \in (c, \infty)$.

Let us choose $\epsilon > 0$. Since $\lim_{x\to\infty} f(x) = l$, there exists a real number $G_1 > c$ such that $|f(x) - l| < \frac{\epsilon}{2}$ for all $x > G_1$.

Since $\lim_{x\to\infty} g(x) = m$ there exists a real number $G_2 > c$ such that $| g(x) - m |< \frac{\epsilon}{2}$ for all $x > G_2$

Let $G = \max\{G_1, G_2\}$. Then $| f(x) - l |< \frac{\epsilon}{2}$ and $| g(x) - m |< \frac{\epsilon}{2}$ for all $x > G$.

$$\text{We have } | h(x) - (m - l) | \begin{array}{cl} = & | g(x) - m - f(x) + l | \\ \leq & | f(x) - l | + | g(x) - m | \\ < & \frac{\epsilon}{2} + \frac{\epsilon}{2} \text{ for all } x > G. \end{array}$$

Therefore $| h(x) - (m - l) |< \epsilon$ for all $x > G$.

That is, $\lim_{x\to\infty} h(x) = m - l$.

Since $h(x) > 0$ for all $x \in (c, \infty)$ and $\lim_{x\to\infty} h(x) = m - l$, $m - l \geq 0$, by Theorem 7.4.1. That is, $l \leq m$.

Theorem 7.4.3. Let $D \subset \mathbb{R}$ and $(c, \infty) \subset D$. Let f, g, h be functions on D to $\mathbb{R}$ such that $f(x) < g(x) < h(x)$ for all $x \in (c, \infty)$ and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} h(x) = l$ then $\lim_{x\to\infty} g(x) = l$.

Proof left to the reader.

Theorem 7.4.4. Cauchy criterion

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $(c, \infty) \subset D$ for some $c \in \mathbb{R}$. Then $\lim_{x\to\infty} f(x) = l(\in \mathbb{R})$ if and only if for a pre-assigned positive ϵ there exists a positive number $G > c$ such that $| f(x') - f(x'') |< \epsilon$ for every pair of points $x', x'' > G$.

Worked Example.

1. Using Cauchy's principle prove that $\lim\limits_{x\to\infty} \cos x$ does not exist.

Let $f(x) = \cos x, x \in \mathbb{R}$. Here the domain of f is $\mathbb{R}$.

Let us choose $\epsilon = \frac{1}{2}$. In order that $\lim\limits_{x\to\infty} f(x)$ should exist, it is necessary that there exists a positive G such that

$| f(x') - f(x'') | < \frac{1}{2}$ for every pair of points $x', x'' > G$.

For a given positive G we can find a natural number n such that $2n\pi > G$, say $2k\pi > G$.

Let $x' = (2k+1)\pi$ and $x'' = 2k\pi$. Then $x', x'' > G$ and $f(x') = -1, f(x'') = 1$. Therefore $|f(x') - f(x'')| \not< \epsilon$ for some pair of points $x', x'' > G$ for every $G > 0$.

This shows that Cauchy's condition for the existence of $\lim\limits_{x\to\infty} \cos x$ is not satisfied. Therefore $\lim\limits_{x\to\infty} \cos x$ does not exist.

7.5. Infinite limits at infinity.

Definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $(c, \infty) \subset D$ for some $c \in \mathbb{R}$. If corresponding to a pre-assigned positive number G there exists a real number $k > c$ such that

$f(x) > G$ (or $< -G$) for all $x > k$

then we say that f tends to ∞ (or, $-\infty$) as $x \to \infty$.

In this case we write $\lim\limits_{x\to\infty} f(x) = \infty$ (or, $-\infty$).

Definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $(-\infty, c) \subset D$ for some $c \in \mathbb{R}$. If corresponding to a pre-assigned positive number G there exists a real number $k < c$ such that

$f(x) > G$ (or $< -G$) for all $x < k$

then we say that f tends to ∞ (or, $-\infty$) as $x \to -\infty$.

In this case we write $\lim\limits_{x\to-\infty} f(x) = \infty$ (or, $-\infty$).

Sequential criterion.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $(c, \infty) \subset D$ for some $c \in \mathbb{R}$. Then $\lim\limits_{x\to\infty} f(x) = \infty$ (or, $-\infty$) if and only if for every sequence $\{x_n\}$ in (c, ∞) diverging to ∞, the sequence $\{f(x_n)\}$ diverges to ∞ (or, $-\infty$).

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $(-\infty, c) \subset D$ for some $c \in \mathbb{R}$. Then $\lim\limits_{x\to-\infty} f(x) = \infty$ (or, $-\infty$) if and only if for every sequence

$\{x_n\}$ in $(-\infty, c)$ diverging to $-\infty$, the sequence $\{f(x_n)\}$ diverges to ∞ (or, $-\infty$).

Theorem 7.5.1. Let $g : D \to \mathbb{R}$ be a function on D and $(c, \infty) \subset D$ for some $c \in \mathbb{R}$. Then $\lim_{x\to\infty} g(x) = l$ $(l \in \mathbb{R})$ if and only if $\lim_{x\to 0+} g(\frac{1}{x}) = l$.

Proof. Let $f(x) = \frac{1}{x}$. Then $\lim_{x\to 0+} f(x) = \infty$.

Let $\lim_{x\to\infty} g(x) = l$. Then for a pre-assigned positive ϵ there exists a positive number $d > c$ such that $| g(x) - l |< \epsilon$ for all $x > d$.

Since $\lim_{x\to 0+} f(x) = \infty$, for the chosen positive number d, there exists a positive number δ such that $f(x) > d$ for all $x \in (0, \delta)$.

Therefore $| gf(x) - l |< \epsilon$ for all $x \in (0, \delta)$.

This implies that $\lim_{x\to 0+} gf(x) = l$, i.e., $\lim_{x\to 0+} g(\frac{1}{x}) = l$.

Conversely, let $\lim_{x\to 0+} gf(x) = l$.

Then for a pre-assigned positive ϵ there exists a positive δ such that $l - \epsilon < gf(x) < l + \epsilon$ for all $x \in (0, \delta) \cap D$, D being the domain of gf.
$x \in (0, \delta) \Rightarrow f(x) > \frac{1}{\delta}$. Hence $l - \epsilon < g(x) < l + \epsilon$ for all $x > \frac{1}{\delta}$.
This implies that $\lim_{x\to\infty} g(x) = l$.

This completes the proof.

Note 1. The thoerem can be generalised to include the cases $l = \infty$ and $l = -\infty$.

Note 2. The theorem in the generalised form says that in order to evaluate $\lim_{x\to\infty} f(x)$ it is sufficient to evaluate $\lim_{y\to 0+} f(\frac{1}{y})$.

Theorem 7.5.2. Let $g : D \to \mathbb{R}$ be a function on D and $(-\infty, c) \subset D$ for some $c \in \mathbb{R}$.

Then $\lim_{x\to -\infty} g(x) = l$ if and only if $\lim_{x\to 0-} g(\frac{1}{x}) = l$.

Proof left to the reader.

Worked Examples.

1. Find $\lim_{x\to\infty} \frac{\sqrt{x}-x}{\sqrt{x}+x}$.

$$\lim_{x\to\infty} \frac{\sqrt{x}-x}{\sqrt{x}+x} = \lim_{y\to 0+} \frac{\frac{1}{\sqrt{y}} - \frac{1}{y}}{\frac{1}{\sqrt{y}} + \frac{1}{y}} = \lim_{y\to 0+} \frac{\sqrt{y}-1}{\sqrt{y}+1} = -1.$$

2. Find $\lim_{x\to\infty} \frac{\sin x}{x}$.

$$\lim_{x\to\infty} \frac{\sin x}{x} = \lim_{y\to 0+} \frac{\sin \frac{1}{y}}{\frac{1}{y}} = \lim_{y\to 0+} y \sin \frac{1}{y} = 0.$$

7.6. Limits of monotone functions.

Theorem 7.6.1. Let $I = (a, b)$ be a bounded open interval and $f : I \to \mathbb{R}$ be a monotone increasing function on I.

(i) If f is bounded above on I then $\lim\limits_{x \to b-} f(x) = \sup\limits_{x \in (a,b)} f(x)$

(ii) If f is bounded below on I then $\lim\limits_{x \to a+} f(x) = \inf\limits_{x \in (a,b)} f(x)$

(iii) If f is unbounded above on I then $\lim\limits_{x \to b-} f(x) = \infty$

(iv) If f is unbounded below on I then $\lim\limits_{x \to a+} f(x) = -\infty$.

Proof. (i) Let $M = \sup\limits_{x \in (a,b)} f(x)$.

Let $\epsilon > 0$. Then $f(x) \leq M$ for all $x \in (a, b)$ and there exists a point $x_o \in (a, b)$ such that $M - \epsilon < f(x_o) \leq M$.

Let $\delta = b - x_o$. Then $f(b - \delta) = f(x_o) > M - \epsilon$.
Since f is monotone increasing, $f(x) > M - \epsilon$ for all $x \in (b - \delta, b)$.
Hence $M - \epsilon < f(x) \leq M < M + \epsilon$ for all $x \in (b - \delta, b)$
or, $| f(x) - M | < \epsilon$ for all $x \in (b - \delta, b)$.

This implies $\lim\limits_{x \to b-} f(x) = M$. That is, $\lim\limits_{x \to b-} f(x) = \sup\limits_{x \in (a,b)} f(x)$.

(ii) Let $m = \inf\limits_{x \in (a,b)} f(x)$.

Let $\epsilon > 0$. Then $f(x) \geq m$ for all $x \in (a, b)$ and there exists a point $x_o \in (a, b)$ such that $m \leq f(x_o) < m + \epsilon$.

Let $\delta = x_o - a$. Then $f(a + \delta) = f(x_o) < m + \epsilon$.
Since f is monotone increasing, $f(x) < m + \epsilon$ for all $x \in (a, a + \delta)$.
Hence $m - \epsilon < m \leq f(x) < m + \epsilon$ for all $x \in (a, a + \delta)$
or, $| f(x) - m | < \epsilon$ for all $x \in (a, a + \delta)$.

This implies $\lim\limits_{x \to a+} f(x) = m$. That is, $\lim\limits_{x \to a+} f(x) = \inf\limits_{x \in (a,b)} f(x)$.

(iii) Let $G > 0$. Since f is unbounded above on (a, b), there exists a point x_o in (a, b) such that $f(x_o) > G$.

Let $\delta = b - x_o$. Then $f(b - \delta) = f(x_o) > G$.
Since f is monotone increasing, $f(x) > G$ for all $x \in (b - \delta, b)$.

This implies $\lim\limits_{x \to b-} f(x) = \infty$.

(iv) Proof similar to (iii).

Theorem 7.6.2. Let $I = (a, b)$ be a bounded open interval and $f : I \to \mathbb{R}$ be a monotone decreasing function on I.

(i) If f is bounded above on I then $\lim\limits_{x \to a+} f(x) = \sup\limits_{x \in (a,b)} f(x)$

(ii) If f is bounded below on I then $\lim\limits_{x \to b-} f(x) = \inf\limits_{x \in (a,b)} f(x)$

(iii) If f is unbounded above on I then $\lim\limits_{x \to a+} f(x) = \infty$

(iv) If f is unbounded below on I then $\lim\limits_{x \to b-} f(x) = -\infty$.

Proof left to the reader.

Theorem 7.6.3. Let $I = (a, b)$ be a bounded open interval and $c \in (a, b)$. If $f : I \to \mathbb{R}$ be a monotone function on I then $\lim\limits_{x \to c-} f(x)$ and $\lim\limits_{x \to c+} f(x)$ both exist.

Proof. **Case 1.** Let f be monotone increasing on I. Then f is monotone increasing on (a, c) and f is bounded above on (a, c), $f(c)$ being an upper bound. Let M be the supremum of f on (a, c). Then $M \le f(c)$.

By Theorem 7.6.1, $\lim\limits_{x \to c-} f(x) = M \le f(c)$.

Also f is monotone increasing on (c, b) and f is bounded below on (c, b), $f(c)$ being a lower bound. Let m be the infimum of f on (c, b). Then $f(c) \le m$.

By Theorem 7.6.1, $\lim\limits_{x \to c+} f(x) = m \ge f(c)$.

Therefore $\lim\limits_{x \to c-} f(x)$ and $\lim\limits_{x \to c+} f(x)$ both exist and $\lim\limits_{x \to c-} f(x) \le \lim\limits_{x \to c+} f(x)$.

Case 2. Let f be monotone decreasing on I. Then f is monotone decreasing on (a, c) and f is bounded below on (a, c), $f(c)$ being a lower bound. Let m be the infimum of f on (a, c). Then $f(c) \le m$.

By Theorem 7.6.2, $\lim\limits_{x \to c-} f(x) = m \ge f(c)$.

Also f is monotone decreasing on (c, b) and f is bounded above on (c, b), $f(c)$ being an upper bound. Let M be the supremum of f on (c, b). Then $M \le f(c)$.

By Theorem 7.6.2, $\lim\limits_{x \to c+} f(x) = M \le f(c)$.

Therefore $\lim\limits_{x \to c-} f(x)$ and $\lim\limits_{x \to c+} f(x)$ both exist and $\lim\limits_{x \to c-} f(x) \le \lim\limits_{x \to c+} f(x)$.

Theorem 7.6.4. Let $a \in \mathbb{R}$ and $I = (a, \infty)$.

Let $f : I \to \mathbb{R}$ be a monotone increasing function on I.

(i) If f is bounded above on I then $\lim\limits_{x \to \infty} f(x) = \sup\limits_{x \in (a,\infty)} f(x)$.

(ii) If f is unbounded above on I then $\lim\limits_{x \to \infty} f(x) = \infty$.

(iii) If f is bounded below on I then $\lim\limits_{x \to a+} f(x) = \inf\limits_{x \in (a,\infty)} f(x)$.

(iv) If f is unbounded below on I then $\lim\limits_{x \to a+} f(x) = -\infty$.

Proof left to the reader.

Theorem 7.6.5. Let $a \in \mathbb{R}$ and $I = (-\infty, a)$.

Let $f : I \to \mathbb{R}$ be a monotone increasing function on I.

(i) If f is bounded above on I then $\lim\limits_{x \to a-} f(x) = \sup\limits_{x \in (-\infty,a)} f(x)$.

(ii) If f is unbounded above on I then $\lim\limits_{x \to a-} f(x) = \infty$.

(iii) If f is bounded below on I then $\lim\limits_{x \to -\infty} f(x) = \inf\limits_{x \in (-\infty,a)} f(x)$.

(iv) If f is unbounded below on I then $\lim\limits_{x \to -\infty} f(x) = -\infty$.

Proof left to the reader.

Similar theorems can be formulated for a monotone decreasing function f.

Examples.

1. Let $f(x) = \tan x, x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

f is a monotone increasing function on $(-\frac{\pi}{2}, \frac{\pi}{2})$. f is unbounded above on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore $\lim\limits_{x \to \frac{\pi}{2}-} \tan x = \infty$.

f is unbounded below on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore $\lim\limits_{x \to -\frac{\pi}{2}+} \tan x = -\infty$.

2. Let $f(x) = \log x, x > 0$.

f is a monotone increasing function on $(0, \infty)$. f is unbounded above on $(0, \infty)$. Therefore $\lim\limits_{x \to \infty} f(x) = \infty$.

f is unbounded below on $(0, \infty)$. Therefore $\lim\limits_{x \to 0+} \log x = -\infty$.

3. Let $f(x) = \frac{1}{x}, x > 0$.

f is a monotone decreasing function on $(0, \infty)$. f is bounded below on $(0, \infty)$. Therefore $\lim\limits_{x \to \infty} f(x) = \inf\limits_{x \in (0,\infty)} f(x) = 0$.

f is unbounded above on $(0,\infty)$. Therefore $\lim\limits_{x\to 0+} f(x) = \infty$.

4. Let $f(x) = \frac{1}{x}, x < 0$.

f is a monotone decreasing function on $(-\infty, 0)$. f is bounded above on $(-\infty, 0)$. $\sup\limits_{x\in(-\infty,0)} f(x) = 0$.

Therefore $\lim\limits_{x\to-\infty} f(x) = \sup\limits_{x\in(-\infty,0)} f(x) = 0$.

f is unbounded below on $(-\infty, 0)$. Therefore $\lim\limits_{x\to 0-} f(x) = -\infty$.

5. Let $f(x) = \frac{1}{x^2}, x \in \mathbb{R} - \{0\}$.

f is a monotone decreasing function on $(0,\infty)$. f is bounded below and unbounded above on $(0,\infty)$.

$\sup\limits_{x\in(0,\infty)} f(x) = \infty$, $\inf\limits_{x\in(0,\infty)} f(x) = 0$.

$\lim\limits_{x\to 0+} f(x) = \infty$, $\lim\limits_{x\to\infty} f(x) = 0$.

f is a monotone increasing function on $(-\infty, 0)$.f is unbounded above and bounded below on $(-\infty, 0)$.

$\sup\limits_{x\in(-\infty,0)} f(x) = \infty$, $\inf\limits_{x\in(-\infty,0)} f(x) = 0$.

$\lim\limits_{x\to 0-} f(x) = \infty$, $\lim\limits_{x\to-\infty} f(x) = 0$.

7.7. Some important limits.

1. Prove that $\lim\limits_{x\to\infty}(1+\frac{1}{x})^x = e$.

We have $\lim\limits_{n\to\infty}(1+\frac{1}{n})^n = e$.

Since $x \to \infty$, we assume $x > 1$.

Let $[x] = k$. Then $k \le x < k+1$ and $1 + \frac{1}{k+1} < 1 + \frac{1}{x} \le 1 + \frac{1}{k}$.

It follows that $(1+\frac{1}{k+1})^k < (1+\frac{1}{x})^x < (1+\frac{1}{k})^{k+1}$.

Taking limit as $x\to\infty$ and noting that as $x\to\infty, k\to\infty$

we have $\lim\limits_{x\to\infty}(1+\frac{1}{x})^x \ge \lim\limits_{k\to\infty}(1+\frac{1}{k+1})^k$

$$= \lim_{k\to\infty} \frac{1}{(1+\frac{1}{k+1})^{k+1}}(1+\frac{1}{k+1}) = e.$$

Also $\lim\limits_{x\to\infty}(1+\frac{1}{x})^x \le \lim\limits_{k\to\infty}(1+\frac{1}{k})^{k+1}$

$$= \lim_{k\to\infty}(1+\tfrac{1}{k})^k.(1+\tfrac{1}{k}) = e.$$

It follows that $\lim\limits_{x\to\infty}(1+\frac{1}{x})^x = e$.

2. Prove that $\lim_{x\to -\infty}(1+\frac{1}{x})^x = e$.

Let $x = -y$. As $x \to \infty, y \to -\infty$.

Then $\lim_{x\to-\infty}(1+\frac{1}{x})^x = \lim_{y\to\infty}(1-\frac{1}{y})^{-y}$

$= \lim_{y\to\infty}(\frac{y}{y-1})^y$

$= \lim_{t\to\infty}(1+\frac{1}{t})^{t+1}$ where $t = y-1$

$= \lim_{t\to\infty}[(1+\frac{1}{t})^t \cdot (1+\frac{1}{t})]$

$= e.$

3. Prove that $\lim_{x\to 0}(1+x)^{1/x} = e$.

We have $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$.

Let $y = \frac{1}{x}$. As $x \to \infty, y \to 0+$.

Then $e = \lim_{x\to\infty}(1+\frac{1}{x})^x = \lim_{y\to 0+}(1+y)^{1/y}$ ... ... (i)

Also we have $e = \lim_{x\to-\infty}(1+\frac{1}{x})^x$.

Let $y = \frac{1}{x}$. As $x \to -\infty, y \to 0-$.

Then $e = \lim_{x\to-\infty}(1+\frac{1}{x})^x = \lim_{y\to 0-}(1+y)^{1/y}$ ... ... (ii)

From (i) and (ii), $\lim_{y\to 0}(1+y)^{1/y} = e$.

That is, $\lim_{x\to 0}(1+x)^{1/x} = e$.

Exercises 11

1. Use sequential criterion for limits to show that the following limits do not exist.

(i) $\lim_{x\to 0}\cos\frac{1}{x^2}$, (ii) $\lim_{x\to 0}\frac{1}{x}\sin\frac{1}{x}$, (iii) $\lim_{x\to\infty} x^{1+\sin x}$ (iv) $\lim_{x\to\infty} x^2 \text{sgn} \cos x$.

Hint. (iii) Take $x_n = \frac{3\pi}{2} + 2n\pi; y_n = \frac{\pi}{2} + 2n\pi$.

(iv) Take $x_n = 2n\pi; y_n = \pi + 2n\pi$.

2. Let $f(x) = x,\ x \in \mathbb{Q}$

$= 2 - x,\ x \in \mathbb{R} - \mathbb{Q}$.

Show that (i) $\lim_{x\to 1} f(x) = 1$; (ii) $\lim_{x\to c} f(x)$ does not exist, if $c \neq 1$.

3. Show that the following limits do not exist.

(i) $\lim_{x\to 0}\frac{|\sin x|}{x}$, (ii) $\lim_{x\to 0}\frac{1}{e^{1/x}+1}$, (iii) $\lim_{x\to 0}\frac{2x+|x|}{2x-|x|}$.

4. Evaluate the limits

(i) $\lim\limits_{x\to 0+} \sqrt{x-[x]}$, $\lim\limits_{x\to 0-} \sqrt{x-[x]}$;

(ii) $\lim\limits_{x\to 0+} \dfrac{1}{e^{\frac{1}{x}}+1}$, $\lim\limits_{x\to 0-} \dfrac{1}{e^{\frac{1}{x}}+1}$;

(iii) $\lim\limits_{x\to 0+} \{[x]+[1-x]\}$, $\lim\limits_{x\to 0-} \{[x]+[1-x]\}$.

(iv) $\lim\limits_{x\to 0+} x[\frac{1}{x}]$, $\lim\limits_{x\to 0-} x[\frac{1}{x}]$.

(v) $\lim\limits_{x\to 0+} [\frac{\sin x}{x}]$, $\lim\limits_{x\to 0-} [\frac{\sin x}{x}]$.

(vi) $\lim\limits_{x\to 0+} \{\frac{\sin x}{x}\}$, $\lim\limits_{x\to 0-} \{\frac{\sin x}{x}\}$, where $\{x\} = x-[x]$ = fractional part of x for all $x \in \mathbb{R}$.

Hint. (iv) For all non-zero real x, $[\frac{1}{x}] = \frac{1}{x} - \theta(x)$, where $0 \leq \theta(x) < 1$.

(v) For all $x \in (0, \frac{\pi}{2})$, $0 < \sin x < x \Rightarrow 0 < \frac{\sin x}{x} < 1 \Rightarrow [\frac{\sin x}{x}] = 0$.

(vi) For all $x \in \mathbb{R}$, $[x] + \{x\} = x$. For all $x \in (0, \frac{\pi}{2})$, $[\frac{\sin x}{x}] = 0$ and this implies $\{\frac{\sin x}{x}\} = \frac{\sin x}{x}$.

5. Evaluate the limits

(i) $\lim\limits_{x\to\infty} \frac{x^2+3x}{x^2+x+1}$, (ii) $\lim\limits_{x\to\infty} \frac{\sin x}{x+\cos x}$, (iii) $\lim\limits_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x}+3}$,

(iv) $\lim\limits_{x\to\infty} (1+\frac{2}{x})^x$, (v) $\lim\limits_{x\to\infty} (x-\sqrt{x^2-1})$, (vi) $\lim\limits_{x\to\infty} (\sqrt[3]{x+1}-\sqrt[3]{x})$.

8. CONTINUITY

8.1. Continuity.

Definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $c \in D$.f is said to be *continuous* at c if given any neighbourhood V of $f(c)$ there exists a neighbourhood W of c such that for all $x \in W \cap D, f(x) \in V$.

Equivalent definitions.

1. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $c \in D$. f is said to be *continuous* at c if for a pre-assigned positive ϵ there exists a positive δ such that

$| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$.

2. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $c \in D$. f is said to be *continuous* at c if for a pre-assigned positive ϵ there exists a positive δ such that

$| f(c+h) - f(c) | < \epsilon$ for all h satisfying $| h | < \delta$ and $c + h \in D$.

Note. In order that we may enquire if a function f is continuous at a point c, c must belong to the domain of f.

Theorem 8.1.1. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. If c be an isolated point of D then f is continuous at c.

Proof. Since c is an isolated point of D, there exists a neighbourhood $N(c, \delta)$ of c such that $N(c, \delta) \cap D = \{c\}$.

Let $\epsilon > 0$. Then $| f(x) - f(c) | < \epsilon$ holds for $x = c$.
Therefore there exists a positive δ such that
$| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$.

This shows that f is continuous at c.

Theorem 8.1.2. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. If $c \in D \cap D'$ then f is continuous at c if and only if $\lim\limits_{x \to c} f(x) = f(c)$.

Proof. Let f be continuous at c. Then for a pre-assigned positive ϵ there exists a positive δ such that

$| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$.

We also have $| f(x) - f(c) | < \epsilon$ for all $x \in N'(c, \delta) \cap D$.

This implies that $\lim_{x \to c} f(x) = f(c)$.

Conversely, let $\lim_{x \to c} f(x) = f(c)$. Then for a pre-assigned positive ϵ there exists a positive δ such that

$| f(x) - f(c) | < \epsilon$ for all $x \in N'(c, \delta) \cap D$.

Also for $x = c$, $| f(x) - f(c) | = | f(c) - f(c) | = 0 < \epsilon$.

Combining, we have $| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$.

This shows that f is continuous at c.

Continuity on a set.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $A \subset D$. f is said to be *continuous on A* if f be continuous at every point of A.

Continuity on an interval.

Let I be an interval and $f : I \to \mathbb{R}$ be a function.

(i) Let c be an interior point of I.

f is said to be continuous at c if for a pre-assigned positive ϵ there exists a positive δ such that

$| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta) \cap I$.

Equivalently, f is said to be continuous at c if for a pre-assigned positive ϵ there exists a positive δ such that

$| f(c + h) - f(c) | < \epsilon$ for all h satisfying $| h | < \delta$ and $c + h \in I$.

(ii) Let a be the left end point of I and $a \in I$.

f is said to be continuous at a if for a pre-assigned positive ϵ there exists a positive δ such that

$| f(x) - f(a) | < \epsilon$ for all $x \in N(a, \delta) \cap I$

i.e., $| f(x) - f(a) | < \epsilon$ for all $x \in [a, a + \delta) \cap I$.

Equivalently, f is said to be continuous at a if for a pre-assigned positive ϵ there exists a positive δ such that

$| f(a + h) - f(a) | < \epsilon$ for all h satisfying $| h | < \delta$ and $a + h \in I$.

(iii) Let b be the right end point of I and $b \in I$.

f is said to be continuous at b if for a pre-assigned positive ϵ there exists a positive δ such that

$| f(x) - f(b) | < \epsilon$ for all $x \in N(b, \delta) \cap I$

i.e., $| f(x) - f(b) | < \epsilon$ for all $x \in (b - \delta, b] \cap I$.

Equivalently, f is said to be continuous at b if for a pre-assigned positive ϵ there exists a positive δ such that

$| f(b + h) - f(b) | < \epsilon$ for all h satisfying $| h | < \delta$ and $b + h \in I$.

Theorem 8.1.3. (Sequential criterion for continuity)

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $c \in D \cap D'$. f is continuous at c if and only if for every sequence $\{x_n\}$ in D converging to c, the sequence $\{f(x_n)\}$ converges to $f(c)$.

Proof. Let f be continuous at c. Let $\{x_n\}$ be a sequence in D such that $\lim x_n = c$.

Since f is continuous at c, for a pre-assigned positive ϵ, there exists a positive δ such that $| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$.

Since $\lim x_n = c$, there exists a natural number m such that $| x_n - c | < \delta$ for all $n \geq m$.

Therefore for all $n \geq m$, $x_n \in N(c, \delta)$ and this implies $x_n \in N(c, \delta) \cap D$ for all $n \geq m$, since $x_n \in D$ for all $n \in \mathbb{N}$.

We have $| f(x_n) - f(c) | < \epsilon$ for all $n \geq m$.

This shows that $\lim f(x_n) = f(c)$.

Conversely, let $\lim f(x_n) = f(c)$ for every sequence $\{x_n\}$ in D converging to c. We prove that f is continuous at c.

Let f be not continuous at c. Then there exists a neighbourhood $N(f(c), \epsilon_o)$ of $f(c)$ such that no matter what neighbourhood $N(c, \delta)$ of c we consider, there will exist at least one point $p(\delta) \in N(c, \delta) \cap D$ such that $f(p) \notin N(f(c), \epsilon_o)$.

Let $\delta = 1$. Then there is a point, say x_1 in $N(c, 1) \cap D$ such that $f(x_1) \notin N(f(c), \epsilon_o)$.

Let $\delta = \frac{1}{2}$. Then there is a point x_2 in $N(c, \frac{1}{2}) \cap D$ such that $f(x_2) \notin N(f(c), \epsilon_o)$.

... ...

Thus we obtain a sequence of points $\{x_n\}$ in D such that $|x_n - c| < \frac{1}{n}$ for all $n \in \mathbb{N}$ and therefore $\lim x_n = c$. But the sequence $\{f(x_n)\}$ does not converge to $f(c)$ since $f(x_n) \notin N(f(c), \epsilon_o)$ for all $n \in \mathbb{N}$.

This is a contradiction to the hypothesis that for every sequence $\{x_n\}$ in D converging to c, the sequence $\{f(x_n)\}$ converges to $f(c)$.

Therefore our assumption is not tenable and f is continuous at c.

Worked Examples.

1. Let $k \in \mathbb{R}$. Prove that the function f defined by $f(x) = k, x \in \mathbb{R}$ is continuous on $\mathbb{R}$.

Let $c \in \mathbb{R}$. $| f(x) - f(c) | = | k - k | = 0$, for all $x \in \mathbb{R}$.

Let us choose $\epsilon > 0$.

Then $| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta)$ for every positive δ.

This implies f is continuous at c. Since c is arbitrary, f is continuous on $\mathbb{R}$.

2. Prove that the function f defined by $f(x) = x, x \in \mathbb{R}$ is continuous on $\mathbb{R}$.

Let $c \in \mathbb{R}$. $\mid f(x) - f(c) \mid = \mid x - c \mid$.
Let us choose $\epsilon > 0$.
Then $\mid f(x) - f(c) \mid < \epsilon$ whenever $\mid x - c \mid < \epsilon$.
That is, $\mid f(x) - f(c) \mid < \epsilon$ for all $x \in N(c, \delta)$ where $\delta = \epsilon$.
So f is continuous at c. Since c is arbitrary, f is continuous on $\mathbb{R}$.

3. A function f is defined on $\mathbb{R}$ by $\begin{aligned} f(x) &= \cos\tfrac{1}{x}, x \neq 0 \\ &= 0, \ \ x = 0. \end{aligned}$

Prove that f is not continuous at 0.

Let us consider a sequence $\{x_n\}$ where $x_n = \frac{1}{2\pi n}, n \in \mathbb{N}$. Then $\lim x_n = 0$; $f(x_n) = 1$ for all $n \in \mathbb{N}$. Therefore $\lim f(x_n) = 1$.

We have a sequence $\{x_n\}$ in $\mathbb{R}$ that converges to 0 but $\lim f(x_n) \neq f(0)$, proving that f is not continuous at 0.

4. A function f is defined on $\mathbb{R}$ by $\begin{aligned} f(x) &= 1, x \in \mathbb{Q} \\ &= 0, x \in \mathbb{R} - \mathbb{Q}. \end{aligned}$

Prove that f is continuous at no point $c \in \mathbb{R}$.

Case 1. Let c be a rational point. Let $\{x_n\}$ be a sequence of irrational points such that $\lim x_n = c$.

Then $f(x_n) = 0$ for all $n \in \mathbb{N}$. Therefore $\lim f(x_n) = 0$.

But $f(c) = 1$. Thus there exists a sequence $\{x_n\} \in \mathbb{R}$ that converges to c but the sequence $\{f(x_n)\}$ does not converge to $f(c)$. By the sequential criterion for continuity, f is not continuous at c.

Case 2. Let c be an irrational point. Let $\{y_n\}$ be a sequence of rational points such that $\lim y_n = c$.

Then $f(y_n) = 1$ for all $n \in \mathbb{N}$. Therefore $\lim f(y_n) = 1$.

But $f(c) = 0$. Thus there exists a sequence $\{y_n\} \in \mathbb{R}$ that converges to c but the sequence $\{f(y_n)\}$ does not converge to $f(c)$. By the sequential criterion for continuity, f is not continuous at c.

Since every real number is either a rational number or an irrational number, it follows that f is not continuous at any point $c \in \mathbb{R}$.

Note. This function f is called *Dirichlet's function*. Dirichlet's function is everywhere discontinuous on $\mathbb{R}$.

5. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If $f(1) = k$ prove that $f(x) = kx$ for all $x \in \mathbb{R}$.

Taking $x = y = 0$, we have $f(0) = 2f(0)$.

This implies $f(0) = 0$ ... ... (i)

Taking $y = -x$, we have $f(x) + f(-x) = 0$.
This implies $f(-x) = -f(x)$ ... ... (ii)

Let x be a positive integer, say n.

$$\begin{aligned} \text{Then } f(x) &= f(1+1+\cdots+1) \\ &= f(1)+f(1)+\cdots+f(1) \ (n \text{ times }) \\ &= nf(1) = kn = kx. \end{aligned}$$

So $f(x) = kx$ if x be a positive integer ... ...(iii)

Let x be a negative integer, say $-n$.

$$\begin{aligned} f(x) = f(-n) &= -f(n) \text{ by (ii)} \\ &= -kn = kx. \end{aligned}$$

So $f(x) = kx$ if x be a negative integer ... ...(iv)
From (i), (iii) and (iv) it follows that $f(x) = kx$ if x be an integer ...(v)

Let x be a rational number, say $\frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.
$f(qx) = f(p) = kp$ by (v).

$$\begin{aligned} \text{Also } f(qx) &= f(x+x+\cdots+x) \\ &= f(x)+f(x)+\cdots+f(x)[q \text{ times }] \\ &= qf(x). \end{aligned}$$

Therefore $qf(x) = kp$
or, $f(x) = \frac{kp}{q} = kx$.
So $f(x) = kx$ if x be a rational number ... ... (vi)

Let x be an irrational number α.
Let us consider a sequence of rational points $\{x_n\}$ converging to α.
Since f is continuous at $\alpha, \lim f(x_n) = f(\alpha)$.
But $\lim f(x_n) = \lim kx_n$, since x_n is rational.
As $\lim kx_n = k\alpha$, it follows that $f(\alpha) = k\alpha$.
So $f(x) = kx$ if x be an irrational number ... ...(vii)

From (v), (vi) and (vii) it follows that $f(x) = kx$ for all $x \in \mathbb{R}$.

6. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Prove that either $f(x) = 0$ for all $x \in \mathbb{R}$, or $f(x) = a^x$ for all $x \in \mathbb{R}$, where a is some positive real number.

Taking $x = y = 0$, we have $f(0) = f(0)f(0)$.
This implies either $f(0) = 0$, or $f(0) = 1$.

Case 1. $f(0) = 0$.
For any real $c, f(c) = f(c+0) = f(c)f(0) = 0$.
Thus in this case $f(x) = 0$ for all $x \in \mathbb{R}$.

Case 2. $f(0) = 1$.

For any real $c, f(c) = f(\frac{c}{2} + \frac{c}{2}) = [f(\frac{c}{2})]^2 \geq 0 \ldots \quad \ldots$(i)

Also $1 = f(0) = f(c - c) = f(c)f(-c) \ldots \quad \ldots$(ii)

From (i) and (ii) it follows that $f(c) > 0$ for all real c in this case.

Let x be a positive integer, say n.

$$\begin{aligned} \text{Then } f(x) &= f(1 + 1 + \cdots + 1) \\ &= f(1)f(1)\cdots f(1) \ (n \text{ factors }) \\ &= [f(1)]^n = a^n \text{ where } a = f(1) > 0. \end{aligned}$$

So $f(x) = a^x$ if x be a positive integer ... ...(iii)

Let x be a negative integer, say $-n$.

$f(x) = f(-n) = \frac{1}{f(n)}$ by (ii) $= \frac{1}{a^n} = a^{-n} = a^x$.

So $f(x) = a^x$ if x be a negative integer ... ...(iv)

From (iii) and (iv) it follows that $f(x) = a^x$ if x be an integer ... ...(v)

Let x be a rational number, say $\frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

$f(p) = f(\frac{p}{q} + \frac{p}{q} + \cdots + \frac{p}{q})$ (q times) $= [f(\frac{p}{q})]^q$.

or, $a^p = [f(x)]^q$

or, $f(x) = a^{\frac{p}{q}} = a^x$.

So $f(x) = a^x$ if x be a rational number ... ...(vi)

Let x be an irrational number α.

Let us consider a sequence of rational points $\{x_n\}$ converging to α. Since f is continuous at $\alpha, \lim f(x_n) = f(\alpha)$.

But $\lim f(x_n) = \lim a^{x_n}$, since x_n is rational.

As $\lim a^{x_n} = a^\alpha$ by Corollary 1 of Example 3, 5.8, it follows that $f(\alpha) = a^\alpha$. So $f(x) = a^x$ if x be an irrational number ... ...(vii)

From (v), (vi) and (vii) it follows that $f(x) = a^x$ for all $x \in \mathbb{R}$.

Theorem 8.1.4. Let $D \subset \mathbb{R}$ and f and g are functions on D to $\mathbb{R}$. Let $c \in D$ and f and g are continuous at c. Then

(i) $f + g$ is continuous at c

(ii) if $k \in \mathbb{R}, kf$ is continuous of c

(iii) fg is continuous at c

(iv) if $g(x) \neq 0$ for all $x \in D, f/g$ is continuous at c.

Proof. (i) $|(f + g)(x) - (f + g)(c)|$

$= |f(x) + g(x) - \overline{f(c) + g(c)}|$

$\leq |f(x) - f(c)| + |g(x) - g(c)|$.

Let us choose $\epsilon > 0$.
Since f is continuous at c, there exists a positive δ_1 such that $|f(x) - f(c)| < \frac{\epsilon}{2}$ for all $x \in N(c, \delta_1) \cap D$.

Since g is continuous at c, there exists a positive δ_2 such that $|g(x) - g(c)| < \frac{\epsilon}{2}$ for all $x \in N(c, \delta_2) \cap D$.

Let $\delta = \min\{\delta_1, \delta_2\}$.
Then $| f(x) - f(c) | < \frac{\epsilon}{2}$ and $| g(x) - g(c) | < \frac{\epsilon}{2}$ for all $x \in N(c, \delta) \cap D$.

Therefore $| (f + g)(x) - (f + g)(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$.

This shows that $f + g$ is continuous at c.

(ii) Proof left to the reader.

(iii) $| fg(x) - fg(c) | = | f(x).g(x) - f(c).g(c) |$
$= | f(x)\{g(x) - g(c)\} + g(c)\{f(x) - f(c)\} |$
$\leq | f(x) || g(x) - g(c) | + | g(c) || f(x) - f(c) |$.

Let us choose $\epsilon > 0$.
Since f is continuous at c, there exists a positive δ_1, such that $| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta_1) \cap D$.

Since $| \ |f(x)| - |f(c)| \ | \leq | f(x) - f(c) |$, it follows that $| \ |f(x)| - |f(c)| \ | < \epsilon$ for all $x \in N(c, \delta_1) \cap D$
or, $| f(c) | - \epsilon < | f(x) | < | f(c) | + \epsilon$ for all $x \in N(c, \delta_1) \cap D$.

Let $| f(c) | + \epsilon = B_1$. Then $| f(x) | < B_1$ for all $x \in N(c, \delta_1) \cap D$.
Let $B = \max\{B_1, | g(c) |\}$. Then $B > 0$ and
$| fg(x) - fg(c) | < B \ | g(x) - g(c) | + B \ | f(x) - f(c) |$.

Since f is continuous at c, there exists a positive δ_2 such that $| f(x) - f(c) | < \frac{\epsilon}{2B}$ for all $x \in N(c, \delta_2) \cap D$.

Since g is continuous at c, there exists a positive δ_3 such that $| g(x) - g(c) | < \frac{\epsilon}{2B}$ for all $x \in N(c, \delta_3) \cap D$.

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$.
Then $| fg(x) - fg(c) | < (\frac{\epsilon}{2B} + \frac{\epsilon}{2B})B$ for all $x \in N(c, \delta) \cap D$
i.e., $| fg(x) - fg(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$.

This shows that fg is continuous at c.

(iv) First we prove that $\frac{1}{g}$ is continuous at c if $g(x) \neq 0$ for all $x \in D$.
$| \frac{1}{g}(x) - \frac{1}{g}(c) | = | \frac{1}{g(x)} - \frac{1}{g(c)} | = \frac{|g(x)-g(c)|}{|g(x)||g(c)|}$.

Let us choose $\epsilon = \frac{1}{2} \ | g(c) |$. Since g is continuous at c, there exists a positive δ_1 such that
$| g(x) - g(c) | < \frac{1}{2} \ | g(c) |$ for all $x \in N(c, \delta_1) \cap D$.

Now $\mid |g(x)| - |g(c)| \mid \leq |g(x) - g(c)| < \frac{1}{2}|g(c)|$.
Therefore $| g(x) | > \frac{1}{2} | g(c) |$ for all $x \in N(c, \delta_1) \cap D$.

Let us choose $\epsilon > 0$. Since g is continuous at c, there exists a positive δ_2 such that
$| g(x) - g(c) | < \frac{|g(c)|^2 \epsilon}{2}$ for all $x \in N(c, \delta_2) \cap D$.
Let $\delta = \min\{\delta_1, \delta_2\}$.
Then $| \frac{1}{g}(x) - \frac{1}{g}(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$.
This shows that $\frac{1}{g}$ is continuous at c.
The proof of the theorem is completed by considering the product of two functions f and $\frac{1}{g}$.

Note 1. If $f_1, f_2, \ldots, f_n$ be n functions on D and each of them be continuous at $c \in D$, then $f_1 + f_2 + \cdots + f_n$ is continuous at c.

Note 2. (iv) holds if $g(x) \neq 0$ in some neighbourhood $N(c, \delta) \subset D$. The existence of such a neighbourhood is guaranteed if $g(c) \neq 0$, by the neighbourhood property of continuity to be discussed later. Therefore (iv) holds under the single condition $g(c) \neq 0$.

Theorem 8.1.5. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}, g : D \to \mathbb{R}$ be both continuous on D. Then

(i) $f + g$ is continuous on D

(ii) if $k \in \mathbb{R}, kf$ is continuous on D

(iii) fg is continuous on D

(iv) if $g(x) \neq 0$ on $D, f/g$ is continuous on D.

Immediate consequence of the Theorem 8.1.4.

Remark. The set of all real functions continuous on the closed and bounded interval $[a, b]$ is denoted by $C[a, b]$.

Theorem 8.1.6. Let $D \subset \mathbb{R}$ and a function $f : D \to \mathbb{R}$ be continuous at $c \in D$. Then $|f|$ is continuous at c.

Proof. $|f| : D \to \mathbb{R}$ is defined by $|f|(x) = |f(x)|$, $x \in D$.
$\mid |f|(x) - |f|(c) \mid = \mid |f(x)| - |f(c)| \mid \leq |f(x) - f(c)|$.

Let us choose $\epsilon > 0$.
Since f is continuous at c, there exists a positive δ such that
$| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$.
Therefore $\mid |f|(x) - |f|(c) \mid < \epsilon$ for all $x \in N(c, \delta) \cap D$.

This shows that $|f|$ is continuous at c.

An immediate consequence of this theorem is the following theorem.

Theorem 8.1.7. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be continuous on D. Then $|f|$ is continuous on D.

Note. If $|f|$ be continuous on D then f may not be continuous on D. For example, let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$\begin{aligned} f(x) &= 1, x \in \mathbb{Q}, \\ &= -1, x \in \mathbb{R} - \mathbb{Q}. \end{aligned}$$

Then $|f| : \mathbb{R} \to \mathbb{R}$ is defined by $|f|(x) = 1, x \in \mathbb{R}$.

Here $|f|$ is continuous on $\mathbb{R}$ but f is not continuous on $\mathbb{R}$.

Definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}, g : D \to \mathbb{R}$ be functions. We define the function $\sup(f,g) : D \to \mathbb{R}$ by

$\sup(f,g)(x) = \sup\{f(x), g(x)\}, x \in D$.

We define the function $\inf(f,g) : D \to \mathbb{R}$ by

$\inf(f,g)(x) = \inf\{f(x), g(x)\}, x \in D$.

Theorem 8.1.8. Let $D \subset \mathbb{R}$ and let $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$ be continuous at $c \in D$. Then $\sup(f,g)$ and $\inf(f,g)$ are continuous at c.

Proof.
$$\begin{aligned} \sup(f,g)(x) &= \sup\{f(x), g(x)\} \\ &= \tfrac{1}{2}(f(x) + g(x)) + \tfrac{1}{2} \mid f(x) - g(x) \mid \\ &= \tfrac{1}{2}(f+g)(x) + \tfrac{1}{2} \mid f - g \mid (x), x \in D. \end{aligned}$$

$$\begin{aligned} \inf(f,g)(x) &= \inf\{f(x), g(x)\} \\ &= \tfrac{1}{2}(f(x) + g(x)) - \tfrac{1}{2} \mid f(x) - g(x) \mid \\ &= \tfrac{1}{2}(f+g)(x) - \tfrac{1}{2} \mid f - g \mid (x), x \in D. \end{aligned}$$

Since f and g are continuous at c, $f+g, f-g, \mid f-g \mid$ are continuous at c. It follows that $\sup(f,g)$ and $\inf(f,g)$ are continuous at c.

Theorem 8.1.9. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be continuous on D. Then $\sup(f,g)$ and $\inf(f,g)$ are continuous on D.

Immediate consequence of the Theorem 8.1.8.

Theorem 8.1.10. Let $A, B \subset \mathbb{R}$ and $f : A \to \mathbb{R}, g : B \to \mathbb{R}$ be functions such that $f(A) \subset B$.

Let $c \in A$ and f is continuous at c and g is continuous at $f(c) \in B$. Then the composite function $g \circ f : A \to \mathbb{R}$ is continuous at c.

Proof. Let W be a neighbourhood of $g \circ f(c)$.

Since g is continuous at $f(c)$, there exists a neighbourhood V of $f(c)$ such that $y \in V \cap B \Rightarrow g(y) \in W$.

Since f is continuous at c, corresponding to the neighbourhood V of $f(c)$ there exists a neighbourhood U of c such that $x \in U \cap A \Rightarrow f(x) \in V$.

Since $f(A) \subset B, f(x) \in V \Rightarrow f(x) \in V \cap B$.

$$\begin{aligned} \text{Therefore } x \in U \cap A \quad &\Rightarrow \quad f(x) \in V \\ &\Rightarrow \quad f(x) \in V \cap B \\ &\Rightarrow \quad gof(x) \in W. \end{aligned}$$

Thus corresponding to a chosen neighbourhood W of $g \circ f(c)$ there exists a neighbourhood U of c such that for all $x \in U \cap A$, $g \circ f(x) \in W$.

This proves that $g \circ f$ is continuous at c.

Theorem 8.1.11. Let $A, B \subset \mathbb{R}$ and let $f : A \to \mathbb{R}$ be continuous on A and let $g : B \to \mathbb{R}$ be continuous on B and $f(A) \subset B$.

Then the composite function $g \circ f : A \to \mathbb{R}$ is continuous on A.

8.2. Continuity of some important functions.

1. Polynomial function.

Let $f(x) = a_o x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$ for all $x \in \mathbb{R}$, where $a_o, a_1, \ldots, a_n$ are real numbers. Then f is a polynomial function.

f is the sum of $n+1$ functions $f_0, f_1, f_2, \ldots, f_n$ where $f_i = a_i x^{n-i}, i = 0, 1, 2, \ldots, n$. Each f_i is continuous on $\mathbb{R}$. Therefore by Theorem 8.1.5, f is continuous on $\mathbb{R}$.

2. Rational function.

Let $p(x)$ and $q(x)$ be polynomial functions on $\mathbb{R}$.

There are at most a finite number of real roots, say $\alpha_1, \alpha_2, \ldots, \alpha_m$ of $q(x)$. If $x \neq \alpha_1, \alpha_2, \ldots, \alpha_m$ then we can define a function f by

$f(x) = \frac{p(x)}{q(x)}, x \neq \alpha_1, \alpha_2, \ldots, \alpha_m$.

By Theorem 8.1.4, if $q(c) \neq 0$ then f is continuous at c.

That is, if c be not a root of $q(x)$ then f is continuous at c.

So a rational function is continuous for all $x \in \mathbb{R}$ for which the function is defined.

3. Trigonometric functions.

(a) Let $f(x) = \sin x, x \in \mathbb{R}$. Let $c \in \mathbb{R}$.

$$\begin{aligned} |\sin x - \sin c| &= 2 \left| \cos \tfrac{x+c}{2} \sin \tfrac{x-c}{2} \right| \\ &\leq 2 \left| \sin \tfrac{x-c}{2} \right|, \text{ since } |\cos x| \leq 1 \\ &\leq 2 \left| \tfrac{x-c}{2} \right|, \text{ since } |\sin x| \leq |x| \\ &= |x - c|. \end{aligned}$$

Let us choose $\epsilon > 0$.

Then $|\sin x - \sin c| < \epsilon$ for all x satisfying $|x - c| < \epsilon$.

So f is continuous at c. Since c is arbitrary, f is continuous on $\mathbb{R}$.

(b) Let $f(x) = \cos x, x \in \mathbb{R}$. Let $c \in \mathbb{R}$.

$$\begin{aligned} |\cos x - \cos c| &= 2\,|\sin\tfrac{x+c}{2}\sin\tfrac{x-c}{2}| \\ &\le 2\,|\sin\tfrac{x-c}{2}|, \text{ since } |\sin\tfrac{x+c}{2}| \le 1 \\ &\le 2\,|\tfrac{x-c}{2}|, \text{ since } |\sin x| \le |x| \\ &= |x-c| \end{aligned}$$

Let us choose $\epsilon > 0$.
Then $|\cos x - \cos c| < \epsilon$ for all x satisfying $|x-c| < \epsilon$.

So f is continuous at c. Since c is arbitrary, f is continuous on $\mathbb{R}$.

(c) Let $f(x) = \tan x$.

f is not defined at the points $(2n+1)\frac{\pi}{2}$ (n being an integer) where the denominator $\cos x = 0$.

Let $c \in \mathbb{R}$ and $c \neq (2n+1)\frac{\pi}{2}$. Then $\lim\limits_{x\to c} \tan x = \tan c$.

So f is continuous at c when $c \neq (2n+1)\frac{\pi}{2}$.

Thus f is continuous on its domain.

(d) The functions $\cot x$, $\operatorname{cosec} x, \sec x$ are continuous on their respective domains.

4. Exponential function.

Let $a > 0$ and $f(x) = a^x, x \in \mathbb{R}$.

Let $c \in \mathbb{R}$. Let $\{x_n\}$ be any sequence in $\mathbb{R}$ converging to c. Then $\lim a^{x_n} = a^c$, by the corollary of the worked out limit 3 in art. 5.8.

So f is continuous at c. Since c is arbitrary, f is continuous on $\mathbb{R}$.

Corollary. The function f defined by $f(x) = e^x, x \in \mathbb{R}$ is continuous on $\mathbb{R}$.

5. Logarithmic function.

Let $f(x) = \log x, x > 0$.

Let $c > 0$. Let $\{x_n\}$ be any sequence such that $x_n > 0$ for all $n \in \mathbb{N}$ and $\lim x_n = c$, then $\lim \log x_n = \log c$, by the corollary of the limit 4 in art.5.8.

So f is continuous at c. Since c is arbitrary, f is continuous on $(0, \infty)$.

6. Square root function.

Let $f(x) = \sqrt{x}, x \geq 0$. The domain of f is $D = \{x \in \mathbb{R} : x \geq 0\}$
Let $c > 0, f(c) = \sqrt{c}$.

$$\begin{aligned} \text{When } x \geq 0, |f(x) - f(c)| &= |\sqrt{x} - \sqrt{c}| \\ &= \left|\frac{x-c}{\sqrt{x}+\sqrt{c}}\right| \le \frac{1}{\sqrt{c}}\,|x-c|. \end{aligned}$$

Let us choose $\epsilon > 0$.
Then $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \sqrt{c}\epsilon$ and $x \geq 0$.

That is, $| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$. $[\delta = \sqrt{c}\epsilon$
So f is continuous at c.
Also $\lim_{x \to 0} f(x) = f(0)$, showing that f is continuous at 0.

Thus f is continuous for all $x \geq 0$.

7. Some composite functions.

(a) Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be such that $f(x) \geq 0$ for all $x \in D$ and f is continuous on D. Then $\sqrt{f}$ is continuous on D.

To prove this, let $g(x) = \sqrt{x}$.

Then the composite function $gf : D \to \mathbb{R}$ is defined by $gf(x) = \sqrt{f(x)}, x \in D$.

Since f is continuous on D and g is continuous on $f(D)$, the composite function gf, i.e., $\sqrt{f}$ is continuous on D.

Worked Examples.

(i) Prove that the function $h(x) = \sqrt{x^2 + 3}, x \in \mathbb{R}$ is continuous on $\mathbb{R}$.

h is the composite function gf where $f(x) = x^2 + 3, x \in \mathbb{R}$ and $g(x) = \sqrt{x}, x \geq 0$. $f(x) > 0$ for $x \in \mathbb{R}$. f is continuous on $\mathbb{R}$ and g is continuous on $f(\mathbb{R})$.

So gf is continuous on $\mathbb{R}$. That is, h is continuous on $\mathbb{R}$.

(ii) Prove that the function $h(x) = \sqrt{\sin x}, x \in [0, \pi]$ is continuous on $[0, \pi]$.

h is the composite function gf where $f(x) = \sin x, x \in [0, \pi]$ and $g(x) = \sqrt{x}, x \geq 0$.

$f(x) \geq 0$ for $x \in [0, \pi]$. f is continuous on $[0, \pi] = D$, say. g is continuous on $f(D)$.

So gf is continuous on $[0, \pi]$. That is, h is continuous on $[0, \pi]$.

(iii) Prove that the function $h(x) = \sqrt{x + \sqrt{x}}, x \geq 0$ is continuous on $[0, \infty)$.

h is the composite function gf where $f(x) = x + \sqrt{x}, x \geq 0$ and $g(x) = \sqrt{x}, x \geq 0$.

$f(x) \geq 0$ for $x \geq 0$. f is continuous on $[0, \infty) = D$, say. g is continuous on $f(D)$.

So gf is continuous on $[0, \infty)$. That is, h is continuous on $[0, \infty)$.

(b) Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be such that $f(x) > 0$ for all $x \in D$ and f is continuous on D. Then $\log f$ is continuous on D.

To prove this, let $g(x) = \log x, x > 0$

The composite function $gf : D \to \mathbb{R}$ is defined by $g(x) = \log f(x), x \in D$.

Since f is continuous on D and g is continuous on $f(D)$, the composite function gf, i.e., $\log f$ is continuous on D.

Worked Examples (continued).

(iv) Prove that the function $h(x) = \log(x^2 + 3)$ is continuous on $\mathbb{R}$.

h is the composite function gf where $f(x) = x^2 + 3, x \in \mathbb{R}$ and $g(x) = \log x, x > 0$.

$f(x) > 0$ for $x \in \mathbb{R}$. f is continuous on $\mathbb{R}$ and g is continuous on $f(\mathbb{R})$.

So gf is continuous on $\mathbb{R}$. That is, h is continuous on $\mathbb{R}$.

(v) Prove that the function $h(x) = \log \sin x$ is continuous on $(0, \pi)$.

h is the composite function gf where $f(x) = \sin x, x \in (0, \pi)$ and $g(x) = \log x, x > 0$.

gf is continuous on $(0, \pi)$. That is, h is continuous on $(0, \pi)$.

(c) Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ is continuous on D. Then e^f is continuous on D.

To prove this let $g(x) = e^x$.

The composite function $gf : D \to \mathbb{R}$ is defined on D and $gf(x) = e^{f(x)}, x \in D$.

Since f is continuous on D and g is continuous on $f(D)$ the composite function gf, i.e., e^f is continuous on D.

Worked Example (continued).

(vi) Prove the function $h(x) = e^{\sin x}$ is continuous on $\mathbb{R}$.

h is the composite function gf where $f(x) = \sin x, x \in \mathbb{R}$ and $g(x) = e^x, x \in \mathbb{R}$. $f(x) \in [-1, 1]$ for $x \in \mathbb{R}$.

f is continuous on $\mathbb{R}$ and g is continuous on $f(\mathbb{R})$, i.e., on $[-1, 1]$.

So gf is continuous on $\mathbb{R}$. That is, h is continuous on $\mathbb{R}$.

8.3. Limits of composite functions.

Theorem 8.3.1. Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$. Let $g : D \to \mathbb{R}$ where $f(A) \subset D$.

Let c be a limit point of A and $\lim_{x \to c} f(x) = l$.

(i) If $l \in D$ and g is continuous at l then $\lim_{x \to c} gf(x) = g(l)$.

(ii) If $l \notin D$ but $l \in D'$ and $\lim_{y \to l} g(y) = m$ then $\lim_{x \to c} gf(x) = m$.

Proof. (i) Since g is continuous at l, for a pre-assigned positive ϵ there exists a positive δ such that

$| g(y) - g(l) | < \epsilon$ for all $y \in N(l, \delta) \cap D$.

Since $\lim_{x \to c} f(x) = l$, there exists a positive δ_1 such that

$f(x) \in N(l, \delta)$ for all $x \in N'(c, \delta_1) \cap A$.

Since $f(A) \subset D, x \in N'(c, \delta_1) \cap A \Rightarrow f(x) \in N(l, \delta) \cap D$.

Therefore $x \in N'(c, \delta_1) \cap A \Rightarrow | g(y) - g(l) | < \epsilon$

i.e., $x \in N'(c, \delta_1) \cap A \Rightarrow | gf(x) - g(l) | < \epsilon$.

This proves $\lim_{x \to c} gf(x) = g(l)$.

(ii) Since $\lim_{y \to l} g(y) = m$, for a pre-assigned positive ϵ there exists a positive δ such that $| g(y) - m | < \epsilon$ for all $y \in N'(l, \delta) \cap D$.

Since $\lim_{x \to c} f(x) = l$, there exists a positive δ_1 such that $f(x) \in N(l, \delta)$ for all $x \in N'(c, \delta_1) \cap A$.

Since $f(A) \subset D, x \in N'(c, \delta_1) \cap A \Rightarrow f(x) \in N(l, \delta) \cap D$
$\Rightarrow f(x) \in N'(l, \delta) \cap D$, since $l \notin D$.

Therefore $x \in N'(c, \delta_1) \cap A \Rightarrow | g(y) - m | < \epsilon$

i.e., $| gf(x) - m | < \epsilon$. This proves $\lim_{x \to c} gf(x) = m$.

Note. In case (ii) if $\lim_{y \to l} g(y) = \infty$ (or $-\infty$) then $\lim_{x \to c} gf(x) = \infty$ (or $-\infty$).

As an immediate corollary it follows that

(i) if $\lim_{x \to c} f(x) = l > 0$ then $\lim_{x \to c} \log f(x) = \log \lim_{x \to c} f(x)$, since $\log x$ is continuous on its domain $(0, \infty)$;

(ii) if $\lim_{x \to c} f(x) = l \geq 0$ then $\lim_{x \to c} \sqrt{f(x)} = \sqrt{\lim_{x \to c} f(x)}$, since $\sqrt{x}$ is continuous on its domain $[0, \infty)$;

(iii) if $\lim_{x \to c} f(x) = l (\in \mathbb{R})$ then $\lim_{x \to c} e^{f(x)} = e^{\lim_{x \to c} f(x)}$, since e^x is continuous on its domain $(-\infty, \infty)$.

Extension of the theorem.

Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$. Let $g : D \to \mathbb{R}$ where $f(A) \subset D$.

(a) Let c be a limit point of A and $\lim_{x \to c} f(x) = \infty$.

If for some $b \in \mathbb{R}, (b, \infty) \subset D$ and $\lim_{y \to \infty} g(y) = m$ then $\lim_{x \to c} gf(x) = m$, where $m \in \mathbb{R}$, or $m = \infty$, or $m = -\infty$.

(b) Let c be a limit point of A and $\lim_{x \to c} f(x) = -\infty$.

If for some $b \in \mathbb{R}$, $(-\infty, b) \subset D$ and $\lim_{y \to -\infty} g(y) = m$ then $\lim_{x \to c} gf(x) = m$, where $m \in \mathbb{R}$, or $m = \infty$, or $m = -\infty$.

(c) For some $a \in \mathbb{R}$, let $(a, \infty) \subset A$ and $\lim_{x \to \infty} f(x) = l$.

(i) If $l \in D$ and g is continuous at l then $\lim_{x \to \infty} gf(x) = g(l)$.

(ii) If $l \notin D$ but $l \in D'$ and $\lim_{y \to l} g(y) = m$ then $\lim_{x \to \infty} gf(x) = m$ where $m \in \mathbb{R}$, or $m = \infty$, or $m = -\infty$.

(d) For some $a \in \mathbb{R}$, let $(-\infty, a) \subset A$ and $\lim_{x \to -\infty} f(x) = l$.

(i) If $l \in D$ and g is continuous at l then $\lim_{x \to -\infty} gf(x) = g(l)$.

(ii) If $l \notin D$ but $l \in D'$ and $\lim_{y \to l} g(y) = m$ then $\lim_{x \to -\infty} gf(x) = m$, where $m \in \mathbb{R}$, or $m = \infty$, or $m = -\infty$.

Some other similar extensions of the theorem can be formulated.

A *word of caution* : $m = \infty(-\infty)$ stands for the phrase "$\lim gf(x) = \infty(-\infty)$".

Examples.

1. $\lim_{x \to 0} \sin \sqrt{x} = 0$.

Let $f(x) = \sqrt{x}, x \geq 0; g(x) = \sin x, x \in \mathbb{R}$.
Here $A = \{x \in \mathbb{R} : x \geq 0\}, D = \mathbb{R}$ and $f(A) \subset D$.
$gf(x) = \sin \sqrt{x}, x \geq 0$.
$0 \in A'$ and $\lim_{x \to 0} f(x) = 0.0 \in \mathbb{R}$ and g is continuous at 0.
Therefore $\lim_{x \to 0} \sin \sqrt{x} = \lim_{x \to 0} gf(x) = g(0) = 0$.

2. $\lim_{x \to 0} \sqrt{1 + \sqrt{x}} = 1$.

Let $f(x) = 1 + \sqrt{x}, x \geq 0; g(x) = \sqrt{x}, x \geq 0$.
Here $A = \{x \in \mathbb{R} : x \geq 0\}, D = \{x \in \mathbb{R}/x\}, f(A) \subset D$.
$gf(x) = \sqrt{1 + \sqrt{x}}, x \geq 0$.
$0 \in A'$ and $\lim_{x \to 0} f(x) = 1.1 \in D$ and g is continuous at 1.
Therefore $\lim_{x \to 0} \sqrt{1 + \sqrt{x}} = \lim_{x \to 1} gf(x) = g(1) = 1$.

3. $\lim_{x \to 0+} e^{1/x} = \infty, \lim_{x \to 0-} e^{1/x} = 0, \lim_{x \to \infty} e^{1/x} = 1, \lim_{x \to -\infty} e^{1/x} = 1$.

Let $f(x) = \frac{1}{x}, x \neq 0; g(x) = e^x, x \in \mathbb{R}$.
Here $A = \{x \in \mathbb{R} : x \neq 0\}, D = \mathbb{R}, f(A) \subset D$. $gf(x) = e^{1/x}, x \neq 0$.

$0 \in A'$, $\lim_{x\to 0} f(x)$ does not exist.
But $\lim_{x\to 0+} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$.
Therefore $\lim_{x\to 0+} gf(x) = \infty$, i.e., $\lim_{x\to 0+} e^{1/x} = \infty$.

$\lim_{x\to 0-} f(x) = -\infty$ and $\lim_{x\to -\infty} g(x) = 0$.
Therefore $\lim_{x\to 0-} gf(x) = 0$, i.e., $\lim_{x\to 0-} e^{1/x} = 0$.

$\lim_{x\to\infty} f(x) = 0, 0 \in D$ and g is continuous at 0.
Therefore $\lim_{x\to\infty} gf(x) = g(0) = 1$, i.e., $\lim_{x\to\infty} e^{1/x} = 1$.

$\lim_{x\to -\infty} f(x) = 0, 0 \in D$ and g is continuous at 0.
Therefore $\lim_{x\to -\infty} gf(x) = g(0) = 1$, i.e., $\lim_{x\to -\infty} e^{1/x} = 1$.

4. $\lim_{x\to 0+} e^{-1/x} = 0$, $\lim_{x\to 0-} e^{-1/x} = \infty$, $\lim_{x\to\infty} e^{-1/x} = 1$, $\lim_{x\to -\infty} e^{-1/x} = 1$.

Let $f(x) = \frac{1}{x}, x \neq 0$; $g(x) = e^{-x}, x \in \mathbb{R}$.
Here $A = \{x \in \mathbb{R} : x \neq 0\}, D = \mathbb{R}, f(A) \subset D$.
$gf(x) = e^{-1/x}, x \neq 0$.

$0 \in A'$, $\lim_{x\to 0} f(x)$ does not exist.
But $\lim_{x\to 0+} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = 0$.
Therefore $\lim_{x\to 0+} gf(x) = 0$, i.e., $\lim_{x\to 0+} e^{-1/x} = 0$.

$\lim_{x\to 0-} f(x) = -\infty$ and $\lim_{x\to -\infty} g(x) = \infty$.
Therefore $\lim_{x\to 0-} gf(x) = 0$, i.e., $\lim_{x\to 0-} e^{-1/x} = 0$.

$\lim_{x\to\infty} f(x) = 0, 0 \in D$ and g is continuous at 0.
Therefore $\lim_{x\to\infty} gf(x) = g(0) = 1$, i.e., $\lim_{x\to\infty} e^{-1/x} = 1$.

$\lim_{x\to -\infty} f(x) = 0, 0 \in D$ and g is continuous at 0.
Therefore $\lim_{x\to -\infty} gf(x) = g(0) = 1$, i.e., $\lim_{x\to -\infty} e^{-1/x} = 1$.

5. **Some important limits.**

(i) $\lim_{x\to 0} \dfrac{\log(1+x)}{x} = 1$, (ii) $\lim_{x\to 0} \dfrac{e^x - 1}{x} = 1$,

(iii) $\lim_{x\to 0} \dfrac{a^x - 1}{x} = \log_e a, a > 0$.

(i) Let $f(x) = (1+x)^{1/x}, x > -1$ but $x \neq 0; g(x) = \log x, x > 0$.

Here $A = (-1,0) \cup (0,\infty)$ $D = \{x \in \mathbb{R} : x > 0\}$.$f(A) \subset D$.
$gf(x) = \frac{\log(1+x)}{x}, x \in A$.

$0 \in A'$ and $\lim\limits_{x\to 0}(1+x)^{1/x} = e$. $e \in D$ and g is continuous at e.
Therefore $\lim\limits_{x\to e} g(x) = g(e) = 1$.

Hence $\lim\limits_{x\to 0} gf(x) = 1$, i.e., $\lim\limits_{x\to 0} \dfrac{\log(1+x)}{x} = 1$.

(ii) Let $\log(1+x) = y$. Then $1 + x = e^y$, i.e., $x = e^y - 1$.
As $x \to 0, y \to 0$.
From (i) $1 = \lim\limits_{x\to 0} \dfrac{\log(1+x)}{x} = \lim\limits_{x\to 0} \dfrac{y}{e^y - 1}$
Therefore $\lim\limits_{y\to 0} \dfrac{e^y - 1}{y} = 1$, i.e., $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1$.

(iii) $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x} = \lim\limits_{x\to 0} \dfrac{e^{x \log_e a} - 1}{x} = \lim\limits_{x\to 0} \dfrac{e^{x \log_e a} - 1}{x \log_e a} . \log_e a$
$= \lim\limits_{y\to 0} \dfrac{e^y - 1}{y} . \log_e a$ [let $x \log_e a = y$; as $x \to 0, y \to 0$]
$= \log_e a$.

8.4. Discontinuity.

We have seen that a function defined on a domain D may be continuous at all points of D, or may be continuous at some points of D and discontinuous at the other points of D, or may be discontinuous at every point of D.

If c does not belong to the domain of f, it is certain that f is discontinuous at c but the nature of discontinuity at c depends on the behaviour of the function f in the immediate neighbourhood of c. Even if c belongs to the domain of f, a discontinuity of f at c can occur in a variety of ways.

We now discuss different types of discontinuity of a finction f at a point c irrespective of the cases whether c belongs to or does not belong to the domain of f.

Discontinuity at an end point of an interval.

I. Let c be the left end point of the interval I and let f be continuous on (c, d) but discontinuous at c, $(c, d) \subset I$.

Three cases may arise.

(a) $\lim\limits_{x\to c} f(x)$ exists but $f(c) \neq \lim\limits_{x\to c} f(x)$.

In this case f is discontinuous at c. This type of discontinuity is called a *removable discontinuity.* c is said to be a *point of removable discontinuity.*

Note. The discontinuity at c can be removed by suitably defining f at c. That is why the discontinuity is called a removable discontinuity.

Example.

1. Let $f(x) = \frac{x^2-4}{x-2}, x > 2$
$\qquad\quad = 10, x = 2.$

Here 2 is the left end point of the interval $[2,\infty)$, the domain of f. $f(2) = 10$, $\lim_{x\to 2} f(x) = 4 \neq f(2)$.

f is not continuous at 2. 2 is a point of removable discontinuity.

Note. If we define f by $f(x) = \frac{x^2-4}{x-2}, x > 2$
$\qquad\qquad\qquad\qquad = 4, x = 2$

then f becomes continuous at 2.

(b) $\lim_{x\to c} f(x)$ does not exist but there exists a neighbourhood $N(c,\delta)$ of c such that f is bounded on $N'(c,\delta) \cap I$.

In this case f is discontinuous at c whether f is defined at c or not. This type of discontinuity is called an *oscillatory discontinuity.* c is said to be a *point of oscillatory discontinuity.*

Example (continued).

2. Let $f(x) = \sin\frac{1}{x}, x > 0$
$\qquad\quad = 0, x = 0.$

Here 0 is the left end point of the interval $[0,\infty)$, the domain of f. $f(0) = 0$. $\lim_{x\to 0} f(x)$ does not exist. f is discontinuous at 0. f is bounded on $(0,\delta)$ for $\delta > 0$. 0 is a point of oscillatory discontinuity.

(c) f is unbounded on every neighbourhood of c. In this case f is discontinuous at c whether f is defined at c or not.

This type of discontinuity is called an *infinite discontinuity.* c is said to be a *point of infinite discontinuity.*

If $\lim_{x\to c} f(x) = \infty$ (or $-\infty$), c is said to be a point of infinite discontinuity.

If $\lim_{x\to c} f(x)$ does not exist in $\mathbb{R}^*$, c is said to be a point of infinite oscillatory discontinuity.

Examples (continued).

3. Let $f(x) = \log x, x > 0$.

Here $\lim\limits_{x \to 0} f(x) = -\infty$. f is discontinuous at 0. 0 is a point of infinite discontinuity.

4. Let $f(x) = \frac{1}{\sqrt{x-3}}, x > 3$.

Here $\lim\limits_{x \to 3} f(x) = \infty$. f is discontinuous at 3. 3 is a point of infinite discontinuity.

5. Let $f(x) = \frac{1}{x} \sin \frac{1}{x}, x > 0$
$\quad = 0, x = 0$.

Here $\lim\limits_{x \to 0} f(x)$ does not exist. f is unbounded in $(0, \delta)$ for each $\delta > 0$.
$\overline{\lim\limits_{x \to 0}} f(x) = \infty, \underline{\lim}_{x \to 0} f(x) = -\infty$.

f is discontinuous at 0. 0 is a point of infinite oscillatory discontinuity.

6. Let $f(x) = \frac{1}{x} \mid \sin \frac{1}{x} \mid, x > 0$
$\quad = 0, x = 0$.

f is unbounded on $(0, \delta)$ for each $\delta > 0$.

Here $\overline{\lim\limits_{x \to 0}} f(x) = \infty$, $\underline{\lim}_{x \to 0} f(x) = 0$.

f is discontinuous at 0. 0 is a point of infinite oscillatory discontinuity.

II. Let c be the right end point of the interval I and let f be continuous on (a, c) but discontinuous at $c, (a, c) \subset I$.

Three cases as in **I** may arise and we have three types of discontinuity at c.

Discontinuity at an interior point of an interval.

Let c be an interior point of the interval I and let f be continuous on (a, c) and (c, b), but discontinuous at $c, (a, b) \subset I$.

Three cases may arise.

(a) (i) $\lim\limits_{x \to c-} f(x)$ and $\lim\limits_{x \to c+}$ both exist and $\lim\limits_{x \to c-} f(x) = \lim\limits_{x \to c+} f(x)$.

Subcase (i'). If f is not defined at c then f is discontinuous at c.

Subcase (i''). If f is defined at c but $f(c) \neq \lim\limits_{x \to c} f(x)$, then f is discontinuous at c.

This type of discontinuity [either in (i') or in (i'')] is called a *removable discontinuity.* c is said to be a *point of removable discontinuity.*

Note. The discontinuity at c can be removed by suitably defining f at c.

(ii) $\lim\limits_{x \to c-} f(x)$ and $\lim\limits_{x \to c+} f(x)$ both exist and $\lim\limits_{x \to c-} f(x) \neq \lim\limits_{x \to c+} f(x)$.

In this case f is discontinuous at c whether f is defined at c or not.

This type of discontinuity is called a *jump discontinuity.* c is said to be a *point of jump discontinuity.*

$\lim\limits_{x \to c+} f(x) - \lim\limits_{x \to c-} f(x)$ [i.e., $f(c+0) - f(c-0)$] is defined to be the *total jump* of f at c and it is denoted by $J_f(c)$.

If f is defined at c, $f(c+0) - f(c)$ is defined to be the *right hand jump* of f at c; and $f(c) - f(c-0)$ is defined to be the *left hand jump* of f at c.

The discontinuities discussed in (a) are called *simple discontinuities* or *discontinuities of the first kind.*

Examples (continued).

7. Let $f(x) = \frac{x^2-4}{x-2}, x \neq 2$
$\quad = 10, x = 2.$

Here $\lim\limits_{x \to 2} f(x) = 4, \quad f(2) = 10.$

f is discontinuous at 2. 2 is a point of removable discontinuity.

Note. If we define f by $f(x) = \frac{x^2-4}{x-2}, x \neq 2$
$\quad = 4, x = 2$

then f becomes continuous at 2.

8. Let $f(x) = [x], 0 < x < 2.$

$$\begin{aligned} f(x) &= 0, 0 < x < 1 \\ &= 1, 1 \leq x < 2. \end{aligned}$$

Here $\lim\limits_{x \to 1-} f(x) = 0, \lim\limits_{x \to 1+} f(x) = 1, f(1) = 1.$

f is discontinuous at 1. 1 is a point of jump discontinuity.

Total jump of f at $1 = f(1+0) - f(1-0) = 1 - 0 = 1.$

Right hand jump at $1 = f(1+0) - f(1) = 1 - 1 = 0.$

Left hand jump at $1 = f(1) - f(1-0) = 1 - 0 = 1.$

9. Let $f(x) = x - [x], 0 < x < 2$

$$\begin{aligned} f(x) &= x, 0 < x < 1 \\ &= x - 1, 1 \leq x < 2 \end{aligned}$$

Here $\lim\limits_{x \to 1-} f(x) = 1, \lim\limits_{x \to 1+} f(x) = 0, \quad f(1) = 0.$

f is discontinuous at 1. 1 is a point of jump discontinuity.

Total jump of f at $1 = f(1+0) - f(1-0) = 0 - 1 = -1.$

Right hand jump at $1 = f(1+0) - f(1) = 0.$

Left hand jump at $1 = f(1) - f(1-0) = 0 - 1 = -1.$

(b) At least one of $\lim_{x \to c-} f(x)$ and $\lim_{x \to c+} f(x)$ does not exist. But f is bounded on some deleted neighbourhood $N'(c, \delta)$ of c.

In this case f is discontinuous at c whether f is defined at c or not. This type of discontinuity is called an *oscillatory discontinuity.*

Since f is bounded on *some* deleted neighbourhood $N'(c, \delta)$ of c,each of $\overline{\lim}_{x \to c+} f(x)$ $(\overline{f(c+0)})$, $\underline{\lim}_{x \to c+} f(x)$ $(\underline{f(c+0)})$, $\overline{\lim}_{x \to c-} f(x)$ $(\overline{f(c-0)})$, $\underline{\lim}_{x \to c-} f(x)$ $(\underline{f(c-0)})$ exist finitely.

Let $M_f(c) = \max\{\overline{f(c+0)}, \underline{f(c+0)}, \overline{f(c-0)}, \underline{f(c-0)}\}$;

$m_f(c) = \min\{\overline{f(c+0)}, \underline{f(c+0)}, \overline{f(c-0)}, \underline{f(c-0)}\}$.

Then both $M_f(c)$ and $m_f(c)$ are finite. $M_f(c) - mf(c)$ is defined to be the *saltus* of f at c. It is denoted by $s_f(c)$.

Note. In particular, if both $f(c+0)$ and $f(c-0)$ exist,
$S_f(c) = | f(c+0) - f(c-0) |$.

The discontinuities discussed in **(b)** are called *discontinuities of the second kind.*

Examples (continued).

10. Let $f(x) = \sin \frac{1}{x}, x \neq 0$
$= 0, x = 0.$

Here $\lim_{x \to 0} f(x)$ does not exist. f is bounded on $N(0, \delta)$ for $\delta > 0$.
$\overline{f(0+0)} = 1, \underline{f(0+0)} = -1, \overline{f(0-0)} = 1, \underline{f(0-0)} = -1.$

f is discontinuous 0. 0 is a point of oscillatory discontinuity. The saltus at $0 = 1 - (-1) = 2$.

11. Let $f(x) = | \sin \frac{1}{x} |, x \neq 0$
$= 0, x = 0.$

Here $\overline{f(0+0)} = 1, \underline{f(0+0)} = 0, \overline{f(0-0)} = 1, \underline{f(0-0)} = 0.$

f is discontinuous 0. 0 is a point of oscillatory discontinuity. The saltus at $0 = 1 - 0 = 1$.

12. Let $f(x) = \operatorname{sgn} x \, | \sin \frac{1}{x} |, x \neq 0$
$= 0, x = 0.$

Here $\overline{f(0+0)} = 1, \underline{f(0+0)} = 0, \overline{f(0-0)} = 0, \underline{f(0-0)} = -1.$

f is discontinuous 0. 0 is a point of oscillatory discontinuity. The saltus at $0 = 1 - (-1) = 2$.

(c) f is unbounded on every neighbourhood of c.

Since f unbounded on every neighbourhood of c, at least one of $\overline{f(c+0)}, \underline{f(c+0)}, \overline{f(c-0)}, \underline{f(c-0)}$ is ∞ or $-\infty$.

If each of $\overline{f(c+0)}$ and $\underline{f(c+0)}$ is ∞ (or $-\infty$) f is said to have an *infinite discontinuity* at the right of c.

If each of $\overline{f(c-0)}$ and $\underline{f(c-0)}$ is ∞ (or $-\infty$) f is said to have an *infinite discontinuity* at the left of c.

In either case c is said to be a point of *infinite discontinuity.*

If one or both of $\overline{f(c+0)}$ and $\underline{f(c+0)}$ be infinite and $\overline{f(c+0)} \neq \underline{f(c+0)}$, f is said to have an *infinite oscillatory discontinuity* at the right of c.

Similar definition for an *infinite oscillatory discontinuity* at the left of c.

Let $M_f(c) = \max\{\overline{f(c+0)}, \underline{f(c+0)}, \overline{f(c-0)}, \underline{f(c-0)}\}$;

$$m_f(c) = \min\{\overline{f(c+0)}, \underline{f(c+0)}, \overline{f(c-0)}, \underline{f(c-0)}\}.$$

If f be unbounded above on every neighbourhood of c, $M_f(c) = \infty$.
If f be unbounded below on every neighbourhood of c, $m_f(c) = -\infty$.

$M_f(c) - m_f(c)$ is said to be the *oscillation* of f at c. It is denoted by $w_f(c)$. $w_f(c)$ is infinite if at least one of $M_f(c)$ and $m_f(c)$ be ∞ (or $-\infty$) and $M_f(c) \neq m_f(c)$.

Examples (continued).

13. Let $f(x) = \frac{1}{x}$.

$\overline{f(0+0)} = \underline{f(0+0)} = \infty, \quad \overline{f(0-0)} = \underline{f(0-0)} = -\infty.$

f has an infinite discontinuity at the right of 0.
f has an infinite discontinuity at the left of 0.
0 is a point of infinite discontinuity.

14. Let $f(x) = \frac{1}{x^2}$.

$\overline{f(0+0)} = \underline{f(0+0)} = \overline{f(0-0)} = \underline{(0-0)} = \infty.$
0 is a point of infinite discontinuity.

15. Let $f(x) = \tan x$.

$\overline{f(\frac{\pi}{2}+0)} = \underline{f(\frac{\pi}{2}+0)} = -\infty, \overline{f(\frac{\pi}{2}-0)} = \underline{f(\frac{\pi}{2}-0)} = \infty.$
$\frac{\pi}{2}$ is a point of infinite discontinuity.

16. Let $f(x) = \frac{1}{x}\sin\frac{1}{x}$.

$\overline{f(0+0)} = \infty, \underline{f(0+0)} = -\infty, \overline{f(0-0)} = \infty, \underline{f(0-0)} = -\infty.$

f has an oscillatory infinite discontinuity at the right of 0

f has an oscillatory infinite discontinuity at the left of 0.

17. Let $f(x) = \frac{1}{x} \mid \sin \frac{1}{x} \mid$.

$\overline{f(0+0)} = \infty, \underline{f(0+0)} = 0, \quad \overline{f(0-0)} = 0, \underline{f(0-0)} = -\infty.$

f has an oscillatory infinite discontinuity at the right of 0

f has an oscillatory infinite discontinuity at the left of 0.

Worked Examples.

1. A function $f : [0,1] \to \mathbb{R}$ is defined by

$f(x) = x$, x is rational in $[0,1]$

$\quad = 1-x$, x is irrational in $[0,1]$.

Show that (i) f is injective on $[0,1]$, (ii) f assumes every real number in $[0,1]$, (iii) f is continuous at $\frac{1}{2}$ and discontinuous at every other point in $[0,1]$.

(i) Let $x_1, x_2 \in [0,1]$ and $x_1 \neq x_2$.

Case 1. x_1, x_2 are both rational. Then $f(x_1) = x_1, f(x_2) = x_2$. As $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$.

Case 2. x_1 is rational, x_2 is irrational. Then $f(x_1) = x_1, f(x_2) = 1 - x_2$. As $f(x_1)$ is rational and $f(x_2)$ is irrational, $f(x_1) \neq f(x_2)$.

Case 3. x_1 is irrational, x_2 is rational. Then $f(x_1) = 1 - x_1, f(x_2) = x_2$. As $f(x_1)$ is irrational and $f(x_2)$ is rational, $f(x_1) \neq f(x_2)$.

Case 4. x_1, x_2 are both irrational. Then $f(x_1) = 1 - x_1, f(x_2) = 1 - x_2$. As $x \neq x_2$, $f(x_1) \neq f(x_2)$.

Therefore $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$. Hence f is injective on $[0,1]$.

(ii) Let $p \in [0,1]$.

If p be rational then the pre-image of p is p, since $p \in [0,1]$ and $f(p) = p$. If p be irrational then the pre-image of p is $1-p$, since $1-p \in [0,1]$ and $f(1-p) = p$.

Thus every element in $[0,1]$ has a pre-image. In other words, f assumes every real number in $[0,1]$.

(iii) Let $c = \frac{1}{2}$. Then $f(c) = \frac{1}{2}$.

$|f(x) - f(c)| = |x - \frac{1}{2}|$, if x be rational in $[0,1]$

$\qquad = |(1-x) - \frac{1}{2}| = |x - \frac{1}{2}|$, if x be irrational in $[0,1]$.

Therefore $|f(x) - f(\frac{1}{2})| = |x - \frac{1}{2}|$ for all $x \in [0,1]$.

Let us choose $\epsilon > 0$. Then $|f(x) - f(\frac{1}{2})| < \epsilon$ holds for all x satisfying $|x - \frac{1}{2}| < \epsilon$. This proves that f is continuous at $\frac{1}{2}$.

Let $c \in [0,1], c \neq \frac{1}{2}$.

Let us consider a sequence of rational points $\{x_n\}$ in $[0,1]$ such that $\lim x_n = c$. Then $\lim f(x_n) = \lim x_n = c$.

Let us consider a sequence of irrational points $\{y_n\}$ in $[0,1]$ such that $\lim y_n = c$. Then $\lim f(y_n) = \lim(1-y_n) = 1-c$.

Since $c \neq 1-c$, f is not continuous at c by sequential criterion for continuity.

Thus f is discontinuous at every point other than $\frac{1}{2}$ in $[0,1]$.

2. A function $f : \mathbb{R}$ is defined by $f(x) = x, x \in \mathbb{Q}$
$= 0, x \in \mathbb{R} - \mathbb{Q}$.

Show that f is continuous at 0 and f has a discontinuity of the second kind at every other point in $\mathbb{R}$.

$$\begin{aligned} \mid f(x) - f(0) \mid = \mid f(x) \mid &= \mid x \mid \text{ if } x \in \mathbb{Q} \\ &= 0 \text{ if } x \in \mathbb{R} - \mathbb{Q}. \end{aligned}$$

Let $\epsilon > 0$. Then $\mid f(x) - f(0) \mid < \epsilon$ for all x in $(0-\epsilon, 0+\epsilon)$.

Therefore f is continuous at 0.

Let $c \in \mathbb{R}$ and $c \neq 0$.

Let us take a sequence of rational numbers $\{x_n\}$ such that $x_n > c$ for all $n \in \mathbb{N}$ and $\lim x_n = c$. Then $\lim f(x_n) = \lim x_n = c$.

Let us take a sequence of irrational numbers $\{y_n\}$ such that $y_n > c$ for all $n \in \mathbb{N}$ and $\lim y_n = c$. Then $\lim f(y_n) = 0$.

$\lim\limits_{x \to c+} f(x)$ does not exist since for two different sequences $\{x_n\}$ and $\{y_n\}$ in (c, ∞) both converging to c, the sequences $\{f(x_n)\}$ and $\{f(y_n)\}$ converge to two different limits.

By similar arguments, $\lim\limits_{x \to c-} f(x)$ does not exist.

It follows that f is discontinuous at c and it is a discontinuity of the second kind.

3. Find the points of discontinuities of the function f defined by

$$f(x) = \lim_{n \to \infty} \frac{(1+\sin \pi x)^n - 1}{(1+\sin \pi x)^n + 1}, x \in \mathbb{R}.$$

Case I. Let x be an integer.

Then $\sin \pi x = 0$ and therefore $f(x) = 0$.

Case II. Let $2m < x < 2m+1, m$ being an integer.

Then $2m\pi < \pi x < (2m+1)\pi$.

Therefore $0 < \sin \pi x < 1$ and $1 < 1 + \sin \pi x < 2$.

$$f(x) = \lim_{n \to \infty} \frac{1 - \frac{1}{(1+\sin \pi x)^n}}{1 + \frac{1}{(1+\sin \pi x)^n}} = 1, \text{ since } \lim_{n \to \infty} (1+\sin \pi x)^n = \infty.$$

Case III. Let $2m - 1 < x < 2m, m$ being an integer.

Then $(2m-1)\pi < \pi x < 2m\pi$.

Therefore $-1 < \sin \pi x < 0$ and $0 < 1 + \sin \pi x < 1$.

$f(x) = \lim\limits_{n \to \infty} \dfrac{(1+\sin \pi x)^n - 1}{(1+\sin \pi x)^n + 1} = -1$, since $\lim\limits_{n\to\infty}(1+\sin \pi x)^n = 0$.

$$\begin{aligned} \text{Thus } f(x) &= 0 \text{ if } x = 0, \pm 1, \pm 2, \cdots \\ &= 1 \text{ if } 2m < x < 2m+1, m \text{ being an integer} \\ &= -1 \text{ if } 2m-1 < x < 2m, m \text{ being an integer.} \end{aligned}$$

Let us examine continuity of f at $x = 2m, m$ being an integer.

$\lim\limits_{x \to 2m-} f(x) = -1$, $\lim\limits_{x \to 2m+} f(x) = 1$ and $f(2m) = 0$.

Therefore f is discontinuous at $2m$.

Let us examine continuity of f at $x = 2m-1, m$ being an integer.

$\lim\limits_{x \to (2m-1)-} f(x) = 1$, $\lim\limits_{x \to (2m-1)+} f(x) = 1$ and $f(2m-1) = 0$.

Therefore f is discontinuous at $2m-1$.

Clearly, f is continuous at c if c be not an integer.

Thus f is discontinuous at $c \in \mathbb{R}$ when c is an integer.

Note. Each point of discontinuity is a point of jump discontinuity.

4. Find the points of discontinuity of the function f defined by

$$f(x) = \lim_{n\to\infty} [\lim_{t \to 0} \frac{\sin^2(n!\pi x)}{\sin^2(n!\pi x) + t^2}], x \in \mathbb{R}.$$

Case I. Let x be rational.

Then by taking n sufficiently large, $n!x$ can be made an integer, so that $\sin(n!\pi x) = 0$.

Therefore $f(x) = \lim\limits_{t\to 0} \dfrac{0}{0+t^2} = 0$.

Case II. Let x be irrational. Then $0 < \sin^2(n!\pi x) < 1$.

Therefore $f(x) = \lim\limits_{n\to\infty} [\lim\limits_{t\to 0} \dfrac{1}{1 + \frac{t^2}{\sin^2(n!\pi x)}}] = 1$.

Thus f is defined by $f(x) = 0$ if x be rational
$= 1$ if x be irrational.

f is discontinuous at all points in $\mathbb{R}$.

5. A function is defined on $[0,1]$ by $f(0) = 1$ and

$$\begin{aligned} f(x) &= 0, \text{ if } x \text{ be irrational} \\ &= \tfrac{1}{n}, \text{ if } x = \tfrac{m}{n} \text{ where } m, n \text{ are positive integers prime to each other.} \end{aligned}$$

Prove that f is continuous at every irrational point in $[0,1]$ and discontinuous at every rational point in $[0,1]$.

Let $a \in [0,1]$ be rational. Let $\{x_n\}$ be a sequence of irrational points such that $x_n \in (0,1)$ for all $n \in \mathbb{N}$ and $\lim x_n = a$. Then $\lim f(x_n) = 0$.

But $f(a) > 0$. So f is discontinuous at a.

Let $\alpha \in (0,1)$ be irrational. Let us choose a positive ϵ. There is a natural number k such that $0 < \frac{1}{k} < \epsilon$ (by Archimedean property).

In $(0,1)$ there are only a finite number of rational points $\frac{m}{n}$ with n less than k.

Hence there exists a positive δ such that the neighbourhood $(\alpha - \delta, \alpha + \delta) \subset (0,1)$ contains no rational point $\frac{m}{n}$ with n less than k.

Therefore for all $x \in N(\alpha, \delta) \subset (0,1)$,
$| f(x) - f(\alpha) | = 0$, if x be irrational
$| f(x) - f(\alpha) | = \frac{1}{n} \leq \frac{1}{k} < \epsilon$, if $x = \frac{m}{n}$.

Thus $| f(x) - f(\alpha) | < \epsilon$ for all $x \in N(\alpha, \delta) \subset (0,1)$.

This proves that f is continuous at α.

Thus f is continuous at every irrational point in $[0,1]$ and discontinuous at every rational point in $[0,1]$.

Exercises 12

1. Give an example of functions f and g which are not continuous at a point $c \in \mathbb{R}$ but the sum $f + g$ is continuous at c.

2. Give an example of functions f and g which are not continuous at a point $c \in \mathbb{R}$ but the product fg is continuous at c.

3. Let $f(x) = \text{sgn } x$, $g(x) = x(1 - x^2)$.
Show that the composite function $g \circ f$ is continuous at 0.

Note. Here f is discontinuous at 0 and g is continuous at $f(0)$, but still the composite gf is continuous at a 0. The converse implication of the theorem 8.1.10 is not true.

4. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at one point $c \in \mathbb{R}$, prove that f is continuous at every point in $\mathbb{R}$.

[**Hint.** Let f be continuous at c. Then $\lim_{h \to 0} f(c+h) = f(c)$, i.e., $\lim_{h \to 0} f(h) = 0$.]

5. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at $x = 0$, prove that f is continuous on $\mathbb{R}$.

6. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $f(x) = 0$ for all $x \in \mathbb{Q}$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

[**Hint.** Let $c \in \mathbb{R}$. Consider a sequence of rational points $\{c_n\}$ converging to c. Use sequential criterion for continuity.]

7. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $f(\frac{x+y}{2}) = \frac{f(x)+f(y)}{2}$ for all $x, y \in \mathbb{R}$. Prove that $f(x) = ax + b, (a, b \in \mathbb{R})$ for all $x \in \mathbb{R}$.

[**Hint.** $f(x) = \frac{1}{2}[f(2x)+f(0)], f(x)+f(y) = \frac{1}{2}[f(2x)+f(2y)]+f(0) = f(x+y)+f(0)$. Let $\phi(x) = f(x) - f(0)$. Then ϕ is continuous on $\mathbb{R}$ and $\phi(x + y) = \phi(x) + \phi(y)$. Worked Ex.5, page 248.]

8. Let $f : (-1, 1) \to \mathbb{R}$ be continuous at 0. If $f(x) = f(x^2)$ for all $x \in (-1, 1)$, prove that $f(x) = f(0)$ for all $x \in (-1, 1)$.

[**Hint.** Let $c \in (-1, 1)$. Consider the sequence $\{c^{2^n}\}$ in $(-1, 1)$ converging to 0. Use sequential criterion for continuity.]

9. Prove that the function f is continuous on the indicated interval.

(i) $f(x) = e^{\sqrt{x}}, x \in [0, \infty)$; (ii) $f(x) = \log \sin x, x \in (0, \pi)$.

10. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by
$$\begin{aligned} f(x) &= 2x, x \in \mathbb{Q} \\ &= 1 - x, x \in \mathbb{R} - \mathbb{Q}. \end{aligned}$$
Prove that f is continuous at $\frac{1}{3}$ and discontinuous at every other point.

11. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by
$$\begin{aligned} f(x) &= x^2 + 1, x \in \mathbb{Q} \\ &= x, x \in \mathbb{R} - \mathbb{Q}. \end{aligned}$$
Prove that f has a discontinuity of the second kind at every point c in $\mathbb{R}$.

12. Find the points of discontinuity of the functions.

(i) $f(x) = [\sin x], x \in \mathbb{R}$; (ii) $f(x) = (-1)^{[x]}, x \in \mathbb{R}$;

(iii) $f(x) = [x] + [-x], x \in \mathbb{R}$; (iv) $f(x) = \lim\limits_{n\to\infty} \dfrac{(1 + \sin\frac{\pi}{x})^n - 1}{(1 + \sin\frac{\pi}{x})^n + 1}, x \in (0, 1)$;

(v) $f(x) = \lim\limits_{n\to\infty} \dfrac{\log(2 + x) - x^{2n} \sin x}{1 + x^{2n}}, x \in \mathbb{R}$.

13. Examine the nature of discontinuity of f at 0.

(i) $\begin{aligned} f(x) &= \tfrac{1}{\sqrt{x}}, x > 0, \\ &= 0, x = 0. \end{aligned}$ (ii) $\begin{aligned} f(x) &= \tfrac{\sin x}{x}, x \neq 0 \\ &= 0, x = 0. \end{aligned}$

(iii) $\begin{aligned} f(x) &= \log \sin x, 0 < x < \pi \\ &= 0, x = 0. \end{aligned}$ (iv) $f(x) = [x] + [1 - x]$.

(v) $\begin{aligned} f(x) &= \cos \tfrac{1}{x}, x \neq 0 \\ &= 0, x = 0. \end{aligned}$ (vi) $\begin{aligned} f(x) &= \tfrac{1}{x} \sin \tfrac{1}{x}, x \neq 0 \\ &= 0, x = 0. \end{aligned}$

(vii) $\begin{aligned} f(x) &= \tfrac{e^{1/x}-1}{e^{1/x}+1}, x \neq 0 \\ &= 0, x = 0. \end{aligned}$ (viii) $\begin{aligned} f(x) &= 1 + 2^{1/x}, x \neq 0 \\ &= 0, x = 0. \end{aligned}$

14. A function f is said to be *piecewise continuous* on an interval I if f be continuous on I except at a finite number of points of jump discontinuity.

Show that f is piecewise continuous on the indicated interval I.

(i) $f(x) = [x]$, $I = [0, 3]$; (ii) $f(x) = x - [x]$, $I = [0, 3]$;

(iii) $f(x) = [2x]$, $I = [0, 3]$; (iv) $f(x) = sgn\ x$, $I = [-2, 2]$.

8.5. Properties of continuous functions.

Theorem 8.5.1. (Neighbourhood property)

Let $D \subset \mathbb{R}$ and a function $f : D \to \mathbb{R}$ be continuous on D. Let $c \in D$. If $f(c) \neq 0$ then there exists a suitable $\delta > 0$ such that for all $x \in N(c, \delta) \cap D$, $f(x)$ keeps the same sign as $f(c)$.

Proof. **Case 1.** $f(c) > 0$. Let us choose a positive ϵ such that $f(c) - \epsilon > 0$.
Since f is continuous at c, there exists a positive δ such that
$| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$.
or, $f(c) - \epsilon < f(x) < f(c) + \epsilon$ for all $x \in N(c, \delta) \cap D$.
Therefore $f(x) > f(c) - \epsilon > 0$ for all $x \in N(c, \delta) \cap D$.

Case 2. $f(c) < 0$. Let us choose a positive ϵ such that $f(c) + \epsilon < 0$.
Since f is continuous at c, there exists a positive δ such that
$| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$.
or, $f(c) - \epsilon < f(x) < f(c) + \epsilon$ for all $x \in N(c, \delta) \cap D$.
Therefore $f(x) < f(c) + \epsilon < 0$ for all $x \in N(c, \delta) \cap D$.

So in any case $f(x)$ keeps the same sign as $f(c)$ for all $x \in N(c, \delta) \cap D$ for some $\delta > 0$.

Note. This property is a *local property* of a continuous function. It is also called the *sign preserving property* of a continuous function.

Corollary. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and $c \in \mathbb{R}$. If $f(c) \neq 0$ then there exists a positive δ such that $f(x)$ keeps the same sign as $f(c)$ for all $x \in N(c, \delta)$.

Worked Examples.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Prove that the set $S = \{x \in \mathbb{R} : f(x) > 0\}$ is an open set in $\mathbb{R}$.

Case I. Let $f(x) \leq 0$ for all $x \in \mathbb{R}$. Then $S = \phi$ and S is an open set.

Case II. Let $f(x) > 0$ for all $x \in \mathbb{R}$. Then $S = \mathbb{R}$ and S is an open set.

Case III. Let S be a proper subset of $\mathbb{R}$.

Let $c \in S$. Then $f(c) > 0$. Since f is continuous on $\mathbb{R}$ and $f(c) > 0$, by the neighbourhood property there exists a positive δ such that for all $x \in N(c, \delta)$, $f(x) > 0$. Therefore $N(c, \delta) \subset S$.

Thus $c \in S \Rightarrow N(c, \delta) \subset S$.

This shows that c is an interior point of S.

Since c is arbitrary, every point of S is an interior point of S and therefore S is an open set.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Prove that the set $S = \{x \in \mathbb{R} : f(x) < 0\}$ is an open set in $\mathbb{R}$.

Similar proof as in Example 1.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Prove that the set $S = \{x \in \mathbb{R} : f(x) \neq 0\}$ is an open set in $\mathbb{R}$.

Similar proof as in Example 1.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Prove that the set $S = \{x \in \mathbb{R} : f(x) = 0\}$ is a closed set in $\mathbb{R}$.

Let $P = \{x \in \mathbb{R} : f(x) > 0\}$, $T = \{x \in \mathbb{R} : f(x) < 0\}$. Then $P \cup T \cup S = \mathbb{R}$.

But $P \cup T$ is an open set, since P and T are open sets in $\mathbb{R}$ by Examples 1 and 2.

Therefore S being the complement of an open set in $\mathbb{R}$, is a closed set.

Theorem 8.5.2. Let $I = [a, b]$ be a closed and bounded inteval and a function $f : I \to \mathbb{R}$ be continuous on I. Then f is bounded on I.

Proof. If possible, let f be not bounded on I. Then for each natural number n, there exists a point $x_n \in [a, b]$ such that $|f(x_n)| > n$. Thus we obtain a sequence $\{x_n\}$ such that $x_n \in [a, b]$ for all $n \in \mathbb{N}$ and $| f(x_n) |> n$ for all $n \in \mathbb{N}$.

Since $[a, b]$ is a bounded interval, the sequence $\{x_n\}$ is bounded.

By Bolzano-Weierstrass theorem, there is a convergent subsequence $\{x_{r_n}\}$ converging to l, say.

Since $[a, b]$ is a closed set and each element of the convergent sequence $\{x_{r_n}\}$ belongs to $[a, b]$, the limit l belongs to $[a, b]$.

Since $l \in [a, b], f$ is continuous at l. Since the sequence $\{x_{r_n}\}$ converges to l and f is continuous at l, the sequence $\{f(x_{r_n}\}$ must converge to $f(l)$, by the sequential criterion for continuity.

Therefore the sequence $\{f(x_{r_n})\}$ must be bounded. But by construction, $| f(x_{r_n}) |> r_n$ and since $\{r_n\}$ is a strictly increasing sequence of natural numbers, $r_n \geq n$.

So $| f(x_{r_n}) |> n$ and this implies that the sequence $\{f(x_{r_n})\}$ is not bounded. Thus we arrive at a contradiction.

Therefore f is bounded on I and the theorem is proved.

Another Proof.

Let $c \in [a, b]$. Then f is continuous at c.

Let $\epsilon > 0$. Then there exists a positive δ such that $|f(x) - f(c)| < \epsilon$ for all $x \in (c - \delta, c + \delta) \cap [a, b]$.

But $|f(x)| - |f(c)| \leq |f(x) - f(c)|$ and this gives $|f(x)| < |f(c)| + \epsilon$

for all $x \in (c-\delta, c+\delta) \cap [a,b]$. This shows that f is bounded on $(c-\delta, c+\delta) \cap [a,b]$.

Thus for every $x \in [a,b]$ there exists an open interval $I_x = (x-\delta_x, x+\delta_x)$ such that f is bounded on $I_x \cap [a,b]$.

The set of all open intervals $\{I_x : x \in [a,b]\}$ forms an open cover of the closed and bounded interval $[a,b]$.

By Heine-Borel theorem, there exists a finite number of these open intervals, say $I_{x_1}, I_{x_2}, \ldots, I_{x_m}$, such that $[a,b] \subset I_{x_1} \cup I_{x_2} \cup \cdots \cup I_{x_m}$.

For each $i = 1, 2, \ldots, m$, f is bounded on $I_i \cap [a,b]$ and therefore there exists a positive real number M_i such that $|f(x)| \leq M_i$ for all $x \in I_i \cap [a,b]$.

Let $M = \max\{M_1, M_2, \ldots, M_m\}$. Then $|f(x)| \leq M$ for all $x \in (I_{x_1} \cup I_{x_2} \cup \cdots \cup I_{x_m}) \cap [a,b]$, i.e., for all $x \in [a,b]$.

Therefore f is bounded on $[a,b]$ and the proof is complete.

Note. Since f is bounded on $[a,b]$, the set $\{f(x) : x \in [a,b]\}$ is a non-empty bounded subset of $\mathbb{R}$. Therefore there exist real numbers M, m such that $M = \sup\limits_{x \in [a,b]} f(x), m = \inf\limits_{x \in [a,b]} f(x)$.

Theorem 8.5.3. Let $I = [a,b]$ be a closed and bounded interval and a function $f : I \to \mathbb{R}$ be continuous on I. Then there is a point c in I such that $f(c) = \sup\limits_{x \in I} f(x)$ and also there is a point d in I such that $f(d) = \inf\limits_{x \in I} f(x)$.

Proof. The set $f(I) = \{f(x) : x \in I\}$ is a bounded set. Since this is a non-empty bounded subset of $\mathbb{R}$, $\sup f(I)$ and $\inf f(I)$ exist.

Let $M = \sup f(I)$. Then
there exists a point $x_1 \in I$ such that $M - 1 < f(x_1) \leq M$,
there exists a point $x_2 \in I$ such that $M - \frac{1}{2} < f(x_2) \leq M$,
there exists a point $x_3 \in I$ such that $M - \frac{1}{3} < f(x_3) \leq M$,
... ...

We obtain a sequence of points $\{x_n\}$ in I such that
$M - \frac{1}{n} < f(x_n) \leq M$ for all $n \in \mathbb{N}$.

Since I is bounded, the sequence $\{x_n\}$ is bounded and therefore there exists a convergent subsequence $\{x_{r_n}\}$ that converges to a limit c, say.

Since I is a closed set and the elements of the convergent sequence $\{x_{r_n}\}$ belong to I, the limit $c \in I$. Therefore f is continuous at c.

Since $\{x_{r_n}\}$ is a sequence in I converging to c and f is continuous at c, the sequence $\{f(x_{r_n})\}$ converges to $f(c)$.

Now $M - \frac{1}{r_n} < f(x_{r_n}) \leq M$ for all $n \in \mathbb{N}$ and $\lim(M - \frac{1}{r_n}) = M$.

By Sandwich theorem, $\lim f(x_{r_n}) = M$. That is, $f(c) = M$.

Taking $m = \inf f(I)$ it can be proved in a similar manner that there exists a point d in I such that $f(d) = m$.

Another Proof.

The set $f(I) = \{f(x) : x \in I\}$ is a bounded set. Since this is a non-empty bounded subset of $\mathbb{R}$, $\sup f(I)$ and $\inf f(I)$ exist.

Let $M = \sup f(I), m = \inf f(I)$. Then for all $x \in [a,b], m \leq f(x) \leq M$.

We shall prove that $f(x) = M$ for some $c \in [a,b]$.

If not, let $f(x) < M$ for all $x \in [a,b]$. Then $M - f(x) > 0$ for all $x \in [a,b]$.

Let $\phi(x) = \frac{1}{M-f(x)}, x \in [a,b]$. Then ϕ is continuous on $[a,b]$ and therefore ϕ is bounded on $[a,b]$.

Let B be an upper bound of ϕ on $[a,b]$. Then $B > 0$ and $0 < \frac{1}{M-f(x)} < B$ for all $x \in [a,b]$.

Therefore $f(x) < M - \frac{1}{B}$ for all $x \in [a,b]$. This contradicts that $M = \sup f(I)$.

Hence there exists a point c in $[a,b]$ such that $f(c) = M$.

In a similar manner it can be proved that there exists a point d in $[a,b]$ such that $f(d) = m$.

This completes the proof.

Note 1. $M(= \sup\limits_{x\in[a,b]} f(x))$ is called the *global maximum* of the function f on $[a,b]$. $m(= \inf\limits_{x\in[a,b]} f(x))$ is called the *global minimum* of the function f on $[a,b]$.

Note 2. A function f continuous on a bounded open interval I may not be bounded on I. Even if it is bounded on I, it may not attain the supremum or the infimum of f at some point of I.

For example, let $I = (0,1)$ and $f : I \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}, x \in (0,1)$. Then f is continuous on I and f is not bounded on I.

Let $I = (2,3)$ and $f : I \to \mathbb{R}$ be defined by $f(x) = x^2, x \in (2,3)$. Then f is continuous on I and f is bounded on I. $\sup\limits_{x\in I} f(x) = 9$, $\inf\limits_{x\in I} f(x) = 4$. But there is no point $c \in I$ such that $f(c) = 9$ and there is no point $d \in I$ such that $f(d) = 4$.

Note 3. A function f continuous on a closed interval I may not be bounded on I.

For example, let $f : [0,\infty) \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}, x \geq 0$. f is continuous on $[0,\infty)$ but f is not bounded on $[0,\infty)$.

Theorem 8.5.4. (Bolzano)

Let $[a, b]$ be a closed and bounded interval and $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If $f(a)$ and $f(b)$ are of opposite signs then there exists at least a point c in the open interval (a, b) such that $f(c) = 0$.

Proof. Let $I_1 = [a, b] = [a_1, b_1]$, say. Without loss of generality, let us assume that $f(a_1) < 0$ and $f(b_1) > 0$.

Let $c_1 = \frac{a_1+b_1}{2}$. Then $f(c_1)$ is either 0 or $\neq 0$. If $f(c_1) = 0$ the theorem is proved. If $f(c_1) \neq 0$ then either $f(c_1) < 0$, or $f(c_1) > 0$.

If $f(c_1) < 0$ we consider the closed interval $[c_1, b_1]$ and call it $I_2 = [a_2, b_2]$. If $f(c_1) > 0$ we consider the closed interval $[a_1, c_1]$ and call it $I_2 = [a_2, b_2]$.

Thus if $f(c_1) \neq 0$, the closed interval $I_2 = [a_2, b_2]$ is such that

(i) f is continuous on I_2 and $f(a_2) < 0, f(b_2) > 0$;

(ii) $I_2 \subset I_1$;

(iii) $| I_2 |= \frac{1}{2}(b - a)$.

Let $c_2 = \frac{a_2+b_2}{2}$. Then $f(c_2)$ is either 0 or $\neq 0$.

If $f(c_2) = 0$ the theorem is proved. If $f(c_2) \neq 0$, then either $f(c_2) < 0$, or $f(c_2) > 0$.

If $f(c_2) < 0$ we consider the closed interval $[c_2, b_2]$ and call it $I_3 = [a_3, b_3]$. If $f(c_2) > 0$ we consider the closed interval $[a_2, c_2]$ and call it $I_3 = [a_3, b_3]$.

Thus if $f(c_2) \neq 0$ the closed interval $I_3 = [a_3, b_3]$ is such that

(i) f is continuous on I_3 and $f(a_3) < 0, f(b_3) > 0$;

(ii) $I_3 \subset I_2 \subset I_1$;

(iii) $| I_3 |= \frac{b-a}{2^2}$.

Let $c_3 = \frac{a_3+b_3}{2}$. Then $f(c_2)$ is either 0 or $\neq 0$.

Continuing in this manner, either we obtain a point c_k in (a, b) such that $f(c_k) = 0$, in which case the theorem is proved, or we obtain a sequence of closed and bounded intervals $\{I_n\}$ such that

(i) f is continuous on I_n and $f(a_n) < 0, f(b_n) > 0$ for all $n \in \mathbb{N}$;

(ii) $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$;

(iii) $| I_n |= \frac{b-a}{2^{n-1}}$ and therefore $\lim | I_n |= 0$.

Thus $\{I_n\}$ is a sequence of nested closed and bounded intervals with $\lim | I_n |= 0$.

By *nested intervals theorem* there exists one and only one point α such that (i) $\alpha \in [a_n, b_n]$ for all $n \in \mathbb{N}$, and (ii) $\lim a_n = \alpha = \lim b_n$.

From (i) it follows that $\alpha \in [a, b]$. Since f is continuous on $[a, b]$, f is continuous at α.

From (ii) it follows that $\{a_n\}$ is a sequence of points in $[a, b]$ converging to α. Since f is continuous at α, $\lim f(a_n) = f(\alpha)$.

But $f(a_n) < 0$ for all $n \in \mathbb{N}$ and this implies $\lim f(a_n) \leq 0$.

That is, $f(\alpha) \leq 0$ ... ... (A)

Also from (ii) it follows that $\{b_n\}$ is a sequence of points in $[a, b]$ converging to α. Since f is continuous at α, $\lim f(b_n) = f(\alpha)$.

But $f(b_n) > 0$ for all $n \in \mathbb{N}$ and this implies $\lim f(b_n) \geq 0$.

That is, $f(\alpha) \geq 0$ ... ... (B)

From (A) and (B) it follows that $f(\alpha) = 0$.

Since $\alpha \in [a, b]$ and $f(a) < 0, f(b) > 0$ it follows that $\alpha \in (a, b)$.

Thus $\alpha = c$ and the theorem is proved.

Theorem 8.5.5. (Intermediate value theorem)

Let $[a, b]$ be a closed an bounded interval and a function $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If $f(a) \neq f(b)$ then f attains every value between $f(a)$ and $f(b)$ at least once in the open interval (a, b).

Proof. Without loss of generality, we assume that $f(a) < f(b)$.

Let μ be a real number such that $f(a) < \mu < f(b)$.

Let us consider the function $\phi : [a, b] \to \mathbb{R}$ defined by $\phi(x) = f(x) - \mu, x \in [a, b]$.

ϕ is continuous on $[a, b]$, since f is continuous on $[a, b]$.

$\phi(a) = f(a) - \mu < 0, \phi(b) = f(b) - \mu > 0$.

As $\phi(a)$ and $\phi(b)$ are of opposite signs, by Bolzano's theorem there exists at least one point c in (a, b) such that $\phi(c) = 0$.

Therefore $f(c) - \mu = 0$, i.e., $f(c) = \mu$.

Thus f attains μ at a point c in (a, b) and the theorem is done.

Note. Let $I = [a, b]$ be a closed and bounded interval and $f : [a, b] \to \mathbb{R}$ be such that $f(a) \neq f(b)$ and f attains every value between $f(a)$ and $f(b)$ at least once in (a, b). Still f may not be continuous on $[a, b]$.

For example, let $f : [0, 2] \to \mathbb{R}$ be defined by $f(0) = 0, f(2) = 2$ and

$$\begin{aligned} f(x) &= x, 0 < x \leq 1 \\ &= 3 - x, 1 < x < 2. \end{aligned}$$

f assumes every value between 0 and 2 on $[0, 2]$. But f is not continuous on $[0, 2]$ since f is not continuous at 1 and 2.

Theorem 8.5.6. Let $[a, b]$ be a closed and bounded interval and a function $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If $\sup_{x\in[a,b]} f(x) \neq \inf_{x\in[a,b]} f(x)$ and μ be a real number lying between $\sup_{x\in[a,b]} f(x)$ and $\inf_{x\in[a,b]} f(x)$ then there is a point p in (a, b) such that $f(p) = \mu$.

Proof. Since f is continuous on the closed and bounded interval $[a, b]$, there is a point c in $[a, b]$ such that $f(c) = \sup_{x\in[a,b]} f(x)$ and there is a point d in $[a, b]$ such that $f(d) = \inf_{x\in[a,b]} f(x)$.

Without loss of generality, let us assume that $c < d$.

Therefore $[c, d] \subset [a, b]$ and f is continuous on $[c, d]$.

Since μ lies between $f(c)$ and $f(d)$, by the intermediate value theorem there is a point p in (c, d) such that $f(p) = \mu$.

Therefore $p \in (a, b)$ and the theorem is proved.

Theorem 8.5.7. Let $I = [a, b]$ be a closed and bounded interval and $f : [a, b] \to \mathbb{R}$ be continuous on I. Then $f(I) = \{f(x) : x \in I\}$ is a closed and bounded interval.

Proof. Since f is continuous on I, f is bounded on I.

Let $M = \sup_{x\in I} f(x), m = \inf_{x\in I} f(x)$. Then there is a point c in I such that $f(c) = M$ and a point d in I such that $f(d) = m$.

Therefore $M \in f(I), m \in f(I)$ and $m \leq M$.

Case 1. $m = M$. In this case f is a constant and $f(I)$ reduces to the point m and $f(I)$ is the closed interval $[m, m]$.

Case 2. $m < M$. Let $J = [m, M]$. We prove that $J = f(I)$.

Let $p \in f(I)$. Then there is a point x_0 in I such that $f(x_0) = p$.

Since $M = \sup_{x\in I} f(x)$, and $m = \inf_{x\in I} f(x), m \leq f(x_0) \leq M$.

Thus $p \in f(I) \Rightarrow p \in [m, M]$ and therefore $f(I) \subset J$ ... ... (i)

Let $q \in J$ and $q \neq m, q \neq M$.

Since f is continuous on (c, d) (or (d, c)) and $f(d) < q < f(c)$, there is a point x_1 in (c, d) (or (d, c)) such that $f(x_1) = q$.

Therefore $q \in f(I)$. Also $m \in f(I)$ and $M \in f(I)$.

Thus $x \in J$ implies $x \in f(I)$ and therefore $J \subset f(I)$ ... ... (ii)

From (i) and (ii) it follows that $J = f(I)$. This completes the proof.

Note 1. The continuous image of a closed and bounded interval $[a, b]$ is the closed and bounded interval $[m, M]$. In particular, if f be a constant (and hence continuous) on $[a, b]$, the image reduces to a point.

Note 2. The continuous image of an open bounded interval may not be an open bounded interval. For example, let $I = (-1, 1)$ and $f(x) = x^2, x \in I$. Then f is continuous on I but $f(I) = [0, 1)$, which is not an open interval. Let $I = (0, 1)$ and $f(x) = x^2, x \in I$. Then $f(I) = (0, 1)$, an open interval.

Theorem 8.5.8. Let I be an interval and $f : I \to \mathbb{R}$ be continuous on I. Then $f(I)$ is an interval.

[A subset S of $\mathbb{R}$ is an interval if for any two points $c, d \in S$ with $c < d$, the closed interval $[c, d] \subset S$]

Proof. Let $p, q \in f(I)$ and $p < q$. There exist points c, d in I such that $f(c) = p, f(d) = q$.

Let $r \in (p, q)$. Then $p < r < q$.

By the intermediate value theorem, there exists a point x_0 in (c, d) [or (d, c)] such that $f(x_0) = r$.

Thus $r \in (p, q) \Rightarrow r \in f(I)$ and therefore $(p, q) \subset f(I)$.

Also $p \in f(I)$ and $q \in f(I)$. So $[p, q] \subset f(I)$.

This proves that $f(I)$ is an interval.

Worked Examples (continued).

5. A function $f : [0, 1] \to [0, 1]$ is continuous on $[0, 1]$. Prove that there exists a point c in $[0, 1]$ such that $f(c) = c$.

If $f(0) = 0$ or $f(1) = 1$, the existence is proved.

We assume $f(0) \neq 0$ and $f(1) \neq 1$.

Let us consider the function $g : [0, 1] \to \mathbb{R}$ defined by $g(x) = f(x) - x, x \in [0, 1]$.

g is continuous on $[0, 1]$ and $g(0) = f(0) > 0$, since $f(0) \in [0, 1]$ and $f(0) \neq 0$. Also $g(1) = f(1) - 1 < 0$, since $f(1) \in [0, 1]$ and $f(1) \neq 1$.

By the Intermediate value theorem there exists a point c in $(0, 1)$ such that $g(c) = 0$. Therefore $f(c) = c$.

Note. c is said to be a *fixed point* of the continuous map f.

6. A function $f : [0, 1] \to \mathbb{R}$ is continuous on $[0, 1]$ and assumes only rational values. If $f(\frac{1}{2}) = \frac{1}{2}$, prove that $f(x) = \frac{1}{2}$ for all $x \in [0, 1]$.

Let us take a point x_1 such that $0 \leq x_1 < \frac{1}{2}$ and consider the closed interval $[x_1, \frac{1}{2}]$. f is continuous on $[x_1, \frac{1}{2}]$.

Let $f(x_1) = p$. Then p is rational.

We prove that $p = \frac{1}{2}$. If not, let $p \neq \frac{1}{2}$.

Then f is continuous on $[x_1, \frac{1}{2}]$ and $f(x_1) \neq f(\frac{1}{2})$.

Let q be an irrational number lying between p and $\frac{1}{2}$.

By the Intermediate value theorem, $f(x) = q$ at some point c in $(x_1, \frac{1}{2})$ and this is a contradiction to the hypothesis that f assumes only rational values.

Hence $p = \frac{1}{2}$ and therefore $f(x) = \frac{1}{2}$ in $[0, \frac{1}{2})$.

Let us take a point x_2 such that $\frac{1}{2} < x_2 \leq 1$ and consider the closed interval $[\frac{1}{2}, x_2]$.

Proceeding with similar arguments we can prove $f(x) = \frac{1}{2}$ in $(\frac{1}{2}, 1]$.

Also $f(\frac{1}{2}) = \frac{1}{2}$. It follows that $f(x) = \frac{1}{2}$ for all $x \in [0, 1]$.

Theorem 8.5.9. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Then for every open subset G of $\mathbb{R}$, $f^{-1}(G)$ is open in $\mathbb{R}$.

Proof. If $f^{-1}(G)$ be empty, then it is open in $\mathbb{R}$.

Let $f^{-1}(G)$ be non-empty and let $c \in f^{-1}(G)$. Then $f(c) \in G$.

Since G is an open set and $f(c) \in G$, $f(c)$ is an interior point of G and so there exists a positive ϵ such that $N(f(c), \epsilon) \subset G$.

Since f is continuous at c, there exists a $\delta > 0$ such that for all $x \in N(c, \delta), f(x) \in N(f(c), \epsilon) \subset G$.

This implies $N(c, \delta) \subset f^{-1}(G)$. Hence c is an interior point of $f^{-1}(G)$.

Thus every point of $f^{-1}(G)$ is an interior point of $f^{-1}(G)$ and therefore $f^{-1}(G)$ is an open set.

Note. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ implies that the inverse image of any open subset is open in $\mathbb{R}$. The converse implication is also true.

We have the following theorem in this respect.

Theorem 8.5.10. A function $f : \mathbb{R} \to \mathbb{R}$ is such that $f^{-1}(G)$ is open in $\mathbb{R}$ whenever G is open in $\mathbb{R}$. Then f is continuous on $\mathbb{R}$.

Proof. Let $c \in \mathbb{R}$. Then $f(c) \in \mathbb{R}$.

Let $\epsilon > 0$. Then the neighbourhood $G = (f(c) - \epsilon, f(c) + \epsilon)$ is an open set in $\mathbb{R}$ and by hypothesis, $f^{-1}(G)$ is an open set in $\mathbb{R}$.

$c \in f^{-1}(G)$, since $f(c) \in G$. Since $f^{-1}(G)$ is an open set, c is an interior point of $f^{-1}(G)$ and therefore there exists a positive δ such that the neighbourhood $N(c, \delta) \subset f^{-1}(G)$, i.e., for all $x \in N(c, \delta)$, $f(x) \in G$, i.e., $|f(x) - f(c)| < \epsilon$ for all $x \in N(c, \delta)$.

This proves that f is continuous at c. Since c is arbitrary, f is continuous on $\mathbb{R}$.

Note. We observe that a continuous function $f : \mathbb{R} \to \mathbb{R}$ pulls back open sets of $\mathbb{R}$ into open sets of $\mathbb{R}$. But f may not map an open set of $\mathbb{R}$ into an open set of $\mathbb{R}$. For example, let f be a constant function on $\mathbb{R}$, say

$f(x) = 2$ for all $x \in \mathbb{R}$. Then f is continuous on $\mathbb{R}$, but the image of an open set, say $\{x \in \mathbb{R} : 0 < x < 1\}$ is the singleton set $\{2\}$ which is not an open set in $\mathbb{R}$.

Theorem 8.5.11. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ if and only if $f^{-1}(F)$ is closed in $\mathbb{R}$ whenever F is closed in $\mathbb{R}$.

Proof. Let f be continuous on $\mathbb{R}$ and let F be a closed set in $\mathbb{R}$. Then $\mathbb{R} - F$ is open in $\mathbb{R}$ and $f^{-1}(\mathbb{R} - F)$ is open in $\mathbb{R}$, by Theorem 8.5.9.

Since $\mathbb{R} - f^{-1}(F) = f^{-1}(\mathbb{R} - F)$, it follows that $f^{-1}(F)$ is closed in $\mathbb{R}$.

Conversely, let $f^{-1}(F)$ is closed in $\mathbb{R}$ whenever F is closed in $\mathbb{R}$. We shall prove that f is continuous on $\mathbb{R}$.

Let G be an open set in $\mathbb{R}$. Then $\mathbb{R} - G$ is closed in $\mathbb{R}$ and by hypothesis, $f^{-1}(\mathbb{R} - G)$ is closed in $\mathbb{R}$.

Since $\mathbb{R} - f^{-1}(G) = f^{-1}(\mathbb{R} - G)$, it follows that $\mathbb{R} - f^{-1}(G)$ is closed in $\mathbb{R}$ and therefore $f^{-1}(G)$ is open in $\mathbb{R}$.

f is continuous on $\mathbb{R}$, by Theorem 8.5.10.

Theorem 8.5.12. The functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both continuous on $\mathbb{R}$. Then the set $S = \{x \in \mathbb{R} : f(x) < g(x)\}$ is an open set in $\mathbb{R}$.

Proof. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = f(x) - g(x)$, $x \in \mathbb{R}$. Then h is continuous on $\mathbb{R}$ and $S = \{x \in \mathbb{R} : h(x) < 0\}$.

Using the neighbourhood property of the function h and proceeding as in the worked Example 1, Art.8.5, the theorem can be established.

Another proof.

If $S = \phi$ then S is an open set.

Let $S \neq \phi$ and let $a \in S$. Then $f(a) < g(a)$. let us choose a real number b such that $f(a) < b < g(a)$.

Let us consider the open set $I = (b, \infty)$. Since g is continuous on $\mathbb{R}$, $g^{-1}(I)$ is an open set in $\mathbb{R}$. $a \in g^{-1}(I)$ and $g(x) > b$ for all $x \in g^{-1}(I)$.

Let us consider the open set $J = (-\infty, b)$. Since f is continuous on $\mathbb{R}$, $f^{-1}(J)$ is an open set in $\mathbb{R}$. $a \in f^{-1}(J)$ and $f(x) < b$ for all $x \in f^{-1}(J)$.

$f^{-1}(J) \cap g^{-1}(I)$ is an open set in $\mathbb{R}$ containing a and $f(x) < b < g(x)$ for all $x \in f^{-1}(J) \cap g^{-1}(I)$.

Therefore $f^{-1}(J) \cap g^{-1}(I) \subset S$. Thus $a \in S \Rightarrow f^{-1}(J) \cap g^{-1}(I) \subset S$. So a is an interior point of S. Therefore S is an open set.

This completes the proof.

Note. The set $S = \{x \in \mathbb{R} : f(x) \neq g(x)\}$ is an open set in $\mathbb{R}$.

Theorem 8.5.13. The functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both continuous on $\mathbb{R}$. Then the set $S = \{x \in \mathbb{R} : f(x) = g(x)\}$ is a closed set in $\mathbb{R}$.

Proof. Let $A = \{x \in \mathbb{R} : f(x) \neq g(x)\}$. Then A is the complement of S in $\mathbb{R}$.

If $A = \phi$ then A is an open set.

Let $A \neq \phi$ and let $a \in A$. Then $f(a) \neq g(a)$. Let $|f(a) - g(a)| = k > 0$. Let $\epsilon = \frac{k}{3}$. Since f and g are continuous at a, there exists a positive δ such that $|f(x) - f(a)| < \epsilon$ and $|g(x) - g(a)| < \epsilon$ for all $x \in N(a, \delta)$.

$|f(a) - g(a)| \leq |f(x) - f(a)| + |f(x) - g(x)| + |g(x) - g(a)|$
or, $|f(x) - g(x)| \geq \epsilon$ for all $x \in N(a, \delta)$.

Therefore $N(a, \delta) \subset A$. Thus $a \in A \Rightarrow N(a, \delta) \subset A$. Therefore a is an interior point of A. So A is an open set in $\mathbb{R}$.

Consequently, S is a closed set in $\mathbb{R}$.

Corollary 1. If f and g are continuous on $\mathbb{R}$ and $f(x) = g(x)$ at all rational points, then $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Let $S = \{x \in \mathbb{R} : f(x) = g(x)\}$. Then by hypothesis, $\mathbb{Q} \subset S \subset \mathbb{R}$.
S is a closed subset in $\mathbb{R}$, by the theorem.
$\mathbb{Q} \subset S \Rightarrow \bar{\mathbb{Q}} \subset \bar{S} \Rightarrow \mathbb{R} \subset S$, since $\bar{\mathbb{Q}} = \mathbb{R}$ and $\bar{S} = S$.
Therefore $S = \mathbb{R}$, i.e., $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Corollary 2. If f is continuous on $\mathbb{R}$ and $f(x) = k$, a constant, at all rational points, then $f(x) = k$ for all $x \in \mathbb{R}$.

Considering the continuous function g defined by $g(x) = k$ for all $x \in \mathbb{R}$, this can be established.

Alternative proof of Intermediate value theorem.

Let $[a, b]$ be a closed and bounded interval and $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$. If $f(a) \neq f(b)$ then for every real number r lying between $f(a)$ and $f(b)$ there is a point c in (a, b) such that $f(c) = r$.

Proof. Suppose on the contrary, there does not exist a point c in (a, b) such that $f(c) = r$.

Let us define a function $g : \mathbb{R} \to \mathbb{R}$ by
$$\begin{aligned} g(x) &= f(a), x \in (-\infty, a) \\ &= f(x), x \in [a, b] \\ &= f(b), x \in (b, \infty). \end{aligned}$$

Then g is continuous on $\mathbb{R}$ and $g = f$ on $[a, b]$.

Let $G_1 = (-\infty, r)$, $G_2 = (r, \infty)$. Then $\mathbb{R} = G_1 \cup \{r\} \cup G_2$.

G_1 and G_2 are open sets in $\mathbb{R}$. Since g is continuous on $\mathbb{R}$, $g^{-1}(G_1)$ and $g^{-1}(G_2)$ are both open sets in $\mathbb{R}$.

Since $g^{-1}(r) = \phi$, $g^{-1}(G_1) = \mathbb{R} - g^{-1}(G_2)$. Since $g^{-1}(G_2)$ is open, $g^{-1}(G_1)$ is closed. Thus $g^{-1}(G_1)$ is both open and closed in $\mathbb{R}$.

But $g^{-1}(G_1)$ is non-empty, since $a \in g^{-1}(G_1)$ and $g^{-1}(G_1) \neq \mathbb{R}$, since $b \notin g^{-1}(G_1)$.

So $g^{-1}(G_1)$ is neither $\mathbb{R}$ nor ϕ and at the same time $g^{-1}(G_1)$ is both open and closed. This is a contradiction, since the only subsets in $\mathbb{R}$ which are both open and closed are ϕ and $\mathbb{R}$.

Hence our assumption is wrong and the theorem is proved.

Definition. A function $f : [a, b] \to \mathbb{R}$ is said to satisfy the *intermediate-value property* on $[a, b]$ if for every x_1, x_2 satisfying $a \leq x_1 < x_2 \leq b$ and for every k between $f(x_1)$ and $f(x_2)$ there exists a $c \in (x_1, x_2)$ such that $f(c) = k$.

Note. A function $f : [a, b] \to \mathbb{R}$ which satisfies the intermediate-value property on $[a, b]$ need not be continuous on $[a, b]$.

For example, the function f defined on $[-1, 1]$ by $f(x) = \sin \frac{1}{x}, x \neq 0$
$= 0, x = 0$

is discontinuous on $[-1, 1]$ but it satisfies the intermediate-value property on $[-1, 1]$.

Theorem 8.5.14. If a function $f : [a, b] \to \mathbb{R}$ satisfies the intermediate-value property on $[a, b]$ then f has no simple discontinuity on $[a, b]$.

Ist kind,

Proof. Let f has a simple discontinuity at $c \in [a, b]$.

If c be an interior point of $[a, b]$ then $\lim\limits_{x \to c+} f(x)$ and $\lim\limits_{x \to c-} f(x)$ both exist finitely.

If $c = a$ then $\lim\limits_{x \to c+} f(x)$ exists finitely.

If $c = b$ then $\lim\limits_{x \to c-} f(x)$ exists finitely.

It is sufficient to prove that

(i) $\lim\limits_{x \to c+} f(x) = f(c)$ for all $c \in [a, b)$; (ii) $\lim\limits_{x \to c-} f(x) = f(c)$ for all $c \in (a, b]$.

Let $c \in [a, b)$ and let $\lim\limits_{x \to c+} f(x) = l$.

Case 1. Let $l < f(c)$. Let us choose a positive ϵ such that $l + \epsilon < f(c)$.

Since $\lim\limits_{x \to c+} f(x) = l$, there exists a positive δ such that $l - \epsilon < f(x) < l + \epsilon$ for all $x \in (c, c + \delta) \cap [a, b]$.

Therefore $l - \epsilon < f(x) < l + \epsilon < f(c)$ for all $x \in (c, c + \delta) \cap [a, b] \ldots \quad \ldots$(i)

Let $x_1 \in (c, c + \delta) \cap [a, b]$. Then $l - \epsilon < f(x_1) < l + \epsilon < f(c)$, by (i).

Since f satisfies intermediate value property on $[c, x_1] \subset [a, b]$, there exists a point p in (c, x_1) such that $f(p) = l + \epsilon$.

But $p \in (c, x_1) \Rightarrow p \in (c, c+\delta) \cap [a, b] \Rightarrow f(p) < l + \epsilon$, a contradiction. Therefore $l \geq f(c)$.

Case 2. Let $l > f(c)$. Let us choose a positive ϵ such that $l - \epsilon > f(c)$.

Since $\lim\limits_{x \to c+} f(x) = l$, there exists a positive δ such that $l - \epsilon < f(x) < l + \epsilon$ for all $x \in (c, c + \delta) \cap [a, b]$.

Therefore $f(c) < l - \epsilon < f(x) < l + \epsilon$ for all $x \in (c, c + \delta) \cap [a, b]$... ...(ii)

Let $x_2 \in (c, c + \delta) \cap [a, b]$. Then $f(c) < l - \epsilon < f(x_2) < l + \epsilon$, by (ii).

Since f satisfies intermediate value property on $[c, x_2] \subset [a, b]$, there exists a point q in (c, x_2) such that $f(q) = l - \epsilon$.

But $q \in (c, x_2) \Rightarrow q \in (c, c+\delta) \cap [a, b] \Rightarrow l - \epsilon < f(q)$, a contradiction. Therefore $l \leq f(c)$.

Combining the cases, we have $\lim\limits_{x \to c+} f(x) = f(c)$ for all $c \in [a, b)$.

Similarly, it can be proved that $\lim\limits_{x \to c-} f(x) = f(c)$ for all $c \in (a, b]$.

This completes the proof.

8.6. Monotone functions and continuity.

Theorem 8.6.1. Let $I = (a, b)$ be an interval. Let $f : I \to \mathbb{R}$ be monotone increasing on I. Then at any point $c \in I$,

(i) $f(c - 0) = \sup\limits_{x \in (a,c)} f(x)$, **(ii)** $f(c + 0) = \inf\limits_{x \in (c,b)} f(x)$,

(iii) $f(c - 0) \leq f(c) \leq f(c + 0)$.

Proof. (i) If $x \in I$ and $x < c$ then $f(x) \leq f(c)$.

Hence the set $\{f(x) : a < x < c\}$ is bounded above, $f(c)$ being an upper bound. The set, being non-empty, has a least upper bound, say u.

Then $u \leq f(c)$, and for a pre-assigned positive ϵ, there exists a point x_0 in (a, c) such that $u - \epsilon < f(x_0) \leq u$.

Let $x_0 = c - \delta, 0 < \delta < c - a$.

Since f is monotonic increasing on (a, c),

$u - \epsilon < f(x_0) \leq f(x) \leq u < u + \epsilon$ for all x in $x_0 < x < c$.

Consequently, $| f(x) - u | < \epsilon$ for all x in $x_0 < x < c$.

This implies that $\lim\limits_{x \to c-} f(x) = u$, i.e., $f(c - 0) = u = \sup\limits_{x \in (a,c)} f(x)$

(ii) If $x \in I$ and $x > c$, then $f(x) \geq f(c)$.

Hence the set $\{f(x) : c < x < b\}$ is bounded below, $f(c)$ being a lower bound. The set being non-empty, has a greatest lower bound, say l.

Then $l \geq f(c)$, and for a pre-assigned positive ϵ, there exists a point x_1 in (c, b) such that $l \leq f(x_1) < l + \epsilon$.

Let $x_1 = c + \delta, 0 < \delta < b - c$.

Since f is monotone increasing on (c, b), for all x in $c < x < x_1$

$l - \epsilon < l \leq f(x) \leq f(x_1) < l + \epsilon$ for all x in $c < x < x_1$

Consequently, $| f(x) - l |< \epsilon$ for all x in $c < x < x_1 + \delta$

This implies that $\lim\limits_{x \to c+} f(x) = l$, i.e., $f(c + 0) = l = \inf\limits_{x \in (c,b)} f(x)$

(iii) We have $f(c - 0) = u \leq f(c)$ and $f(c + 0) = l \geq f(c)$.

Therefore $f(c - 0) \leq f(c) \leq f(c + 0)$.

Note. If $f : I \to \mathbb{R}$ be monotone decreasing on $I = (a, b)$ then at any point $c \in I$, **(i)** $f(c - 0) = \inf\limits_{x \in (a,c)} f(x)$, **(ii)** $f(c + 0) = \sup\limits_{x \in (c,b)} f(x)$,

(iii) $f(c - 0) \geq f(c) \geq f(c + 0)$.

Corollary 1. If $f : (a, b) \to \mathbb{R}$ be monotone on (a, b) then at every point $c \in (a, b), f(c-0)$ and $f(c+0)$ both exist. Therefore a monotone function f cannot have a discontinuity of the second kind in its domain.

Corollary 2. If f be monotone increasing on $I = [a, b]$ then for any two points $c, d \in I$ with $c < d, f(c + 0) \leq f(d - 0)$.

Theorem 8.6.2. Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be monotone increasing on I. Then **(i)** $f(a + 0) = \inf\limits_{x \in (a,b)} f(x)$, **(ii)** $f(b - 0) = \sup\limits_{x \in (a,b)} f(x)$,

(iii) $f(a) \leq f(a + 0); f(b - 0) \leq f(b)$.

Proof left to the reader.

Theorem 8.6.3. Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be monotone decreasing on I. Then **(i)** $f(a + 0) = \sup\limits_{x \in (a,b)} f(x)$, **(ii)** $f(b - 0) = \inf\limits_{x \in (a,b)} f(x)$,

(iii) $f(a) \geq f(a + 0)$; $f(b - 0) \geq f(b)$.

Proof left to the reader.

Let $f : [a, b] \to \mathbb{R}$ be monotone on $[a, b]$ and $c \in (a, b)$.
Since $f(c - 0)$ and $f(c + 0)$ both exist,
the *jump* of f at c is defined by $J(c) = f(c + 0) - f(c - 0)$.

The jump at a is defined by $J(a) = f(a + 0) - f(a)$.

The jump at b is defined by $J(b) = f(b) - f(b - 0)$.

(i) If f be monotone increasing on I and $c \in I, J(c) \geq 0$.

(ii) If f be monotone decreasing on I and $c \in I, J(c) \leq 0$.

Theorem 8.6.4. If a function $f : [a, b] \to \mathbb{R}$ be monotone on $[a, b]$ then the set of points of discontinuities of f in $[a, b]$ is a countable set.

Proof. First let us consider the case when f is monotone increasing on $[a, b]$. Let S be the set of all points of discontinuity of f. Since f is increasing on $[a, b]$, $J(x) \geq 0$ for all $x \in [a, b]$ and $S = \{x \in [a, b] : J(x) > 0\}$.

Let T be the set of points of discontinuity of f on (a, b).

For any two points c, d in T with $c < d$,

$f(c - 0) < f(c + 0) \leq f(d - 0) < f(d + 0)$.

Thus the open intervals $(f(c - 0), f(c + 0))$ and $(f(d - 0), f(d + 0))$ are disjoint.

Let us choose a rational point r_x in $(f(x - 0), f(x + 0))$ for every $x \in T$ and define a mapping $\phi : T \to \mathbb{Q}$ by $\phi(x) = r_x$ for all $x \in T$.

Then ϕ is injective because, for any two distinct elements x_1, x_2 in T the intervals $(f(x_1 - 0), f(x_1 + 0))$ and $(f(x_2 - 0), f(x_2 + 0))$ are disjoint and therefore $r_{x_1} \neq r_{x_2}$.

Since $\mathbb{Q}$ is an enumerable set, $\phi(T)$ being a proper subset of $\mathbb{Q}$ is countable. Since ϕ is injective and $\phi(T)$ is countable, T is countable. Therefore $T \cup \{a, b\}$ is also a countable set.

Thus S, being a subset of $T \cup \{a, b\}$, is a countable set.

The case when f is monotone decreasing on $[a, b]$ can be similarly dealt with and the theorem is done.

Corollary. Let $f : \mathbb{R} \to \mathbb{R}$ be monotone on $\mathbb{R}$. Then the set of points of discontinuity of f is a countable set.

$\mathbb{R}$ can be considered as the union of an enumerable number of closed intervals $\{[0, 1] \cup [1, 2] \cup \cdots\} \cup \{[-1, 0] \cup [-2, -1] \cup \cdots\}$.

Theorem 8.6.5. If a function $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and injective on $[a, b]$ then f is strictly monotone on $[a, b]$.

Proof. $f(a) \neq f(b)$ since f is injective on $[a, b]$.

Case 1. $f(a) < f(b)$. Let $x_1 \in (a, b)$. We prove that $f(a) < f(x_1) < f(b)$.

If not, then either (i) $f(x_1) < f(a) < f(b)$, or (ii) $f(a) < f(b) < f(x_1)$.

If $f(x_1) < f(a) < f(b)$ then by the intermediate value theorem on the interval $[x_1, b]$, there is a point $x' \in (x_1, b)$ such that $f(x') = f(a)$ and this contradicts that f is injective on $[a, b]$.

If $f(a) < f(b) < f(x_1)$ then by the intermediate value theorem on the interval $[a, x_1]$, there is a point $x'' \in (a, x_1)$ such that $f(x'') = f(b)$ and this again contradicts that f is injective on $[a, b]$.

Therefore $f(a) < f(x_1) < f(b)$ when $a < x_1 < b$.

Let $x_2 \in (a, b)$ such that $a < x_1 < x_2 < b$.
Since $a < x_1 < x_2, f(a) < f(x_1) < f(x_2)$, by what we have proved.
Similarly, since $x_1 < x_2 < b, f(x_1) < f(x_2) < f(b)$.
Therefore $a < x_1 < x_2 < b \Rightarrow f(a) < f(x_1) < f(x_2) < f(b)$.

Hence f is strictly increasing on $[a, b]$.

Case 2. $f(a) > f(b)$. By similar arguments we can prove that f is strictly decreasing on $[a, b]$.

The theorem can be extended to hold on any kind of interval.

Theorem 8.6.5(a). Let I be an interval and $f : I \to \mathbb{R}$ is continuous and injective on I. Then f is strictly monotone on I.

Proof. Let $r, s \in I$ and $r < s$. Since f is injective on I, $f(r) \neq f(s)$.

Case 1. $f(r) < f(s)$.

Let c, d be arbitrary points in I with $c < d$. Let $a = \min\{c, r\}, b = \max\{d, s\}$. Then $[a, b]$ is a closed and bounded interval contained in I and containing the points r, s, c, d.

Since $[a, b] \subset I, f$ is continuous and injective on $[a, b]$ and by the Theorem 8.6.5, f is strictly monotone on $[a, b]$. Since $r, s \in [a, b]$ and $f(r) < f(s)$, f must be strictly increasing on $[a, b]$ and therefore $f(c) < f(d)$. As c, d are arbitrary points in I, f is strictly increasing on I.

Case 2. $f(r) > f(s)$.

In this case we can prove f is strictly decreasing on I.

Theorem 8.6.6. Let $I = [a, b]$ be closed and bounded interval and $f : I \to \mathbb{R}$ be strictly monotone and continuous on I. Then there exists an inverse function $g : J \to \mathbb{R}$ where $J = f(I)$, such that

(i) g is strictly monotone on J and **(ii)** g is continuous on J.

We prove the theorem for the case when f is strictly increasing on I. The proof for the other case (when f is strictly decreasing) is similar.

Proof. f is strictly increasing on I. Since f is continuous on I, f is bounded on I. Since f is strictly increasing on $I, \sup f = f(b)$ and $\inf f = f(a)$. Therefore $J = [f(a), f(b)]$.

Since f is strictly increasing on I, $a \leq x_1 < x_2 \leq b \Rightarrow f(x_1) < f(x_2)$. So $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

This proves that f is injective on I. Also since $f(I) = J, f$ is surjective.

Therefore f is bijective and hence there exists an inverse function $g : J \to I$ such that $x \in I$ and $f(x) = y \Leftrightarrow y \in J$ and $g(y) = x$.

First we prove that g is strictly increasing on J.

Let $y_1, y_2 \in J$ with $y_1 < y_2$. Then there exist x_1, x_2 in I such that $y_1 = f(x_1), y_2 = f(x_2)$ and since f is strictly increasing on $I, y_1 < y_2 \Rightarrow x_1 < x_2$. That is, $y_1 < y_2$ in $J \Rightarrow g(y_1) < g(y_2)$ in I.

This proves that g is strictly increasing on J.

To prove that g is continuous on J, let $d \in J$ and $d = f(c)$. Then $g(d) = c$.

Let $\{y_n\}$ be a sequence in J converging to d.

Let $f(x_n) = y_n$, $n = 1, 2, 3, \ldots$

Then $g(y_n) = x_n$ and $\{x_n\}$ is a sequence in I. I being a bounded set, the sequence $\{x_n\}$ is a bounded sequence and therefore it must have a convergent subsequence.

Let $\{x_{r_n}\}$ be a convergent subsequence of the sequence $\{x_n\}$. I being a closed set, $\lim x_{r_n} \in I$.

The continuity of f at $\lim x_{r_n}$ gives $f(\lim x_{r_n}) = \lim f(x_{r_n}) = \lim y_{r_n}$.

But $\lim y_{r_n} = d$, since $\lim y_n = d$. Therefore $f(\lim x_{r_n}) = d = f(c)$.

Since f is injective on I, it follows that $\lim x_{r_n} = c$.

Thus every convergent subsequence of $\{x_n\}$ converges to c.

It follows that $\underline{\lim}\ x_n = \overline{\lim}\ x_n = c$ and therefore $\lim\ x_n = c$, i.e., $\lim\ g(y_n) = g(d)$.

Thus every sequence $\{y_n\}$ in J converging to d, the sequence $\{g(y_n)\}$ converges to $g(d)$. This proves that g is continuous at d.

Since $d \in J$, it follows that g is continuous on J.

This completes the proof.

Note. The theorem can be extended to any kind of interval. If I be an interval and $f : I \to \mathbb{R}$ be strictly monotone and continuous on I then the inverse function f^{-1} is continuous and strictly monotone on $J[= f(I)]$.

Theorem 8.6.7. If a function $f : [a, b] \to \mathbb{R}$ satisfies the intermediate-value property on $[a, b]$ and f is injective on $[a, b]$ then f is strictly monotone on $[a, b]$.

Proof. In the proof of the Theorem 8.6.5, the injectivity of f on $[a, b]$ and the intermediate-value property of f on $[a, b]$ were only utilised. Therefore it follows from the proof of the Theorem 8.6.5 that f is strictly monotone on $[a, b]$.

Theorem 8.6.8. If $f : [a, b] \longrightarrow \mathbb{R}$ satisfies the intermediate-value property on $[a, b]$ and f is injective on $[a, b]$ then f is continuous on $[a, b]$.

Proof. Since f satisfies the intermediate-value property on $[a, b]$ and f is injective on $[a, b]$ then f is strictly monotone on $[a, b]$, by the Theorem 8.6.7. Since f is strictly monotone on $[a, b]$, f cannot have a discontinuity of the second second kind in $[a, b]$, by the Corollary 1 of the Theorem 8.6.1 and since f satisfies the intermediate-value property on $[a, b]$, f has no discontinuity of the first kind on $[a, b]$ by the Theorem 8.5.10.

Therefore f is continuous on $[a, b]$.

Examples.

1. The exponential function and its inverse.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = e^x, x \in \mathbb{R}$. The range of f is $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. f is continuous and strictly increasing on $\mathbb{R}$.

Hence there exists an inverse function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}$ such that g is continuous and strictly increasing on $\mathbb{R}^+$.

g is defined by $g(y) = \log y, y \in \mathbb{R}^+$. The range of g is $\mathbb{R}$.

$gf(x) = x$ for all $x \in \mathbb{R}$, i.e., $\log(e^x) = x$ for all $x \in \mathbb{R}$, and

$fg(y) = y$ for all $y \in \mathbb{R}^+$, i.e., $e^{\log y} = y$ for all $y \in \mathbb{R}^+$.

g is called the *logarithm function.*

2. The nth power function and its inverse.

Case 1. Let n be an even positive integer and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = x^n, x \in \mathbb{R}$. The range of f is $[0, \infty)$.

f is not injective on $\mathbb{R}$, but f is continuous on $\mathbb{R}$ and strictly increasing on $[0, \infty)$

Let $I = [0, \infty)$ and let $f : I \longrightarrow \mathbb{R}$ be defined by $f(x) = x^n, x \in I$.

Then f is continuous and strictly increasing on I and $f(I) = [0, \infty)$.

Hence there exists an inverse function $g : [0, \infty) \longrightarrow [0, \infty)$ such that g is continuous and strictly increasing on $[0, \infty)$.

g is defined by $g(y) = \sqrt[n]{y}, y \in [0, \infty)$.
$gf(x) = x$ for all $x \in [0, \infty)$, i.e., $\sqrt[n]{x^n} = x$ for all $x \in [0, \infty)$, and
$fg(y) = y$ for all $y \in [0, \infty)$, i.e., $(\sqrt[n]{y})^n = y$ for all $y \in [0, \infty)$.

g is called the *nth root function* (n even) and it is defined on $[0, \infty)$.

Case 2. Let n be an odd positive integer.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^n, x \in \mathbb{R}$. Then f is continuous and srtictly increasing on $\mathbb{R}$. The range of f is $\mathbb{R}$.

Hence there exists an inverse function $g : \mathbb{R} \to \mathbb{R}$ such that g is continuous and strictly increasing on $\mathbb{R}$.

g is defined by $g(y) = \sqrt[n]{y}$, $y \in \mathbb{R}$.

g is called the *nth root function* (n odd) and it is defined on $\mathbb{R}$.

3. Sine function and its inverse.

Let $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$ be defined by $f(x) = \sin x, x = [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then f is strictly increasing and continuous on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The range of f is $[-1, 1]$.

Hence there exists an inverse function $g : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that g is continuous and strictly increasing on $[-1, 1]$.
g is defined by $g(y) = \sin^{-1} y, y \in [-1, 1]$.

g is called the *principal inverse sine function.*

4. Cosine function and its inverse.

Let $f : [0, \pi] \to \mathbb{R}$ be defined by $f(x) = \cos x, x \in [0, \pi]$. Then f is continuous and strictly decreasing on $[0, \pi]$. The range of f is $[-1, 1]$.

Hence there exists an inverse function $g : [-1, 1] \to [0, \pi]$ such that g is continuous and strictly decreasing on $[-1, 1]$.
g is defined by $g(y) = \cos^{-1} y, y \in [-1, 1]$. The range of g is $[0, \pi]$.

g is called the *principal inverse cosine function.*

5. Tangent function and its inverse.

Let $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ be defined by $f(x) = \tan x, x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then f is continuous and strictly increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$. The range of f is $\mathbb{R}$.
Hence there exists an inverse function $g : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ such that g is continuous and strictly increasing on $\mathbb{R}$.
g is defined by $g(y) = \tan^{-1} y, y \in \mathbb{R}$. The range of g is $(-\frac{\pi}{2}, \frac{\pi}{2})$.

g is called the *principal inverse tangent function.*

6. Cotangent function and its inverse.

Let $f : (0, \pi) \to \mathbb{R}$ be defined by $f(x) = \cot x, x \in (0, \pi)$. Then f is continuous and strictly increasing on $(0, \pi)$. The range of f is $\mathbb{R}$.
Hence there exists an inverse function $g : \mathbb{R} \to (0, \pi)$ such that g is continuous and strictly decreasing on $\mathbb{R}$.
g is defined by $g(y) = \cot^{-1} y, y \in \mathbb{R}$. The range of g is $(0, \pi)$.

g is called the *principal inverse cotangent function.*

8.7. Uniform continuity.

Let I be an interval and a function $f : I \to \mathbb{R}$ be continuous on I. Let $c \in I$. Then for a pre-assigned positive ϵ there exists a positive δ such that for all $x \in N(c, \delta) \cap I$, $|f(x) - f(c)| < \epsilon$.

If we move to another point $c' \in I$ and keep the same ϵ fixed then it may happen that the same δ does not work but a smaller δ may be necessary for c' to fulfil the requirement of the condition for continuity.

Thus δ depends not only on ϵ but also on the point c and therefore δ can be expressed as $\delta(\epsilon, c)$. Let $\delta_0 = \inf\{\delta(\epsilon, c) : c \in I\}$. $\delta_0 \geq 0$ since $\delta(\epsilon, c) > 0$ for all $c \in I$.

If $\delta_0 > 0$, then for all $c \in I$ and $x \in N(c, \delta_0) \cap I$, $| f(x) - f(c) | < \epsilon$. That is, δ_0 works uniformly over the entire interval I in the sense that for any two points $x_1, x_2 \in I$ satisfying $|x_1 - x_2| < \delta_0$, $|f(x_1) - f(x_2)| < \epsilon$ holds. In this case f is said to be *uniformly continuous* on I.

Every function continuous on an interval I may not be uniformly continuous on I, because a positive δ_0 as $\inf\{\delta(\epsilon, c) : c \in I\}$ may not be available.

Definition. A function $f : I \to \mathbb{R}$ is said to be *uniformly continuous* on I if corresponding to a pre-assigned positive ϵ there exists a positive δ such that for any two points x_1, x_2 in I

$$| x_2 - x_1 | < \delta \Rightarrow | f(x_2) - f(x_1) | < \epsilon.$$

Note 1. The definition of uniform continuity shows that uniform continuity is a property of the function on an interval (or on a set) but continuity is a property of the function at a point. This is expressed by saying that continuity of a function is a *local property* while uniform continuity of a function is a *global property*.

2. It follows from the definition of uniform continuity that if a function f be uniformly continuous on an interval I, then it is also uniformly continuous on any subinterval $I_1 \subset I$.

Worked Examples.

1. Show that the function f defined by $f(x) = \frac{1}{x}$, $x \in [1, \infty)$ is uniformly continuous on $[1, \infty)$.

Let $c \geq 1$. Then for all $x \geq 1$,
$| f(x) - f(c) | = | \frac{1}{x} - \frac{1}{c} | = | \frac{x-c}{cx} | \leq | x - c |$, since $| cx | \geq 1$.

Let us choose $\epsilon > 0$. Then for all $x \geq 1$, satisfying $| x - c | < \epsilon$, $| f(x) - f(c) | < \epsilon$, whatever $c (\geq 1)$ may be.

This shows that f is uniformly continuous on $[1, \infty)$.

2. Show that the function f defined by $f(x) = \sin x, x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

Let $c \in \mathbb{R}$. Then for all $x \in \mathbb{R}$,

$$\begin{aligned} \mid f(x) - f(c) \mid = \mid \sin x - \sin c \mid &= 2 \mid \sin \tfrac{x-c}{2} \mid \mid \cos \tfrac{x+c}{2} \mid \\ &\leq 2 \mid \sin \tfrac{x-c}{2} \mid \\ &\leq 2 . \tfrac{|x-c|}{2}, \text{ since } \mid \sin x \mid \leq \mid x \mid \\ &\qquad \text{for all } x \in \mathbb{R}. \end{aligned}$$

Let us choose $\epsilon > 0$. Then for all $x \in \mathbb{R}$, satisfying $\mid x - c \mid < \epsilon$, $\mid f(x) - f(c) \mid < \epsilon$, whatever $c (\in \mathbb{R})$ may be.
This shows that f is uniformly continuous on $\mathbb{R}$.

3. Let $f(x) = x^2, x \in \mathbb{R}$. Show that f is uniformly continuous on any closed interval $[a, b], a \geq 0$; but f is not uniformly continuous on $[a, \infty), a \geq 0$.

First part. Let us choose $\epsilon > 0$. f will be uniformly continuous on $[a, b]$ if we can find a $\delta > 0$ such that for any two points x_1, x_2 in $[a, b]$, $\mid x_2 - x_1 \mid < \delta \Rightarrow \mid f(x_2) - f(x_1) \mid < \epsilon$.

$\mid f(x_2) - f(x_1) \mid = \mid x_2^2 - x_1^2 \mid = \mid x_2 - x_1 \mid \mid x_2 + x_1 \mid < 2b \mid x_2 - x_1 \mid$, since $0 \leq x_1 \leq b, 0 \leq x_2 \leq b$.

If we choose $\delta = \frac{\epsilon}{2b}$, then for any two points x_1, x_2 in $[a, b]$ satisfying $\mid x_2 - x_1 \mid < \delta$, the inequality $\mid f(x_2) - f(x_1) \mid < \epsilon$ holds.

This shows that f is uniformly continuous on $[a, b], a \geq 0$.

Second part. Let us choose $\epsilon > 0$. Then for any two points x_1, x_2 in $[a, b]$ satisfying $\mid x_2 - x_1 \mid < \delta$, the inequality $\mid f(x_2) - f(x_1) \mid < \epsilon$ will hold if we choose $\delta = \frac{\epsilon}{2b}$.

But as b takes larger and larger values, δ gets smaller and smaller. So it is not possible to find a *single* positive δ which will work for *all* $b > a$. It follows that f is not uniformly continuous on $[a, \infty), a \geq 0$.

Theorem 8.7.1. Let I be an interval and a function $f : I \to \mathbb{R}$ be uniformly continuous on I. Then f is continuous on I.

Proof. Since f is uniformly continuous on I, for a pre-asigned positive ϵ there exists a positive δ such that for any two points x_1, x_2 in I,

$$\mid x_2 - x_1 \mid < \delta \Rightarrow \mid f(x_2) - f(x_1) \mid < \epsilon.$$

Let $c \in I$. Taking $x_1 = c$, the condition yields
$\mid f(x_2) - f(c) \mid < \epsilon$ for all $x_2 \in I$ satisfying $\mid x_2 - c \mid < \delta$,
i.e., $\mid f(x) - f(c) \mid < \epsilon$ for all $x \in I$ satisfying $\mid x - c \mid < \delta$.
This proves that f is continuous at c.
Since c is arbitrary, f is continuous on I. This completes the proof.

Theorem 8.7.2. Let $I = [a, b]$ be a closed and bounded interval and and a function $f : I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.

Proof. If possible, let f be not uniformly continuous on I.

Then there exists a positive ϵ_0 for which no positive δ will work, i.e., for each positive δ there exist points x, y in $[a, b]$ such that $| x - y | < \delta$ but $| f(x) - f(y) | \not< \epsilon_0$.

Let $\delta = 1$. Then there exist points x_1, y_1 in $[a, b]$ such that $| x_1 - y_1 | < 1$ but $| f(x_1) - f(y_1) | \geq \epsilon_0$.
Let $\delta = \frac{1}{2}$. Then there exist points x_2, y_2 in $[a, b]$ such that $| x_2 - y_2 | < \frac{1}{2}$ but $| f(x_2) - f(y_2) | \geq \epsilon_0$.
... ...

Thus we obtain two sequences $\{x_n\}$ and $\{y_n\}$ in $[a, b]$ such that $| x_n - y_n | < \frac{1}{n}$ but $| f(x_n) - f(y_n) | \geq \epsilon_0$ for all $n \in \mathbb{N}$.

Since $a \leq x_n \leq b, a \leq y_n \leq b$, both the sequences $\{x_n\}$ and $\{y_n\}$ are bounded sequences.

By Bolzano-Weierstrass theorem, there exists a convergent subsequence of $\{x_n\}$, say $\{x_{r_n}\}$ and let l be the limit of the subsequence $\{x_{r_n}\}$. Since $[a, b]$ is a closed interval, $l \in [a, b]$.

Let us consider the subsequence $\{y_{r_n}\}$ of the sequence $\{y_n\}$.

Since $| x_{r_n} - y_{r_n} | < \frac{1}{r_n}$ for all $n \in \mathbb{N}$ and since $\lim \frac{1}{r_n} = 0$ and $\lim x_{r_n} = l$, it follows that the subsequence $\{y_{r_n}\}$ converges to l.

Since $l \in [a, b], f$ is continuous at l.

Since $\{x_{r_n}\}$ converges to l and f is continuous at $l, \lim f(x_{r_n}) = f(l)$. Since $\{y_{r_n}\}$ converges to l and f is continuous at $l, \lim f(y_{r_n}) = f(l)$.

Thus both the sequences $\{f(x_{r_n})\}$ and $\{f(y_{r_n})\}$ converge to a common limit. But this is contradicted by the condition $| f(x_{r_n}) - f(y_{r_n}) | \geq \epsilon_0$ for all $n \in \mathbb{N}$.

So our assumption that f is not uniformly continuous on $[a, b]$ is not tenable. Therefore f is uniformly continuous on $[a, b]$ and this completes the proof.

Theorem 8.7.3. Let $D \subset \mathbb{R}$ and a function $f : D \to \mathbb{R}$ be uniformly continuous on D. If $\{x_n\}$ be a Cauchy sequence in D then $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$.

Proof. Since f is uniformly continuous on D, for a pre-assigned positive ϵ there exists a positive δ such that for every pair of points x', x'' in D satisfying $| x' - x'' | < \delta$, $| f(x') - f(x'') | < \epsilon$ in $\mathbb{R}$.

Since $\{x_n\}$ is a Cauchy sequence, there exists a natural number k such that $| x_m - x_n | < \delta$ for all $m, n > k$.

It follows that for all $m, n > k$, $| f(x_m) - f(x_n) | < \epsilon$ in $\mathbb{R}$. This shows that $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$.

This completes the proof.

Note. If $f : D \to \mathbb{R}$ be continuous on D but not uniformly continuous on D and $\{x_n\}$ be a Cauchy sequence in D then $\{f(x_n)\}$ may not be a Cauchy sequence in $\mathbb{R}$.

For example, let $f(x) = \frac{1}{x}$, $x \in (0, 1]$.

Then f is continuous on $(0, 1]$. Let us consider the sequence $\{x_n\}$ in $(0, 1]$ where $x_n = \frac{1}{n}, n \in \mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence in $(0, 1]$. But $\{f(x_n)\} = \{1, 2, 3, 4, \ldots\}$. This is not a Cauchy sequence in $\mathbb{R}$.

Theorem 8.7.4. Let I be a bounded interval and a function $f : I \to \mathbb{R}$ be uniformly continuous on I. Then f is bounded on I.

Proof. Let us assume that f is not bounded on I. Then there is a sequence $\{x_n\}$ in I such that $| f(x_n) | > n$ for $n = 1, 2, 3, \ldots$

Since $\{x_n\}$ is a sequence in a bounded interval I, it is a bounded sequence and therefore it has a convergent subsequence, say $\{x_{r_n}\}$ in I. Since $\{x_{r_n}\}$ is a convergent sequence in I, it is a Cauchy sequence in I.

Since f is uniformly continuous on $I, \{f(x_{r_n})\}$ must be a Cauchy sequence in $\mathbb{R}$. But by construction, $| f(x_{r_n}) | > r_n > n$, for $n = 1, 2, 3, \ldots$ and this shows that $\{f(x_{r_n})\}$ can not be a Cauchy sequence and we arrive at a contradiction. This proves that f is bounded on I.

Note. If I be a bounded interval and $f : I \to \mathbb{R}$ be continuous on I, then f may not be bounded on I. For example, let $f(x) = \frac{1}{x}, x \in (0, 1)$. Then f is continuous on the bounded interval $(0, 1)$ but f is not bounded on $(0, 1)$.

If I be a bounded interval and f is continuous and bounded on I, f may not be uniformly continuous on I. For example, let $f(x) = \sin \frac{1}{x}, x \in (0, 1)$. f is continuous on $(0, 1)$ and bounded on $(0, 1)$. But f is not uniformly continuous on $(0, 1)$.

Worked Examples (continued).

4. Prove that the function $f(x) = \sin \frac{1}{x}$, $x \in (0, 1)$ is not uniformly continuous on $(0, 1)$.

Let us assume that f is uniformly continuous on $(0, 1)$. Then for every Cauchy sequence $\{x_n\}$ in $(0, 1)$, the sequence $\{f(x_n)\}$ must be a Cauchy sequence in $\mathbb{R}$.

Let us consider the sequence $\{x_n\}$ where $x_n = \frac{2}{n\pi}, n \in \mathbb{N}$. This is a Cauchy sequence in $(0, 1)$. The sequence $\{f(x_n)\}$ is $\{1, 0, -1, 0, \ldots \ldots\}$.

This is a divergent sequence and therefore this is not a Cauchy sequence in $\mathbb{R}$. Therefore f is not uniformly continuous on $(0, 1)$.

5. Prove that the function $f(x) = \frac{1}{x}$, $x \in (0,1)$ is not uniformly continuous on $(0, 1)$.

Let us assume that f is uniformly continuous on $(0, 1)$. Then for every Cauchy sequence $\{x_n\}$ in $(0, 1)$, the sequence $\{f(x_n)\}$ must be a Cauchy sequence in $\mathbb{R}$.

Let us consider the sequence $\{x_n\}$ where $x_n = \frac{1}{n+1}, n \in \mathbb{N}$. This is a Cauchy sequence in $(0, 1)$. The sequence $\{f(x_n)\}$ is $\{2, 3, 4, \ldots\}$. This is not a Cauchy sequence in $\mathbb{R}$. Therefore f is not uniformly continuous on $(0, 1)$.

Note. This example shows that a function continuous on an open bounded interval may not be uniformly continuous on that interval.

The following theorem gives a necessary and sufficient condition under which a function continuous on an open bounded interval will be uniformly continuous on that interval.

Theorem 8.7.5. Let a function f be continuous on an open bounded interval (a, b). Then f is uniformly continuous on (a, b) if and only if $\lim\limits_{x \to a+} f(x)$ and $\lim\limits_{x \to b-} f(x)$ both exist finitely.

Proof. Let f be continuous on an open bounded interval (a, b) and let $\lim\limits_{x \to a+} f(x)$ and $\lim\limits_{x \to b-} f(x)$ both exist finitely.

Let us define a function g on $[a, b]$ by $g(x) = f(x)$, for all $x \in (a, b)$ and $g(a) = \lim\limits_{x \to a+} f(x)$, $g(b) = \lim\limits_{x \to b-} f(x)$.

g is continuous on (a, b), since f is continuous on (a, b).

$g(a) = \lim\limits_{x \to a+} f(x)$ (by definition) $= \lim\limits_{x \to a+} g(x)$ and

$g(b) = \lim\limits_{x \to b-} f(x)$ (by definition) $= \lim\limits_{x \to b-} g(x)$.

Therefore g is right continuous at a and left continuous at b and consequently, g is continuous on $[a, b]$.

By Theorem 8.7.2, g is uniformly continuous on $[a, b]$.

By Theorem 8.7.1, g is uniformly continuous on (a, b). Since $g = f$ on (a, b), it follows that f is uniformly continuous on (a, b).

Conversely, let f be uniformly continuous on (a, b). We prove that both the limits $\lim\limits_{x \to a+} f(x)$ and $\lim\limits_{x \to b-} f(x)$ exist finitely.

Let $\{x_n\}$ be a sequence in (a, b) converging to a. Then $\{x_n\}$ is a Cauchy sequence in (a, b). Since f is uniformly continuous on (a, b), the sequence $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$ and therefore it is convergent. Let $\lim_{n\to\infty} f(x_n) = l$.

Let $\{y_n\}$ be another sequence in (a, b) converging to a. Then the sequence $\{x_n - y_n\}$ is a sequence in (a, b) converging to 0.

Let $\epsilon > 0$. Since f is uniformly continuous on (a, b), there exists a positive δ such that for any two points $x_1, x_2 \in (a, b)$

$$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \tfrac{\epsilon}{2}.$$

Since $\{x_n - y_n\}$ is a sequence in (a, b) converging to 0, there exists a natural number k such that $|x_n - y_n| < \delta$ for all $n \geq k$.

Therefore $|f(x_n) - f(y_n)| < \frac{\epsilon}{2}$ for all $n \geq k$.

$|f(y_n) - l| \leq |f(y_n) - f(x_n)| + |f(x_n) - l| < \epsilon$ for all $n \geq k$. This proves that $\lim_{n\to\infty} f(y_n) = l$.

Thus for every sequence $\{x_n\}$ in (a, b) converging to a, the sequence $\{f(x_n)\}$ converges to the limit l. This implies $\lim_{x\to a+} f(x) = l$.

In a similar manner it can be proved that $\lim_{x\to b-} f(x)$ exists finitely.

This completes the proof.

Definition. Let a function f be continuous on an interval I. A function g is said to be a *continuous extension* of f to $\mathbb{R}$ if g be continuous on $\mathbb{R}$ and $g(x) = f(x)$ for all $x \in I$.

If a function f be continuous on a closed and bounded interval $[a, b]$, then the function g defined on $\mathbb{R}$ by

$$\begin{aligned} g(x) &= f(a), \text{ for } x < a \\ &= f(x), \text{ for } x \in [a, b] \\ &= f(b), \text{ for } x > b \end{aligned}$$

is clearly a continuous extension of f to $\mathbb{R}$.

If a function f be continuous on an open interval (a, b), then f may not have a continuous extension to $\mathbb{R}$.

The following theorem specifies the conditions under which a function f continuous on an open bounded interval (a, b) may have a continuous extension to $\mathbb{R}$.

Theorem 8.7.6. Let a function f be continuous on an open bounded interval (a, b). Then f admits of a continuous extension to $\mathbb{R}$ if and only if f be uniformly continuous on (a, b).

Proof. Let f be uniformly continuous on (a, b). Then both the limits $\lim_{x\to a+} f(x)$ and $\lim_{x\to b-} f(x)$ exist finitely.

Let $\lim_{x \to a+} f(x) = l$ and $\lim_{x \to b-} f(x) = m$.

Let us define a function g on $\mathbb{R}$ by $g(x) = l$, for $x \leq a$
$= f(x)$, for $x \in (a, b)$
$= m$, for $x \geq b$.

Since $g(x) = f(x)$ for all $x \in (a, b)$, $l = \lim_{x \to a+} f(x) = \lim_{x \to a+} g(x)$ and $m = \lim_{x \to b-} f(x) = \lim_{x \to b-} g(x)$.

But by definition, $l = g(a)$ and $m = g(b)$. Therefore g is right continuous at a and left continuous at b and consequently, g is continuous at a and continuous at b.

Since f is uniformly continuous on (a, b), f is continuous on (a, b) and since $g = f$ on (a, b), g is continuous on (a, b).

Also by definition, g is continuous on $(-\infty, a)$ and on (b, ∞).

Consequently, g is continuous on $\mathbb{R}$.

Conversely, let f be continuous on an open bounded interval (a, b) and g be a continuous extension of f to $\mathbb{R}$. Then g is continuous on $\mathbb{R}$ and $g(x) = f(x)$ for all $x \in (a, b)$.

Since g is continuous on $\mathbb{R}$, g is continuous at a.

Therefore $\lim_{x \to a+} g(x) = \lim_{x \to a-} g(x) = g(a)$.

Since $f(x) = g(x)$ for all $x \in (a, b)$, $g(a) = \lim_{x \to a+} g(x) = \lim_{x \to a+} f(x)$.

This shows that the limit $\lim_{x \to a+} f(x)$ exists finitely.

Since g is continuous at b, it can be shown that the limit $\lim_{x \to b-} f(x)$ exists finitely. Consequently, f is uniformly continuous on (a, b).
This completes the proof.

Worked Examples (continued).

6. If $f : [0, \infty) \to \mathbb{R}$ be continuous on $[0, \infty)$ and $\lim_{x \to \infty} f(x) = 0$, prove that f is uniformly continuous on $[0, \infty)$.

Let $\epsilon > 0$. Then there exists a real number $p > 0$ such that $|f(x)| < \frac{\epsilon}{2}$ for all $x \geq p$ ... ... (i)

Since f is continuous at p, there exists a positive δ_1 such that $|f(x) - f(p)| < \frac{\epsilon}{2}$ for all x satisfying $|x - p| < \delta_1$ ... ... (ii)

Since f is continuous on $[0, p]$, it is uniformly continuous on $[0, p]$. Hence there exists a positive δ_2 such that for all $x_1, x_2 \in [0, p]$ with $|x_1 - x_2| < \delta_2$, $|f(x_1) - f(x_2)| < \epsilon$ ... ... (iii)

Let $\delta = \min\{\delta_1, \delta_2\}$ and let $a, b \in [0, \infty)$ with $|a - b| < \delta$.

Case (i). Let $a, b \in [p, \infty)$. Then by (i), $|f(a)| < \frac{\epsilon}{2}$ and $|f(b)| < \frac{\epsilon}{2}$; and therefore $|f(a) - f(b)| < |f(a)| + |f(b)| < \epsilon$.

Case (ii). Let $a, b \in [0, p]$. Then $|a - b| < \delta \Rightarrow |a - b| < \delta_2$ and by (iii), $|f(a) - f(b)| < \epsilon$.

Case (iii). Let $a \in [0, p], b \in [p, \infty)$ with $|a - b| < \delta$.

$|a - b| < \delta \Rightarrow |a - p| < \delta_1, |b - p| < \delta_1$. Then $|f(a) - f(p)| < \frac{\epsilon}{2}, |f(b) - f(p)| < \frac{\epsilon}{2}$ by (ii) and therefore $|f(a) - f(b)| < |f(a) - f(p)| + |f(b) - f(p)| < \epsilon$.

Therefore we have $|f(a) - f(b)| < \epsilon$, whenever $a, b \in [0, \infty)$ with $|a - b| < \delta$. This proves that f is uniformly continuous on $[0, \infty)$.

Q 7. If $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and f is continuous at a point of $\mathbb{R}$, prove that f is uniformly continuous on $\mathbb{R}$.

Let f be continuous at a point $c \in \mathbb{R}$.
Let us choose $\epsilon > 0$. There exists a positive δ such that
$| f(c + h) - f(c) | < \epsilon$ for all h satisfying $| h | < \delta$.
But $| f(c + h) - f(c) | = | f(c) + f(h) - f(c) | = | f(h) |$.
Continuity of f at c implies $| f(h) | < \epsilon$ for all h satisfying $| h | < \delta$.

Let x_1, x_2 be any two points in $\mathbb{R}$ such that $| x_1 - x_2 | < \delta$.
Then $| f(x_1 - x_2) | < \epsilon$.

$f(x + y) = f(x) + f(y)$ gives $f(0 + 0) = f(0) + f(0)$
or, $f(0) = 2f(0)$ or, $f(0) = 0$.
Also $0 = f(0) = f(x + (-x)) = f(x) + f(-x)$.
Therefore $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.
$| f(x_1 - x_2) | = | f(x_1) + f(-x_2) | = | f(x_1) - f(x_2) |$.

Thus $| f(x_1) - f(x_2) | < \epsilon$ for any two points x_1, x_2 in $\mathbb{R}$ satisfying $| x_1 - x_2 | < \delta$. δ depends on ϵ only and not on the points x_1, x_2 in $\mathbb{R}$. This proves that f is uniformly continuous on $\mathbb{R}$.

Q 8. Let A be a non-empty subset of $\mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f_A(x) = \inf\{| x - a | : a \in A\}$. Prove that f_A is uniformly continuous on $\mathbb{R}$.

Let $x_1, x_2 \in \mathbb{R}$. $f_A(x_1) = \inf\{| x_1 - a | : a \in A\}$ and $f_A(x_2) = \inf\{| x_2 - a | : a \in A\}$.

$| x_1 - a | \leq | x_1 - x_2 | + | x_2 - a |$ for all $a \in A$.
This implies $\inf\{| x_1 - a | : a \in A\} \leq | x_1 - x_2 | + \inf\{| x_2 - a | : a \in A\}$
or, $f_A(x_1) \leq | x_1 - x_2 | + f_A(x_2)$
or, $f_A(x_1) - f_A(x_2) \leq | x_1 - x_2 |$.

Similarly, $f_A(x_2) - f_A(x_1) \leq | x_1 - x_2 |$.

We have, $-\mid x_1 - x_2 \mid \leq f_A(x_2) - f_A(x_1) \leq \mid x_1 - x_2 \mid$
or, $\mid f_A(x_1) - f_A(x_2) \mid \leq \mid x_1 - x_2 \mid$.

Let $\epsilon > 0$. Then $\mid f_A(x_1) - f_A(x_2) \mid < \epsilon$ for all $x_1, x_2 \in \mathbb{R}$ satisfying $\mid x_1 - x_2 \mid < \epsilon$. This proves that f_A is uniformly continuous on $\mathbb{R}$.

~~Note.~~ Since f_A is uniformly continuous on $\mathbb{R}$, f_A is continuous on $\mathbb{R}$. The set $\{x \in \mathbb{R} : f_A(x) = 0\}$ is a closed set. (worked Ex.4, Page 277).

The set $\{x \in \mathbb{R} : f_A(x) = 0\} = \bar{A}$, by worked Ex.1, Page 65. Therefore if A be a non-empty closed set in $\mathbb{R}$, then the continuous function f_A defined by $f_A(x) = \inf\{\mid x - a \mid : a \in A\}$ is such that the set $\{x \in \mathbb{R} : f_A(x) = 0\} = A$.

Thus for a given closed set $A \subset \mathbb{R}$ there always exists a continuous function f on $\mathbb{R}$ such that $f(x) = 0$ on A.

Lipschitz function.

Definition. Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to satisfy a *Lipschitz condition* on I if there exists a positive real number M such that $|f(x_1) - f(x_2)| \leq M|x_1 - x_2|$ for any two points $x_1, x_2 \in I$.

In this case f is also said to be a *Lipschitz function* on I.

For example, let $f(x) = x^2$, $x \in [0, 2]$. Then
$|f(x_1) - f(x_2)| = |{x_1}^2 - {x_2}^2| \leq 4|x_1 - x_2|$ for all $x_1, x_2 \in [0, 2]$.
Therefore f satisfies Lipschitz condition with $M = 4$ on $[0, 2]$.
$M \in [4, \infty)$

Theorem 8.7.7. Let $f : I \to \mathbb{R}$ be a Lipschitz function on I. Then f is uniformly continuous on I.

Proof. Since f is a Lipschitz function on I, there exists a positive real number k such that $\mid f(x_1) - f(x_2) \mid \leq k \mid x_1 - x_2 \mid$ for all $x_1, x_2 \in I$.

Let $\epsilon > 0$. Then for all points x_1, x_2 in I satisfying $\mid x_1 - x_2 \mid < \frac{\epsilon}{k}$, $\mid f(x_1) - f(x_2) \mid < k.\frac{\epsilon}{k} = \epsilon$.

This proves that f is uniformly continuous on I.

Worked Example (continued).

9. Let $f(x) = \log x$, $x \in (0, \infty)$. Show that f is uniformly continuous on $[a, \infty)$, where $a > 0$.

Let $x_1, x_2 \in [a, \infty)$, where $a > 0$.
If $x_1 < x_2$ then $\frac{x_2}{x_1} > 1$ and therefore

$$0 < \log \frac{x_2}{x_1} < \frac{x_2}{x_1} - 1, \text{ since } \log(1 + x) < x \text{ if } x > 0$$
$$\leq \frac{x_2 - x_1}{a}.$$

If $x_2 < x_1$ then $\frac{x_1}{x_2} > 1$ and therefore

$$0 < \log \frac{x_1}{x_2} < \frac{x_1}{x_2} - 1, \text{ since } \log(1+x) < x \text{ if } x > 0$$
$$\leq \frac{x_1 - x_2}{a}.$$

In either case, $|\log x_2 - \log x_1| \leq \frac{1}{a}|x_2 - x_1|$.

This shows that f is a Lipschitz function on $[a, \infty)$ with $M = \frac{1}{a}$ and therefore f is uniformly continuous on $[a, \infty)$.

8.8. Continuity on a compact set.

Theorem 8.8.1. Let $D \subset \mathbb{R}$ be a compact set and a function $f : D \to \mathbb{R}$ be continuous on D. Then $f(D)$ is a compact set in $\mathbb{R}$.

The theorem says that continuous image of a compact set in $\mathbb{R}$ is a compact set.

Proof. Let $\mathcal{G}$ be a family of open intervals $\{I_\alpha : \alpha \in \Lambda\}$, Λ being the index set, such that $f(D) \subset \bigcup_{\alpha \in \Lambda} I_\alpha$. Then $\mathcal{G}$ is an open cover of $f(D)$.

Let $c \in D$. Then $f(c) \in f(D)$ and there exists an open interval of the family $\mathcal{G}$, say I_c, such that $f(c) \in I_c$.

Since I_c is an open interval, it is open set.

So $f(c)$ is an interior point of I_c and there exists a neighborhood of $f(c)$, say $N(f(c), \epsilon_c)$ such that $N(f(c), \epsilon_c) \subset I_c$.

Since f is continuous at c, there exists a $\delta_c > 0$ such that $f(x) \in N(f(c), \epsilon_c)$ for all $x \in N(c, \delta_c) \cap D$.

Clearly, the set of neighbourhoods $\{N(c, \delta_c) : c \in D\}$ covers D.

Since D is compact, there exists a finite subcollection of the family of the neighbourhoods $\{N(c, \delta_c) : c \in D\}$ which also covers D. Therefore there exists a finite number of points $c_1, c_2, \ldots, c_m$ in D such that $D \subset N(c_1, \delta_{c_1}) \cup N(c_2, \delta_{c_2}) \cup \cdots \cup N(c_m, \delta_{c_m})$.

Let $p \in f(D)$. Then there exists a point $q \in D$ such that $f(q) = p$.

Since $D \subset N(c_1, \delta_{c_1}) \cup N(c_2, \delta_{c_2}) \cup \cdots \cup N(c_m, \delta_{c_m})$, $q \in N(c_k, \delta_{c_k})$ for some natural number $k \leq m$.

But $x \in N(c_i, \delta_{c_i}) \Rightarrow f(x) \in N(f(c_i), \epsilon_{c_i}) \subset I_{c_i}$ for each $i = 1, 2, \ldots, m$.

As $q \in N(c_k, \delta_{c_k}), p \in N(f(c_k), \epsilon_{c_k}) \subset I_{c_k}$.

Since p is arbitrary, $f(D) \subset I_{c_1} \cup I_{c_2} \cup \cdots \cup I_{c_m}$.

Thus a finite subcollection of $\mathcal{G}$ covers $f(D)$ and therefore $f(D)$ is compact.

This completes the proof.

Another proof.

Let $\{y_n\}$ be a sequence in $f(D)$. For each $n \in \mathbb{N}$, let us choose $x_n \in D$ such that $f(x_n) = y_n$.

Since D is compact and $\{x_n\}$ is a sequence in D, there is a subsequence $\{x_{r_n}\}$ of $\{x_n\}$ such that $\{x_{r_n}\}$ converges to a point, say c, of D.

Since f is continuous at c, the sequence $\{f(x_{r_n})\}$ converges to $f(c)$. That is, the subsequence $\{y_{r_n}\}$ of $\{y_n\}$ converges to a point $f(c)$ of $f(D)$.

Therefore every sequence in $f(D)$ has a subsequence that converges to a point of $f(D)$. Consequently, $f(D)$ is compact.

Corollary. Since $f(D)$ is a compact set in $\mathbb{R}$, it is closed and bounded.

Since $f(D)$ is a closed and bounded set, $\sup f(D)$ and $\inf f(D)$ both exist and both belong to $f(D)$. (worked Example 1, Page 63)

Therefore there exists a point x^* in D such that $f(x^*) = \sup f(D)$ and there exists a point x_* in D such that $f(x_*) = \inf f(D)$.

Theorem 8.8.2. Let $D \subset \mathbb{R}$ be a compact set and a function $f : D \to \mathbb{R}$ be one-to-one and continuous on D. Then $f^{-1} : E \to D$ is continuous on E where $E = f(D) \subset \mathbb{R}$.

Proof. Since D is a compact set and f is continuous on D, E is compact. Let $b \in E$. Since f is one-to-one on D and $f(D) = E$, the inverse function $f^{-1} : E \to \mathbb{R}$ exists. Let $f^{-1}(b) = a$.

Let us choose a positive ϵ. Let $A = N(a, \epsilon) \cap D$.

Then $D - A = D - N(a, \epsilon)$.

Since D is a closed set and $N(a, \epsilon)$ is an open set, $D - A$ is a closed subset of D. Since D is compact and $D - A$ is a closed subset of D, $D - A$ is compact. (worked Ex.10, Page 97)

Let $D_1 = D - A$. Since $f : D \to E$ is continuous on D, the restriction function $\frac{f}{D_1} : D_1 \to E$ is also continuous on D_1.

Therefore $\frac{f}{D_1}(D_1)(= f(D_1))$ is compact.

Let $f(D_1) = E_1$. Now $a \notin D_1$ and f is one-to-one on D implies $f(a) \notin E_1$.

Since E_1 is compact and $b \notin E_1$ it follows that b is not a limit point of E_1. So there exists a neighbourhood $N(b, \delta)$ of b such that $[N(b, \delta) \cap E] \cap E_1 = \phi$.

Thus for every point $y \in N(b, \delta) \cap E$, $f^{-1}(y) \in N(a, \epsilon) \cap D$. Therefore f^{-1} is continuous at b.

Since b is arbitrary, f^{-1} is continuous on E.
This completes the proof.

Another proof. Since f is one-to-one, the inverse mapping $f^{-1}: f(D) \to D$ exists. Since D is compact and f is continuous on D, $f(D)$ is compact.

Case 1. Let $y_0 \in f(D)$ and y_0 be a limit point point of $f(D)$. Then there exists a sequence of distinct points $\{y_n\}$ in $f(D)$ such that $\lim y_n = y_0$.

Since f is injective, there exists a sequence of distinct points $\{x_n\}$ in D such that $f(x_n) = y_n$ for all $n \in \mathbb{N}$. Then $x_n = f^{-1}(y_n)$ for all $n \in \mathbb{N}$. f^{-1} will be continuous at y_0 if we can prove that $\lim f^{-1}(y_n) = f^{-1}(y_0)$. i.e., if we can prove that $\lim x_n = x_0$ where $x_0 = f^{-1}(y_0)$.

Let us assume on the contrary, that the sequence $\{x_n\}$ does not converge to x_0. Then there exists a positive ϵ and a subsequence $\{x_{r_n}\}$ of $\{x_n\}$ such that $| x_{r_n} - x_0 | \geq \epsilon$ for all $n \in \mathbb{N}$.

Since D is compact and $\{x_{r_n}\}$ is a sequence in D, there exists a convergent subsequence $\{x'_n\}$ of the sequence $\{x_{r_n}\}$ such that $\{x'_n\}$ converges to a point, say x' in D.

As $| x'_n - x_0 | \geq \epsilon$ holds for all $n \in \mathbb{N}$, it follows that $x' \neq x_0$.

Since f is continuous on D, f is continuous at x' and $\lim f(x'_n) = f(x')$. Since f is injective, $x' \neq x_0 \Rightarrow f(x') \neq y_0$.

Let $f(x'_n) = y'_n$. Then $\{y'_n\}$ is a subsequence of the sequence $\{y_n\}$ and as $\lim y_n = y_0, \lim y'_n$ must be y_0.

Thus we arrive at a contradiction. Therefore our assumption that the sequence $\{x_n\}$ does not converge to x_0 is not tenable. We conclude $\lim x_n = x_0$ and thereby f^{-1} is continuous at y_0.

Case 2. Let $y_0 \in f(D)$ and y_0 be an isolated point of $f(D)$. Then f^{-1} is continuous at y_0.

Since y_0 is an arbitrary point of $f(D)$, f^{-1} is continuous on $f(D)$.
This completes the proof.

Theorem 8.8.3. Let $D \subset \mathbb{R}$ be a compact set and a function $f : D \to \mathbb{R}$ is continuous on D. Then f is uniformly continuous on D.

Proof. Let $c \in D$. Then f is continuous at c. Therefore for a pre-assigned positive ϵ there exists a positive δ_c such that

$| f(x) - f(c) | < \frac{\epsilon}{2}$ for all $x \in N(c, \delta_c) \cap D$.

Let $\mathcal{G}$ be the family of neighbourhoods $\{N(c, \frac{1}{2}\delta_c) : c \in D\}$. Clearly, $\mathcal{G}$ is an open cover of D. Since D is compact, there is a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers D.

Let $\mathcal{G}' = \{N(c_1, \frac{1}{2}\delta_{c_1}), N(c_2, \frac{1}{2}\delta_{c_2}), \cdots, N(c_m, \frac{1}{2}\delta_{c_m})\}$.

Let $\delta = \min\{\frac{1}{2}\delta_{c_1}, \frac{1}{2}\delta_{c_2}, \cdots, \frac{1}{2}\delta_{c_m}\}$ and let $x_1, x_2 \in D$ such that $| x_1 - x_2 |< \delta$.

Since $x_1 \in D, x_1 \in N(c_k, \frac{1}{2}\delta_{c_k})$ for some natural number $k \leq m$.

Therefore $| x_1 - c_k |< \frac{1}{2}\delta_{c_k}$.

$$\begin{aligned} | x_2 - c_k | &\leq | x_2 - x_1 | + | x_1 - c_k | \\ &< \delta + \tfrac{1}{2}\delta_{c_k} \\ &\leq \tfrac{1}{2}\delta_{c_k} + \tfrac{1}{2}\delta_{c_k} = \delta_{c_k}. \end{aligned}$$ This shows that $x_2 \in N(c_k, \delta_{c_k})$.

Since $x_1 \in N(c_k, \frac{1}{2}\delta_{c_k}) \cap D, | f(x_1) - f(c_k) |< \frac{\epsilon}{2}$.

Since $x_2 \in N(c_k, \delta_{c_k}) \cap D, | f(x_2) - f(c_k) |< \frac{\epsilon}{2}$.

Hence $| f(x_2) - f(x_1) |< \epsilon$.

Thus for all $x_1, x_2 \in D$ satisfying $| x_2 - x_1 |< \delta, | f(x_2) - f(x_1) |< \epsilon$.
This shows that f is uniformly continuous on D.
This completes the proof.

Corollary. If $D \subset \mathbb{R}$ be compact and a map $f : D \to \mathbb{R}$ be continuous on D, then f maps a Cauchy sequence in D to a Cauchy sequence in $\mathbb{R}$.

Exercises 13

1. (i) Give an example of a function f which satisfies the intermediate-value property on a closed and bounded interval $[a, b]$ but is not continuous on $[a, b]$.

(ii) Give an example of a function f which is monotone increasing on a closed and bounded interval $[a, b]$ but does not satisfy the intermediate-value property on $[a, b]$.

2. Let $c \in \mathbb{R}$ and a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at c. If for every positive δ there is a point y in $(c - \delta, c + \delta)$ such that $f(y) = 0$, prove that $f(c) = 0$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $c \in \mathbb{R}$ such that $f(c) > \mu$. Prove that there exists a neighbourhood U of c such that $f(x) > \mu$ for all $x \in U$.

4. Let a function $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Prove that the set $Z(f) = \{x \in \mathbb{R} : f(x) = 0\}$ is a closed set in $\mathbb{R}$.

Give an example of a function f continuous on $\mathbb{R}$ such that
(i) $Z(f)$ is a bounded enumerable set; (ii) $Z(f)$ is an unbounded enumerable set.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. A point $c \in \mathbb{R}$ is said to be a *fixed point* of f if $f(c) = c$ holds. Prove that the set of all fixed points of f is a closed set.

6. Let $I = [a, b]$ be a closed and bounded interval and a function $f : I \to \mathbb{R}$ be continuous on I and $f(x) > 0$ for all $x \in I$. Prove that there exists a positive number α such that $f(x) \geq \alpha$ for all $x \in I$.

7. A function $f : [0, 1] \to \mathbb{R}$ is continuous on $[0, 1]$ and f assumes only rational values on $[0, 1]$. Prove that f is a constant.

8. Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and let $f(a) < g(a)$, $f(b) > g(b)$. Show that there exists a point c in (a, b) such that $f(c) = g(c)$.

Deduce that $\cos x = x^2$ for some $x \in (0, \frac{\pi}{2})$.

9. A function $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and $x_1, x_2, x_3 \in [a, b]$. Prove that there is a point $c \in [a, b]$ such that $f(c) = \frac{f(x_1)+f(x_2)+f(x_3)}{3}$.

[**Hint.** There exist p, q in $[a, b]$ such that $f(p) \leq f(x) \leq f(q)$ for all $x \in [a, b]$. Then $f(p) \leq \frac{f(x_1)+f(x_2)+f(x_3)}{3} \leq f(q)$. Apply intermediate value theorem to the function f on $[p, q]$ or $[q, p]$.]

10. A function $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and $x_1, x_2, \ldots, x_n \in [a, b]$. Prove that there is a point $c \in [a, b]$ such that $f(c) = \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}$.

11. Give an example of a function f which is continuous on a closed interval I but (i) f is not bounded on I; (ii) $f(I)$ is not a closed interval.

12. A function $f : [0, 1] \to [0, 1]$ is defined by
$$f(x) = x, \; x \in \mathbb{Q}$$
$$= 1 - x, \; x \in \mathbb{R} - \mathbb{Q}.$$

Prove that (i) f is injective as well as surjective and $f^{-1} = f$;
(ii) f is not continuous on $[0, 1]$.

[**Note.** This example shows that continuity of f is not necessary for the existence of f^{-1}.]

13. If $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$ be both continuous functions on $[a, b]$ having the same range $[0, 1]$, prove that $f(c) = g(c)$ for some $c \in [a, b]$.

[**Hint.** $f(p) = 0, f(q) = 1$ for some $p, q \in [a, b]$. If $g(p) \neq 0$ and $g(q) \neq 1$, consider $f - g$. Then $(f - g)(p) < 0, (f - g)(q) > 0$. If $g(p) = 0, c = p$. If $g(q) = 1, c = q$.]

14. A real function f is continuous on $[0, 2]$ and $f(0) = f(2)$. Prove that there exists at least a point c in $[0, 1]$ such that $f(c) = f(c + 1)$.

[**Hint.** If $f(0) = f(1)$ then $c = 0, 1$. If $f(0) \neq f(1)$, consider g on [0,1] defined by $g(x) = f(x) - f(x + 1)$.]

15. If $f : (-\infty, 0] \to \mathbb{R}$ be continuous on $(-\infty, 0]$ and $\lim_{x \to -\infty} f(x) = 0$, prove that f is uniformly continuous on $(-\infty, 0]$.

16. Prove that the following functions are uniformly continuous on the indicated interval.

(i) $f(x) = \sqrt{x}$, on $[1, \infty)$;

(ii) $f(x) = \frac{1}{1+x^2}$, on $\mathbb{R}$;

(iii) $f(x) = x \sin \frac{1}{x}$, $x \neq 0$
$= 0$, $x = 0$, on $[-1, 1]$;

(iv) $f(x) = \tan x$, on $[a, b]$ where $-\frac{\pi}{2} < a < b < \frac{\pi}{2}$.

9. DIFFERENTIATION

9.1. Differentiability. Derivative.

Let $I = [a, b]$ be an interval and a function $f : I \longrightarrow \mathbb{R}$.

(i) Let c be an interior point of I.

f is said to be *differentiable* at c if $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists.

If the limit be l, l is said to be the *derivative* of f at c and is denoted by $f'(c)$.

Since c is an interior point of the domain of f, in order that $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ may exist, both the limits $\lim\limits_{x \to c+} \dfrac{f(x) - f(c)}{x - c}$ and $\lim\limits_{x \to c-} \dfrac{f(x) - f(c)}{x - c}$ should exist and should be equal.

(ii) Let c be the left end point a.

f is said to be differentiable at a if $\lim\limits_{x \to a+} \dfrac{f(x) - f(a)}{x - a}$ exists. If l be the limit, l is called the derivative of f at a and is denoted by $f'(a)$.

(iii) Let c be the right end point b.

f is said to be differentiable at b if $\lim\limits_{x \to b-} \dfrac{f(x) - f(b)}{x - b}$ exists. If l be the limit, l is called the derivative of f at b and is denoted by $f'(b)$.

Note. If $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} = \infty$ (or $-\infty$) then f is said to have the derivative ∞ (or $-\infty$) at c and we write $f'(c) = \infty$ (or $-\infty$). However, f is said to be differentiable at c if $f'(c)$ is *finite*. This is also expressed by saying that "$f'(c)$ exists".

Right hand derivative, Left hand derivative.

Let I be an interval and $f : I \longrightarrow \mathbb{R}$. Let $c \in I$.

If $\lim\limits_{x \to c+} \dfrac{f(x) - f(c)}{x - c}$ exists and equals l, l is called the *right hand derivative* of f at c and it is denoted by $Rf'(c)$ (or by $f'_+(c)$).

If $\lim\limits_{x \to c-} \dfrac{f(x) - f(c)}{x - c}$ exists and equals l, l is called the *left hand derivative* of f at c and it is denoted by $Lf'(c)$ (or by $f'_-(c)$).

Therefore if c be an interior point of the domain of f, the derivative of f at c exists if and only if $Rf'(c)$ and $Lf'(c)$ both exist and be equal.

If c be the left end point of the interval I, the derivative of f at c, if it exists, is the right hand derivative of f at c.

If c be the right end point of the interval I, the derivative of f at c, if it exists, is the left hand derivative of f at c.

Theorem 9.1.1. Let I be an interval and a function $f : I \to \mathbb{R}$ be differentiable at a point $c \in I$. Then f is continuous at c.

Proof. For all $x \in I$, but $x \neq c$, $f(x) - f(c) = \frac{f(x)-f(c)}{x-c}.(x-c)$.

Since f is differentiable at c, $\lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c}$ exists and the limit is finite. Also, since $\lim\limits_{x\to c}(x-c) = 0$,

$$\begin{aligned}\lim_{x\to c}[f(x)-f(c)] &= \lim \tfrac{f(x)-f(c)}{x-c}.(x-c)\\ &= f'(c).0\\ &= 0, \text{ since } f'(c) \text{ is finite.}\end{aligned}$$

Therefore $\lim\limits_{x\to c} f(x) = f(c)$ and this shows that f is continuous at c.

Note. The continuity of f at a point $c \in I$ does not ensure differentiability of f at c.

For example, let $f(x) = |x|, x \in \mathbb{R}$.

At $x = 0, f(x) = 0$. Also f is continuous at 0.

$$\lim_{x\to 0+} \tfrac{f(x)-f(0)}{x-0} = \lim_{x\to 0+} \tfrac{|x|}{x} = \lim_{x\to 0+} \tfrac{x}{x} = 1$$

$$\lim_{x\to 0-} \tfrac{f(x)-f(0)}{x-0} = \lim_{x\to 0-} \tfrac{|x|}{x} = \lim_{x\to 0-} \tfrac{-x}{x} = -1.$$

As $Rf'(0) \neq Lf'(0)$, f is not differentiable at 0.

Hence continuity at a point c does not imply differentiability at c.

Remark. If $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$, it is possible to define differentiability of f at a point $c \in D$, provided $c \in D'$ also.

If $c \in D \cap D', f$ is said to be differentiable at c if $\lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c}$ exists. If l be the limit, then l is called the derivative of f at c and is denoted by $f'(c)$.

If $D_1 = D \cap [c, \infty)$ and c be a limit point of D_1, then $\lim\limits_{x\to c+} \dfrac{f(x)-f(c)}{x-c}$, if it exists, is said to be the right hand derivative of f at c and it is denoted by $Rf'(c)$.

If $D_2 = D \cap (-\infty, c]$ and c be a limit point of D_2, then

$\lim_{x \to c-} \dfrac{f(x) - f(c)}{x - c}$, if it exists, is said to be the left hand derivative of f at c and it is denoted by $Lf'(c)$.

If c be a limit point of both D_1 and D_2, then f is differentiable at c if and only if $Rf'(c)$ and $Lf'(c)$ both exist and are real.

Let I be an interval and $f : I \to \mathbb{R}$ is differentiable at a point $c \in I$. Then $f'(c)$ exists. Let A be a subset of I such that at every point of A, f is differentiable.

Then $f'(x)$ exists for each $x \in A. f'$ can be considered as a function on A. f' is said to be the *derived function* of f on A. If f' be a function of x, then $f'(x)$ is expressed by $f'(x) = \frac{d}{dx} f(x), x \in A$ or by $f'(x) = Df(x), x \in A$.

Examples.

1. Let $k \in \mathbb{R}$ and $f(x) = k, x \in \mathbb{R}$. Find the derived function f' and its domain.

Let $c \in \mathbb{R}$. When $x \neq c$, $\frac{f(x)-f(c)}{x-c} = \frac{k-k}{x-c} = 0$.

Therefore $\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c} = 0$. That is, $f'(c) = 0$ for all $c \in \mathbb{R}$.

The derived function f' is defined by $f'(x) = 0, x \in \mathbb{R}$. The domain of f' is $\mathbb{R}$.

2. Let $f(x) = x^2, x \in \mathbb{R}$. Find the derived function f' and its domain.

Let $c \in \mathbb{R}$. When $x \neq c$, $\frac{f(x)-f(c)}{x-c} = \frac{x^2-c^2}{x-c} = x + c$.

Therefore $\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c} = \lim_{x \to c}(x + c) = 2c$.

That is, $f'(c) = 2c$ for all $c \in \mathbb{R}$.

The derived function f' is defined by $f'(x) = 2x, x \in \mathbb{R}$. The domain of f' is $\mathbb{R}$.

3. Let $f(x) = \sqrt{x}, x \in [0, \infty)$. Find the derived function f' and its domain.

Let $c \in [0, \infty)$. When $x \geq 0$ but $\neq c$, $\frac{f(x)-f(c)}{x-c} = \frac{\sqrt{x}-\sqrt{c}}{x-c} = \frac{1}{\sqrt{x}+\sqrt{c}}$.

$\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c} = \lim_{x \to c} \dfrac{1}{\sqrt{x} + \sqrt{c}} = \dfrac{1}{2\sqrt{c}}$ provided $c \neq 0$.

That is, $f'(c) = \frac{1}{2\sqrt{c}}$ if $c \in (0, \infty)$.

The domain of f' is $\{x \in \mathbb{R} : x > 0\}$.

The derived function f' is defined by $f'(x) = \frac{1}{2\sqrt{x}}, x \in (0, \infty)$.

Note. Here the domain of f' is a proper subset of the domain of f.

f' $D = (0, \infty)$
f $D = [0, \infty)$

$f(x) = x, \ 0 \le x \le 1$
$= 2 - x^2, \ 1 < x < 2.$
$= x - x^2, \ 2 \le x \le 3.$

Find the derived function f' and its domain.

$$f'(x) = 1 \text{ for } x \in (0,1)$$
$$= -2x \text{ for } x \in (1,2)$$
$$= 1 - 2x \text{ for } x \in (2,3).$$

$\lim_{x \to 0+} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0+} \frac{x}{x} = 1$. Therefore $Rf'(0) = 1$.

Hence f is differentiable at 0 and $f'(0) = 1$.

$\lim_{x \to 1-} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1-} \frac{x-1}{x-1} = 1$. Therefore $Lf'(1) = 1$.

$\lim_{x \to 1+} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1+} \frac{(2-x^2)-1}{x-1} = -2$. Therefore $Rf'(1) = -2$.

Hence f is not differentiable at 1.

$\lim_{x \to 2-} \frac{f(x)-f(2)}{x-2} = \lim_{x \to 2-} \frac{(2-x^2)-(-2)}{x-2} = \lim_{x \to 2-} -(x+2) = -4$. Therefore $Lf'(2) = -4$.

$\lim_{x \to 2+} \frac{f(x)-f(2)}{x-2} = \lim_{x \to 2+} \frac{(x-x^2)-(-2)}{x-2} = \lim_{x \to 2+} -(x+1) = -3$. Therefore $Rf'(2) = -3$.

Hence f is not differentiable at 2.

$\lim_{x \to 3-} \frac{f(x)-f(3)}{x-3} = \lim_{x \to 3-} \frac{x-x^2-(-6)}{x-3} = -5$. Therefore $Lf'(3) = -5$.

Hence f is differentiable at 3 and $f'(3) = -5$.

The derived function f' is defined by $f'(x) = 1, \ 0 \le x < 1$
$= -2x, \ 1 < x < 2$
$= 1 - 2x, \ 2 < x \le 3.$

The domain of f' is $[0,1) \cup (1,2) \cup (2,3]$.

Note. The domain of f' is a proper subset of the domain of f.

Theorem 9.1.2. Let I be an interval and $c \in I$. Let the functions $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be differentiable at c. Then

(i) $f + g$ is differentiable at c and $(f+g)'(c) = f'(c) + g'(c)$

(ii) if $k \in \mathbb{R}$, kf is differentiable at c and $(kf)'(c) = kf'(c)$

(iii) $f.g$ is differentiable at c and $(f.g)'(c) = f'(c)g(c) + f(c)g'(c)$

(iv) if $g(c) \neq 0, f/g$ is differentiable at c and $(f/g)'(c) = \frac{f'(c)g(c)-f(c)g'(c)}{[g(c)]^2}$.

Proof. Proofs of **(i)** and **(ii)** are left to the reader.

(iii) Let $h = f.g$. Then for $x \in I, x \neq c$,

$$\frac{h(x)-h(c)}{x-c} = \frac{f(x)g(x)-f(c)g(c)}{x-c}$$
$$= \frac{f(x)g(x)-f(c)g(x)+f(c)g(x)-f(c)g(c)}{x-c}$$
$$= \frac{f(x)-f(c)}{x-c}.g(x) + f(c).\frac{g(x)-g(c)}{x-c}.$$

Since g is continuous at c by Theorem 9.1.1, $\lim\limits_{x \to c} g(x) = g(c)$.

Since f is differentiable at c, $\lim\limits_{x \to c} \dfrac{f(x)-f(c)}{x-c} = f'(c)$.

Since g is differentiable at c, $\lim\limits_{x \to c} \dfrac{g(x)-g(c)}{x-c} = g'(c)$.

Therefore $\lim\limits_{x \to c} \dfrac{h(x)-h(c)}{x-c}$

$$= \lim_{x \to c} \frac{f(x)-f(c)}{x-c}. \lim_{x \to c} g(x) + f(c). \lim_{x \to c} \frac{g(x)-g(c)}{x-c}$$
$$= f'(c)g(c) + f(c)g'(c).$$

Therefore h is differentiable at c and $g'(c) = f'(c)g(c) + f(c)g'(c)$.

(iv) Let $h = f/g$. Since g is differentiable at c, g is continuous at c. Since $g(c) \neq 0$, there exists a neighbourhood $N(c)$ of c such that $g(x) \neq 0$ for all $x \in N(c) \cap I$. Therefore for $x \in N(c) \cap I, x \neq c$,

$$\frac{h(x)-h(c)}{x-c} = \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x-c} = \frac{f(x)g(c)-f(c)g(x)}{g(x)g(c).(x-c)}$$
$$= \frac{f(x)g(c)-f(c)g(c)+f(c)g(c)-f(c)g(x)}{g(x)g(c)(x-c)}$$
$$= \frac{1}{g(x)g(c)}\left[\frac{f(x)-f(c)}{x-c}.g(c) - f(c).\frac{g(x)-g(c)}{x-c}\right].$$

Since g is continuous at c by Theorem 9.1.1, $\lim\limits_{x \to c} g(x) = g(c)$.

Since f and g are differentiable at c,

$\lim\limits_{x \to c} \dfrac{f(x)-f(c)}{x-c} = f'(c)$ and $\lim\limits_{x \to c} \dfrac{g(x)-g(c)}{x-c} = g'(c)$.

Therefore $\lim\limits_{x \to c} \dfrac{h(x)-h(c)}{x-c} = \dfrac{1}{g(c).g(c)}[f'(c)g(c) - f(c)g'(c)]$.

Therefore $h(x)$ is differentiable at c and $h'(c) = \frac{f'(c)g(c)-f(c)g'(c)}{[g(c)]^2}$.

Theorem 9.1.3. Let I be an interval and the functions $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be both differentiable on a subset $D \subset I$. Then

(i) $f+g$ is differentiable on D and $(f+g)'(x) = f'(x) + g'(x), x \in D$

(ii) if $k \in \mathbb{R}, kf$ is differentiable on D and $(kf)'(x) = kf'(x), x \in D$

(iii) $f.g$ is differentiable on D and $(f.g)'(x) = f'(x)g(x) + f(x)g'(x), x \in D$

(iv) if $g'(x) \neq 0$ on $D, f/g$ is differentiable on D and $(f/g)'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}, x \in D$.

This is an immediate consequence of the Theorem 9.1.2.

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Theorem 9.1.4. Let I and J be intervals. Let $f : I \to \mathbb{R}, g : J \to \mathbb{R}$ and $f(I) \subset J$. Let $c \in I$ and f is differentiable at c and g is differentiable at $f(c)$. Then the composite function gf is differentiable at c and $(gf)'(c) = g'(f(c)).f'(c)$.

Proof. Let $f(c) = d$. Since g is differentiable at d, $\lim_{y \to d} \frac{g(y) - g(d)}{y - d} = g'(d)$. Since f is differentiable at c, $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$.

Let us define a function $G : J \to \mathbb{R}$ by

$$\begin{aligned} G(y) &= \tfrac{g(y)-g(d)}{y-d} \text{ if } y \in J \text{ and } y \neq d \\ &= g'(d) \text{ if } y = d. \end{aligned}$$

$$\begin{aligned} \text{Then } \lim_{y \to d} G(y) &= \lim_{y \to d} \frac{g(y) - g(d)}{y - d} \\ &= g'(d), \text{ since } g \text{ is differentiable at } d \\ &= G(d), \text{ by definition .} \end{aligned}$$

This shows that G is continuous at d.

Since f is continuous at c and G is continuous at $d(= f(c))$, the composite function Gf is continuous at c. Hence $\lim_{x \to c} Gf(x) = Gf(c)$.

$$\begin{aligned} \text{But } \lim_{x \to c} Gf(x) &= \lim_{x \to c} \frac{g(f(x)) - g(d)}{f(x) - d}, \text{ by definition of } G \\ &= \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)}. \end{aligned}$$

Therefore $\lim_{x \to c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} = g'(d)$, since $Gf(c) = g'(d)$.

We also have $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$.

Hence $\lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c} = g'(d).f'(c)$.

Therefore the function gf is differentiable at c and $(gf)'(c) = g'(d)f'(c) = g'(f(c))f'(c)$.

This completes the proof.

As an immediate consequence of the theorem we have the following theorem.

Theorem 9.1.5. Let I and J be intervals and $f : I \longrightarrow \mathbb{R}, g : J \longrightarrow \mathbb{R}$ be functions such that $f(I) \subset J$. If f is differentiable on I and g is differentiable on $f(I)$ then the composite function gf is differentiable on I and $(gf)'(x) = g'(f(x)).f'(x), x \in I$.

Example (continued).

5. Find the derived function of $f(x) = x^\alpha, x > 0$ and $\alpha \in \mathbb{R}$.

$f(x) = x^\alpha = e^{\alpha \log x}$.
Let $g(x) = \alpha \log x, x > 0$ and $h(x) = e^x; x \in \mathbb{R}$.

Then $f(x) = hg(x)$, $x > 0$ and $f'(x) = h'(g(x)).g'(x)$, $x > 0$.

But $h'(x) = e^x$ and hence $h'(g(x)) = e^{\alpha \log x} = x^\alpha$; $g'(x) = \frac{\alpha}{x}$.

Therefore $f'(x) = x^\alpha.\frac{\alpha}{x} = \alpha x^{\alpha-1}, x > 0$.

Theorem 9.1.6. Let $I \subset \mathbb{R}$ be an interval and a function $f : I \longrightarrow \mathbb{R}$ be strictly monotone and continuous on I. Let $J = f(I)$ and let $g : J \longrightarrow \mathbb{R}$ be the inverse to f. If f is differentiable at $c \in I$ and $f'(c) \neq 0$ then g is differentiable at $d(= f(c))$ and $g'(d) = \frac{1}{f'(c)}$.

Proof. Let $y \in J, y \neq d$. Let $g(y) = x \in I$. Then $f(x) = y$ and since f is strictly monotone on $I, x \neq c$.

$\frac{g(y)-g(d)}{y-d} = \frac{g(f(x))-g(f(c))}{f(x)-f(c)}$.

Since f is strictly monotone and continuous on I, g is continuous on J. As $y \longrightarrow d, g(y) \longrightarrow g(d)$.

Since $gf(x) = x$ for all $x \in I$, it follows that $x \longrightarrow c$ as $y \longrightarrow d$.

Therefore $\lim\limits_{y \to d} \frac{g(y)-g(d)}{y-d} = \lim\limits_{x \to c} \frac{g(f(x))-g(f(c))}{f(x)-f(c)} = \lim\limits_{x \to c} \frac{x-c}{f(x)-f(c)}$.

Since f is differentiable at c, $\lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c} = f'(c)$ and since $f'(c) \neq 0$, $\lim\limits_{x \to c} \frac{x-c}{f(x)-f(c)} = \frac{1}{f'(c)}$.

Therefore $\lim\limits_{y \to d} \frac{g(y)-g(d)}{y-d} = \frac{1}{f'(c)}$. That is, $g'(d) = \frac{1}{f'(c)}$.

Examples (continued).

6. Let $f(x) = x^2, x \in [0, \infty)$. f is strictly increasing and continuous on $[0, \infty)$. Let $I = [0, \infty)$. Then $f(I) = [0, \infty)$.

The inverse function g defined by $g(y) = \sqrt{y}, y \in [0, \infty)$ is continuous on $[0, \infty)$.

f is differentiable on $[0, \infty)$ and $f'(x) = 2x, x \in [0, \infty)$.

$f'(x) \neq 0$ on $(0, \infty)$. Let $I_1 = (0, \infty)$. Then $f(I_1) = (0, \infty)$.

Hence $g'(y)$ exists for all $y \in (0, \infty)$ and $g'(y) = \frac{1}{f'(x)} = \frac{1}{2x} = \frac{1}{2g(y)} = \frac{1}{2\sqrt{y}}$, $y \in (0, \infty)$.

7. Let $f(x) = e^x, x \in \mathbb{R}$. f is strictly increasing and continuous on $\mathbb{R}$. $f(\mathbb{R}) = (0, \infty)$.

The inverse function g defined by $g(y) = \log y, y \in (0, \infty)$ is continuous on $(0, \infty)$.

f is differentiable on $\mathbb{R}$ and $f'(x) = e^x \neq 0$ on $\mathbb{R}$.

Hence $g'(y)$ exists for all $y \in (0, \infty)$ and $g'(y) = \frac{1}{f'(x)} = \frac{1}{e^x} = \frac{1}{e^{\log y}} = \frac{1}{y}$, $y \in (0, \infty)$.

8. Let $f(x) = \sin x, x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.$f$ is strictly increasing and continuous on $I = [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then $f(I) = [-1, 1]$. The inverse function g defined by $g(y) = \sin^{-1} y, y \in [-1, 1]$ is continuous on $[-1, 1]$.

f is differentiable on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $f'(x) = \cos x, x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

$f'(x) \neq 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Let $I_1 = (-\frac{\pi}{2}, \frac{\pi}{2})$. Then $f(I_1) = (-1, 1)$.

Hence $g'(y)$ exists for all $y \in (-1, 1)$ and

$g'(y) = \frac{1}{f'(x)} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}}$, $y \in (-1, 1)$.

That is, $D \sin^{-1} y = \frac{1}{\sqrt{1-y^2}}$, $y \in (-1, 1)$.

9. Let $f(x) = \cos x, x \in [0, \pi]$. f is strictly decreasing and continuous on $I = [0, \pi]$. $f(I) = [-1, 1]$. The inverse function g defined by $g(y) = \cos^{-1} y, y \in [-1, 1]$ is continuous on $[-1, 1]$.

f is differentiable on $[0, \pi]$ and $f'(x) = -\sin x, x \in [0, \pi]$.

$f'(x) \neq 0$ on $(0, \pi)$. Let $I_1 = (0, \pi)$. Then $f(I_1) = (-1, 1)$.

Hence $g'(y)$ exists for all $y \in (-1, 1)$ and

$$\begin{aligned} g'(y) = \frac{1}{f'(x)} = \frac{1}{-\sin x} &= \frac{1}{-\sqrt{1-\cos^2 x}} \text{ since } \sin x > 0 \text{ in } (0, \pi) \\ &= \frac{1}{-\sqrt{1-y^2}}, y \in (-1, 1). \end{aligned}$$

That is, $D \cos^{-1} y = \frac{-1}{\sqrt{1-y^2}}$, $y \in (-1, 1)$.

10. Let $f(x) = \tan x, x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. f is strictly increasing and continuous on $I = (-\frac{\pi}{2}, \frac{\pi}{2})$. $f(I) = \mathbb{R}$. The inverse function g defined by $g(y) = \tan^{-1} y, y \in \mathbb{R}$, is continuous on $\mathbb{R}$.

f is differentiable on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $f'(x) = \sec^2 x, x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

$f'(x) \neq 0$ on $I = (-\frac{\pi}{2}, \frac{\pi}{2})$. Hence $g'(y)$ exists for all $y \in \mathbb{R}$ and $g'(y) = \frac{1}{f'(x)} = \frac{1}{\sec^2 x} = \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2}, y \in \mathbb{R}$.

That is, $D \tan^{-1} y = \frac{1}{1+y^2}$, $y \in \mathbb{R}$.

11. Let $f(x) = \sec x$, $x \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.

Let $I_1 = [0, \frac{\pi}{2}), I_2 = (\frac{\pi}{2}, \pi]$.

f is strictly increasing and continuous on I_1. $f(I_1) = [1, \infty)$.

The inverse function $g : [1, \infty) \to \mathbb{R}$ defined by $g(y) = \sec^{-1} y$ is continuous on $[1, \infty)$.

f is differentiable on I_1 and $f'(x) = \sec x \tan x, x \in I_1$.

$f'(x) \neq 0$ on $(0, \frac{\pi}{2})$.

Hence $g'(y)$ exists for all $y \in (1, \infty)$ and $g'(y) = \frac{1}{f'(x)} = \frac{1}{\sec x \tan x}$.

f is strictly increasing and continuous on I_2. $f(I_2) = (-\infty, -1]$.

The inverse function $g : (-\infty, -1] \to \mathbb{R}$ defined by $g(y) = \sec^{-1} y$ is continuous on $(-\infty, -1]$.

f is differentiable on I_2 and $f'(x) = \sec x \tan x, x \in I_2$.

$f'(x) \neq 0$ on $(\frac{\pi}{2}, \pi)$. Hence $g'(y)$ exists for all $y \in (-\infty, -1)$ and $g'(y) = \frac{1}{f'(x)} = \frac{1}{\sec x \tan x}$.

But $\sec x \tan x = y\sqrt{y^2 - 1}, x \in (0, \frac{\pi}{2})$ and $\sec x \tan x = -y\sqrt{y^2 - 1}$, $x \in (\frac{\pi}{2}, \pi)$.

That is, $D \sec^{-1} y = \frac{1}{|y|\sqrt{y^2-1}}, y \in (-\infty, -1) \cup (1, \infty)$.

12. Let $f(x) = \operatorname{cosec} x, x \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.

Let $I_1 = [-\frac{\pi}{2}, 0)$, $I_2 = (0, \frac{\pi}{2}]$.

f is strictly decreasing and continuous on I_1. $f(I_1) = (-\infty, -1]$.

The inverse function $g : (-\infty, -1] \to \mathbb{R}$ defined by $g(y) = \operatorname{cosec}^{-1} y$ is continuous on $(-\infty, -1]$.

f is differentiable on I_1 and $f'(x) = -\operatorname{cosec} x \cot x, x \in [-\frac{\pi}{2}, 0)$

$f'(x) \neq 0$ on $(-\frac{\pi}{2}, 0)$. Hence $g'(y)$ exists for all $y \in (-\infty, -1)$ and $g'(y) = \frac{1}{f'(x)} = \frac{1}{-\operatorname{cosec} x \cot x}$.

f is strictly decreasing and continuous on I_2. $f(I_2) = [1, \infty)$.

The inverse function $g : [1, \infty) \to \mathbb{R}$ defined by $g(y) = \operatorname{cosec}^{-1} y$ is continuous on $[1, \infty)$.

f is differentiable on I_2 and $f'(x) = -\operatorname{cosec} x \cot x, x \in I_2$.

$f'(x) \neq 0$ on $(0, \frac{\pi}{2})$. Hence $g'(y)$ exists for all $y \in (1, \infty)$ and $g'(y) = \frac{1}{f'(x)} = \frac{1}{-\operatorname{cosec} x \cot x}$.

But $\operatorname{cosec} x \cot x = -y\sqrt{y^2 - 1}, x \in [-\frac{\pi}{2}, 0)$ and $\operatorname{cosec} x \cot x = y\sqrt{y^2 - 1}$, $x \in (0, \frac{\pi}{2}]$.

That is, $D \operatorname{cosec}^{-1} y = \frac{-1}{|y|\sqrt{y^2-1}}, y \in (-\infty, -1) \cup (1, \infty)$.

13. Let $f(x) = \cot x, x \in (0, \pi)$. f is strictly decreasing and continuous on $I = (0, \pi)$. $f(I) = \mathbb{R}$.

The inverse function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(y) = \cot^{-1} y, y \in \mathbb{R}$ is continuous on $\mathbb{R}$.

f is differentiable on $(0, \pi)$ and $f'(x) = -\operatorname{cosec}^2 x, x \in (0, \pi)$.

$f'(x) \neq 0$ on $(0, \pi)$. Hence $g'(y)$ exists for all $y \in \mathbb{R}$ and $g'(y) = \frac{1}{f'(x)} = \frac{-1}{\operatorname{cosec}^2 x} = -\frac{1}{1+y^2}$.

That is, $D \cot^{-1} y = -\frac{1}{1+y^2}, y \in \mathbb{R}$.

Worked Examples.

1. A function f is defined on some neighbourhood of c and f is differentiable at c. Prove that $\lim\limits_{h \to 0} \frac{f(c+h) - f(c-h)}{2h} = f'(c)$.

Show by an example that the limit may exist even if $f'(c)$ does not exist.

$$\begin{aligned}\lim_{h \to 0+} \frac{f(c+h)-f(c-h)}{2h} &= \lim_{h \to 0+} \left[\frac{f(c+h)-f(c)}{2h} + \frac{f(c-h)-f(c)}{-2h}\right] \\ &= \lim_{h \to 0+} \frac{f(c+h)-f(c)}{2h} + \lim_{k \to 0-} \frac{f(c+k)-f(c)}{2k} \\ &= \tfrac{1}{2} Rf'(c) + \tfrac{1}{2} Lf'(c), \text{ since } f'(c) \text{ exists} \\ &= f'(c).\end{aligned}$$

$$\begin{aligned}\lim_{h \to 0-} \frac{f(c+h)-f(c-h)}{2h} &= \lim_{h \to 0-} \left[\frac{f(c+h)-f(c)}{2h} + \frac{f(c-h)-f(c)}{-2h}\right] \\ &= \lim_{h \to 0-} \frac{f(c+h)-f(c)}{2h} + \lim_{k \to 0+} \frac{f(c+k)-f(c)}{2k} \\ &= \tfrac{1}{2} Lf'(c) + \tfrac{1}{2} Rf'(c) \\ &= f'(c).\end{aligned}$$

Therefore $\lim\limits_{h \to 0} \dfrac{f(c+h) - f(c-h)}{2h} = f'(c)$.

Second part. Let $f(x) = |\, x \,|, c = 0$.

$$\begin{aligned}\text{Then } \lim_{h \to 0} \frac{f(c+h)-f(c-h)}{2h} &= \lim_{h \to 0} \frac{f(h)-f(-h)}{2h} \\ &= \lim_{h \to 0} \frac{|h| - |-h|}{2h} = 0.\end{aligned}$$

But $f'(0)$ does not exist.

2. A function f is defined by $f(x) = x^2 \sin \frac{1}{x}, x \neq 0$
$= 0, x = 0$.

Show that f is differentiable at 0 but f' is not continuous at 0.

$\lim\limits_{x \to 0} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x \to 0} x \sin \frac{1}{x} = 0$, since $\lim\limits_{x \to 0} x = 0$ and $\sin \frac{1}{x}$ is bounded on some deleted neighbourhood of 0.

Hence $f'(0) = 0$. When $x \neq 0, f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$.

Thus the derived function f' is defined by

$$\begin{aligned} f'(x) &= 2x\sin\tfrac{1}{x} - \cos\tfrac{1}{x}, x \neq 0 \\ &= 0,\ x = 0. \end{aligned}$$

$\lim\limits_{x\to 0} f'(x)$ does not exist, since $\lim\limits_{x\to 0}\cos\frac{1}{x}$ does not exist.

Therefore f' is not continuous at 0.

3. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(0) = 0$ and

$$\begin{aligned} f(x) &= 0, \text{ if } x \text{ is irrational} \\ &= \tfrac{1}{q}, \text{ if } x = \tfrac{p}{q}, \text{ where } p \in \mathbb{Z}, q \in \mathbb{N} \text{ and } gcd(p,q) = 1. \end{aligned}$$

Show that f is not differentiable at 0.

$\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0}\frac{f(x)}{x}$.

Let $\phi(x) = \frac{f(x)}{x}$. Let $\{x_n\}$ be a sequence of rational points converging to 0 where $x_n = \frac{1}{n}, n \in \mathbb{N}$.

Then $\lim\limits_{n\to\infty}\phi(x_n) = \lim\limits_{n\to\infty} n.\frac{1}{n} = 1$.

Let $\{y_n\}$ be a sequence of irrational points converging to 0.

$\lim\limits_{n\to\infty}\phi(y_n) = \lim\limits_{n\to\infty}\dfrac{f(y_n)}{y_n} = 0$, since $f(y_n) = 0$ for all $n \in \mathbb{N}$.

Therefore $\lim\limits_{x\to 0}\phi(x)$ does not exist, since for two sequences $\{x_n\}$ and $\{y_n\}$ both converging to 0, the sequences $\{\phi(x_n)\}$ and $\{\phi(y_n)\}$ converge to two different limits.

Hence f is not differentiable at 0.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $c \in \mathbb{R}$ and $f(c) \neq 0$. Let $g(x) = |f(x)|, x \in \mathbb{R}$. Show that g is differentiable at c and

$$\begin{aligned} g'(c) &= f'(c), \text{ if } f(c) > 0 \\ &= -f'(c), \text{ if } f(c) < 0. \end{aligned}$$

Let $h(x) = |x|, x \in \mathbb{R}$. Then $g(x) = h(f(x)), x \in \mathbb{R}$.

$g'(c) = h'(f(c)).f'(c)$, provided h is differentiable at $f(c)$.

If $f(c) > 0$, then $f(x) > 0$ for all x in some neighbourhood of c, since f is continuous at c. Therefore $\lim\limits_{x\to c}\frac{|f(x)|-|f(c)|}{f(x)-f(c)} = \lim\limits_{x\to c}\frac{f(x)-f(c)}{f(x)-f(c)} = 1$.

That is, $h'(f(c)) = 1$ if $f(c) > 0$. Similarly, $h'(f(c)) = -1$ if $f(c) < 0$.

Therefore
$$\begin{aligned} g'(c) &= f'(c), \text{ if } f(c) > 0 \\ &= -f'(c), \text{ if } f(c) < 0. \end{aligned}$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $c \in \mathbb{R}$ and $f(c) = 0$. Let $g(x) = |f(x)|, x \in \mathbb{R}$. Show that g is differentiable at c if and only if $f'(c) = 0$.

Let $h(x) = |x|, x \in \mathbb{R}$. Then $g(x) = h(f(x)), x \in \mathbb{R}$.

$\lim\limits_{x\to c}\frac{g(x)-g(c)}{x-c} = \lim\limits_{x\to c}\frac{|f(x)|-|f(c)|}{f(x)-f(c)}.\frac{f(x)-f(c)}{x-c} = \lim\limits_{x\to c}\frac{|f(x)|}{f(x)}.\frac{f(x)-f(c)}{x-c}$.

$\lim\limits_{x \to c} \frac{|f(x)|}{f(x)}$ does not exist, in general. But $\frac{|f(x)|}{f(x)}$ is bounded in some deleted neighbourhood $N'(c)$ of c, since $|\frac{|f(x)|}{f(x)}| = 1$ for all $x \in N'(c)$.

Therefore $\lim\limits_{x \to c} \frac{|f(x)|}{f(x)} \cdot \frac{f(x)-f(c)}{x-c}$ exists if and only if $\lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c} = 0$, i.e., if and only if $f'(c) = 0$ and in this case $\lim\limits_{x \to c} \frac{g(x)-g(c)}{x-c} = 0$, i.e., $g'(c) = 0$.

6. Let $f(x) = \sin^{-1} \frac{2x}{1+x^2}$, $x \in \mathbb{R}$. Find the derived function f'.

Let $g(x) = \frac{2x}{1+x^2}$, $x \in \mathbb{R}$, $h(x) = \sin^{-1}x$, $|x| \leq 1$.

Since $|g(x)| \leq 1$ for all $x \in \mathbb{R}$, the composite function hg is defined for all $x \in \mathbb{R}$.

Let $c \in \mathbb{R}$. Then $|g(c)| \leq 1$ and $|g(c)| = 1$ if $c = \pm 1$. Since h is differentiable for all $x \in (-1, 1)$, h is not differentiable at $g(c)$ if $c = \pm 1$.

Let $c \in \mathbb{R}$ and $c \neq \pm 1$.

Then $f'(c) = h'(g(c)).g'(c) = \frac{1}{\sqrt{1-(\frac{2c}{1+c^2})^2}} \cdot \frac{2(1-c^2)}{(1+c^2)^2} = \frac{2(1-c^2)}{\sqrt{(1-c^2)^2}.(1+c^2)}$.

Therefore $f'(c) = \frac{2}{1+c^2}$, if $c^2 < 1$, i.e., if $|c| < 1$

$= \frac{-2}{1+c^2}$, if $c^2 > 1$, i.e., $|c| > 1$.

Hence $f'(x) = \frac{2}{1+x^2}$, if $|x| < 1$; $f'(x) = \frac{-2}{1+x^2}$, if $|x| > 1$; and f is not differentiable at ± 1.

7. Let $f(x) = x^3 + 2x + 3$, $x \in \mathbb{R}$. Show that f has an inverse function g on $\mathbb{R}$. Find the derivative of g at the points corresponding to $x = 0$, $x = -1$.

$f'(x) = 3x^2 + 2 > 0$ for all $x \in \mathbb{R}$. Therefore f is a continuous and strictly increasing function on $\mathbb{R}$.

Let $y \in \mathbb{R}$ has a pre-image x in $\mathbb{R}$. Then $x^3 + 2x + (3 - y) = 0$. This is a cubic equation in x and it has a real root. This means that each y has a pre-image and therefore $f(\mathbb{R}) = \mathbb{R}$.

f admits of an inverse function g on $\mathbb{R}$.

$f'(x) \neq 0$ on $\mathbb{R}$. Therefore g is differentiable at every point in $\mathbb{R}$ and $g'(y) = \frac{1}{f'(x)}$, where $f(x) = y$.

$f(0) = 3$. $g'(3) = \frac{1}{f'(0)} = \frac{1}{2}$. $f(-1) = 0$. $g'(0) = \frac{1}{f'(-1)} = \frac{1}{5}$.

9.2. Higher order derivatives.

Let I be an interval and a function $f : I \to \mathbb{R}$ be differentiable at a point $c \in I$. If f be differentiable at every point of some subinterval $I_1(c)$ such that $c \in I_1(c) \subset I$, then $f' : I_1(c) \to \mathbb{R}$ is a function on $I_1(c)$.

If f' be differentiable at c then the derivative of f' at c is called the *second order derivative* of f at c and is denoted by $f''(c)$ or by $f^{(2)}(c)$.

This is to note that c may also be an end point of the sub-interval $I_1(c)$.

If f' be differentiable at every point of some sub-interval $I_2(c)$ such that $c \in I_2(c) \subset I_1(c)$, then $f'' : I_2(c) \longrightarrow \mathbb{R}$ is a function on $I_2(c)$.

If f'' be differentiable at c then the derivative of f'' at c is called the *third order derivative* of f at c and is denoted by $f'''(c)$ or by $f^{(3)}(c)$.

In a similar manner we define the nth order derivative $f^{(n)}(c)$ whenever the derivative exists.

This is to emphasize that in order that the nth derivative of f may exist at c, $f^{(n-1)}$ must be defined on some sub-interval containing c, allowing the possibility of c to be an end point also of such subinterval.

Exercises 14

1. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by
$$\begin{aligned} f(x) &= x, x < 1 \\ &= 2 - x, 1 \le x \le 2 \\ &= x^2 - 3x + 2, x > 2. \end{aligned}$$
Show that $f'(x)$ does not exist at 1 and 2.

2. A function f is defined on some neighbourhood $N(0)$ of 0 by
$$\begin{aligned} f(x) &= \tfrac{x}{1+e^{1/x}}, x \neq 0 \\ &= 0, \; x = 0. \end{aligned}$$
Find $Lf'(0)$ and $Rf'(0)$. Show that f is not differentiable at 0.

3. A function f is defined by
$$\begin{aligned} f(x) &= x\left(\tfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}\right), x \neq 0 \\ &= 0, \; x = 0. \end{aligned}$$
Show that f is continuous at 0 but not differentiable at 0.

4. $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = |\, x \,| + |\, x - 1 \,| + |\, x - 2 \,|$, $x \in \mathbb{R}$.
Find the derived function f' and specify the domain of f'.

5. Find $f'(x)$ if (i) $f(x) = \sin^{-1} 2x\sqrt{1 - x^2}$, $|x| \le 1$,
(ii) $f(x) = \sin^{-1}(3x - 4x^3)$, $|x| \le 1$,
(iii) $f(x) = \cos^{-1}(8x^4 - 8x^2 + 1)$, $|x| \le 1$.

6. $f : \mathbb{R} \to \mathbb{R}$ is defined by
$$\begin{aligned} f(x) &= e^{-1/x^2} \sin \tfrac{1}{x}, x \neq 0 \\ &= 0, \; x = 0. \end{aligned}$$
Show that f' is continuous at 0.

7. A function f is defined on $(-1, 1)$ by
$$\begin{aligned} f(x) &= x^\alpha \sin \tfrac{1}{x^\beta}, x \neq 0 \\ &= 0, x = 0. \end{aligned}$$
Prove that (i) if $0 < \beta < \alpha - 1$, f' is continuous at 0;
(ii) if $0 < \alpha - 1 \le \beta$, f' is discontinuous at 0.

8. $f(x) = x^2 \sin \frac{1}{x}, x \neq 0$ and $g(x) = x,\ x \in \mathbb{R}$.
$\quad = 0,\ x = 0;$

Show that $\lim\limits_{x \to 0} \dfrac{f'(x)}{g'(x)}$ does not exist, but $\lim\limits_{x \to 0} \dfrac{f(x)}{g(x)} = \dfrac{f'(0)}{g'(0)}$.

9. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$\begin{aligned} f(x) &= 0, \text{ if } x = 0 \text{ or } x \text{ is irrational} \\ &= \tfrac{1}{q^3}, \text{ if } x = \tfrac{p}{q}, \text{ where } p \in \mathbb{Z}, q \in \mathbb{N} \text{ and } gcd(p,q) = 1. \end{aligned}$$

Show that f is differentiable at 0 and $f'(0) = 0$.

[**Hint.** For $x \neq 0, 0 \leq \left| \frac{f(x)}{x} \right| \leq x^2$.]

10. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$, if x is rational
$= 0$, if x is irrational.

Show that f is differentiable at 0 and $f'(0) = 0$.

[**Hint.** $0 \leq \frac{f(x)}{x} \leq x$ for $x > 0$ and $x \leq \frac{f(x)}{x} \leq 0$ for $x < 0$.]

11. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x$, if x is rational
$= \sin x$, if x is irrational.

Show that f is differentiable at 0 and $f'(0) = 1$.

[**Hint.** $\cos x \leq \frac{f(x)}{x} \leq 1$ for all $x \in N'(0, \frac{\pi}{2})$.]

12. Let $f(x) = x^5 + 4x + 1, x \in \mathbb{R}$.
(i) Show that f has an inverse function g differentiable on $\mathbb{R}$.
(ii) Find $g'(1)$, $g'(6)$.

9.3. Sign of the derivative.

Let $I \subset \mathbb{R}$ be an interval and a function $f : I \to \mathbb{R}$.

Let c be an *interior point* of I.

f is said to be *increasing* at c if there exists a positive δ such that $f(x) < f(c)$ for all $x \in I$ satisfying $c - \delta < x < c$ and $f(x) > f(c)$ for all $x \in I$ satisfying $c < x < c + \delta$.

f is said to be *decreasing* at c if there exists a positive δ such that $f(x) > f(c)$ for all $x \in I$ satisfying $c - \delta < x < c$ and $f(x) < f(c)$ for all $x \in I$ satisfying $c < x < c + \delta$.

Let c be the *left end point* of I.

f is said to be *increasing* at c if there exists a positive δ such that $f(x) > f(c)$ for all $x \in I$ satisfying $c < x < c + \delta$;

f is said to be *decreasing* at c if there exists a positive δ such that $f(x) < f(c)$ for all $x \in I$ satisfying $c < x < c + \delta$.

Let c be the *right end point* of I.

f is said to be *increasing* at c if there exists a positive δ such that $f(x) < f(c)$ for all $x \in I$ satisfying $c - \delta < x < c$;

f is said to be *decreasing* at c if there exists a positive δ such that $f(x) > f(c)$ for all $x \in I$ satisfying $c - \delta < x < c$.

Theorem 9.3.1. Let $I \subset \mathbb{R}$ be an interval and a function $f : I \to \mathbb{R}$ be differentiable at $c \in I$.

(i) If $f'(c) > 0$ then f is increasing at c

(ii) if $f'(c) < 0$ then f is decreasing at c.

Proof. (i) $\lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c} > 0$. Therefore there exists a positive δ such that $\frac{f(x)-f(c)}{x-c} > 0$ for all $x \in N'(c, \delta) \cap I$.

Let c be an interior point of I.

Then $\frac{f(x)-f(c)}{x-c} > 0$ for all $x \in (c-\delta, c) \cap I$ and for all $x \in (c, c+\delta) \cap I$.

Therefore $f(x) < f(c)$ for all $x \in (c - \delta, c) \cap I$ and $f(x) > f(c)$ for all $x \in (c, c + \delta) \cap I$. This proves that f is increasing at c.

Let c be the left end point of I.

Then $\frac{f(x)-f(c)}{x-c} > 0$ for all $x \in I$ such that $c < x < c + \delta$.

Therefore $f(x) > f(c)$ for all $x \in I$ such that $c < x < c + \delta$.

This proves that f is increasing at c.

Let c be the right end point of I.

Then $\frac{f(x)-f(c)}{x-c} > 0$ for all $x \in I$ such that $c - \delta < x < c$.

Therefore $f(x) < f(c)$ for all $x \in I$ satisfying $c - \delta < x < c$.

This proves that f is increasing at c.

(ii) Similar proof.

Note 1. A function f may be increasing (or decreasing) at a point c in its domain without being differentiable at c. For example, the function f defined by
$$\begin{aligned} f(x) &= x, x < 1 \\ &= 2x - 1, x \geq 1 \end{aligned}$$
is increasing at 1 but f is not differentiable at 1.

The function f defined by
$$\begin{aligned} f(x) &= 1 - x, x < 0 \\ &= 1 - 2x, x \geq 0 \end{aligned}$$
is decreasing at 0 but f is not differentiable at 0.

Note 2. If f is increasing at a point c then $f'(c)$ may not be positive. For example, let $f(x) = x^3, x \in \mathbb{R}$. f is increasing at 0, but $f'(0) = 0$.

If f is decreasing at a point c then $f'(c)$ may not be negative. For example, let $f(x) = -x^3, x \in \mathbb{R}$. f is decreasing at 0, but $f'(0) = 0$.

Note 3. If $f'(x) > 0$ at c, it does not follow that f increases monotonically in some neighbourhood of c. For if x_1, x_2 be any two points in a small neighbourhood of c such that $c < x_1 < x_2$ then it has only been proved that $f(c) < f(x_1)$ and $f(c) < f(x_2)$ and we cannot conclude that $f(x_1) < f(x_2)$.

For example, let $f(x) = \frac{x}{2} + x^2 \sin \frac{1}{x}, x \neq 0$
$= 0, x = 0.$

Then f is increasing at 0. But in every neighbourhood of $0, f'(x)$ assumes both positive and negative values. Therefore f is not monotonic in any neighbourhood of 0.

9.4. Properties of the derivative.

We have seen that if a function f be continuous on a closed and bounded interval $[a, b]$ and $f(a) \neq f(b)$, then f assumes every value between $f(a)$ and $f(b)$. We have a similar theorem for a derived function.

Theorem 9.4.1. (Darboux)

Let $I = [a, b]$ and a function $f : I \to \mathbb{R}$ be differentiable on I. Let $f'(a) \neq f'(b)$. If k be a real number lying between $f'(a)$ and $f'(b)$ then there exists a point c in (a, b) such that $f'(c) = k$.

Proof. Without loss of generality, let $f'(a) < k < f'(b)$.

Let us define $g : [a, b] \to \mathbb{R}$ by $g(x) = f(x) - kx, x \in [a, b]$.

g is differentiable on $[a, b]$ and therefore g is continuous on $[a, b]$. Consequently, g will attain the minimum value (the greatest lower bound) at some point c in $[a, b]$.

$g'(a) = f'(a) - k < 0$. This implies g is decreasing at a.

Therefore there exists a positive δ such that $g(x) < g(a)$ for all $x \in [a, b]$ satisfying $a < x < a + \delta$.

This shows that $g(a)$ is not the minimum value of g on $[a, b]$.

$g'(b) = f'(b) - k > 0$. This implies g is increasing at b.

Therefore there exists a positive δ such that $g(x) < g(b)$ for all $x \in [a, b]$ satisfying $b - \delta < x < b$.

This shows that $g(b)$ is not the minimum value of g on $[a, b]$.

Thus $c \neq a, c \neq b$ and therefore $a < c < b$.

Since $c \in (a, b)$, $g'(c)$ exists. We prove that $g'(c) = 0$.

Let $g'(c) > 0$. Then there exists a positive δ such that $g(x) < g(c)$ for all $x \in [a, b]$ satisfying $c - \delta < x < c$. This contradicts that $g(c)$ is the minimum value of g on $[a, b]$. Therefore $g'(c) \not> 0$.

Let $g'(c) < 0$. Then there exists a positive δ such that $g(x) < g(c)$ for all $x \in [a, b]$ satisfying $c < x < c + \delta$. This contradicts that $g(c)$ is the minimum value of g on $[a, b]$. Therefore $g'(c) \not< 0$.

Consequently, $g'(c) = 0$, i.e., $f'(c) = k$ and the theorem is established.

Corollary. Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be differentiable on I. If $f'(a)f'(b) < 0$ then there exists a point c in (a, b) such that $f'(c) = 0$.

Note. Darboux's theorem is the intermediate-value property of the derived function f' on $[a, b]$. Although a derived function f' may not be a continuous function on $[a, b]$ (Ex.2, Page 315) the intermediate-value property holds for a derived function.

Worked Example.

1. Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) = 0, x \in [-1, 0]$
$= 1, x \in (0, 1]$.

Does there exist a function g such that $g'(x) = f(x), x \in [-1, 1]$?

If possible, let there exist a function $g : [-1, 1] \to \mathbb{R}$ such that $g'(x) = f(x), x \in [-1, 1]$.

Then g is differentiable on $[-1, 1]$ and
$$\begin{aligned} g'(x) &= 0, x \in [-1, 0] \\ &= 1, x \in (0, 1]. \end{aligned}$$

Since g is differentiable on $[-1, 1]$ and $g'(-1) \neq g'(1)$, by Darboux's theorem g' must assume every real number lying between $g'(-1)$ and $g'(1)$, i.e., between 0 and 1 on $[-1, 1]$. But this is not so and therefore g does not exist.

Theorem 9.4.2. Let I be an interval and a function $f : I \to \mathbb{R}$ be differentiable on I. Then $f'(I)$ is an interval.

[A subset S of $\mathbb{R}$ is an interval if for any two points $c, d \in S$ with $c < d$, the closed interval $[c, d] \subset S$].

Proof. Let $p, q \in f'(I)$ and $p < q$. There exists points $c, d \in I$ such that $f'(c) = p, f'(d) = q$. Let $r \in (p, q)$. Then $p < r < q$.

By Darboux's theorem, there exists a point x_0 in (c, d) [or (d, c)] such that $f'(x_0) = r$. Therefore $r \in f'(I)$ and that implies $(p, q) \subset f'(I)$. Also $p \in f'(I)$ and $q \in f'(I)$. Hence $[p, q] \subset f'(I)$.

Therefore $f'(I)$ is an interval.

Theorem 9.4.3. If $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ then the derived function f' cannot have a jump discontinuity on $[a, b]$.

Proof. If for some $c \in (a, b]$ $\lim\limits_{x \to c-} f'(x)$ does not exist, or for some $c \in [a, b)$ $\lim\limits_{x \to c+} f'(x)$ does not exist then clearly c cannot be a point of jump discontinuity of f'.

Therefore let us assume that for each $c \in (a, b]$, $\lim_{x \to c-} f'(x)$ exists and also for each $c \in [a, b)$, $\lim_{x \to c+} f'(x)$ exists. Now it is sufficient to prove

(i) $\lim_{x \to c-} f'(x) = f'(c)$ for all $c \in (a, b]$, and

(ii) $\lim_{x \to c+} f'(x) = f'(c)$ for all $c \in [a, b)$.

Let $c \in (a, b]$ and let $\lim_{x \to c-} f'(x) = l$. We prove that $f'(c) = l$.

If not, let $l < f'(c)$. Let us choose $\epsilon > 0$ such that $l + \epsilon < f'(c)$.

Since $\lim_{x \to c-} f'(x) = l$, there exists a positive δ such that

$l - \epsilon < f'(x) < l + \epsilon$ for all $x \in (c - \delta, c) \cap (a, b]$.

Let $d \in (c - \delta, c) \cap (a, b]$. Then $l - \epsilon < f'(d) < l + \epsilon < f'(c)$.

By Darboux's theorem on $[d, c] \subset [a, b]$, there exists a point ξ in (d, c) such that $f'(\xi) = l + \epsilon$.

But $\xi \in (d, c) \Rightarrow \xi \in (c - \delta, c) \cap (a, b]$ and this implies $f'(\xi) < l + \epsilon$, a contradiction. Therefore $l \not< f'(c)$.

Next let $l > f'(c)$. Let us choose $\epsilon_1 > 0$ such that $l - \epsilon_1 > f'(c)$.

Since $\lim_{x \to c-} f'(x) = l$, there exists a positive δ_1 such that

$l - \epsilon_1 < f'(x) < l + \epsilon_1$ for all $x \in (c - \delta_1, c) \cap (a, b]$.

Let $d_1 \in (c - \delta_1, c) \cap (a, b]$. Then $f'(c) < l - \epsilon_1 < f'(d_1)$.

By Darboux's theorem on $[d_1, c] \subset [a, b]$, there exists a point ξ_1 in (d_1, c) such that $f'(\xi_1) = l - \epsilon_1$.

But $\xi_1 \in (d_1, c) \Rightarrow \xi_1 \in (c - \delta_1, c) \cap (a, b]$ and this implies $f'(\xi_1) > l - \epsilon_1$, a contradiction. Therefore $l \not> f'(c)$.

Thus $\lim_{x \to c-} f'(x) = f'(c)$ for all $c \in (a, b]$.

In a similar manner it can be proved that $\lim_{x \to c+} f'(x) = f'(c)$ for all $c \in [a, b)$. This completes the proof.

Note 1. A derived function on an interval can have a discontinuity of second kind. (Ex.3.)

2. If f' exists in some deleted neighbourhood of c and $\lim_{x \to c-} f'(x) \neq \lim_{x \to c+} f'(x)$ then f cannot be differentiable at c. (Ex.4.)

3. If a function f be differentiable on an interval I and f' is monotonic on I then f' is continuous on I.

It follows from the property that a monotone function can have only jump discontinuities in its domain.

Worked Examples (continued).

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 \sin \frac{1}{x}, x \neq 0$
$= 0, \quad x = 0.$

Show that f is differentiable on $\mathbb{R}$ but f' is not continuous on $\mathbb{R}$.

For $x \neq 0, f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$.

$\lim\limits_{x \to 0} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x \to 0} x \sin \frac{1}{x} = 0$, by Theorem 7.1.7.

Therefore $f'(0) = 0$. Hence f is differentiable on $\mathbb{R}$ and the derived function f' is defined by $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}, x \neq 0$
$= 0, x = 0.$

Now $\lim\limits_{x \to 0} f'(x)$ does not exist, since $\lim\limits_{x \to 0} \cos \frac{1}{x}$ does not exist. This proves that f' is not continuous at 0.

Note. f' is bounded on any neighbourhood $(-\delta, \delta)$ of 0. 0 is a point of oscillatory discontinuity of f'.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 \sin \frac{1}{x^2}, x \neq 0$
$= 0, \quad x = 0.$

Show that f is differentiable on $\mathbb{R}$ but f' is not continuous on $\mathbb{R}$.

For $x \neq 0, f'(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}$.

$\lim\limits_{x \to 0} \frac{f(x)-f(c)}{x-0} = \lim\limits_{x \to 0} \dfrac{x^2 \sin \frac{1}{x^2}}{x} = \lim\limits_{x \to 0} x \sin \frac{1}{x^2} = 0.$

Therefore $f'(0) = 0$. Hence f is differentiable on $\mathbb{R}$ and the derived function f' is defined by

$$\begin{aligned} f'(x) &= 2x \sin \tfrac{1}{x^2} - \tfrac{2}{x} \cos \tfrac{1}{x^2}, x \neq 0 \\ &= 0, x = 0. \end{aligned}$$

Now $\lim\limits_{x \to 0} f'(x)$ does not exist, since $\lim\limits_{x \to 0} \frac{2}{x} \cos \frac{1}{x^2}$ does not exist. This proves that f' is not continuous at 0.

f' is unbounded on any neighbourhood of 0. 0 is a point of infinite discontinuity of f'.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |\, x \,|, x \in \mathbb{R}$.

$$\begin{aligned} \text{Then } f(x) &= x, x > 0 \\ &= 0, x = 0 \\ &= -x, x < 0. \end{aligned}$$

For $x > 0, f'(x) = 1$ and when $x < 0, f'(x) = -1$. $\lim\limits_{x \to 0-} f'(x) \neq \lim\limits_{x \to 0+} f'(x)$. This does not contradict the Theorem 9.4.3, because f is not differentiable at 0.

9.5. Rolle's theorem and Mean value theorems.

Theorem 9.5.1. (Rolle's theorem)

Let a function $f : [a, b] \to \mathbb{R}$ be such that

(i) f is continuous on $[a, b]$,

(ii) f is differentiable at every point of (a, b), and

(iii) $f(a) = f(b)$.

Then there exists at least one point ξ in (a, b) such that $f'(\xi) = 0$.

Proof. Since f is continuous on $[a, b]$, f is bounded on $[a, b]$.

Let $\sup\limits_{x\in[a,b]} f(x) = M$, $\inf\limits_{x\in[a,b]} f(x) = m$.

By the property of continuity there exists a point c in $[a, b]$ such that $f(c) = M$ and there exists point d in $[a, b]$ such that $f(d) = m$.

Two cases arise.

Case 1. $M = m$.

In this case $f(x) = M$ for all $x \in [a, b]$. Therefore $f'(x) = 0$ for all $x \in [a, b]$. The theorem holds trivially in this case.

Case 2. $M \neq m$.

In this case at least one of M and m, if not both, must be unequal to $f(a)$ (and $f(b)$).

Let $M \neq f(a)$. Then $c \neq a, c \neq b$. $\therefore$ $a < c < b$.

By condition (ii), $f'(c)$ exists.

If possible, let $f'(c) > 0$. Then $\lim\limits_{x\to c} \dfrac{f(x) - f(c)}{x - c} > 0$.

Hence there exists a positive δ such that $\frac{f(x)-f(c)}{x-c} > 0$ for all $x \in [a, b]$ satisfying $0 < | x - c | < \delta$.

This implies $f(x) - f(c) > 0$ for all $x \in [a, b]$ satisfying $c < x < c + \delta$ i.e., $f(x) > M$ for all $x \in [a, b]$ satisfying $c < x < c + \delta$.

This contradicts that M is the supremum of f on $[a, b]$.

Consequently, $f'(c) \not> 0$ ... ... (i)

If possible, let $f'(c) < 0$. Then $\lim\limits_{x\to c} \dfrac{f(x) - f(c)}{x - c} < 0$.

Hence there exists a positive δ such that $\frac{f(x)-f(c)}{x-c} < 0$ for all $x \in [a, b]$ satisfying $0 < | x - c | < \delta$.

This implies $f(x) > f(c)$ for all $x \in [a, b]$ satisfying $c - \delta < x < c$ i.e., $f(x) > M$ for all $x \in [a, b]$ satisfying $c - \delta < x < c$.

This contradicts that M is the supremum of f on $[a, b]$.

Consequently, $f'(c) \not< 0$ ... ... (ii)

From (i) and (ii) it follows that $f'(c) = 0$. Therefore $\xi = c$.

If however, $M = f(a) = f(b)$, then $m \neq f(a)$ and therefore $d \neq a, d \neq b$ and $a < d < b$.

Proceeding with similar arguments we can prove $f'(d) = 0$. Therefore $\xi = d$.

This completes the proof of the theorem.

Note. The set of conditions in Rolle's theorem is a set of sufficient conditions in the sense that $f'(x)$ may be zero at some point ξ in (a, b) even if the conditions of the theorem do not hold together. To establish this, let us consider the function

$$f(x) = | x | + | x - 1 |, x \in [-1, 2].$$

$$\begin{aligned} f(x) &= 2x - 1, 1 < x \leq 2 \\ &= 1, \quad 0 \leq x \leq 1 \\ &= 1 - 2x, -1 \leq x < 0. \end{aligned}$$

f is continuous on $[-1, 2]$; f is not differentiable at 0 and 1; $f(-1) = f(2) = 3$.

Therefore f does not satisfy the second condition of Rolle's theorem.

But $f'(x) = 0$ for all $x \in (0, 1)$.

Although f does not satisfy all the conditions of Rolle's theorem together on $[-1, 2], f'(x) = 0$ at some points in $(-1, 2)$.

Corollary. Rolle's theorem for polynomials.

If a polynomial function f has at least two real roots, then between any two real roots there exists at least one real root of the derived polynomial function f'.

Let α, β be two real roots of the polynomial function $f, \alpha < \beta$.

Then (i) f is continuous on $[\alpha, \beta]$, (ii) f is differentiable on (α, β) and (iii) $f(\alpha) = f(\beta)$.

Therefore by Rolle's theorem, there exists at least one real number ξ in (α, β) such that $f'(\xi) = 0$. That is, ξ is a real root of the derived polynomial function f'.

Geometrical Interpretation.

If a function f has a graph which is a continuous curve on the interval $[a, b]$; and the curve has a tangent at every point on it with abscissa between a and b; and the ordinates $f(a), f(b)$ are equal, then there exists at least one point ξ in (a, b) such that the tangent to the curve at $(\xi, f(\xi))$ is parallel to the x-axis.

Worked Examples.

1. If $p(x)$ is a polynomial of degree > 1 and $k \in \mathbb{R}$, prove that between any two real roots of $p(x) = 0$ there is a real root of $p'(x) + kp(x) = 0$.

Let $f(x) = e^{kx}p(x), x \in \mathbb{R}$.
Then $f'(x) = e^{kx}[kp(x) + p'(x)], x \in \mathbb{R}$.
Let α, β be two real roots of $p(x)$ and $\alpha < \beta$. Then $p(\alpha) = 0, p(\beta) = 0$.
Therefore $f(\alpha) = e^{k\alpha}p(\alpha) = 0$. $f(\beta) = e^{k\beta}p(\beta) = 0$.
f is continuous on $[\alpha, \beta]$; $f'(x)$ exists for all $x \in (\alpha, \beta)$; and $f(\alpha) = f(\beta)$.
By Rolle's theorem, $f'(\gamma) = 0$ for some γ in (α, β).
or, $e^{k\gamma}[kp(\gamma) + p'(\gamma)] = 0$.
This implies $kp(\gamma) + p'(\gamma) = 0$, since $e^{k\gamma} \neq 0$.
That is, γ is a root of $kp(x) + p'(x) = 0$, where $\alpha < \gamma < \beta$.

2. The functions u, v, u' and v' are all continuous on $\mathbb{R}$ and $uv' - u'v \neq 0$ in $\mathbb{R}$. Prove that between any two consecutive real roots of $u = 0$ lies one real root of $v = 0$ and between any two consecutive real roots of $v = 0$ lies one real root of $u = 0$.

Let α, β be any two consecutive real roots of $u = 0, \alpha < \beta$. We prove that there exists a real root of $v = 0$ in (α, β)

Since $u(\alpha) = 0$ and $u(\beta) = 0, v(\alpha) \neq 0$ and $v(\beta) \neq 0$ by the given condition.

If possible, let $v = 0$ has no real root in (α, β). Then $v \neq 0$ in $[\alpha, \beta]$. Let $f = \frac{u}{v}$ in $[\alpha, \beta]$.

Then the function f is continuous on $[\alpha, \beta]$; $f' = \frac{vu' - v'u}{v^2}$ exists in (α, β); and $f(\alpha) = 0 = f(\beta)$.

By Rolle's theorem, there exists a point ξ in (α, β) such that $f'(\xi) = 0$.

This implies $uv' - u'v = 0$ at ξ. This is a contradiction to the hypothesis. Therefore $v = 0$ has a real root lying in (α, β).

Let γ, δ be two consecutive real roots of $v = 0, \gamma < \delta$. Similar arguments will establish that $u = 0$ has a real root lying in (γ, δ).

Observation. Taking $u = \sin x, v = \cos x$ it follows that between any two consecutive zeroes of $\sin x$ there is a zero of $\cos x$, and conversely.

3. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and $f''(x)$ exists for all $x \in (a, b)$. Let $a < c < b$. Prove that there exists a point ξ in (a, b) such that

$$f(c) = \frac{b-c}{b-a}f(a) + \frac{c-a}{c-b}f(b) + \frac{1}{2}(c-a)(c-b)f''(\xi).$$

Let $\phi : [a, b] \to \mathbb{R}$ be defined by

$$\phi(x) = f(x) - \frac{(x-b)(x-c)}{(a-b)(a-c)}f(a) - \frac{(x-c)(x-a)}{(b-c)(b-a)}f(b) - \frac{(x-a)(x-b)}{(c-a)(c-b)}f(c), \ x \in [a, b].$$

ϕ is continuous on $[a, b]$, since f is continuous on $[a, b]$.

Since $f''(x)$ exists for all $x \in (a, b)$, f' is continuous on (a, b).

Therefore $\phi''(x)$ exists for all $x \in (a, b)$ and ϕ' is continuous on (a, b) and hence ϕ is differentiable on (a, b).

$\phi(a) = 0, \phi(b) = 0, \phi(c) = 0$.

Applying Rolle's theorem to the function ϕ on $[a, c]$ and $[c, b]$,
$\phi'(\xi_1) = 0$ for some $\xi_1 \in (a, c)$ and $\phi'(\xi_2) = 0$ for some $\xi_2 \in (c, b)$.

Applying Rolle's theorem to the function ϕ' on $[\xi_1, \xi_2]$,
$\phi''(\xi) = 0$ for some $\xi \in (\xi_1, \xi_2)$. That is, $\phi''(\xi) = 0$ for some $\xi \in (a, b)$.
But $\phi''(\xi) = f''(\xi) - \frac{2f(a)}{(a-b)(a-c)} - \frac{2f(b)}{(b-c)(b-a)} - \frac{2f(c)}{(c-a)(c-b)}$.

Hence $f(c) = \frac{b-c}{b-a} f(a) + \frac{c-a}{b-a} f(b) + \frac{1}{2}(c-a)(c-b)f''(\xi), a < \xi < b$.

Theorem 9.5.2. Mean value theorem (Lagrange)

Let a function $f : [a, b] \to \mathbb{R}$ be such that

(i) f is continuous on $[a, b]$, and

(ii) f is differentiable at every point of (a, b).

Then there exists at least a point ξ in (a, b) such that

$$\frac{f(b)-f(a)}{b-a} = f'(\xi).$$

Proof. Let us define $\phi : [a, b] \to \mathbb{R}$ by
$\phi(x) = f(x) + Ax, x \in [a, b]$ where A is a constant.

Clearly, ϕ is continuous on $[a, b]$, since f is continuous on $[a, b]$; and ϕ is differentiable at every point of (a, b), since f is differentiable at every point of (a, b).

Let us choose A such that $\phi(a) = \phi(b)$.

Then $f(a) + Aa = f(b) + Ab$ and this determines $A = \frac{f(b)-f(a)}{a-b}$.

For this choice of A, ϕ satisfies all conditions of Rolle's theorem on $[a, b]$. Therefore there exists at least a point ξ in (a, b) such that $\phi'(\xi) = 0$.

But $\phi'(\xi) = f'(\xi) + A$ and therefore $0 = f'(\xi) + \frac{f(b)-f(a)}{a-b}$
or, $f'(\xi) = \frac{f(b)-f(a)}{b-a}$.

Another form.

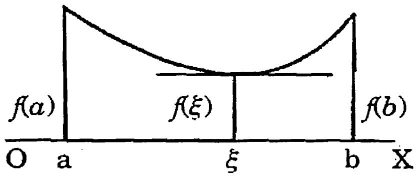
If $b = a + h$ then $b - a = h$ and $\xi = a + \theta h$ for some real number θ satisfying $0 < \theta < 1$. The theorem takes the following form.

Let $f : [a, a+h] \to \mathbb{R}$ be such that (i) f is continuous on $[a, a+h]$, and (ii) f is differentiable at every point of $(a, a+h)$.

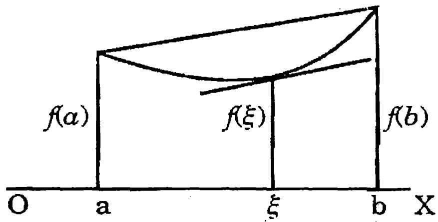
Then there exists a real number θ satisfying $0 < \theta < 1$ such that $f(a+h) = f(a) + hf'(a+\theta h)$.

Geometrical interpretation.

If a function f has a graph which is a continuous curve on the interval $[a, b]$ and the curve has a tangent at every point on it with abscissa between a and b then there exists a point ξ in (a, b) such that the tangent to the curve at $(\xi, f(\xi))$ is parallel to the line segment joining the points $(a, f(a))$ and $(b, f(b))$.



Rolle's theorem



Mean value theorem

Remark.

Rolle's theorem is a particular case of Mean value theorem. If $f(a) = f(b)$ holds in addition to the two conditions of Mean value theorem, then $f(b) - f(a) = 0$ and consequently $f'(\xi) = 0$.

In the particular case, the geometrical interpretation is that there is a point $(\xi, f(\xi))$ on the curve, the tangent at which is parallel to the x-axis.

Theorem 9.5.3. Let a function $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = 0$ for all $x \in (a, b)$ then f is a constant on $[a, b]$.

Proof. Let $x_1, x_2 \in [a, b]$ and $a \leq x_1 < x_2 \leq b$.

Then f is continuous on $[x_1, x_2]$, and f is differentiable on (x_1, x_2).

By the Mean value theorem there exists a point ξ in (x_1, x_2) such that $f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$.

But $f'(\xi) = 0$, by hypothesis. Therefore $f(x_2) = f(x_1)$.

Since x_1 and x_2 are arbitrary points in $[a, b]$, f is a constant on $[a, b]$.

Corollary. Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be both continuous on $[a, b]$ and they are both differentiable on (a, b). If $f'(x) = g'(x)$ for all $x \in (a, b)$ then $f = g + c$, where $c (\in \mathbb{R})$ is a constant.

Theorem 9.5.4. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and $f'(x) \geq 0$ for all $x \in (a, b)$. Then f is a monotone increasing function on $[a, b]$.

Proof. Let $x_1, x_2 \in [a,b]$ and $a \le x_1 < x_2 \le b$.
Then f is continuous on $[x_1, x_2]$, and f is differentiable on (x_1, x_2).

By the Mean value theorem there exists a point ξ in (x_1, x_2) such that $f'(\xi) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. But $f'(\xi) \ge 0$.
Therefore $f(x_2) \ge f(x_1)$, since $x_2 - x_1 > 0$.

Since x_1 and x_2 are arbitrary points in $[a,b]$, f is a monotone increasing function on $[a,b]$.

Note. If $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and $f'(x) > 0$ for all $x \in (a,b)$ then f is a strictly increasing function on $[a,b]$.

Theorem 9.5.5. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and $f'(x) \le 0$ for all $x \in (a,b)$. Then f is a monotone decreasing function on $[a,b]$.

The proof is similar and left to the reader.

Note. If $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and $f'(x) < 0$ for all $x \in (a,b)$ then f is a strictly decreasing function on $[a,b]$.

Worked Examples.

1. Prove that $\frac{2x}{\pi} < \sin x$ for $0 < x < \frac{\pi}{2}$.

Let $f(x) = \frac{\sin x}{x}$, $0 < x < \frac{\pi}{2}$.
f is continuous on $[\delta, \frac{\pi}{2}]$ for some $\delta > 0$. $f'(x) = \frac{x \sin x - \cos x}{x}$ on $[\delta, \frac{\pi}{2}]$.
Because $x < \tan x$ in $0 < x < \frac{\pi}{2}$, $f'(x) < 0$ in $\delta < x < \frac{\pi}{2}$.
Therefore f is a strictly decreasing function on $(0, \frac{\pi}{2}]$.

Because $f(\frac{\pi}{2}) = \frac{2}{\pi}$, it follows that $f(x) > \frac{2}{\pi}$ for $0 < x < \frac{\pi}{2}$, i.e., $\frac{2x}{\pi} < \sin x$ for $0 < x < \frac{\pi}{2}$.

2. Prove that $\frac{x}{1+x} < \log(1+x) < x$ for all $x > 0$.

Let $f(x) = \log(1+x) - \frac{x}{1+x}, x \ge 0$.
f is continuous on $[0,\infty)$. $f'(x) = \frac{x}{(1+x)^2} > 0$ for all $x > 0$.

Therefore f is a strictly increasing function on $[0,\infty)$.
So $f(x) > f(0)$ for all $x > 0$.

Consequently, $\log(1+x) > \frac{x}{1+x}$ for all $x > 0$ ... ... (i)

Let $g(x) = x - \log(1+x), x \ge 0$.
g is continuous on $[0,\infty)$. $g'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0$ for all $x > 0$.

Therefore g is a strictly increasing function on $[0,\infty)$.
So $g(x) > g(0)$ for all $x > 0$.

Consequently, $x > \log(1+x)$ for all $x > 0$ ... ... (ii)

From (i) and (ii), $\frac{x}{1+x} < \log(1+x) < x$ for all $x > 0$.

3. Prove that $(1+\frac{1}{x})^x > (1+\frac{1}{y})^y$ if $x, y \in \mathbb{R}$ and $x > y > 0$.

Let $f(x) = (1+\frac{1}{x})^x, x > 0$.

$$\begin{aligned} \text{Then } f'(x) &= (1+\tfrac{1}{x})^x[\log(1+\tfrac{1}{x}) + x.\tfrac{1}{1+\frac{1}{x}}.(-\tfrac{1}{x^2})] \\ &= (1+\tfrac{1}{x})^x[\log(1+\tfrac{1}{x}) - \tfrac{\frac{1}{x}}{1+\frac{1}{x}}]. \end{aligned}$$

Let $\phi(x) = \log(1+x) - \frac{x}{1+x}, x \geq 0$.

Then ϕ is continuous on $[0,\infty)$ and $\phi'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{1+x^2} > 0$ for $x > 0$.

Hence ϕ is a strictly increasing function on $[0,\infty)$.

Since $\phi(0) = 0, \phi(x) > 0$ for $x > 0$. That is, $\log(1+x) > \frac{x}{1+x}$ for $x > 0$.

It follows that $\log(1+\frac{1}{x}) > \frac{\frac{1}{x}}{1+\frac{1}{x}}$ for $x > 0$.

We also have $(1+\frac{1}{x})^x > 0$ for $x > 0$. Therefore $f'(x) > 0$ for $x > 0$, showing that f is a strictly increasing function for $x > 0$.

Hence $x > y > 0 \Rightarrow (1+\frac{1}{x})^x > (1+\frac{1}{y})^y$.

4. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition $| f(x) - f(y) | \leq (x-y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function on $\mathbb{R}$.

By the condition,
$-(x-y)^2 \leq f(x) - f(y) \leq (x-y)^2$ for all $x, y \in \mathbb{R}$.

Let $c \in \mathbb{R}$. Then $-h^2 \leq f(c+h) - f(c) \leq h^2$ for all $h \in \mathbb{R}$.

That is, $-h \leq \frac{f(c+h)-f(c)}{h} \leq h$ if $h > 0$ and $h \leq \frac{f(c+h)-f(c)}{h} \leq -h$ if $h < 0$.

By Sandwich theorem, it follows from the first that $\lim\limits_{h\to 0+} \frac{f(c+h)-f(c)}{h} = 0$; and it follows from the second that $\lim\limits_{h\to 0-} \frac{f(c+h)-f(c)}{h} = 0$.

Consequently, $\lim\limits_{h\to 0} \dfrac{f(c+h)-f(c)}{h} = 0$. That is, $f'(c) = 0$ for all $c \in \mathbb{R}$.

This proves that f is a constant function on $\mathbb{R}$.

5. Use Mean value theorem to prove $0 < \frac{1}{x}\log\frac{e^x-1}{x} < 1$, for $x > 0$.

Let $x_0 > 0$. Let $f(x) = e^x, x \in [0, x_0]$.

Then f is continuous on $[0, x_0]$ and f is differentiable on $(0, x_0)$.

By Mean value theorem, there exists a real number θ satisfying $0 < \theta < 1$ such that $\frac{f(x_0)-f(0)}{x_0} = f'(\theta x_0)$

or, $e^{x_0} - 1 = x_0 e^{\theta x_0}$
or, $\log \frac{e^{x_0}-1}{x_0} = \theta x_0$ or, $\frac{1}{x_0} \log \frac{e^{x_0}-1}{x_0} = \theta$.

Since $0 < \theta < 1$, we have $0 < \frac{1}{x_0} \log \frac{e^{x_0}-1}{x_0} < 1$.

Since $x_0 (> 0)$ is arbitrary, it follows that $0 < \frac{1}{x} \log \frac{e^x - 1}{x} < 1$ for all $x > 0$.

6. A function f is twice differentiable on $[a, b]$ and $f(a) = f(b) = 0$ and $f(c) < 0$ for some c in (a, b). Prove that there is at least one point ξ in (a, b) for which $f''(\xi) > 0$.

Since f'' exists on $[a, b]$, f and f' are both continuous on $[a, b]$.

Applying Mean value theorem on $[a, c]$ and $[c, b]$, we have
$\frac{f(c)-f(a)}{c-a} = f'(\xi_1), a < \xi_1 < c$ and $\frac{f(b)-f(c)}{b-c} = f'(\xi_2), c < \xi_2 < b$.

But $f(a) = f(b) = 0$. Therefore $f'(\xi_1) = \frac{f(c)}{c-a}, f'(\xi_2) = \frac{f(c)}{c-b}$.

Applying Mean value theorem to the function f' on $[\xi_1, \xi_2]$,
$\frac{f'(\xi_2)-f'(\xi_1)}{\xi_2-\xi_1} = f''(\xi)$ for some ξ in (ξ_1, ξ_2).

But $f'(\xi_2) - f'(\xi_1) = \frac{f(c)(b-a)}{(c-b)(c-a)} > 0$, since $f(c) < 0$.
Therefore $f''(\xi) > 0$, since $\xi_2 - \xi_1 > 0$.

Since $\xi_1 < \xi < \xi_2$, ξ lies between a and b and $f''(\xi) > 0$.

7. Let $a, b \in \mathbb{R}$ and $a < b$. If a function f has finite derivative at each point in (a, b), and for $c \in (a, b)$ if $\lim\limits_{x \to c+} f'(x)$ is finite $(= l)$, prove that $f'(c) = l$.

[That is, if $f'(c+0)$ and $f'(c)$ both exist finitely, then $f'(c+0)$ cannot be different from $f'(c)$.]

Let us choose $\delta > 0$ such that $(c, c+\delta) \subset (a, b)$.
Since f is differentiable on (a, b), f is differentiable on $[c, c+\delta]$.

Let $\{h_n\}$ be a sequence of points such that $h_n > 0$ for all $n \in \mathbb{N}$ and $\lim h_n = 0$. Then there exists a natural number k such that $0 < h_n < \delta$ for all $n \geq k$.

By Mean value theorem applied to f on $[c, c+h_n]$ for $n \geq k$,
$\frac{f(c+h_n)-f(c)}{h_n} = f'(c+\theta_n h_n)$ where $0 < \theta_n < 1$.

As $\theta_n > 0, \theta_n h_n > 0$ for all $n \in \mathbb{N}$. As $0 < \theta_n < 1$ and $h_n \to 0$, $\lim \theta_n h_n = 0$.

Since $\lim\limits_{x \to c+} f'(x) = l$, we have $\lim\limits_{\theta_n h_n \to 0+} f'(c+\theta_n h_n) = l$ by sequential criterion.

It follows that $\lim_{h_n \to 0+} \frac{f(c+h_n)-f(c)}{h_n} = l$.

Since f has a finite derivative at c, $\lim_{h \to 0+} \frac{f(c+h)-f(c)}{h} = Rf'(c)$.

By sequential criterion, $\lim_{h_n \to 0+} \frac{f(c+h_n)-f(c)}{h_n} = Rf'(c)$.

Therefore $Rf'(c) = l$.

Since $f'(c)$ exists, $f'(c) = Rf'(c) = l$.

Note. If $\lim_{x \to c-} f'(x)$ and $f'(c)$ be both finite, then $f'(c-0) = f'(c)$.

8. Let $a, b \in \mathbb{R}$ and $a < b$. If a function f be continuous on (a, b) and f has finite derivative at each point in (a, b) except possibly at c. If $\lim_{x \to c} f'(x)$ is finite $(= l)$ then prove that $f'(c) = l$.

[That is, if $f'(c+0)$ and $f'(c-0)$ both exist finitely and be equal, then f has a derivative at c and $f'(c) = f'(c+0) = f'(c-0)$.]

Let us choose $\delta > 0$ such that $(c, c+\delta) \subset (a, b)$.

f is continuous on $[c, c+\delta]$ and f is differentiable on $(c, c+\delta)$.

Let $\{h_n\}$ be a sequence of points such that $h_n > 0$ for all $n \in \mathbb{N}$ and $\lim h_n = 0$. Then there exists a natural number k such that $0 < h_n < \delta$ for all $n \geq k$.

By Mean value theorem applied to f on $[c, c+h_n]$ for $n \geq k$,

$\frac{f(c+h_n)-f(c)}{h_n} = f'(c+\theta_n h_n), 0 < \theta_n < 1$.

As $\theta_n > 0, \theta_n h_n > 0$ for all $n \in \mathbb{N}$. As $0 < \theta_n < 1$ and $h_n \to 0, \lim \theta_n h_n \to 0$.

Since $\lim_{x \to c} f'(x) = l$, $\lim_{x \to c+} f'(x) = l$.

By sequential criterion, $\lim_{\theta_n h_n \to 0+} f'(c+\theta_n h_n) = l$.

It follows that $\lim_{h_n \to 0+} \frac{f(c+h_n)-f(c)}{h_n} = l$.

Since $\{h_n\}$ is an arbitrary sequence converging to 0,

$\lim_{h \to 0+} \frac{f(c+h)-f(c)}{h} = l$. That is, $Rf'(c) = l$.

Also since $\lim_{x \to c-} f'(x)$ exists and equals l, proceeding with similar arguments we can establish that $Lf'(c) = l$. Hence $f'(c) = l$.

Note. We have shown that under the given conditions one-sided derivative $Rf'(c)$ exists if $f'(c+0)$ exists, and $Lf'(c)$ exists if $f'(c-0)$ exists.

The next example shows that if the right hand limit (one-sided limit) of the derivative f' exists at a and f be right continuous at a, then the right hand derivative at a exists.

9. Let $a \in \mathbb{R}$ and a function f be differentiable for $x > a$, right continuous at a and $f'(a+0)$ exists. Show that $Rf'(a)$ exists and $Rf'(a) = f'(a+)$.

Let us consider the interval $[a, a+\delta]$ for some $\delta > 0$.

As f is differentiable on $(a, a+\delta]$, f is continuous on $(a, a+\delta]$. Since f is right continuous at a, f is continuous on $[a, a+\delta]$.

Let us consider a sequence of points $\{h_n\}$ such that $h_n > 0$ for all $n \in \mathbb{N}$ and $\lim h_n = 0$. Then there exists a natural number k such that $h_n < \delta$ for all $n \geq k$. Then $[a, a+h_n] \subset [a, a+\delta]$ for all $n \geq k$.

By Lagrange's Mean value theorem, for $n \geq k$,
$f(a+h_n) - f(a) = h_n f'(a + \theta_n h_n)$ for some θ_n satisfying $0 < \theta_n < 1$.

Therefore $\frac{f(a+h_n)-f(a)}{h_n} = f'(a+\theta_n h_n)$.

Since $h_n \to 0+$ and $0 < \theta_n < 1, \theta_n h_n \to 0+$.

Since $f'(a+)$ exists, $\lim\limits_{\theta_n h_n \to 0+} f'(a+\theta_n h_n) = \lim\limits_{x \to a+} f'(x) = f'(a+0)$.

Therefore $\lim\limits_{h_n \to 0+} \dfrac{f(a+h_n) - f(a)}{h_n}$ exists and equals $f'(a+0)$.

Since $\{h_n\}$ is an arbitrary sequence converging to 0 (from the right), by sequential criterion for limits it follows that
$\lim\limits_{h \to 0+} \frac{f(a+h)-f(a)}{h}$ exists and equals $f'(a+0)$.

But $\lim\limits_{h \to 0+} \frac{f(a+h)-f(a)}{h} = Rf'(a)$, since the limit exists.

Hence $Rf'(a) = f'(a+0)$.

Note. If the left hand limit of the derivative f' exists at b and f be left continuous at b, then the left hand derivative $Lf'(b)$ exists and $Lf'(b) = f'(b-0)$.

However these conditions are sufficient conditions for the existence of the one-sided derivatives. For example, let us consider the function f defined on $\mathbb{R}$ by $f(x) = x^2 \sin \frac{1}{x}, x \neq 0$
$= 0, x = 0$.

Then $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}, x \neq 0$
$= 0, x = 0$.

Here $f'_+(0) = f'_-(0) = 0$, but neither $f'(0+)$ nor $f'(0-)$ exists.

10. Let $I = [a, b]$ be a closed and bounded interval and $f : I \to \mathbb{R}$ be differentiable on I. Let $J = f(I)$ and $g : J \to \mathbb{R}$ be differentiable on J. Use Mean value theorem to prove that the composite function $g \circ f$ is differentiable on I and $(g \circ f)'(x) = g'(f(x))f'(x)$ for all $x \in I$, assuming that f' and g' are continuous on I and J respectively.

Let c be an interior point of I.

$g \circ f$ is differentiable at c if $\lim\limits_{h \to 0} \dfrac{g \circ f(c+h) - g \circ f(c)}{h}$ exists.

f satisfies all conditions of Mean value theorem on $[c, c+h]$ or on $[c+h, c]$. By Mean value theorem, $f(c+h) = f(c) + hf'(c+\theta h)$ for some real number θ satisfying $0 < \theta < 1$.

Let $f(c) = d \in J, f(c+h) = d + k \in J$.

g satisfies all conditions of Mean value theorem on $[d, d+k]$ or on $[d+k, d]$. By Mean value theorem, $g(d+k) = g(d) + kg'(d+\theta' k)$ for some real number θ' satisfying $0 < \theta' < 1$.

i.e.,$g \circ f(c+h) = g \circ f(c) + hf'(c+\theta h)g'[f(c) + \theta' hf'(c+\theta h)]$.

$$\text{Therefore } \lim_{h \to 0} \frac{g \circ f(c+h) - g \circ f(c)}{h}$$
$$= \lim_{h \to 0} f'(c+\theta h).g'[f(c) + \theta' hf'(c+\theta h)]$$
$$= f'(c).g'(f(c)), \text{ since } f' \text{ and } g' \text{ are both continuous at } c.$$

Therefore $(g \circ f)'(c) = f'(c).g'(f(c))$.

Similar proof for $c = a$ and $c = b$.

Since c is arbitrary, $(g \circ f)'(x) = g'(f(x)).f'(x)$ for all $x \in [a, b]$.

Note. This is an alternative proof for differentiability of the composite function $g \circ f$ under wider conditions.

Theorem 9.5.6. Let I be an interval. If a function $f : I \to \mathbb{R}$ be such that f' exists and is bounded on I then f is uniformly continuous on I.

Proof. Let $x_1, x_2 \in I$ with $x_1 < x_2$.

Since f' is bounded on I, there exists a positive real number k such that $| f'(x) | \leq k$ for all $x \in I$.

f satisfies both the conditions of Mean value theorem on $[x_1, x_2]$ and therefore there exists a point ξ in (x_1, x_2) such that $f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$.

Therefore $|\frac{f(x_2) - f(x_1)}{x_2 - x_1}| \leq k$. That is, $|f(x_2) - f(x_1)| \leq k|x_2 - x_1|$.

It follows that $| f(x_2) - f(x_1) | \leq k | x_2 - x_1 |$ for all $x_1, x_2 \in I$.

Let us choose $\epsilon > 0$. Then there exists a positive $\delta (= \frac{\epsilon}{k})$ such that $|f(x_2) - f(x_1)| < \epsilon$ for all x_1, x_2 in I satisfying $|x_2 - x_1| < \delta$.

Hence f is uniformly continuous on I.

Note. Under the stated conditions f satisfies a Lipschitz's condition on I.

Worked Example.

1. Prove that the function f defined on $\mathbb{R}$ by $f(x) = \frac{1}{x^2+1}, x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

$f'(x) = -\frac{2x}{(x^2+1)^2}, x \in \mathbb{R}$. Therefore $| f'(x) |< 2$ for all $x \in \mathbb{R}$.

Let x_1, x_2 be any two points in $\mathbb{R}$ such that $x_1 < x_2$.

f is continuous on $[x_1, x_2]$ and f is differentiable on (x_1, x_2). By the Mean value theorem, there exists a point ξ in (x_1, x_2) such that $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(\xi)$.

Since $|f'(x)| < 2$ for all $x \in \mathbb{R}$, $|f(x_1) - f(x_2)| < 2|x_2 - x_1|$.

Let us choose $\epsilon > 0$. There exists a positive $\delta(= \frac{\epsilon}{2})$ such that $| f(x_1) - f(x_2) |< \epsilon$ for all x_1, x_2 in $\mathbb{R}$ satisfying $| x_2 - x_1 |< \delta$.

This proves that f is uniformly continuous on $\mathbb{R}$.

Theorem 9.5.7. (Generalised Mean value theorem)

Let the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be such that

(i) f and g are both continuous on $[a, b]$, and

(ii) f and g are both differentiable on (a, b).

Then there exists a point ξ in (a, b) such that $[g(b) - g(a)]f'(\xi) = [f(b) - f(a)]g'(\xi)$.

Proof. Let us define $\phi : [a, b] \to \mathbb{R}$ by $\phi(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)], x \in [a, b]$.

ϕ is continuous on $[a, b]$, since f and g are continuous on $[a, b]$.

ϕ is differentiable on (a, b), since f and g are differentiable on (a, b). Also $\phi(a) = f(a)g(b) - g(a)f(b) = \phi(b)$.

By Rolle's theorem, $\phi'(\xi) = 0$ for some ξ in (a, b).
But $\phi'(x) = f'(x)[g(b) - g(a)] - g'(x)[f(b) - f(a)]$.
$\phi'(\xi) = 0$ gives $f'(\xi)[g(b) - g(a)] = g'(\xi)[f(b) - f(a)]$.

This completes the proof.

Note. If $g'(x) \neq 0$ for all $x \in (a, b)$, then $g(b) \neq g(a)$; because the condition $g(b) = g(a)$ together with the conditions satisfied by g in the theorem would imply $g'(c) = 0$ for some $c \in (a, b)$, by Rolle's theorem. Therefore if $g'(x) \neq 0$ for all $x \in (a, b)$, the conclusion of the theorem can be expressed as $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$ for some ξ in (a, b).

Theorem 9.5.8. Mean value theorem (Cauchy)

Let the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be such that

(i) f and g are both continuous on $[a, b]$,

(ii) f and g are both differentiable on (a, b), and

(iii) $g'(x) \neq 0$ for all $x \in (a, b)$.

Then there exists a point ξ in (a, b) such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$.

Proof. $g(a) \neq g(b)$. Because, if $g(a) = g(b)$ then g would satisfy all conditions of Rolle's theorem on $[a, b]$ and consequently $g'(x)$ would be equal to 0 for some point c in (a, b) and this would contradict the condition (iii) of the theorem.

Let us define $\phi : [a, b] \to \mathbb{R}$ by $\phi(x) = f(x) + Ag(x), x \in [a, b]$, where A is a constant.

ϕ is continuous on $[a, b]$, since f and g are both continuous on $[a, b]$; ϕ is differentiable on (a, b), since f and g are both differentiable on (a, b).

Let us choose A such that $\phi(a) = \phi(b)$. Then $f(a)+Ag(a) = f(b)+Ag(b)$.

or, $A[g(a) - g(b)] = f(b) - f(a)$. Since $g(a) \neq g(b), A = \frac{f(b)-f(a)}{g(a)-g(b)}$.

For this choice of A, ϕ satisfies all conditions of Rolle's theorem on $[a, b]$. Therefore there exists a point ξ in (a, b) such that $\phi'(\xi) = 0$.

But $\phi'(\xi) = f'(\xi) - \frac{f(b)-f(a)}{g(a)-g(b)} g'(\xi)$.

As $a < \xi < b, g'(\xi) \neq 0$ and therefore $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$.

This completes the proof.

Note 1. Lagrange's Mean value theorem can be deduced from Cauchy's Mean value theorem by taking $g(x) = x, x \in [a, b]$.

Note 2. Both f and g satisfy the conditions of Lagrange's Mean value theorem. Consequently, there exist points c and d in (a, b) such that $\frac{f(b)-f(a)}{b-a} = f'(c)$ and $\frac{g(b)-g(a)}{b-a} = g'(d)$.

c and d are different points in (a, b) in general, and therefore a single point ξ in (a, b) may not be found to satisfy the conclusion of Cauchy's Mean value theorem, unless the third condition "$g'(x) \neq 0$ for all $x \in (a, b)$" is imposed on the function g.

Exercises 15

1. Show that there does not exist a function ϕ such that $\phi'(x) = f(x)$ where

(i) $f(x) = [x], x \in [0, 2]$, (ii) $f(x) = x - [x], x \in [0, 2]$.

2. Let I be an interval and a function $f : I \to \mathbb{R}$ is differentiable on I. If f' be monotonic on I prove that f' is continuous on I.

3. Verify Rolle's theorem for the following funcfions on the indicated intervals.

(i) $f(x) = x^2 - 5x + 10$ on $[2, 3]$, (ii) $f(x) = (x - a)^3(x - b)^4$ on $[a, b]$,

(iii) $f(x) = x^2 + \cos x$ on $[-\frac{\pi}{4}, \frac{\pi}{4}]$, (iv) $f(x) = \sin(\frac{1}{x})$ on $[\frac{1}{3\pi}, \frac{1}{2\pi}]$.

4. Show that the following functions do not satisfy the conditions of Rolle's theorem on the indicated intervals.

(i) $f(x) = 1 - \mid x - 1 \mid$ on $[0, 2]$, (ii) $f(x) = 1 - (x - 1)^{2/3}$ on $[0, 2]$.

5. Verify the hypothesis and the conclusion of Mean value theorem for the following functions on the indicated intervals.

(i) $f(x) = x(x - 1)(x - 2)$ on $[0, \frac{1}{2}]$, (ii) $f(x) = \frac{x}{x-1}$ on $[2, 4]$,

(iii) $f(x) = x^3 - 3x + 1$ on $[1, 3]$, (iv) $f(x) = 4 - (6 - x)^{2/3}$ on $[5, 7]$,

(v) $f(x) = \cos(\frac{1}{x}), x \neq 0$
$\quad = 0,\ x = 0$ on $[-1, 1]$.

6. Calculate ξ in Cauchy's mean value theorem for each of the following pairs of functions.

(i) $f(x) = \sin x,\ g(x) = \cos x$ on $[\frac{\pi}{4}, \frac{3\pi}{4}]$,

(ii) $f(x) = \sqrt{x},\ g(x) = \frac{1}{\sqrt{x}}$ on $[1, 3]$,

(iii) $f(x) = \log x,\ g(x) = \frac{1}{x}$ on $[1, e]$,

(iv) $f(x) = (1 + x)^{3/2},\ g(x) = \sqrt{1 + x}$ on $[0, \frac{1}{2}]$.

7. A function f is differentiable on $[0, 2]$ and $f(0) = 0, f(1) = 2, f(2) = 1$. Prove that $f'(c) = 0$ for some c in $(0, 2)$.

[**Hint.** Apply Lagrange's Mean value theorem to f on $[0, 1]$ and on $[1, 2]$. Then $f'(c_1) > 0$ and $f'(c_2) < 0$ for some $c_1 \in (0, 1)$ and some $c_2 \in (1, 2)$. Apply Darboux's theorem to f' on $[c_1, c_2]$.]

8. Prove that the equation $(x - 1)^3 + (x - 2)^3 + (x - 3)^3 + (x - 4)^3 = 0$ has only one real root.

9. If $c_0 + \frac{c_1}{2} + \frac{c_2}{3} + \cdots + \frac{c_n}{n+1} = 0$ where $c_0, c_1, c_2, \cdots, c_n$ are real, show that the equation $c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = 0$ has at least one real root between 0 and 1.

10. Prove that between any two real roots of the equation $e^x \cos x + 1 = 0$ there is at least one real root of the equation $e^x \sin x + 1 = 0$.

[**Hint.** Let $f(x) = e^{-x} + \cos x, x \in \mathbb{R}$. Apply Rolle's theorem on $[c, d]$ where c and d $(c < d)$ are any two real roots of the equation $f(x) = 0$.]

11. Prove that between any two real roots of the equation $e^x \sin x + 1 = 0$ there is at least one real root of the equation $\tan x + 1 = 0$.

[**Hint.** Let $f(x) = e^x \sin x + 1, x \in \mathbb{R}$. Proceed as in Ex.10]

12. If f is differentiable on $[0,1]$ show by Cauchy's Mean value theorem that the equation $f(1)-f(0)=\frac{f'(x)}{2x}$ has at least one solution in $(0,1)$.

13. A function $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and $f''(x)$ exists for all $x\in(a,b)$. If $a<c<b$ and $f(a)=f(b)=0$, prove that there exists a point ξ in (a,b) such that $f(c)=\frac{1}{2}(c-a)(c-b)f''(\xi)$.

14. A function f is twice differentiable on $[a,b]$ and $f(a)=f(b)=0$. If $f(c)>0$ for some $c\in(a,b)$, prove that there exists a point ξ in (a,b) such that $f''(\xi)<0$.

15. A function $f:[a,b]\to\mathbb{R}$ is said to satisfy a *Lipschitz condition* of order α on $[a,b]$ if there exists a real number $M>0$ such that

$|f(x)-f(y)|<M\,|x-y|^\alpha$ for all x,y in $[a,b]$.

If f satisfies a Lipschitz condition of order $\alpha>1$ on $[a,b]$, prove that f is a constant on $[a,b]$.

16. A function f is thrice differentiable on $[a,b]$ and $f(a)=f(b)=0, f'(a)=f'(b)=0$. Prove that $f'''(c)=0$ for some $c\in(a,b)$.

17. If $f''(x)\geq 0$ on $[a,b]$ prove that $f(\frac{x_1+x_2}{2})\leq\frac{1}{2}[f(x_1)+f(x_2)]$ for any two points x_1,x_2 in $[a,b]$.

Use the principle of induction to prove that

$f(\frac{x_1+x_2+\cdots+x_n}{n})\leq\frac{1}{n}[f(x_1)+f(x_2)+\cdots+f(x_n)]$ for any n points $x_1,x_2,\ldots,x_n$ in $[a,b]$.

[**Hint.** Let $x_1<x_2$. Apply Mean value theorem on $[x_1,\frac{x_1+x_2}{2}]$ and on $[\frac{x_1+x_2}{2},x_2]$.]

18. A function f is continuous on $[a,b]$ and $f''(x)$ is finite for every $x\in(a,b)$. If the line segment joining the points $A(a,f(a))$ and $B(b,f(b))$ intersects the graph of f at some point P different from A and B, prove that $f''(\xi)=0$ for some ξ in (a,b).

[**Hint.** Let $P=(c,f(c))$. Then $\frac{f(c)-f(a)}{c-a}=\frac{f(b)-f(c)}{b-c}$.]

19. If $\phi(x)=f(x)+f(1-x), x\in[0,1]$ and $f''(x)<0$ for all $x\in[0,1]$, show that ϕ is an increasing function on $[0,\frac{1}{2}]$ and a decreasing function on $[\frac{1}{2},1]$.

[**Hint.** f' is a strictly decreasing function on $[0,1]$.

$\phi'(0)=f'(0)-f'(1)>0, \phi'(\frac{1}{2})=0, \phi'(1)=f'(1)-f'(0)<0.$

$0<x<\frac{1}{2}\Rightarrow 0<x<1-x<1\Rightarrow f'(x)>f'(1-x)\Rightarrow\phi'(x)>0.$

$\frac{1}{2}<x<1\Rightarrow 0<1-x<x<1\Rightarrow f'(1-x)>f'(x)\Rightarrow\phi'(x)<0.$]

20. Let $f(x)=\frac{\sin x}{x}, x\in(0,\frac{\pi}{2})$, $\phi(x)=\frac{\tan x}{x}, x\in(0,\frac{\pi}{2})$. Show that f is a strictly decreasing function on $(0,\frac{\pi}{2})$ and ϕ is a strictly increasing function on $(0,\frac{\pi}{2})$.

21. Prove that

(i) $\frac{2x}{\pi}<\sin x<x$ for $0<x<\frac{\pi}{2}$, (ii) $x<\tan x<\frac{4x}{\pi}$ for $0<x<\frac{\pi}{4}$,

(iii) $\frac{x}{\sin x} < \frac{\tan x}{x}$ for $x \in (0, \frac{\pi}{2})$, (iv) $x < \sin^{-1} x < \frac{x}{\sqrt{1-x^2}}$ for $0 < x < 1$.

22. Use Mean value theorem to prove $0 < \frac{1}{\log(1+x)} - \frac{1}{x} < 1$ for $x > 0$.

23. Use Mean value theorem to prove that the function f defined by

(i) $f(x) = \sin x,\ x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$,

(ii) $f(x) = \tan^{-1} x,\ x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

24. Prove that $\frac{x}{1+x} < \log(1+x) < x$ for all $x > 0$. Deduce that

$\log \frac{2n+1}{n+1} < \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < \log 2$, n being a positive integer.

9.6. The nth order derivatives.

1. Let $f(x) = x^\alpha, \alpha \in \mathbb{R}$.

$$\begin{aligned}
\text{Then } f'(x) &= \alpha x^{\alpha-1} \\
f''(x) &= \alpha(\alpha-1)x^{\alpha-2} \\
f'''(x) &= \alpha(\alpha-1)(\alpha-2)x^{\alpha-3} \\
\cdots & \quad \cdots \\
f^n(x) &= \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)x^{\alpha-n}.
\end{aligned}$$

Note 1. Here the result has been obtained by inference. But it can be proved by the principle of induction.

2. The domain D of the function f is $(0,\infty)$ when α is irrational; $D = [0,\infty)$ when $\alpha = \frac{m}{n}, m \in \mathbb{N}, n(>1) \in \mathbb{N}$; $D = (0,\infty)$ when $\alpha = -\frac{m}{n}, m \in \mathbb{N}, n(>1) \in \mathbb{N}$; $D = \mathbb{R}$ when α is an integer ≥ 0 and $D = \mathbb{R} - \{0\}$ when α is an integer < 0.

The domain D of the function f^n is $(0,\infty)$ when α is not an integer; $D = \mathbb{R}$ when α is an integer ≥ 0 and $D = \mathbb{R} - \{0\}$ when α is an integer < 0.

Corollary. Let $f(x) = x^m$ where m is a positive integer.

$$\begin{aligned}
\text{Then } f'(x) &= mx^{m-1} \\
f''(x) &= m(m-1)x^{m-2} \\
f'''(x) &= m(m-1)(m-2)x^{m-3} \\
\cdots & \quad \cdots \\
f^m(x) &= m! \text{ and} \\
f^{m+r}(x) &= 0 \text{ for } r = 1,2,3,\ldots
\end{aligned}$$

In the following discussion the functions and their deriatives of different orders are assumed to be defined on their respective domains.

In each of the following cases, the nth derivative is obtained by inference. But in every case it can be proved by the principle of induction.

2. Let $f(x) = \frac{1}{ax+b}$, $a \in \mathbb{R}, b \in \mathbb{R}, ax + b \neq 0$.

Then
$$\begin{aligned} f'(x) &= -1(ax+b)^{-2}.a \\ f''(x) &= (-1)^2 2!(ax+b)^{-3}.a^2 \\ f'''(x) &= (-1)^3 3!(ax+b)^{-4}.a^3 \\ \cdots & \quad \cdots \\ f^n(x) &= (-1)^n n!(ax+b)^{-n-1}.a^n. \end{aligned}$$

3. Let $f(x) = e^{ax+b}$, $a \in \mathbb{R}, b \in \mathbb{R}$.

Then
$$\begin{aligned} f'(x) &= e^{ax+b}.a \\ f''(x) &= e^{ax+b}.a^2 \\ \cdots & \quad \cdots \\ f^n(x) &= e^{ax+b}.a^n \end{aligned}$$

4. Let $f(x) = \log(ax+b)$, $a \in \mathbb{R}, b \in \mathbb{R}, ax + b > 0$.

Then
$$\begin{aligned} f'(x) &= (ax+b)^{-1}.a \\ f''(x) &= -1.(ax+b)^{-2}.a^2 \\ f'''(x) &= (-1)^2 2!.(ax+b)^{-3}.a^3 \\ \cdots & \quad \cdots \\ f^n(x) &= (-1)^{n-1}(n-1)!(ax+b)^{-n}.a^n. \end{aligned}$$

5. Let $f(x) = \sin ax$, $a \in \mathbb{R}$.

Then
$$\begin{aligned} f'(x) &= \cos ax.a = \sin(\pi/2 + ax).a \\ f''(x) &= -\sin ax.a^2 = \sin(2\pi/2 + ax).a^2 \\ f'''(x) &= -\cos ax.a^3 = \sin(3\pi/2 + ax).a^3 \\ \cdots & \quad \cdots \\ f^n(x) &= \sin(n\pi/2 + ax).a^n. \end{aligned}$$

6. Let $f(x) = \cos ax$, $a \in \mathbb{R}$. Then $f^n(x) = \cos(n\pi/2 + ax).a^n$.

7. Let $f(x) = e^{ax}\sin bx$, $a \in \mathbb{R}, b \in \mathbb{R}$.
Then $f'(x) = e^{ax}(a \sin bx + b \cos bx)$.
Let $a = r\cos\theta, b = r\sin\theta; -\pi < \theta \leq \pi$. Then $r^2 = a^2 + b^2$.

$$\begin{aligned} f'(x) &= re^{ax}(\sin bx \cos\theta + \cos bx \sin\theta) = re^{ax}\sin(bx+\theta) \\ f''(x) &= r^2 e^{ax}\sin(bx + 2\theta) \\ \cdots & \quad \cdots \\ f^n(x) &= r^n e^{ax}\sin(bx + n\theta), \text{ where } r\cos\theta = a,\ r\sin\theta = b. \end{aligned}$$

8. Let $f(x) = e^{ax}\cos bx$, $a \in \mathbb{R}, b \in \mathbb{R}$.
Then $f'(x) = e^{ax}(a\cos bx - b\sin bx)$.
Let $a = r\cos\theta, b = r\sin\theta; -\pi < \theta \leq \pi$. Then $r^2 = a^2 + b^2$.

$$\begin{aligned} f'(x) &= re^{ax}(\cos bx\cos\theta - \sin bx \sin\theta) = re^{ax}\cos(bx+\theta) \\ \cdots & \quad \cdots \\ f^n(x) &= r^n e^{ax}\cos(bx + n\theta), \text{ where } r\cos\theta = a, r\sin\theta = b. \end{aligned}$$

Worked Examples.

1. If $f(x) = \frac{x^3}{x^2-1}$, prove that for $n > 1$, $f^n(0) = 0$ if n be even
$= -n!$ if n be odd.

$f(x) = x + \frac{x}{x^2-1} = x + \frac{1}{2}[\frac{1}{x+1} + \frac{1}{x-1}]$.
$f'(x) = 1 + \frac{1}{2}[(-1)(x+1)^{-2} + (-1)(x-1)^{-2}]$.
$f''(x) = \frac{1}{2}[(-1)^2 2!(x+1)^{-3} + (-1)^2 2!(x-1)^{-3}]$.
... ...
Therefore $f^n(x) = \frac{1}{2}[\frac{(-1)^n n!}{(x+1)^{n+1}} + \frac{(-1)^n n!}{(x-1)^{n+1}}]$ for $n > 1$.
That is , for $n > 1, f^n(0) = \frac{1}{2}(-1)^n n![1 + (-1)^{n+1}]$.
Therefore for $n > 1, f^n(0) = 0$ if n be even
$= -n!$ if n be odd.

2. If $y = \frac{1}{x^2+a^2}$, prove that $y_n = \frac{(-1)^n n!}{a^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta$, where $\cot\theta = \frac{x}{a}$.

Let $x = a\cot\theta$. Then $\frac{dx}{d\theta} = -a \operatorname{cosec}^2\theta$, $y = \frac{1}{a^2}\sin^2\theta$.

Therefore $y_1 = \frac{1}{a^2}\sin 2\theta.\frac{d\theta}{dx} = \frac{1}{a^2}\sin 2\theta(-\frac{\sin^2\theta}{a}) = -\frac{1}{a^3}\sin^2\theta\sin 2\theta$.

$y_2 = -\frac{1}{a^3}[2\sin\theta\cos\theta\sin 2\theta + 2\sin^2\theta\cos 2\theta].\frac{d\theta}{dx}$
$= \frac{-2}{a^3}\sin\theta[\sin 2\theta\cos\theta + \cos 2\theta\sin\theta].(-\frac{\sin^2\theta}{a}) = \frac{(-1)^2 2!}{a^4}\sin^3\theta\sin 3\theta$.

$y_3 = \frac{2!}{a^4}[3\sin^2\theta\cos\theta\sin 3\theta + 3\sin^3\theta\cos 3\theta].(-\frac{\sin^2\theta}{a})$
$= \frac{(-1)^3 3!}{a^5}\sin^4\theta(\cos\theta\sin 3\theta + \sin\theta\cos 3\theta) = \frac{(-1)^3 3!}{a^5}\sin^4\theta\sin 4\theta$.
... ...
$y_n = \frac{(-1)^n n!}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta$, where $\cot\theta = \frac{x}{a}$.

3. If $y = \tan^{-1}x$, prove that $y_n = (-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$ where $\cot\theta = x$.

$y_1 = \frac{1}{1+x^2}$. Let $y_1 = z$. Then by Example 2,

$z_n = (-1)^n n!\sin^{n+1}\theta\sin(n+1)\theta$, where $\cot\theta = x$.
Hence $y_n = z_{n-1} = (-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$, where $\cot\theta = x$.

4. If $y = \cot^{-1}x$, prove that $y_n = (-1)^n(n-1)!\sin^n y\sin ny$.

$y_1 = -\frac{1}{1+x^2} = -\frac{1}{cosec^2 y} = -\sin^2 y = -\sin y.\sin y$.
$y_2 = -2\sin y\cos y\frac{dy}{dx} = -\sin 2y(-\sin^2 y) = (-1)^2\sin^2 y\sin 2y$.

$y_3 = (-1)^2[2\sin y\cos y\sin 2y + \sin^2 y.2\cos 2y].\frac{dy}{dx}$
$= (-1)^2 2\sin y[\sin 2y\cos y + \sin y\cos 2y](-\sin^2 y)$
$= (-1)^3.2!\sin^3 y\sin 3y$.
... ...
$y_n = (-1)^n(n-1)!\sin^n y\sin ny$.

Theorem 9.6.1. (Leibnitz)

If f and g be two functions each differentiable n times at a, then the nth derivative of the product function fg at a is given by

$(fg)^n(a) = f^n(a)g(a) + n_{C_1} f^{n-1}(a)g'(a) + n_{C_2} f^{n-2}(a)g''(a) + \cdots + n_{C_r} f^{n-r}(a)g^r(a) + \cdots + f(a)g^n(a)$

$= \sum_{r=0}^{n} n_{C_r} D^{n-r} f(a) D^r g(a)$, where $D^r f(a) = f^r(a)$, $r \geq 1$; $D^0 f(a) = f(a)$.

Proof. $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.

This shows that the theorem is true for $n = 1$.

Let us assume the theorem to be true for $n = m$, where m is a natural number.

Then $(fg)^m(a) = f^m(a)g(a) + m_{C_1} f^{m-1}(a)g'(a) + + m_{C_r} f^{m-r}(a)g^r(a) + \cdots + f(a)g^m(a)$.

Differentiating the function $(fg)^m$ at a in L.H.S., and the function $f^m g + m_{C_1} f^{m-1} g' + \cdots + m_{C_r} f^{m-r} g^r + \cdots + fg^m$ at a in R.H.S., we have

$(fg)^{m+1}(a) = [f^{m+1}(a)g(a) + f^m(a)g'(a)] + m_{C_1}[f^m(a)g'(a) + f^{m-1}(a)g''(a)] + \cdots + m_{C_r}[f^{m-r+1}(a)g^r(a) + f^{m-r}(a)g^{r+1}(a)] + \cdots + [f'(a)g^m(a) + f(a)g^{m+1}(a)]$

$= f^{m+1}(a)g(a) + (1 + m_{C_1})[f^m(a)g'(a)] + (m_{c_1} + m_{c_2})[f^{m-1}(a)g''(a)] + \cdots + (m_{c_{r-1}} + m_{C_r})[f^{m+1-r}(a)g^r(a)] + \cdots + f(a)g^{m+1}(a)$

$= f^{m+1}(a)g(a) + (m+1)_{C_1} f^m(a)g'(a) + (m+1)_{c_2} f^{m-1}(a)g''(a) + \cdots + (m+1)_{C_r} f^{m+1-r}(a)g^r(a) + \cdots + f(a)g^{m+1}(a)$.

This shows that the theorem is true for $n = m + 1$, if it is true for $n = m$. And the theorem is true for $n = 1$.

By the principle of induction, the theorem is true for all natural numbers n.

Worked Examples.

1. If $x + y = 1$ prove that the nth derivative of $x^n y^n$ is

$n!\{y^n - (n_{C_1})^2 y^{n-1}x + (n_{C_2})^2 y^{n-2}x^2 - (n_{C_3})^2 y^{n-3}x^3 + \cdots + (-1)^n x^n\}$.

By Leibnitz's theorem,

$D^n(x^n y^n) = D^n(x^n)y^n + n_{C_1} D^{n-1}(x^n)D(y^n) + n_{C_2} D^{n-2}(x^n)D^2(y^n) + n_{C_3} D^{n-3}(x^n)D^3(y^n) + \cdots + x^n D(y^n)$.

If $r \leq n, D^r(x^n) = n(n-1)\cdots(n-r+1)x^{n-r} = \frac{n!}{(n-r)!}x^{n-r}$.

If $r \leq n, D^r(y^n) = D^r(1-x)^n = \frac{n!}{(n-r)!}y^{n-r}.(-1)^r$.

Therefore $D^n(x^ny^n) = n!y^n - n_{C_1}\frac{n!}{1!}x.\frac{n!}{(n-1)!}y^{n-1} + n_{C_2}\frac{n!}{2!}x^2.\frac{n!}{(n-2)!}y^{n-2} - n_{C_3}\frac{n!}{3!}x^3.\frac{n!}{(n-3)!}y^{n-3} + \cdots + x^n.n!(-1)^n$

$= n![y^n - (n_{C_1})^2xy^{n-1} + (n_{C_2})^2x^2y^{n-2} - (n_{C_3})^2x^3y^{n-3} + \cdots + (-1)^nx^n]$.

2. If $y = \frac{x^n}{1+x^2}$ prove that

$y_n = n!\sin\theta[\sin\theta - n_{C_1}\cos\theta\sin 2\theta + n_{C_2}\cos^2\theta\sin 3\theta - \cdots + (-1)^n\cos^n\theta.\sin(n+1)\theta]$, where $x = \cot\theta$.

Let $u = x^n, v = \frac{1}{1+x^2}$.

By Leibnitz's theorem, $y_n = u_nv + n_{C_1}u_{n-1}v_1 + n_{C_2}u_{n-2}v_2 + \cdots + uv_n$.

If $r \leq n, u_r = n(n-1)\cdots(n-r+1)x^{n-r} = \frac{r!}{(n-r)!}x^{n-r}$.

If $r \leq n, v_r = (-1)^r r!\sin^{r+1}\theta\sin(r+1)\theta$, where $x = \cot\theta$.

$y_n = n!\sin^2\theta + n_{C_1}.\frac{n!}{1!}x(-\sin^2\theta\sin 2\theta) + n_{C_2}.\frac{n!}{2!}x^2.(2!\sin^3\theta\sin 3\theta) + n_{C_3}\frac{n!}{3!}x^3(-3!\sin^4\theta\sin 4\theta) + \cdots + x^n.(-1)^nn!\sin^{n+1}\theta\sin(n+1)\theta$

$= n![\sin^2\theta - n_{C_1}\cot\theta\sin^2\theta\sin 2\theta + n_{C_2}\cot^2\theta\sin^3\theta\sin 3\theta - \cdots + (-1)^n\cot^n\theta\sin^{n+1}\theta\sin(n+1)\theta]$

$= n!\sin\theta[\sin\theta - n_{C_1}\cos\theta\sin 2\theta + n_{C_2}\cos^2\theta\sin 3\theta - \cdots + (-1)^n\cos^n\theta\sin(n+1)\theta]$.

3. If $y = (x + \sqrt{1+x^2})^m$ find the value of $y_n(0)$.

$y_1 = m(x+\sqrt{1+x^2})^{m-1}(1 + \frac{x}{\sqrt{1+x^2}}) = \frac{my}{\sqrt{1+x^2}}$.

$y_2 = \frac{my_1}{\sqrt{1+x^2}} - \frac{mxy}{(1+x^2)^{3/2}} = \frac{m^2y}{1+x^2} - \frac{xy_1}{1+x^2}$

or, $(1+x^2)y_2 + xy_1 - m^2y = 0$.

Differentiating n times by Leibnitz's theorem, we have

$[(1+x^2)y_{n+2} + n_{C_1}.2xy_{n+1} + n_{C_2}.2y_n] + [xy_{n+1} + ny_n] - m^2y_n = 0$

or, $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$.

At $x = 0, y_{n+2}(0) = (m^2 - n^2)y_n(0)$ ... ... (i)

But $y(0) = 1, y_1(0) = m, y_2(0) = m^2$. Using (i) we have

$y_3(0) = (m^2 - 1^2)y_1(0) = m(m^2 - 1^2)$.

$y_4(0) = (m^2 - 2^2)y_2(0) = m^2(m^2 - 2^2)$.

... ...

Therefore

$y_n(0) = m(m^2 - 1^2)(m^2 - 3^2)\cdots[m^2 - (n-2)^2]$, if n be odd.

$y_n(0) = m^2(m^2 - 2^2)(m^2 - 4^2)\cdots[m^2 - (n-2)^2]$, if n be even.

Exercises 16

1. If $y = \tan^{-1} x$, prove that $y_n = (-1)^{n-1}(n-1)! \sin^n \theta \sin n\theta$, where $\cot\theta = x$.

2. If $y = \tan^{-1} \frac{1+x}{1-x}$, prove that $y_n = (-1)^{n-1}(n-1)! \sin^n \theta \sin n\theta$, where $\cot\theta = x$.

3. If $y = \frac{1}{1+x+x^2+x^3}$, prove that
$y_n = \frac{1}{2}(-1)^n n! \sin^{n+1}\theta[\sin(n+1)\theta - \cos(n+1)\theta + (\sin\theta + \cos\theta)^{-n-1}]$, where $\cot\theta = x$.

4. If $y = \frac{\log x}{x}$, prove that $y_n = \frac{(-1)^n n!}{x^{n+1}}[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots \cdots - \frac{1}{n}]$.

5. If $y = x \log \frac{x-1}{x+1}$, prove that $y_n = (-1)^n(n-2)![\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}]$.

6. If $y = x^{n-1}\log x$, prove that $y_n = \frac{(n-1)!}{x}$.

7. If $y = x^n \log x$, prove that $y_n = n![\log x + 1 + \frac{1}{2} + \cdots + \frac{1}{n}]$.

8. If $y = x^{n-1}e^{1/x}$, prove that $y_n = \{(-1)^n e^{1/x}\}/x^{n+1}$.

9. If $y = \tan^{-1} x$, prove that

(i) $(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$;

(ii) $(y_n)_0 = 0$, if n be even
$= (-1)^{\frac{1}{2}(n-1)}(n-1)!$, if n odd.

10. If $y = \cos(m\sin^{-1} x)$, prove that

(i) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$;

(ii) $(y_n)_0 = 0$, if n be odd
$= -m^2(2^2 - m^2)(4^2 - m^2)\cdots\{(n-2)^2 - m^2\}$, if n be even.

11. If $y = e^{a\sin^{-1} x}$, prove that

(i) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + a^2)y_n = 0$;

(ii) $(y_n)_0 = \{(n-2)^2 + a^2\}\{(n-4)^2 + a^2\}\cdots(2^2 + a^2)a^2$, if n be even
$= \{(n-2)^2 + a^2\}\{(n-4)^2 + a^2\}\cdots(1^2 + a^2)a$, if n be odd.

12. If $y^{1/m} + y^{-1/m} = 2x$, prove that
$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$.

13. If $y = a\cos(\log x) + b\sin(\log x)$, prove that
$x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$.

14. If $y = \log(x + \sqrt{1+x^2})$, prove that
$y_{2n}(0) = 0$; $y_{2n+1}(0) = (-1)^n.1^2.3^2.5^2\cdots(2n-1)^2$.

15. If $y = (\sinh^{-1} x)^2$, prove that
$y_{2n+1}(0) = 0$; $y_{2n}(0) = (-1)^{n-1}2.2^2.4^2.6^2\cdots(2n-2)^2$.

16. If $y = (1+x^2)^{m/2}\sin(m\tan^{-1} x)$, show that

(i) $(1+x^2)y_{n+2} + 2(n-m+1)xy_{n+1} + (n-m)(n+1-m)y_n = 0$;

(ii) $y_{2n}(0) = 0$; $y_{2n+1}(0) = (-1)^n m(m-1)\cdots(m-2n)$.

17. If $y = e^{-x}x^n$, prove that $xy_{n+2} + (x+1)y_{n+1} + (n+1)y_n = 0$.

Deduce that $xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0$, where $L_n(x) = \frac{1}{n!}e^x D^n(e^{-x}x^n)$.

9.7. Taylor's theorem and expansion of functions.

Theorem 9.7.1. (Taylor's theorem)

Let a function $f : [a, a+h] \to \mathbb{R}$ be such that

(i) f^{n-1} is continuous on $[a, a+h]$, and

(ii) f^{n-1} is differentiable on $(a, a+h)$.

Then there exists a real number θ satisfying $0 < \theta < 1$ such that $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h)$, where p is a positive integer $\leq n$ ... ... (I)

Proof. Since f^{n-1} is continuous on $[a, a+h], f, f', f'', \ldots, f^{n-1}$ are all continuous on $[a, a+h]$.

Since f^{n-1} is differentiable on $(a, a+h), f, f', f'', \ldots, f^{n-1}$ are all differentiable on $(a, a+h)$.

Let us consider the function $\phi : [a, a+h] \to \mathbb{R}$ defined by $\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \cdots + \frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x) + A(a+h-x)^p$, for $x \in [a, a+h]$ where A is a constant to be determined under the condition $\phi(a) = \phi(a+h)$.

$\phi(a+h) = \phi(a)$ gives

$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + Ah^p$.

or, $A = \frac{1}{h^p}[f(a+h) - f(a) - hf'(a) - \cdots - \frac{h^{n-1}}{(n-1)!}f^{n-1}(a)]$ ... ... (i)

ϕ is continuous on $[a, a+h]$, since $f, f', \ldots, f^{n-1}$ are continuous on $[a, a+h]$.

ϕ is differentiable on $(a, a+h)$, since $f, f', \ldots, f^{n-1}$ are differentiable on $(a, a+h)$. Also $\phi(a) = \phi(a+h)$.

By Rolle's theorem, there exists a real number θ satisfying $0 < \theta < 1$ such that $\phi'(a+\theta h) = 0$.

But $\phi'(x) = f'(x) + [-f'(x) + (a+h-x)f''(x)] + [-(a+h-x)f''(x) + \frac{(a+h-x)^2}{2!}f'''(x)] + \cdots + [-\frac{(a+h-x)^{n-2}}{(n-2)!}f^{n-1}(x) + \frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x)] - pA(a+h-x)^{p-1}$

$= \frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x) - pA(a+h-x)^{p-1}$.

$\phi'(a+\theta h) = 0$ gives $\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta h) = pAh^{p-1}(1-\theta)^{p-1}$

or, $A = \frac{h^{n-p}(1-\theta)^{n-p}}{p(n-1)!} \cdot f^n(a+\theta h)$.

Therefore from (i) $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h), 0 < \theta < 1$.

The last term $\frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h)$ is called the *remainder after n terms* and it is denoted by R_n. R_n in this form is called **Schlomilch-Roche's form of remainder.**

If $p = 1, R_n = \frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta h)$. R_n in this form is called **Cauchy's form of remainder.**

If $p = n, R_n = \frac{h^n}{n!}f^n(a+\theta h)$. R_n in this form is called **Lagrange's form of remainder.**

Note 1. Taylor's theorem with any particular form of remainder can be proved independently by suitably defining the function ϕ. The last term of $\phi(x)$ [i.e., $A(x+h-x)^p$] is to be chosen as $A(a+h-x)$, or as $A(a+h-x)^n$ according as R_n is desired in Cauchy's form or in Lagrange's form.

Note 2. Taylor's theorem is the nth Mean value theorem. Lagrange's mean value theorem is a particular case, corresponding to $n = 1$.

Note 3. The theorem also holds if $h < 0$. In this case the interval $[a, a+h]$ is to be replaced by $[a+h, a]$.

Note 4. For a given function f and the given interval $[a, a+h]$, θ appearing in the expansion of $f(a+h)$ depends on n. This dependence can be properly indicated by writing $R_n = \frac{h^n(1-\theta_n)^{n-p}}{p(n-1)!}f^n(a+\theta_n h)$, $0 < \theta_n < 1$.

Another form of Taylor's theorem.

Let a function $f : [a, a+h] \to \mathbb{R}$ be such that

(i) f^{n-1} is continuous on $[a, a+h]$, and

(ii) f^{n-1} is differentiable on $(a, a+h)$.

Then for any $x \in (a, a+h]$ there exists a real number θ satisfying $0 < \theta < 1$ such that

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)$$
$$+ \frac{(x-a)^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta(x-a)), \text{ where } p \text{ is a positive integer} \le n \quad \ldots \text{ (II)}$$

Since f satisfies the conditions of Taylor's theorem on $[a, a+h]$ and since $x \in (a, a+h]$, f satisfies the conditions on $[a, x]$ also.

Therefore replacing $a+h$ by x in (I) we have (II).

Note 1. For an independent proof of the theorem in this form, we are to consider the function $\phi : [a, x] \to \mathbb{R}$ defined by

$$\phi(t) = f(t)+(x-t)f'(t)+\frac{(x-t)^2}{2!}f''(t)+\cdots+\frac{(x-t)^n}{(n-1)!}f^{n-1}(t)+A(x-t)^p,$$

for $t \in [a, x]$.

Note 2. Taking $a = 0$, the form (II) reduces to

$$f(x) = f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\cdots+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n(1-\theta)^{n-p}}{p(n-1)!}f^n(\theta x).$$

This is **Maclaurin's** theorem.

Theorem 9.7.2. (Maclaurin's theorem)

Let a function $f : [0, h] \to \mathbb{R}$ be such that

(i) f^{n-1} is continuous on $[0, h]$, and

(ii) f^{n-1} is differentiable on $(0, h)$.

Then for any $x \in (0, h]$ there exists a real number θ satisfying $0 < \theta < 1$ such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}(0) + \frac{x^n(1-\theta)^{n-p}}{p(n-1)!}f^n(\theta x),$$

where p is a positive integer $\leq n$.

The last term $\frac{x^n(1-\theta)^{n-p}}{p(n-1)!}f^n(\theta x)$ is called the remainder after n terms and it is denoted by R_n.

R_n in this form is called Schlomich-Roche's form of remainder.

If $p = 1$, $R_n = \frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(a + \theta x)$. (Cauchy's form)

If $p = n$, $R_n = \frac{x^n}{n!}f^n(a + \theta x)$. (Lagrange's form)

For an independent proof of the theorem we are to consider the function $\phi : [0, x] \to \mathbb{R}$ defined by

$\phi(t) = f(t) + (x - t)f'(t) + \frac{(x-t)^2}{2!}f''(t) + \cdots + \frac{(x-t)^{n-1}}{(n-1)!}f^{n-1}(t) + A(x - t)^p$, for $t \in [0, x]$, where A is a constant to be determined under the condition $\phi(0) = \phi(x)$.

For the proof of the theorem with a particular form of remainder, the last term of $\phi(t)$ [i.e., $A(x-t)^p$] is to be chosen as $A(x-t)$ or as $A(x-t)^n$ according as R_n is desired in Cauchy's form or in Lagrange's form.

Theorem 9.7.3. (General form of Taylor's theorem)

Let $a \in \mathbb{R}$. Let a real function f defined on some neighbourhood $N(a)$ of a be such that f^{n-1} is differentiable on $N(a)$.

Then for any $x \in N'(a)$ [i.e., $N(a) - \{a\}$] there exists a real number θ satisfying $0 < \theta < 1$ such that

$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{(x-a)^n(1-\theta)^{n-p}}{(n-1)!}f^n(a+\theta(x-a))$, where p is a positive integer $\leq n$.

Proof. Since f^{n-1} is differentiable on $N(a), f, f', f'', \ldots, f^{n-1}$ are all differentiable on $N(a)$ and therefore $f, f', f'', \ldots, f^{n-1}$ are all continuous on $N(a)$.

Let us consider the function $\phi : I \to \mathbb{R}$ defined by

$\phi(t) = f(t) + (x-t)f'(t) + \frac{(x-t)^2}{2!}f''(t) + \cdots + \frac{(x-t)^{n-1}}{(n-1)!}f^{n-1}(t) + A(x-t)^p$, for $t \in I, I$ being $[a,x] \subset N(a)$ or $[x,a] \subset N(a)$, where A is a constant to be determined under the condition $\phi(x) = \phi(a)$.

$\phi(x) = \phi(a)$ gives $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a) + A(x-a)^p$.

or, $A = \frac{1}{(x-a)^p}[f(x) - f(a) - (x-a)f'(a) - \cdots - \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)]$.

ϕ is continuous on I since $f, f', f'', \ldots, f^{n-1}$ are continuous on I. ϕ is differentiable on I since $f, f', f'', \ldots, f^{n-1}$ are differentiable on I. Also $\phi(a) = \phi(x)$.

By Rolle's theorem, there exists a real number θ satisfying $0 < \theta < 1$ such that $\phi'(a+\theta(x-a)) = 0$.

But $\phi'(t) = f'(t) + [-f'(t) + (x-t)f''(t)] + [-(x-t)f''(t) + \frac{(x-t)^2}{2!}f'''(t)]$
$+ \cdots + [-\frac{(x-t)^{n-2}}{(n-2)!}f^{n-1}(t) + \frac{(x-t)^{n-1}}{(n-1)!}f^n(t)] - Ap(x-t)^{p-1}$
$= \frac{(x-t)^{n-1}}{(n-1)!}f^n(t) - Ap(x-t)^{p-1}$.

$\phi'(a+\theta(x-a)) = 0$ gives
$\frac{(x-a)^{n-1}(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta(x-a)) = Ap(x-a)^{p-1}(1-\theta)^{p-1}$
or, $A = \frac{(x-a)^{n-p}(1-\theta)^{n-p}}{p(n-1)!}.f^n(a+\theta(x-a))$.

It follows that $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{(x-a)^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta(x-a)), 0 < \theta < 1$.

The last term is the remainder R_n in Schlomilch-Roche's form.

If $p = 1, R_n = \frac{(x-a)^n(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta(x-a))$ (Cauchy's form).

If $p = n, R_n = \frac{(x-a)^n}{n!}f^n(a+\theta(x-a))$ (Lagrange's form).

In particular, if $a = 0$, for any $x \in N'(0)$ there exists a real number θ satisfying $0 < \theta < 1$ such that
$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + R_n$,

where $R_n = \frac{x^n(1-\theta)^{n-p}}{p(n-1)!} f^n(\theta x)$ (Schlomilch-Roche's form)

$= \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^n(\theta x)$ (Cauchy's form)

$= \frac{x^n}{n!} f^n(\theta x)$ (Lagrange's form).

This is the **general form of Maclaurin's theorem.**

Worked Examples.

1. Use Taylor's theorem to prove that

$1 + \frac{x}{2} - \frac{x^3}{8} < \sqrt{1+x} < 1 + \frac{x}{2}$, if $x > 0$.

Let $f(x) = \sqrt{1+x}$, $x \geq 0$.

Then $f'(x) = \frac{1}{2\sqrt{1+x}}$, $f''(x) = -\frac{1}{4(1+x)^{3/2}}$, $f'''(x) = \frac{3}{8(1+x)^{5/2}}$.

By Taylor's theorem with Lagrange's form of remainder (after 3 terms), for any $x > 0$,

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(c)$, for some $c \in (0, x)$.

or, $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16(1+c)^{5/2}}$.

Therefore for $x > 0$, $\sqrt{1+x} > 1 + \frac{x}{2} - \frac{x^3}{8}$, since $\frac{x^3}{16(1+c)^{5/2}} > 0$.

By Taylor's theorem with Lagrange's form of remainder (after 2 terms), for any $x > 0$,

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(d)$, for some $d \in (0, x)$.

or, $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8(1+d)^{3/2}}$.

Therefore for $x > 0$, $\sqrt{1+x} < 1 + \frac{x}{2}$, since $\frac{x^2}{8(1+d)^{3/2}} > 0$.

From (i) and (ii), $1 + \frac{x}{2} - \frac{x^3}{8} < \sqrt{1+x} < 1 + \frac{x}{2}$, if $x > 0$.

2. Let $c \in \mathbb{R}$ and a real function f be such that f'' is continuous on some neighbourhood of c. Prove that $\lim\limits_{h \to 0} \dfrac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c)$.

Let f'' be continuous on $(c - \delta, c + \delta)$ for some $\delta > 0$.

By Taylor's theorem with Lagrange's form of remainder (after 2 terms), for any h satisfying $0 < h < \delta$

$f(c+h) = f(c) + hf'(c) + \frac{h^2}{2!} f''(c + \theta h), 0 < \theta < 1$

and $f(c-h) = f(c) - hf'(c) + \frac{h^2}{2!} f''(c - \theta' h), 0 < \theta' < 1$.

Therefore $f(c+h) + f(c-h) - 2f(c) = \frac{h^2}{2}[f''(c+\theta h) + f''(c - \theta' h)]$

or, $\frac{f(c+h)-2f(c)+f(c-h)}{h^2} = \frac{1}{2}[f''(c+\theta h) + f''(c - \theta' h)]$.

Since f'' is continuous at c,

$$\lim_{h\to 0} f''(c+\theta h) = f''(c), \lim_{h\to 0} f''(c-\theta' h) = f''(c).$$

Hence $\lim_{h\to 0} \dfrac{f(c+h)-2f(c)+f(c-h)}{h^2} = f''(c)$.

3. Let $a \in \mathbb{R}$ and a real function f defined on some neighbourhood $N(a)$ of a be such that f'' is continuous at a and $f''(a) \neq 0$. Prove that $\lim_{h\to 0} \theta = \frac{1}{2}$, where θ is given by $f(a+h) = f(a) + hf'(a+\theta h)(0 < \theta < 1)$.

Since f'' is a continuous at a, f'' exists in some neighbourhood $(a-\delta, a+\delta) \subset N(a)$.

Case 1. Let $0 < h < \delta$.

By Taylor's theorem with Lagrange's form of remainder after 2 terms, $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a+\theta' h), 0 < \theta' < 1$.

Therefore $f'(a+\theta h) = f'(a) + \frac{h}{2}f''(a+\theta' h)$.

Applying Mean value theorem to f' on the interval $[a, a+\theta h]$, we have $f'(a+\theta h) = f'(a) + \theta h f''(a+\theta\theta'' h), 0 < \theta'' < 1$.

Therefore $\frac{1}{2}f''(a+\theta' h) = \theta f''(a+\theta\theta'' h)$.

Taking limit as $h \to 0+$, we have

$\frac{1}{2}\lim_{h\to 0+} f''(a+\theta' h) = \lim_{h\to 0+}[\theta . f''(a+\theta\theta'' h)]$.

Since f'' is continuous at a, $\lim_{h\to 0+} f''(a+\theta' h) = f''(a)$, $\lim_{h\to 0+} f''(a+\theta\theta'' h) = f''(a)$. Therefore $\lim_{h\to 0+} \theta = \frac{1}{2}$, since $f''(a) \neq 0$.

Case 2. Let $-\delta < h < 0$. Proceeding similarly, we have $\lim_{h\to 0-} \theta = \frac{1}{2}$.

Hence $\lim_{h\to 0} \theta = \frac{1}{2}$.

4. Let a function $f : [a, \infty) \to \mathbb{R}$ be twice differentiable on $[a, \infty)$ and there exist positive real numbers A and B such that $|f(x)| \leq A, |f''(x)| \leq B$ for all $x \in [a, \infty)$. Prove that $| f'(x) | \leq 2\sqrt{AB}$ for all $x \in [a, \infty)$.

Let $x \geq a$ and $h > 0$.

By Taylor's theorem with Lagrange's form of remainder after 2 terms, $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(c)$ for some $c \in (x, x+h)$.

Therefore $|hf'(x)| = |f(x+h) - f(x) - \frac{h^2}{2}f''(c)|$

$\leq |f(x+h)| + |f(x)| + |\frac{h^2}{2}f''(c)|$. Therefore $| f'(x) | \leq \frac{2A}{h} + \frac{h}{2}B$, for all $x \geq a$ and for all $h > 0$.

Let $\phi(h) = \frac{2A}{h} + \frac{h}{2}B, h > 0$. Then $\phi(h) \geq 2\sqrt{\frac{2A}{h}.\frac{h}{2}B}$, i.e., $\geq 2\sqrt{AB}$.

Consequently, $|f'(x)| \leq 2\sqrt{AB}$ for all $x \in [a, \infty)$.

5. If $f(x) = \sin x$ prove that $\lim_{h\to 0} \theta = \frac{1}{\sqrt{3}}$, where θ is given by $f(h) = f(0) + hf'(\theta h)$, $0 < \theta < 1$.

$f(h) = f(0) + hf'(\theta h)$ gives $\sin h = h \cos \theta h$, $0 < \theta < 1$ ... (i)

Also $f(h) = f(0) + hf'(0) + \frac{h^2}{2} f''(0) + \frac{h^3}{6} f'''(\theta' h)$, $0 < \theta' < 1$

and $f'(\theta h) = f'(0) + \theta h f''(0) + \frac{\theta^2 h^2}{2} f'''(\theta'' \theta h)$, $0 < \theta'' < 1$.

Therefore $\sin h = h - \frac{h^3}{6} \cos(\theta' h)$, $0 < \theta' < 1$ (ii)

and $\cos \theta h = 1 - \frac{\theta^2 h^2}{2} \cos(\theta'' \theta h)$, $0 < \theta'' < 1$ (iii)

From (i) and (ii) $\cos \theta h = 1 - \frac{h^2}{6} \cos(\theta' h)$ ... (iv)

From (iii) and (iv) $3\theta^2 \cos(\theta'' \theta h) = \cos(\theta' h)$.

$\lim_{h\to 0} \theta = \frac{1}{\sqrt{3}}$, since $\lim_{h\to 0} \cos(\theta' h) = 1$ and $\lim_{h\to 0} \cos(\theta'' \theta h) = 1$.

Note. This does not contradict worked Ex.3, since here $f''(0) = 0$.

6. Let $f : [a, b] \to \mathbb{R}$ be such that $f''(x)$ exists in $[a, b]$ and $f'(a) = f'(b)$. Prove that $f(\frac{a+b}{2}) = \frac{1}{2}[f(a) + f(b)] + \frac{1}{8}(b-a)^2 f''(c)$ for some $c \in (a, b)$.

By Mean value theorem applied to f on $[a, \frac{a+b}{2}]$ and on $[\frac{a+b}{2}, b]$,

$f(\frac{a+b}{2}) = f(a) + \frac{b-a}{2} f'(a) + \frac{(b-a)^2}{8} f''(\xi_1)$ for some $\xi_1 \in (a, \frac{a+b}{2})$ ... (i)

$f(\frac{a+b}{2}) = f(b) - \frac{b-a}{2} f'(b) + \frac{(b-a)^2}{8} f''(\xi_2)$ for some $\xi_2 \in (\frac{a+b}{2}, b)$ ... (ii)

From (i) and (ii) $f(\frac{a+b}{2}) = \frac{1}{2}[f(a) + f(b)] + \frac{1}{8}(b-a)^2[\frac{f''(\xi_1)+f''(\xi_2)}{2}]$.

If $f''(\xi_1) \neq f''(\xi_2)$, by intermediate value property of the derived function f'', $\frac{f''(\xi_1)+f''(\xi^2)}{2} = f''(c)$ for some $c \in (\xi_1, \xi_2)$.

Therefore $f(\frac{a+b}{2}) = \frac{1}{2}[f(a) + f(b)] + \frac{1}{8}(b-a)^2 f''(c)$ for some $c \in (a, b)$.

If however, $f''(\xi_1) = f''(\xi_2)$, then $c = \xi_1$ and the result holds.

7. Let $a \in \mathbb{R}$ and a real function f be such that $f''(x)$ exists in $[a - h, a + h]$ for some $h > 0$. Prove that $\frac{f(a+h)-2f(a)+f(a-h)}{h^2} = f''(c)$ for some $c \in (a - h, a + h)$.

By Mean value theorem applied to f on $[a, a+h]$ and on $[a-h, a]$,

$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2} f''(\xi_1)$ for some $\xi_1 \in (a, a+h)$ ... (i)

$f(a-h) = f(a) - hf'(a) + \frac{h^2}{2} f''(\xi_2)$ for some $\xi_2 \in (a-h, a)$ ... (ii)

From (i) and (ii) $f(a+h) + f(a-h) - 2f(a) = h^2[\frac{f''(\xi_1)+f''(\xi_2)}{2}]$.

If $f''(\xi_1) \neq f''(\xi_2)$, by intermediate value property of the derived function f'', $\frac{f(a+h)-2f(a)+f(a-h)}{h^2} = f''(c)$ for some $c \in (a-h, a+h)$.

If however, $f''(\xi_1) = f''(\xi_2)$, then $c = \xi_1$ and the result holds.

9.7.4. Taylor's infinite series.

Let $a \in \mathbb{R}$. Let a real function f defined on some neighbourhood $N(a)$ of a be such that f^{n-1} is differentiable on $N(a)$.

Then for any $x \in N'(a)$ $[= N(a) - \{a\}]$, $f(x) = P_n(x) + R_n(x)$, where $R_n(x)$ is the remainder after n terms and

$$P_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a).$$

$P_n(x)$ is a polynomial of degree $n-1$. $P_n(x)$ is such that

$$P_n(a) = f(a), P_n'(a) = f'(a), P_n''(a) = f''(a), \ldots, P_n^{n-1}(a) = f^{n-1}(a).$$

$P_n(x)$ is called the n-th **Taylor polynomial** of f about the point a.

If f be a function such that for all $n \in \mathbb{N}$, f^n exists on $N(a)$, then the polynomial $P_n(x)$ for any $x \in N'(a)$ takes the form of an infinite series

$$f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots \quad \cdots \text{(i)}$$

The questions now arise:

1. Does the infinite series (i) converge?

2. If it be convergent, does it converge to $f(x)$?

The infinite series (i) will be convergent if and only if the sequence of partial sums [i.e., the sequence $\{P_n(x)\}$] be convergent.

In order that the sum of the series (i) may be $f(x)$, the limit of the sequence $\{P_n(x)\}$ must be $f(x)$.

Since $f(x) = P_n(x) + R_n(x)$ for any n, the series (i) will converge to $f(x)$ if and only if $\lim\limits_{n\to\infty} R_n(x) = 0$.

Therefore if $f : N(a) \to \mathbb{R}$ be such that $f^n(x)$ exists on $N(a)$ for all $n \in \mathbb{N}$, then for any x belonging to some subset $A \subset N'(a)$ for which $\lim\limits_{n\to\infty} R_n(x) = 0$, $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots \quad \cdots$ (ii)

The equality (ii) also holds for $x = a$ trivially.

The infinite series in the right hand side of (ii) is called **Taylor's infinite series** for the function f about the point a, the region of convergence of the series being $A \cup \{a\}$.

In particular, if $a = 0$, then Taylor's infinite series for the function f about 0 takes the form $f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots \cdots$

This is called **Maclaurin's infinite series** for f.

9.7.5. Expansion of some functions.

1. Let $f(x) = e^x, x \in \mathbb{R}$.

For all $n \in \mathbb{N}$, $f^n(x) = e^x$ for all $x \in \mathbb{R}$.

By Taylor's theorem with Lagrange's form of remainder after n terms, for a non-zero $x \in \mathbb{R}$,
$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + R_n(x)$, where $R_n(x) = \frac{x^n}{n!}f^n(\theta x)$ for some real number θ satisfying $0 < \theta < 1$ ... ... (i)

Since for all $n \in \mathbb{N}$, $f^n(x)$ exists for all real x, the right hand polynomial in (i) takes the form of an infinite series as $n \to \infty$. The infinite series will converge to $f(x)$ for those non-zero real x for which $\lim\limits_{n\to\infty} R_n = 0$.

$R_n = \frac{x^n}{n!}e^{\theta x}$.

Let $u_n = \frac{|x|^n}{n!}, x \neq 0$. Then $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{|x|}{n+1} = 0$. This proves $\lim u_n = 0$. [Theorem 5.8.1.]

For all real $x, e^{\theta x}$ is bounded. Hence $\lim\limits_{n\to\infty} | R_n |= 0$ for all real $x \neq 0$ and this implies $\lim R_n = 0$ for all real $x \neq 0$. [Theorem 5.6.1.]

Consequently, the infinite series $1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$ converges to e^x for all real $x \neq 0$.

At $x = 0$, the convergence holds trivially.

So $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$ for all $x \in \mathbb{R}$.

2. Let $f(x) = \sin x, x \in \mathbb{R}$.

For all $n \in \mathbb{N}, f^n(x) = \sin(\frac{n\pi}{2} + x)$ exists for all $x \in \mathbb{R}$.

By Taylor's theorem with Lagrange's form of remainder after n terms, for any non-zero $x \in \mathbb{R}$,
$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + R_n(x)$, where $R_n(x) = \frac{x^n}{n!}f^n(\theta x)$ for some real number θ satisfying $0 < \theta < 1$ ... ... (i)

$$\begin{aligned} f(0) = 0, f^n(0) = \sin \tfrac{n\pi}{2} &= 0 \text{ if } n \text{ be even} \\ &= 1 \text{ if } n = 4k+1, k \text{ being an integer} \\ &= -1 \text{ if } n = 4k+3, k \text{ being an integer.} \end{aligned}$$

Therefore $f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{x^n}{n!}\sin(\frac{n\pi}{2} + \theta x)$.

Since for all $n \in \mathbb{N}$, $f^n(x)$ exists for all real x, the right hand polynomial in (i) takes the form of an infinite series as $n \to \infty$. The infinite series will converge to $f(x)$ for those non-zero real x for which $\lim\limits_{n\to\infty} R_n = 0$.

Now $| R_n |=| \frac{x^n}{n!}\sin(\frac{n\pi}{2} + \theta x) |\leq \frac{|x|^n}{n!}$.

Let $u_n = \frac{|x|^n}{n!}, x \neq 0$. Then $\lim \frac{u_{n+1}}{u_n} = \lim \frac{|x|}{n+1} = 0$. This proves $\lim u_n = 0$. [Theorem 5.8.1.]

Hence $\lim | R_n |= 0$ for all real $x \neq 0$ and this implies $\lim R_n =$ (all real $x \neq 0$. [Theorem 5.6.1.]

Consequently, the infinite series $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ converges to s for all real $x \neq 0$.

At $x = 0$, the convergence holds trivially.

Therefore $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ for all real x.

3. Let $f(x) = (1+x)^m, x \in \mathbb{R}$.

Case 1. Let m be a positive integer.

$$\begin{aligned} \text{Then } f^n(x) &= m(m-1)\cdots(m-n+1)(1+x)^{m-n} \text{ if } 1 \leq n \cdot \\ &= m! \text{ if } n = m \\ &= 0 \text{ if } n > m. \end{aligned}$$

By Taylor's theorem with remainder after $\underline{m+1}$ terms, for any n zero $x \in \mathbb{R}$,

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^m}{m!}f^m(0)$.

But $f(0) = 1, f^n(0) = m_{c_n}$ for $1 \leq n \leq m$. Therefore

$(1+x)^m = 1 + m_{c_1}x + m_{c_2}x^2 + \cdots + m_{c_m}x^m$ for all non-zero $x \in$

At $x = 0$, the equality holds trivially.

Therefore $(1+x)^m = 1 + m_{c_1}x + m_{c_2}x^2 + \cdots + m_{c_m}x^m$ for all $x \in$

Thus we obtain a *finite series* expansion in this case.

Case 2. Let m be not a positive integer.

In this case f is defined for all $x \neq -1$, if m be a negative integer a f is defined for all $x > -1$, if m be not an integer.

Considering all cases, f is defined for all $x > -1$; and for all n $\mathbb{N}, f^n(x) = m(m-1)\cdots(m-n+1)(1+x)^{m-n}$ for all $x > -1$.

By Taylor's theorem with Cauchy's form of remainder after n term for any real non-zero $x > -1$,

$f(x) = f(0) + xf(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + R_n(x)$, whe $R_n(x) = \frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x)$ for some real θ satisfying $0 < \theta < 1$ ... ... (

But $f(0) = 1, f^n(0) = m(m-1)\cdots(m-n+1), f^n(\theta x) = m(m$ $1)\cdots(m-n+1)(1+\theta x)^{m-n}$.

Therefore $f(x) = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots + \frac{m(m-1)\cdots(m-n+1)}{(n-1)!}x^{n-1}$ $R_n(x)$, where $R_n(x) = \frac{x^n(1-\theta)^{n-1}}{(n-1)!}m(m-1)\cdots(m-n+1)(1+\theta x)^{m-}$

Since for all $n \in \mathbb{N}$, $f^n(x)$ exists for all $x > -1$, the right hand sid

$$R_n = (-1)^{n-1}.(1-\theta)^{n-1}.\tfrac{x^n}{(1+\theta x)^n} \text{ (Cauchy's form)}.$$

Since for all $n \in \mathbb{N}$, $f^n(x)$ exists for all $x > -1$, the right hand polynomial takes the form of an infinite series as $n \to \infty$.

The infinite series will converge to $f(x)$ for those non-zero $x > -1$ for which $\lim\limits_{n\to\infty} R_n = 0$.

Case 1. Let $0 < x \leq 1$. We take R_n in Lagrange's form.

$R_n = \frac{x^n}{n!}\frac{(-1)^{n-1}.(n-1)!}{(1+\theta x)^n} = \frac{(-1)^{n-1}}{n}(\frac{x}{1+\theta x})^n$.

In $0 < x < 1$, $1 + \theta x > x > 0$. Therefore $0 < \frac{x}{1+\theta x} < 1$.

When $x = 1$, $\frac{x}{1+\theta x} = \frac{1}{1+\theta} < 1$.
So in $0 < x \leq 1$, $0 < \frac{x}{1+\theta x} < 1$ and hence $0 < (\frac{x}{1+\theta x})^n < 1$.

Therefore $\lim |R_n| = \lim \frac{1}{n}|\frac{x}{1+\theta x}|^n = 0$, since $\lim \frac{1}{n} = 0$ and $|\frac{x}{1+\theta x}|^n$ is bounded. Hence $\lim\limits_{n\to\infty} R_n = 0$. [Theorem 5.6.1.]

Case 2. Let $-1 < x < 0$. We take R_n in Cauchy's form.

Here $R_n = \frac{(-1)^{n-1}x^n(1-\theta)^{n-1}}{(1+\theta x)^n}$. $|R_n| = |x|^n|\frac{1-\theta}{1+\theta x}|^{n-1}.\frac{1}{|1+\theta x|}$.

In $-1 < x < 0$, $0 < 1 - \theta < 1 + \theta x < 1$.
Therefore $0 < \frac{1-\theta}{1+\theta x} < 1$ and hence $0 < |\frac{1-\theta}{1+\theta x}|^{n-1} < 1$.

In $-1 < x < 0$, $\lim |x|^n = 0$.

For all real x, $-|x| \leq x \leq |x|$. Hence $-|x| < -\theta|x| \leq \theta x \leq \theta|x| < |x|$, since $0 < \theta < 1$.

In $-1 < x < 0$, $0 < 1 - |x| < 1 + \theta x$. Therefore $\frac{1}{|1+\theta x|} < \frac{1}{1-|x|}$.
Hence $\lim\limits_{n\to\infty} |R_n| = 0$ and this implies $\lim\limits_{n\to\infty} R_n = 0$. [Theorem 5.6.1.]

Thus the infinite series $x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ converges to $\log(1 + x)$ for all non-zero $x \in (-1, 1]$.

At $x = 0$, the convergence holds trivially.

So $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ for all $x \in (-1, 1]$.

Worked Example.

1. Use Taylor's theorem to the function $f(x) = e^x, x \in \mathbb{R}$ to prove that e is irrational.

f satisfies all conditions of Taylor's theorem on $\mathbb{R}$.

By Taylor's theorem with Lagrange's form of remainder, for any real $x \neq 0$ there exists a real number θ satisfying $0 < \theta < 1$ such that

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n}{n!}f^n(\theta x)$
or, $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}e^{\theta x}$.

Therefore $e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{(n-1)!} + \frac{e^\theta}{n!}$.
Since $e^\theta > 0, e > 2$.

If possible, let e be rational and $e = \frac{p}{q}$ where p, q are positive integers prime to each other. Let us choose $n > q$ and $n > 2$.

Then $\frac{p}{q} = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{(n-1)!} + \frac{e^\theta}{n!}$

or, $\frac{p(n-1)!}{q} - (n-1)!\{1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{(n-1)!}\} = \frac{e^\theta}{n}$.

Since $n > q$, $\frac{p(n-1)!}{q}$ is an integer. And $(n-1)!(1+1+\frac{1}{2!} + \cdots + \frac{1}{(n-1)!})$ is also an integer. Therefore $\frac{e^\theta}{n}$ turns out to be an integer.

Since $0 < \theta < 1$, we have $0 < e^\theta < e < 3$ and therefore $0 < \frac{e^\theta}{n} < 1$, since $n > 2$. This shows that $\frac{e^\theta}{n}$ is a proper fraction.

Thus we arrive at a contradiction that an integer is equal to a proper fraction. Therefore e must be irrational.

Note. e lies between 2 and 3. The approximation of e upto ten places of decimal is given by $e = 2.7182818284$.

Exercises 17

1. Stating all conditions to be satisfied by f for the expansion, expand the polynomial $f(x)$ in powers of $x - 1$.

   (i) $f(x) = x^4 + x^3 + x^2 + x + 1$; (ii) $f(x) = x^5 + x^3 + x$.

2. Use Taylor's theorem to prove that

   (i) $\cos x \geq 1 - \frac{x^2}{2}$ for $-\pi < x < \pi$;

   (ii) $x - \frac{x^3}{6} < \sin x < x$ for $0 < x < \pi$;

   (iii) $x - \frac{x^2}{2} < \log(1+x) < x$ for $x > 0$.

[**Hint.** (i) Let $x \in (0, \pi)$. By Taylor's theorem, $\cos x = 1 - \frac{x^2}{2} + \frac{x^3}{6}\sin c, 0 < c < x < \pi$. Therefore $\cos x > 1 - \frac{x^2}{2}$.

Let $x \in (-\pi, 0)$. By Taylor's theorem, $\cos x = 1 - \frac{x^2}{2} + \frac{x^3}{6}\sin c, -\pi < x < c < 0$. Therefore $\cos x > 1 - \frac{x^2}{2}$.]

3. If $x \in [0,1]$ prove that $|\log(1+x) - (x - \frac{x^2}{2} + \frac{x^3}{3})| < \frac{1}{4}$.

4. If $x \in [-1,1]$ prove that $|\sin x - (x - \frac{x^3}{3!} + \frac{x^5}{5!})| < \frac{1}{7!}$.

5. Let $a \in \mathbb{R}$ and a real function f defined on some neighbourhood $N(a)$ of a be such that f^n is continuously differentiable on $N(a)$ and $f^{n+1}(a) \neq 0$. If for $a+h \in N(a)$, $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}f^n(a+\theta h), (0 < \theta < 1)$, prove that $\lim\limits_{h \to 0} \theta = \frac{1}{n+1}$.

6. Verify Maclaurin's infinite series expansion of the following functions on the indicated intervals.

(i) $\cos^2 x = 1 - \frac{2x^2}{2!} + \frac{2^3x^4}{4!} - \frac{2^5x^6}{6!} + \cdots$ on $\mathbb{R}$;

(ii) $\sin^2 x = \frac{2x^2}{2!} - \frac{2^3x^4}{4!} + \frac{2^5x^6}{6!} - \cdots$ on $\mathbb{R}$;

(iii) if $a > 0, a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \cdots$ on $\mathbb{R}$;

(iv) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$ on $\mathbb{R}$;

(v) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$ on $\mathbb{R}$;

(vi) $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ for $-1 < x < 1$;

(vii) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ for $-1 < x < 1$;

(viii) $\log(1+2x) = 2x - \frac{2^2x^2}{2} + \frac{2^3x^3}{3} - \cdots$ for $-\frac{1}{2} < x \leq \frac{1}{2}$;

(ix) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ for $-1 \leq x \leq 1$.

9.8. Maxima and minima.

Let I be an interval.

A function $f : I \to \mathbb{R}$ is said to have a *global maximum* (or an *absolute maximum*) on I if there exists a point $c \in I$ such that $f(c) \geq f(x)$ for all $x \in I$. c is said to be a *global maximum point* for f on I.

f is said to have a *global minimum* (or an *absolute minimum*) on I if there exists a point $c \in I$ such that $f(c) \leq f(x)$ for all $x \in I$. c is said to be a *global minimum point* for f on I.

A function $f : I \to \mathbb{R}$ is said to have a *local maximum* (or a *relative maximum*) at a point $c \in I$ if there exists a neighbourhood $N(c, \delta)$ of c such that $f(c) \geq f(x)$ for all $x \in N(c, \delta) \cap I$.

f is said to have a *local minimum* (or a *relative minimum*) at a point $c \in I$ if there exists a neighbourhood $N(c, \delta)$ of c such that $f(c) \leq f(x)$ for all $x \in N(c, \delta) \cap I$.

We say that f has a *local extremum* (or a *relative extremum*) at a point $c \in I$ if f has either a local maximum or a local minimum at c.

Note. If $f : I \to \mathbb{R}$ has a local maximum (a local minimum) at a point $c \in I$ then c is a global maximum point (a global minimum point) for f on $N(c, \delta) \cap I$ for some suitable $\delta > 0$

Theorem 9.8.1. Let $f : I \to \mathbb{R}$ be such that f has a local extremum at an interior point c of I. If $f'(c)$ exists then $f'(c) = 0$.

Proof. We prove the theorem for the case when f has a local maximum at c. The proof of the other case is similar.

Since $f'(c)$ exists, either $f'(c) > 0$, or $f'(c) < 0$, or $f'(c) = 0$.

Let $f'(c) > 0$. Then $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} > 0$.

Therefore there exists a positive δ such that $\frac{f(x)-f(c)}{x-c} > 0$ for all $x \in N'(c, \delta) \subset I$.

Let $c < x < c + \delta$. Then $x - c > 0$ and therefore $f(x) > f(c)$ for all $x \in (c, c + \delta)$. This contradicts that f has a local maximum at c.

Consequently, $f'(c) \not> 0$ ... ... (i)

Let $f'(c) < 0$. Then $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} < 0$.

Therefore there exists a positive δ such that $\frac{f(x)-f(c)}{x-c} < 0$ for all $x \in N'(c, \delta) \subset I$.

Let $c - \delta < x < c$. Then $x - c < 0$ and therefore $f(x) > f(c)$ for all $x \in (c - \delta, c)$. This contradicts that f has a local maximum at c.

Consequently, $f'(c) \not< 0$ ... ... (ii)

From (i) and (ii) we have $f'(c) = 0$.

This proves the theorem.

Corollary. Let $f : I \to \mathbb{R}$ and c be an interior point of I, where f has a local extremum. Then either $f'(c)$ does not exist, or $f'(c) = 0$.

Note 1. The theorem says that if the derivative $f'(c)$ exists at an interior point c of local extremum, $f'(c)$ must be 0. A function may, however have a local extremum at an interior point c of its domain without being differentiable at c. For example, the function defined by $f(x) = |x|, x \in \mathbb{R}$ has a local minimum at 0 but $f'(0)$ does not exist.

Note 2. The condition $f'(c) = 0$ (when $f'(c)$ exists) is only a necessary condition for an interior point c to be a point of local extremum of the function f.

For example, for the function f defined by $f(x) = x^3, x \in \mathbb{R}$, 0 is an interior point of the domain of f. $f'(0) = 0$ but 0 is neither a point of local maximum nor a point of local minimum of the function f.

Note 3. The theorem holds if c is an interior point of I.

Let a function f be defined on $[0, 1]$ by $f(x) = x, x \in [0, 1]$. Then f has a local maximum at 1 (not an interior point of I), f is differentiable at 1, but $f'(1) \neq 0$.

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Theorem 9.8.2. (First derivative test for extrema)

Let f be continuous on $I = [a, b]$ and c be an interior point of I. Let f be differentiable on (a, c) and (c, b).

1. If there exists a neighbourhood $(c-\delta, c+\delta) \subset I$ such that $f'(x) \geq 0$ for $x \in (c-\delta, c)$ and $f'(x) \leq 0$ for $x \in (c, c+\delta)$ then f has a local maximum at c.

2. If there exists a neighbourhood $(c-\delta, c+\delta) \subset I$ such that $f'(x) \leq 0$ for $x \in (c-\delta, c)$ and $f'(x) \geq 0$ for $x \in (c, c+\delta)$ then f has a local minimum at c.

3. If $f'(x)$ keeps the same sign on $(c-\delta, c)$ and $(c, c+\delta)$ then f has no extremum at c.

Proof. **1.** Let $x \in (c-\delta, c)$. Applying Mean value theorem to the function f on $[x, c]$, there exists a point ξ in (x, c) such that $f(c) - f(x) = (c-x)f'(\xi)$.

Since $f'(\xi) \geq 0$, we have $f(x) \leq f(c)$ for $x \in (c-\delta, c)$.

Let $x \in (c, c+\delta)$. Applying Mean value theorem to the function f on $[c, x]$, there exists a point η in (c, x) such that $f(x) - f(c) = (x-c)f'(\eta)$.

Since $f'(\eta) \leq 0$, we have $f(x) \leq f(c)$ for $x \in (c, c+\delta)$.

It follows that $f(c) \geq f(x)$ for all $x \in N(c, \delta) \cap I$.

Therefore f has a local maximum at c.

2. Similar proof.

3. Let $f'(x) > 0$ for $x \in (c-\delta, c)$ and for $x \in (c, c+\delta)$.

Then $f(x) < f(c)$ for $x \in (c-\delta, c)$ and $f(c) < f(x)$ for $x \in (c, c+\delta)$. Therefore f has neither a maximum nor a minimum at c.

Similar proof if $f'(x) < 0$ for $x \in (c-\delta, c)$ and for $(c, c+\delta)$.

Note. The converse of the theorem is not true.

For example, let $f(x) = 2x^2 + x^2 \sin \frac{1}{x}, x \neq 0$
$= 0, \quad x = 0.$

Then f has a local minimum at 0.

$f'(x) = 4x + 2x \sin \frac{1}{x} - \cos \frac{1}{x}, x \neq 0$
$= 0, \quad x = 0.$

f' takes both positive and negative values on both sides of 0 (in the immediate neighbourhood).

Examples.

1. Let $f(x) = |x|, x \in \mathbb{R}$.

f is continuous on $\mathbb{R}$. f is not differentiable at 0.

$f'(x) < 0$ for $x \in (-\delta, 0)$ and $f'(x) > 0$ for $x \in (0, \delta)$ for some $\delta > 0$. Therefore f has a local minimum at 0.

2. Let $f(x) = |x-1| + |x-2|, x \in [0,3]$.

$$\begin{aligned}\text{Then } f(x) &= 3-2x, \text{ if } 0 \leq x < 1\\ &= 1, \text{ if } 1 \leq x \leq 2\\ &= 2x-3, \text{ if } 2 < x \leq 3.\end{aligned}$$

f is continuous on $[0,3]$. f is not differentiable at 1 and 2.

$f'(x) < 0$ for $x \in (1-\delta, 1)$, $f'(x) = 0$ for $x \in (1, 1+\delta)$ for some δ satisfying $0 < \delta < 1$. Therefore f has a local minimum at 1.

$f'(x) = 0$ for $x \in (2-\delta, 2)$, $f'(x) > 0$ for $x \in (2, 2+\delta)$ for some δ satisfying $0 < \delta < 1$. Therefore f has a local minimum at 2.

3. $f(x) = (x-1)^2(x-3)^3, x \in \mathbb{R}$.

$$\begin{aligned}f'(x) &= 2(x-1)(x-3)^3 + 3(x-1)^2(x-3)^2\\ &= (x-1)(x-3)^2(5x-9), x \in \mathbb{R}.\end{aligned}$$

f is continuous on $\mathbb{R}$. $f'(x) = 0$ at the points $1, 3, \frac{9}{5}$.

$f'(x) > 0$ for $x \in (1-\delta, 1)$ and $f'(x) < 0$ for $x \in (1, 1+\delta)$ for some $\delta > 0$. Therefore f has a local maximum at 1.

$f'(x) > 0$ for $x \in (3-\delta, 3)$ and $f'(x) > 0$ for $x \in (3, 3+\delta)$, for some $\delta > 0$. Therefore f has neither a maximum nor a minimum at 3.

$f'(x) < 0$ for $x \in (\frac{9}{5} - \delta, \frac{9}{5})$ and $f'(x) > 0$ for $x \in (\frac{9}{5}, \frac{9}{5} + \delta)$ for some $\delta > 0$. Therefore f has a local minimum at $\frac{9}{5}$.

Theorem 9.8.3. (Higher order derivative test for extrema)

$n \geq 2$

Let $f : I \to \mathbb{R}$ and c be an interior point of I.

If $f'(c) = f''(c) = \cdots = f^{n-1}(c) = 0$ and $f^n(c) \neq 0$, then f has

(i) no extremum at c if n be odd, and

(ii) a local extremum at c if n be even:

a local maximum if $f^n(c) < 0$, a local minimum if $f^n(c) > 0$.

Proof. Since $f^n(c) \neq 0$, $f^n(c)$ is either positive or negative.

If $f^n(c) > 0$, then f^{n-1} is increasing at c.

Therefore there exists a positive δ such that $f^{n-1}(x) < f^{n-1}(c)$ for $x \in (c-\delta, c)$ and $f^{n-1}(c) < f^{n-1}(x)$ for $x \in (c, c+\delta)$.

That is, $f^{n-1}(x) < 0$ for $x \in (c-\delta, c)$ and $f^{n-1}(x) > 0$ for $x \in (c, c+\delta)$ ... (i)

If $f^n(c) < 0$, then f^{n-1} is decreasing at c.

By similar arguments there exists a positive δ such that $f^{n-1}(x) > 0$ for $x \in (c-\delta, c)$ and $f^{n-1}(x) < 0$ for $x \in (c, c+\delta)$ ... ... (ii)

Since $f^n(c)$ exists, then $f', f'', \cdots, f^{n-1}$ all exist in some δ-neighbourhood of c.

Let $x \in (c-\delta, c)$. By Taylor's theorem with Lagrange's form of remainder after $n-1$ terms, there exists a point ξ such that $f(x) = f(c) + (x-c)f'(c) + \cdots + \frac{(x-c)^{n-2}}{(n-2)!} f^{n-2}(c) + \frac{(x-c)^{n-1}}{(n-1)!} f^{n-1}(\xi), x < \xi < c$. This gives $f(x) - f(c) = \frac{(x-c)^{n-1}}{(n-1)!} f^{n-1}(\xi)$, $x < \xi < c$ ... (iii)

Let $x \in (c, c+\delta)$. By Taylor's theorem with Lagrange's form of remainder after $n-1$ terms, there exists a point η such that $f(x) = f(c) + (x-c)f'(c) + \cdots + \frac{(x-c)^{n-2}}{(n-2)!} f^{n-2}(c) + \frac{(x-c)^{n-1}}{(n-1)!} f^{n-1}(\eta), c < \eta < x$. This gives $f(x) - f(c) = \frac{(x-c)^{n-1}}{(n-1)!} f^{n-1}(\eta)$, $c < \eta < x$ ... (iv)

Case 1. Let n be odd.

Then $\frac{(x-c)^{n-1}}{(n-1)!} > 0$ for all $x \in (c-\delta, c)$ and for all $x \in (c, c+\delta)$.

Subcase (*i*). If $f^n(c) > 0$, then $f^{n-1}(\xi) < 0$ and $f^{n-1}(\eta) > 0$, by (i).

Using (iii) and (iv) we have $f(x) < f(c)$ for all $x \in (c-\delta, c)$ and $f(x) > f(c)$ for all $x \in (c, c+\delta)$.

Therefore f has neither a maximum nor a minimum at c.

Subcase (*ii*). If $f^n(c) < 0$, then $f^{n-1}(\xi) > 0$ and $f^{n-1}(\eta) < 0$, by (ii).

Using (iii) and (iv) we have $f(x) > f(c)$ for all $x \in (c-\delta, c)$ and $f(x) < f(c)$ for all $x \in (c, c+\delta)$.

Therefore f has neither a maximum nor a minimum at c.

Case 2. Let n be even.

Then $\frac{(x-c)^{n-1}}{(n-1)!} < 0$ for all $x \in (c-\delta, c)$ and $\frac{(x-c)^{n-1}}{(n-1)!} > 0$ for all $x \in (c, c+\delta)$.

Subcase (*i*). If $f^n(c) > 0$, then $f^{n-1}(\xi) < 0$ and $f^{n-1}(\eta) > 0$, by (i).

Using (iii) and (iv) we have $f(x) > f(c)$ for all $x \in (c-\delta, c)$ and also for all $x \in (c, c+\delta)$.

Therefore f has a minimum at c.

Subcase (*ii*). If $f^n(c) < 0$, then $f^{n-1}(\xi) > 0$ and $f^{n-1}(\eta) < 0$, by (ii).

Using (iii) and (iv) we have $f(x) < f(c)$ for all $x \in (c-\delta, c)$ and also for all $x \in (c, c+\delta)$.

Therefore f has a maximum at c.

This completes the proof.

Worked Examples.

1. $f(x) = x^5 - 5x^4 + 5x^3 + 10$.

Show that f has a maximum at 1 and a minimum at 3 and f has neither a maximum nor a minimum at 0.

For an extremum $f'(x) = 0$. $f'(x) = 0$ at $x = 1, 3, 0$.
$f''(x) = 20x^3 - 60x^2 + 30x$. Therefore $f''(1) < 0$, $f''(3) > 0$, $f''(0) = 0$.

Since $f'(1) = 0$ and $f''(1) < 0, f$ has a local maximum at 1.
Since $f'(3) = 0$ and $f''(3) > 0, f$ has a local minimum at 3.

Since $f'(0)$ and $f''(0) = 0$, in order to decide the nature of f at 0, we are to examine derivatives of higher order at 0.
$f'''(x) = 60x^2 - 120x + 30$. $f'''(0) = 30 \neq 0$.

Therefore f has neither a maximum nor a minimum at 0.

2. If $f'(x) = (x-a)^{2n}(x-b)^{2m+1}$ where m, n are positive integers, show that f has neither a maximum nor a minimum at a and f has a minimum at b.

a is a multiple root of order $2n$ of the polynomial $f'(x)$.
Therefore a is a multiple root of order $2n-1$ of the polynomial $f''(x)$, a multiple root of order $2n-2$ of the polynomial $f'''(x), \cdots$, a simple root of the polynomial $f^{2n}(x)$. And a is not a root of $f^{2n+1}(x)$.
Therefore $f'(a) = f''(a) = \cdots = f^{2n}(a) = 0$ and $f^{2n+1}(a) \neq 0$.
Since $2n+1$ is odd, f has neither a maximum nor a minimum at a.
Let h be an arbitrarily small positive number.
$f'(b-h) = (b-h-a)^{2n}(-h)^{2m+1} < 0$.
$f'(b+h) = (b+h-a)^{2n}(h)^{2m+1} > 0$.

f is continuous at b. $f'(x) < 0$ for $x \in (b-\delta, b)$ and $f'(x) > 0$ for $x \in (b, b+\delta)$ for some $\delta > 0$. Hence f has a local minimum at b.

3. Find the local extremum points of the function $f(x) = \frac{x^2}{(1-x)^3}$.

$f'(x) = \frac{2(1-x)^3x+3x^2(1-x)^2}{(1-x)^6} = \frac{x(1-x)^2(x+2)}{(1-x)^6} = \frac{x(x+2)}{(1-x)^4}$.
$f'(x) = 0$ at $x = -2, 0$.

Let h be an arbitrarily small positive number.
$f'(-2-h) > 0$, $f'(-2) = 0$, $f'(-2+h) < 0$.
$f'(0-h) < 0$, $f'(0) = 0$, $f'(0+h) > 0$.

f is continuous at -2. $f'(x) > 0$ for $x \in (-2-\delta, -2)$ and $f'(x) < 0$ for $x \in (-2, -2+\delta)$ for some $\delta > 0$.
f is continuous at 0. $f'(x) < 0$ for $x \in (-\delta, 0)$ and $f'(x) > 0$ for $x \in (0, \delta)$ for some $\delta > 0$.

Hence f has a local maximum at -2 and a local minimum at 0.

4. Find the global maximum and the global minimum of the function f on $\mathbb{R}$, where $f(x) = \frac{x^2-2x+4}{x^2+2x+4}$, $x \in \mathbb{R}$.

$f'(x) = \frac{4(x^2-4)}{(x^2+2x+4)^2}$.

$f'(x) = 0$ at $x = \pm 2$. $f'(x) < 0$ for $|x| < 2$ and $f'(x) > 0$ for $|x| > 2$.

f is continuous at 2. $f'(2+h) > 0$ and $f'(2-h) < 0$ for sufficiently small $h > 0$. Therefore f has a local minimum at 2 and $f(2) = \frac{1}{3}$.

f is continuous at -2. $f'(-2+h) < 0$ and $f'(-2-h) > 0$ for suffiociently small $h > 0$. Therefore f has a local maximum at -2 and $f(-2) = 3$.

As $f'(x) > 0$ for $x > 2$ and f is continuous at 2, f is an increasing function on $[2, \infty)$ and $\lim\limits_{x\to\infty} f(x) = 1$.

Therefore $\sup\limits_{x\in[2,\infty)} f(x) = 1$ and $\inf\limits_{x\in[2,\infty)} f(x) = f(2) = \frac{1}{3}$.

As $f'(x) > 0$ for $x < -2$ and f is continuous at -2, f is an increasing function on $(-\infty, -2]$ and $\lim\limits_{x\to-\infty} f(x) = 1$.

Therefore $\sup\limits_{x\in(-\infty,-2]} f(x) = f(-2) = 3$ and $\inf\limits_{x\in(-\infty,-2]} = 1$.

$\sup\limits_{x\in[-2,2]} f(x) = 3$ and $\inf\limits_{x\in[-2,2]} f(x) = \frac{1}{3}$.

Therefore $\sup\limits_{x\in\mathbb{R}} f(x) = f(-2) = 3$ and $\inf\limits_{x\in\mathbb{R}} f(x) = f(2) = \frac{1}{3}$.

Exercises 18

1. Examine if f has a local maximum or a local minimum at 0.

(i) $f(x) = 2x + 3, x > 0$
$\quad\;\; = -3x + 1, x \le 0$

(ii) $f(x) = 2x + 3, x \ge 0$
$\quad\;\; = -3x + 1, x < 0$

(iii) $f(x) = 2x + 3, x < 0$
$\quad\;\; = -3x + 1, x \ge 0$

(iv) $f(x) = 2x + 3, x \le 0$
$\quad\;\; = -3x + 1, x > 0$

(v) $f(x) = 2x + 3, x > 0$
$\quad\;\; = -3x + 3, x \le 0$

(vi) $f(x) = 2x + 3, x \le 0$
$\quad\;\; = -3x + 3, x > 0$

(vii) $f(x) = x - [x]$,

(viii) $f(x) = |x| + |x - 1|$.

2. Find the points of local maximum and local minimum of the function f.

(i) $f(x) = 12x^5 - 45x^4 + 40x^3 + 1, x \in \mathbb{R}$

(ii) $f(x) = \frac{x^2-3x+4}{x^2+3x+4}, x \in \mathbb{R}$

(iii) $f(x) = 8x^5 - 10x^3 + 5x^2 + 1, x \in \mathbb{R}$

(iv) $f(x) = \frac{x^2+x+1}{x^2-x+1}, x \in \mathbb{R}$

(v) $f(x) = 4x + 2 - 5\log(1 + x^2), x \in \mathbb{R}$ (vi) $f(x) = \frac{x}{(1+x^2)^2}, x \in \mathbb{R}$

(vii) $f(x) = (x - 1)^4(x - 2)^2, x \in \mathbb{R}$ (viii) $f(x) = (x - 1)^5(x - 2)^4, x \in \mathbb{R}$

(ix) $f(x) = \sin^{-1} 2x\sqrt{1 - x^2}, x \in (-1, 1)$ (x) $f(x) = \sin^{-1}(3x - 4x^3), x \in (-1, 1)$.

Find the global maximum and the global minimum of the function f in (ii), (iv) and (vi) on $\mathbb{R}$.

3. Find the maximum and the minimum values of

(i) $\sin x(1 + \cos x)$ in $[0, 2\pi]$ (ii) $\cos x + \cos 2x$ in $[-\frac{\pi}{4}, \frac{5\pi}{4}]$

(iii) $\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x$ in $[0, \pi]$

(iv) $\cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 2x$ in $[0, \pi]$.

4. Find the extreme values of the function f in its domain.

(i) $f(x) = x^x$, (ii) $f(x) = x^{\frac{1}{x}}$,

(iii) $f(x) = \frac{\log x}{x}$, (iv) $f(x) = 2^x - x$.

5. If $ax^2 + 2hxy + by^2 = 1$, show that the maximum and the minimum values of $x^2 + y^2$ are given by the roots of the quadratic equation $(t - a)(t - b) = h^2$.

[**Hint.** Let $x = r\cos\,\theta, y = r\sin\,\theta$.]

6. (i) Divide the number 10 into two parts such that the sum of their cubes is the least possible.

(ii) Decompose the number 36 into two factors such that the sum of their squares is the least possible.

7. (i) The perimeter of an isosceles triangle is $2s$. What must its sides be so that the volume of the solid generated by revolving the triangle about the base is the greatest possible?

(ii) The perimeter of an isosceles triangle is $2s$. What must its sides be so that the volume of the solid generated by revolving the triangle about the altitude upon the base is the greatest possible?

8. (i) Determine the altitude of a right circular cylinder of greatest possible volume that can be inscribed in a sphere of radius r.

(ii) Determine the altitude of a right circular cone of greatest possible volume that can be inscribed in a sphere of radius r.

9. (i) Show that the semi-vertical angle of a right circular cone of maximum possible volume and of the given curved surface is $\sin^{-1}(\frac{1}{\sqrt{3}})$.

(ii) Show that the semi-vertical angle of a right circular cone of minimum possible curved surface and of the given volume is $\sin^{-1}(\frac{1}{\sqrt{3}})$.

10. (i) Show that the semi-vertical angle of a right circular cone of maximum

possible volume and of the given surface is $\sin^{-1}(\frac{1}{3})$.

(ii) Show that the semi-vertical angle of a right circular cone of minimum possible surface and of the given volume is $\sin^{-1}(\frac{1}{3})$.

11. (i) One corner of a rectangular sheet of paper is folded over so as to reach the opposite edge (lengthwise) of the sheet. If the area of the folded part be minimum , show that the crease divides the width in the ratio 2 : 3.

(ii) One corner of a long rectangular sheet of paper of width b is folded over so as to reach the opposite edge (lengthwise) of the sheet. Show that the minimum length of the crease is $\frac{3\sqrt{3}b}{4}$.

12. A line is drawn through a fixed point $(a, b)[a > 0, b > 0]$ to meet the positive direction of the co-ordinate axes at P and Q respectively. Show that

(i) the minimum value of PQ is $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$;

(ii) the minimum value of $OP + OQ$ is $(\sqrt{a} + \sqrt{b})^2$, O being the origin.

13. p is the length of perpendicular from the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the normal at a variable point on the ellipse. Show that the greatest value of p is $a - b$.

14. A tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the major axis and the minor axis at P, Q respectively. Show that the least value of PQ is $a + b$.

9.9. Indeterminate forms.

In the chapter on limits it was shown that if $\lim_{x\to c} f(x) = l$ and $\lim_{x\to c} g(x) = m \neq 0$ then $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{l}{m}$.

If however, $m = 0$ then the limit could not be evaluated. The case when $l = 0$ and $m = 0$ was not covered in earlier chapters. In this case the limit of the quotient $\frac{f}{g}$ is said to take the *indeterminate form* $\frac{0}{0}$

We will see that in this case the limit may be finite or infinite, or even the limit may not exist.

The other indeterminate forms are represented by the symbols $\frac{\infty}{\infty}$, $\infty - \infty$, $0.\infty$, 0^0, 1^∞, $1^{-\infty}$, ∞^0.

We now discuss several theorems concerning evaluation of indeterminate forms.

Theorem 9.9.1. Case $\frac{0}{0}$

Let $c \in \mathbb{R}$. Let f and g be two functions such that $f(c) = g(c) = 0, g(x) \neq 0$ in some deleted neighbourhood $N'(c, \delta)$; f and g are differentiable at c and $g'(c) \neq 0$. Then $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{f'(c)}{g'(c)}$.

Proof. Let $x \in (c, c+\delta)$. Then $\frac{f(x)}{g(x)} = \frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}}$.

Since f and g are differentiable at c, $\lim\limits_{x \to c+} \dfrac{f(x)-f(c)}{x-c} = Rf'(c) = f'(c)$ and $\lim\limits_{x \to c+} \dfrac{g(x)-g(c)}{x-c} = Rg'(c) = g'(c)$.

Therefore $\lim\limits_{x \to c+} \dfrac{f(x)}{g(x)} = \dfrac{f'(c)}{g'(c)}$ since $g'(c) \neq 0$ ... ... (i)

Let $x \in (c-\delta, c)$.

Since f and g are differentiable at c, $\lim\limits_{x \to c-} \dfrac{f(x)-f(c)}{x-c} = Lf'(c) = f'(c)$ and $\lim\limits_{x \to c-} \dfrac{g(x)-g(c)}{x-c} = Lg'(c) = g'(c)$.

Therefore $\lim\limits_{x \to c-} \dfrac{f(x)}{g(x)} = \dfrac{f'(c)}{g'(c)}$ since $g'(c) \neq 0$ ... ... (ii)

From (i) and (ii) we have $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{f'(c)}{g'(c)}$.

Corollary. Let f and g be functions on $[a, b]$ such that $f(a) = g(a) = 0, g(x) \neq 0$ on (a, b); f and g are differentiable at a and $g'(a) \neq 0$.

Then $\lim\limits_{x \to a+} \dfrac{f(x)}{g(x)} = \dfrac{f'(a)}{g'(a)}$.

Examples.

1. Let $f(x) = x^2 \sin \frac{1}{x}, x \neq 0$
   $= 0, \; x = 0;$
   and $g(x) = \sin x, x \in \mathbb{R}$.
   Then $f(0) = g(0) = 0$. $g(x) \neq 0$ in some deleted neighbourhood of 0.
   $f'(0)$ and $g'(0)$ both exist and $g'(0) = 1 \neq 0$.
   Therefore $\lim\limits_{x \to 0} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to 0} \dfrac{\frac{f(x)-f(0)}{x-0}}{\frac{g(x)-g(0)}{x-0}} = \frac{f'(0)}{g'(0)} = 0$.

2. Let $f(x) = \sin x, x \in \mathbb{R}, g(x) = \sqrt{x}, x \in [0, \infty]$.
   The theorem can not be applied here, since $g'(0)$ does not exist.

We now come to the limit theorem known as L'Hospital's Rule where differentiability of the functions f and g at the point c are not assumed. The theorem asserts that the limiting behaviour of $\frac{f}{g}$ at c is same as that of $\frac{f'}{g'}$ under certain conditions.

Theorem 9.9.2. L'Hospital's rule. Case $\frac{0}{0}$

Let $c \in \mathbb{R}$. Let the functions f and g be continuous on some neighbourhood $N(c, \delta)$ and f, g are differentiable on the deleted neighbourhood $N'(c, \delta)$.

Let $f(c) = g(c) = 0$ and $g(x) \neq 0, g'(x) \neq 0$ on $N'(c, \delta)$. Then

(a) if $\lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)} = l (l \in \mathbb{R})$ then $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = l$.

(b) if $\lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)} = \infty$ (or, $-\infty$) then $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \infty$ (or, $-\infty$).

Proof. **(a)** Let us choose $\epsilon > 0$. Then there exists a positive $\delta_1 < \delta$ such that $|\frac{f'(x)}{g'(x)} - l| < \epsilon$ for all $x \in N'(c, \delta_1)$... ... (i)

Let $x \in (c, c + \delta_1)$. Applying Cauchy's Mean value theorem to f and g on $[c, x]$ we have $\frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f'(\xi)}{g'(\xi)}$ for some ξ in (c, x).

That is, $\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}$.

Therefore for all $x \in (c, c + \delta_1), \mid \frac{f(x)}{g(x)} - l \mid = \mid \frac{f'(\xi)}{g'(\xi)} - l \mid$.

Since $\xi \in (c, x), \mid \frac{f'(\xi)}{g'(\xi)} - l \mid < \epsilon$ from (i).

Therefore for all $x \in (c, c + \delta_1), \mid \frac{f(x)}{g(x)} - l \mid < \epsilon$.

This proves that $\lim\limits_{x \to c+} \dfrac{f(x)}{g(x)} = l$ ... ... (ii)

Let $x \in (c - \delta_1, c)$. Applying Cauchy's Mean value theorem to f and g on $[x, c]$ we have $\frac{f(c) - f(x)}{g(c) - g(x)} = \frac{f'(\xi)}{g'(\xi)}$ for some $\xi \in (x, c)$.

That is, $\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}$.

Therefore for all $x \in (c - \delta_1, c), \mid \frac{f(x)}{g(x)} - l \mid = \mid \frac{f'(\xi)}{g'(\xi)} - l \mid$.

Since $\xi \in (x, c), \mid \frac{f'(\xi)}{g'(\xi)} - l \mid < \epsilon$, from (i).

Therefore for all $x \in (c - \delta_1, c), \mid \frac{f(x)}{g(x)} - l \mid < \epsilon$.

This proves that $\lim\limits_{x \to c-} \dfrac{f(x)}{g(x)} = l$ ... . (iii)

From (ii) and (iii) it follows that $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = l$.

(b) Let us choose $G > 0$. Then there exists a positive $\delta_1 < \delta$ such that $\frac{f'(x)}{g'(x)} > G$ for all $x \in N'(c, \delta_1)$.

Let $x \in (c, c+\delta_1)$. Applying Cauchy's Mean value theorem to f and g on $[c, x]$, we have

$\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}$ for some $\xi \in (c, x)$.

Therefore for all $x \in (c, c+\delta_1)$, $\frac{f(x)}{g(x)} > G$, since $\xi \in (c, x)$.

This proves that $\lim\limits_{x \to c+} \dfrac{f(x)}{g(x)} = \infty$ ... ... (i)

Let $x \in (c-\delta_1, c)$. Applying Cauchy's Mean value theorem to f and g on $[x, c]$, we have

$\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}$ for some $\xi \in (x, c)$.

Therefore for all $x \in (c-\delta_1, c)$, $\frac{f(x)}{g(x)} > G$. since $\xi \in (x, c)$.

This proves that $\lim\limits_{x \to c-} \dfrac{f(x)}{g(x)} = \infty$ ... ...(ii)

From (i) and (ii) it follows that $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \infty$.

Similar proof for the case when $\lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)} = -\infty$.

Corollary. If f and g be continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = g(a) = 0; g(x) \neq 0$ and $g'(x) \neq 0$ on (a, b) then

(a) if $\lim\limits_{x \to a+} \dfrac{f'(x)}{g'(x)} = l (l \in \mathbb{R})$ then $\lim\limits_{x \to a+} \dfrac{f(x)}{g(x)} = l$.

(b) if $\lim\limits_{x \to a+} \dfrac{f'(x)}{g'(x)} = \infty$ (or $-\infty$) then $\lim\limits_{x \to a+} \dfrac{f(x)}{g(x)} = \infty$ (or $-\infty$).

We now extend the results to the case of limits at infinity. We consider the case when $x \to \infty$. The case when $x \to -\infty$ is similar.

Theorem 9.9.3. Let f and g be continuous and differentiable on $[c, \infty)$ for some positive c. Let $\lim\limits_{x \to \infty} f(x) = 0$, $\lim\limits_{x \to \infty} g(x) = 0$ and $g(x) \neq 0, g'(x) \neq 0$ on (c, ∞). Then if

$\lim\limits_{x \to \infty} \dfrac{f'(x)}{g'(x)}$ exists in $\mathbb{R}^*$ then $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to \infty} \dfrac{f'(x)}{g'(x)}$.

Proof. Let us define functions F and G on $[0, \frac{1}{c}]$ by

$$\begin{aligned} F(t) &= f(\tfrac{1}{t}), 0 < t \leq \tfrac{1}{c} & G(t) &= g(\tfrac{1}{t}), 0 < t \leq \tfrac{1}{c} \\ &= 0, t = 0; & &= 0, t = 0. \end{aligned}$$

We have $\lim\limits_{t \to 0+} F(t) = \lim\limits_{x \to \infty} f(x)$ and $\lim\limits_{t \to 0+} G(t) = \lim\limits_{x \to \infty} g(x)$.

F and G are continuous on $[0, \frac{1}{c}]$, differentiable on $(0, \frac{1}{c})$ and $F(0) = 0, G(0) = 0$.

$F'(t) = -(\frac{1}{t^2})f'(\frac{1}{t}), G'(t) = -(\frac{1}{t^2})g'(\frac{1}{t})$ for $0 < t < \frac{1}{c}$.

Therefore $G(t) \neq 0, G'(t) \neq 0$ on $0 < t < \frac{1}{c}$.

By the corollary of the theorem 9.9.2,

if $\lim\limits_{t \to 0+} \dfrac{F'(t)}{G'(t)}$ exists in $\mathbb{R}^*$ then $\lim\limits_{t \to 0+} \dfrac{F(t)}{G(t)} = \lim\limits_{t \to 0+} \dfrac{F'(t)}{G'(t)}$.

But $\frac{F'(t)}{G'(t)} = \frac{f'(\frac{1}{t})}{g'(\frac{1}{t})}$ for $0 < t < \frac{1}{c}$ and $\frac{F(t)}{G(t)} = \frac{f(\frac{1}{t})}{g(\frac{1}{t})}$ for $0 < t < \frac{1}{c}$.

Therefore $\lim\limits_{t \to 0+} \dfrac{F'(t)}{G'(t)} = \lim\limits_{t \to 0+} \dfrac{f'(\frac{1}{t})}{g'(\frac{1}{t})} = \lim\limits_{x \to \infty} \dfrac{f'(x)}{g'(x)}$

$$\lim_{t \to 0+} \frac{F(t)}{G(t)} = \lim_{t \to 0+} \frac{f(\frac{1}{t})}{g(\frac{1}{t})} = \lim_{x \to \infty} \frac{f(x)}{g(x)}.$$

Therefore if $\lim\limits_{x \to \infty} \dfrac{f'(x)}{g'(x)}$ exists in $\mathbb{R}^*$, then $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to \infty} \dfrac{f'(x)}{g'(x)}$.

Theorem 9.9.4. Another form of the rule. Case $\frac{0}{0}$

Let $c \in \mathbb{R}$. Let the functions f and g be differentiable on some deleted neighbourhood $N'(c, \delta)$ and $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} g(x) = 0, g'(x) \neq 0$ on $N'(c, \delta)$. Then

(a) if $\lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)} = l (l \in \mathbb{R})$ then $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = l$.

(b) if $\lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)} = \infty$ (or $-\infty$) then $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \infty$ (or $-\infty$).

Proof. (a) Let us choose $\epsilon > 0$. Then there exists a positive $\delta_1 < \delta$ such that

$| \frac{f'(x)}{g'(x)} - l | < \epsilon$ for all $x \in N'(c, \delta_1)$ ... ... (i)

Let us define functions F and G by

$F(x) = f(x)$ for $x \in N'(c, \delta_1)$ $\quad$ $G(x) = g(x)$ for $x \in N'(c, \delta_1)$

$= 0$ for $x = c$; $\quad$ $= 0$ for $x = c$.

F and G are differentiable on $N'(c, \delta_1)$.

Since $\lim\limits_{x \to c} F(x) = \lim\limits_{x \to c} f(x) = 0 = F(c), F$ is continuous at c.

Similarly G is continuous at c.

Let $x \in (c, c + \delta_1)$. Applying Cauchy's Mean value theorem to F and G on $[c, x]$, we have $\frac{F(x) - F(c)}{G(x) - G(c)} = \frac{F'(\xi)}{G'(\xi)}$ for some ξ in (c, x).

That is, $\frac{F(x)}{G(x)} = \frac{F'(\xi)}{G'(\xi)}$, for some ξ in (c, x).

Hence $\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}$ for $x \in (c, c+\delta_1)$.

Therefore for all $x \in (c, c+\delta_1)$, $\left| \frac{f(x)}{g(x)} - l \right| = \left| \frac{f'(\xi)}{g'(\xi)} - l \right|$.

Since $\xi \in (c, x)$, $\left| \frac{f'(x)}{g'(x)} - l \right| < \epsilon$ from (i).

Therefore for all $x \in (c, c+\delta_1)$, $\left| \frac{f(x)}{g(x)} - l \right| < \epsilon$.

This proves that $\lim\limits_{x\to c+} \dfrac{f(x)}{g(x)} = l$ ... ... (ii)

Let $x \in (c-\delta_1, c)$. Applying Cauchy's Mean value theorem to F and G on $[x, c]$ we can prove in a similar manner

$\lim\limits_{x\to c-} \dfrac{f(x)}{g(x)} = l$ ... ... (iii)

From (ii) and (iii) it follows that $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = l$.

(b) Let us choose $B > 0$. Then there exists a positive $\delta_1 < \delta$ such that

$\left| \frac{f'(x)}{g'(x)} \right| > B$ for all $x \in N'(c, \delta_1)$ ... ... (i)

Let us define functions F and G as in **(a)**.

Let $x \in (c, c+\delta_1)$. Applying Cauchy's Mean value theorem to F and G on $[c, x]$, we have $\frac{F(x)}{G(x)} = \frac{F'(\xi)}{G'(\xi)}$ for some ξ in (c, x).

Therefore $\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}$ for $x \in (c, c+\delta_1)$.

Since $\xi \in (c, x)$, $\frac{f'(\xi)}{g'(\xi)} > B$ from (i).

Therefore $\frac{f(x)}{g(x)} > B$ for all $x \in (c, c+\delta_1)$.

This proves that $\lim\limits_{x\to c+} \dfrac{f(x)}{g(x)} = \infty$ ... ... (ii)

Let $x \in (c-\delta_1, c)$. Applying Cauchy's Mean value theorem to F and G on $[x, c]$ we can prove in a similar manner

$\lim\limits_{x\to c-} \dfrac{f(x)}{g(x)} = \infty$ ... ... (iii)

From (ii) and (iii) it follows that $\lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)} = \infty$.

Similar proof for the case when $\lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)} = -\infty$.

Corollary. If f and g be continuous on $[a, b]$ and differentiable on (a, b) and $\lim\limits_{x\to a+} f(x) = 0$, $\lim\limits_{x\to a+} g(x) = 0$; $g(x) \neq 0, g'(x) \neq 0$ on (a, b) then

(a) if $\lim\limits_{x\to a+} \dfrac{f'(x)}{g'(x)} = l (l \in \mathbb{R})$ then $\lim\limits_{x\to a+} \dfrac{f(x)}{g(x)} = l$;

(b) if $\lim\limits_{x\to a+} \dfrac{f'(x)}{g'(x)} = \infty$ (or $-\infty$) then $\lim\limits_{x\to a+} \dfrac{f(x)}{g(x)} = \infty$ (or $-\infty$).

Note. Under the conditions, stated in the theorem, satisfied by f and g, if $\lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)}$ exists then $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)}$ exists and equals $\lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)}$.

However, $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)}$ may exist even if $\lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)}$ does not exist.

For example, let $f(x) = x^2 \sin(\frac{1}{x}), x \neq 0$
$= 0, x = 0$

and $g(x) = x, x \in \mathbb{R}$.

Then $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} = 0$ but $\lim\limits_{x\to 0} \dfrac{f'(x)}{g'(x)}$ does not exist.

9.9.5. Generalised L'Hospital's rule. Case $\frac{0}{0}$

Let $c \in \mathbb{R}$. Let f and g be such that $f^n(x), g^n(x)$ exist on some deleted neighbourhood $N'(c,\delta), g^n(x) \neq 0$ on $N'(c,\delta)$ and

$$\lim_{x\to c} f(x) = \lim_{x\to c} f'(x) = \cdots = \lim_{x\to c} f^{n-1}(x) = 0,$$
$$\lim_{x\to c} g(x) = \lim_{x\to c} g'(x) = \cdots = \lim_{x\to c} g^{n-1}(x) = 0.$$

Then if $\lim\limits_{x\to c} \dfrac{f^n(x)}{g^n(x)}$ exists in $\mathbb{R}^*$, then $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to c} \dfrac{f^n(x)}{g^n(x)}$.

Examples.

1. Let $f(x) = \sin x, x \in \mathbb{R}$, $g(x) = \sqrt{x}, x \in [0,\infty)$.

Then
$$\begin{aligned}\lim_{x\to 0+} \frac{f(x)}{g(x)} &= \lim_{x\to 0+} \frac{f'(x)}{g'(x)} \quad \text{[by L'Hospital's rule, } \tfrac{0}{0}] \\ &= \lim_{x\to 0+} 2\sqrt{x}\cos x \\ &= 0.\end{aligned}$$

2. Evaluate the limit
$$\lim_{x\to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x\sin x}.$$

Here the limit takes the indeterminate form $\frac{0}{0}$. We have to apply L'Hospital's rule successively. The evaluation of the limit can be exhibited as follows-

$$\lim_{x\to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x\sin x} \quad (= \tfrac{0}{0})$$
$$= \lim_{x\to 0} \frac{e^x + e^{-x} - \frac{2}{1+x}}{x\cos x + \sin x} \quad (= \tfrac{0}{0})$$

$$= \lim_{x\to 0} \frac{e^x - e^{-x} + \frac{2}{(1+x)^2}}{-x\sin x + 2\cos x} = 1.$$

Form $\frac{\infty}{\infty}$.

Theorem 9.9.6. Let $c \in \mathbb{R}$. Let f and g be differentiable in some deleted neighbourhood $N'(c,\delta)$ of c. Let $\lim_{x\to c} f(x) = \infty, \lim_{x\to c} g(x) = \infty$ and $g(x) \neq 0, g'(x) \neq 0$ on $N'(c,\delta)$. Then

(a) if $\lim_{x\to c} \frac{f'(x)}{g'(x)} = l (l \in \mathbb{R})$ then $\lim_{x\to c} \frac{f(x)}{g(x)} = l$;

(b) if $\lim_{x\to c} \frac{f'(x)}{g'(x)} = \infty$ (or $-\infty$) then $\lim_{x\to c} \frac{f(x)}{g(x)} = \infty$ (or $-\infty$).

Proof. (a) Let us choose ϵ such that $0 < \epsilon < \frac{1}{2}$. Then there exists a positive $\delta_1 < \delta$ such that $| \frac{f'(x)}{g'(x)} - l | < \epsilon$ for all $x \in N'(c,\delta_1)$.

Let us choose $c_1 \in (c, c+\delta_1)$. Then $c < c_1 < c+\delta_1$

Since $\lim_{x\to c+} f(x) = \infty$, we can choose c_2 in (c, c_1) such that $f(x) \neq f(c_1)$ for all $x \in (c, c_2)$.

Since $g'(x) \neq 0$ on $N'(c,\delta), g(x) \neq g(c_1)$ for all $x \in (c, c_2)$.

Let us define a function ϕ on $[c, c_2]$ by $\phi(x) = \frac{1-\frac{f(c_1)}{f(x)}}{1-\frac{g(c_1)}{g(x)}}$.

Since $\lim_{x\to c+} f(x) = \infty and \lim_{x\to c+} g(x) = \infty$, $\lim_{x\to c+} \phi(x) = 1$.

Therefore there exists a c_3 in (c, c_2) such that

$1-\epsilon < \phi(x) < 1+\epsilon$ for all $x \in (c, c_3)$

or, $\frac{1}{2} < \phi(x) < \frac{3}{2}$ for all $x \in (c, c_3)$.

Now $\frac{f(x)}{g(x)} = \frac{f(x)-f(c_1)}{g(x)-g(c_1)} \cdot \frac{1}{\phi(x)}$ for all $x \in (c, c_2)$.

Applying Cauchy's Mean value theorem to f and g on $[c, c_1]$, $\frac{f(x)-f(c_1)}{g(x)-g(c_1)} = \frac{f'(\xi)}{g'(\xi)}$ for some ξ in (c, c_1).

Therefore $\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)} \cdot \frac{1}{\phi(x)}$ for all $x \in (c, c_2)$.

$$\begin{aligned} \text{Now } \left| \frac{f(x)}{g(x)} - l \right| &= \left| \frac{f'(\xi)}{g'(\xi)} \cdot \frac{1}{\phi(x)} - l \right| \\ &= \left| \frac{f'(\xi)}{g'(\xi)} - l\phi(x) \right| \cdot \frac{1}{|\phi(x)|} \\ &\leq \left\{ \left| \frac{f'(\xi)}{g'(\xi)} - l \right| + |\phi(x) - 1| \cdot |l| \right\} \cdot \frac{1}{|\phi(x)|}. \end{aligned}$$

Therefore for all $x \in (c, c_3)$, $| \frac{f(x)}{g(x)} - l | < 2\epsilon(1 + |l|)$.

Since ϵ is arbitrary, $\lim_{x\to c+} \dfrac{f(x)}{g(x)} = l$ ... ... (i)

Let us choose c_4 in $(c-\delta_1, c)$. Then $c - \delta_1 < c_4 < c$.

Since $\lim\limits_{x \to c-} f(x) = \infty$, we can choose c_5 in (c_4, c) such that $f(x) \neq f(c_4)$ for all $x \in (c_5, c)$.

Since $g'(x) \neq 0$ on $N'(c, \delta), g(x) \neq g(c_4)$ for all $x \in (c_5, c)$.

Defining $\phi(x) = \frac{1 - \frac{f(c_4)}{f(x)}}{1 - \frac{g(c_4)}{g(x)}}$ for $x \in (c_5, c)$ and proceeding similarly as above we can prove $\lim\limits_{x \to c-} \dfrac{f(x)}{g(x)} = l$ ... ... (ii)

From (i) and (ii) $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = l$.

(b) Let us choose $G > 0$. Then there exists a positive $\delta_1 < \delta$ such that $\frac{f'(x)}{g'(x)} > G$ for all $x \in N'(c, \delta_1)$.

Let us choose $c_1 \in (c, c + \delta_1)$. Then $c < c_1 < c + \delta_1$.

Since $\lim\limits_{x \to c+} f(x) = \infty$, we can choose c_2 in (c, c_1) such that $f(x) \neq f(c_1)$ for all $x \in c, c_2)$.

Since $g'(x) \neq 0$ on $N'(c, \delta), g(x) \neq g(c_1)$ for all $x \in (c, c_2)$.

Let us define ϕ on $[c, c_2]$ by $\phi(x) = \frac{1 - \frac{f(c_1)}{f(x)}}{1 - \frac{g(c_1)}{g(x)}}$.

Since $\lim\limits_{x \to c+} f(x) = \infty and \lim\limits_{x \to c+} g(x) = \infty$, we have $\lim\limits_{x \to c+} \phi(x) = 1$.

Let us choose $\epsilon = \frac{1}{2}$. Then there exists a c_3 in (c, c_2) such that $| \phi(x) - 1 | < \frac{1}{2}$ for all $x \in (c, c_3)$. Therefore $\frac{1}{2} < \phi(x) < \frac{3}{2}$.

Now $\frac{f(x)}{g(x)} = \frac{f(x) - f(c_1)}{g(x) - g(c_1)} \cdot \frac{1}{\phi(x)}$ for all $x \in (c, c_2)$.

Applying Cauchy's Mean value theorem to f and g on $[c, c_1]$,

$\frac{f(x) - f(c_1)}{g(x) - g(c_1)} = \frac{f'(\xi)}{g'(\xi)}$ for some ξ in (c, c_1).

Therefore $\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)} \cdot \frac{1}{\phi(x)}$ for all $x \in (c, c_2)$.

Consequently, $\frac{f(x)}{g(x)} > \frac{2}{3} G$ for all $x \in (c, c_3)$.

This proves that $\lim\limits_{x \to c+} \dfrac{f(x)}{g(x)} = \infty$ ... ... (i)

In a similar manner we can prove that $\lim\limits_{x \to c-} \dfrac{f(x)}{g(x)} = \infty$ ... ... (ii)

From (i) and (ii) $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \infty$.

The case when $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = -\infty$ can be similarly dealt with.

Corollary. Let f and g be continuous on $[a,b]$ and differentiable on (a,b). Let $\lim_{x\to a+} f(x) = \infty$, $\lim_{x\to a+} g(x) = \infty$ and $g(x) \neq 0, g'(x) \neq 0$ on (a,b). Then

(a) if $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = l (l \in \mathbb{R})$ then $\lim_{x\to a+} \frac{f(x)}{g(x)} = l$;

(b) if $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = \infty$ (or $-\infty$) then $\lim_{x\to a+} \frac{f(x)}{g(x)} = \infty$ (or $-\infty$).

Other indeterminate forms.

Indeterminate forms such as $\infty - \infty$, $0.\infty$, 1^∞, 0^0, ∞^0 can be reduced to either of the forms $\frac{0}{0}$, $\frac{\infty}{\infty}$ by algebraic manipulations and use of logarithmic and exponential functions.

Worked Examples.

1. Evaluate $\lim_{x\to 0+} (\frac{1}{x} - \frac{1}{\sin x})$.

Let $f(x) = \frac{1}{x}, x \in (0,1), g(x) = \frac{1}{\sin x}, x \in (0,1)$.

$\lim_{x\to 0+} [f(x) - g(x)]$ takes the indeterminate form $\infty - \infty$.

$$\begin{aligned}\text{We have } \lim_{x\to 0+} \left(\frac{1}{x} - \frac{1}{\sin x}\right) &= \lim_{x\to 0+} \frac{\sin x - x}{x \sin x} \quad \left(= \frac{0}{0}\right)\\ &= \lim_{x\to 0+} \frac{\cos x - 1}{\sin x + x\cos x} \quad \left(= \frac{0}{0}\right)\\ &= \lim_{x\to 0+} \frac{-\sin x}{2\cos x - x\sin x} = 0.\end{aligned}$$

2. Evaluate $\lim_{x\to 0+} x \log x$.

Let $f(x) = x, x \in (0,\infty), g(x) = \log x, x \in (0,\infty)$.
Then $\lim_{x\to 0+} x = 0$, $\lim_{x\to 0+} \log x = \infty$.
$\lim_{x\to 0+} x\log x$ takes the indeterminate form $0.\infty$.

$$\begin{aligned}\text{We have } \lim_{x\to 0+} x\log x &= \lim_{x\to 0+} \frac{\log x}{\frac{1}{x}} \quad \left(= \frac{\infty}{\infty}\right)\\ &= \lim_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0+}(-x) = 0.\end{aligned}$$

3. Evaluate $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$.

Let $f(x) = \frac{\sin x}{x}$, $x \neq 0$, $g(x) = \frac{1}{x}$, $x \neq 0$.

Then $\lim_{x\to 0+} f(x) = 1$, $\lim_{x\to 0+} g(x) = \infty$.

$\lim_{x\to 0+} f(x)^{g(x)}$ takes the indeterminate form 1^∞.

$$\lim_{x\to 0+} \log\left(\frac{\sin x}{x}\right)^{\frac{1}{x}} = \lim_{x\to 0+} \frac{\log \frac{\sin x}{x}}{x} \quad (= \tfrac{0}{0})$$

$$= \lim_{x\to 0+} \frac{\frac{x}{\sin x}\cdot\frac{x\cos x - \sin x}{x^2}}{1}$$

$$= \lim_{x\to 0+} \frac{x\cos x - \sin x}{x \sin x} \quad (= \tfrac{0}{0})$$

$$= \lim_{x\to 0+} \frac{-x\sin x}{x\cos x + \sin x} \ (= \tfrac{0}{0})$$

$$= \lim_{x\to 0+} \frac{-x\cos x - \sin x}{-x\sin x + 2\cos x} = 0.$$

Therefore $\lim_{x\to 0+} \left(\frac{\sin x}{x}\right)^{\frac{1}{x}} = e^0 = 1$.

Also we have $\lim_{x\to 0-} f(x) = 1$ and $\lim_{x\to 0-} g(x) = -\infty$.

$\lim_{x\to 0-} f(x)^{g(x)}$ takes the indeterminate form $1^{-\infty}$.

Proceeding similarly, we have $\lim_{x\to 0-} \left(\frac{\sin x}{x}\right)^{\frac{1}{x}} = 1$.

Consequently, $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x}} = 1$.

4. Evaluate $\lim_{x\to 0+} x^x$.

Let $f(x) = x, x > 0$; $g(x) = x, x > 0$. Then $\lim_{x\to 0+} f(x) = 0$, $\lim_{x\to 0+} g(x) = 0$.

$\lim_{x\to 0+} [f(x)]^{g(x)}$ takes the indeterminate form 0^0.

$$\lim_{x\to 0+} \log x^x = \lim_{x\to 0+} \frac{\log x}{\frac{1}{x}} \quad (= \tfrac{\infty}{\infty}) = \lim_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0+} -x = 0.$$

Therefore $\lim_{x\to 0+} x^x = 1$.

5. Evaluate $\lim_{x\to\infty} \left(\frac{3x}{3x+1}\right)^{3x+1}$.

Let $f(x) = \left(\frac{3x}{3x+1}\right)^{3x+1}$, $x \in [0,\infty)$. Then $f(x) = e^{(3x+1)\log\frac{3x}{3x+1}}$.

$$\lim_{x\to\infty} (3x+1)\log\frac{3x}{3x+1} = \lim_{x\to\infty} \frac{\log\frac{3x}{3x+1}}{\frac{1}{3x+1}} \quad (= \tfrac{0}{0})$$

$$= \lim_{x\to\infty} \frac{\frac{3x+1}{3x}\cdot\frac{3}{(3x+1)^2}}{-\frac{1}{(3x+1)^2}} = \lim_{x\to\infty} -\frac{3x+1}{x} = -3.$$

Since the exponential function is continuous,

$\lim_{x\to\infty} e^{(3x+1)\log\frac{3x}{3x+1}} = e^{-3}$, i.e., $\lim_{x\to\infty} \left(\frac{3x}{3x+1}\right)^{3x+1} = e^{-3}$.

6. Evaluate $\lim\limits_{n\to\infty}(1-\frac{1}{2n})^{n+1}$.

Let $f(x)=(1-\frac{1}{2x})^{x+1}, x>1$. Then $\log f(x)=(x+1)\log(1-\frac{1}{2x})$.

$$\lim_{x\to\infty}\log f(x)=\lim_{x\to\infty}\frac{\log(1-\frac{1}{2x})}{\frac{1}{x+1}}\quad(=\tfrac{0}{0})$$

$$=\lim_{x\to\infty}\frac{\frac{1}{1-\frac{1}{2x}}\cdot\frac{1}{2x^2}}{-(\frac{1}{x+1})^2}$$

$$=\lim_{x\to\infty}\frac{2x(x+1)^2}{-(2x-1).2x^2}=\lim_{x\to\infty}\frac{x^2+2x+1}{-(2x^2-x)}=-\tfrac{1}{2}.$$

Hence $\lim\limits_{x\to\infty}f(x)=e^{-\frac{1}{2}}$.

Let us consider the sequence $\{n\}$ that diverges to ∞.
By sequential criterion for limits, $\lim\limits_{n\to\infty}f(n)=e^{-\frac{1}{2}}$.
That is, $\lim\limits_{n\to\infty}(1-\frac{1}{2n})^{n+1}=e^{-\frac{1}{2}}$.

7. Evaluate $\lim\limits_{n\to\infty}(1+\frac{3}{n})^{2n}$.

Let $f(x)=(1+\frac{3}{x})^{2x},\ x\in(0,\infty)$.
$\lim\limits_{x\to\infty}f(x)$ takes the indeterminate form 0^∞.

$$\lim_{x\to\infty}\log f(x)=\lim_{x\to\infty}\frac{\log(1+\frac{3}{x})}{\frac{1}{2x}}\quad(=\tfrac{\infty}{\infty})$$

$$=\lim_{x\to\infty}\frac{\frac{1}{1+\frac{3}{x}}\cdot\frac{-3}{x^2}}{-\frac{1}{2x^2}}=\lim_{x\to\infty}\frac{6}{1+\frac{3}{x}}=6.$$

Hence $\lim\limits_{x\to\infty}f(x)=e^6$, i.e., $\lim\limits_{x\to\infty}(1+\frac{3}{x})^{2x}=e^6$.

Let us consider the sequence $\{n\}$ that diverges to ∞.
By sequential criterion for limits, $\lim\limits_{n\to\infty}f(n)=e^6$.
That is, $\lim\limits_{n\to\infty}(1+\frac{3}{n})^{2n}=e^6$.

Exercises 19

1. Prove that

(i) $\lim\limits_{x\to0}\dfrac{xe^x-\log(1+x)}{x^2}=\dfrac{3}{2}$, (ii) $\lim\limits_{x\to0}\dfrac{x-\log(1+x)}{1-\cos x}=1$,

(iii) $\lim\limits_{x\to0}\dfrac{\tan x-x}{x^2\tan x}=1/3$ (iv) $\lim\limits_{x\to0}\dfrac{\cosh x-\cos x}{x\sin x}=\cancel{\dfrac{1}{3}}$, 1 ,

(v) $\lim\limits_{x\to0}\dfrac{e^x-e^{\sin x}}{x-\sin x}=1$, (vi) $\lim\limits_{x\to0}\dfrac{e^x-e^{-x}-2x}{x^2\sin x}=\dfrac{1}{3}$.

10. FUNCTIONS OF BOUNDED VARIATION

10.1. Introduction.

Let $[a, b]$ be a closed and bounded interval. A *partition* P of $[a, b]$ is a finite ordered set $(x_0, x_1, \ldots, x_n)$ of points of $[a, b]$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

The family of all partitions of $[a, b]$ is denoted by $\mathcal{P}[a, b]$ and the partition $P = (x_0, x_1, \ldots, x_n)$ is a member of $\mathcal{P}[a, b]$.

For example, $P = (0, \frac{1}{2}, 1)$ is a partition of $[0, 1]$, $Q = (0, \frac{1}{8}, \frac{1}{2}, \frac{7}{8}, 1)$ is another partition of $[0, 1]$.

The partition $P = (x_0, x_1, \ldots, x_n)$ of $[a, b]$ divides the interval $[a, b]$ into non-overlapping subintervals $[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b]$.

Definition. Let $[a, b]$ be a closed and bounded interval and $f : [a, b] \to \mathbb{R}$ be a function. Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a, b]$. Let us consider the sum

$$V(P, f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})|.$$

For different partitions $P \in \mathcal{P}[a, b]$, $V(P, f)$ gives a set of non-negative real numbers. If the set $\{V(P, f) : P \in \mathcal{P}[a, b]\}$ be bounded above, then f is said to be a *function of bounded variation* (or a *BV-function*) on $[a, b]$.

The supremum of the set $\{V(P, f) : P \in \mathcal{P}[a, b]\}$ is said to be the *total variation* of f on $[a, b]$ and is denoted by $V_f[a, b]$ (or by V_f, if there is no confusion regarding the interval).

Note. Since each sum $V(P, f) \geq 0$, it follows that $V_f[a, b] = 0$ if and only if f is a constant function on $[a, b]$.

Examples.

1. Let $k \in \mathbb{R}$ and $f(x) = k$, $x \in [a, b]$.

Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a, b]$. Then

$$V(P, f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})| = 0.$$

For each partition P of $[a, b]$, $V(P, f) = 0$. Therefore the set $\{V(P, f) : P \in \mathcal{P}[a, b]\}$ is bounded above and the supremum of the set is 0, a finite real number.

Consequently, f is a function of bounded variation on $[a, b]$ and the total variation $V_f[a, b]$ is 0.

2. Let $f(x) = x, x \in [a, b]$. P: ordered set

Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a, b]$. Then

$$\begin{aligned} V(P, f) &= |x_1 - x_0| + |x_2 - x_1| + \cdots + |x_n - x_{n-1}| \\ &= (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) \\ &= b - a. \end{aligned}$$

For each partition P of $[a, b]$, $V(P, f) = b - a$. Therefore the set $\{V(P, f) : P \in \mathcal{P}[a, b]\}$ is bounded above and the supremum of the set is $b - a$, a finite real number.

Therefore f is a function of bounded variation on $[a, b]$ and the total variation $V_f[a, b]$ is $b - a$.

3. Let $f(x) = \sin\, x, x \in [a, b]$.

Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a, b]$. Then

$V(P, f) = |\sin\, x_1 - \sin\, x_0| + |\sin\, x_2 - \sin\, x_1| + \cdots + |\sin\, x_n - \sin\, x_{n-1}|$.

By Mean value theorem, $|f(x_r) - f(x_{r-1})| = |x_r - x_{r-1}||\cos\, \xi_r|$ for some ξ_r satisfying $x_{r-1} < \xi_r < x_r$. This holds for $r = 1, 2, \ldots, n$.

Therefore $|f(x_r) - f(x_{r-1})| \le |x_r - x_{r-1}|$, since $|\cos\, \xi_r| \le 1$.

$$\begin{aligned} V(P, f) &\le |x_1 - x_0| + |x_2 - x_1| + \cdots + |x_n - x_{n-1}| \\ \text{i.e., } &\le (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) \\ \text{i.e., } &\le (b - a). \end{aligned}$$

This holds for every partition P of $[a, b]$. Therefore the set $\{V(P, f) : P \in \mathcal{P}[a, b]\}$ is bounded above and $V(P, f) \le b - a$.

Therefore f is a function of bounded variation on $[a, b]$.

4. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = 1$, if x be rational
$= 0$, if x be irrational.

Let $P = (x_0, x_1, \ldots, x_{2n})$ be a partition of $[a, b]$ such that $x_0, x_2, \ldots, x_{2n}$ are all rational and $x_1, x_3, \ldots, x_{2n-1}$ are all irrational. Then

$V(P, f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_{2n}) - f(x_{2n-1})| = 2n$.

Clearly, the set $\{V(P, f) : P \in \mathcal{P}[0, 1]\}$ is not bounded above and therefore f is not a function of bounded variation on $[0, 1]$.

Theorem 10.1.1. Let $[a, b] \subset \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. Then f is bounded on $[a, b]$.

Proof. Let P be a partition of $[a, b]$. Since f is a function of bounded variation on $[a, b]$, $\sup\{V(P, f) : P \in \mathcal{P}[a, b]\}$ is finite.

Let $\sup\{V(P, f) : P \in \mathcal{P}[a, b]\} = M$, where M is a non-negative real number.

Let $x \in (a, b)$. Let us consider the partition $P_0 = (a, x, b)$ of $[a, b]$.

Then $V(P_0, f) = |f(x) - f(a)| + |f(b) - f(x)| \leq M$. This gives $|f(x) - f(a)| \leq M$ and therefore $|f(x)| \leq |f(a)| + M$.

If however, $x = a$, then $|f(x)| = |f(a)| \leq |f(a)| + M$ and also if $x = b$, then $V(P_0, f) = |f(x) - f(a)| + |f(b) - f(b)| = |f(x) - f(a)|$ and this implies $|f(x)| \leq |f(a)| + M$.

Thus for all $x \in [a, b]$, $|f(x)| \leq |f(a)| + M$ and this proves that f is bounded on $[a, b]$.

Note 1. The converse of the theorem is not true. A function f bounded on $[a, b]$ may not be a function of bounded variation on $[a, b]$. For example, let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = x \cos \frac{\pi}{2x}, x \neq 0$
$= 0, x = 0.$

Then f is bounded on $[0, 1]$, since $|f(x)| \leq 1$ for all $x \in [0, 1]$.

Let $P = (0, \frac{1}{2n}, \frac{1}{2n-1}, \cdots, \frac{1}{2}, 1)$ be a partition of $[0, 1]$.

Then $f(\frac{1}{2r}) = \frac{1}{2r} \cos \frac{2r\pi}{2} = \frac{1}{2r}.(-1)^r$, for $r = 1, 2, \ldots, n$
and $f(\frac{1}{2r-1}) = \frac{1}{2r-1} \cos \frac{(2r-1)\pi}{2} = 0$, for $r = 1, 2, \ldots, n$.

Then $V(P, f) = |f(\frac{1}{2n}) - f(0)| + |f(\frac{1}{2n-1}) - f(\frac{1}{2n})| + \cdots + |f(\frac{1}{2}) - f(1)|$
$= \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n-2} + \frac{1}{2n-2} + \cdots + \frac{1}{2} + \frac{1}{2} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$

Since $1 + \frac{1}{2} + \cdots + \frac{1}{n}$ tends to ∞ as n tends to ∞, the set $\{V(P, f) : P \in \mathcal{P}[0, 1]\}$ is not bounded above and therefore f is not a function of bounded variation on $[0, 1]$.

Note 2. It follows from the theorem that a function f, not bounded on $[a, b]$, cannot be a function of bounded variation on $[a, b]$.

Theorem 10.1.2. Let $f : [a, b] \to \mathbb{R}$ be monotone on $[a, b]$. Then f is a function of bounded variation on $[a, b]$.

Proof. **Case 1.** Let f be monotone increasing on $[a, b]$.

Let $P = (x_0, x_1, \ldots, x_n)$ be a partition of $[a, b]$. Then
$V(P, f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})|$
$= f(b) - f(a).$

This holds for every partition P of $[a,b]$. Therefore the supremum of the set $\{V(P,f) : P \in \mathcal{P}[a,b]\}$ is $f(b) - f(a)$, a finite real number.

This proves that f is a function of bounded variation on $[a,b]$.

Case 2. Let f be monotone decreasing on $[a,b]$.

In a similar manner it can be proved that $V(P,f) = f(a) - f(b)$ for all partitions of $[a,b]$ and f is a function of bounded variation on $[a,b]$ in this case.

This completes the proof.

Note. If f be monotone increasing on $[a,b]$, then $V_f[a,b] = f(b) - f(a)$; if f be monotone decreasing on $[a,b]$, then $V_f[a,b] = f(a) - f(b)$.

Lf

Definition. A function $f : [a,b] \to \mathbb{R}$ is said to satisfy *Lipschitz condition* on $[a,b]$ if there exists a positive real number M such that $|f(x_1) - f(x_2)| \le M|x_1 - x_2|$ for *any two points* x_1, x_2 in $[a,b]$. In this case f is also said to be a *Lipschitz function* on $[a,b]$.

Theorem 10.1.3. Let $f : [a,b] \to \mathbb{R}$ be a Lipschitz function on $[a,b]$. Then f is a function of bounded variation on $[a,b]$.

Proof. Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a,b]$. Since f is a Lipschitz function on $[a,b]$, there is a positive real number M such that

$|f(x_r) - f(x_{r-1})| \le M|x_r - x_{r-1}|$, for $r = 1, 2, \ldots, n$.

Therefore $V(P,f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})| \le M[|x_1 - x_0| + |x_2 - x_1| + \cdots + |x_n - x_{n-1}|] = M(b-a)$.

For each partition P of $[a,b]$, $V(P,f) \le M(b-a)$. Therefore the supremum of the set $\{V(P,f) : P \in \mathcal{P}[a,b]\}$ is a finite real number.

Consequently, f is a function of bounded variation on $[a,b]$

Note. The converse of the theorem is not true. A function f of bounded variation on $[a,b]$ may not be a Lipschitz function on $[a,b]$. For example, let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}, x \in [0,1]$.

Then f being a monotone increasing function on $[0,1]$, is a function of bounded variation on $[0,1]$. But f is not a Lipschitz function on $[0,1]$, because if $x_1 = 0$, no positive real number M can be found to satisfy the condition "$|f(x_2) - f(x_1)| \le M|x_2 - x_1|$ for all $x_2 \in (0,1]$".

Theorem 10.1.4. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, f' exists and be bounded on (a,b). Then f is a function of bounded variation on $[a,b]$.

Proof. There exists a positive real number k such that $|f'(x)| \le k$ for all $x \in (a,b)$.

Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a, b]$. Then
$V(P, f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})|$.

By Mean value theorem, we have $f(x_r) - f(x_{r-1}) = (x_r - x_{r-1})f'(\xi_r)$ for some ξ_r satisfying $x_{r-1} < \xi_r < x_r$.
Therefore $|f(x_r) - f(x_{r-1})| \leq k|x_r - x_{r-1}|$, for $r = 1, 2, \ldots, n$.
This implies $V(P, f) \leq k(b - a)$.

Therefore the set $\{V(P, f) : P \in \mathcal{P}[a, b]\}$ is bounded above and therefore the supremum of the set is a finite real number.

Consequently, f is a function of bounded variation on $[a, b]$

Note. Boundedness of f' on (a, b) is not necessary for the function f to be of bounded variation on $[a, b]$. For example, let $f(x) = \sqrt{x}, x \in [0, 1]$. Then f is a monotone increasing function on $[0, 1]$ and therefore it is a function of bounded variation on $[0, 1]$. But f' is not bounded on $(0, 1)$.

Remark. A function f continuous on a closed and bounded interval $[a, b]$ may not be a function of bounded variation on $[a, b]$. For example, let $f(x) = x \cos \frac{\pi}{2x}$, if $x \in (0, 1]$
$= 0$, if $x = 0$.

Then f continuous on $[0, 1]$. But f is not a function of bounded variation on $[0, 1]$. [Worked out in Note 1 of Theorem 10.1.1.]

Worked Example.

1. A function $f : [0, 1] \to \mathbb{R}$ is defined by $f(x) = x^2 \cos \frac{1}{x}$, if $x \neq 0$
$= 0$, if $x = 0$.

Show that f is a function of bounded variation on $[0, 1]$.

f is continuous on $[0, 1]$. $f'(x) = 2x \cos \frac{1}{x} + \sin \frac{1}{x}, x \in (0, 1]$.
f' is bounded on $(0, 1)$, since $|f'(x)| < 3$ for all $x \in (0, 1)$.
Therefore f is a function of bounded variation on $[0, 1]$.

Theorem 10.1.5. Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be functions of bounded variation on $[a, b]$. Then

(i) $f + g$ is a function of bounded variation on $[a, b]$ and $V_{f+g} \leq V_f + V_g$;

(ii) $f - g$ is a function of bounded variation on $[a, b]$ and $V_{f-g} \leq V_f + V_g$;

(iii) cf $(c \in \mathbb{R})$ is a function of bounded variation on $[a, b]$.

Proof. (i) Let $h(x) = f(x) + g(x), x \in [a, b]$.
Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a, b]$. Then

$$V(P,f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})|;$$
$$V(P,g) = |g(x_1) - g(x_0)| + |g(x_2) - g(x_1)| + \cdots + |g(x_n) - g(x_{n-1})|;$$
$$V(P,h) = |h(x_1) - h(x_0)| + |h(x_2) - h(x_1)| + \cdots + |h(x_n) - h(x_{n-1})|.$$

Now $|h(x_r) - h(x_{r-1})| = |f(x_r) + g(x_r) - f(x_{r-1}) - g(x_{r-1})|$
$\leq |f(x_r) - f(x_{r-1})| + |g(x_r) - g(x_{r-1})|$.

Therefore $V(P,h) \leq V(P,f) + V(P,g)$.

Since f and g are functions of bounded variation on $[a,b]$, $V(P,f) \leq V_f[a,b]$, $V(P,g) \leq V_g[a,b]$ for all partitions P of $[a,b]$.

Therfore $V(P,h) \leq V_f[a,b] + V_g[a,b]$ for all partitions P of $[a,b]$.

This shows that the set $\{V(P,h) : P \in \mathcal{P}[a,b]\}$ is bounded above and $\sup\{V(P,h) : P \in \mathcal{P}[a,b]\} \leq V_f[a,b] + V_g[a,b]$, a finite number.

Hence h (i.e.,$f+g$) is a function of bounded variation on $[a,b]$ and $V_{f+g} \leq V_f + V_g$.

Note. Strict inequality in the above relation holds for some functions f and g. For example, let $f(x) = x$, $x \in [1,2]$, $g(x) = 1 - x$, $x \in [1,2]$. Then $(f+g)(x) = 1$, $x \in [1,2]$.

Since f is a monotone increasing function on $[1,2]$, $V_f[1,2] = f(2) - f(1) = 1$. Since g is a monotone decreasing function on $[1,2]$, $V_g[1,2] = g(1) - g(2) = 1$. Since $f+g$ is a constant function on $[1,2]$, $V_{f+g}[1,2] = 0$.

Clearly, $V_{f+g}[1,2] < V_f[1,2] + V_g[1,2]$.

(ii) Similar proof.

(iii) Similar proof.

Note. The class S of all BV-functions on $[a,b]$ form a *real vector space*, since $f \in S, g \in S \Rightarrow f+g \in S$ and $c \in \mathbb{R}, f \in S \Rightarrow cf \in S$.

Theorem 10.1.6. Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be functions of bounded variation on $[a,b]$. Then fg is a function of bounded variation on $[a,b]$ and $V_{fg} \leq AV_g + BV_f$, where $A = \sup\{|f(x)| : x \in [a,b]\}$, $B = \sup\{|g(x)| : x \in [a,b]\}$.

Proof. Let $h(x) = f(x).g(x)$, $x \in [a,b]$.

Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a,b]$. Then

$$V(P,f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})|;$$
$$V(P,g) = |g(x_1) - g(x_0)| + |g(x_2) - g(x_1)| + \cdots + |g(x_n) - g(x_{n-1})|;$$
$$V(P,h) = |h(x_1) - h(x_0)| + |h(x_2) - h(x_1)| + \cdots + |h(x_n) - h(x_{n-1})|.$$
$$|h(x_r) - h(x_{r-1})| = |f(x_r).g(x_r) - f(x_{r-1}).g(x_{r-1})|$$

$= |f(x_r)[g(x_r) - g(x_{r-1})] + g(x_{r-1})[f(x_r) - f(x_{r-1})]|$
$\leq |f(x_r)||g(x_r) - g(x_{r-1})| + |g(x_{r-1})||f(x_r) - f(x_{r-1})|.$

Since f and g are functions of bounded variation on $[a,b]$, f and g are bounded on $[a,b]$. There exist positive real numbers A, B such that $|f(x)| \leq A$, $|g(x)| \leq B$, for all $x \in [a,b]$.

Then $|h(x_r) - h(x_{r-1})| \leq A|g(x_r) - g(x_{r-1})| + B|f(x_r) - f(x_{r-1})|$.

Since f and g are functions of bounded variation on $[a,b]$, $V(P,f) \leq V_f[a,b]$, $V(P,g) \leq V_g[a,b]$ for all partitions P of $[a,b]$.
Therefore $V(P,h) \leq AV_g[a,b] + BV_f[a,b]$ for all partitions P of $[a,b]$.

This shows that the set $\{V(P,h) : P \in \mathcal{P}[a,b]\}$ is bounded above and therefore $\sup\{V(P,h) : P \in \mathcal{P}[a,b]\}$ is a finite real number.

Hence h (i.e., $f.g$) is a function of bounded variation on $[a,b]$ and $V_{f.g} \leq AV_g + BV_f$.

Theorem 10.1.7. Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$. If there exists a positive real number k such that $0 < k \leq f(x)$ for all $x \in [a,b]$, then $1/f$ is a function of bounded variation on $[a,b]$ and $V_{\frac{1}{f}} \leq \frac{V_f}{k^2}$.

Proof. Let $h(x) = 1/f(x), x \in [a,b]$.
Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a,b]$. Then

$V(P,f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})|;$
$V(P,h) = |h(x_1) - h(x_0)| + |h(x_2) - h(x_1)| + \cdots + |h(x_n) - h(x_{n-1})|.$

Now $|h(x_r) - h(x_{r-1})| = |\frac{1}{f(x_r)} - \frac{1}{f(x_{r-1})}| = \frac{|f(x_{r-1}) - f(x_r)|}{|f(x_r)f(x_{r-1})|}$.

Since $0 < k \leq f(x)$ for all $x \in [a,b]$, $|f(x_r)f(x_{r-1})| > k^2$ for all $x \in [a,b]$.
Therefore $V(P,h) < \frac{1}{k^2}.V(P,f)$.

Since f is a function of bounded variation on $[a,b]$, $V(P,f) \leq V_f[a,b]$ for all partitions P of $[a,b]$.
Therfore $V(P,h) < \frac{1}{k^2}.V_f[a,b]$ for all partitions P of $[a,b]$.

This shows that the set $\{V(P,h) : P \in \mathcal{P}[a,b]\}$ is bounded above and therefore $\sup\{V(P,h) : P \in \mathcal{P}[a,b]\}$ is a finite real number.

Hence h (i.e.,$1/f$) is a function of bounded variation on $[a,b]$ and $V_{1/f} \leq \frac{V_f}{k^2}$.

Theorem 10.1.8. Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$. Then $|f|$ is a function of bounded variation on $[a,b]$;

Proof. Let $h(x) = |f(x)|, x \in [a, b]$.

Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a, b]$. Then

$V(P, f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})|$;

$V(P, h) = |h(x_1) - h(x_0)| + |h(x_2) - h(x_1)| + \cdots + |h(x_n) - h(x_{n-1})|$.

Now $|h(x_r) - h(x_{r-1})| = ||f(x_r)| - |f(x_{r-1})|| \leq |f(x_r) - f(x_{r-1})|$.

Therefore $V(P, h) \leq V(P, f)$.

Since f is a function of bounded variation on $[a, b]$, $V(P, f) \leq V_f[a, b]$ for all partitions P of $[a, b]$.

Therefore $V(P, h) \leq V_f[a, b]$ for all partitions P of $[a, b]$.

This shows that the set $\{V(P, h) : P \in \mathcal{P}[a, b]\}$ is bounded above and therefore $\sup\{V(P, h) : P \in \mathcal{P}[a, b]\}$ is a finite real number.

Hence h (i.e., $|f|$) is a function of bounded variation on $[a, b]$.

Refinement of a partition.

Let $[a, b]$ be a closed and bounded interval. Let $P = (x_0, x_1, x_2, \ldots, x_n)$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a, b]$.

A partition Q of $[a, b]$ is said to be a *refinement* of P if P be a proper subset of Q. That is, Q is obtained by adjoining a *finite number* of additional points to P.

For example, if $P = (0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1)$ be a partition of $[0, 1]$ and $Q = (0, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, 1)$ then Q is a refinement of P.

If $R = (0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{3}{4}, 1)$ then R is a refinement of P but not a refinement of Q.

Theorem 10.1.9. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$ and P be a partition of $[a, b]$. If Q be a refinement of P then $V(Q, f) \geq V(P, f)$.

Proof. Let $P = (x_0, x_1, x_2, \ldots, x_n)$, where $a = x_0, x_n = b$.

First we examine the effect of adjoining one additional point y to P.

Let $P_1 = (x_0, x_1, \ldots, x_{k-1}, y, x_k, \ldots, x_n)$.

The subinterval $[x_{k-1}, x_k]$ is divided into two smaller subintervals $[x_{k-1}, y]$ and $[y, x_k]$.

$V(P, f) = |f(x_1) - f(x_0)| + \cdots + |f(x_k) - f(x_{k-1})| + \cdots + |f(x_n) - f(x_{n-1})|$.
$V(P_1, f) = |f(x_1) - f(x_0)| + \cdots + |f(y) - f(x_{k-1})| + |f(x_k) - f(y)| + \cdots + |f(x_n) - f(x_{n-1})|$.

Since $|f(x_k) - f(x_{k-1})| = |f(x_k) - f(y) + f(y) - f(x_{k-1})| \leq |f(y) - f(x_{k-1})| + |f(x_k) - f(y)|$, it follows that $V(P_1, f) \geq V(P, f)$.

If Q be any refinement of P then Q can be obtained from P by adjoining a finite number of additional points to P, one at a time.

By repeating the argument a finite number of times, we have $V(Q, f) \geq V(P, f)$.

Theorem 10.1.10. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$ and $c \in (a, b)$. Then

(i) f is of bounded variation on $[a, c]$ and on $[c, b]$; and

(ii) $V_f[a, b] = V_f[a, c] + V_f[c, b]$.

Proof. (i) Let P_1 be a partition of $[a, c]$ and P_2 be a partition of $[c, b]$. Let $P = P_1 \cup P_2$. Then P is a partition of $[a, b]$

Clearly, $V(P_1, f) + V(P_2, f) = V(P, f)$.

Since f is a function of bounded variation on $[a, b]$, $V(P, f) \leq V_f[a, b]$ for all partitions P of $[a, b]$.

Since each of $V(P_1, f)$ and $V(P_2, f)$ is non-negative, it follows that $V(P_1, f) \leq V_f[a, b]$ for all partitions P_1 of $[a, c]$ and $V(P_2, f) \leq V_f[a, b]$ for all partitions P_2 of $[c, b]$.

Hence f is a function of bounded variation on $[a, c]$ and on $[c, b]$

(ii) We use here an important property of bounded sets in $\mathbb{R}$.

If S_1 and S_2 be subsets of $\mathbb{R}$ both bounded above and $T = \{x + y : x \in S_1, y \in S_2\}$, then $\sup T = \sup S_1 + \sup S_2$.

Here both the sets $S_1 = \{V(P_1, f) : P_1 \in \mathcal{P}[a, c]\}$ and $S_2 = \{V(P_2, f) : P_2 \in \mathcal{P}[c, b]\}$ are bounded above and $\sup S_1 = V_f[a, c]$, $\sup S_2 = V_f[c, b]$.

The supremum of the set $\{V(P_1, f) + V(P_2, f) : P_1 \in \mathcal{P}[a, c], P_2 \in \mathcal{P}[c, b]\} = \sup\{V(P_1, f) : P_1 \in \mathcal{P}[a, c]\} + \sup\{V(P_2, f) : P_2 \in \mathcal{P}[c, b]\}$ $= V_f[a, c] + V_f[c, b]$.

Since the set $\{V(P_1, f) + V(P_2, f) : P_1 \in \mathcal{P}[a, c], P_2 \in \mathcal{P}[c, b]\}$ is a proper subset of the set $\{V(P, f) : P \in \mathcal{P}[a, b]\}$, we have $V_f[a, c] + V_f[c, b] \leq V_f[a, b] \ldots \ldots$ (i)

To obtain the reverse inequality, let P be a partition of $[a, b]$ and let $P_0 = P \cup \{c\}$. Then P_0 is a refinement of P if $c \notin P$, and $P_0 = P$ if $c \in P$.

Let $P_1 = P_0 \cap [a, c]$, $P_2 = P_0 \cap [c, b]$. Then P_1 is a partition of $[a, c]$ and P_2 is a partition of $[c, b]$ and $V(P_0, f) = V(P_1, f) + V(P_2, f)$.

Since P_0 is a refinement of the partition P, $V(P_0, f) \geq V(P, f)$.

We have $V(P,f) \leq V(P_0,f) = V(P_1,f)+V(P_2,f) \leq V_f[a,c]+V_f[c.b]$ and this holds for all partitions P of $[a,b]$.

This shows that $V_f[a,c] + V_f[c,b]$ is an upper bound of the set $\{V(P,f) : P \in \mathcal{P}[a,b]\}$. Therefore $V_f[a,b] \leq V_f[a,c] + V_f[c.b]$ ... (ii)

From (i) and (ii) we have $V_f[a,b] = V_f[a,c] + V_f[c,b]$.

This completes the proof.

Theorem 10.1.11. Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a;c]$ and on $[c,b]$ where $c \in (a,b)$. Then

(i) f is of bounded variation on $[a,b]$, and

(ii) $V_f[a,c] + V_f[c,b] = V_f[a,b]$.

Proof. (i) let P be a partition of $[a,b]$ and let $P_0 = P \cup \{c\}$. Then P_0 is a refinement of P if $c \notin P$; and $P_0 = P$ if $c \in P$.

Let $P_1 = P_0 \cap [a,c]$, $P_2 = P_0 \cap [c,b]$. Then P_1 is a partition of $[a,c]$ and P_2 is a partition of $[c,b]$ and $V(P_0,f) = V(P_1,f) + V(P_2,f)$.

We also have either $P_0 = P$ or P_0 is a refinement of P. Therefore $V(P_0,f) \geq V(P,f)$

We have $V(P,f) \leq V(P_0,f) = V(P_1,f) + V(P_2,f)$.

Since f is of bounded variation on $[a,c]$ and also on $[c,b]$, $V(P_1,f) \leq V_f[a,c]$ and $V(P_2,f) \leq V_f[c,b]$.

Therefore the set $\{V(P,f) : P \in \mathcal{P}[a,b]\}$ is bounded above and therefore f is of bounded variation on $[a,b]$.

(ii) Since $V(P,f) \leq V_f[a,c]+V_f[c,b]$, $V_f[a,c]+V_f[c,b]$ is an upper bound of the set $\{V(P,f) : P \in \mathcal{P}[a,b]\}$ and since $\sup\{V(P,f) : P \in \mathcal{P}[a,b]\}$ is $V_f[a,b]$, it follows that $V_f[a,b] \leq V_f[a,c] + V_f[c,b]$ ... ... (i)

Let $\epsilon > 0$.

Since $V_f[a,c]$ is the supremum of the set $\{V(P,f) : P \in \mathcal{P}[a,c]\}$, there exists a partition Q_1 of $[a,c]$ such that $V(Q_1,f) > V_f[a,c] - \frac{\epsilon}{2}$.

Since $V_f[c,b]$ is the supremum of the set $\{V(P,f) : P \in \mathcal{P}[c,b]\}$, there exists a partition Q_2 of $[c,b]$ such that $V(Q_2,f) > V_f[c,b] - \frac{\epsilon}{2}$.

Let $Q = Q_1 \cup Q_2$. Then Q is a partition of $[a,b]$ and $V(Q,f) = V(Q_1,f) + V(Q_2,f) > V_f[a,c] + V_f[c,b] - \epsilon$.

But $V_f[a,b] \geq V(Q,f)$. Therefore $V_f[a,b] > V_f[a,c] + V_f[c,b] - \epsilon$.

Since ϵ is an arbitrarily small positive number, it follows that

$V_f[a,b] \geq V_f[a,c] + V_f[c,b]$ ... ... (ii)

Using (i) and (ii) the proof is complete.

Corollary. If $f : [a,b] \to \mathbb{R}$ and if it be possible to divide the interval $[a,b]$ into a finite number of subintervals in each of which f is monotone, then f is a BV-function on $[a,b]$.

Worked Example (continued).

2. Let $f : [0,3] \to \mathbb{R}$ be defined by $f(x) = x^2 - 4x + 3$, $x \in [0,3]$. Show that f is a function of bounded variation on $[0,3]$. Calculate $V_f[0,3]$.

f is continuous on $[0,3]$. $f'(x) < 0$ if $x \in (0,2)$; $f'(x) > 0$ if $x \in (2,3)$. Therefore f is a decreasing function on $[0,2]$ and is an increasing function on $[2,3]$.

Hence f is a BV-function on $[0,2]$ and on $[2,3]$ and therefore f is a BV-function on $[0,3]$.

$V_f[0,2] = f(0) - f(2) = 4$, since f is a decreasing function on $[0,2]$;
$V_f[2,3] = f(3) - f(2) = 1$, since f is an increasing function on $[2,3]$.
Therefore $V_f[0,3] = V_f[0,2] + V_f[2,3] = 5$.

Theorem 10.1.12. Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$ and let $\phi : [a,b] \to \mathbb{R}$ be such that ϕ is bounded on $[a,b]$ and $\phi(x) = f(x)$ except at only one point in $[a,b]$. Then ϕ is a function of bounded variation on $[a,b]$.

Proof. Let $\phi(c) \neq f(c)$, $c \in [a,b]$. Let $\phi(c) = f(c) + \mu$, $\mu \in \mathbb{R}, \mu \neq 0$.

Case 1. $c = a$.

Let us take a partition $P = (x_0, x_1, x_2, \ldots, x_n)$ of $[a,b]$. Then
$V(P,f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})|$.
$V(P,\phi) = |\phi(x_1) - \phi(x_0)| + |\phi(x_2) - \phi(x_1)| + \cdots + |\phi(x_n) - \phi(x_{n-1})|$.

$V(P,\phi) - V(P,f) = |\phi(x_1) - \phi(x_0)| - |f(x_1) - f(x_0)| = |f(x_1) + \mu - \phi(x_0)| - |f(x_1) - f(x_0)| \leq |\mu|$.

Therefore $V(P,\phi) \leq V(P,f) + |\mu|$ and this holds for every partition P of $[a,b]$.

Consequently, $\sup\{V(P,\phi) : P \in \mathcal{P}[a,b]\} \leq |\mu| + V_f[a,b]$, a finite positive number. Hence ϕ is a function of bounded variation on $[a,b]$.

Case 2. $c = b$.

Similar proof.

Case 3. $a < c < b$.

Since f is a function of bounded variation on $[a,b]$ and $a < c < b$, f is a BV-function on $[a,c]$ and on $[c,b]$.

Since ϕ is bounded on $[a,c]$ and $\phi(x) = f(x)$ for all $x \in [a,c]$ except at c, ϕ is a function of bounded variation on $[a,c]$, by Case 2.

Since ϕ is bounded on $[c,b]$ and $\phi(x) = f(x)$ for all $x \in [c,b]$ except at c, ϕ is a function of bounded variation on $[c,b]$, by Case 1.

Since ϕ is a BV-function on $[a,c]$ and on $[c,b]$, ϕ is a function of bounded variation on $[a,b]$. This completes the proof.

Note. If $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$ and $\phi : [a,b] \to \mathbb{R}$ be such that ϕ is bounded on $[a,b]$ and $\phi(x) = f(x)$ except at a finite number of points in $[a,b]$, then ϕ is a a function of bounded variation on $[a,b]$.

Worked Example (continued).

3. Let $f : [1,3] \to \mathbb{R}$ be defined by $f(x) = x - [x]$, $x \in [1,3]$. Show that f is a function of bounded variation on $[1,3]$. Calculate $V_f[1,3]$.

$$\begin{aligned} f(x) &= x - 1, \text{ if } 1 \le x < 2 \\ &= x - 2, \text{ if } 2 \le x < 3 \\ &= 0, \text{ if } x = 3. \end{aligned}$$

Let us define a function ϕ_1 on $[1,2]$ by $\phi_1(x) = x - 1$, $x \in [1,2]$. Then ϕ_1 is a monotone increasing function on $[1,2]$ and therefore ϕ_1 is a BV-function on $[1,2]$. Since f is bounded on $[1,2]$ and $f(x) = \phi_1(x)$ for all $x \in [1,2]$ except at 2, f is a BV-function on $[1,2]$.

Let us define a function ϕ_2 on $[2,3]$ by $\phi_2(x) = x - 2$, $x \in [2,3]$. Then ϕ_2 is a monotone increasing function on $[2,3]$ and therefore ϕ_2 is a BV-function on $[2,3]$. Since f is bounded on $[2,3]$ and $f(x) = \phi_2(x)$ for all $x \in [2,3]$ except at 3, f is a BV-function on $[2,3]$.

Since f is a BV-function on $[1,2]$ and on $[2,3]$, f is a BV-function on $[1,3]$.

$$V_f[1,2] = \sup_{x \in (1,2)} [V_f[1,x] + |f(2) - f(x)|] = \sup_{x \in (1,2)} [x - 1 + |1 - x|] = 2.$$
$$V_f[2,3] = \sup_{x \in (2,3)} [V_f[2,x] + |f(3) - f(x)|] = \sup_{x \in (2,3)} [x - 2 + |2 - x|] = 2.$$
$$V_f[1,3] = V_f[1,2] + V_f[2,3] = 4.$$

10.2. Variation function.

Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$ and $x \in (a,b]$. Then the total variation of f on $[a,x]$, i.e., $V_f[a,x]$ is a function of x for all $x \in (a,b]$. Let us define a function V on $[a,b]$ by

$$V(a) = 0 \text{ and } V(x) = V_f[a,x], \text{ if } a < x \le b.$$

V is called the *variation function* of f on $[a,b]$. The variation function of f is also denoted by V_f.

Therefore $V_f(x) = V_f[a,x]$, if $a < x \le b$ and $V_f(a) = 0$.

Theorem 10.2.1. Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$. Then the variation function V defined by

$$V(a) = 0 \text{ and } V(x) = V_f[a,x], \text{ if } a < x \leq b$$

is a monotone increasing function on $[a,b]$.

Proof. If $a < x < y \leq b$, then $V(y) - V(x) = V_f[a,y] - V_f[a,x] = V_f[x,y] \geq 0$.

If $a = x < y \leq b$, then $V(y) - V(x) = V_f[a,y] - V(a) = V_f[a,y] \geq 0$.

Therefore $a \leq x < y \leq b \Rightarrow V(x) \leq V(y)$ and this proves that V is a monotone increasing function on $[a,b]$.

Theorem 10.2.2. Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$ and V be the variation function of f on $[a,b]$. Then

(i) $V + f$ is a monotone increasing function on $[a,b]$;

(ii) $V - f$ is a monotone increasing function on $[a,b]$.

Proof. (i) Let $F(x) = V(x) + f(x), x \in [a,b]$.

If $a < x < y \leq b$ then $F(y) - F(x) = V(y) - V(x) + [f(y) - f(x)] = V_f[a,y] - V_f[a,x] + [f(y) - f(x)] = V_f[x,y] - [f(x) - f(y)]$.

But $V_f(x,y) \geq |f(y) - f(x)|$, by definition.

Therefore if $a < x < y \leq b$ then $F(y) - F(x) \geq 0$.

If $a = x < y \leq b$ then $F(y) - F(x) = V(y) - V(x) + [f(y) - f(x)] = V_f[a,y] - V(a) + [f(y) - f(a)] = V_f[a,y] - [f(a) - f(y)] \geq 0$, by the foregoing argument.

Therefore $a \leq x < y \leq b \Rightarrow F(x) \leq F(y)$ and this proves that F, i.e., $V + f$ is a monotone increasing function on $[a,b]$.

(ii) Let $G(x) = V(x) - f(x), x \in [a,b]$.

If $a < x < y \leq b$ then $G(y) - G(x) = V(y) - V(x) - [f(y) - f(x)] = V_f[a,y] - V_f[a,x] - [f(y) - f(x)] = V_f[x,y] - [f(y) - f(x)]$.

But $V_f(x,y) \geq |f(y) - f(x)|$, by definition.

Therefore if $a < x < y \leq b$ then $G(y) - G(x) \geq 0$.

If $a = x < y \leq b$ then $G(y) - G(x) = V(y) - V(x) - [f(y) - f(x)] = V_f[a,y] - V(a) - [f(y) - f(a)] = V_f[a,y] - [f(y) - f(a)] \geq 0$, by the foregoing argument.

Therefore $a \leq x < y \leq b \Rightarrow V(x) \leq V(y)$ and this proves that G, i.e., $V - f$ is a monotone increasing function on $[a,b]$.

This completes the proof.

With the help of the two previous theorems we have an important characterisation of a function of bounded variation.

Theorem 10.2.3. Let $f : [a,b] \to \mathbb{R}$. Then f is a function of bounded variation on $[a,b]$ if and only if f can be expressed as the difference of two monotone increasing functions on $[a,b]$.

Proof. Let f be a function of bounded variation on $[a,b]$.

Then the variation function V is defined on $[a,b]$ by $V(x) = V_f[a,x]$, for $a < x \leq b$ and $V(a) = 0$. We prove that V and $V + f$ are both monotone increasing functions on $[a,b]$.

If $a < x < y \leq b$ then $V(y) - V(x) = V_f[a,y] - V_f[a,x] = V_f[x,y] \geq 0$.
If $a = x < y \leq b$ then $V(y) - V(x) = V_f[a,y] - V(a) = V_f[a,y] \geq 0$.

Therefore $a \leq x < y \leq b \Rightarrow V(x) \leq V(y)$ and this proves that V is a monotone increasing function on $[a,b]$.

Let $F(x) = V(x) + f(x), x \in [a,b]$.

If $a < x < y \leq b$ then $F(y) - F(x) = V(y) - V(x) + [f(y) - f(x)] = V_f[a,y] - V_f[a,x] + [f(y) - f(x)] = V_f[x,y] - [f(x) - f(y)] \geq 0$, since $V_f(x,y) \geq |f(y) - f(x)|$, by definition.

If $a = x < y \leq b$ then $F(y) - F(x) = V(y) - V(x) + [f(y) - f(x)] = V_f[a,y] - V(a) + [f(y) - f(a)] = V_f[a,y] - [f(a) - f(y)] \geq 0$, by the foregoing argument.

Therefore $a \leq x < y \leq b \Rightarrow F(x) \leq F(y)$ and this proves that F, i.e., $V + f$ is a monotone increasing function on $[a,b]$.

f can be expressed as $f = (V + f) - V$. Thus f is expressed as the difference of two monotone increasing functions $V + f$ and V.

Conversely, let f be expressed as the difference of two monotone increasing functions on $[a,b]$.

Since a monotone increasing function on $[a,b]$ is a function of bounded variation on $[a,b]$ and the difference of two BV-functions on $[a,b]$ is a BV-function on $[a,b]$, it follows that f is a function of bounded variation on $[a,b]$.

This completes the proof.

Note. f can also be expressed as $f = V - (V - f)$, where V is the variation function of f on $[a,b]$. Since V and $V - f$ are both monotone increasing functions on $[a,b]$, f is expressed as the difference of two monotone increasing functions on $[a,b]$. This shows that the representation of f as the difference of two monotone increasing functions is not unique.

Worked Examples (continued).

4. $f(x) = x^2, x \in [-1, 1]$. Show that f is a function of bounded variation on $[-1, 1]$. Find the variation function V on $[-1, 1]$. Express f as the difference of two monotone increasing functions on $[-1, 1]$.

f is continuous on $[-1, 1]$. $f'(x) < 0$ on $(-1, 0)$ and $f'(x) > 0$ on $(0, 1)$. Therefore f is a decreasing function on $[-1, 0]$ and is an increasing function on $[0, 1]$.

Hence f is a BV-function on $[-1, 0]$ and on $[0, 1]$ and therefore f is a BV-function on $[-1, 1]$.

$V(-1) = 0$.

If $-1 < x \leq 0$, then $V(x) = V_f[-1, x]$
$= f(-1) - f(x)$, since f is decreasing on $[-1, 0]$
$= 1 - x^2$.

If $0 < x \leq 1$, then $V(x) = V_f[-1, x] = V_f[-1, 0] + V_f[0, x]$
$= [f(-1) - f(0)] + [f(x) - f(0)]$, since f is a decreasing on $[-1, 0]$
$= 1 + x^2$. and increasing on $[0, 1]$

Therefore $V(x) = 1 - x^2$, if $-1 \leq x \leq 0$
$= 1 + x^2$, if $0 < x \leq 1$.

V is a monotone increasing function on $[-1, 1]$.

$(V + f)(x) = 1$, if $-1 \leq x \leq 0$
$= 1 + 2x^2$, if $0 < x \leq 1$.

$V + f$ is a monotone increasing function on $[-1, 1]$.

f can be expressed as $(V + f) - V$, the difference of two monotone increasing functions.

Note. Here $(V - f)(x) = 1 - 2x^2$, if $-1 \leq x \leq 0$
$= 1$, if $0 < x \leq 1$.

$V - f$ is a monotone increasing function on $[-1, 1]$.

f can also be expressed as $V - (V - f)$, the difference of two monotone increasing functions on $[-1, 1]$.

5. $f(x) = [x], x \in [1, 3]$. Show that f is a function of bounded variation on $[1, 3]$. Find the variation function V on $[1, 3]$. Express f as the difference of two monotone increasing functions on $[1, 3]$.

$f(x) = 1$, if $1 \leq x < 2$
$= 2$, if $2 \leq x < 3$
$= 3$, if $x = 3$.

f is a monotone increasing function on $[1, 3]$. Therefore f is a function of bounded variation on $[1, 3]$.

$V(1) = 0$.
If $1 < x < 2$ then $V(x) = V_f[-1, x]$
$= f(x) - f(1)$, since f is increasing on $[1, 2]$
$= 0$.

$V(2) = V_f[1, 2] = f(2) - f(1) = 1$.
If $2 < x < 3$ then $V(x) = V_f[1, x] = V_f[1, 2] + V_f[2, x]$
$= [f(2) - f(1)] + [f(x) - f(2)]$, since f is a increasing on $[1, 3]$
$= (2 - 1) + 0 = 1$.

$V(3) = V_f[1, 3] = V_f[1, 2] + V_f[2, 3] = [f(2) - f(1)] - [f(3) - f(2)] = 2$.
Therefore $V(x) = 0$, if $1 \le x < 2$
$= 1$, if $2 \le x < 3$
$= 2$, if $x = 3$.

V is a monotone increasing function on $[1, 3]$.

$(V + f)(x) = 1$, if $1 \le x < 2$
$= 3$, if $2 \le x < 3$
$= 5$, if $x = 3$.

$V + f$ is a monotone increasing function on $[1, 3]$.

f can be expressed as $(V + f) - V$, the difference of two monotone increasing functions.

Theorem 10.2.4. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$ then f can have only discontinuities of the first kind and the points of discontinuity of f form a countable set.

Proof. Since f is a function of bounded variation on $[a, b]$, f can be expressed as $f(x) = g(x) - h(x)$, where g and h are monotone increasing functions on $[a, b]$.

A monotone function can have only discontinuities of the first kind.

Let $c \in (a, b)$. Then each of $g(c + 0), g(c - 0), h(c + 0), h(c - 0)$ exists and therefore each of $f(c + 0), f(c - 0)$ exists.

By similar arguments, each of $f(a + 0), f(b - 0)$ exists.

It follows that f can have only discontinuities of the first kind on $[a, b]$.

Let E_1, E_2 be respectively the sets of points of discontinuity of g and h. Then $E_1 \cup E_2$ is the set of points of discontinuity of f.

By Theorem 8.6.4, E_1 and E_2 are both countable sets. Therefore the set $E_1 \cup E_2$ is countable.

This completes the proof.

Theorem 10.2.5. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$ and let V be the variation function on $[a, b]$. If f be continuous at a point $c \in [a, b]$ then V is continuous at c, and conversely.

Proof. Let $c \in [a, b)$ and f is right continuous at c.

Let us choose a positive ϵ. There exists a positive δ_1 such that $|f(x) - f(c)| < \frac{\epsilon}{2}$ for all $x \in [c, c + \delta_1) \cap [c, b]$.

Since $V_f[c, b]$ is the supremum of the set $\{V(P, f) : P \in \mathcal{P}[c, b]\}$, there exists a partition P_0 of $[c, b]$ such that $V_f[c, b] - \frac{\epsilon}{2} < V(P_0, f) \leq V_f[c, b]$.

Let $P_0 = (c, x_1, x_2, \ldots, x_{n-1}, b)$. Let $\delta < \min\{x_1 - c, \delta_1\}$.

Let us choose a point x_0 in $(c, c + \delta)$.

Let $P_1 = P_0 \cup \{x_0\}$. Then the partition P_1 is a refinement of P_0 and $V(P_1, f) \geq V(P_0, f)$.

Therefore for all $x_0 \in (c, c + \delta)$, we have $V_f[c, b] - \frac{\epsilon}{2} < V(P_0, f) \leq V(P_1, f) \leq V_f[c, b]$ and also we have $|f(x_0) - f(c)| < \frac{\epsilon}{2}$.

But $$\begin{aligned} V(P_1, f) &= |f(x_0) - f(c)| + |f(x_1) - f(x_0)| + \cdots + |f(b) - f(x_{n-1}| \\ &< \tfrac{\epsilon}{2} + V_f[x_0, b]. \end{aligned}$$

Therefore for all $x_0 \in (c, c + \delta)$, $V(b) - V(c) - \frac{\epsilon}{2} < \frac{\epsilon}{2} + V(b) - V(x_0)$.
or, $V(x_0) - V(c) < \epsilon$ for all $x_0 \in (c, c + \delta)$.

Also we have $V(x_0) \geq V(c)$ for all $x_0 \in [c, c+\delta)$, since V is a monotone increasing function on $[a, b]$. Therefore $|V(x_0) - V(c)| < \epsilon$ for all $x_0 \in [c, c + \delta)$ and this proves that V is right continuous at c.

By similar arguments, if f be left continuous at $c \in (a, b]$ then V is left continuous at c.

It follows that V is continuous at every point $c \in [a, b]$ where f is continuous.

Conversely, let $c \in [a, b)$ and V be right continuous at c.

Let us choose a positive ϵ. Then there exists a positive δ such that $|V(x) - V(c)| < \epsilon$ for all $x \in [c, c + \delta) \cap [c, b]$.

For all $c \in [a, b)$ and for all $x \in (c, c + \delta)$, we have $|f(x) - f(c)| \leq V_f[c, x] \leq |V(x) - V(c)|$.

Therefore $|f(x) - f(c)| < \epsilon$ for all $x \in [c, c + \delta) \cap [c, b]$.

This proves that f is right continuous at c.

By similar arguments, if $c \in (a, b]$ and V be left continuous at c then f is left continuous at c.

It follows that f is continuous at every point $c \in [a, b]$ where V is continuous.

This completes the proof.

Corollary. If $f : [a, b] \to \mathbb{R}$ be continuous and be of bounded variation on $[a, b]$ then f can be expressed as the difference of two monotone increasing and continuous functions on $[a, b]$, and conversely.

10.3. Positive variation, Negative variation.

Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. Let $P = (x_0, x_1, x_2, \ldots, x_n)$ be a partition of $[a, b]$. Let us consider the sum

$$\Delta f_1 + \Delta f_2 + \cdots + \Delta f_n, \text{ where } \Delta f_r = f(x_r) - f(x_{r-1}) \quad \ldots \ldots \quad \text{(i)}$$

We have $V(P, f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})| = |\Delta f_1| + |\Delta f_2| + \cdots + |\Delta f_n|$.

Let $V_+(P, f)$ denote the sum $\sum\limits_{\Delta f_i > 0} \Delta f_i$ (i.e., the sum of all positive differences in (i)) and $V_-(P, f)$ denote the sum $\sum\limits_{\Delta f_i < 0} |\Delta f_i|$ (i.e., the sum of absolute values of all negative differences in (i)).

Then $V_+(P, f) - V_-(P, f) = f(b) - f(a)$ and
$V_+(P, f) + V_-(P, f) = V(P, f)$.

Therefore $2V_+(P, f) = V(P, f) + f(b) - f(a)$ and
$2V_-(P, f) = V(P, f) - f(b) + f(a) \quad \ldots \ldots \quad$ (ii)

Since f is a function of bounded variation on $[a, b]$, the set $\{V(P, f) : P \in \mathcal{P}[a, b]\}$ is bounded above. It follows from (ii) that the both the sets $\{V_+(P, f) : P \in \mathcal{P}[a, b]\}$ and $\{V_-(P, f) : P \in \mathcal{P}[a, b]\}$ are bounded above.

The supremum of the set $\{V_+(P, f) : P \in \mathcal{P}[a, b]\}$ is said to be the *positive variation* of f on $[a, b]$ and is denoted by $(V_+)_f[a, b]$ or by $p_f[a, b]$.

The supremum of the set $\{V_-(P, f) : P \in \mathcal{P}[a, b]\}$ is said to be the *negative variation* of f on $[a, b]$ and is denoted by $(V_-)_f[a, b]$ or by $n_f[a, b]$.

Note. It follows from (ii) that $2p_f[a, b] = V_f[a, b] + f(b) - f(a)$ and
$2n_f[a, b] = V_f[a, b] - f(b) + f(a)$.

Therefore $p_f[a, b] + n_f[a, b] = V_f[a, b]$ and $p_f[a, b] - n_f[a, b] = f(b) - f(a)$.

Positive variation function, Negative variation function.

Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$ and let $x \in (a, b]$.

The positive variation of f on $[a, x]$ is a function of x for all $x \in (a, b]$. Let us define a function V_+ on $[a, b]$ by

$V_+(a) = 0$ and $V_+(x) = p_f[a, x]$, if $x \in (a, b]$.

V_+ is called the *positive variation function* of f on $[a, b]$. It is also denoted by p. Therefore $p(x) = p_f[a, x]$, if $x \in (a, b]$ and $p(a) = 0$.

The negative variation of f on $[a, x]$ is a function of x for all $x \in (a, b]$. Let us define a function V_- on $[a, b]$ by

$V_-(a) = 0$ and $V_-(x) = n_f[a, x]$, if $x \in (a, b]$.

V_- is called the *negative variation function* of f on $[a, b]$. It is also denoted by n. Therefore $n(x) = n_f[a, x]$, if $x \in (a, b]$ and $n(a) = 0$.

Theorem 10.3.1. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. Then

(i) $p(x) + n(x) = V(x)$ for all $x \in [a, b]$,

(ii) $p(x) - n(x) = f(x) - f(a)$ for all $x \in [a, b]$,

where V is the variation function, p is the positive variation function and n is the negative variation function of f on $[a, b]$.

Proof. (i) We have $p(a) = 0$, $n(a) = 0$ and $V(a) = 0$.

Therefore $V(a) = p(a) + n(a)$ ... ... (i)

If $a < x \leq b$, then $V(x) = V_f[a, x]$, $p(x) = p_f[a, x]$, $n(x) = n_f[a, x]$.

Therefore $p(x) + n(x) = V(x)$ for all $x \in (a, b]$, since $p_f[a, x] + n_f[a, x] = V_f[a, x]$ for all $x \in (a, b]$ ... ... (ii)

From (i) and (ii) $p(x) + n(x) = V(x)$ for all $x \in [a, b]$.

(ii) Similar proof.

Theorem 10.3.2. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. Then the positive variation function p is a monotone increasing function on $[a, b]$.

Proof. By definition, $2p(x) = V(x) + f(x) - f(a)$.

If $a < x < y \leq b$, then $p(y) - p(x) = \frac{1}{2}\{V(y) - V(x)\} + \frac{1}{2}[f(y) - f(x)]$
$= \frac{1}{2}\{V_f[a, y] - V_f[a, x]\} + \frac{1}{2}[f(y) - f(x)]$
$= \frac{1}{2}[V_f[x, y] - (f(x) - f(y))] \geq 0$, since $V_f[x, y] \geq |f(y) - f(x)|$.

If $a = x < y \leq b$, then $p(y) - p(x) = \frac{1}{2}\{V(y) - V(x)\} + \frac{1}{2}[f(y) - f(x)]$
$= \frac{1}{2}\{V_f[a, y]\} + \frac{1}{2}[f(y) - f(a)]$, since $V(x) = V(a) = 0$
$= \frac{1}{2}[V_f[a, y] - (f(a) - f(y))] \geq 0$, since $V_f[a, y] \geq |f(y) - f(a)|$.

Therefore $a \leq x < y \leq b \Rightarrow p(x) \leq p(y)$ and this proves that p is a monotone increasing function on $[a, b]$.

Theorem 10.3.3. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. Then the negative variation function n is a monotone increasing function on $[a, b]$.

Similar proof.

Note 1. In view of the theorem 10.2.5, the functions p and n are both continuous at the point where f is continuous.

2. Since $p(x) - n(x) = f(x) - f(a)$ for all $x \in [a, b]$, we have $f(x) = [p(x)+f(a)]-n(x)$. Since p and n are both monotone increasing functions on $[a, b]$, f is expressed as the difference of two monotone increasing functions $p + f(a)$ and n.

Worked Examples (continued).

6. $f(x) = |x|, x \in [-1, 2]$. Show that f is a function of bounded variation on $[1, 3]$. Calculate the positive variation, the negative variation and the total variation of f on $[-1, 2]$. Find the functions p and n on $[-1, 2]$.

f is a monotone decreasing function on $[-1, 0]$ and a monotone increasing function on $[0, 2]$. Therefore f is a function of bounded variation on $[-1, 0]$ and on $[0, 2]$ and consequently, f is a BV-function on $[-1, 2]$.

$V_f[-1, 2] = V_f[-1, 0] + V_f[0, 2] = [f(-1) - f(0)] + [f(2) - f(0)]$. since f is monotone decreasing on $[-1, 0]$ and monotone increasing on $[0, 2]$.

Therefore $V_f[-1, 2] = 1 + 2 = 3$.

We have $2p_f[-1, 2] = V_f[-1, 2] + f(2) - f(-1) = 3 + (2 - 1) = 4$ and $2n_f[-1, 2] = V_f[-1, 2] - f(2) + f(-1) = 3 + (-2 + 1) = 2$.

or, $p_f[-1, 2] = 2$ and $n_f[-1, 2] = 1$.

$V(-1) = 0$.

If $-1 < x \leq 0$, then $V(x) = V_f[-1, x]$
$= f(-1) - f(x)$, since f is decreasing on $[-1, 0]$
$= 1 + x$.

If $0 < x \leq 2$, then $V(x) = V_f[-1, x] = V_f[-1, 0] + V_f[0, x]$
$= [f(-1) - f(0)] + [f(x) - f(0)]$, since f is decreasing on $[-1, 0]$
$= 1 + x.$ and increasing on $[0, 2]$

Therefore $V(x) = 1 + x$, if $-1 \leq x \leq 0$
$= 1 + x$, if $0 < x \leq 2$.

That is, $V(x) = 1 + x, \ -1 \leq x \leq 2$.

The positive variation function p is given by $2p(x) = V(x) + f(x) - f(0)$ for all $x \in [-1, 2]$.

Therefore $p(x) = \frac{1}{2}[(1 + x) + (-x)] = \frac{1}{2}$, if $-1 \leq x < 0$
$= \frac{1}{2}[(1 + x) + (x)] = \frac{1}{2}(1 + 2x)$, if $0 \leq x \leq 2$.

The negative variation function n is given by $2n(x) = V(x) - f(x) + f(0)$ for all $x \in [-1, 2]$.

Therefore $n(x) = \frac{1}{2}[(1+x)+x] = \frac{1}{2}(1+2x)$, if $-1 \leq x < 0$
$= \frac{1}{2}[(1+x)-(x)] = \frac{1}{2}$, if $0 \leq x \leq 2$.

7. $f(x) = x - [x], x \in [1,3]$. Show that f is a function of bounded variation on $[1,3]$. Find the positive variation function, the negative variation function and express f as the difference of two monotone increasing functions on $[1,3]$.

$f(x) = x - 1$, if $1 \leq x < 2$
$= x - 2$, if $2 \leq x < 3$
$= 0$, if $x = 3$.

f is a function of bounded variation on $[1,3]$. [worked Ex. 3]

$V(1) = 0$. For $1 < x < 2$, $V(x) = V_f[1,x] = f(x) - f(1) = x - 1$.
$V(2) = V_f[1,2] = 2$. [worked Ex. 3]

For $2 < x < 3$, $V(x) = V_f[0,2] + V_f[2,x] = V(2) + f(x) - f(2) = 2 + (x-2) = x$.
$V(3) = V_f[1,3] = V_f[1,2] + V_f[2,3] = 2 + 2 = 4$. [worked Ex.3]

The variation function V is given by $V(x) = x - 1$, if $1 \leq x < 2$
$= x$, if $2 \leq x < 3$
$= 4$, if $x = 3$.

The positive variation function p is given by $2p(x) = V(x) + f(x) - f(1)$ for all $x \in [1,3]$. Therefore
$p(x) = \frac{1}{2}[(x-1)+(x-1)] = x - 1$, if $1 \leq x < 2$
$= \frac{1}{2}[x + (x-2)] = x - 1$, if $2 \leq x < 3$
$= \frac{1}{2}[4+0] = 2$, if $x = 3$.
That is, $p(x) = x - 1,\ x \in [1,3]$.

The negative variation function n is given by $2n(x) = V(x) - f(x) + f(1)$ for all $x \in [1,3]$. Therefore
$n(x) = \frac{1}{2}[(x-1)-(x-1)] = 0$, if $1 \leq x < 2$
$= \frac{1}{2}[(x)-(x-2)] = 1$, if $2 \leq x < 3$
$= \frac{1}{2}[4-0] = 2$, if $x = 3$.
Clearly, p and n are monotone increasing functions on $[1,3]$.

Since $p(x) - n(x) = f(x) - f(1),\ x \in [1,3]$ and $f(1) = 0$, f can be expressed as $f(x) = p(x) - n(x),\ x \in [1,3]$.

8. Let $x_1, x_2, \ldots, x_n, \ldots$ be an enumeration of all rational points in $[0,1]$ and let $f : [0,1] \to \mathbb{R}$ be defined by $f(x_n) = \frac{1}{n^2}$, $n = 1, 2, 3, \ldots$
$= 0$, elsewhere.
Prove that that f is a function of bounded variation on $[0,1]$.

Let us take a partition $P = (y_0, y_1, y_2, \ldots, y_n)$ of $[0,1]$.

Then $V(P, f) = |f(y_1) - f(y_0)| + |f(y_2) - f(y_1)| + \cdots + |f(y_n) - f(y_{n-1})|$
$\leq 2[|f(y_0)| + |f(y_1)| + |f(y_2)| + \cdots + |f(y_n)|]$.

Let us choose a natural number m such that the rational points in P form a proper subset of the set $\{x_1, x_2, \ldots, x_m\}$

Then $V(P, f) \leq 2[\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{m^2}]$

$\leq 2s$, where s is the sum of the convergent series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$.

Therefore $V(P, f) \leq 2s$ for all partitions P of $[a, b]$ and therefore the supremum of the set $\{V(P, f) : P \in \mathcal{P}[a, b]\}$ is finite.

Hence f is a function of bounded variation on $[0, 1]$.

Exercises 20

1. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$ and $[c.d] \subset [a, b]$. Prove that f is a function of bounded variation on $[c, d]$.

2. Give an example of a function f continuous on a closed interval $[a, b]$ but f is not a function of bounded variation on $[a, b]$.

3. Give an example of a function f not continuous on a closed interval $[a, b]$ but f is a function of bounded variation on $[a, b]$.

4. Show that the function f is not of bounded variation on $[0, 1]$.

(i) $f(x) = x \sin \frac{\pi}{x}$, if $x \in (0, 1]$ and $f(0) = 0$.

(ii) $f(x) = \sin \frac{\pi}{x}$, if $x \in (0, 1]$ and $f(0) = 0$.

Hint. (i) Consider the partition $(0, \frac{2}{2n+1}, \frac{2}{2n-1}, \cdots, \frac{2}{5}, \frac{2}{3}, 1)$ of $[0, 1]$.

5. Show that f is a function of bounded variation on $[0, \frac{\pi}{2}]$. Find the variation function V on $[0, \frac{\pi}{2}]$.

(i) $f(x) = \sin 2x$, (ii) $f(x) = \sin x + \cos x$, (iii) sgn $\cos 2x$.

6. Show that f is a function of bounded variation on $[0, 2]$. Find the variation function V on $[0, 2]$. Express f as the difference of two monotone increasing functions on $[0, 2]$.

(i) $f(x) = x^2 - 2x + 2$, (ii) $f(x) = [x] - x$, (iii) $f(x) = |x - 1|$.

7. Show that f is a function of bounded variation on $[0, 3]$. Calculate the total variation, the positive variation and the negative variation of f on $[0, 3]$.

(i) $f(x) = x^2 - 4x + 1$, (ii) $f(x) = \text{sgn } (x - 1)$, (iii) $f(x) = |x - 2|$.

8. Show that f is a function of bounded variation on $[0, 2]$. Find the positive variation function p and the negative variation function n on $[0, 2]$. Hence express f as the difference of two monotone increasing functions on $[0, 2]$.

(i) $f(x) = x^2 - 2x + 2$, (ii) $f(x) = \text{sgn } (x - 1)$, (iii) $f(x) = |x - 1|$.

11.1. Partition.

Let $[a,b]$ be a closed and bounded interval. A partition of $[a,b]$ is a finite ordered set $P = (x_0, x_1, \ldots, x_n)$ of points of $[a,b]$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

The family of all partitions of $[a,b]$ is denoted by $\mathcal{P}[a,b]$ and the partition $P = (x_0, x_1, \ldots, x_n)$ is a member of $\mathcal{P}[a,b]$.

For example, $P = (0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1)$ is a partition of $[0,1], Q = (0, \frac{1}{4}, \frac{3}{8}, \frac{2}{4}, \frac{3}{4}, \frac{7}{8}, 1)$ is another partition of $[0,1]$.

Let $P \in \mathcal{P}[a,b]$ where $P = (x_0, x_1, x_2, \ldots, x_n)$ such that $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. Then P divides the interval $[a,b]$ into non-overlapping subintervals $[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b]$.

11.2. Riemann integrability.

Let $[a,b]$ be a closed and bounded interval. Let $f : [a,b] \to \mathbb{R}$ be a bounded function on $[a,b]$. Let us take a partition P of $[a,b]$ defined by $P = (x_0, x_1, x_2, \ldots, x_n)$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. Since f is bounded on $[a,b]$, f is bounded on $[x_{r-1}, x_r]$, for $r = 1, 2, \ldots, n$.

Let $M = \sup\limits_{x \in [a,b]} f(x)$, $\quad m = \inf\limits_{x \in [a,b]} f(x)$;

$M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $\quad m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

Then $m \le m_r \le M_r \le M$, for $r = 1, 2, \ldots, n$. ... ... (i)

The sum $M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1})$ is said to be the *upper Darboux sum* or the *upper sum* of f corresponding to the partition P and is denoted by $U(P, f)$;

and the sum $m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1})$ is said to be the *lower Darboux sum* or the *lower sum* of f corresponding to the partition P and is denoted by $L(P, f)$.

Each $P \in \mathcal{P}[a,b]$ determines two numbers $U(P, f)$ and $L(P, f)$.

By the inequality (i) we have

$$m(x_r - x_{r-1}) \le m_r(x_r - x_{r-1}) \le M_r(x_r - x_{r-1}) \le M(x_r - x_{r-1}),$$

for $r = 1, 2, \ldots, n$. Therefore

$$\begin{aligned} m \sum_{r=1}^{n} (x_r - x_{r-1}) &\le \sum_{r=1}^{n} m_r(x_r - x_{r-1}) \\ &\le \sum_{r=1}^{n} M_r(x_r - x_{r-1}) \le M \sum_{r=1}^{n} (x_r - x_{r-1}) \end{aligned}$$

or, $m(b-a) \le L(P, f) \le U(P, f) \le M(b-a)$ ... ... (ii)

Let us the consider the set $\mathcal{P}[a, b]$ of all partitions of $[a, b]$.

We have two sets of real numbers $\{U(P, f) : P \in \mathcal{P}[a, b]\}$ and $\{L(P, f) : P \in \mathcal{P}[a, b]\}$.

The inequality (ii) shows that both these sets are bounded sets. $M(b-a)$ is an upper bound and $m(b-a)$ is a lower bound in respect of both the sets.

The supremum (the least upper bound) of the set $\{L(P, f) : P \in \mathcal{P}[a, b]\}$ exists and it is called the *lower integral* of f on $[a, b]$ and is denoted by $\underline{\int}_a^b f(x)dx$, or by $\underline{\int}_a^b f dx$, or by $\underline{\int}_a^b f$.

The infimum (the greatest lower bound) of the set $\{U(P, f) : P \in \mathcal{P}[a, b]\}$ exists and it is called the *upper integral* of f on $[a, b]$ and is denoted by $\overline{\int}_a^b f(x)dx$, or by $\overline{\int}_a^b f dx$, or by $\overline{\int}_a^b f$.

f is said to be *Riemann integrable* on $[a, b]$ if $\underline{\int}_a^b f = \overline{\int}_a^b f$.

The common value $\underline{\int}_a^b f$ or $\overline{\int}_a^b f$ is called the *Riemann integral* of f on $[a, b]$ and it is denoted by $\int_a^b f(x)dx$, or by $\int_a^b f dx$, or by $\int_a^b f$.

In addition, we define $\int_a^a f = 0$, $\int_b^a f = -\int_a^b f$.

Note 1. It follows from the definition of the supremum and the infimum of a bounded set that

$$m(b-a) \le \underline{\int}_a^b f \le M(b-a), m(b-a) \le \overline{\int}_a^b \le M(b-a).$$

Note 2. The class of all Riemann integrable functions on $[a, b]$ is denoted by $\mathcal{R}[a, b]$. The class of all functions bounded on $[a, b]$ is denoted by $\mathcal{B}[a, b]$. From the definition of an integrable function it follows that $\mathcal{R}[a, b]$ is a subset of $\mathcal{B}[a, b]$.

We shall see that there are functions in $\mathcal{B}[a, b]$ which do not belong to $\mathcal{R}[a, b]$, i.e., $\mathcal{R}[a, b]$ is a proper subset of $\mathcal{B}[a, b]$.

From now on we shall often drop the adjective 'Riemann' and simply use the term 'integral' to mean Riemann integral, the term 'integrability' to mean Riemann integrability.

Worked Examples.

1. Let $[a,b]$ be a closed and bounded interval and $c \in \mathbb{R}$. A function $f : [a,b] \to \mathbb{R}$ is defined by $f(x) = c, x \in [a,b]$. Prove that $f \in \mathcal{R}[a,b]$.

f is bounded on $[a,b]$.

Let us take a partition P of $[a,b]$ defined by $P = (x_0, x_1, x_2, \ldots, x_n)$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

Let $M = \sup_{x \in [a,b]} f(x)$, $\quad m = \inf_{x \in [a,b]} f(x)$;

$M_r = \sup_{x \in [x_{r-1}, x_r]} f(x)$, $\quad m_r = \inf_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

Then $M = c, m = c$; $M_r = c, m_r = c$, for $r = 1, 2, \ldots, n$.

Now $U(P,f) = M_1(x_1 - a) + M_2(x_2 - x_1) + \cdots + M_n(b - x_{n-1})$
$= c(b-a)$;

$L(P,f) = m_1(x_1 - a) + m_2(x_2 - x_1) + \cdots + m_n(b - x_{n-1})$
$= c(b-a)$.

Let us consider the set $\mathcal{P}[a,b]$ of all partitions of $[a,b]$.

The set $\{L(P,f) : P \in \mathcal{P}[a,b]\}$ is the singleton set $\{c(b-a)\}$. The least upper bound of the set is $c(b-a)$, i.e., $\underline{\int}_a^b f = c(b-a)$.

Also the set $\{U(P,f) : P \in \mathcal{P}[a,b]\}$ is the singleton set $\{c(b-a)\}$. The greatest lower bound of the set is $c(b-a)$, i.e., $\overline{\int}_a^b = c(b-a)$.

As $\underline{\int}_a^b f = \overline{\int}_a^b f = c(b-a)$, f is integrable on $[a,b]$ and $\int_a^b f = c(b-a)$.

2. A function f is defined on $[0,1]$ by

$$\begin{aligned} f(x) &= 1, \text{ if } x \text{ is rational} \\ &= 0, \text{ if } x \text{ is irrational.} \end{aligned}$$

Show that f is not integrable on $[0,1]$.

f is bounded on $[0,1]$.

Let us take a partition P of $[0,1]$ defined by $P = (x_0, x_1, x_2, \ldots, x_n)$, where $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$.

Let $M = \sup_{x \in [0,1]} f(x)$, $\quad m = \inf_{x \in [0,1]} f(x)$;

$M_r = \sup_{x \in [x_{r-1}, x_r]} f(x)$, $\quad m_r = \inf_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

Then $M = 1, m = 0$; $M_r = 1, m_r = 0$, for $r = 1, 2, \ldots, n$.

$U(P,f) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1}) = 1$.
$L(P,f) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(b - x_{n-1}) = 0$.
Let us consider the set $\mathcal{P}[0,1]$ of all partitions of $[0,1]$.

The set $\{L(P,f) : P \in \mathcal{P}[0,1]\}$ is the singleton set $\{0\}$. The least upper bound of the set is 0, i.e., $\underline{\int}_0^1 f = 0$.

The set $\{U(P,f) : P \in \mathcal{P}[0,1]\}$ is the singleton set $\{1\}$. The greatest lower bound of the set is 1, i.e., $\overline{\int_0^1} f = 1$.

Since $\underline{\int_0^1} f \neq \overline{\int_0^1} f$, f is not integrable on $[0,1]$.

Lemma 11.2.1. Let $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$ and P be any partition of $[a,b]$. Then $L(P,f) \leq U(P,f)$.

Proof. Let $P = (x_0, x_1, x_2, \ldots, x_n)$ be a partition of $[a,b]$. Then $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $\quad m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

Then $m_r \leq M_r$, for $r = 1, 2, \ldots, n$.

Since $x_r - x_{r-1} > 0$, for $r = 1, 2, \ldots, n$ it follows that

$m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1})$

$\leq M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1})$.

That is, $L(P,f) \leq U(P,f)$. This proves the lemma.

11.3. Refinement of a partition.

Let $[a,b]$ be a closed and bounded interval. Let $P = (x_0, x_1, x_2, \ldots, x_n)$ be a partition of $[a,b]$.

A partition Q of $[a,b]$ is said to be a *refinement* of P if P be a proper subset of Q. That is, Q is obtained by adjoining a *finite number* of additional points to P.

For example, if $P = (0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1)$ be a partition of $[0,1]$ and $Q = (0, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, 1)$ then Q is a refinement of P.

If $R = (0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{3}{4}, 1)$ then R is a refinement of P but not a refinement of Q.

Lemma 11.3.1. Let a function $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$ and P be a partition of $[a,b]$. If Q be a refinement of P then

$$U(P,f) \geq U(Q,f) \text{ and } L(P,f) \leq L(Q,f).$$

Proof. Let $P = (x_0, x_1, x_2, \ldots, x_n)$. First we examine the effect of adjoining one additional point y to P.

Let $P_1 = (x_0, x_1, \ldots, x_{k-1}, y, x_k, \ldots, x_n)$.

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $\quad m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

The subinterval $[x_{k-1}, x_k]$ is divided into two smaller subintervals $[x_{k-1}, y]$ and $[y, x_k]$.

Let $M_k' = \sup\limits_{x \in [x_{k-1}, y]} f(x)$, $m_k' = \inf\limits_{x \in [x_{k-1}, y]} f(x)$,

$$M_k'' = \sup_{x\in[y,x_k]} f(x), m_k'' = \inf_{x\in[y,x_k]} f(x).$$

Then $M_k' \le M_k, M_k'' \le M_k; m_k' \ge m_k, m_k'' \ge m_k$.

Therefore
$$\begin{aligned} M_k(x_k - x_{k-1}) &= M_k[(x_k - y + y - x_{k-1})] \\ &\ge M_k''(x_k - y) + M_k'(y - x_{k-1}); \\ m_k(x_k - x_{k-1}) &= m_k[x_k - y + y - x_{k-1}] \\ &\le m_k''(x_k - y) + m_k'(y - x_{k-1}). \end{aligned}$$

$$\begin{aligned} U(P,f) - U(P_1,f) &= M_k(x_k - x_{k-1}) - [M_k'(y - x_{k-1}) \\ &\quad + M_k''(x_k - y)] \ge 0; \\ L(P,f) - L(P_1,f) &= m_k(x_k - x_{k-1}) - [m_k'(y - x_{k-1}) \\ &\quad + m_k''(x_k - y)] \le 0. \end{aligned}$$

Therefore $U(P,f) \ge U(P_1,f)$ and $L(P,f) \le L(P_1,f)$.

If Q be any refinement of P then Q can be obtained from P by adjoining a finite number of additional points to P, one at a time.

By repeating the argument a finite number of times we have
$U(P,f) \ge U(Q,f)$; $L(P,f) \le L(Q,f)$.
This completes the proof.

Note. By the Lemma 11.2.1, it follows that
$$L(P,f) \le L(Q,f) \le U(Q,f) \le U(P,f).$$

11.4. Norm of a partition.

Let $[a,b]$ be a closed and bounded interval and $P = (x_0, x_1, \ldots, x_n)$ be a partition of $[a,b]$.

The interval $[a,b]$ is divided into n subintervals $[a,x_1]$, $[x_1,x_2]$, $\ldots$, $[x_{n-1},x_n]$. The norm of the partition P, denoted by $\|P\|$, is defined by
$$\|P\| = \max\{(x_1 - x_0), (x_2 - x_1), \ldots, (x_n - x_{n-1})\}.$$

In other words, $\|P\|$ is the maximum length of the subintervals into which $[a,b]$ is divided by the partition P.

If Q be a refinement of P then $\|Q\| \le \|P\|$. But the converse is not true.

For example, let $P = (0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1)$, $Q = (0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1)$ be two partitions of $[0,1]$. Then $\|P\| = \frac{1}{4}, \|Q\| = \frac{1}{6}$. Here $\|Q\| < \|P\|$ but Q is not a refinment of P.

Lemma 11.4.1. Let $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$ and P is a partition of $[a,b]$ with $\|P\| = \delta$. If P_k be a refinement of P with k additional points of partition, then
$$\begin{aligned} 0 &\le U(P,f) - U(P_k,f) \le (M-m)k\delta, \\ 0 &\le L(P_k,f) - L(P,f) \le (M-m)k\delta \end{aligned}$$

where $M = \sup_{x \in [a,b]} f(x), \quad m = \inf_{x \in [a,b]} f(x).$

Proof. Let $P = (x_0, x_1, \ldots, x_n)$. First we examine the effect of adjoining one additional point y to P and let $P_1 = (x_0, x_1, \ldots, x_{k-1}, y, x_k, \ldots, x_n)$.

Let $M_r = \sup_{x \in [x_{r-1}, x_r]} f(x), m_r = \inf_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

The subinterval $[x_{k-1}, x_k]$ is divided into two small subintervals $[x_{k-1}, y]$ and $[y, x_k]$.

Let $M'_k = \sup_{x \in [x_{k-1}, y]} f(x), m'_k = \inf_{x \in [x_{k-1}, y]} f(x),$

$M''_k = \sup_{x \in [y, x_k]} f(x), m''_k = \inf_{x \in [y, x_k]} f(x).$

Then $m \le m_k \le m'_k \le M'_k \le M_k \le M$;

$m \le m_k \le m''_k \le M''_k \le M_k \le M$.

$U(P, f) - U(P_1, f) = M_k(x_k - x_{k-1}) - [M'_k(y - x_{k-1}) + M''_k(x_k - y)]$
$= (M_k - M'_k)(y - x_{k-1}) + (M_k - M''_k)(x_k - y)$.

Since $0 \le M_k - M'_k \le M - m$ and $0 \le M_k - M''_k \le M - m$, it follows that $0 \le U(P, f) - U(P_1, f) \le (M - m)[y - x_{k-1}) + (x_k - y)]$

$\le (M - m)\delta$, since $(x_k - x_{k-1}) \le \delta$.

$L(P, f) - L(P_1, f) = m_k(x_k - x_{k-1}) - [m'_k(y - x_{k-1}) + m''_k(x_k - y)]$
$= (m_k - m'_k)(y - x_{k-1}) + (m_k - m''_k)(x_k - y)$.

Since $0 \le m'_k - m_k \le M - m$ and $0 \le m''_k - m_k \le M - m$, it follows that $0 \le L(P_1, f) - L(P, f) \le (M - m)[(y - x_{k-1}) + (x_k - y)]$

$\le (M - m)\delta$, since $(x_k - x_{k-1}) \le \delta$.

By introducing k additional points one by one in the partition P we obtain the partition P_k and it follows from above that

$0 \le U(P, f) - U(P_k, f) \le (M - m)k\delta,$

$0 \le L(P_k, f) - L(P, f) \le (M - m)k\delta.$

This proves the lemma.

Lemma 11.4.2. Let a function $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and P, Q be any two partitions of $[a, b]$. Then

$L(P, f) \le U(Q, f); L(Q, f) \le U(P, f).$

Proof. Let $S = P \cup Q$. Then the partition S is a refinement of P as well as a refinement of Q.

By Lemmas 11.2.1 and 11.3.1,

$L(P, f) \le L(S, f) \le U(S, f) \le U(P, f)$

and $L(Q, f) \le L(S, f) \le U(S, f) \le U(Q, f)$.

Therefore $L(P, f) \le L(S, f) \le U(S, f) \le U(Q, f)$

and $L(Q, f) \le L(S, f) \le U(S, f) \le U(P, f)$.

Note. The Lemma states that for any two partitions in $\mathcal{P}[a,b]$, the lower sum corresponding to one does not exceed the upper sum corresponding to the other.

Theorem 11.4.3. Let a function $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$. Then $\underline{\int}_a^b f \le \overline{\int}_a^b f$.

Proof. Let P, Q be any two partitions of $[a,b]$. Then $L(P,f) \le U(Q,f)$. Keeping Q fixed, this inequality holds for every partition $P \in \mathcal{P}[a,b]$.

Therefore $U(Q,f)$ is an upper bound of the set $\{L(P,f) : P \in \mathcal{P}[a,b]\}$. But the least upper bound of the set $\{L(P,f) : P \in \mathcal{P}[a,b]\}$ is $\underline{\int}_a^b f$. Therefore $U(Q,f) \ge \underline{\int}_a^b f$.

This holds for every partition $Q \in \mathcal{P}[a,b]$. Therefore $\underline{\int}_a^b f$ is a lower bound of the set $\{U(Q,f) : Q \in \mathcal{P}[a,b]\}$. But the greatest lower bound of the set $\{U(Q,f) : Q \in \mathcal{P}[a,b]\}$ is $\overline{\int}_a^b f$.

Therefore $\underline{\int}_a^b f \le \overline{\int}_a^b f$.

Note. It follows from Note 1 of 11.2 that
$$m(b-a) \le \underline{\int}_a^b f \le \overline{\int}_a^b f \le M(b-a).$$

Worked Example.

Prove that the function f defined on $[a,b]$ by $f(x) = x, x \in [a,b]$ is integrable on $[a,b]$ by showing that $\underline{\int}_a^b f = \overline{\int}_a^b f$. Evaluate $\int_a^b f$.

f is bounded on $[a,b]$.

Let us take the partition $P_n = (a, a+h, a+2h, \ldots, a+nh)$ where $nh = b-a$. Then P_n divides the interval $[a,b]$ into n subintervals of equal length.

Let $M_r = \sup\limits_{x \in [a+(r-1)h, a+rh]} f(x), m_r = \inf\limits_{x \in [a+(r-1)h, a+rh]} f(x)$, for $r = 1, 2, \ldots, n$.

Since f is an increasing function on $[a,b]$, $M_r = a + rh, m_r = a + (r-1)h$, for $r = 1, 2, \ldots, n$.

$$\begin{aligned}
U(P_n, f) &= h[(a+h) + (a+2h) + (a+nh)] \\
&= h[na + h(1 + 2 + \cdots + n)] = nha + \tfrac{nh(nh+h)}{2} \\
&= (b-a)a + \tfrac{1}{2}(b-a)^2(1 + \tfrac{1}{n}). \\
L(P_n, f) &= h[a + (a+h) + \cdots + (a + \overline{n-1}h)] \\
&= h[na + h(1 + 2 + \cdots + \overline{n-1}] = nha + \tfrac{nh(nh-h)}{2} . \\
&= (b-a)a + \tfrac{1}{2}(b-a)^2(1 - \tfrac{1}{n}).
\end{aligned}$$

$$\sup\{L(P_n, f)) : n \in \mathbb{N}\} = (b-a)a + \tfrac{(b-a)^2}{2} = \tfrac{b^2-a^2}{2}$$
and $\inf\{U(P_n, f)) : n \in \mathbb{N}\} = (b-a)a + \frac{(b-a)^2}{2} = \frac{b^2-a^2}{2}$.

[illegible] of $[a,b]$ such that [illegible] there exists a partition P of $[a,b]$ [illegible]

Let $f \in \mathcal{R}[a,b]$. Then $\underline{\int}_a^b f = \overline{\int}_a^b f$.

Proof. Let us choose $\epsilon > 0$.

Since $\underline{\int}_a^b f$ is the least upper bound of the set $\{L(P,f) : P \in \mathcal{P}[a,b]\}$, there exists a partititon P' of $[a,b]$ such that

$$\underline{\int}_a^b f - \frac{\epsilon}{2} < L(P',f) \leq \underline{\int}_a^b f.$$

Since $\overline{\int}_a^b f$ is the greatest lower bound of the set $\{U(P,f) : P \in \mathcal{P}[a,b]\}$, there exists a partititon P'' of $[a,b]$ such that

$$\overline{\int}_a^b f \leq U(P'',f) < \overline{\int}_a^b f + \frac{\epsilon}{2}.$$

Let $P = P' \cup P''$. Then P is a refinement of both P' and P''.
Therefore $L(P',f) \leq L(P,f)$ and $U(P,f) \leq U(P'',f)$.
Also we have $L(P,f) \leq U(P,f)$.
Combining, we have
$\underline{\int}_a^b f - \frac{\epsilon}{2} < L(P',f) \leq L(P,f) \leq U(P,f) \leq U(P'',f) < \overline{\int}_a^b f + \frac{\epsilon}{2}$.
Therefore $U(P,f) - L(P,f) < (\overline{\int}_a^b f + \frac{\epsilon}{2}) - (\underline{\int}_a^b f - \frac{\epsilon}{2})$

or, $U(P,f) - L(P,f) < \epsilon$, since $\overline{\int}_a^b f = \underline{\int}_a^b f$.

To prove the converse, we first observe that for any partition P of $[a,b]$, $L(P,f) \leq \underline{\int}_a^b f \leq \overline{\int}_a^b f \leq U(P,f)$.

Hence $\overline{\int}_a^b f - \underline{\int}_a^b f \leq U(P,f) - L(P,f)$.

Let us choose $\epsilon > 0$. By the condition there exists a partition P_ϵ of $[a,b]$ such that $U(P_\epsilon,f) - L(P_\epsilon,f) < \epsilon$.

We have $\overline{\int}_a^b f - \underline{\int}_a^b f \leq U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.
Also we have $\overline{\int}_a^b f \geq \underline{\int}_a^b f$, by Theorem 11.4.3.

Therefore $0 \leq \overline{\int}_a^b f - \underline{\int}_a^b f < \epsilon$. This holds for every positive ϵ.
It follows that $\overline{\int}_a^b f = \underline{\int}_a^b f$ and hence f is integrable on $[a, b]$.
This proves the theorem.

Theorem 11.4.5. (Darboux)

Let $[a, b]$ be a closed and bounded interval and a function $f[a, b] \to \mathbb{R}$ be bounded on $[a, b]$. Then

(i) to each pre-assigned positive ϵ there corresponds a positive δ such that $U(P, f) < \overline{\int}_a^b f + \epsilon$ for all partitions P of $[a, b]$ satisfying $||P|| \leq \delta$; and

(ii) to each pre-assigned positive ϵ there corresponds a positive δ such that $L(P, f) > \underline{\int}_a^b f - \epsilon$ for all partitions P of $[a, b]$ satisfying $||P|| \leq \delta$.

Proof. (i) Since f is bounded on $[a, b]$, there exists a positive real number B such that $| f(x) | \leq B$ for all $x \in [a, b]$.

Since $\overline{\int}_a^b f$ is the infimum of the set $\{U(P, f) : P \in \mathcal{P}[a, b]\}$, for a pre-assigned positive ϵ there exists a partition Q of $[a, b]$ such that

$$U(Q, f) < \overline{\int}_a^b f + \tfrac{\epsilon}{2}.$$

Let $Q = (x_0, x_1, x_2, \ldots, x_n)$ and let $\delta = \frac{\epsilon}{4B(n-1)}$.
Let P be any partition of $[a, b]$ such that $||P|| \leq \delta$.

Let $P' = P \cup Q$. Then P' is a refinement of P by adjoining $n - 1$ additional points $x_1, x_2, \ldots, x_{n-1}$ at most.

Therefore $0 \leq U(P, f) - U(P', f) \leq 2(n-1)B\delta$, by Lemma 11.4.1.
or, $U(P, f) \leq U(P', f) + \frac{\epsilon}{2}$.
Since P' is also refinement of Q, $U(P', f) \leq U(Q, f)$.
It follows that $U(P, f) \leq U(Q, f) + \frac{\epsilon}{2} < \overline{\int}_a^b f + \epsilon$.

(ii) Since f is bounded on $[a, b]$ there exists a positive real number B such that $| f(x) | \leq B$ for all $x \in [a, b]$.

Since $\underline{\int}_a^b f$ is the supremum of the set $\{L(P, f) : P \in \mathcal{P}[a, b]\}$, for a pre-assigned positive ϵ there exists a partition Q of $[a, b]$ such that

$$L(Q, f) > \underline{\int}_a^b f - \tfrac{\epsilon}{2}.$$

Let $Q = (x_0, x_1, x_2, \ldots, x_n)$ and let $\delta = \frac{\epsilon}{4B(n-1)}$.

Let P be any partition of $[a, b]$ such that $||P|| \leq \delta$.

Let $P' = P \cup Q$. Then P' is a refinement of P by adjoining $n - 1$ additional points $x_1, x_2, \ldots, x_{n-1}$ at most.

Therefore $0 \leq L(P', f) - L(P, f) \leq 2(n-1)B\delta$, by Lemma 11.4.1.
or, $L(P, f) \geq L(P', f) - \frac{\epsilon}{2}$.
Since P' is also refinement of Q, $L(P', f) \geq L(Q, f)$.
It follows that $L(P, f) \geq L(Q, f) - \frac{\epsilon}{2} > \underline{\int}_a^b f - \epsilon$.

This completes the proof.

Theorem 11.4.6. Let a function $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$. If $\{P_n\}$ be a sequence of partitions of $[a, b]$ such that the sequence $\{||P_n||\}$ converges to 0, then

(i) $\lim_{n\to\infty} U(P_n, f) = \overline{\int}_a^b f$; and (ii) $\lim_{n\to\infty} L(P_n, f) = \underline{\int}_a^b f$.

Proof. (i) Let us choose $\epsilon > 0$. Since f is bounded on $[a, b]$, by Darboux theorem there exists a positive δ such that $\overline{\int}_a^b f \leq U(P, f) < \overline{\int}_a^b f + \epsilon$ for all partitions P satisfying $||P|| < \delta$.

Since $\lim ||P_n|| = 0$, there exists a natural number k such that $||P_n|| < \delta$ for all $n \geq k$.

Therefore $\overline{\int}_a^b f \leq U(P_n, f) < \overline{\int}_a^b f + \epsilon$ for all $n \geq k$.

Hence the inequality $| U(P_n, f) - \overline{\int}_a^b f | < \epsilon$ holds for all $n \geq k$.

This implies $\lim_{n\to\infty} U(P_n, f) = \overline{\int}_a^b f$.

(ii) Similar proof.

Worked Examples.

1. A function f is defined by $f(x) = x^2, x \in [a, b]$, where $a > 0$. Find $\overline{\int}_a^b f$ and $\underline{\int}_a^b f$. Deduce that f is integrable on $[a, b]$.

f is bounded on $[a, b]$. Let $P_n = (a, a + h, a + 2h, \ldots, a + nh)$ where $h = \frac{b-a}{n}$. Then P_n is a partition of $[a, b]$ dividing $[a, b]$ into subintervals of equal length. $||P_n|| = \frac{b-a}{n}$.

Let $M_r = \sup_{x\in[a+(r-1)h, a+rh]} f(x)$, $m_r = \inf_{x\in[a+(r-1)h, a+rh]} f(x)$, for $r = 1, 2, \ldots, n$.

Since f is an increasing function on $[a, b]$,
$M_r = (a + rh)^2$, $m_r = (a + \overline{r-1}h)^2$, for $r = 1, 2, \ldots, n$.

$$\begin{aligned} U(P_n, f) &= h[(a+h)^2 + (a+2h)^2 + \cdots + (a+nh)^2] \\ &= h[na^2 + 2ah.\tfrac{n(n+1)}{2} + h^2.\tfrac{n(n+1)(2n+1)}{6}] \end{aligned}$$

$= nha^2 + a.nh(nh+h) + \frac{nh(nh+h)(2nh+h)}{6}$
$= (b-a)a^2 + a(b-a)^2(1+\frac{1}{n}) + \frac{1}{6}(b-a)^3(1+\frac{1}{n})(2+\frac{1}{n})$.

$L(P_n, f) = h[a^2 + (a+h)^2 + \cdots + (a+\overline{n-1}h^2]$
$= h[na^2 + 2ah.\frac{n(n-1)}{2} + h^2.\frac{n(n-1)(2n-1)}{6}]$
$= nha^2 + a.nh(nh-h) + \frac{nh(nh-h)(2nh-h)}{6}$
$= (b-a)a^2 + a(b-a)^2(1-\frac{1}{n}) + \frac{1}{6}(b-a)^3(1-\frac{1}{n})(2-\frac{1}{n})$.

Let us consider the sequence of partitions $\{P_n\}$ of $[a,b]$.
Here $\lim\limits_{n\to\infty} \|P_n\| = \lim \frac{b-a}{n} = 0$.
Then $\overline{\int_a^b} = \lim\limits_{n\to\infty} U(P_n, f), \underline{\int_a^b} f = \lim\limits_{n\to\infty} L(P_n, f)$.
Therefore $\overline{\int_a^b} f = (b-a)a^2 + a(b-a)^2 + \frac{(b-a)^3}{3} = \frac{b^3-a^3}{3}$,
and $\underline{\int_a^b} f = (b-a)a^2 + a(b-a)^2 + \frac{(b-a)^3}{3} = \frac{b^3-a^3}{3}$.
As $\overline{\int_a^b} f = \underline{\int_a^b} f$, f is integrable on $[a,b]$ and $\int_a^b f = \frac{b^3-a^3}{3}$.

2. A function f is defined on $[a,b]$ by $f(x) = e^x$. Find $\overline{\int_a^b} f$ and $\underline{\int_a^b} f$. Deduce that f is integrable on $[a,b]$.

f is bounded on $[a,b]$.

Let $P_n = (a+h, a+2h, \ldots, a+nh)$ where $nh = b-a$. Then P_n is a partition of $[a,b]$ dividing $[a,b]$ into n subintervals of equal length. $\|P_n\| = \frac{b-a}{n}$.

Let $M_r = \sup\limits_{x\in[a+(r-1)h, a+rh]} f(x)$, $m = \inf\limits_{x\in[a+(r-1)h, a+rh]} f(x)$, for $r = 1, 2, \ldots, n$.

Then $M_r = e^{a+rh}$, $m_r = e^{a+(r-1)h}$, for $r = 1, 2, \ldots, n$.

$U(P_n, f) = h[e^{a+h} + e^{a+2h} + \cdots + e^{a+nh}]$
$= h.e^{a+h}.\frac{e^{nh}-1}{e^h-1} = h.e^{a+h}.\frac{e^{b-a}-1}{e^h-1}$
$= \frac{h.e^h}{e^h-1}.(e^b - e^a)$;

$L(P_n, f) = h[e^a + e^{a+h} + \cdots + e^{a+(n-1)h}]$
$= h.e^a(\frac{e^{nh}-1}{e^h-1})$
$= \frac{h}{e^h-1}(e^b - e^a)$.

Let us consider the sequence of partitions $\{P_n\}$ of $[a,b]$. Here $\lim \|P_n\| = \lim \frac{b-a}{n} = 0$.

Then $\overline{\int_a^b} f = \lim\limits_{n\to\infty} U(P_n, f)$ and $\underline{\int_a^b} f = \lim\limits_{n\to\infty} L(P_n, f)$.

So $\overline{\int_a^b} f = \lim\limits_{n\to\infty} \dfrac{he^h}{e^h-1}(e^b - e^a) = e^b - e^a$

and $\underline{\int}_a^b f = \lim\limits_{n\to\infty} \dfrac{e^h}{e^h - 1}(e^b - e^a) = e^b - e^a$.

As $\overline{\int}_a^b f = \underline{\int}_a^b f$, f is integrable on $[a, b]$ and $\int_a^b = e^b - e^a$.

3. A function f is defined on $[0, 1]$ by $f(x) = x, x \in [0,1] \cap \mathbb{Q}$
$= 0, x \in [0,1] - \mathbb{Q}$.

Find $\underline{\int}_0^1 f$ and $\overline{\int}_0^1 f$. Deduce that f is not integrable on $[0, 1]$.

f is bounded on $[0, 1]$. Let us take the partition P_n of $[0, 1]$ defined by $P_n = (0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n})$.

Let $M_r = \sup\limits_{x\in[\frac{r-1}{n}, \frac{r}{n}]} f(x), m_r = \inf\limits_{x\in[\frac{r-1}{n}, \frac{r}{n}]} f(x)$, for $r = 1, 2, \ldots, n$.

Then $M_r = \frac{r}{n}, m_r = 0$, for $r = 1, 2, \ldots, n$.

$$U(P_n, f) = M_1(\tfrac{1}{n} - 0) + M_2(\tfrac{2}{n} - \tfrac{1}{n}) + \cdots + M_n(\tfrac{n}{n} - \tfrac{n-1}{n})$$
$$= \tfrac{1}{n}[\tfrac{1}{n} + \tfrac{2}{n} + \cdots + \tfrac{n}{n}] = \tfrac{n+1}{2n};$$
$$\text{and } L(P_n, f) = m_1(\tfrac{1}{n} - 0) + m_2(\tfrac{2}{n} - \tfrac{1}{n}) + \cdots + m_n(\tfrac{n}{n} - \tfrac{n-1}{n})$$
$$= 0, \text{ since each } m_r = 0.$$

Let us consider the sequence of partitions $\{P_n\}$ of $[0, 1]$. $||P_n|| = \frac{1}{n}$ and $\lim\limits_{n\to\infty} ||P_n|| = 0$.

Then $\lim\limits_{n\to\infty} U(P_n, f) = \overline{\int}_0^1 f$ and $\lim\limits_{n\to\infty} L(P_n, f) = \underline{\int}_0^1 f$.

Therefore $\overline{\int}_0^1 f = \lim\limits_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$ and $\underline{\int}_0^1 f = 0$.

Since $\underline{\int}_0^1 f \neq \overline{\int}_0^1 f$, f is not integrable on $[0, 1]$.

4. A function f is defined on $[0, 1]$ by

$$f(x) = x \text{ if } x \text{ be rational}$$
$$= x^2 \text{ if } x \text{ be irrational}.$$

Find $\underline{\int}_0^1 f$ and $\overline{\int}_0^1 f$. Deduce that f is not integrable on $[0, 1]$.

f is bounded on $[0, 1]$. For all $x \in (0, 1), x > x^2$.
Let $I = [0, 1]$. $f/(I \cap \mathbb{Q})$ is monotone increasing on $I \cap \mathbb{Q}$.
$f/(I - \mathbb{Q})$ is monotone increasing on $I - \mathbb{Q}$.

Let P_n be the partition of $[0, 1]$ defined by $P_n = (x_0, x_1, \ldots, x_n)$ where $x_0 = 0, x_r = \frac{r}{n}, r = 1, 2, \ldots, n$.

Let $M_r = \sup\limits_{x\in[x_{r-1}, x_r]} f(x), m_r = \inf\limits_{x\in[x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

Since $f/(I \cap \mathbb{Q})$ is monotone increasing on $[x_{r-1}, x_r] \cap \mathbb{Q}$,
$\sup\limits_{x\in[x_{r-1}, x_r]\cap\mathbb{Q}} f(x) = f(x_r) = \frac{r}{n}$.

Since $f/(I-\mathbb{Q})$ is monotone increasing on $[x_{r-1},x_r]-\mathbb{Q}$ and x_r is rational, $\sup\limits_{x\in[x_{r-1},x_r]-\mathbb{Q}} f(x) = \lim\limits_{n\to\infty} f(u_n)$, where $\{u_n\}$ is a sequence of irrational points in $[x_{r-1},x_r]$ converging to x_r

$$= x_r^2 = (\tfrac{r}{n})^2.$$

Since $\frac{r}{n} \geq (\frac{r}{n})^2$, $\sup\limits_{x\in[x_{r-1},x_r]} f(x) = \frac{r}{n}$. Hence $M_r = \frac{r}{n}$, for $r = 1,2,\ldots,n$.

Since $f/(I\cap\mathbb{Q})$ is monotone increasing on $[x_{r-1},x_r]\cap\mathbb{Q}$, $\inf\limits_{x\in[x_{r-1},x_r]} f(x) = f(x_{r-1}) = \frac{r-1}{n}$.

Since $f/(I-\mathbb{Q})$ is monotone increasing on $[x_{r-1},x_r]-\mathbb{Q}$ and x_{r-1} is rational, $\inf\limits_{x\in[x_{r-1},x_r]-\mathbb{Q}} f(x) = \lim\limits_{n\to\infty} f(v_n)$, where $\{v_n\}$ is a sequence of irrational points in $[x_{r-1},x_r]$ converging to x_{r-1}

$$= x_{r-1}^2 = (\tfrac{r-1}{n})^2.$$

Since $\frac{r-1}{n} \geq (\frac{r-1}{n})^2$, $\inf\limits_{x\in[x_{r-1},x_r]} f(x) = (\frac{r-1}{n})^2$. Hence $m_r = (\frac{r-1}{n})^2$, for $r = 1,2,\ldots,n$.

$$\begin{aligned} U(P_n,f) &= M_1(x_1-x_0)+M_2(x_2-x_1)+\cdots+M_n(x_n-x_{n-1}) \\ &= \tfrac{1}{n}[\tfrac{1}{n}+\tfrac{2}{n}+\cdots+\tfrac{n}{n}] = \tfrac{n+1}{2n}. \\ L(P_n,f) &= m_1(x_1-x_0)+m_2(x_2-x_1)+\cdots+m_n(x_n-x_{n-1}) \\ &= \tfrac{1}{n}[0+\tfrac{1^2}{n^2}+\tfrac{2^2}{n^2}+\cdots+\tfrac{(n-1)^2}{n^2}] = \tfrac{(n-1)(2n-1)}{6n^2}. \end{aligned}$$

Let us consider the sequence of partitions $\{P_n\}$ of $[0,1]$. $\|P_n\| = \frac{1}{n}$ and $\lim\limits_{n\to\infty}\|P_n\| = 0$.

Then $\overline{\int}_0^1 f = \lim\limits_{n\to\infty} U(P_n,f)$ and $\underline{\int}_0^1 f = \lim\limits_{n\to\infty} L(P_n,f)$.

So $\overline{\int}_0^1 f = \lim\limits_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2}$ and $\underline{\int}_0^1 f = \lim\limits_{n\to\infty}\frac{(n-1)(2n-1)}{6n^2} = \frac{1}{3}$.

Since $\overline{\int}_0^1 f \neq \underline{\int}_0^1 f$, f is not integrable on $[0,1]$.

Theorem 11.4.7. Another condition for integrability.

Let a function $f:[a,b]\to\mathbb{R}$ be bounded on $[a,b]$. Then f is integrable on $[a,b]$ if and only if for each $\epsilon>0$ there exists a positive δ such that

$$U(P,f)-L(P,f)<\epsilon$$

for every partition P of $[a,b]$ satisfying $\|P\|\leq\delta$.

Proof. Let $f\in\mathcal{R}[a,b]$. Then $\underline{\int}_a^b f = \overline{\int}_a^b f$.

Let us choose $\epsilon>0$.

Since f is bounded on $[a,b]$, by Darboux theorem, there corresponds a positive δ_1 such that $U(P, f) < \overline{\int}_a^b f + \frac{\epsilon}{2}$ for all partitions P of $[a, b]$ satisfying $||P|| \leq \delta_1$.

Also there corresponds a positive δ_2 such that $L(P, f) > \underline{\int}_a^b f - \frac{\epsilon}{2}$ for all partitions P of $[a, b]$ satisfying $||P|| \leq \delta_2$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $U(P, f) < \overline{\int}_a^b f + \frac{\epsilon}{2}$ and $L(P, f) > \underline{\int}_a^b f - \frac{\epsilon}{2}$ for all partitions P of $[a, b]$ satisfying $||P|| \leq \delta$.

So $U(P, f) - L(P, f) < \epsilon$ for all partitions P of $[a, b]$ satisfying $||P|| \leq \delta$.

To prove the converse, we first observe that for any partition P of $[a, b]$, $L(P, f) \leq \underline{\int}_a^b \leq \overline{\int}_a^b f \leq U(P, f)$.

That is, $\overline{\int}_a^b f - \underline{\int}_a^b f \leq U(P, f) - L(P, f)$ for any partiton P of $[a, b]$.

Let us choose $\epsilon > 0$. By the condition, there exists a positive δ such that for all partitions P of $[a, b]$ satisfying $||P|| \leq \delta, U(P, f) - L(P, f) < \epsilon$ holds.

Therefore exists a partition, say P_ϵ of $[a, b]$ such that $||P_\epsilon|| < \delta$ and $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.

Also we have $\overline{\int}_a^b f \geq \underline{\int}_a^b f$, by Theorem 11.4.3.

Therefore $0 \leq \overline{\int}_a^b f - \underline{\int}_a^b f \leq U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.

This holds for each positive ϵ. It follows that $\overline{\int}_a^b f = \underline{\int}_a^b f$ and hence f is integrable on $[a, b]$.

This completes the proof.

11.5. Some Riemann integrable functions.

Theorem 11.5.1. Let a function $f : [a, b] \to \mathbb{R}$ be monotone on $[a, b]$. Then f is integrable on $[a, b]$.

Proof. Let f be monotone increasing on $[a, b]$. Clearly, f is bounded on $[a, b]$, $f(a)$ being a lower bound and $f(b)$ being an upper bound of f on $[a, b]$. $f(b) - f(a) \geq 0$.

Let us choose $\epsilon > 0$.

Let P be a partition of $[a, b]$ with $||P|| < \epsilon/\{f(b) - f(a) + 1\}$. Let $P = (x_0, x_1, x_2, \ldots, x_n)$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

Then $M_r = f(x_r)$ and $m_r = f(x_{r-1})$, for $r = 1, 2, \ldots, n$.

$$\begin{aligned}\text{We have } U(P,f)-L(P,f) &= \sum_{r=1}^{n}(M_r-m_r)(x_r-x_{r-1})\\ &= \sum_{r=1}^{n}\{f(x_r)-f(x_{r-1})\}(x_r-x_{r-1})\\ &\le \|P\|\sum_{r=1}^{n}\{f(x_r)-f(x_{r-1})\}\\ &= \|P\|\{f(b)-f(a)\}<\epsilon.\end{aligned}$$

Therefore for a chosen positive ϵ, there exists a partition P of $[a,b]$ such that $U(P,f)-L(P,f)<\epsilon$.

This being a sufficient condition for integrability, f is integrable on $[a,b]$.

Proceeding in a similar manner it can be proved that if f be monotone decreasing on $[a,b]$, then f is integrable on $[a,b]$.

This completes the proof.

Theorem 11.5.2. Let a function $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$. Then f is integrable on $[a,b]$.

Proof. Since f is continuous on $[a,b]$, f is bounded on $[a,b]$. Since f is continuous on $[a,b]$, f is uniformly continuous on $[a,b]$.

Therefore for a chosen $\epsilon>0$, there exists a positive δ such that $|f(x')-f(x'')|<\frac{\epsilon}{b-a}$ for any two ponts x',x'' in $[a,b]$ satisfying $|x'-x''|<\delta$ ... ... (i)

Let P be a partition of $[a,b]$ with $\|P\|<\delta$. Let $P=(x_0,x_1,x_2,\ldots,x_n)$, where $a=x_0<x_1<x_2<\cdots<x_n=b$.

Let $M_r=\sup\limits_{x\in[x_{r-1},x_r]} f(x)$, $m_r=\inf\limits_{x\in[x_{r-1},x_r]} f(x)$, for $r=1,2,\ldots,n$.

Since f is continuous on $[x_{r-1},x_r]$ there exist points ξ_r,η_r in $[x_{r-1},x_r]$ such that $f(\xi_r)=M_r$, $f(\eta_r)=m_r$.

From (i) $|M_r-m_r|<\frac{\epsilon}{b-a}$, for $r=1,2,\ldots,n$.

$$\begin{aligned}\text{We have } U(P,f)-L(P,f) &= \sum_{r=1}^{n}(M_r-m_r)(x_r-x_{r-1})\\ &< \frac{\epsilon}{b-a}\sum_{r=1}^{n}(x_r-x_{r-1})=\epsilon.\end{aligned}$$

Therefore for a chosen positive ϵ there exists a partition P of $[a,b]$ such that $U(P,f)-L(P,f)<\epsilon$.

This being a sufficient condition for integrability, f integrable on $[a,b]$.
This completes the proof.

Note 1. The class of all functions continuous on $[a,b]$ is denoted by $C[a,b]$. It follows from the theorem that $f\in C[a,b]\Rightarrow f\in\mathcal{R}[a,b]$.

Note 2. We shall see that $C[a,b]$ is a proper subset of $\mathcal{R}[a,b]$. There are some discontinuous functions which belong to $\mathcal{R}[a,b]$.

Theorem 11.5.3. Let a function $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and let f be continuous $[a, b]$ except for a finite number of points in $[a, b]$. Then f is integrable on $[a, b]$.

Proof. Since f is bounded on $[a, b]$, there exists a positive real number k such that $| f(x) | < k$ for all $x \in [a, b]$. Let f be discontinuous at m points $x_1, x_2, \ldots, x_m$ in $[a, b]$ such that $x_1 < x_2 < \cdots < x_m$.

Case 1. Let $a < x_1 < x_2 < \cdots < x_m < b$.

Let $\epsilon > 0$. Let us enclose m points $x_1, x_2, \ldots, x_m$ in m non-overlapping subintervals$[x_1 - \frac{\delta_1}{2}, x_1 + \frac{\delta_1}{2}], \cdots, [x_m - \frac{\delta_m}{2}, x_m + \frac{\delta_m}{2}]$ of $[a, b]$ such that $a < x_1 - \frac{\delta_1}{2}, b > x_m + \frac{\delta_m}{2}$ and $\delta_1 + \delta_2 + \cdots + \delta_m < \frac{\epsilon}{4k}$.

Let $M^{(r)} = \sup\limits_{x \in [x_r - \frac{\delta_r}{2}, x_r + \frac{\delta_r}{2}]} f(x)$, $m^{(r)} = \inf\limits_{x \in [x_r - \frac{\delta_r}{2}, x_r + \frac{\delta_r}{2}]} f(x)$, for $r = 1, 2, \ldots, m$.

Then $M^{(r)} - m^{(r)} < 2k$, for $r = 1, 2, \ldots, m$.

f is continuous on the remaining $m + 1$ subintervals $[a, x_1 - \frac{\delta_1}{2}], [x_1 + \frac{\delta_1}{2}, x_2 - \frac{\delta_2}{2}], \cdots, [x_m + \frac{\delta_m}{2}, b]$.

Then there exist partitions P_1 of $[a, x_1 - \frac{\delta_1}{2}], P_2$ of $[x_1 + \frac{\delta_1}{2}, x_2 - \frac{\delta_2}{2}], \cdots, P_{m+1}$ of $[x_m + \frac{\delta_m}{2}, b]$ such that
$U(P_k, f) - L(P_k, f) < \frac{\epsilon}{2(m+1)}$, for $k = 1, 2, \ldots, m + 1$.

The partitions $P_1, P_2, \ldots, P_{m+1}$ are disjoint.

Let $P = P_1 \cup P_2 \cup \cdots \cup P_{m+1}$. Then P is a partition of $[a, b]$.

$U(P, f) - L(P, f) = [U(P_1, f) - L(P_1, f)] + [U(P_2, f) - L(P_2, f)] + \cdots + [U(P_{m+1}, f) - L(P_{m+1}, f)] + (M^{(1)} - m^{(1)})\delta_1 + (M^{(2)} - m^{(2)})\delta_2 + \cdots + (M^{(m)} - m^{(m)})\delta_m < \frac{\epsilon}{2(m+1)} \cdot (m + 1) + 2k(\delta_1 + \delta_2 \cdots + \delta_m) < \epsilon$,
since $\delta_1 + \delta_2 + \cdots + \delta_m < \frac{\epsilon}{4k}$.

Thus for a chosen positive ϵ there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$.

This being a sufficient condition for integrability, f is integrable on $[a, b]$.

Case 2. Either $a = x_1$, or $x_m = b$, or both.

If $a = x_1$, the subinterval enclosing the point x_1 can be chosen as $[a, a + \delta_1]$. If $x_m = b$, the last subinterval can be chosen as $[b - \delta_m, b]$. In any case, proceeding with similar arguments it can be proved that f is integrable on $[a, b]$.

This completes the proof.

Corollary. If $f : [a, b] \to \mathbb{R}$ be piecewise continuous on $[a, b]$ then f is integrable on $[a, b]$.

Theorem 11.5.4. Let $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$ and let f be continuous on $[a,b]$ except on a infinite subset $S \subset [a,b]$ such that the number of limit points of S is finite. Then f is integrable on $[a,b]$.

Proof. Since f is bounded on $[a,b]$, there exists a positive number k such that $| f(x) | < k$ for all $x \in [a,b]$. Let S' (the derived set of S) = $\{x_1, x_2, \ldots, x_m\}$ such that $x_1 < x_2 < \cdots < x_m$.

Case 1. Let $a < x_1 < x_2 < \cdots < x_m < b$.

Let us choose $\epsilon > 0$. Let the points $x_1, x_2, \ldots, x_m$ be enclosed by m non-overlapping subintervals $[x_1 - \frac{\delta_1}{2}, x_1 + \frac{\delta_1}{2}], [x_2 - \frac{\delta_2}{2}, x_2 + \frac{\delta_2}{2}], \cdots [x_m - \frac{\delta_m}{2}, x_m + \frac{\delta_m}{2}]$ of $[a,b]$ such that $a < x_1 - \frac{\delta_1}{2}, b > x_m + \frac{\delta_m}{2}$ and $\delta_1 + \delta_2 + \cdots + \delta_m < \frac{\epsilon}{4k}$.

Let $M^{(r)} = \sup\limits_{x \in [x_r - \frac{\delta_r}{2}, x_r + \frac{\delta_r}{2}]} f(x)$, $m^{(r)} = \inf\limits_{x \in [x_r - \frac{\delta_r}{2}, x_r + \frac{\delta_r}{2}]} f(x)$, for $r = 1, 2, \ldots, m$.

Then $M^{(r)} - m^{(r)} < 2k$, for $r = 1, 2, \ldots, m$.

On each of the remaining $m+1$ subintervals $[a, x_1 - \frac{\delta_1}{2}], [x_1 + \frac{\delta_1}{2}, x_2 - \frac{\delta_2}{2}], \cdots, [x_m + \frac{\delta_m}{2}, b]$, f is continuous except for a finite number of points. So f is integrable on each of these intervals, by Theorem 11.5.3.

Therefore there exist partitions P_1 of $[a, x_1 - \frac{\delta_1}{2}]$, P_2 of $[x_1 + \frac{\delta_1}{2}, x_2 - \frac{\delta_2}{2}], \cdots, P_{m+1}$ of $[x_m + \frac{\delta_m}{2}, b]$ such that
$U(P_k, f) - L(P_k, f) < \frac{\epsilon}{2(m+1)}$, for $k = 1, 2, \ldots, m+1$.

The partitions $P_1, P_2, \cdots, P_{m+1}$ are disjoint.

Let $P = P_1 \cup P_2 \cup \cdots \cup P_{m+1}$. Then P is a partition of $[a,b]$.

$$U(P,f) - L(P,f) = [U(P_1,f) - L(P_1,f)] + [U(P_2,f) - L(P_2,f)] +$$
$$\cdots + [U(P_{m+1},f) - L(P_{m+1},f)] + (M^{(1)} - m^{(1)})\delta_1 + (M^{(2)} - m^{(2)})\delta_2 +$$
$$\cdots + (M^{(m)} - m^{(m)})\delta_m < \frac{\epsilon}{2(m+1)} . (m+1) + 2k(\delta_1 + \delta_2 + \cdots \delta_m) < \epsilon.$$

Thus for a chosen positive ϵ there exists a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \epsilon$.

This being a sufficient condition for integrability, f is integrable on $[a,b]$.

Case 2. Either $a = x_1$, or $x_m = b$, or both.

If $a = x_1$, the subinterval enclosing the point x_1 can be chosen as $[a, a+\delta_1]$. If $x_m = b$, the last subinterval can be chosen as $[b - \delta_m, b]$. In any case, proceeding with similar arguments it can be proved that f is integrable on $[a,b]$.

This completes the proof.

Note. Since the set S is bounded and infinite, the derived set S' cannot be the null set, by Bolzano-Weierstrass theorem.

Examples.

1. Let $f(x) = \operatorname{sgn} x$, $x \in [-2, 2]$.

Then
$$\begin{aligned} f(x) &= -1, \ -2 \le x < 0. \\ &= 0, \ x = 0 \\ &= 1, \ 0 < x \le 2. \end{aligned}$$

f is bounded on $[-2, 2]$, since $|f(x)| \le 1$ for all $x \in [-2, 2]$. f is continuous on $[-2, 2]$ except at only one point, 0. Therefore f is integrable on $[-2, 2]$.

2. Let $f(x) = [x], x \in [0, 2]$.

$$\begin{aligned} f(x) &= 0, \ 0 \le x < 1 \\ &= 1, \ 1 \le x < 2 \\ &= 2, \ x = 2. \end{aligned}$$

f is bounded on $[0, 2]$, since $|f(x)| \le 2$ for all $x \in [0, 2]$. f is continuous on $[0, 2]$ except for the points 1, 2. So f is integrable on $[0, 2]$.

3. Let f be defined on $[0, 1]$ by

$f(0) = 0$ and

$f(x) = \frac{1}{2^{n-1}}, \frac{1}{2^n} < x \le \frac{1}{2^{n-1}}$, for $n = 1, 2, 3, \ldots$

f is monotone increasing and bounded on $[0, 1]$. Therefore f is integrable on $[0, 1]$.

4. Let f be defined on $[0, 1]$ by

$f(0) = 0$ and

$f(x) = (-1)^{r-1}, \ \frac{1}{r+1} < x \le \frac{1}{r}$, for $r = 1, 2, 3, \ldots$

f is continuous on $[0, 1]$ except at the points $0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ The set of points of discontinuity of f has only one limit point. Also f is bounded on $[0, 1]$. Therefore f is integrable on $[0, 1]$.

5. A function f is defined on $[0, 1]$ by $f(0) = 0$ and

$$\begin{aligned} f(x) &= 0, \text{ if } x \text{ be irrational} \\ &= \tfrac{1}{q}, \text{ if } x = \tfrac{p}{q} \text{ where } p, q \text{ are positive integers prime} \\ &\quad \text{to each other.} \end{aligned}$$

Show that f is integrable on $[0, 1]$ and $\int_0^1 f = 0$.

f is bounded on $[0, 1]$. Let us choose a positive ϵ such that $0 < \epsilon < 2$. Then there exists a natural number k such that $k < \frac{2}{\epsilon} \le k + 1$, by Archimedean property of $\mathbb{R}$. Let the rational numbers in $(0, 1)$ be arranged as

$$\tfrac{1}{1}; \ \tfrac{1}{2}; \ \tfrac{1}{3}, \tfrac{2}{3}; \ \tfrac{1}{4}, \tfrac{3}{4}; \ \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}; \ \ldots; \ \tfrac{1}{k}, \ldots, \tfrac{k-1}{k}; \ \tfrac{1}{k+1}, \ldots, \tfrac{k}{k+1}; \ \ldots$$

There are only a finite number of rational numbers of the form $\frac{p}{q}$ in $[0, 1]$ with denominator $\leq k$. At every such point $f(x) \geq \frac{1}{k} > \frac{\epsilon}{2}$; and at all other rational points in $[0, 1]$, $f(x) \leq \frac{\epsilon}{2}$.

Let the finite number of rational points for which $f(x) > \frac{\epsilon}{2}$ be $x_1, x_2, \cdots, x_m$ where $x_1 < x_2 < \cdots < x_m$.

Let us enclose the points by subintervals $[x_1 - \frac{\delta_1}{2}, x_1 + \frac{\delta_1}{2}]$, $[x_2 - \frac{\delta_2}{2}, x_2 + \frac{\delta_2}{2}], \cdots, [x_m - \frac{\delta_m}{2}, x_m + \frac{\delta_m}{2}]$ such that $\delta_1 + \delta_2 + \cdots + \delta_m < \frac{\epsilon}{2}$.

Since each of these subintervals contain rational as well as irrational points, the oscillation of f in each of these subintervals is less than 1.

Let $P = (0, x_1 - \frac{\delta_1}{2}, x_1 + \frac{\delta_1}{2}, x_2 - \frac{\delta_2}{2}, \cdots, x_m + \frac{\delta_m}{2}, 1)$. Then P is a partition of $[0, 1]$ dividing $[0, 1]$ into $2m + 1$ subintervals, m of which enclose the points $x_1, x_2, \ldots, x_m$. In each of the remaining $m + 1$ subintervals, the oscillation of f is less than $\frac{\epsilon}{2}$ and the sum of these $m + 1$ subintervals is less than 1.

So $U(P, f) - L(P, f) < 1.\frac{\epsilon}{2} + \frac{\epsilon}{2}.1. = \epsilon$.

Therefore there exists a partition P of $[0, 1]$ such that $U(P, f) - L(P, f) < \epsilon$.

This being a sufficient condition for integrability, f is integrable on $[0, 1]$.

Let $P = (x_0, x_1, \ldots, x_n)$, where $0 = x_0 < x_1 < \cdots < x_n = 1$ be an arbitrary partition of $[0, 1]$. Let $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, $r = 1, 2, \ldots, n$.

Since every subinterval $[x_{r-1}, x_r]$ contains irrational points, $m_r = 0$ for $r = 1, 2, \ldots, n$. Therefore $L(P, f) = 0$.

Consequently, $\underline{\int}_0^1 f = \sup\{L(P, f) : P \in \mathcal{P}[a, b]\} = 0$.

Since $f \in \mathcal{R}[0, 1]$, $\underline{\int}_0^1 f = \int_0^1 f$ and therefore $\int_0^1 f = 0$.

Note. This function f is continuous at 0 and at every irrational point in $[0, 1]$ and discontinuous at every every non-zero rational point in $[0, 1]$. This example shows that a function bounded and continuous on a closed and bounded interval $[a, b]$ except for an infinite set of points $S \subset [a, b]$ having infinite number of limit points, is also $\mathcal{R}$- integrable on $[a, b]$.

Remarks.

We have seen that a bounded function f

(i) continuous on a closed interval $[a, b]$ is integrable on $[a, b]$;

(ii) continuous on $[a, b]$ except at a finite number of points of discontinuity is integrable on $[a, b]$;

(iii) continuous on $[a, b]$ with infinite number of points of discontinuity, is integrable on $[a, b]$ provided the set of points of discontinuity has only a finite number of limit points.

The reader may think that (iii) describes the most general type of bounded discontinuous functions f that are integrable on $[a, b]$. That this is not true, is established by the worked out Example 5 of 11.5.

In this respect there is a theorem of Lebesgue that a necesary and sufficient condition for a function bounded on $[a, b]$ to be Riemann integrable on $[a, b]$ is that the set of points of discontinuity of f is a set of measure zero.

Definition. A set $S \subset \mathbb{R}$ is said to be a *set of measure zero* if for each $\epsilon > 0$ there exists a countable collection of open intervals $\{I_n\}$ such that $S \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} |I_n| < \epsilon$.

Some examples of a set of measure zero.

1. A finite set $S \subset \mathbb{R}$ is a set of measure zero.

Let $S = \{x_1, x_2, \ldots, x_m\} \subset \mathbb{R}$.

Let $\epsilon > 0$. Let x_r be enclosed by the open interval $I_r = (x_r - \frac{\epsilon}{2(m+1)}, x_r + \frac{\epsilon}{2(m+1)})$, for $r = 1, 2, \ldots, m$. Then $S \subset \bigcup_{r=1}^{m} I_r$ and $|I_1| + |I_2| + \cdots + |I_m| = \frac{m\epsilon}{m+1} < \epsilon$.

Therefore S is covered by a finite collection of open intervals such that the sum of the length of the intervals is less than ϵ, proving that S is a set of measure zero.

2. An enumerable subset of $\mathbb{R}$ is a set of measure zero.

Let S be an enumerable subset of $\mathbb{R}$. The elements of S can be described as $x_1, x_2, x_3, \ldots$

Let $\epsilon > 0$. For each r, let x_r be enclosed by the open interval $I_r = (x_r - \frac{\epsilon}{2^{r+2}}, x_r + \frac{\epsilon}{2^{r+2}})$. Then $S \subset \bigcup_{r=1}^{\infty} I_r$.

$$\begin{aligned} \text{Now } |I_1| + |I_2| + |I_3| + \cdots &= \tfrac{\epsilon}{2^2} + \tfrac{\epsilon}{2^3} + \tfrac{\epsilon}{2^4} + \cdots \\ &= \tfrac{\epsilon}{2^2}[1 + \tfrac{1}{2} + \tfrac{1}{2^2} + \cdots] \\ &= \tfrac{\epsilon}{2^2}.2 = \tfrac{\epsilon}{2} < \epsilon. \end{aligned}$$

Therefore S is covered by a countable number of open intervals I_n such that the sum of their lengths is less than ϵ.

This proves that S is a set of measure zero.

Corollary. Since $\mathbb{Q}$ is an ennumerable set, the set $\mathbb{Q}$ is a set of measure zero.

3. Let S be a bounded infinite subset of $\mathbb{R}$ having a finite number of limit points. Then S is a set of measure zero.

Let the limit points of S be $x_1, x_2, \ldots, x_m$.

Let the points $x_1, x_2, \ldots, x_m$ be enclosed by m disjoint open intervals $(x_1 - \frac{\delta_1}{2}, x_1 + \frac{\delta_1}{2}), (x_2 - \frac{\delta_2}{2}, x_2 + \frac{\delta_2}{2}), \cdots, (x_m - \frac{\delta_m}{2}, x_m + \frac{\delta_m}{2})$ such that $\delta_1 + \delta_2 + \cdots + \delta_m < \frac{\epsilon}{2}$.

Outside these open intervals there lie at most a finite number of points of S and those can be covered by a finite number of open intervals, the sum of whose lengths is less than $\frac{\epsilon}{2}$.

Considering the two finite families of open intervals, S is covered by a countable collection of open intervals, the sum of whose lengths is less than ϵ.

Therefore S is a set of measure zero.

11.6. Properties of Riemann integrable functions.

Theorem 11.6.1. Let $f : [a, b] \to \mathbb{R}, g : [a, b] \to \mathbb{R}$ be both integrable on $[a, b]$. Then $f + g$ is integrable on $[a, b]$ and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

Proof. Since $f \in \mathcal{R}[a, b]$ and $g \in \mathcal{R}[a, b]$, f and g are both bounded on $[a, b]$. Therefore there exist positive real numbers k_1, k_2 such that $| f(x) | < k_1$ and $| g(x) | < k_2$ for all $x \in [a, b]$.

$| f(x) + g(x) | \leq | f(x) | + | g(x) | < k_1 + k_2$ for all $x \in [a, b]$.

This shows that $f + g$ is bounded on $[a, b]$.

Let us choose $\epsilon > 0$.

Since $f \in \mathcal{R}[a, b]$, there exists a partition P_1 of $[a, b]$ such that

$$U(P_1, f) - L(P_1, f) < \tfrac{\epsilon}{2}.$$

Since $g \in \mathcal{R}[a, b]$, there exists a partition P_2 of $[a, b]$ such that

$$U(P_2, g) - L(P_2, g) < \tfrac{\epsilon}{2}.$$

Let $P_0 = P_1 \cup P_2$. Then P_0 is a refinement of P_1 as well as of P_2 and $L(P_1, f) \leq L(P_0, f) \leq U(P_0, f) \leq U(P_1, f)$;

$$L(P_2, g) \leq L(P_0, g) \leq U(P_0, g) \leq U(P_2, g).$$

So $U(P_0, f) - L(P_0, f) \leq U(P_1, f) - L(P_1, f) < \frac{\epsilon}{2}$
and $U(P_0, g) - L(P_0, g) \leq U(P_2, g) - L(P_2, g) < \frac{\epsilon}{2}$.

Let $P_0 = (x_0, x_1, \ldots, x_n)$, where $a = x_0 < x_1 < \cdots < x_n = b$.

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} (f + g)(x),\ m_r = \inf\limits_{x \in [x_{r-1}, x_r]} (f + g)(x)$

$M'_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x),\ m'_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$

$M''_r = \sup\limits_{x \in [x_{r-1}, x_r]} g(x),\ m''_r = \inf\limits_{x \in [x_{r-1}, x_r]} g(x)$, for $r = 1, 2, \ldots, n$.

Then $M_r \leq M'_r + M''_r,\ m_r \geq m'_r + m''_r$, for $r = 1, 2, \ldots, n$.

$$U(P_0, f + g) = M_1(x_1 - x_0) + \cdots + M_n(x_n - x_{n-1})$$
$$\leq [M'_1(x_1 - x_0) + \cdots + M'_n(x_n - x_{n-1})]$$

$$+[M_1''(x_1 - x_0) + \cdots + M_n''(x_n - x_{n-1})]$$
$$= U(P_0, f) + U(P_0, g).$$

Similarly, $L(P_0, f+g) \geq L(P_0, f) + L(P_0, g)$.

Hence $U(P_0, f+g) - L(P_0, f+g) \leq [U(P_0, f) - L(P_0, f)] + [U(P_0, g) - L(P_0, g)] < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Therefore for a chosen $\epsilon > 0$ there exists a partition P_0 of $[a, b]$ such that $U(P_0, f+g) - L(P_0, f+g) < \epsilon$.

This being a sufficient condition for integrability, $f + g$ is integrable on $[a, b]$.

Second part.

Let $\epsilon > 0$. Since $f \in \mathcal{R}[a, b]$, there exists a partition P_1 of $[a, b]$ such that $U(P_1, f) < \int_a^b f + \frac{\epsilon}{2}$.

Since $g \in \mathcal{R}[a, b]$, there exists a partition P_2 of $[a, b]$ such that $U(P_2, g) < \int_a^b g + \frac{\epsilon}{2}$.

Let $P_0 = P_1 \cup P_2$. Then P_0 is a refinement of P_1 and P_2 and $U(P_0, f) \leq U(P_1, f) < \int_a^b f + \frac{\epsilon}{2}$; $U(P_0, g) \leq U(P_2, g) < \int_a^b g + \frac{\epsilon}{2}$.

Therefore $U(P_0, f+g) \leq U(P_0, f) + U(P_0, g) < \int_a^b f + \int_a^b g + \epsilon$.

Since $f + g$ is integrable on $[a, b]$, $\int_a^b (f+g) \leq U(P_o, f+g)$.

Hence $\int_a^b (f+g) < \int_a^b f + \int_a^b g + \epsilon$. ... ...(i)

Considering the lower sums, by similar arguments we have

$\int_a^b (f+g) > \int_a^b f + \int_a^b g - \epsilon$. ... ...(ii)

From (i) and (ii) we have $| \int_a^b (f+g) - \int_a^b f - \int_a^b g | < \epsilon$.

This holds for every positive ϵ. Therefore $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

Note. For a finite number of functions $f_1, f_2, \ldots, f_n$ each integrable on $[a, b]$, $f_1 + f_2 + \cdots + f_n$ is integrable on $[a, b]$ and

$$\int_a^b (f_1 + f_2 + \cdots + f_n) = \int_a^b f_1 + \int_a^b f_2 + \cdots + \int_a^b f_n.$$

Theorem 11.6.2. Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and $c \in \mathbb{R}$. Then cf is integrable on $[a, b]$ and $\int_a^b cf = c \int_a^b f$.

Proof. Since $f \in \mathcal{R}[a, b]$, f is bounded on $[a, b]$. Therefore cf is bounded on $[a, b]$.

Case 1. $c = 0$.

$cf(x) = 0$ for all $x \in [a, b]$. cf is integrable on $[a, b]$ and $\int_a^b cf = 0$ and therefore $\int_a^b cf = c \int_a^b f$.

Case 2. $c > 0$.

$\underline{\int}_a^b f =$ the supremum of the set $\{L(P, f) : P \in \mathcal{P}[a,b]\}$.

$\overline{\int}_a^b f =$ the infimum of the set $\{U(P, f) : P \in \mathcal{P}[a,b]\}$.

$$\begin{aligned}\underline{\int}_a^b cf &= \text{the supremum of the set } \{L(P, cf) : P \in \mathcal{P}[a,b]\}\\ &= \text{the supremum of the set } \{cL(P, f) : P \in \mathcal{P}[a,b]\}\\ &= c. \text{ the supremum of the set } \{L(P, f) : P \in \mathcal{P}[a,b]\}\\ &= c.\underline{\int}_a^b f.\end{aligned}$$

$$\begin{aligned}\overline{\int}_a^b cf &= \text{the infimum of the set } \{U(P, cf) : P \in \mathcal{P}[a,b]\}\\ &= \text{the infimum of the set } \{cU(P, f) : P \in \mathcal{P}[a,b]\}\\ &= c. \text{ the infimum of the set } \{U(P, f) : P \in \mathcal{P}[a,b]\}\\ &= c.\overline{\int}_a^b f.\end{aligned}$$

Since $f \in \mathcal{R}[a,b]$, $\underline{\int}_a^b f = \overline{\int}_a^b f = \int_a^b f$.

Hence $\underline{\int}_a^b cf = \overline{\int}_a^b cf = c\int_a^b f$.

This shows that cf is integrable on $[a,b]$ and $\int_a^b cf = c\int_a^b f$.

Case 3. $c < 0$.

Similar proof.

Note. The two theorems 11.6.1 and 11.6.2 together establish that Riemann integrals satisfy linearity property.

Theorem 11.6.3. Let a function $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Then $|f|$ is integrable on $[a,b]$.

Proof. Since $f \in \mathcal{R}[a,b]$, f is bounded on $[a,b]$. Therefore there exists a positive real number k such that $|f(x)| < k$ for all $x \in [a,b]$.

Now $|\,|f|(x)\,| = |f|(x) = |f(x)| < k$ for all $x \in [a,b]$. This shows that $|f|$ is bounded on $[a,b]$.

Let us choose $\epsilon > 0$.

Since f is integrable on $[a,b]$, there exists a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \epsilon$.

Let $P = (x_0, x_1, \ldots, x_n)$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

Let $M_r = \sup\limits_{x\in[x_{r-1},x_r]} f(x)$, $m_r = \inf\limits_{x\in[x_{r-1},x_r]} f(x)$,

$M'_r = \sup\limits_{x\in[x_{r-1},x_r]} |f|(x)$, $m'_r = \inf\limits_{x\in[x_{r-1},x_r]} |f|(x)$, for $r = 1, 2, \ldots, n$.

For any two points α, β in $[x_{r-1}, x_r]$, we have

$|\,|f|(\alpha) - |f|(\beta)\,| = |\,|f(\alpha)| - |f(\beta)|\,| \leq |f(\alpha) - f(\beta)|$ ... ... (i)

We use here an important property of a bounded subset of $\mathbb{R}$.

If S be a non-empty bounded subset of $\mathbb{R}$ with sup $S = M$ and inf $S = m$, then the supremum of the set $\{|x-y| : x \in S, y \in S\}$ is $M - m$. [worked Ex.5, page 33]

Since f is bounded on $[x_{r-1}, x_r]$ with $\sup\limits_{x\in[x_{r-1},x_r]} f(x) = M_r$ and $\inf\limits_{x\in[x_{r-1},x_r]} f(x) = m_r$, the supremum of the set $\{|f(\alpha) - f(\beta)| : \alpha \in [x_{r-1}, x_r], \beta \in [x_{r-1}, x_r]\}$ is $M_r - m_r$.

Since $|f|$ is bounded on $[x_{r-1}, x_r]$ with $\sup\limits_{x\in[x_{r-1},x_r]} |f|(x) = M'_r$ and $\inf\limits_{x\in[x_{r-1},x_r]} |f|(x) = m'_r$, the supremum of the set $\{|\,|f|(\alpha) - |f|(\beta)\,| : \alpha \in [x_{r-1}, x_r], \beta \in [x_{r-1}, x_r]\}$ is $M'_r - m'_r$.

From the inequality (i) it follows that $M_r - m_r$ is an upper bound of the set $\{|\,|f|(\alpha) - |f|(\beta)\,| : \alpha \in [x_{r-1}, x_r], \beta \in [x_{r-1}, x_r]\}$.

Therefore $M'_r - m'_r \leq M_r - m_r$. This holds for $r = 1, 2, \ldots, n$.

So $U(P, |f|) - L(P, |f|)$
$= (M'_1 - m'_1)(x_1 - x_0) + \cdots + (M'_n - m'_n)(x_n - x_{n-1})$
$\leq (M_1 - m_1)(x_1 - x_0) + \cdots + (M_n - m_n)(x_n - x_{n-1})$
$= U(P, f) - L(P, f) < \epsilon$.

This being a sufficient condition for integrability, $|f|$ is integrable on $[a, b]$. This completes the proof.

Note. The converse of the theorem is not true. For example,
let $f : [a, b] \to \mathbb{R}$ be defined by
$$f(x) = 1, x \in [a, b] \cap \mathbb{Q}$$
$$= -1, x \in [a, b] - \mathbb{Q}.$$

Then f is not integrable on $[a, b]$.
But $|f|(x) = 1$ for all $x \in [a, b]$ and $|f|$ is integrable on $[a, b]$.

Theorem 11.6.4. Let a function $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. Then f^2 is integrable on $[a, b]$

Proof. Since $f \in \mathcal{R}[a, b]$, f is bounded on $[a, b]$. Therefore there exists a positive real number k such that $|f(x)| \leq k$ for all $x \in [a, b]$. So $|f^2(x)| \leq k^2$ for all $x \in [a, b]$. This shows that f^2 is bounded on $[a, b]$.

Let us choose $\epsilon > 0$.

Since f is integrable on $[a, b]$, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \frac{\epsilon}{2k}$.

Let $P = (x_0, x_1, x_2, \ldots, x_n)$, where $a = x_0 < x_1 < \cdots < x_n = b$.
Let $M_r = \sup\limits_{x\in[x_{r-1},x_r]} f(x)$, $m_r = \inf\limits_{x\in[x_{r-1},x_r]} f(x)$,

$M'_r = \sup\limits_{x\in[x_{r-1},x_r]} f^2(x)$, $m'_r = \inf\limits_{x\in[x_{r-1},x_r]} f^2(x)$, for $r = 1, 2, \ldots, n$.

For any two points α, β in $[x_{r-1}, x_r]$, we have

$$\begin{aligned} | f^2(\alpha) - f^2(\beta) | &= | \{f(\alpha)\}^2 - \{f(\beta)\}^2 | \\ &= | f(\alpha) + f(\beta) || f(a) - f(\beta) | \\ &\leq 2k \mid f(\alpha) - f(\beta) \mid \ldots \quad \ldots \text{(i)} \end{aligned}$$

Since f is bounded on $[x_{r-1}, x_r]$ with $\sup\limits_{x\in[x_{r-1},x_r]} f(x) = M_r$ and $\inf\limits_{x\in[x_{r-1},x_r]} f(x) = m_r$, the supremum of the set $\{| f(\alpha) - f(\beta) |: \alpha \in [x_{r-1}, x_r], \beta \in [x_{r-1}, x_r]\}$ is $M_r - m_r$. [worked Ex.5, page 33]

Since f^2 is bounded on $[x_{r-1}, x_r]$ with $\sup\limits_{x\in[x_{r-1},x_r]} f^2(x) = M'_r$ and $\inf\limits_{x\in[x_{r-1},x_r]} f^2(x) = m'_r$, the supremum of the set $\{| f^2(\alpha) - f^2(\beta) |: \alpha \in [x_{r-1}, x_r], \beta \in [x_{r-1}, x_r]\} = M'_r - m'_r$. [worked Ex. 5, page 32]

It follows from the inequality (i) that $2k(M_r - m_r)$ is an upper bound of the set $\{| f^2(\alpha) - f^2(\beta) |: \alpha, \beta \in [x_{r-1}, x_r]\}$.

Therefore $M'_r - m'_r \leq 2k(M_r - m_r)$. This holds for $r = 1, 2, \ldots, n$.

$$\begin{aligned} U(P, f^2) - L(P, f^2) &= \sum_{r=1}^{n} (M'_r - m'_r)(x_r - x_{r-1}) \\ &\leq 2k \sum_{r=1}^{n} (M_r - m_r)(x_r - x_{r-1}) \\ &= 2k[U(P, f) - L(P, f)] < \epsilon. \end{aligned}$$

This being a sufficient condition for integrability, f^2 is integrable on $[a, b]$. This completes the proof.

Theorem 11.6.5. Let the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be both integrable on $[a, b]$. Then fg is integrable on $[a, b]$.

Proof. Since $f \in \mathcal{R}[a, b]$, f is bounded on $[a, b]$. Since $g \in \mathcal{R}[a, b]$, g is bounded on $[a, b]$. Therefore fg is bounded on $[a, b]$.

Now $fg = \frac{1}{2}(f + g)^2 - \frac{1}{2}f^2 - \frac{1}{2}g^2$.

Since $f \in \mathcal{R}[a, b]$ and $g \in \mathcal{R}[a, b]$, $\frac{1}{2}(f + g)^2, \frac{1}{2}f^2$ and $\frac{1}{2}g^2$ are all integrable on $[a, b]$ by Theorems 11.6.1, 11.6.2 and 11.6.4.

Hence fg is integrable on $[a, b]$. This completes the proof.

Theorem 11.6.6. Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. If there exists a positive real number k such that such that $f(x) \geq k$ for all $x \in [a, b]$ then $\frac{1}{f}$ is integrable on $[a, b]$.

Proof. $|\frac{1}{f}(x)| = |\frac{1}{f(x)}| \leq \frac{1}{k}$ for all $x \in [a, b]$. This shows that $\frac{1}{f}$ is bounded on $[a, b]$.

Let us choose $\epsilon > 0$.

Since f is integrable on $[a,b]$ there exists a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < k^2\epsilon$.

Let $P = (x_0, x_1, \ldots, x_n)$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

Let $M_r = \sup\limits_{x\in[x_{r-1},x_r]} f(x)$, $m_r = \inf\limits_{x\in[x_{r-1},x_r]} f(x)$;

$M'_r = \sup\limits_{x\in[x_{r-1},x_r]} \frac{1}{f}(x)$, $m'_r = \inf\limits_{x\in[x_{r-1},x_r]} \frac{1}{f}(x)$, for $r = 1, 2, \ldots, n$.

For any two points α, β in $[x_{r-1}, x_r]$, we have

$|\frac{1}{f}(\alpha) - \frac{1}{f}(\beta)| = |\frac{1}{f(\alpha)} - \frac{1}{f(\beta)}| = \frac{1}{|f(\alpha)|\,|f(\beta)|}|f(\alpha) - f(\beta)|$

$\leq \frac{1}{k^2}|f(\alpha) - f(\beta)|$ ... ... (i)

Since f is bounded on $[x_{r-1}, x_r]$ with $\sup\limits_{x\in[x_{r-1},x_r]} f(x) = M_r$ and $\inf\limits_{x\in[x_{r-1},x_r]} f(x) = m_r$, the supremum of the set $\{|f(\alpha) - f(\beta)| : \alpha \in [x_{r-1}, x_r], \beta \in [x_{r-1}, x_r]\}$ is $M_r - m_r$. [worked Ex.5, page 33]

Since $\frac{1}{f}$ is bounded on $[x_{r-1}, x_r]$ with $\sup\limits_{x\in[x_{r-1},x_r]} \dfrac{1}{f}(x) = M'_r$ and $\inf\limits_{x\in[x_{r-1},x_r]} \dfrac{1}{f}(x) = m'_r$, the supremum of the set $\{| \frac{1}{f}(\alpha) - \frac{1}{f}(\beta) | : \alpha \in [x_{r-1}, x_r], \beta \in [x_{r-1}, x_r]\} = M'_r - m'_r$.

It follows from the inequality (i) that $\frac{1}{k^2}(M_r - m_r)$ is an upper bound of the set $\{| \frac{1}{f}(\alpha) - \frac{1}{f}(\beta) | : \alpha, \beta \in [x_{r-1}, x_r]\}$.

Therefore $M'_r - m'_r \leq \frac{1}{k^2}(M_r - m_r)$. This holds for $r = 1, 2, \ldots, n$.

$$\begin{aligned} U(P, \tfrac{1}{f}) - L(P, \tfrac{1}{f}) &= \sum_{r=1}^{n} (M'_r - m'_r)(x_r - x_{r-1}) \\ &\leq \sum_{r=1}^{n} \frac{1}{k^2}(M_r - m_r)(x_r - x_{r-1}) \\ &= \tfrac{1}{k^2}[U(P,f) - L(P,f)] < \epsilon. \end{aligned}$$

Therefore for a chosen $\epsilon > 0$ there exists a partition P of $[a,b]$ such that $U(P, \frac{1}{f}) - L(P, \frac{1}{f}) < \epsilon$.

This being a sufficient condition for integrability, $\frac{1}{f}$ is integrable on $[a,b]$. This completes the proof.

Note 1. If $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and $f(x) > 0$ for all $x \in [a,b]$, then $\frac{1}{f}$ may not be integrable on $[a,b]$. For example, let

$$f : [0,1] \to \mathbb{R} \text{ be defined by } f(x) \begin{aligned}[t] &= x, 0 < x \leq 1 \\ &= 1, x = 0. \end{aligned}$$

Then f is bounded on $[0,1]$, f is continuous on $[0,1]$ except at only one point, 0. So f is integrable on $[0,1]$. Also $f(x) > 0$ for all $x \in [0,1]$.

$\frac{1}{f}$ is unbounded on $[0,1]$ and therefore $\frac{1}{f}$ is not integrable on $[0,1]$.

Note 2. If however, $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and $f(x) > 0$ for all $x \in [a, b]$, then there exists a positive real number k such that $f(x) \geq k$ for all $x \in [a, b]$. [Ex.6, Exercises 13.]

In this case, $\frac{1}{f}$ is integrable on $[a, b]$, by the theorem.

Theorem 11.6.7. Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be both integrable on $[a, b]$ and there exists a positive real number k such that $g(x) \geq k$ for all $x \in [a, b]$. Then f/g is integrable on $[a, b]$.

Proof. Since $f \in \mathcal{R}[a, b]$, f is bounded on $[a, b]$. Therefore there exists a positive real number B such that $| f(x) | < B$ for all $x \in [a, b]$.
Since $g(x) \geq k$ for all $x \in [a, b]$, $\frac{1}{g(x)} \leq \frac{1}{k}$ for all $x \in [a, b]$.

Therefore $| \frac{f}{g}(x) | = | \frac{f(x)}{g(x)} | < \frac{B}{k}$ for all $x \in [a, b]$. This shows that f/g is bounded on $[a, b]$.

Since $g \in \mathcal{R}[a, b]$ and $g(x) \geq k > 0$ for all $x \in [a, b]$, $\frac{1}{g} \in \mathcal{R}[a, b]$.

Since f and $\frac{1}{g}$ are both integrable on $[a, b]$, $\frac{f}{g}$ is integrable on $[a, b]$ by Theorem 11.6.5.

Theorem 11.6.8. Let $I = [a, b] \subset \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and $J = [c, d] \subset \mathbb{R}$ such that $f(I) \subset J$ and $\phi : [c, d] \to \mathbb{R}$ be continuous on $[c, d]$. Then the composite function $\phi \circ f$ is integrable on $[a, b]$.

Proof. Since ϕ is continuous on $[c, d]$, ϕ is bounded on $[c, d]$ and therefore there exists a positive real number k such that $| \phi(t) | \leq k$ for all $t \in [c, d]$.

Let us choose $\epsilon > 0$. Since ϕ is continuous on $[c, d]$, ϕ is uniformly continuous on $[c, d]$ and therefore there exists a positive δ such that for all $s, t \in [c, d]$, $| s - t | < \delta \Rightarrow | \phi(s) - \phi(t) | < \frac{\epsilon}{2(b-a)}$ ... ... (i)

Since f is integrable on $[a, b]$, there exists a partition $P = (a = x_0, x_1, \cdots, x_n = b)$ of $[a, b]$ such that $U(P, f) - L(P, f) < \frac{\epsilon}{4k}.\delta$ ... (ii)

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$;

$M'_r = \sup\limits_{x \in [x_{r-1}, x_r]} \phi \circ f(x)$, $m'_r = \inf\limits_{x \in [x_{r-1}, x_r]} \phi \circ f(x)$, for $r = 1, 2, \ldots, n$.

Let us consider two subsets A and B of the set $S = \{1, 2, \ldots, n\}$ (the indices of the points of partition P), where $A = \{r : M_r - m_r < \delta\}$, $B = \{r : M_r - m_r \geq \delta\}$.

Let $r \in A$. Then for $x, y \in [x_{r-1}, x_r]$, $| f(x) - f(y) | < \delta$ and by (i) this implies $| \phi(f(x)) - \phi(f(y)) | < \frac{\epsilon}{2(b-a)}$.

Therefore if $r \in A$, $M'_r - m'_r \leq \frac{\epsilon}{2(b-a)}$, since $M'_r - m'_r$ is the supremum of the set $\{\phi(f(x)) - \phi(f(y)) : x, y \in [x_{r-1}, x_r]\}$.

Consequently, $\sum\limits_{r \in A} (M'_r - m'_r)(x_r - x_{r-1}) \leq \dfrac{\epsilon}{2(b-a)}.(b-a)$

$$\text{i.e., } \leq \tfrac{\epsilon}{2}. \quad \ldots \text{ (iii)}$$

Let $r \in B$. Then $\sum\limits_{r \in B}(M'_r - m'_r)(x_r - x_{r-1}) \leq 2k(x_r - x_{r-1})$

$$\leq 2k.(x_r - x_{r-1})\frac{(M_r - m_r)}{\delta}$$
$$\leq \frac{2k}{\delta}[U(P, f) - L(P, f)]$$
$$< \frac{2k}{\delta} \cdot \frac{\epsilon\delta}{4k}, \text{ using (i)}$$
$$\text{i.e., } < \frac{\epsilon}{2} \ldots \ldots \text{ (iv)}$$

Therefore $U(P, \phi \circ f) - L(P, \phi \circ f) = \sum\limits_{r=1}^{n}(M'_r - m'_r)(x_r - x_{r-1})$

$$< \epsilon, \text{ using (iii) and (iv).}$$

This proves that $\phi \circ f$ is integrable on $[a, b]$.

Note. If $f : I \to \mathbb{R}$ be integrable on I and $\phi : J \to \mathbb{R}$ be integrable on J, then $\phi \circ f$ may not be integrable on I. For example, let $\phi : [0, 1] \to \mathbb{R}$ be defined by $\phi(x) = 1$, if $x \neq 0$

$= 0$, if $x = 0$.

Then ϕ is integrable on $[0, 1]$.

Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = 0$, if x be irrational

$= \frac{1}{n}$, if $x = \frac{m}{n}$ where m, n are positive integers and $gcd(m, n) = 1$.

Then f is integrable on $[0, 1]$.

$\phi \circ f : [0, 1] \to \mathbb{R}$ is defined by $\phi \circ f(x) = 0$, if x be irrational

$= 1$, if x be rational.

$\phi \circ f$ is not integrable on $[0, 1]$.

Theorem 11.6.9. Let a function $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and $c \in (a, b)$. Then f is integrable on $[a, c]$ and also integrable on $[c, b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$.

Proof. Since $f \in \mathcal{R}[a, b]$, f is bounded on $[a, b]$. Therefore f is bounded on $[a, c]$ as well as on $[c, b]$.

Let us choose $\epsilon > 0$. Since f is integrable on $[a, b]$, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$.

If $c \notin P$, let P_c be the refinement of P by adjoining the point c to P, i.e., $P_c = P \cup \{c\}$.

Then $L(P, f) \leq L(P_c, f) \leq U(P_c, f) \leq U(P, f)$.

If however, $c \in P$, then $P_c = P$.

In any case, $U(P_c, f) - L(P_c, f) \leq U(P, f) - L(P, f) < \epsilon$.

Let $P_c = P_1 \cup P_2$ where $P_1 = [a, c] \cap P_c, P_2 = [c, b] \cap P_c$.

Then P_1 is a partition of $[a, c]$ and P_2 is a partition of $[c, b]$ and

$U(P_c, f) = U(P_1, f) + U(P_2, f)$, $L(P_c, f) = L(P_1, f) + L(P_2, f)$.

Therefore $[U(P_1, f) - L(P_1, f)] + [U(P_2, f) - L(P_2, f)] < \epsilon$

Since $U(P_1, f) - L(P_1, f) \geq 0$ and $U(P_2, f) - L(P_2, f) \geq 0$, it follows that $U(P_1, f) - L(P_1, f) < \epsilon$ and $U(P_2, f) - L(P_2, f) < \epsilon$.

By the condition for integrability, f is integrable on $[a, c]$ and also integrable on $[c, b]$.

Second part.

For any partition P of $[a, b]$,

$U(P, f) \geq U(P_c, f)$, where $P_c = P \cup \{c\}$
$= U(P_1, f) + U(P_2, f)$, where $P_1 = [a, c] \cap P_c$ and $P_2 = [c, b] \cap P_c$
$\geq \overline{\int}_a^c f + \overline{\int}_c^b f$.

This shows that $\overline{\int}_a^c f + \overline{\int}_c^b f$ is a lower bound of the set $\{U(P, f) : P \in \mathcal{P}[a, b]\}$. Since the infimum of the set is $\overline{\int}_a^b f$, it follows that

$\overline{\int}_a^c f + \overline{\int}_c^b f \leq \overline{\int}_a^b f$ ... ... (i)

For any partition P of $[a, b]$,

$L(P, f) \leq L(P_c, f)$, where $P_c = P \cup \{c\}$
$= L(P_1, f) + L(P_2, f)$, where $P_1 = [a, c] \cap P_c$ and $P_2 = [c, b] \cap P_c$
$\leq \underline{\int}_a^c f + \underline{\int}_c^b f$.

This shows that $\underline{\int}_a^c f + \underline{\int}_c^b f$ is an upper bound of the set $\{L(P, f) : P \in \mathcal{P}[a, b]\}$. Since the supremum of the set is $\underline{\int}_a^b f$, it follows that

$\underline{\int}_a^c f + \underline{\int}_c^b f \geq \underline{\int}_a^b f$ ... ... (ii)

But $\underline{\int}_a^b f = \overline{\int}_a^b f = \int_a^b f$, $\underline{\int}_a^c f = \overline{\int}_a^c f = \int_a^c f$, $\underline{\int}_c^b f = \overline{\int}_c^b f = \int_c^b f$.

From (i) and (ii) we have $\int_a^c f + \int_c^b f \leq \int_a^b f \leq \int_a^c f + \int_c^b f$.

Consequently, $\int_a^b f = \int_a^c f + \int_c^b f$. This completes the proof.

Corollary. If $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and $a < c < d < b$, then f is integrable on $[c, d]$.

Proof. $f \in \mathcal{R}[a, b] \Rightarrow f \in \mathcal{R}[c, b]$, since $a < c < b$
$\Rightarrow f \in \mathcal{R}[c, d]$, since $c < d < b$.

Theorem 11.6.10. Let $f : [a, b] \to \mathbb{R}$ and $c \in (a, b)$. If f is integrable on $[a, c]$ as well as on $[c, b]$ then f is integrable on $[a, b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof. Since $f \in \mathcal{R}[a, c]$, f is bounded on $[a, c]$. Since $f \in \mathcal{R}[c, b]$, f is bounded on $[c, b]$. Therefore f is bounded on $[a, b]$.

Let us choose $\epsilon > 0$. Since f is integrable on $[a, c]$, there exists a partition P_1 of $[a, c]$ such that $U(P_1, f) - L(P_1, f) < \frac{\epsilon}{2}$.

Since f is integrable on $[c, b]$, there exists a partition P_2 of $[c, b]$ such that $U(P_2, f) - L(P_2, f) < \frac{\epsilon}{2}$.

Let $P = P_1 \cup P_2$. Then P is a partition of $[a, b]$ and

$U(P, f) = U(P_1, f) + U(P_2, f)$, $L(P, f) = L(P_1, f) + L(P_2, f)$.

So $U(P, f) - L(P, f) = [U(P_1, f) - L(P_1, f)] + [U(P_2, f) - L(P_2, f)] < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Therefore for a chosen $\epsilon > 0$ there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$.

This being a sufficient condition for integrability, f is integrable on $[a, b]$.

Second part. (Same · Previous)

For any partition P of $[a, b]$,

$U(P, f) \geq U(P_c, f)$, where $P_c = P \cup \{c\}$
$= U(P_1, f) + U(P_2, f)$, where $P_1 = [a, c] \cap P_c$ and $P_2 = [c, b] \cap P_c$
$\geq \overline{\int}_a^c f + \overline{\int}_c^b f$.

This shows that $\overline{\int}_a^c f + \overline{\int}_c^b f$ is a lower bound of the set $\{U(P, f) : P \in \mathcal{P}[a, b]\}$. Since the infimum of the set is $\overline{\int}_a^b f$, it follows that

$\overline{\int}_a^c f + \overline{\int}_c^b f \leq \overline{\int}_a^b f$ ... ... (i)

For any partition P of $[a, b]$,

$L(P, f) \leq L(P_c, f)$, where $P_c = P \cup \{c\}$
$= L(P_1, f) + L(P_2, f)$, where $P_1 = [a, c] \cap P_c$ and $P_2 = [c, b] \cap P_c$
$\leq \underline{\int}_a^c f + \underline{\int}_c^b f$.

This shows that $\underline{\int}_a^c f + \underline{\int}_c^b f$ is an upper bound of the set $\{L(P, f) : P \in \mathcal{P}[a, b]\}$. Since the supremum of the set is $\underline{\int}_a^b f$, it follows that

$\underline{\int}_a^c f + \underline{\int}_c^b f \geq \underline{\int}_a^b f$ ... ... (ii)

But $\underline{\int}_a^b f = \overline{\int}_a^b f$, $\underline{\int}_a^c f = \overline{\int}_a^c f$, $\underline{\int}_c^b f = \overline{\int}_c^b f$.

From (i) and (ii) we have $\int_a^c f + \int_c^b f \leq \int_a^b f \leq \int_a^c f + \int_c^b f$.

Consequently, $\int_a^b f = \int_a^c f + \int_c^b f$.

This completes the proof.

Theorem 11.6.11. Let $f : [a, b] \to \mathbb{R}$, $\phi : [a, b] \to \mathbb{R}$ be both bounded on $[a, b]$ such that $f(x) = \phi(x)$ except for a finite number of points in

$[a, b]$. If f be integrable on $[a, b]$ then ϕ is also integrable on $[a, b]$ and $\int_a^b \phi = \int_a^b f$.

Proof. Since ϕ is bounded on $[a, b]$, there exists a positive number B such that $|\phi(x)| < B$ for all $x \in [a, b]$. Let $f(x) = \phi(x)$ except at p points $x_1, x_2, \ldots, x_p$ such that $x_1 < x_2 < \cdots < x_p$.

Case 1. Let $a < x_1 < x_2 < \cdots < x_p < b$.

Let $\epsilon > 0$. Let the points $x_1, x_2, \ldots, x_p$ be enclosed by p non-overlapping subintervals $[x_1 - \frac{\delta_1}{2}, x_1 + \frac{\delta_1}{2}], [x_2 - \frac{\delta_2}{2}, x_2 + \frac{\delta_2}{2}], \cdots, [x_p - \frac{\delta_p}{2}, x_p + \frac{\delta_p}{2}]$ such that $\delta_1 + \delta_2 + \cdots + \delta_p < \frac{\epsilon}{4B}$.

Let $M^{(r)} = \sup\limits_{x \in [x_r - \frac{\delta_r}{2}, x_r + \frac{\delta_r}{2}]} \phi(x)$, $m^{(r)} = \inf\limits_{x \in [x_r - \frac{\delta_r}{2}, x_r + \frac{\delta_r}{2}]} \phi(x), r = 1, 2, \ldots, p$.

Then $M^{(r)} - m^{(r)} < 2B$, for $r = 1, 2, \ldots, p$.

On each of the remaining $p+1$ subintervals $[a, x_1 - \frac{\delta_1}{2}], [x_1 + \frac{\delta_1}{2}, x_2 - \frac{\delta_2}{2}], \cdots, [x_p + \frac{\delta_p}{2}, b], f = \phi$ and since f is integrable on these subintervals, there exist partitions P_1 of $[a, x_1 - \frac{\delta_1}{2}], P_2$ of $[x_1 + \frac{\delta_1}{2}, x_2 - \frac{\delta_2}{2}], \cdots, P_{p+1}$ of $[x_p - \frac{\delta_p}{2}, b]$ such that

$U(P_k, \phi) - L(P_k, \phi) < \frac{\epsilon}{2(p+1)}$, for $k = 1, 2, \ldots, p+1$.

The partitions $P_1, P_2, \ldots, P_{p+1}$ are disjoint.

Let $P = P_1 \cup P_2 \cup \cdots \cup P_{p+1}$. Then P is a partition of $[a, b]$.

$$\begin{aligned} U(P, \phi) - L(P, \phi) &= [U(P_1, \phi) - L(P_1, \phi)] + [U(P_2, \phi) - L(P_2, \phi)] \\ &\quad + \cdots + [U(P_{p+1}, \phi) - L(P_{p+1}, \phi)] + (M^{(1)} - m^{(1)})\delta_1 + \cdots + (M^{(p)} - m^{(p)})\delta_p \\ &< \tfrac{\epsilon}{2(p+1)}.(p+1) + 2B.\tfrac{\epsilon}{4B} \\ &= \epsilon. \end{aligned}$$

Therefore there exists a patition P of $[a, b]$ such that $U(P, \phi) - L(P, \phi) < \epsilon$. Hence ϕ is integrable on $[a, b]$.

Case 2. Either $a = x_1$, or $x_p = b$, or both.

If $a = x_1$, the subinterval enclosing the point x_1 can be chosen as $[a, a + \delta_1]$.

If $x_p = b$, the subinterval enclosing the point x_p can be chosen as $[b - \delta_p, b]$.

In any case, proceeding with similar arguments it can be proved that ϕ is integrable on $[a, b]$.

Second part.

Let $g(x) = f(x) - \phi(x)$, $x \in [a, b]$.

Then g is bounded on $[a, b]$ and $g(x) = 0$ on $[a, b]$ except at p points. Hence g is integrable on $[a, b]$, by Theorem 11.5.3.

Let $g_+(x) = \max\{g(x), 0\}$, $g_-(x) = \min\{g(x), 0\}$, i.e., $g_+(x) = \frac{1}{2}[g(x) + |g(x)|]$ and $g_-(x) = \frac{1}{2}[g(x) - |g(x)|]$.

Then g_+ and g_- are both integrable on $[a, b]$. $g_+(x) \geq 0$ and $-g_-(x) \geq 0$ for all $x \in [a, b]$.

Let $P = (a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b)$ be a partition of $[a, b]$.

Let $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} g_+(x), r = 1, 2, \ldots, n.$

As $g_+(x) = 0$ in $[x_{r-1}, x_r]$ for all x except at p points at most, $m_r = 0$. This holds for $r = 1, 2, \ldots, n$.

Therefore $L(P, g_+) = 0$ and this holds for every partition $P \in P[a, b]$. Therefore $\underline{\int}_a^b g_+ = 0$.

Since g_+ is integrable on $[a, b]$, $\int_a^b g_+ = \underline{\int}_a^b g_+ = 0$.

By similar arguments, $\underline{\int}_a^b g_- = 0$ and hence $\int_a^b g_- = 0$.

As $g(x) = g_+(x) + g_-(x)$ for all $x \in [a, b]$, it follows that $\int_a^b g = 0$.

Since f and ϕ are both integrable on $[a, b]$, $\int_a^b f - \int_a^b \phi = \int_a^b g = 0$.

Therefore $\int_a^b f = \int_a^b \phi$.

This completes the proof.

Note. If f and ϕ be both bounded on $[a, b]$, $f(x) = \phi(x)$ except for an enumerable number of points on $[a, b]$ and f is integrable on $[a, b]$, then ϕ may not be integrable on $[a, b]$. For example,

let $f(x) = 1,\ x \in [0, 1]$ and $\phi(x) \begin{array}{l} = 0,\ x \in [0,1] \cap \mathbb{Q} \\ = 1,\ x \in [0,1] - \mathbb{Q}. \end{array}$

Then f and ϕ are both bounded on $[0, 1]$. $f = \phi$ on $[0, 1]$ except for an enumerable number of points. f is integrable on $[0, 1]$ but ϕ is not integrable on $[0, 1]$.

Definition.

Piecewise continuous function. A function $f : [a, b] \to \mathbb{R}$ is said to be a *piecewise continuous* function on $[a, b]$ if there exists a partition $P = (x_0, x_1, \ldots, x_n)$ of $[a, b]$ where $a = x_0 < x_1 < \cdots < x_n = b$, such that f is continuous on the open interval (x_{k-1}, x_k) for $1 \leq k \leq n$ and each of $f(a+0), f(b-0), f(x_k+0), f(x_k-0)$ is finite for $1 \leq k \leq n-1$.

Clearly, a piecewise continuous function on $[a, b]$ is continuous on $[a, b]$ except for a finite number of points of jump discontinuity.

A step function on $[a, b]$ is an example of a piecewise continuous function on $[a, b]$. [page 100]

As a direct application of the Theorem 11.6.10, the evaluation of the definite integral $\int_a^b f$ becomes very simple when f is a step function or a piecewise continuous function on $[a, b]$.

Let $f : [a, b] \to \mathbb{R}$ be a step function on $[a, b]$ and $P = (x_0, x_1, \ldots, x_n)$ be a partition of $[a, b]$ where $a = x_0 < x_1 < \cdots < x_n = b$, such that $f(x) = c_k$ on (x_{k-1}, x_k) for $1 \le k \le n$ and $f(x_k) = \mu_k$ for $0 \le k \le n$.

Then f is bounded on $[a, b]$ and is continuous on $[a, b]$ except for the points $x_0, x_1, \ldots, x_n$ at most.

Let us define n functions $\phi_1, \phi_2, \ldots, \phi_n$ on $[a, x_1], [x_1, x_2], \cdots, [x_{n-1}, b]$ respectively by $\phi_k(x) = c_k, 1 \le k \le n$.

$$\begin{aligned} \text{Then } \int_a^b f &= \int_a^{x_1} \phi_1 + \int_{x_1}^{x_2} \phi_2 + \cdots + \int_{x_{n-1}}^b \phi_n, \\ &\qquad \text{by Theorems 11.6.9 and 11.6.11} \\ &= c_1(x_1 - a) + c_2(x_2 - x_1) + \cdots + c_n(b - x_{n-1}) \\ &= \sum_{k=1}^{n} c_k(x_k - x_{k-1}). \end{aligned}$$

Let $f : [a, b] \to \mathbb{R}$ be a piecewise continuous function on $[a, b]$ and $P = (x_0, x_1, \ldots, x_n)$ be a partition of $[a, b]$ where $a = x_0 < x_1 < \cdots < x_n = b$, such that f is continuous on (x_{k-1}, x_k) for $1 \le k \le n$ and each of $f(a + 0), f(b - 0), f(x_k - 0), f(x_k + 0)$ (for $1 \le k \le n - 1$) is finite.

f is continuous on $[a, b]$ except for the points $x_0, x_1, \ldots, x_n$, at most.

Let us define n functions $\phi_1, \phi_2, \ldots, \phi_n$ on $[a, x_1]$, $[x_1, x_2]$, $\cdots$, $[x_{n-1}, b]$ respectively by

$$\begin{aligned} &\phi_1(a) = f(a + 0); \quad \phi_1(x) = f(x), x \in (a, x_1); \phi_1(x_1) = f(x_1 - 0) \\ &\phi_2(x_1) = f(x_1 + 0); \phi_2(x) = f(x), x \in (x_1, x_2); \phi_2(x_2) = f(x_2 - 0) \\ &\qquad \cdots \qquad \cdots \qquad \cdots \\ &\phi_n(x_{n-1}) = f(x_{n-1} + 0); \quad \phi_n(x) = f(x), x \in (x_{n-1}, b); \phi_n(b) = f(b - 0). \end{aligned}$$

Then $\int_a^b f = \int_a^{x_1} \phi_1 + \int_{x_1}^{x_2} \phi_2 + \cdots + \int_{x_{n-1}}^b \phi_n$, by Theorems 11.6.9 and 11.6.11.

Worked Examples.

1. A function $f : [a, b] \to \mathbb{R}$ is bounded on $[a, b]$ and for every $c \in (a, b)$, f is integrable on $[c, b]$. Prove that f is integrable on $[a, b]$.

Let M be the supremum and m be the infimum of f on $[a, b]$.

Let us consider a sequence of points $\{c_n\}$ in (a, b) such that $\lim c_n = a$.

Let $\epsilon > 0$. Then there exists a natural number k such that $| c_n - a | < \frac{\epsilon}{2(M-m)}$ for all $n \ge k$.

Then $| c_k - a | < \frac{\epsilon}{2(M-m)}$.

Since f is integrable on $[c, b]$ for every $c \in (a, b)$, f is integrable on $[c_k, b]$. Therefore there exists a partition Q of $[c_k, b]$ such that $U(Q, f) - L(Q, f) < \frac{\epsilon}{2}$.

Let P be a partition of $[a, b]$ defined by $P = \{a\} \cup Q$.

Then $U(P, f) - L(P, f) < (M - m)(c_k - a) + [U(Q, f) - L(Q, f)]$

$$< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2}, \text{ i.e., } < \epsilon.$$

This proves that f is integrable on $[a, b]$.

Note 1. Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and for every $d \in (a, b)$, f is integrable on $[a, d]$. Then f is integrable on $[a, b]$.

The proof is similar.

Note 2. Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and for every c, d satisfying $a < c < d < b$, f is integrable on $[c, d]$. Then f is integrable on $[a, b]$.

Proof. Taking $d = \frac{a+b}{2}$, f is integrable on $[c, \frac{a+b}{2}]$ for every $c \in (a, \frac{a+b}{2})$. Therefore f is integrable on $[a, \frac{a+b}{2}]$.

Taking $c = \frac{a+b}{2}$, f is integrable on $[\frac{a+b}{2}, d]$ for every $d \in (\frac{a+b}{2}, b)$. Therefore f is integrable on $[\frac{a+b}{2}, b]$.

Consequently, f is integrable on $[a, b]$, by Theorem 11.6.10.

2. Let $f(x) = [x], x \in [0, 3]$. Prove that f is integrable on $[0, 3]$. Evaluate $\int_0^3 f$.

$$\begin{aligned} f(x) &= 0, \ 0 \le x < 1 \\ &= 1, \ 1 \le x < 2 \\ &= 2, \ 2 \le x < 3 \\ &= 3, \ x = 3. \end{aligned}$$

f is bounded on $[0, 3]$. f is continuous on $[0, 3]$ except for the points 1, 2, 3. So f is integrable on $[0, 3]$.

Let us define functions ϕ_1, ϕ_2, ϕ_3 on $[0, 1], [1, 2], [2, 3]$ respectively by $\phi_1(x) = 0, x \in [0, 1]$; $\phi_2(x) = 1, x \in [1, 2]$; $\phi_3(x) = 2, x \in [2, 3]$.

$$\begin{aligned} \text{Then } \int_0^3 f &= \int_0^1 f + \int_1^2 f + \int_2^3 f, \text{ by Theorem 11.6.9} \\ &= \int_0^1 \phi_1 + \int_1^2 \phi_2 + \int_2^3 \phi_3, \text{ by Theorem 11.6.11} \\ &= \int_0^1 0dx + \int_1^2 1dx + \int_2^3 2dx \\ &= 0 + (2 - 1) + 2.(3 - 2) = 3. \end{aligned}$$

3. $f(x) = \frac{1}{n}, \frac{1}{n+1} < x \le \frac{1}{n} (n = 1, 2, 3, \cdots)$
$= 0, x = 0.$

Prove that f is integrable on $[0, 1]$. Evaluate $\int_0^1 f$.

$$\begin{aligned} f(x) &= 1, \tfrac{1}{2} < x \le 1 \\ &= \tfrac{1}{2}, \tfrac{1}{3} < x \le \tfrac{1}{2} \\ &= \tfrac{1}{3}, \tfrac{1}{4} < x \le \tfrac{1}{3} \\ &\cdots\cdots \\ &= 0, x = 0. \end{aligned}$$

f is bounded on $[0,1]$ and is continuous on $[0,1]$ except at $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$

The set of points of discontnuity of f on $[0,1]$ is an infinite set having only one limit point. Therefore f is integrable on $[0,1]$.

Let us define functions $\phi_1, \phi_2, \ldots$ on $[\frac{1}{2},1], [\frac{1}{3},\frac{1}{2}], \cdots$ respectively by $\phi_1(x) = 1, \frac{1}{2} \leq x \leq 1;\ \phi_2(x) = \frac{1}{2}, \frac{1}{3} \leq x \leq \frac{1}{2}, \cdots$

$\int_{\frac{1}{2}}^{1} \phi_1 = 1(1-\frac{1}{2}) = \frac{1}{2},\ \int_{\frac{1}{3}}^{\frac{1}{2}} \phi_2 = \frac{1}{2}(\frac{1}{2}-\frac{1}{3}), \cdots \int_{\frac{1}{n+1}}^{\frac{1}{n}} \phi_n = \frac{1}{n}(\frac{1}{n}-\frac{1}{n+1}), \cdots$

Now $\int_0^1 f$

$= \lim\limits_{n\to\infty} [1(1-\frac{1}{2}) + \frac{1}{2}(\frac{1}{2}-\frac{1}{3}) + \frac{1}{3}(\frac{1}{3}-\frac{1}{4}) + \cdots + \frac{1}{n}(\frac{1}{n}-\frac{1}{n+1})]$

$= \lim\limits_{n\to\infty} [(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}) - (\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{n(n+1)})]$

$= \lim\limits_{n\to\infty} [(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}) - (1-\frac{1}{n+1})]$

$= \frac{\pi^2}{6} - 1$, since $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{6}$ and $\lim(1-\frac{1}{n+1}) = 1$.

11.7. Inequalities.

Theorem 11.7.1. Let a function $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. If M be the supremum and m be the infimum of f on $[a,b]$ then

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

Proof. Let $P_0 = (a = x_0, x_1, x_2, \ldots, x_n = b)$ be a partition of $[a,b]$.

Let M_r be the supremum of f and m_r be the infimum of f on $[x_{r-1}, x_r]$, for $r = 1,2,\ldots,n$. Then $m \leq m_r$, $M \geq M_r$, for $r = 1,2,\ldots,n$.

$$L(P_0,f) = m_1(x_1-x_0) + \cdots + m_n(x_n - x_{n-1})$$
$$\geq m[(x_1-x_0) + \cdots + (x_n-x_{n-1})] = m(b-a).$$

Similarly, $U(P_0,f) \leq M(b-a)$.

Since f is integrable on $[a,b]$, $L(P,f) \leq \int_a^b f \leq U(P,f)$ for all partitions P of $[a,b]$.

Therefore $m(b-a) \leq L(P_0,f) \leq \int_a^b f \leq U(P_0,f) \leq M(b-a)$.
or, $m(b-a) \leq \int_a^b f \leq M(b-a)$.

This completes the proof.

Corollary 1. Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Then there exists a real number μ satisfying $m \leq \mu \leq M$ such that $\int_a^b f = \mu(b-a)$.

Corollary 2. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Then there exists a point $c \in [a,b]$ such that $\int_a^b f = f(c)(b-a)$.

Proof. Since f is continuous on $[a,b]$, f is integrable on $[a,b]$ and f attains every real number μ satisfying $m \leq \mu \leq M$ at least at a point in $[a, b]$. Therefore there exists a point $c \in [a, b]$ such that $f(c) = \mu$ and $\int_a^b f = f(c)(b - a)$.

Theorem 11.7.2. Let a function $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and $f(x) \geq 0$ for all $x \in [a, b]$. Then $\int_a^b f \geq 0$.

Proof. Since f is integrable on $[a, b]$, f is bounded on $[a, b]$.

Let $P = (x_0, x_1, x_2, \ldots, x_n)$ be a partition of $[a, b]$.

Let m_r be the infimum of f on $[x_{r-1}, x_r]$, for $r = 1, 2, \ldots, n$.

Then $m_r \geq 0$, for $r = 1, 2, \ldots, n$.

$L(P, f) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1}) \geq 0$.

Therefore $\underline{\int_a^b} f = \sup\{L(P, f) : P \in \mathcal{P}[a, b]\} \geq 0$.

Since f is integrable on $[a, b]$, $\int_a^b f = \underline{\int_a^b} f$ and therefore $\int_a^b f \geq 0$.

Theorem 11.7.3. Let $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$ be both integrable on $[a, b]$ and $f(x) \geq g(x)$ for all $x \in [a, b]$. Then $\int_a^b f \geq \int_a^b g$.

Proof. Let $\phi : [a, b] \to \mathbb{R}$ be defined by $\phi(x) = f(x) - g(x), x \in [a, b]$.

Since f and g are both integrable on $[a, b]$, ϕ is integrable on $[a, b]$ and $\int_a^b \phi = \int_a^b f - \int_a^b g$.

But $\int_a^b \phi \geq 0$ by the previous theorem. Therefore $\int_a^b f \geq \int_a^b g$.

Corollary. Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and $M = \sup\limits_{x \in [a,b]} f(x)$, $m = \inf\limits_{x \in [a,b]} f(x)$. Then $m(b - a) \leq \int_a^b f \leq M(b - a)$.

Proof. Let $\phi(x) = M$ and $\psi(x) = m, x \in [a, b]$. Then ϕ and ψ are both integrable on $[a, b]$ and $\int_a^b \phi = M(b - a)$ $\int_a^b \psi = m(b - a)$ [Ex.1, 11.2.]

Since $\psi(x) \leq f(x) \leq \phi(x)$ on $[a, b]$, $\int_a^b \psi \leq \int_a^b f \leq \int_a^b \phi$.

Theorem 11.7.4. Let a function $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and $f(x) \geq 0$ for all $x \in [a, b]$. Let there exist a point c in $[a, b]$ such that f is continuous at c and $f(c) > 0$, then $\int_a^b f > 0$.

Proof. **Case 1.** Let $a < c < b$.

Let $\epsilon = \frac{1}{2} f(c) > 0$. Since f is continuous at c, there exists a positive δ such that $f(c) - \epsilon < f(x) < f(c) + \epsilon$ for all $x \in [c - \delta, c + \delta] \subset [a, b]$.

$\int_a^b f = \int_a^{c-\delta} f + \int_{c-\delta}^{c+\delta} f + \int_{c+\delta}^b f$.

Since $f(x) \geq 0$ for all $x \in [a, c - \delta]$, $\int_a^{c-\delta} f \geq 0$.

Since $f(x) \geq 0$ for all $x \in [c + \delta, b]$, $\int_{c+\delta}^b f \geq 0$.

Since $f(x) > \frac{1}{2} f(c)$ for all $x \in [c - \delta, c + \delta]$, $\int_{c-\delta}^{c+\delta} f \geq \frac{1}{2} f(c).2\delta > 0$.

Consequently, $\int_a^b f > 0$.

Case 2. Let $c = a$.

Let $\epsilon = \frac{1}{2} f(a) > 0$. As f is continuous at a, there exists a positive δ such that $f(a) - \epsilon < f(x) < f(a) + \epsilon$ for all $x \in [a, a+\delta] \subset [a, b]$.

$\int_a^b f = \int_a^{a+\delta} f + \int_{a+\delta}^b f$.

Since $f(x) \geq 0$ for all $x \in [a+\delta, b]$, $\int_{a+\delta}^b f \geq 0$.

Since $f(x) > \frac{1}{2} f(a)$ for all $x \in [a, a+\delta]$, $\int_a^{a+\delta} f \geq \frac{1}{2} f(a).\delta > 0$.

Consequently, $\int_a^b f > 0$.

Case 3. Let $c = b$.

Proceeding as in case 2, $\int_a^b f > 0$. This completes the proof.

Note 1. If f is continuous on $[a, b]$ and $f(x) > 0$ on $[a, b]$ then $\int_a^b f > 0$.

Note 2. If f is integrable on $[a, b]$ and $f(x) > 0$ on $[a, b]$ then also $\int_a^b f > 0$, because there exists at least a point of continuity $c \in [a, b]$ of f. [Remark after Example 5, 11.5.].

Theorem 11.7.5. Let the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be both integrable on $[a, b]$ and $f(x) \geq g(x)$ for all $x \in [a, b]$. Let there exist a point c in $[a, b]$ such that f and g are both continuous at c and $f(c) > g(c)$. Then $\int_a^b f > \int_a^b g$.

Proof. Let $\phi : [a, b] \to \mathbb{R}$ be defined by $\phi(x) = f(x) - g(x), x \in [a, b]$. Then $\phi(= f - g)$ is integrable on $[a, b]$ and $\phi(x) \geq 0$ for all $x \in [a, b]$. Also ϕ is continuous at c and $\phi(c) > 0$.

Therefore by the previous theorem, $\int_a^b \phi > 0$.

But $\int_a^b \phi = \int_a^b (f - g) = \int_a^b f - \int_a^b g$. Therefore $\int_a^b f > \int_a^b g$.

This completes the proof.

Theorem 11.7.6. Let a function $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. Then

$$\int_a^b | f | \geq | \int_a^b f |.$$

Proof. If $a \in \mathbb{R}$, then $- | a | \leq a \leq | a | \; \ldots \; \ldots$ (i)

If $a, b \in \mathbb{R}$, then $| a | \leq b \Leftrightarrow -b \leq a \leq b \; \ldots \; \ldots$ (ii)

For all $x \in [a, b]$, $- | f(x) | \leq f(x) \leq | f(x) |$, by (i)

That is, $- | f | (x) \leq f(x) \leq | f | (x)$, for all $x \in [a, b]$.

$f \in \mathcal{R}[a, b] \Rightarrow | f | \in \mathcal{R}[a, b]$. Therefore $- \int_a^b | f | \leq \int_a^b f \leq \int_a^b | f |$.

This implies $| \int_a^b f | \leq \int_a^b | f |$, by (ii). This completes the proof.

Worked Examples.

1. Prove that $\frac{\pi^2}{9} < \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx < \frac{2\pi^2}{9}$.

$1 \leq \frac{1}{\sin x} \leq 2$ for all $x \in [\frac{\pi}{6}, \frac{\pi}{2}]$. Therefore $x \leq \frac{x}{\sin x} \leq 2x$ for all $x \in [\frac{\pi}{6}, \frac{\pi}{2}]$.

Let $f(x) = \frac{x}{\sin x}$, $\phi(x) = x$, $\psi(x) = 2x, x \in [\frac{\pi}{6}, \frac{\pi}{2}]$.

f and ϕ are both bounded and integrable on $[\frac{\pi}{6}, \frac{\pi}{2}]$ and $f(x) \geq \phi(x)$ for all $x \in [\frac{\pi}{6}, \frac{\pi}{2}]$. Also f and ϕ are both continuous at $\frac{\pi}{3}$ and $f(\frac{\pi}{3}) > \phi(\frac{\pi}{3})$.

Hence $\int_{\pi/6}^{\pi/2} f(x)dx > \int_{\pi/6}^{\pi/2} \phi(x)dx = \int_{\pi/6}^{\pi/2} xdx = \frac{\pi^2}{9}$.

f and ψ are both bounded and integrable on $[\frac{\pi}{6}, \frac{\pi}{2}]$ and $f(x) \leq \psi(x)$ for all $x \in [\frac{\pi}{6}, \frac{\pi}{2}]$. Also f and ψ are both continuous at $\frac{\pi}{3}$ and $f(\frac{\pi}{3}) < \psi(\frac{\pi}{3})$.

Hence $\int_{\pi/6}^{\pi/2} f(x)dx < \int_{\pi/6}^{\pi/2} \psi(x)dx = 2\int_{\pi/6}^{\pi/2} xdx = \frac{2\pi^2}{9}$.

Consequently, $\frac{\pi^2}{9} < \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx < \frac{2\pi^2}{9}$.

2. If a function f is continuous on $[a, b]$, $f(x) \geq 0$ on $[a, b]$ and $\int_a^b f = 0$, prove that $f = 0$ on $[a, b]$ identically.

If f is not identically zero on $[a, b]$ then there exists a point $c \in [a, b]$ such that $f(c) > 0$.

Since f is continuous on $[a, b]$, f is integrable on $[a, b]$.

Since $f(x) \geq 0$ on $[a, b]$ and f is integrable on $[a, b]$, $\int_a^b f \geq 0$.

Since f is continuous at $c \in [a, b]$ and $f(c) > 0$, $\int_a^b f > 0$.

But by hypothesis, $\int_a^b f = 0$. Therefore there exists no point c in $[a, b]$ such that $f(c) > 0$. So f is identically zero on $[a, b]$.

3 A function g is continuous on $[a, b]$ and $g(x) = \int_a^x g(t)dt$. Prove that $g(x) = 0$ for all $x \in [a, b]$.

Since g is continuous on $[a, b], g$ is bounded on $[a, b]$. Therefore there exists a positive real number B such that $| g(x) | \leq B$ for all $x \in [a, b]$ ... ...(i)

$$| g(x) | = | \int_a^x g(t)dt | \leq \int_a^x | g(t) | dt \quad \ldots \quad \ldots\text{(ii)}$$

Since $| g(t) | \leq B$ for all $t \in [a, b]$, it follows from (ii) that $| g(x) | \leq B(x - a)$ for all $x \in [a, b]$ ... ...(iii)

Using (ii) and (iii), $| g(x) | \leq B\frac{(x-a)^2}{2!}$ for all $x \in [a, b]$ ...(iv)

$$\text{For } 0 < x \le 1,\ F(x) = \int_{-1}^{x} f(t)dt = \int_{-1}^{0} f(t)dt + \int_{0}^{x} f(t)dt.$$
$$= 0 + \int_0^x 1dt = x.$$

$$\text{We have } F(x) = 0, -1 \le x \le 0$$
$$= x, 0 < x \le 1.$$

Clearly, F is continuous on $[-1, 1]$.

Note. Although f is not continuous on $[-1, 1]$, F is continuous on $[-1, 1]$.

We observe that the function F is continuous on $[a, b]$ when f is integrable on $[a, b]$.

If, however, f be continuous on $[a, b]$ then F will be differentiable on $[a, b]$ as we shall see in the next theorem.

Theorem 11.8.2. If a function $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, then the function F defined by $F(x) = \int_a^x f(t)dt$, $x \in [a, b]$ is differentiable at any point $c \in [a, b]$ at which f is continuous and $F'(c) = f(c)$.

Proof. Let $c \in [a, b)$. Let $\epsilon > 0$. Since f is continuous at c there exists a positive δ such that $| f(x) - f(c) | < \epsilon$ for all $x \in [c, c + \delta)$.

Let us choose h satisfying $0 < h < \delta$. Then $f(c) - \epsilon < f(x) < f(c) + \epsilon$ for all $x \in [c, c + h]$.

Therefore $\int_c^{c+h} f(c) - \epsilon \le \int_c^{c+h} f \le \int_c^{c+h} f(c) + \epsilon$

or, $[f(c) - \epsilon].h \le F(c + h) - F(c) \le [f(c) + \epsilon].h$

or, $|\frac{F(c+h)-F(c)}{h} - f(c)| < \epsilon$. This holds for all h satisfying $0 < h < \delta$.

This implies $\displaystyle\lim_{h \to 0+} \frac{F(c + h) - F(c)}{h} = f(c)$.

That is, $RF'(c) = f(c)$ ... ... (i)

Let $c \in (a, b]$. Let $\epsilon > 0$. Since f is continuous at c there exists a positive η such that $| f(x) - f(c) | < \epsilon$ for all $x \in (c - \eta, c]$.

Let us choose h satisfying $0 < h < \eta$. Then $f(c) - \epsilon < f(x) < f(c) + \epsilon$ for all $x \in [c - h, c]$.

Therefore $\int_{c-h}^{c} f(c) - \epsilon \le \int_{c-h}^{c} f \le \int_{c-h}^{c} f(c) + \epsilon$

or, $[f(c) - \epsilon].h \le F(c) - F(c - h) \le [f(c) + \epsilon].h$

or, $|\frac{F(c-h)-F(c)}{-h} - f(c)| < \epsilon$. This holds for all h satisfying $0 < h < \delta$.

This implies $\displaystyle\lim_{h \to 0-} \frac{F(c + h) - F(c)}{h} = f(c)$.

That is, $LF'(c) = f(c)$ ... ... (ii)

From (i) and (ii) it follows that f is differentiable at any point $c \in [a, b]$ at which f is continuous and $F'(c) = f(c)$.

Using (ii) and (iv), $|\ g(x)\ | \leq B\frac{(x-a)^3}{3!}$ for all $x \in [a,b]$.

Continuing thus we have $|\ g(x)\ | \leq B\frac{(x-a)^n}{n!}$ for all $x \in [a,b]$ and for all $n \in \mathbb{N}$. Since $g(a) = 0, |\ g(x)\ | \geq 0$ for all $x \in [a,b]$.

Therefore $0 \leq |\ g(x)\ | \leq B\frac{(x-a)^n}{n!}$ for all $x \in [a,b]$ and for all $n \in \mathbb{N}$.

Since $\lim\limits_{n\to\infty} \frac{(x-a)^n}{n!} = 0$, it follows that $g(x) = 0$ for all $x \in [a,b]$.

11.8. Fundamental Theorem.

Let a function $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Then for each $x \in [a,b]$, f is integrable on $[a,x]$. $\int_a^x f(t)dt$ exists and it depends on x. Therefore we can define a function F on $[a,b]$ by

$$F(x) = \int_a^x f(t)dt.$$

Theorem 11.8.1. If $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ then the function F defined by $F(x) = \int_a^x f(t)dt, x \in [a,b]$ is continuous on $[a,b]$.

Proof. Let x_1, x_2 be any two points in $[a,b]$.
$F(x_2) - F(x_1) = \int_a^{x_2} f(t)dt - \int_a^{x_1} f(t)dt = \int_{x_1}^{x_2} f(t)dt$.
Therefore $|\ F(x_2) - F(x_1)\ | = |\ \int_{x_1}^{x_2} f(t)dt\ |$.

Since f is integrable on $[a,b]$, f is bounded on $[a,b]$. Therefore there exists a real number $k > 0$ such that $|\ f(x)\ | < k$ for all $x \in [a,b]$.

If $x_2 > x_1$, $|\ \int_{x_1}^{x_2} f(t)dt\ | \leq \int_{x_1}^{x_2} |\ f(t)\ |\ dt \leq (x_2 - x_1)k$.

If $x_1 > x_2$, $|\ \int_{x_1}^{x_2} f(t)dt\ | = |\ \int_{x_2}^{x_1} f(t)dt\ | \leq \int_{x_2}^{x_1} |\ f(t)\ |\ dt \leq (x_1 - x_2)k$.
Consequently, $|\ F(x_2) - F(x_1)\ | \leq |\ x_2 - x_1\ |\ k$.

Let us choose $\epsilon > 0$. Then $|\ F(x_2) - F(x_1)\ | < \epsilon$ for all x_1, x_2 in $[a,b]$ satisfying $|\ x_2 - x_1\ | < \frac{\epsilon}{k}$.
Let $\delta = \epsilon/k$. Then $|\ F(x_2) - F(x_1)\ | < \epsilon$ for all x_1, x_2 in $[a,b]$ satisfying $|\ x_2 - x_1\ | < \delta$.

This proves that F is uniformly continuous on $[a,b]$ and therefore F is continuous on $[a,b]$.

Worked Example.

1. Let $f(x) \begin{aligned} &= 0, \ -1 \leq x \leq 0 \\ &= 1, 0 < x \leq 1. \end{aligned}$

Prove that f is integrable on $[-1,1]$. Show that the function F defined by $F(x) = \int_{-1}^x f(t)dt$ is continuous on $[-1,1]$.

f is bounded on $[-1,1]$ and is continuous on $[-1,1]$ except at only one point, 0. Therefore f is integrable on $[-1,1]$.
For $-1 \leq x \leq 0$, $F(x) = \int_{-1}^x f(t)dt = 0$.

Corollary. If $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ then F is differentiable on $[a, b]$ and $F'(x) = f(x)$ for all $x \in [a, b]$.

An alternative proof for $F'(x) = f(x)$ on $[a, b]$, assuming continuity of f on $[a, b]$.

Theorem 11.8.3. If a function $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ then the function F defined by $F(x) = \int_a^x f(t)dt, x \in [a, b]$ is differentiable on $[a, b]$ and $F'(x) = f(x)$ for all $x \in [a, b]$.

Proof. **Case 1.** Let $c \in (a, b)$.

Let us choose h such that $c + h \in [a, b]$. Then
$F(c + h) - F(c) = \int_c^{c+h} f(t)dt$.

Let $h > 0$. Since f is continuous on $[c, c+h]$, f is bounded on $[c, c+h]$.

Let $M = \sup\limits_{t \in [c,c+h]} f(t)$, $m = \inf\limits_{t \in [c,c+h]} f(t)$.

Then $m \le f(t) \le M$ for all $t \in [c, c + h]$.

Therefore $mh \le \int_c^{c+h} f(t)dt \le Mh$

or, $\int_c^{c+h} f(t)dt = \mu h$ where $m \le \mu \le M$.

Since f is continuous on $[c, c + h]$, $\mu = f(c + \theta h)$ for some θ satisfying $0 \le \theta \le 1$. Then $\frac{F(c+h)-F(c)}{h} = f(c + \theta h)$.

Since f is continuous at c, $\lim\limits_{h \to 0+} f(c + \theta h) = f(c)$.

Therefore we have $\lim\limits_{h \to 0+} \dfrac{F(c + h) - F(c)}{h} = f(c)$ (i)

Let $h < 0$. Considering the interval $[c + h, c]$, we have
$-mh \le \int_{c+h}^{c} f(t)dt \le -Mh$ where $M = \sup\limits_{t \in [c+h,c]} f(t)$, $m = \inf\limits_{t \in [c+h,c]} f(t)$

or, $\frac{F(c+h)-F(c)}{h} = \mu$, where $m \le \mu \le M$.

Since f is continuous on $[c + h, c]$, $\mu = f(c + \theta h)$ for some θ satisfying $0 \le \theta \le 1$.

Taking limit as $h \to 0-$ and noting that $\lim\limits_{h \to 0-} f(c + \theta h) = f(c)$, we have $\lim\limits_{h \to 0-} \dfrac{F(c + h) - F(c)}{h} = f(c)$ $\cdots$ (ii)

From (i) and (ii) we have $F'(c) = f(c)$.

Case 2. Let $c = a$.

Let us choose h such that $a + h < b$. Then
$F(a + h) - F(a) = \int_a^{a+h} f(t)dt$.

Considering the interval $[a, a + h]$, we have
$mh \le \int_a^{a+h} f(t)dt \le Mh$ where $M = \sup\limits_{t \in [a,a+h]} f(t)$, $m = \inf\limits_{t \in [a,a+h]} f(t)$

or, $\frac{F(a+h)-F(a)}{h} = \mu$, where $m \le \mu \le M$.

Since f is continuous on $[a, a+h]$, $\mu = f(a+\theta h)$ for some θ satisfying $0 \leq \theta \leq 1$.

Taking limit as $h \to 0+$ and noting that $\lim\limits_{h \to 0+} f(a+\theta h) = f(a)$, we have $\lim\limits_{h \to 0+} \dfrac{F(a+h) - F(a)}{h} = f(a)$.

or, $F'(a) = f(a)$.

Case 3. Let $c = b$. Similar proof.

This completes the proof.

Definition. A function ϕ is called an *antiderivative* or a *primitive* of a function f on an interval I, if $\phi'(x) = f(x)$ for all $x \in I$.

If ϕ be an antiderivative of f on I, then $\phi + c$, where $c \in \mathbb{R}$, is obviously an antiderivative of f on I. This shows that if f admits of an antiderivative on I, then there exist many antiderivatives of f on I.

It follows from the previous theorem that if f be continuous on a closed interval $[a, b]$, then f possesses an antiderivative on $[a, b]$ given by F. Therefore continuity of f ensures the existence of an antiderivative of f.

It is worthwhile to note that continuity of f is not a necessary condition for the existence of an antiderivative of f.

For example, let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$\begin{aligned} f(x) &= 2x \sin \tfrac{1}{x} - \cos \tfrac{1}{x}, x \neq 0 \\ &= 0, \quad x = 0. \end{aligned}$$

f is not continuous on $[-1, 1]$, 0 being the point of discontinuity.

Let $\phi : [-1, 1] \to \mathbb{R}$ be defined by

$$\begin{aligned} \phi(x) &= x^2 \sin \tfrac{1}{x}, x \neq 0 \\ &= 0, \quad x = 0. \end{aligned}$$

Then $\phi'(x) = f(x)$ for all $x \in [-1, 1]$.

Thus ϕ is an antiderivative of f on $[-1, 1]$ although f is not continuous on $[-1, 1]$.

Worked Example (continued).

2. Let $f(x) = 1, 0 \leq x \leq 1$
$\qquad\qquad = x, 1 < x \leq 2$.

Verify that the function F defined by $F(x) = \int_0^x f(t)dt, x \in [0, 2]$ is differentiable on $[0, 2]$ and $F'(x) = f(x), x \in [0, 2]$.

f is continuous on $[0, 2]$ and therefore f is integrable on $[0, 2]$. Hence $\int_0^x f(t)dt$ exists for all $x \in [0, 2]$.

For $0 \leq x \leq 1$, $F(x) = \int_0^x f(t)dt = \int_0^x dt = x$.

For $1 < x \le 2$, $F(x) = \int_0^x f(t)dt \;=\; \int_0^1 f(t)dt + \int_1^x f(t)dt$
$= 1 + \int_1^x tdt = \frac{1}{2}(x^2+1)$.

We have $F(x) = x, 0 \le x \le 1$
$= \frac{1}{2}(x^2+1), 1 < x \le 2$.

$\lim\limits_{h\to 0-} \frac{F(1+h)-F(1)}{h} = 1$, $\lim\limits_{h\to 0+} \frac{F(1+h)-F(1)}{h} = 1$. Therefore $F'(1) = 1$.

So we have $F'(x) = 1, 0 \le x \le 1$
$= x, 1 < x \le 2$.

Therefore $F'(x) = f(x)$ for all $x \in [0,2]$.

Theorem 11.8.4. If $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and $\phi : [a,b] \to \mathbb{R}$ be an antiderivative of f on $[a,b]$, then

$$\int_a^b f = \phi(b) - \phi(a).$$

Proof. Since f is continuous on $[a,b]$, f is integrable on $[a,b]$.

Let $F(x) = \int_a^x f(t)dt, x \in [a,b]$.

Since f is continuous on $[a,b]$, F is differentiable on $[a,b]$ and $F'(x) = f(x)$ for all $x \in [a,b]$. So F is an antiderivative of f on $[a,b]$.

Since ϕ is an antiderivative of f on $[a,b]$, for all $x \in [a,b], \phi(x) = F(x) + c$, where c is a constant.

So $\phi(a) = F(a) + c = c$, since $F(a) = 0$.

Therefore $\phi(x) = F(x) + \phi(a)$, for all $x \in [a,b]$.

Consequently, $\int_a^b f = F(b) = \phi(b) - \phi(a)$.

Note. The theorem states that if f be *continuous* on $[a,b]$ then the integral $\int_a^b f$ can be evaluated in terms of an antiderivative of f on $[a,b]$.

Theorem 11.8.5. (Fundamental theorem of Integral calculus)

If (i) $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$, and
(ii) f possesses an antiderivative ϕ on $[a,b]$, then

$$\int_a^b f = \phi(b) - \phi(a).$$

Proof. Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a,b]$.

Let $M_r = \sup\limits_{x\in[x_{r-1},x_r]} f(x)$, $m_r = \inf\limits_{x\in[x_{r-1},x_r]} f(x)$, for $r = 1,2,\ldots,n$.

Since $\phi'(x) = f(x)$ for all $x \in [a,b]$, ϕ satisfies all conditions of Lagrange's Mean value theorem on $[x_{r-1}, x_r]$, for $r = 1,2,\ldots,n$.

Therefore for $r = 1,2,\ldots,n$,

$$\begin{aligned}\phi(x_r) - \phi(x_{r-1}) &= \phi'(\xi_r)(x_r - x_{r-1}) \text{ for some } \xi_r \text{ in } (x_{r-1}, x_r)\\ &= f(\xi_r)(x_r - x_{r-1}).\end{aligned}$$

The summation gives $\sum_{r=1}^{n} f(\xi_r)(x_r - x_{r-1}) = \phi(b) - \phi(a)$.
But $m_r \leq f(\xi_r) \leq M_r$ for $r = 1, 2, \ldots, n$.
Therefore $\sum_{r=1}^{n} m_r(x_r - x_{r-1}) \leq \phi(b) - \phi(a) \leq \sum_{r=1}^{n} M_r(x_r - x_{r-1})$.

Therefore $L(P, f) \leq \phi(b) - \phi(a) \leq U(P, f)$.
This holds for all partitions P of $[a, b]$.
So $\phi(b) - \phi(a)$ is an upper bound of the set $\{L(P, f) : P \in \mathcal{P}[a, b]\}$.

As the supremum of the set is $\underline{\int}_a^b f$, it follows that

$$\underline{\int}_a^b f \leq \phi(b) - \phi(a) \quad \ldots\ldots \text{(i)}$$

Also $\phi(b) - \phi(a)$ is a lower bound of the set $\{U(P, f) : P \in \mathcal{P}[a, b]\}$.

As the infimum of the set is $\overline{\int}_a^b f$, it follows that

$$\overline{\int}_a^b f \geq \phi(b) - \phi(a) \quad \ldots\ldots \text{(ii)}$$

From (i) and (ii) $\underline{\int}_a^b f \leq \phi(b) - \phi(a) \leq \overline{\int}_a^b f$.

Since f is integrable on $[a, b]$, $\overline{\int}_a^b f = \underline{\int}_a^b f = \int_a^b f$.

Consequently, $\int_a^b f = \phi(b) - \phi(a)$.

Corollary. Since $\phi(b) - \phi(a) = (b - a)f(\xi)$ for some $\xi \in (a, b)$, $\int_a^b f = (b - a)f(a + \theta(b - a))$ for some θ satisfying $0 < \theta < 1$.

Note. The evaluation of the integral $\int_a^b f$ in terms of the antiderivative of f is possible if f satisfies the conditions of the theorem. That these conditions are independent of each other can be seen from the following two examples.

(i) Let $f : [-1, 1] \to \mathbb{R}$ be defined by
$$f(x) = 0, \; -1 \leq x < 0$$
$$= 1, \; 0 \leq x \leq 1.$$

f is bounded on $[-1, 1]$ and continuous on $[-1, 1]$ except at only one point, 0. Therefore f is integrable on $[-1, 1]$.

If possible, let g be an antiderivative of f on $[-1, 1]$.

Then
$$g'(x) = 0, \; -1 \leq x < 0$$
$$= 1, \; 0 \leq x \leq 1.$$

g is differentiable on $[-1, 1]$ and $g'(-1) \neq g'(1)$. By Darboux's theorem, g' must assume every real number lying between $g'(-1)$ and $g'(1)$, i.e., between 0 and 1. But it does not do so.

It follows that g does not exist. That is, f has no antiderivative on $[-1, 1]$.

Therefore f has no antiderivaitive on $[-1, 1]$ although f is integrable on $[-1, 1]$.

(ii) Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$\begin{aligned} f(x) &= 2x \sin \tfrac{1}{x^2} - \tfrac{2}{x} \cos \tfrac{1}{x^2}, x \neq 0 \\ &= 0, x = 0. \end{aligned}$$

f is unbounded on every neighbourhood of 0. Hence f is not integrable on $[-1, 1]$.

Let $\phi : [-1, 1] \to \mathbb{R}$ be defined by $\begin{aligned} \phi(x) &= x^2 \sin \tfrac{1}{x^2}, x \neq 0 \\ &= 0, x = 0. \end{aligned}$

Then $\phi'(x) = f(x)$ on $[-1, 1]$. So ϕ is an antiderivative of f on $[-1, 1]$.

Therefore f has an antiderivative on $[-1, 1]$ although f is not integrable on $[-1, 1]$.

Worked Examples (continued).

3. Let f be defined on $[-2, 2]$ by $\begin{aligned} f(x) &= 3x^2 \cos \tfrac{\pi}{x^2} + 2\pi \sin \tfrac{\pi}{x^2}, x \neq 0 \\ &= 0, \quad x = 0. \end{aligned}$

Show that f is integrable on $[-2, 2]$. Evaluate $\int_{-2}^{2} f$.

f is bounded on $[-2, 2]$. f is continuous on $[-2, 2]$ except at 0. Since f is continuous on $[-2, 2]$ except at only one point, f is integrable on $[-2, 2]$.

Let $\phi : [-2, 2] \to \mathbb{R}$ be defined by $\begin{aligned} \phi(x) &= x^3 \cos \tfrac{\pi}{x^2}, \quad x \neq 0. \\ &= 0, \quad x = 0. \end{aligned}$

Then $\phi'(x) = 3x^2 \cos \frac{\pi}{x^2} + 2\pi \sin \frac{\pi}{x^2}$, for all $x (\neq 0) \in [-2, 2]$.

$\lim\limits_{x \to 0} \dfrac{\phi(x) - \phi(0)}{x - 0} = \lim\limits_{x \to 0} x^2 \cos \dfrac{\pi}{x^2} = 0$. Therefore $\phi'(0) = 0$.

Hence ϕ is an antiderivative of f on $[-2, 2]$.

By the fundamental theorem,

$$\begin{aligned} \int_{-2}^{2} f(x)dx &= \phi(2) - \phi(-2) \\ &= 8 \cos \tfrac{\pi}{4} + 8 \cos \tfrac{\pi}{4} = 8\sqrt{2}. \end{aligned}$$

4. A function f be defined on $[0, 3]$ by $f(x) = [x], x \in [0, 3]$.

Show that f is integrable on $[0, 3]$ but $\int_0^3 f$ can not be evaluated by the fundamental theorem.

f is bounded on $[0, 3]$ and is continuous on $[0, 3]$, except at the points 1, 2, 3. f has jump discontinuity at 1, 2, 3.

Therefore f is integrable on $[0, 3]$. $\int_0^3 f(x)dx = 3$ [by worked Example 2, page 434]

Let g be an antiderivative of f on $[0, 3]$. Then $g' = f$ on $[0. 3]$ and g' must have jump discontinuity at 1, 2, 3. Since a derived function cannot have a jump discontinuity in its domain, g does not exist.

Since f has no antiderivative on $[0,3]$, the fundamental theorem cannot be utilised to evaluate $\int_0^3 f$.

5. A function f is defined on $[0,1]$ by

$$f(x) = \sqrt{1-x^2}, x \in [0,1] \cap \mathbb{Q}.$$
$$= 1-x, x \in [0,1] - \mathbb{Q}.$$

Show that f is not integrable on $[0,1]$.

f is bounded on $[0,1]$. For all $x \in (0,1), \sqrt{1-x^2} > 1-x$.

Let $I = [0,1]$. $f/(I \cap \mathbb{Q})$ is monotone decreasing on $I \cap \mathbb{Q}$, $f/1-\mathbb{Q}$ is monotone decreasing on $I - \mathbb{Q}$.

Let us take a partition P_n of $[0,1]$ defined by $P_n = (x_0, x_1, \ldots, x_n)$ where $x_r = \frac{r}{n}$.

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$

Since $f/(I \cap \mathbb{Q})$ is monotone decreasing on $[x_{r-1}, x_r] \cap \mathbb{Q}$,

$$\sup_{x \in [x_{r-1}, x_r] \cap \mathbb{Q}} f(x) = f(x_{r-1}) = \sqrt{1 - x_{r-1}{}^2} = \sqrt{1 - (\tfrac{r-1}{n})^2}.$$

Since $f/(I - \mathbb{Q})$ is monotone decreasing on $[x_{r-1}, x_r] - \mathbb{Q}$ and x_{r-1} is rational, $\sup\limits_{x \in [x_{r-1}, x_r] - \mathbb{Q}} f(x) = \lim\limits_{n \to \infty} f(u_n)$ where $\{u_n\}$ is a sequence of irrational points in $[x_{r-1}, x_r]$ converging to $x_{r-1} = 1 - x_{r-1} = 1 - \frac{r-1}{n}$.

Since $1 - \frac{r-1}{n} < \sqrt{1 - (\frac{r-1}{n})^2}$, $\sup\limits_{x \in [x_{r-1}, x_r]} f(x) = \sqrt{1 - (\frac{r-1}{n})^2}$.

That is, $M_r = \sqrt{1 - (\frac{r-1}{n})^2}$.

Since $f/(I \cap \mathbb{Q})$ is monotone decreasing on $[x_{r-1}, x_r] \cap \mathbb{Q}$,

$$\inf_{x \in [x_{r-1}, x_r] \cap Q} f(x) = f(x_r) = \sqrt{1 - (\tfrac{r}{n})^2}.$$

Since $f/(I - \mathbb{Q})$ is monotone decreasing on $[x_{r-1}, x_r] - \mathbb{Q}$ and x_r is rational, $\inf\limits_{x \in [x_{r-1}, x_r] - \mathbb{Q}} f(x) = \lim\limits_{n \to \infty} f(\nu_n)$ where $\{\nu_n\}$ is a sequence of irrational points in $[x_{r-1}, x_r]$ converging to $x_r = 1 - x_r = 1 - \frac{r}{n}$.

Since $1 - \frac{r}{n} \le \sqrt{1 - (\frac{r}{n})^2}$, $\inf\limits_{x \in [x_{r-1}, x_r]} f(x) = 1 - \frac{r}{n}$.

That is, $m_r = 1 - \frac{r}{n}$.

$$U(P_n, f) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1})$$
$$= \tfrac{1}{n}\{\sqrt{1 - (\tfrac{0}{n})^2} + \sqrt{1 - (\tfrac{1}{n})^2} + \cdots + \sqrt{1 - (\tfrac{n-1}{n})^2}\}.$$

$$L(P_n, f) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1})$$
$$= \tfrac{1}{n}\{(1 - \tfrac{1}{n}) + (1 - \tfrac{2}{n}) + \cdots + (1 - \tfrac{n}{n})\}$$
$$= \tfrac{1}{n^2}[1 + 2 + \cdots + (n-1)] = \tfrac{n-1}{2n}.$$

Let us consider the sequence of partitions $\{P_n\}$ of $[0,1]$.
Here $\|P_n\| = \frac{1}{n}$ and $\lim\limits_{n\to\infty} \|P_n\| = 0$.

$\overline{\int_0^1} f = \lim\limits_{n\to\infty} U(P_n, f) = \lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{r=0}^{n-1} \sqrt{1-(\frac{r}{n})^2}$
$= \int_0^1 \sqrt{1-x^2}dx$, since $\sqrt{1-x^2}$ is integarble on $[0,1]$
$= [\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x]_0^1$ [by the fundamental theorem]
$= \frac{\pi}{4}$;
and $\underline{\int_0^1} = \lim\limits_{n\to\infty} L(P_n, f) = \lim\limits_{n\to\infty} \frac{n-1}{2n} = \frac{1}{2}$.

Since $\overline{\int_0^1} f \neq \underline{\int_0^1} f$, f is not integrable on $[0,1]$.

6. Prove that $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \frac{\pi}{6}$.

In $0 \le x \le 1, 4 - x^2 + x^3 = 4 - (x^2 - x^3) \le 4$;
$4 - x^2 + x^3 = (4 - x^2) + x^3 \ge 4 - x^2$.
Therefore $\frac{1}{2} \le \frac{1}{\sqrt{4-x^2+x^3}} \le \frac{1}{\sqrt{4-x^2}}$ in $0 \le x \le 1$.

Let $f(x) = \frac{1}{2}, x \in [0,1]$; $g(x) = \frac{1}{\sqrt{4-x^2+x^3}}, x \in [0,1]$.

f and g are both continuous on $[0,1]$ and therefore they are integrable on $[0,1]$. We have $f(x) \le g(x)$ on $[0,1]$.
Therefore $\int_0^1 f(x)dx \le \int_0^1 g(x)dx$. Also $f(\frac{1}{2}) < g(\frac{1}{2})$.

Therefore $\int_0^1 f(x)dx < \int_0^1 g(x)dx$, by Theorem 11.7.5
or, $\int_0^1 \frac{1}{2}dx < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}}$.
or, $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}}$ ... ... (i)

Let $h(x) = \frac{1}{\sqrt{4-x^2}}, x \in [0,1]$.
Then h is continuous on $[0,1]$ and therefore h is integrable on $[0,1]$.
We have $g(x) \le h(x)$ on $[0,1]$.
Therefore $\int_0^1 g(x)dx \le \int_0^1 h(x)dx$. Also $g(\frac{1}{2}) < h(\frac{1}{2})$.

Therefore $\int_0^1 g(x)dx < \int_0^1 h(x)dx$, by Theorem 11.7.5.
or, $\int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \int_0^1 \frac{dx}{\sqrt{4-x^2}}$
$= [\sin^{-1} \frac{x}{2}]_0^1$, by the fundamental theorem
$= \frac{\pi}{6}$ ... ... (ii)

From (i) and (ii) $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \frac{\pi}{6}$.

7. Let $I = [a,b] \subset \mathbb{R}$ and let $f : I \to \mathbb{R}$ be continuous on I. Let $J = [c,d] \subset \mathbb{R}$ and let $u : J \to \mathbb{R}$ be differentiable on J and $u(J) \subset I$;

$v : J \to \mathbb{R}$ be differentiable on J and $v(J) \subset I$. If $g : J \to \mathbb{R}$ be defined by $g(x) = \int_{u(x)}^{v(x)} f(t)dt$ for $x \in J$, then prove that

$g'(x) = (f \circ v)(x).v'(x) - (f \circ u)(x).u'(x)$ for all $x \in J$.

Let $u(x) = w, v(x) = z$ for $x \in J$. Then $w \in I, z \in I$.
Let $H(z) = \int_a^z f(t)dt, z \in v(J)$, $G(w) = \int_a^w f(t)dt, w \in u(J)$.
Then $g(x) = H(z) - G(w)$ for $x \in J$.

Since f is continuous on I, f is continuous on $[a, z]$ for all $z \in v(J)$ and f is continuous on $[a, w]$ for all $w \in u(J)$.

Therefore $H'(z) = f(z)$ for all $z \in v(J)$, $G'(w) = f(w)$ for all $w \in u(J)$.

Since u is differentiable on J, $u'(x)$, i.e., $\frac{dw}{dx}$ exists for all $x \in J$.
Since v is differentiable on J, $v'(x)$, i.e., $\frac{dz}{dx}$ exists for all $x \in J$.

$$\begin{aligned} \text{For all } x \in J,\ g'(x) &= \tfrac{d}{dx}H(z) - \tfrac{d}{dx}G(w) \\ &= H'(z)\tfrac{dz}{dx} - G'(w)\tfrac{dw}{dx} \\ &= f(v(x)).v'(x) - f(u(x)).u'(x) \\ &= (f \circ v)(x).v'(x) - (f \circ u)(x).u'(x). \end{aligned}$$

Note. If, in particular, $u(x) = a$ for all $x \in J$, then $g : J \to \mathbb{R}$ is defined by $g(x) = \int_a^{v(x)} f(t)dt$ for $x \in J$ and in that case

$g'(x) = (f \circ v)(x).v'(x)$ for all $x \in J$.

8. Find ϕ' where ϕ is defined on $[0, 1]$ by $\phi(x) = \int_{x^3}^{x^2} \frac{1}{\sqrt[3]{1+t^2}} dt, x \in [0, 1]$.

Let $u = x^3, x \in [0, 1]$; $v = x^2, x \in [0, 1]$; $f(t) = \frac{1}{\sqrt[3]{1+t^2}}, t \in \mathbb{R}$.
Then $\phi(x) = \int_0^v f(t)dt - \int_0^u f(t)dt$.
Let $H(v) = \int_0^v f(t)dt, v \in [0, 1]$. As $x \in [0, 1], v \in [0, 1]$.
Let $G(u) = \int_0^u f(t)dt, u \in [0, 1]$. As $x \in [0, 1], u \in [0, 1]$.

Since f is continuous on $[0, v]$ for all $v \in [0, 1]$, $H'(v) = f(v)$ for all $v \in [0, 1]$.
Since f is continuous on $[0, u]$ for all $u \in [0, 1]$, $G'(u) = f(u)$ for all $u \in [0, 1]$.
For all $x \in [0, 1]$, $\phi'(x) = H'(v)\frac{dv}{dx} - G'(u)\frac{du}{dx} = \frac{2x}{\sqrt[3]{1+x^4}} - \frac{3x^2}{\sqrt[3]{1+x^6}}$.

9. $\phi(x) = \int_{x^2}^{x^3} \frac{1}{(1+t^2)^3} dt, x \in [1, \infty)$. Find $\phi'(x)$.

Let $u = x^2, x \in [1, \infty)$; $v = x^3, x \in [1, \infty)$; $f(t) = \frac{1}{(1+t^2)^3}, t \in \mathbb{R}$.
Then $\phi(x) = \int_0^v f(t)dt - \int_0^u f(t)dt$.
Let $H(v) = \int_0^v f(t)dt, v \in [1, \infty)$. As $x \in [1, \infty), v \in [1, \infty)$.
Let $G(u) = \int_0^u f(t)dt, u \in [1, \infty)$. As $x \in [1, \infty), u \in [1, \infty)$.

Since f is continuous on $[0, v]$ for all $v \in [1, \infty), H'(v) = f(v)$ for all $v \in [1, \infty)$.

Since f is continuous on $[0, u]$ for all $u \in [1, \infty), G'(u) = f(u)$ for all $u \in [1, \infty)$.

For all $x \in [1, \infty)$, $\phi'(x) = H'(v)\frac{dv}{dx} - G'(u)\frac{du}{dx} = \frac{3x^2}{(1+x^6)^3} - \frac{2x}{(1+x^4)^3}$.

10. Evaluate $\lim\limits_{x \to 0} \dfrac{\int_0^{x^2} e^{\sqrt{1+t}} dt}{x^2}$.

Let $y = \int_0^{x^2} e^{\sqrt{1+t}} dt$;
$u = x^2$; $F(x) = \int_0^x e^{\sqrt{1+t}} dt$; $f(t) = e^{\sqrt{1+t}}, t \in \mathbb{R}$.
Then $y = F(u)$, $\frac{dy}{dx} = F'(u)\frac{du}{dx} = 2xF'(u)$.

Since f is continuous on $[0, x]$, $F'(x) = f(x) = e^{\sqrt{1+x}}$.
So $\frac{dy}{dx} = 2xe^{\sqrt{1+x^2}}$.

$$\lim_{x \to 0} \frac{\int_0^{x^2} e^{\sqrt{1+t}} dt}{x^2} \text{ (form } \tfrac{0}{0}) = \lim_{x \to 0} \frac{2xe^{\sqrt{1+x^2}}}{2x} = \lim_{x \to 0} e^{\sqrt{1+x^2}} = e.$$

11. A function f is continuous on $\mathbb{R}$ and $\int_{-x}^x f(t)dt = 2\int_0^x f(t)dt$ for all $x \in \mathbb{R}$. Prove that f is an even function on $\mathbb{R}$.

$2\int_0^x f(t)dt = \int_{-x}^x f(t)dt = \int_{-x}^0 f(t)dt + \int_0^x f(t)dt$
or, $\int_0^x f(t)dt = \int_{-x}^0 f(t)dt = -\int_0^{-x} f(t)dt$. ... (i)

Let $F(x) = \int_0^x f(t)dt, x \in \mathbb{R}$.
Since f is continuous on $\mathbb{R}$, $F'(x) = f(x)$ for all $x \in \mathbb{R}$. ... (ii)

From (i) $F(x) = -\int_0^{-x} f(t)dt$
$= -\int_0^u f(t)dt$, where $u = -x$
$= -F(u) = -F(-x)$.

Therefore $F'(x) = -F'(-x).(-1) = F'(-x)$.

From (ii) $f(x) = f(-x)$ for all $x \in \mathbb{R}$, i.e., f is an even function on $\mathbb{R}$.

As an extension of the fundamenlal theorem we come to the following theorems with stricter conditions.

Theorem 11.8.6. If (i) $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, and
(ii) there exists a function $\phi : [a, b] \to \mathbb{R}$ such that ϕ is continuous on $[a, b]$ and $\phi'(x) = f(x)$ for all $x \in (a, b)$, then

$$\int_a^b f = \phi(b) - \phi(a).$$

Proof. Let $P = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < \cdots < x_n = b$, be a partition of $[a, b]$.

Let $M_r = \sup_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

ϕ satisfies all conditions of Lagrange's Mean value theorem on $[x_{r-1}, x_r]$, for $r = 1, 2, \ldots, n$.

$$\begin{aligned} \phi(x_r) - \phi(x_{r-1}) &= \phi'(\xi_r)(x_r - x_{r-1}) \text{ for some } \xi_r \text{ in } (x_{r-1}, x_r) \\ &= f(\xi_r)(x_r - x_{r-1}). \end{aligned}$$

Therefore $\sum_{r=1}^{n} f(\xi_r)(x_r - x_{r-1}) = \phi(b) - \phi(a)$.

But $m_r \le f(\xi_r) \le M_r$ for $r = 1, 2, \ldots, n$.

Therefore $\sum_{r=1}^{n} m_r(x_r - x_{r-1}) \le \phi(b) - \phi(a) \le \sum_{r=1}^{n} M_r(x_r - x_{r-1})$.

This gives $L(P, f) \le \phi(b) - \phi(a) \le U(P, f)$.

This holds for all partitions P of $[a, b]$.

Proceeding with the same arguments as in the fundamental theorem (Theorem 11.8.5), we have $\int_a^b f = \phi(b) - \phi(a)$.

Worked Example (continued).

12. A function f is defined on $[-2, 1]$ by $f(x) = \operatorname{sgn} x$ and $\phi(x) = | x |$. Show that $\int_{-2}^{1} f(x)dx = \phi(1) - \phi(-2)$ although $\phi'(x) \ne f(x)$ on $[-2, 1]$.

ϕ is continuous on $[0, 1]$ and $\phi'(x) = \operatorname{sgn} x$ on $(0, 1)$.

Therefore $\int_0^1 f(x)dx = \phi(1) - \phi(0)$, by the theorem
$= \phi(1)$, since $\phi(0) = 0$.

ϕ is continuous on $[-2, 0]$ and $\phi'(x) = \operatorname{sgn} x$ on $(-2, 0)$.

Therefore $\int_{-2}^{0} f(x)dx = \phi(0) - \phi(-2)$, by the theorem
$= -\phi(-2)$, since $\phi(0) = 0$.

Since f is integrable on $[-2, 1]$, $\int_{-2}^{1} f(x)dx = \int_{-2}^{0} f(x)dx + \int_0^1 f(x)dx$ $= \phi(1) - \phi(-2)$.

Theorem 11.8.7. If (i) $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, and (ii) there exists a function $\phi : [a, b] \to \mathbb{R}$ such that ϕ is continuous on $[a, b]$ and $\phi'(x) = f(x)$ for all $x \in [a, b] - E$, where E is a finite set $\subset [a, b]$, then

$$\int_a^b f = \phi(b) - \phi(a).$$

Proof. Let $\phi'(x) \ne f(x)$ at the points $x_1, x_2, \ldots, x_m$.

Case 1. Let $a < x_1 < x_2 < \cdots < x_m < b$.

$\phi'(x) = f(x)$ on the open intervals $(a, x_1), (x_1, x_2), \ldots, (x_m, b)$.

Proceeding as in the proof of the previous theorem, we have $\int_{x_{r-1}}^{x_r} f = \phi(x_r) - \phi(x_{r-1})$ for $r = 1, 2, \ldots, m+1$, where $a = x_0, b = x_{m+1}$.

Since f is integrable on $[a,b]$, $\int_a^b f = \int_a^{x_1} f + \int_{x_1}^{x_2} f + \cdots + \int_{x_m}^b f$
$= \phi(b) - \phi(a)$.

Case 2. If $a = x_1$ then we consider the open intervals excluding (a, x_1) and if $b = x_m$ then we consider the open intervals excluding (x_m, b) and proceed along similar lines to obtain the result.

Worked Example (continued).

13. Let f be defined on $[0,3]$ by $f(x) = x$, if $1 < x \le 3$
$= -x$, if $0 \le x \le 1$
and ϕ be defined on $[0,3]$ by $\phi(x) = \frac{1}{2} \mid x^2 - 1 \mid$. Show that $\int_0^3 f = \phi(3) - \phi(0)$.

f is integrable on $[0,3]$. ϕ is continuous on $[0,3]$ and $\phi'(x) = f(x)$ for all $x \in [0,3] - \{1\}$.

Therefore $\int_0^3 f(x)dx = \phi(3) - \phi(0)$, by the theorem.

11.9. Another definition of integrability.

Let $f : [a,b] \to \mathbb{R}$ and let $P = (x_0, x_1, x_2, \ldots, x_n)$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be a partition of $[a,b]$ and $\xi_1, \xi_2, \ldots, \xi_n$ are arbitrarily chosen points such that $x_{r-1} \le \xi_r \le x_r$, for $r = 1, 2, \ldots, n$. Then the sum

$$f(\xi_1)(x_1 - x_0) + f(\xi_2)(x_2 - x_1) + \cdots + f(\xi_n)(x_n - x_{n-1})$$

is called a *Riemann sum* for f corresponding to the partition P and the chosen intermediate points ξ_r. This is denoted by $S(P, f, \xi)$, or by $S(P, f)$.

Definition.

A function $f : [a,b] \to \mathbb{R}$ is said to be *Riemann integrable* on $[a,b]$ if there exists a real number B such that for each $\epsilon > 0$ there exists a positive $\delta(\epsilon)$ satisfying $\mid S(P,f) - B \mid < \epsilon$ for all partitions P of $[a,b]$ with $\|P\| < \delta$, where $S(P,f)$ is a Riemann sum for f corresponding to the partition P and to any choice of intermediate points.

In this case $B = \int_a^b f$.

This condition is expressed by the symbol $\lim\limits_{\|P\| \to 0} S(P, f) = B$.

Note. Since $S(P, f)$ is not a function of $\|P\|$, this limit is not of the type that we usually define.

Using this symbolic notation, the definition of integrability of a function f can be restated as the following:

A function $f : [a,b] \to \mathbb{R}$ is said to be integrable on $[a,b]$ if there exists a real number B such that $\lim_{\|P\| \to 0} S(P, f) = B$,

where $S(P, f)$ is a *Riemann sum* for f corresponding to the partition P of $[a, b]$ and to an arbitrary choice of intermediate points. In this case, $B = \int_a^b f$.

[[**Remark.** A partition $P = (x_0, x_1, \ldots, x_n)$ of $[a, b]$ divides the the interval $[a, b]$ into subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$. If a point ξ_r be selected at random in the subinterval $[x_{r-1}, x_r]$ for $r = 1, 2, \ldots, n$, then the chosen points $\xi_1, \xi_2, \ldots, \xi_n$ are called *tags* of the corresponding subintervals.

The ordered set of ordered pairs ($([x_0, x_1], \xi_1), ([x_1, x_2], \xi_2), \ldots, ([x_{n-1}, x_n], \xi_n)$), where each ordered pair contains a subinterval as the first element and the corresponding tag as the second element, is said to be a *tagged partition* of $[a, b]$ and it is denoted by $\dot{P}$ [P being the underlying partition of $[a, b]$.

Since each tag can be chosen in infinitely many ways, we can have infinitely many tagged partitions corresponding to a single partition P.

Evidently, the norm of a partition P is same as the norm of each tagged partition $\dot{P}$, since each of them have the same set of subintervals.

The definition of Riemann integrability of a function f can be rephrased in terms of tagged partitions.

A function $f : [a, b] \to \mathbb{R}$ is said to be *Riemann integrable* on $[a, b]$ if there exists a real number B such that for each $\epsilon > 0$ there exists a positive $\delta(\epsilon)$ satisfying $| S(\dot{P}, f) - B | < \epsilon$ for all tagged partitions $\dot{P}$ of $[a, b]$ with $\|\dot{P}\| < \delta$.]]

Theorem 11.9.1. If $f : [a, b] \to \mathbb{R}$ be such that $\lim_{\|P\| \to 0} S(P, f) = B$, where $S(P, f)$ is a Riemann sum for f corresponding to the partition P of $[a, b]$, then B is unique.

Proof. Let $\lim_{\|P\| \to 0} S(P, f) = B_1$ and $\lim_{\|P\| \to 0} S(P, f) = B_2$.

Let us choose $\epsilon > 0$.

Since $\lim_{\|P\| \to 0} S(P, f) = B_1$, there exists a positive δ_1 such that if $\dot{P}$ be a tagged partition of $[a, b]$ with $\|\dot{P}\| < \delta_1$ then $|S(\dot{P}, f) - B_1| < \frac{\epsilon}{2}$.

Since $\lim_{\|P\| \to 0} S(P, f) = B_2$, there exists a positive δ_2 such that if $\dot{P}$ be a tagged partition of $[a, b]$ with $\|\dot{P}\| < \delta_2$ then $|S(\dot{P}, f) - B_2| < \frac{\epsilon}{2}$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $|S(\dot{P}, f) - B_1| < \frac{\epsilon}{2}$ and $|S(\dot{P}, f) - B_2| < \frac{\epsilon}{2}$ for all tagged partitions $\dot{P}$ of $[a, b]$ with $\|\dot{P}\| < \delta$

$|B_1 - B_2| \le |B_1 - S(\dot{P}, f)| + |S(\dot{P}, f) - B_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$, i.e., $< \epsilon$ for all tagged partitions $\dot{P}$ of $[a, b]$ with $||\dot{P}|| < \delta$.

Since ϵ is arbitrary, it follows that $B_1 = B_2$ and therefore B is unique.

Theorem 11.9.2. If $f : [a, b] \to \mathbb{R}$ be such that $\lim_{||P|| \to 0} S(P, f)$ exists, where $S(P, f)$ is a Riemann sum for f corresponding to the partition P of $[a, b]$, then f is bounded on $[a, b]$.

Proof. Let $\lim_{||P|| \to 0} S(P, f) = B$. Let us choose $\epsilon = 1$.

Then there exists a positive δ such that $| S(P, f) - B | < 1$ for all partitions P of $[a, b]$ satisfying $||P|| < \delta$.

Let f be not bounded on $[a, b]$. Then there exists at least one subinterval. say $[u, v]$ of $[a, b]$ such that $|v - u| < \delta$ and f is not bounded on $[u, v]$.

Let $P_0 = (x_0, x_1, x_2, \ldots, x_n)$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be a partition of $[a, b]$ such that $||P_0|| < \delta$ and $u = x_{r-1}, v = x_r$ for some $r = 1, 2, \ldots, n$.

$| S(P_0, f) - B | < 1$ for any choice of intermediate points ξ_r satisfying $x_{r-1} < \xi_r < x_r$.

Therefore $|f(\xi_1)(x_1 - x_0) + \cdots + f(\xi_r)(v - u) + \cdots + f(\xi_n)(x_n - x_{n-1}) - B| < 1$ for any intermediate points ξ_r satisfying $x_{r-1} < \xi_r < x_r$.

$|f(\xi_r)(v - u)| - |B - [f(\xi_1)(x_1 - x_0) + \cdots + f(\xi_{r-1})(x_{r-1} - x_{r-2}) + f(\xi_{r+1})(x_{r+1} - x_r) + \cdots + f(\xi_n)(x_n - x_{n-1})]|$

$\le |f(\xi_1)(x_1 - x_0) + \cdots + f(\xi_r)(v - u) + \cdots + f(\xi_n)(x_n - x_{n-1}) - B| < 1$, since $|b| - |a| \le |b - a|$ for all $a, b \in \mathbb{R}$.

Therefore $|f(\xi_r)(v - u)| < 1 + |[f(\xi_1)(x_1 - x_0) + \cdots + f(\xi_{r-1})(x_{r-1} - x_{r-2}) + f(\xi_{r+1})(x_{r+1} - x_r) + \cdots + f(\xi_n)(x_n - x_{n-1}) - B|$.

Let us keep $\xi_1, \xi_2, \ldots, \xi_{r-1}, \xi_{r+1}, \ldots, \xi_n$ fixed. Then

$|f(\xi_r)(v - u)|$ is bounded for every $\xi_r \in [u, v]$ and we have a contradiction.

Therefore f is bounded on $[a, b]$. This completes the proof.

Note. The theorem says that a necessary condition for Riemann integrability of a function f on $[a, b]$ in the new approach is that f is bounded on $[a, b]$. Therefore in developing the theory of integrability we shall be confined to the functions in $\mathcal{B}[a, b]$ only.

Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and let $P = (x_0, x_1, x_2, \ldots, x_n)$ be a partition of $[a, b]$ and $\xi_1, \xi_2, \cdots, \xi_n$ are arbitrarily chosen points such that $x_{r+1} \le \xi_r \le x_r$, for $r = 1, 2, \ldots, n$.

Let $M_r = \sup_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$.
Then $m_r \leq f(\xi_r) \leq M_r$, for $r = 1, 2, \ldots, n$. Therefore

$$\sum_{r=1}^{n} m_r(x_r - x_{r-1}) \leq \sum_{r=1}^{n} f(\xi_r)(x_r - x_{r-1}) \leq \sum_{r=1}^{n} M_r(x_r - x_{r-1})$$

or, $L(P, f) \leq S(P, f) \leq U(P, f)$.

Thus for a bounded function f, any Riemann sum corresponding to a partition P lies between the lower Darboux sum and the upper Darboux sum of f corresponding to P, no matter how we select the intermediate points.

If m_r and M_r be attained by f at some points in $[x_{r-1}, x_r]$ then for a particular choice of the intermediate points $\xi_r, S(P, f) = L(P, f)$ and for some other choice of the intermediate points $\xi_r, S(P, f) = U(P, f)$.

However, in general, the upper Darboux sum and the lower Darboux sum are not Riemann sums. But by proper choice of the intermediate points ξ_r, $S(P, f, \xi)$ can be made arbitrarily close to the upper and the lower Darboux sums.

The equivalence of two definitions of integrability of a function f, one in terms of exact bounds of the lower and the upper Darboux sums and the other in terms of limits of Riemann sums, can be established by the following theorems.

Theorem 11.9.3. Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ in the sense of definition 11.2. Then for each $\epsilon > 0$ there exists a positive δ such that $| S(P, f) - \int_a^b f | < \epsilon$ for all partitions P of $[a, b]$ satisfying $\|P\| < \delta$, where $S(P, f)$ is a Riemann sum for f corresponding to P and to an arbitrary choice of intermediate points.

Proof. Since f is integrable on $[a, b]$ and $\epsilon > 0$, there exists a positive δ such that $U(P, f) - L(P, f) < \epsilon$ for all partitions P of $[a, b]$ satisfying $\|P\| < \delta$.

For every partition P of $[a, b], L(P, f) \leq \int_a^b f \leq U(P, f)$.

For every partition P of $[a, b], L(P, f) \leq S(P, f) \leq U(P, f)$ where $S(P, f)$ is a Riemann sum for f corresponding to P and to any choice of intermediate points. Therefore for every partition P of $[a, b]$,

$| S(P, f) - \int_a^b f | \leq U(P, f) - L(P, f)$, where $S(P, f)$ is any Riemann sum for f corresponding to P and to any choice of intermediate points.

Hence $| S(P, f) - \int_a^b f | < \epsilon$ for all partitions P of $[a, b]$ satisfying $\|P\| < \delta$, where $S(P, f)$ is a Riemann sum for f corresponding to P and to any choice of intermediate points.

Theorem 11.9.4. Let $f : [a, b] \to \mathbb{R}$ be such that there exist a real number B such that $\lim\limits_{\|P\| \to 0} S(P, f) = B$, where $S(P, f)$ is a Riemann sum for f corresponding to the partition P of $[a, b]$ and to any choice of intermediate points. Then f is integrable on $[a, b]$ in the sense of definition 11.2 and $\int_a^b f = B$.

Proof. Let $\epsilon > 0$. There exists a $\delta > 0$ such that
$B - \frac{\epsilon}{2} < S(P, f) < B + \frac{\epsilon}{2}$ for all partitions P of $[a, b]$ such that $\|P\| < \delta$ where $S(P, f)$ is any Riemann sum for f corresponding to P and to any choice of intermediate points.

Since $\lim\limits_{\|P\| \to 0} S(P, f)$ exists, f is bounded on $[a, b]$. Let us take a partition P_0 of $[a, b]$ such that $\|P_0\| < \delta$.

Let $P_0 = (x_0, x_1, x_2, \ldots, x_n)$ where $a = x_0 < x_1 < \cdots < x_n = b$.

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

There exist points α_r, β_r in $[x_{r-1}, x_r]$ such that
$M_r - \frac{\epsilon}{2(b-a)} < f(\alpha_r) \leq M_r$, $m_r \leq f(\beta_r) < m_r + \frac{\epsilon}{2(b-a)}$.

Selecting the points $\alpha_1, \alpha_2, \ldots, \alpha_n$ as intermediate points
$U(P_0, f) - \frac{\epsilon}{2} < S(P_0, f, \alpha) < U(P_0, f)$.

Selecting the points $\beta_1, \beta_2, \ldots, \beta_n$ as intermediate points
$L(P_0, f) \leq S(P_0, f, \beta) < L(P_0, f) + \frac{\epsilon}{2}$

Since $B - \frac{\epsilon}{2} < S(P_0, f) < B + \frac{\epsilon}{2}$ where $S(P_0, f)$ is a Riemann sum for f corresponding to any choice of intermediate points, we have
$U(P_0, f) - \frac{\epsilon}{2} < B + \frac{\epsilon}{2}$ and $B - \frac{\epsilon}{2} < L(P_0, f) + \frac{\epsilon}{2}$.

Hence $B - \epsilon < L(P_0, f) \leq U(P_0, f) < B + \epsilon$ ... ... (i)
Therefore $U(P_0, f) - L(P_0, f) < 2\epsilon$.
This proves that f is integrable on $[a, b]$ in the sense of definition 11.2.

As $L(P, f) \leq \int_a^b f \leq U(P, f)$ for every partition P of $[a, b]$, it follows from (i) that

$B - \epsilon < L(P, f) \leq \int_a^b f \leq U(P, f) < B + \epsilon$ for all partitions P satisfying $\|P\| < \delta$.

Therefore $\left| \int_a^b f - B \right| < \epsilon$. Since ϵ is arbitrary, $\int_a^b f = B$.

Theorem 11.9.5. Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and let $\{P_n\}$ be a sequence of partitions of $[a, b]$ such that the sequence $\{\|P_n\|\}$ converges to 0. If there exists a real number B such that corresponding to each $\epsilon > 0$ there exists a natural number k satisfying $| S(P_n, f) - B | < \epsilon$ for all $n \geq k$, where $S(P_n, f)$ is a Riemann sum for f corresponding to

P_n and to any choice of intermediate points, then f is integrable on $[a, b]$ and $\int_a^b f = B$.

Proof. Let $\epsilon > 0$. By the given condition there exists a natural number k_1 such that $| S(P_n, f) - B | < \frac{\epsilon}{2}$ for all $n \geq k_1$.

Let $\|P_{k_1}\| = \delta$. Since $\lim \|P_n\| = 0$, there exists a natural number k_2 such that $\|P_n\| < \delta$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $| S(P_n, f) - B | < \frac{\epsilon}{2}$ and $\|P_n\| < \delta$ for all $n \geq k$.

It follows that $| S(P_n, f) - B | < \frac{\epsilon}{2}$ for all partitions P_n satisfying $\|P_n\| < \delta$.

Let P_0 be a partition of $[a, b]$ such that $\|P_0\| < \delta$.

Then $| S(P_0, f) - B | < \frac{\epsilon}{2}$.

Let $P_0 = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < \cdots < x_n = b$.

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \ldots, n$.

There exist points α_r, β_r in $[x_{r-1}, x_r]$ such that
$M_r - \frac{\epsilon}{2(b-a)} < f(\alpha_r) \leq M_r$, $m_r \leq f(\beta_r) < m_r + \frac{\epsilon}{2(b-a)}$.

Selecting $\alpha_1, \alpha_2, \ldots, \alpha_n$ as intermediate points, we have
$U(P_0, f) - \frac{\epsilon}{2} < S(P_0, f, \alpha) < U(P_0, f)$.

Selecting $\beta_1, \beta_2, \ldots, \beta_n$ as intermediate points, we have
$L(P_0, f) < S(P_0, f, \beta) < L(P_0, f) + \frac{\epsilon}{2}$.

Since $B - \frac{\epsilon}{2} < S(P_0, f) < B + \frac{\epsilon}{2}$ where $S(P_0, f)$ is a Riemann sum for f corresponding to any choice of intermediate points, we have
$U(P_0, f) - \frac{\epsilon}{2} < B + \frac{\epsilon}{2}$ and $B - \frac{\epsilon}{2} < L(P_0, f) + \frac{\epsilon}{2}$.

Hence $B - \epsilon < L(P_0, f) \leq U(P_0, f) < B + \epsilon$ ... ... (i)

Therefore $U(P_0, f) - L(P_0, f) < 2\epsilon$.

This proves that f is integrable on $[a, b]$.

Since for every partition P of $[a, b]$ $L(P, f) \leq \int_a^b f \leq U(P, f)$, from (i) it follows that

$B - \epsilon < L(P_0, f) \leq \int_a^b f \leq U(P_0, f) < B + \epsilon$.

So we have $| \int_a^b f - B | < \epsilon$.

This holds for each $\epsilon > 0$. Hence $\int_a^b f = B$.

Note. The theorem says that if f be bounded on $[a, b]$ and $\{P_n\}$ be a sequence of partitions of $[a, b]$ such that the sequence $\{\|P_n\|\}$ converges to 0, then if $\lim\limits_{n \to \infty} S(P_n, f) = B$ where $S(P_n, f)$ is a Riemann sum for f corresponding to P_n and to any choice of intermediate points, then f is integrable on $[a, b]$ and $\int_a^b f = B$.

In particular, if $P_n = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < \cdots < x_n = b$ and $x_1 - x_0 = x_2 - x_1 = \cdots = x_n - x_{n-1}$ then $\|P_n\| = \frac{b-a}{n}$ and $\lim \|P_n\| = 0$; and if for every choice of intermediate points $\xi_1, \xi_2, \ldots, \xi_n$, $\lim\limits_{n\to\infty} \frac{b-a}{n} \sum\limits_{r=1}^{n} f(\xi_r) = B$ then $\int_a^b f$ exists and equals B.

Theorem 11.9.6. Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and $\{P_n\}$ be a sequence of partitions of $[a, b]$ such that the sequence $\{\|P_n\|\}$ converges to 0. Then if $\epsilon > 0$ be given, there exists a natural number k such that $| S(P_n, f) - \int_a^b f | < \epsilon$ for all $n \geq k$ where $S(P, f)$ is a Riemann sum of f corresponding to P and any choice of intermediate points.

Proof. Since f is integrable on $[a, b]$ and $\epsilon > 0$ there exists a positive δ such that for all partitions P satisfying $\|P\| < \delta$

$$U(P, f) - L(P, f) < \epsilon.$$

Since $\lim \|P_n\| = 0$, there exists a natural number k such that

$$\|P_n\| < \delta \text{ for all } n \geq k.$$

Therefore $U(P_n, f) - L(P_n, f) < \epsilon$ for all $n \geq k$.

Since f is integrable on $[a, b]$, $L(P_n, f) \leq \int_a^b f \leq U(P_n, f)$ for all $n \in \mathbb{N}$.

Also for each P_n, $L(P_n, f) \leq S(P_n, f) \leq U(P_n, f)$ where $S(P_n, f)$ is a Riemann sum for f corresponding to P_n and to any choice of intermediate points.

Therefore $| S_n(P_n, f) - \int_a^b f | \leq U(P_n, f) - L(P_n, f) < \epsilon$ for all $n \geq k$.

Note. The theorem says that if f be integrable on $[a, b]$ and $\{P_n\}$ be a sequence of partitions of $[a, b]$ such that $\lim \|P_n\| = 0$, then $\lim\limits_{n\to\infty} S(P_n, f) = \int_a^b f$ where $S(P_n, f)$ is a Riemann sum for f corresponding to the partitions P_n and to any choice of intermediate points.

In particular, if $P_n = (x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < \cdots < x_n = b$ and $x_1 - x_0 = x_2 - x_1 = \cdots = x_n - x_{n-1}$ then $\|P_n\| = \frac{b-a}{n}$ and $\lim \|P_n\| = 0$. Then for an *integrable* function f, $\int_a^b f = \lim\limits_{n\to\infty} \frac{b-a}{n} \sum\limits_{r=1}^{n} f(\xi_r)$ for *any particular choice* of points $\xi_1, \xi_2, \ldots, \xi_n$.

Worked Examples.

1. A function f is defined on $[0, 1]$ by

$$\begin{aligned} f(x) &= 1, \text{ if } x \text{ is rational} \\ &= 0, \text{ if } x \text{ is irrational.} \end{aligned}$$

Show that f is not integrable on $[0, 1]$.

f is bounded on $[0, 1]$.

Let P_n be the partition of $[0, 1]$ defined by $P_n = (x_0, x_1, x_2, \ldots, x_n)$ where $x_r = \frac{r}{n}, 0 \le r \le n$.

Let us choose α_r in $[x_{r-1}, x_r]$ by $\alpha_r = x_r$, for $r = 1, 2, \ldots, n$.
Then $S(P_n, f, \alpha) = \frac{1}{n}[f(\alpha_1) + f(\alpha_2) + \cdots + f(\alpha_n)] = 1$.

Let us choose β_r in $[x_{r-1}, x_r]$ by $\beta_r = x_r - \frac{1}{\sqrt{2}n}$, for $r = 1, 2, \ldots, n$.
Then $S(P_n, f, \beta) = \frac{1}{n}[f(\beta_1) + f(\beta_2) + \cdots + f(\beta_n)] = 0$.

Let us consider the sequence of partitions $\{P_n\}$. $\|P_n\| = \frac{1}{n}$. $\lim \|p_n\| = 0$.
$\lim\limits_{n\to\infty} S(P_n, f, \alpha) = 1$, $\lim\limits_{n\to\infty} S(P_n, f, \beta) = 0$.

Since for two different choices of intermediate points ξ_r, the Riemann sums $S(P_n, f, \xi)$ converge to two different limits, f is not integrable on $[0, 1]$.

2. A function f is defined on $[0, 1]$ by

$$\begin{aligned} f(x) &= x, \text{ if } x \text{ is rational} \\ &= 1 - x, \text{ if } x \text{ is irrational.} \end{aligned}$$

Show that f is not integrable on $[0, 1]$.

f is bounded on $[0, 1]$. Let P_n be the partition of $[0, 1]$ defined by $P_n = (x_0, x_1, x_2, \ldots, x_{2n})$ where $x_r = \frac{r}{2n}, 0 \le r \le 2n$.
Let us choose α_r in $[x_{r-1}, x_r]$ by $\alpha_r = x_r$, for $r = 1, 2, \ldots, n$ and $\alpha_r = x_r - \frac{1}{\sqrt{5}n}$, for $r = n+1, \ldots, 2n$.

$$\begin{aligned} \text{Then } S(P_n, f, \alpha) &= \tfrac{1}{2n}[(x_1 + x_2 + \cdots + x_n) + (1 - x_{n+1} + \tfrac{1}{\sqrt{5}n}) \\ &\quad + \cdots + (1 - x_{2n} + \tfrac{1}{\sqrt{5}n})] \\ &= \tfrac{1}{2n}[\tfrac{1+2+\cdots+n}{2n} + n + \tfrac{1}{\sqrt{5}} - \tfrac{(n+1)+\cdots+2n}{2n}] \\ &= \tfrac{1}{2n}[\tfrac{2n}{4} + \tfrac{1}{\sqrt{5}}]. \end{aligned}$$

Let us choose β_r in $[x_{r-1}, x_r]$ by $\beta_r = x_r - \frac{1}{\sqrt{5}n}$, for $r = 1, \ldots, n$ and $\beta_r = x_r$, for $r = n+1, \ldots, 2n$.

$$\begin{aligned} \text{Then } S(P_n, f, \beta) &= \tfrac{1}{2n}[(1 - x_1 + \tfrac{1}{\sqrt{5}n} + \cdots + (1 - x_n + \tfrac{1}{\sqrt{5}n}) \\ &\quad +(x_{n+1} + \cdots + x_{2n})] \\ &= \tfrac{1}{2n}[n - \tfrac{1+2+\cdots+n}{2n} + \tfrac{1}{\sqrt{5}} + \tfrac{(n+1)+\cdots+2n}{2n}] \\ &= \tfrac{1}{2n}[\tfrac{6n}{4} + \tfrac{1}{\sqrt{5}}]. \end{aligned}$$

Let us consider the sequence of partitions $\{P_n\}$. $\|P_n\| = \frac{1}{2n}$. $\lim \|p_n\| = 0$. $\lim\limits_{n\to\infty} S(P_n, f, \alpha) = \frac{1}{4}$, $\lim\limits_{n\to\infty} S(P_n, f, \beta) = \frac{3}{4}$.

Since for two different choices of intermediate points ξ_r, the Riemann sums $S(P_n, f, \xi)$ converge to two different limits, f is not integrable on $[0, 1]$.

3. Let $f : [0,1] \to \mathbb{R}$ be integrable on $[0,1]$. Show that

$$\int_0^1 f = \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n} f\left(\frac{r}{n}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n-1} f\left(\frac{r}{n}\right).$$

Let $P_n = (0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1)$. Then $\|P_n\| = \frac{1}{n}$ and $\{P_n\}$ is a sequence of partition of $[0,1]$ such that $\lim \|P_n\| = 0$.

Since f is integrable on $[0,1]$, $\lim S(P_n, f) = \int_0^1 f$ where $S(P_n, f)$ is the Riemann sum $\frac{1}{n}\{f(\xi_1) + f(\xi_2) + \cdots + f(\xi_n)\}$ for any particular choice of intermediate points $\xi_1, \xi_2, \ldots, \xi_n$, $\frac{r-1}{n} \leq \xi_r \leq \frac{r}{n}$, $r = 1, 2, \ldots, n$.

Let $\xi_r = \frac{r}{n}$. Then $\int_0^1 f = \lim [illegible] + f\left(\frac{n}{n}\right)]$

Let $\xi_r = \frac{r-1}{n}$. Then $\int_0^1 f = \lim [illegible] + \cdots + f\left(\frac{n-1}{n}\right)]$

Note. If f be integrable [illegible] the form $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)$ [illegible] done by the result established [illegible] of the limit [illegible] be eff[illegible]

4. Evaluate the limit [illegible]

$\lim [\frac{1}{n+1} + [illegible]$

$\lim [\frac{1}{n+1} [illegible]$

$\lim \frac{1}{n} [illegible]$

[illegible]

Therefore [illegible]

5. Evaluate the limit
$\lim_{n\to\infty} \{(1+\frac{1^2}{n^2})(1+\frac{2^2}{n^2})\cdots(1+\frac{n^2}{n^2})\}^{1/n}$ as an integral.

Let $P = \{(1+\frac{1^2}{n^2})(1+\frac{2^2}{n^2})\cdots(1+\frac{n^2}{n^2})\}^{1/n}$.
Then $\log P = \frac{1}{n}\sum_{r=1}^{n}\log(1+\frac{r^2}{n^2})$.

$\lim_{n\to\infty}\log P = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log(1+\frac{r^2}{n^2})$

$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f(\frac{r}{n})$, where $f(x) = \log(1+x^2)$

$= \int_0^1 \log(1+x^2)dx$, since f is integrable on $[0,1]$.

Hence $\lim_{n\to\infty} P = e^{\int_0^1 \log(1+x^2)dx}$.

6. Evaluate the limit
$\lim_{n\to\infty}[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+3n}]$ as an integral.

Let $S = \lim_{n\to\infty}[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+3n}]$.
Then $S = \lim_{n\to\infty}\frac{1}{n}[\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\cdots+\frac{1}{1+\frac{3n}{n}}]$

$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{3n}\frac{1}{1+\frac{r}{n}} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{3n} f(\frac{r}{n})$, where $f(x) = \frac{1}{1+x}$.

Let $P_n = (0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{3n}{n})$ be a partition of $[0,3]$ dividing $[0,3]$ into $3n$ subintervals of equal length. $\lim \|P_n\| = \lim \frac{1}{n} = 0$. Let us choose $\xi_r = \frac{r}{n}$, $r = 1, 2, \ldots, 3n$.

Then $\frac{1}{n}\sum_{r=1}^{3n} f(\frac{r}{n})$ is the Riemann sum for f on the interval $[0,3]$ corresponding to the partition P_n and the chosen intermediate points $\xi_1, \xi_2, \ldots, \xi_n$.

As f is continuous on $[0,3]$, f is integrable on $[0,3]$.

Therefore $\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{3n} f(\frac{r}{n}) = \int_0^3 f(x)dx$ and $S = \int_0^3 f(x)dx = \int_0^3 \frac{1}{1+x}dx = [\log(1+x)]_0^1 = \log 2$.

7. Evaluate $\int_a^b x^2 dx$.

Let $f(x) = x^2, x \in [a,b]$. Then f is continuous on $[a,b]$ and therefore f is integrable on $[a,b]$.

Let $P_n = (a, a+h, \ldots, a+nh)$, where $h = \frac{b-a}{n}$. Then $\|P_n\| = h$. $\{P_n\}$ is a sequence of partitions of $[a,b]$ such that $\lim \|P_n\| = 0$.

Since f is integrable on $[a, b]$,

$\int_a^b f = \lim\limits_{n\to\infty} h \sum\limits_{r=1}^{n} f(\xi_r)$ for any particular choice of intermediate points $\xi_1, \xi_2, \ldots, \xi_n$, where $a + \overline{r-1}h \le \xi_r \le a + rh$.

Let $\xi_r = a + rh$. Then $\int_a^b x^2 dx$

$= \lim\limits_{n\to\infty} h[f(a+h) + f(a+2h) + \cdots + f(a+nh)]$

$= \lim\limits_{h\to 0} h[(a+h)^2 + (a+2h)^2 + \cdots + (a+nh)^2]$

$= \lim\limits_{h\to 0} h[na^2 + ah.n(n+1) + h^2.\dfrac{n(n+1)(2n+1)}{6}]$

$= \lim\limits_{h\to 0}[a^2(b-a) + a.(b-a)(b-a+h) + \frac{(b-a)(b-a+h)(2b-2a+h)}{6}]$

$= (b-a)[a^2 + a(b-a) + \frac{(b-a)^2}{3}]$

$= \frac{(b-a)}{3}(b^2 + ab + a^2) = \frac{b^3 - a^3}{3}$.

8. Evaluate $\int_a^b x^{99} dx$ where $0 < a < b$.

Let $f(x) = x^{99}$, $x \in [a, b]$. Then f is continuous on $[a, b]$ and therefore f is integrable on $[a, b]$.

Let $P_n = (a, ah, ah^2, \ldots, ah^{n-1}, b)$ where $h^n = b/a$. Then $\|P_n\| = ah^{n-1}(h-1) = a.(\frac{b}{a})^{\frac{n-1}{n}}[(\frac{b}{a})^{\frac{1}{n}} - 1]$.

$\{P_n\}$ is a sequence of partitions of $[a, b]$ such that $\lim \|P_n\| = 0$.

Since f is integrable on $[a, b]$,

$\int_a^b f = \lim\limits_{n\to\infty}[a(h-1)f(\xi_1) + ah(h-1)f(\xi_2) + \cdots + ah^{n-1}(h-1)f(\xi_n)]$ for any patticular choice of intermediate points $\xi_1, \xi_2, \ldots, \xi_n$, where $ah^{r-1} \le \xi_r \le ah^r$.

Let $\xi_r = ah^{r-1}$.

Then $\int_a^b x^{99} dx = \lim\limits_{h\to 1} a^{100}(h-1)[1 + h^{100} + h^{2.100} + \cdots + h^{(n-1)100}]$

$= \lim\limits_{h\to 1} a^{100}(h-1)[\dfrac{h^{100n} - 1}{h^{100} - 1}]$

$= \lim\limits_{h\to 1} \dfrac{h-1}{h^{100} - 1}[(\frac{b}{a})^{100} - 1].a^{100}$

$= \frac{1}{100}.(b^{100} - a^{100})$.

9. Evaluate $\int_a^b \frac{1}{x} dx, 0 < a < b$.

Let $f(x) = \frac{1}{x}, x \in [a, b]$. Then f is continuous on $[a, b]$. Therefore f is integrable on $[a, b]$.

Let $P_n = (a, ah, ah^2, \ldots, ah^{n-1}, b)$ where $h^n = b/a$ be a partition of $[a, b]$. Then $\|P_n\| = ah^{n-1}(h-1) = a.(\frac{b}{a})^{\frac{n-1}{n}}[(\frac{b}{a})^{\frac{1}{n}} - 1]$.

$\{P_n\}$ is a sequence of partitions of $[a, b]$ such that $\lim \|P_n\| = 0$.

Since f is integrable on $[a, b]$,

$\int_a^b f = \lim\limits_{\|P_n\| \to 0} [a(h-1)f(\xi_1) + ah(h-1)f(\xi_2) + \ldots + ah^{r-1}(h-1)f(\xi_r) + \cdots + ah^{n-1}(h-1)f(\xi_n)]$ for any particular choice of intermediate points $\xi_1, \xi_2, \ldots, \xi_n$ where $ah^{r-1} \leq \xi_r \leq ah^r$.

Let $\xi_r = ah^{r-1}$. Then $\int_a^b (\frac{1}{x})dx$

$= \lim\limits_{h \to 1} [a(h-1).\frac{1}{a} + ah(h-1).\frac{1}{ah} + \cdots + ah^{n-1}(h-1).\frac{1}{ah^{n-1}}]$

$= \lim\limits_{h \to 1} n(h-1) = \lim\limits_{h \to 1} n\{(\frac{b}{a})^{\frac{1}{n}} - 1\}$

$= \lim\limits_{n \to \infty} \dfrac{e^{\frac{1}{n}\log\frac{b}{a}} - 1}{\frac{1}{n}} = \log \frac{b}{a}$.

11.10. Integration by substitution.

Theorem 11.10.1.

Let (i) $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$,

(ii) $\phi : [\alpha, \beta] \to \mathbb{R}$ be differentiable on $[\alpha, \beta]$ such that $\phi(\alpha) = a, \phi(\beta) = b$, and

(iii) $f \circ \phi$ and ϕ' are integrable on $[\alpha, \beta]$ and $\phi'(t) \neq 0$ for all $t \in [\alpha, \beta]$.

Then $\int_a^b f(x)dx = \int_\alpha^\beta f(\phi(t))\phi'(t)dt$.

Since $\phi'(t) \neq 0$ on $[\alpha, \beta]$, it follows from Darboux's theorem that either $\phi'(t) > 0$ for all $t \in [\alpha, \beta]$ or $\phi'(t) < 0$ for all $t \in [\alpha, \beta]$, i.e., either ϕ is strictly increasing on $[\alpha, \beta]$ or ϕ is strictly decreasing on $[\alpha, \beta]$.

Accordingly, the theorem can be stated in two parts.

First part.

Let (i) $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$,

(ii) $\phi : [\alpha, \beta] \to \mathbb{R}$ be differentiable and strictly increasing on $[\alpha, \beta]$ such that $\phi(\alpha) = a, \phi(\beta) = b$, and

(iii) $f \circ \phi$ and ϕ' are integrable on $[\alpha, \beta]$.

Then $\int_a^b f(x)dx = \int_\alpha^\beta f(\phi(t))\phi'(t)dt$.

Proof. Since ϕ is differentiable on $[\alpha, \beta]$, ϕ is continuous on $[\alpha, \beta]$.

Since ϕ is continuous and strictly increasing on $[\alpha, \beta]$ and $\phi(\alpha) = a, \phi(\beta) = b, \phi^{-1}$ is continuous and strictly increasing on $[a, b]$.

Let $P = (x_0, x_1, \ldots, x_n)$ be any partition of $[a, b]$ and $Q = \{y_0, y_1, \ldots, y_n\}$ where $y_i = \phi^{-1}(x_i)$ be the corresponding partition of $[\alpha, \beta]$.

By Lagrange's Mean value theorem for the function ϕ on $[y_{r-1}, y_r]$, $\phi(y_r) - \phi(y_{r-1}) = (y_r - y_{r-1})\phi'(\eta_r)$ for some $\eta_r \in (y_{r-1}, y_r)$.

That is, $x_r - x_{r-1} = (y_r - y_{r-1})\phi'(\eta_r)$, $r = 1, 2, \ldots, n$ ... ... (i)

Let $\phi(\eta_r) = \xi_r$, $r = 1, 2, \ldots, n$.

Now $S(P, f, \xi) = f(\xi_1)(x_1 - x_0) + f(\xi_2)(x_2 - x_1) + \cdots + f(\xi_n)(x_n - x_{n-1})$

$= f(\phi(\eta_1))\phi'(\eta_1)(y_1 - y_0) + \cdots + f(\phi(\eta_n))\phi'(\eta_n)(y_n - y_{n-1})$

$= S(Q, (f \circ \phi).\phi', \eta)$.

Since f is integrable on $[a, b]$, $\lim\limits_{\|P\| \to 0} S(P, f, \xi) = \int_a^b f$.

Since ϕ' is integrable on $[a, b]$, ϕ' is bounded on $[a, b]$. It follows from (i) that $\|Q\| \to 0$ as $\|P\| \to 0$.

Therefore $\int_a^b f(x)dx = \lim\limits_{\|Q\| \to 0} S(Q, (f \circ \phi).\phi', \eta)$

$= \int_\alpha^\beta (f \circ \phi).\phi'(t)dt$, since $f_0\phi$ and ϕ' are both integrable on $[\alpha, \beta]$

$= \int_\alpha^\beta f(\phi(t))\phi'(t)dt$.

Second part

Let (i) $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$,

(ii) $\phi : [\beta, \alpha] \to \mathbb{R}$ be differentiable and strictly decreasing on $[\beta, \alpha]$ such that $\phi(\alpha) = a$, $\phi(\beta) = b$, and

(iii) $f \circ \phi$ and ϕ' are integrable on $[\beta, \alpha]$.

Then $\int_a^b f(x)dx = \int_\alpha^\beta f(\phi(t))\phi'(t)dt$.

Similar proof.

Note 1. The theorem is called *substitution theorem* because under the stated conditions, the integral $\int_a^b f(x)dx$ can be evaluated by the substitution $x = \phi(t)$.

Note 2. If $\phi'(t) = 0$ at a finite number of points in $[\alpha, \beta]$ and all the other conditions remain same, the theorem still holds.

Another version of the theorem with wider conditions is given below.

Theorem 11.10.2. Let $I = [\alpha, \beta]$ be a closed and bounded interval and a function $\phi : I \to \mathbb{R}$ be such that ϕ' is continuous and $\neq 0$ on I.

Let $\phi(\alpha) = a$, $\phi(\beta) = b$ and a function f be continuous on $\phi([\alpha, \beta])$.

Then $\int_{\alpha}^{\beta} f(\phi(t))\phi'(t)dt = \int_{a}^{b} f(x)dx$.

Proof. Since ϕ' is continuous on I, ϕ is continuous on I.

Since $\phi' \neq 0$ on I, it follows from Darboux's theorem that either $\phi'(t) > 0$ for all $t \in [\alpha, \beta]$, or $\phi'(t) < 0$ for all $t \in [\alpha, \beta]$, i.e., either ϕ is strictly increasing on I, or strictly decreasing on I. Therefore $\phi[\alpha, \beta] = [a, b]$ or$[b, a]$ according as

$$\phi'(t) > 0 \text{ on } I, \text{ or } \phi'(t) < 0 \text{ on } I.$$

Case 1. Let $a < b$.

Let $F(x) = \int_{a}^{x} f(u)du, a \leq x \leq b$. Since f is continuous on $[a, b], F'(x) = f(x)$ for all $x \in [a, b]$.

Since ϕ is continuous on $[\alpha, \beta]$ and f is continuous on $[\phi(\alpha).\phi(\beta)]$, $f_o\phi$ is continuous on $[\alpha, \beta]$.

Let $G(x) = \int_{\alpha}^{x} f(\phi(t))\phi'(t)dt, x \in [\alpha, \beta]$.

Since both $f_o\phi$ and ϕ' are continuous on $[\alpha.\beta]$, $G'(x) = f(\phi(x)).\phi'(x)$, for all $x \in [\alpha, \beta]$.

Since ϕ is differentiable on $[\alpha, \beta]$ and F is differentiable on $[\phi(\alpha), \phi(\beta)]$,

$$\begin{aligned}[F(\phi(x))]' &= F'(\phi(x))\phi'(x) \\ &= f(\phi(x))\phi'(x) \text{ for all } x \in [\alpha, \beta].\end{aligned}$$

Therefore $F(\phi(x)) - G(x)$ is a constant for all $x \in [\alpha, \beta]$.

But $F(\phi(\alpha)) = G(\alpha) = 0$. Therefore $F(\phi(x)) = G(x)$ for all $x \in [\alpha, \beta]$.

In particular, $F[\phi(\beta)] = G(\beta)$, i.e., $\int_{a}^{b} f = \int_{\alpha}^{\beta} f(\phi(t))\phi'(t)dt$.

Case II. $a > b$.

Similar proof.

Examples.

1. Evaluate $\int_{-1}^{1} \frac{1}{1+x^2}dx$ by the substitution $x = \tan t$.

Let $\phi(t) = \tan t, t \in [-\frac{\pi}{4}, \frac{\pi}{4}]$.

ϕ is differentiable and strictly increasing on $[-\frac{\pi}{4}, \frac{\pi}{4}]$. $\phi'(t)$ is integrable on $[-\frac{\pi}{4}, \frac{\pi}{4}]$. $\phi(-\frac{\pi}{4}) = -1, \phi(\frac{\pi}{4}) = 1$.

Let $f(x) = \frac{1}{1+x^2}, x \in [-1, 1]$.

f is continuous and therefore integrable on $[-1. 1]$.

$$\begin{aligned}\text{Then } \int_{-1}^{1} f(x)dx &= \int_{-\pi/4}^{\pi/4} f(\phi(t))\phi'(t)dt \\ &= \int_{-\pi/4}^{\pi/4} \frac{1}{1+\tan^2 t} . \sec^2 t dt \\ &= \int_{-\pi/4}^{\pi/4} dt = \frac{\pi}{2}.\end{aligned}$$

2. Evaluate $\int_0^3 \frac{tdt}{\sqrt{1+t^2}}$.

Let $\phi(t) = 1+t^2, t \in [0,3]$. ϕ is differentiable on $[0,3]$, ϕ' is continuous and therefore integrable on $[0,3]$, $\phi'(t) > 0$ on $[0,3]$. $\phi(0) = 1$, $\phi(3) = 10$.

Let $f(x) = \frac{1}{\sqrt{x}}, x \in [1,10]$. Then f is integrable on $[1,10]$.

Then $\int_0^3 f(\phi(t))\phi'(t)dt = \int_1^{10} f(x)dx$
or, $\int_0^3 \frac{1}{\sqrt{1+t^2}} 2tdt = \int_1^{10} \frac{1}{\sqrt{x}} dx$
$= [2\sqrt{x}]_1^{10}$, by the fundamental theorem 11.8.5.
$= 2(\sqrt{10} - 1)$.

So $\int_0^3 \frac{t}{\sqrt{1+t^2}} dt = (\sqrt{10} - 1)$.

3. Evaluate $\int_{-1}^1 \frac{e^{2\tan^{-1} t}}{1+t^2} dt$.

Let $\phi(t) = \tan^{-1} t,\ t \in [-1,1]$.
ϕ is differentiable on $[-1,1]$, ϕ' is continuous and therefore integrable on $[-1,1]$. $\phi'(t) > 0$ on $[-1,1]$. $\phi(-1) = -\frac{\pi}{4}, \phi(1) = \frac{\pi}{4}$.
Let $f(x) = e^{2x}, x \in [-\frac{\pi}{4}, \frac{\pi}{4}]$. f is integrable on $[-\frac{\pi}{4}, \frac{\pi}{4}]$.
Then $\int_{-1}^1 f(\phi(t))\phi'(t)dt = \int_{-\pi/4}^{\pi/4} f(x)dx$

or, $\int_{-1}^1 \frac{e^{2\tan^{-1} t}}{1+t^2} dt = \int_{-\pi/4}^{\pi/4} e^{2x} dx$
$= [\frac{e^{2x}}{2}]_{-\pi/4}^{\pi/4}$, by the fundamental theorem
$= \sinh \frac{\pi}{2}$.

11.11. Integration by parts.

Theorem 11.11.1. Let $f : [a,b] \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ be both differentiable on $[a,b]$ and f', g' are both integrable on $[a,b]$. Then
$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx.$

Proof. Since f and g are both differentiable on $[a,b]$, fg is differentiable on $[a,b]$.

Since f and g are differentiable on $[a,b]$, f and g are continuous on $[a,b]$ and therefore they are both integrable on $[a,b]$.

Therefore $fg' + f'g$ is integrable on $[a,b]$, i.e., $(fg)'$ is integrable on $[a,b]$.

So by the fundamental theorem, $\int_a^b (fg)' = [fg]_a^b = f(b)g(b) - f(a)g(a)$. Also $\int_a^b (fg)' = \int_a^b (fg' + f'g) = \int_a^b fg' + \int_a^b f'g$.

Therefore $\int_a^b f(x)g'(x)dx + \int_a^b f'(x)g(x)dx = f(b)g(b) - f(a)g(a)$
or, $\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx$.

11.12. Mean value theorems.

Theorem 11.12.1. (First Mean value theorem)

If (i) $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be both integrable on $[a, b]$, and
(ii) $g(x)$ has the same sign for all $x \in [a, b]$,

then there is a number μ such that

$\int_a^b f(x)g(x)dx = \mu \int_a^b g(x)dx$ where $m \le \mu \le M$ and

$$m = \inf_{x\in[a,b]} f(x),\ M = \sup_{x\in[a,b]} g(x).$$

If further, f is continuous on $[a, b]$ then there exists a point ξ in $[a, b]$ such that $\int_a^b f(x)g(x)dx = f(\xi) \int_a^b g(x)dx$.

Proof. **Case 1.** Let $g(x) > 0, x \in [a, b]$.

Since $m = \inf\limits_{x\in[a,b]} f(x)$ and $M = \sup\limits_{x\in[a,b]} f(x)$, $m \le f(x) \le M$ for all $x \in [a, b]$. Therefore $mg(x) \le f(x)g(x) \le Mg(x)$ for all $x \in [a, b]$.

Since f and g are both integrable on $[a, b], mg, fg$ and Mg are integrable on $[a, b]$, and

$\int_a^b mg(x)dx \le \int_a^b f(x)g(x)dx \le \int_a^b Mg(x)dx$

or, $m \int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le M \int_a^b g(x)dx$.

Therefore $\int_a^b f(x)g(x)dx = \mu \int_a^b g(x)dx$, where $m \le \mu \le M$.

Case II. Let $g(x) < 0, x \in [a, b]$.

The proof is similar.

Second part. If f be continuous on $[a, b]$ there exists a point ξ in $[a, b]$ such that $f(\xi) = \mu$, where $m \le \mu \le M$.

It follows that $\int_a^b f(x)g(x)dx = f(\xi) \int_a^b g(x)dx$, where $a \le \xi \le b$.

Corollary. If, in particular, $g(x) = 1$ for all $x \in [a, b]$, then

$\int_a^b f(x)dx = \mu \int_a^b dx = \mu(b - a)$, where $m \le \mu \le M$.

If, moreover, f is continuous on $[a, b]$, there exists a point ξ in $[a, b]$ such that $\int_a^b f(x)dx = f(\xi)(b - a)$.

Since $\xi \in [a, b], \xi = a + \theta(b - a)$ for some θ satisfying $0 \le \theta \le 1$.

Therefore $\int_a^b f(x)dx = (b - a)f(a + \theta(b - a)), 0 \le \theta \le 1$.

Examples.

1. If $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and $\int_a^b f(x)dx = 0$, prove that there exists at least a point $c \in [a, b]$ such that $f(c) = 0$.

Since f is continuous on $[a, b], f$ is integrable on $[a, b]$

By the first Mean value theorem there exists a point c in $[a, b]$ such that $\int_a^b f(x)dx = f(c)(b-a)$.

Since $\int_a^b f(x)dx = 0$, it follows that $f(c) = 0$.

2. Use first Mean value theorem to prove that
$\frac{\pi}{6} \le \int_0^{1/2} \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}dx \le \frac{\pi}{6}.\frac{1}{\sqrt{1-k^2/4}}$, $k^2 < 1$.

Let $f(x) = \frac{1}{\sqrt{1-k^2x^2}}, x \in [0, \frac{1}{2}]$; $g(x) = \frac{1}{\sqrt{1-x^2}}, x \in [0, \frac{1}{2}]$. Then f and g are integrable on $[0, \frac{1}{2}]$ and $g(x) > 0$ for all $x \in [0, \frac{1}{2}]$.

Since f is continuous on $[0, \frac{1}{2}]$, by the first Mean value theorem there exists a point ξ in $[0, \frac{1}{2}]$ such that

$$\int_0^{1/2} f(x)g(x)dx = f(\xi) \int_0^{1/2} g(x)dx.$$

or, $\int_0^{1/2} \frac{1}{\sqrt{(1-k^2x^2)(1-x^2)}}dx = \frac{1}{\sqrt{1-k^2\xi^2}}.\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}dx = \frac{\pi}{6}.\frac{1}{\sqrt{1-k^2\xi^2}}$.

Since $0 \le \xi \le \frac{1}{2}, 1 \le \frac{1}{\sqrt{1-k^2\xi^2}} \le \frac{1}{\sqrt{1-k^2/4}}$

Therefore $\frac{\pi}{6} \le \int_0^{1/2} \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}dx \le \frac{\pi}{6}.\frac{1}{\sqrt{1-k^2/4}}$.

Lemma 11.12.2. Abel's Inequality.

If (i) $a_1, a_2, \ldots, a_n$ be n positive real numbers such that $a_1 \ge a_2 \ge \cdots \ge a_n$,

(ii) $\nu_1, \nu_2, \ldots, \nu_n \in \mathbb{R}$ and

(iii) there exist $h, H \in \mathbb{R}$ such that $h < \nu_1 + \nu_2 + \cdots + \nu_p < H$ for $1 \le p \le n$,

then $a_1 h < a_1\nu_1 + a_2\nu_2 + \cdots + a_n\nu_n < a_1 H$.

Proof. Let $s_p = \nu_1 + \nu_2 + \cdots + \nu_p, 1 \le p \le n$.

Then $a_1\nu_1 + a_2\nu_2 + \cdots + a_n\nu_n$
$= a_1s_1 + a_2(s_2 - s_1) + \cdots + a_n(s_n - s_{n-1})$
$= (a_1 - a_2)s_1 + (a_2 - a_3)s_2 + \cdots + (a_{n-1} - a_n)s_{n-1} + a_ns_n$.

Since $a_r - a_{r-1} \ge 0$ for $r = 2, 3, \ldots, n$ and $h < s_r < H$ for $r = 1, 2, \ldots, n$, we have $a_1\nu_1 + a_2\nu_2 + \cdots + a_n\nu_n$
$< H[(a_1 - a_2) + (a_2 - a_3) + \cdots + (a_{n-1} - a_n) + a_n] = a_1H$ and
$> h[(a_1 - a_2) + (a_2 - a_3) + \cdots + (a_{n-1} - a_n) + a_n] = a_1h$.
Combining, we have $a_1h < a_1\nu_1 + a_2\nu_2 + \cdots + a_n\nu_n < a_1H$.

This completes the proof.

Theorem 11.12.3. Second Mean value theorem (Bonnet's form)

If (i) $f : [a,b] \to \mathbb{R}$ and $\phi : [a,b] \to \mathbb{R}$ be both integrable on $[a,b]$, and

(ii) f is monotone decreasing and non-negative on $[a,b]$,

then there exists a point ξ in $[a,b]$ such that

$$\int_a^b f(x)\phi(x)dx = f(a)\int_a^\xi \phi(x)dx.$$

Proof. Let $P = (x_0, x_1, \dots, x_n)$ be a partition of $[a,b]$. Then $a = x_0 < x_1 < \dots < x_n = b$.

Let $M_r = \sup\limits_{x\in[x_{r-1},x_r]} \phi(x)$, $m_r = \inf\limits_{x\in[x_{r-1},x_t]} \phi(x)$, for $r = 1,2,\dots,n$.

Let $\xi_1 = a, \xi_r$ be an arbitrary point in $[x_{r-1}, x_r]$, for $r = 1,2,\dots,n$

Then $m_r(x_r - x_{r-1}) \le \int_{x_{r-1}}^{x_r} \phi(x)dx \le M_r(x_r - x_{r-1})$

and $m_r(x_r - x_{r-1}) \le \phi(\xi_r)(x_r - x_{r-1}) \le M_r(x_r - x_{r-1})$.

We have $\sum\limits_{r=1}^{p} m_r(x_r - x_{r-1}) \le \int_a^p \phi(x)dx \le \sum\limits_{r=1}^{p} M_r(x_r - x_{r-1})$

and $\sum\limits_{r=1}^{p} m_r(x_r - x_{r-1}) \le \sum\limits_{r=1}^{p} \phi(\xi_r)(x_r - x_{r-1}) \le \sum\limits_{r=1}^{p} M_r(x_r - x_{r-1})$.

Therefore $\left| \sum\limits_{r=1}^{p} \phi(\xi_r)(x_r - x_{r-1}) - \int_a^{x_p} \phi(x)dx \right| \le \sum\limits_{r=1}^{p}(M_r - m_r)(x_r - x_{r-1}) \le \sum\limits_{r=1}^{n}(M_r - m_r)(x_r - x_{r-1})$

or, $\int_a^{x_p} \phi(x)dx - \sum\limits_{r=1}^{n}(M_r - m_r)(x_r - x_{r-1}) \le \sum\limits_{r=1}^{p} \phi(\xi_r)(x_r - x_{r-1}) \le \int_a^{x_p} \phi(x)dx + \sum\limits_{r=1}^{n}(M_r - m_r)(x_r - x_{r-1})$.

As ϕ is integrable on $[a,b]$, $\int_a^x \phi(x)dx$ is continuous on $[a,b]$ and therefore $\int_a^x \phi(x)dx$ is bounded on $[a,b]$.

Let $M = \sup\limits_{x\in[a,b]} \int_a^x \phi(x)dx$, $m = \inf\limits_{x\in[a,b]} \int_a^x f(x)dx$.

Then $m - \sum\limits_{r=1}^{n}(M_r - m_r)(x_r - x_{r-1}) \le \sum\limits_{r=1}^{p} \phi(\xi_r)(x_r - x_{r-1}) \le M + \sum\limits_{r=1}^{n}(M_r - m_r)(x_r - x_{r-1})$. This inequality holds for $p = 1,2,\dots,n$.

Let $\nu_r = \phi(\xi_r)(x_r - x_{r-1}), a_r = f(\xi_r)$. Then

(i) $a_1, a_2, \dots, a_n$ are positive numbers and $a_1 \ge a_2 \ge \dots \ge a_n$,

(ii) $\nu_1, \nu_2, \dots, \nu_n$ are n numbers such that $h \le \nu_1 + \nu_2 + \dots + \nu_p \le H$ for $1 \le p \le n$, where

$h = m - \sum\limits_{r=1}^{n}(M_r - m_r)(x_r - x_{r-1}), H = M + \sum\limits_{r=1}^{n}(M_r - m_r)(x_r - x_{r-1})$.

By Abel's Inequality,

$f(a).h \le \sum_{r=1}^{n} f(\xi_r)\phi(\xi_r)(x_r - x_{r-1}) \le f(a).H.$

Let $\|P\| \to 0$. Then $h \to m$, since $\lim_{\|P\| \to 0} [U(P, \phi) - L(P, \phi)] = 0$;

$H \to M$, since $\lim_{\|P\| \to 0} [U(P, \phi) - L(P, \phi)] = 0$;

and $\sum_{r=1}^{n} f(\xi_r)\phi(\xi_r)(x_r - x_{r-1}) \to \int_a^b f(x)\phi(x)dx$, since $f\phi \in \mathcal{R}[a, b]$.

It follows that $mf(a) \le \int_a^b f(x)\phi(x)dx \le Mf(a)$

or, $\int_a^b f(x)\phi(x)dx = \mu f(a)$ where $m \le \mu \le M$.

But M, m are the supremum and the infimum of the continuous function $\int_a^x \phi(t)dt$ on $[a, b]$. Therefore there exists a point ξ in $[a, b]$ such that $\int_a^\xi \phi(t)dt = \mu$.

Consequently, $\int_a^b f(x)\phi(x)dx = f(a) \int_a^\xi \phi(x)dx, a \le \xi \le b$.

This completes the proof.

Theorem 11.12.4. Second Mean value theorem (Weierstrass' form)

If (i) $f : [a, b] \to \mathbb{R}$ and $\phi : [a, b] \to \mathbb{R}$ be both integrable on $[a, b]$, and

(ii) f is monotonic on $[a, b]$,

then there exists a point ξ in $[a, b]$ such that

$\int_a^b f(x)\phi(x)dx = f(a) \int_a^\xi \phi(x)dx + f(b) \int_\xi^b \phi(x)dx.$

Proof. **Case 1.** Let f be monotone decreasing on $[a, b]$.

Let $\psi(x) = f(x) - f(b), x \in [a, b]$.

Then ψ is monotone decreasing on $[a, b]$ and $\psi(x) \ge 0$ on $[a, b]$.

By Bonnet's theorem there exists a point ξ in $[a, b]$ such that

$\int_a^b \psi(x)\phi(x)dx = \psi(a) \int_a^\xi \phi(x)dx$

or, $\int_a^b [f(x) - f(b)]\phi(x)dx = [f(a) - f(b)] \int_a^\xi \phi(x)dx$

or, $\int_a^b f(x)\phi(x)dx = f(a) \int_a^\xi \phi(x)dx + f(b)[\int_a^b \phi(x)dx - \int_a^\xi \phi(x)dx]$

$= f(a) \int_a^\xi \phi(x)dx + f(b) \int_\xi^b \phi(x)dx.$

Case 2. Let f be monotone increasing on $[a, b]$.

Let $\psi(x) = f(b) - f(x), x \in [a, b]$.

Then ψ is monotone decreasing on $[a, b]$ and $\psi(x) \ge 0$ on $[a, b]$.

By Bonnet's theorem there exists a point ξ in $[a, b]$ such that

$\int_a^b \psi(x)\phi(x)dx = \psi(a) \int_a^\xi \phi(x)dx$

or, $\int_a^b [f(b) - f(x)]\phi(x)dx = [f(b) - f(a)] \int_a^\xi \phi(x)dx$

or, $\int_a^b f(x)\phi(x)dx = f(a) \int_a^\xi \phi(x)dx + f(b)[\int_a^b \phi(x)dx - \int_a^\xi \phi(x)dx]$

$$= f(a) \int_a^\xi \phi(x)dx + f(b) \int_\xi^b \phi(x)dx.$$

This completes the proof.

Examples (continued).

3. Show that the second Mean value theorem (Weierstrass' form) is applicable to $\int_a^b \frac{\sin x}{x} dx$ where $0 < a < b < \infty$. Also prove that

$$\left| \int_a^b \frac{\sin x}{x} dx \right| < 4/a.$$

Let $f(x) = \frac{1}{x}, x \in [a,b], \phi(x) = \sin x \in [a,b]$.

Then f and ϕ are both integrable on $[a,b]$ and f is monotone decreasing on $[a,b]$.

By the Mean value theorem (Weierstrass' form) there exists a point ξ in $[a,b]$ such that

$\int_a^b f(x)\phi(x)dx = f(a) \int_a^\xi \phi(x)dx + f(b) \int_\xi^b \phi(b)dx$

or, $\int_a^b \frac{\sin x}{x} dx = (1/a) \int_a^\xi \sin x dx + (1/b) \int_\xi^b \sin x dx$

$= (1/a)[-\cos\xi + \cos a] + (1/b)[-\cos b + \cos\xi]$

Therefore $\left| \int_a^b \frac{\sin x}{x} dx \right| \leq (\frac{1}{a}) \mid -\cos\xi + \cos a \mid + (\frac{1}{b}) \mid -\cos b + \cos\xi \mid$

$\leq (\frac{1}{a})\{\mid -\cos\xi \mid + \mid \cos a \mid\} + (\frac{1}{b})\{\mid -\cos b \mid + \mid \cos\xi \mid\}$

$\leq \frac{1}{a}(1+1) + (\frac{1}{b})(1+1)$

$< 4/a$, since $a < b$.

4. Show that the second Mean value theorem (Bonnet's form) is applicable to $\int_a^b \frac{\sin x}{x} dx$ where $0 < a < b < \infty$. Also prove that $\left| \int_a^b \frac{\sin x}{x} dx \right| \leq \frac{2}{a}$.

Let $f(x) = \frac{1}{x}, x \in [a,b], \phi(x) = \sin x, x \in [a,b]$.

Then f and ϕ are both integrable on $[a,b]$ and f is monotone decreasing on $[a,b]$ and $f(x) > 0$ for all $x \in [a,b]$.

By the Mean value theorem (Bonnet's form) there exists a point ξ in $[a,b]$ such that $\int_a^b \frac{\sin x}{x} dx = (\frac{1}{a}) \int_a^\xi \sin x dx = (\frac{1}{a})\{-\cos\xi + \cos a\}$.

Therefore $\left| \int_a^b \frac{\sin x}{x} dx \right| \leq \frac{2}{a}$.

11.13. Logarithmic function.

Definition. The logarithmic function L (or log) is defined by
$L(x) = \int_1^x \frac{1}{t} dt$, for $x > 0$.

Property 1. $L(1) = 0$.

From definition it follows that $L(1) = \int_1^1 \frac{1}{t} dt = 0$.

Property 2.
$$\begin{aligned} L(x) &< 0 \text{ if } 0 < x < 1 \\ &= 0 \text{ if } x = 1 \\ &> 0 \text{ if } x > 1. \end{aligned}$$

Proof. If $0 < x < 1$, the function f defined by $f(t) = 1/t, t \in [x, 1]$ is continuous on $[x, 1]$ and $f(t) > 0$ for all $t \in [x, 1]$.

Therefore $\int_x^1 f(t)dt > 0$. That is, $L(x) < 0$.

If $x > 1$, the fuction f defined by $f(t) = 1/t, t \in [1, x]$ is continuous on $[1, x]$ and $f(t) > 0$ for all $t \in [1, x]$.

Therefore $\int_1^x f(t)dt > 0$. That is, $L(x) > 0$.

$$\text{Thus } L(x) \begin{array}{ll} < & 0 \text{ if } 0 < x < 1 \\ = & 0 \text{ if } x = 1 \\ > & 0 \text{ if } x > 1. \end{array}$$

Property 3. For $x > 0, y > 0, L(xy) = L(x) + L(y)$.

Proof. Since $xy > 0, L(xy) = \int_1^{xy} \frac{1}{t} dt = \int_1^x (\frac{1}{t})dt + \int_x^{xy} (\frac{1}{t})dt$

$= L(x) + \int_x^{xy} (\frac{1}{t})dt = L(x) + \int_1^y \frac{1}{u} du$ [putting $t = xu$ in the second]

$= L(x) + L(y)$.

Corollary 1. In particular, if $y = \frac{1}{x}$, then $L(x) + L(\frac{1}{x}) = L(1) = 0$. Therefore $L(\frac{1}{x}) = -L(x), x > 0$.

Corollary 2. For $x > 0, y > 0, L(\frac{x}{y}) = L(x.\frac{1}{y}) = L(x) + L(\frac{1}{y}) = L(x) - L(y)$.

Property 4. For $x > 0, L(x^n) = nL(x), n$ being an integer.

Proof. **Case 1.** $n = 0$.

In this case, $L(x^n) = L(1) = 0$ and $nL(x) = 0$. Therefore $L(x^n) = nL(x)$.

Case 2. n is a positive integer.

When $n = 1$, the property holds.

Let the property hold for $n = m$, where m is a positive integer.

$$\begin{aligned} \text{Then } L(x^m) &= mL(x). \\ \text{So } L(x^{m+1}) &= L(x^m) + L(x), \text{ by property 3} \\ &= mL(x) + L(x) \\ &= (m+1)L(x). \end{aligned}$$

This shows that the property holds for $n = m+1$ if it holds for $n = m$. Also the property holds for $n = 1$.

By the principle of induction, the property holds for all positive integers n.

Case 3. n is a negative integer.

Let $n = -m$, where m is a positive integer.

$$\begin{aligned} \text{Then } L(x^n) = L(x^{-m}) &= L\{(\tfrac{1}{x})^m\} \\ &= mL(\tfrac{1}{x}), \text{ by case 2} \\ &= -mL(x) \\ &= nL(x). \end{aligned}$$

Combining all cases, the proof is complete.

Property 5. For $x > 0, L(x^\alpha) = \alpha L(x), \alpha$ being a rational number.

Proof. **Case 1.** α is an integer.

This is property 3.

Case 2. α is a positive fraction.

Let $\alpha = p/q, p$ and q are positive integers, $q > 1$.

$$\begin{aligned} L(x^\alpha) = L(x^{p/q}) &= L\{(x^{\frac{1}{q}})^p\} \\ &= pL(x^{\frac{1}{q}}), \text{ by property 3.} \end{aligned}$$

$$\begin{aligned} \text{Also } L(x) &= L\{(x^{\frac{1}{q}})^q\} \\ &= qL(x^{\frac{1}{q}}), \text{ by property 3.} \end{aligned}$$

$$\begin{aligned} \text{Therefore } L(x^\alpha) &= \tfrac{p}{q}L(x) \\ &= \alpha L(x). \end{aligned}$$

Case 3. α is a negative fraction.

Let $\alpha = -\beta$ where β is a positive fraction.

$$\begin{aligned} \text{Then } L(x^\alpha) &= L(x^{-\beta}) \\ &= L\{(\tfrac{1}{x})^\beta\} \\ &= \beta L(\tfrac{1}{x}), \text{ by case 2} \\ &= -\beta L(x) \\ &= \alpha L(x). \end{aligned}$$

Combining all cases, the proof is complete.

Corollary. If $x > 0$ and α be a real number, $L(x^\alpha) = \alpha L(x)$.

If α be irrational, let us consider a sequence $\{\alpha_n\}$ of rational points converging to α, Then $L(x^{\alpha_n}) = \alpha_n L(x)$ for all $n \in \mathbb{N}$.

Taking limit as $n \to \infty$ and noting that L is continuous, we have $L(x^\alpha) = \alpha L(x)$.

Property 6. The function L defined by $L(x) = \int_1^x \frac{1}{t} dt,\ x > 0$ is strictly increasing on $(0, \infty)$.

Also (i) $\lim\limits_{x \to \infty} L(x) = \infty$, (ii) $\lim\limits_{x \to 0+} L(x) = -\infty$.

Proof. Let $0 < x_1 < x_2$.

Then $L(x_2) - L(x_1) = \int_{x_1}^{x_2} \frac{1}{t} dt > 0$, since $\frac{1}{t}$ is continuous on $[x_1, x_2]$ and $\frac{1}{t} > 0$ on $[x_1, x_2]$.

Therefore $0 < x_1 < x_2 \Rightarrow L(x_2) > L(x_1)$.

This proves that the function L is strictly increasing on $(0, \infty)$.

(i) Let us choose $G > 0$.

Since $0 < 1/G$ and $L(2) > 0$, there exists a natural number m such that $0 < \frac{1}{mL(2)} < \frac{1}{G}$, by Archimedean property of $\mathbb{R}$.

Therefore $L(2^m) > G$.

Since L is strictly increasing on $(0, \infty)$, $L(x) > G$ for all $x > 2^m$.

This proves that $\lim\limits_{x \to \infty} L(x) = \infty$.

(ii) Let us choose $G > 0$. As in case (i), we have $L(2^m) > G$.

or, $L(\frac{1}{2^m}) < -G$.

Since L is strictly increasing on $(0, \infty), L(x) < -G$ for all $x < \frac{1}{2^m}$.

This proves that $\lim\limits_{x \to 0+} L(x) = -\infty$.

Property 7. The function L defined by $L(x) = \int_1^x \frac{1}{t} dt, x > 0$ is continuous on $(0, \infty)$.

As $L(x)$ is defined as an integral, if follows from the Theorem 11.8.1 that L is continuous on $(0, \infty)$.

Property 8. $\frac{d}{dx} L(x) = \frac{1}{x}, x > 0$.

Proof. Let $x_0 > 0$ and let us choose $h > 0$.

Then $\frac{L(x_0+h) - L(x_0)}{h} = \frac{1}{h} \int_{x_0}^{x_0+h} \frac{1}{t} dt$.

For all $t \in [x_0, x_0 + h]$, $\frac{1}{x_0+h} \le \frac{1}{t} \le \frac{1}{x_0}$.

So we have $\int_{x_0}^{x_0+h} \frac{1}{x_0+h} dt \le \int_{x_0}^{x_0+h} \frac{1}{t} dt \le \int_{x_0}^{x_0+h} \frac{1}{x_0} dt$

or, $\frac{1}{x_0+h} \le \frac{L(x_0+h) - L(x_0)}{h} \le \frac{1}{x_0}$.

By sandwich theorem, $\lim\limits_{h \to 0+} \frac{L(x_0+h) - L(x_0)}{h} = \frac{1}{x_0}$ ... (i)

Let us choose $h < 0$ such that $x_0 + h > 0$.

For all $t \in [x_0 + h, x_0]$, $\frac{1}{x_0} \le \frac{1}{t} \le \frac{1}{x_0+h}$.

So we have $\int_{x_0+h}^{x_0} \frac{1}{x_0} dt \leq \int_{x_0+h}^{x_0} \frac{1}{t} dt \leq \int_{x_0+h}^{x_0} \frac{1}{x_0+h} dt$

or, $\frac{-h}{x_0} \leq \int_{x_0+h}^{x_0} \frac{1}{t} dt \leq \frac{-h}{x_0+h}$

or, $\frac{1}{x_0} \leq \frac{1}{h} \int_{x_0}^{x_0+h} \frac{1}{t} dt \leq \frac{1}{x_0+h}$.

By sandwich theorem, $\lim\limits_{h \to 0-} \frac{L(x_0+h)-L(x_0)}{h} = \frac{1}{x_0}$ ... (ii)

From (i) and (ii) $\lim\limits_{h \to 0} \dfrac{L(x_0+h) - L(x_0)}{h} = \frac{1}{x_0}$.

This implies $\frac{d}{dx} L(x) = \frac{1}{x}, x > 0$.

Corollary. $\lim\limits_{x \to 0} \frac{L(1+x)}{x} = 1$.

Property 9. *The logarithmic function L defined by*

$$L(x) = \int_1^x \frac{1}{t} dt, x > 0$$

is a bijective function from $(0, \infty)$ to $(-\infty, \infty)$.

Proof. As L is a strictly increasing function on $(0, \infty)$, it is one-to-one on $(0, \infty)$.

As $L(x) \to \infty$ as $x \to \infty$ and $L(x) \to -\infty$ as $x \to 0+$ and L is continuous and strictly increasing on $(0, \infty)$, the function L assumes every real number in $(-\infty, \infty)$ exactly once.

This proves that L is a bijective function with domain $(0, \infty)$ and range $(-\infty, \infty)$.

Definition. The unique real number x satisfying $L(x) = 1$ is denoted by e, i.e., $L(e) = 1$. Therefore e is defined by

$$1 = \int_1^e \frac{1}{t} dt.$$

Property 10. $e = \lim\limits_{n \to \infty} (1 + \frac{1}{n})^n$.

Proof. We have $\frac{d}{dx} L(x) = \frac{1}{x}, x > 0$. So $L'(1) = 1$.

That is, $\lim\limits_{h \to 0} \frac{L(1+h)-L(1)}{h} = 1$

or, $\lim\limits_{h \to 0} \frac{L(1+h)}{h} = 1$.

Let us consider the sequence $\{h_n\}$ where $h_n = \frac{1}{n}$. $\lim h_n = 0$.

By sequential criterion, $\lim\limits_{h_n \to 0} \frac{L(1+h_n)}{h_n} = 1$

or, $\lim\limits_{n \to \infty} nL(1 + \frac{1}{n}) = 1$

or, $\lim\limits_{n \to \infty} L\{(1 + \frac{1}{n})^n\} = 1$

or, $L\{\lim\limits_{n \to \infty} (1 + \frac{1}{n})^n\} = 1$, since L is continuous.

Since $L(e) = 1$ and and L is a bijective function on $(0, \infty)$ it follows that $\lim_{n \to \infty} (1 + \frac{1}{n})^n = e$.

Property 11. For $x > -1$ and $x \neq 0$, $\frac{x}{1+x} < L(1 + x) < x$.

Proof. **Case 1.** Let $x_0 > 0$.

Then $1 + x_0 > 1$. For all $t \in [1, 1 + x_0]$, $\frac{1}{1+x_0} \leq \frac{1}{t} \leq 1$.

Let $f(t) = \frac{1}{t}, t \in [1, 1 + x_0]$. Then f is continuous on $[1, 1 + x_0]$ and $\frac{1}{1+x_0} < f(t) < 1$ at $t = 1 + \frac{x_0}{2}$.

Therefore $\int_1^{1+x_0} \frac{1}{1+x_0} dt < \int_1^{1+x_0} \frac{1}{t} dt < \int_1^{1+x_0} dt$

or, $\frac{x_0}{1+x_0} < L(1 + x_0) < x_0$.

As $x_0 > 0$ is arbitrary, $\frac{x}{1+x} < L(1 + x_0) < x$ for all $x > 0$.

Case 2. Let $-1 < x_0 < 0$.

Then $0 < x_0 + 1 < 1$.

For all $t \in [x_0 + 1, 1]$, $1 \leq \frac{1}{t} \leq \frac{1}{1+x_0}$.

Let $f(t) = \frac{1}{t}, t \in [x_0 + 1, 1]$. Then f is continuous on $[x_0 + 1, 1]$ and $1 < f(t) < \frac{1}{1+x_0}$ at $t = \frac{x_0}{2} + 1$.

Therefore $\int_{1+x_0}^{1} dt < \int_{1+x_0}^{1} \frac{1}{t} dt < \int_{1+x_0}^{1} \frac{1}{1+x_0} dt$

or, $-x_0 < -L(1 + x_0) < \frac{-x_0}{1+x_0}$.

or, $\frac{x_0}{1+x_0} < L(1 + x_0) < x_0$.

As $x_0 \in (-1, 0)$ is arbitrary, $\frac{x}{1+x} < L(1 + x) < x$ for all $x \in (-1, 0)$.

Hence for all $x > -1$ and $x \neq 0$, $\frac{x}{1+x} < L(1 + x) < x$.

Note. For all $x \in N'(0, \frac{1}{2})$ $\quad \frac{1}{1+x} < \frac{L(1+x)}{x} < 1$. But $\lim_{x \to 0} \frac{1}{1+x} = 1$.

Therefore by sandwich theorem, $\lim_{x \to 0} \frac{L(1+x)}{x} = 1$.

Property 12. $2 < e < 3$.

Proof. $L(2) = \int_1^2 \frac{1}{t} dt$.

For all $t \in [1, 2]$, $\frac{1}{t} \leq 1$. So we have $\int_1^2 \frac{1}{t} dt \leq \int_1^2 1 dt$.

The function $\frac{1}{t}$ is continuous on $[1, 2]$ and also $\frac{1}{t} < 1$ at $t = 2$.

Therefore $\int_1^2 \frac{1}{t} dt < \int_1^2 1 dt = 1$, by Theorem 11.7.5.

Since L is a strictly increasing function on $(0, \infty)$ and $L(2) < 1$ and $L(e) = 1$, it follows that $2 < e$ ... ... (A)

Again $L(3) = \int_1^3 \frac{1}{t} dt = \int_1^2 \frac{1}{t} dt + \int_2^3 \frac{1}{t} dt$.

Now $\int_1^2 \frac{1}{t} dt = \int_0^1 \frac{du}{2-u}$, by the substitution $t = 2 - u$

and $\int_2^3 \frac{1}{t} dt = \int_0^1 \frac{du}{2+u}$, by the substitution $t = 2 + u$.

Hence $\int_1^3 \frac{1}{t}dt = \int_0^1 \frac{du}{2-u} + \int_0^1 \frac{du}{2+u} = 4\int_0^1 \frac{du}{4-u^2}$.

In $0 \le u \le 1$, $\frac{1}{4-u^2}$ is continuous and $\frac{1}{4-u^2} \ge \frac{1}{4}$.

It follows that $\int_0^1 \frac{du}{4-u^2} \ge \int_0^1 \frac{1}{4}du$.

Both the functions f and g where $f(u) = \frac{1}{4-u^2}, u \in [0,1]$; $g(u) = \frac{1}{4}, u \in [0,1]$ are continuous on $[0,1]$ and $f(\frac{1}{2}) > g(\frac{1}{2})$.

Therefore $\int_0^1 \frac{1}{4-u^2} > \frac{1}{4}$, by Theorem 11.7.5.

Consequently, $\int_1^3 \frac{1}{t}dt > 1$. That is, $L(3) > 1$.

Since L is a strictly increasing function on $(0,\infty)$ and $L(3) > 1$ and $L(e) = 1$, it follows that $e < 3$ ... ... (B)

From (A) and (B) it follows that $2 < e < 3$.

11.14. Exponential function.

Since the logarithmic function is a bijective function on $(0,\infty)$ with the range $(-\infty,\infty)$, it admits of an inverse function.

The inverse function of the logarithmic function is said to be the *exponential function* and it is denoted by $E(x)$ or e^x.

The domain of the exponential function is $\mathbb{R}$ and the range is $(0,\infty)$.

Since the logarithmic function is continuous and strictly increasing on $(0,\infty)$ with its range $(-\infty,\infty)$, the exponential function is continuous and strictly increasing on $(-\infty,\infty)$ with its range $(0,\infty)$.

Therefore for all $x > 0$, $EL(x) = x$ and for all $x \in \mathbb{R}$, $LE(x) = x$.

Property 1. $E(0) = 1$.

Since $EL(x) = x$ for all $x > 0$, $EL(1) = 1$ and since $L(1) = 0$, it follows that $E(0) = 1$.

Property 2. $E(x)E(y) = E(x+y)$ for all $x, y \in \mathbb{R}$.

Proof. For all $x, y \in \mathbb{R}, LE(x+y) = x + y = LE(x) + LE(y)$.

Since $E(x) > 0$ for all real x,

$L(E(x)) + L(E(y)) = L(E(x) + E(y))$, by property 3, 11.13.

or, $LE(x+y) = L(E(x) + E(y))$.

Since logarithmic function is one-to-one, it follows that

$E(x+y) = E(x).E(y)$.

Corollary. In particular, if $y = -x$, then $E(x-x) = E(x).E(-x)$ or, $E(x).E(-x) = E(0) = 1$. i.e., $E(-x) = \frac{1}{E(x)}$ for all real x.

Property 3. $E(nx) = \{E(x)\}^n, n$ being an integer.

Proof. **Case 1.** $n = 0$. The property holds trivially.

Case 2. n is a positive integer.

The property holds for $n = 1$.

Let us assume that the property holds for $n = m$, a positive integer. Then $E(mx) = \{E(x)\}^m$.

$$\begin{aligned} E\{(m+1)x\} = E(mx + x) &= E(mx)E(x) \\ &= \{E(x)^m\}E(x) = \{E(x)\}^{m+1}. \end{aligned}$$

This shows that the property holds for $n = m+1$ if it holds for $n = 1$.

By the principle of induction, $E(nx) = \{E(x)\}^n$ for all positive integers n.

Case 3. n is a negative integer, say $n = -m$.

$$\begin{aligned} E(nx) &= E(-mx) \\ &= \tfrac{1}{E(mx)} = \tfrac{1}{\{E(x)\}^m} = \{E(x)\}^{-m} = \{E(x)\}^n. \end{aligned}$$

Combining all case, the proof is complete

Property 4. $E(\alpha x) = \{E(x)\}^\alpha, \alpha$ being a rational number.

Proof. **Case 1.** $n = 0$. The property holds trivially.

Case 2. α is a positive rational number, say $\alpha = \frac{p}{q}$, where p and q are positive integers.

$E(px) = E(q.(\frac{p}{q})x) = [E((\frac{p}{q})x)]^q$, by case 1.

Also $E(px) = [E(x)]^p$. Therefore $[E(x)]^p = [E((\frac{p}{q})x)]^q$.

Since $E(x) > 0$ for all $x, [E(x)]^{\frac{p}{q}} = E((\frac{p}{q})x)$

or, $E(\alpha x) = \{E(x)\}^\alpha$.

Case 3. α is a negative rational number.

Let $\alpha = -\beta$, where β is a positive rational number.

$$\begin{aligned} E(\alpha x) = E(-\beta x) = \tfrac{1}{E(\beta x)} &= \tfrac{1}{\{E(x)\}^\beta}, \text{ by case 1} \\ &= \{E(x)\}^{-\beta} = \{E(x)\}^\alpha. \end{aligned}$$

Combining the cases, the proof is complete.

Property 5. $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$

Proof. Let $y = e^x - 1$. Then $x = \log(1 + y)$.

Since the exponential function is continuous on $(-\infty, \infty), x \to 0$ implies $e^x \to e^0 = 1$. Therefore as $x \to 0, y \to 0$.

Now $\lim\limits_{x\to 0} \frac{e^x-1}{x} = \lim\limits_{y\to 0} \frac{y}{\log(1+y)}$.

But $\lim\limits_{y\to 0} \frac{\log(1+y)}{y} = 1$, by the corollary of property 9, 11.13.

Hence $\lim\limits_{x\to 0} \frac{e^x-1}{x} = 1$.

Property 6. $\frac{d}{dx}(e^x) = e^x$ for all $x \in \mathbb{R}$.

Proof. Let $x_0 \in \mathbb{R}$. Then

$$\lim_{h \to 0} \frac{e^{x_0+h} - e^{x_0}}{h} = \lim_{h \to 0} \frac{e^{x_0}e^h - e^{x_0}}{h} = e^{x_0} \lim_{h \to 0} \frac{e^h - 1}{h} = e^{x_0}, \text{ by property 5.}$$

Hence $\frac{d}{dx}(e^x) = e^x$ for all $x \in \mathbb{R}$.

Worked Examples.

1. Prove that $\lim\limits_{x \to \infty} \frac{L(x)}{x^\alpha} = 0$ for all real $\alpha > 0$.

$\lim\limits_{x \to \infty} L(x) = \infty$ and $\lim\limits_{x \to \infty} x^\alpha = \infty$, for all real $\alpha > 0$.
Let us choose β such that $0 < \beta < \alpha$.
If $t > 1$, then $t^\beta > 1$ and $0 < t^{-1} < t^{-1+\beta}$.
If $x > 1$, then $0 \leq \int_1^x t^{-1}dt \leq \int_1^x t^{-1+\beta}dt$.
t^{-1} and $t^{-1+\beta}$ are both continuous on $[1, x]$ and $0 < t^{-1} < t^{-1+\beta}$ for $1 < t \leq x$.
Therefore if $x > 1$, $0 < \int_1^x t^{-1}dt < \int_1^x t^{-1+\beta}dt$.
or, $0 < L(x) < \frac{x^\beta - 1}{\beta} < \frac{x^\beta}{\beta}$.
Therefore if $x > 1$, $0 < \frac{L(x)}{x^\alpha} < \frac{1}{\beta x^{\alpha-\beta}}$, since $x^\alpha > 0$.
$\lim\limits_{x \to \infty} \frac{1}{x^{\alpha-\beta}} = 0$, since $\alpha > \beta$ and therefore $\lim\limits_{x \to \infty} \frac{L(x)}{x^\alpha} = 0$.

Note. This result says that if $\alpha > 0$ then x^α tends to ∞ with x more rapidly than $L(x)$ does.

2. Prove that $\lim\limits_{x \to \infty} \frac{x^\alpha}{e^x} = 0$ for all real α..

Case 1. $\alpha > 0$.
We have $\lim\limits_{y \to \infty} \frac{L(y)}{y^\beta} = 0$ for all $\beta > 0$.
Let $\alpha = \frac{1}{\beta}$. Then $\lim\limits_{y \to \infty} \frac{L(y)^\alpha}{y} = 0$ for all $\alpha > 0$.
Let $L(y) = x$. Then $x \to \infty$ as $y \to \infty$ and $\lim\limits_{x \to \infty} \frac{x^\alpha}{e^x} = 0$ for all $\alpha > 0$.

Case 2. $\alpha = 0$.
$\lim\limits_{x \to \infty} \frac{x^\alpha}{e^x} = \lim\limits_{x \to \infty} \frac{1}{e^x} = \lim\limits_{x \to \infty} e^{-x} = \lim\limits_{y \to -\infty} e^y = 0$.

Case 3. $\alpha < 0$. Let $\beta = -\alpha$. Then $\beta > 0$.
$\lim\limits_{x \to \infty} \frac{x^\alpha}{e^x} = \lim\limits_{x \to \infty} \frac{1}{x^\beta e^x} = 0$, since $\lim\limits_{x \to \infty} \frac{1}{e^x} = 0$ and $\lim\limits_{x \to \infty} \frac{1}{x^\beta} = 0$.
Combining all the cases, we have the result.

Note. This result says that $E(x)$ tends to ∞ with x more rapidly than any power of x does.

Exercises 21

1. Take the partition $P_n = (0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n})$ of $[0,1]$ and show that $\sup\{L(P_n, f) : n \in \mathbb{N}\} = \inf\{U(P_n, f) : n \in \mathbb{N}\}$ for the function f.

   (i) $f(x) = x^2, x \in [0,1]$, (ii) $f(x) = x^3, x \in [0,1]$.

   Deduce that f is integrable on $[0,1]$.

2. Let $f : [a,b] \to \mathbb{R}$ be bounded and monotone increasing on $[a,b]$. If P_n be the partition of $[a,b]$ dividing into n sub-intervals of equal length prove that

   $\int_a^b f \le U(P_n, f) \le \int_a^b f + \frac{b-a}{n}[f(b) - f(a)]$.

   Consider the sequence of partitions $\{P_n\}$ and deduce that $\lim\limits_{n\to\infty} U(P_n, f) = \int_a^b f$.

   Utilise this result to evaluate

   (i) $\int_0^1 x dx$, (ii) $\int_0^1 x^2 dx$, (iii) $\int_0^1 e^x dx$.

   [**Hint.** $U(P_n, f) - L(P_n, f) = \frac{b-a}{n}[f(b) - f(a)]$, $L(P_n, f) \le \int_a^b f \le U(P_n, f)$.]

3. Let $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$. Let P_n be the partition of $[a,b]$ dividing into n sub intervals of equal length. Consider the sequence of partitions $\{P_n\}$ and prove that $\lim\limits_{n\to\infty} L(P_n, f) = \underline{\int}_a^b f$, $\lim\limits_{n\to\infty} U(P_n, f) = \overline{\int}_a^b f$.

   Evaluate $\underline{\int}_0^1 f$ and $\overline{\int}_0^1 f$ when

   (i) $f(x) = 2x, x \in [0,1]$, (ii) $f(x) = \cos x, x \in [0, \frac{\pi}{2}$,

4. Let $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$ $(0 < a < b)$. Let $P_n = (a, ar, ar^2, \ldots, ar^n)$ where $r^n = b/a$. Consider the sequence of partitions $\{P_n\}$ and prove that $\lim\limits_{n\to\infty} L(P_n, f) = \underline{\int}_a^b f$, $\lim\limits_{n\to\infty} U(P_n, f) = \overline{\int}_a^b f$.

   Evaluate $\underline{\int}_1^2 f$ and $\overline{\int}_1^2 f$ when

   (i) $f(x) = x^9, x \in [1,2]$, (ii) $f(x) = x^{99}, x \in [1,2]$.

5. A function f is defined on [0,1] by $f(x) = x^2$, x is rational
   $= x^3$, x is irrational.

   (i) Evaluate $\underline{\int}_0^1, \overline{\int}_0^1 f$; (ii) Show that f is not integrable on $[0,1]$.

6. A function f is defined on [0,1] by $f(x) = x^2 + x^3$, x is rational
   $= x + x^2$, x is irrational.

   (i) Evaluate $\underline{\int}_0^1 f, \overline{\int}_0^1 f$; (ii) Show that f is not integrable on $[0,1]$.

7. A function f is defined on $[0,1]$ by $f(x) = \sin x$, x is rational
$= x$, x is irrational.

(i) Evaluate $\underline{\int}_0^{\frac{\pi}{2}} f, \overline{\int}_0^{\frac{\pi}{2}} f$; (ii) Show that f is not integrable on $[0, \frac{\pi}{2}]$.

8. Let $a, b \in \mathbb{R}$ and $a < b$. If $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$, prove that f is integrable on $[a, b]$.

Give an example of a function f integrable on $[0, 1]$, but f is not a function of bounded variation on $[0, 1]$.

9. Let $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. Prove that

(i) $\max(f, g) : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$;

(ii) $\min(f, g) : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$.

[**Hint.** $\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2} \mid f - g \mid$; $\min(f, g) = \frac{1}{2}(f + g) - \frac{1}{2} \mid f - g \mid$.]

10. Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. Prove that

(i) $f^+ : [a, b] \to \mathbb{R}$ defined by $f^+(x) = \sup\{f(x), 0\}, x \in [a, b]$ is integrable on $[a, b]$.

(ii) $f^- : [a, b] \to \mathbb{R}$ defined by $f^-(x) = -\inf\{f(x), 0\}, x \in [a, b]$ is integrable on $[a, b]$.

[**Hint.** $f^+ = \frac{1}{2}(\mid f \mid + f)$; $f^- = \frac{1}{2}(\mid f \mid - f)$.]

11. Let $f(x) = x[x], x \in [0, 3]$. Show that f is integrable on $[0, 3]$. Evaluate $\int_0^3 f$.

12. Let $f(x) = x - [x], x \in [0, 3]$. Show that f is integrable on $[0, 3]$. Evaluate $\int_0^3 f$.

13. A function f is defined on $I = [0, 10]$ by $f(x) = 0$ when $x \in I \cap \mathbb{Z}$.
$= 1$ when $x \in I - \mathbb{Z}$.

Prove that f is integrable on I. Evaluate $\int_0^{10} f$.

14. A function f is defined on $[0, 1]$ by $f(0) = 1$,

$f(x) = (-1)^{n-1}$ when $\frac{1}{n+1} < x \leq \frac{1}{n}$ $(n = 1, 2, 3, \ldots)$.

Prove that (i) f is integrable on $[0, 1]$, (ii) $\int_0^1 f = \log(4/e)$.

15. A function f is defined on $[0, 1]$ by $f(0) = 0$,

$f(x) = \frac{1}{2^n}, \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n}$ $(n = 0, 1, 2, \ldots)$.

Prove that (i) f is integrable on $[0, 1]$, (ii) $\int_0^1 f = \frac{2}{3}$.

16. A function f is defined on $[0, 1]$ by $f(0) = 0$,

$f(x) = \lim\limits_{n \to \infty} \frac{(1+\sin \frac{\pi}{x})^n - 1}{(1+\sin \frac{\pi}{x})^n + 1}$, $x \in (0, 1]$.

Prove that (i) f is integrable on $[0, 1]$.

[**Hint.** $\frac{1}{2} < x < 1 \Rightarrow \pi < \frac{\pi}{x} < 2\pi \Rightarrow 0 < 1 + \sin\frac{\pi}{x} < 1 \Rightarrow f(x) = -1$.
$\frac{1}{3} < x < \frac{1}{2} \Rightarrow 2\pi < \frac{\pi}{x} < 3\pi \Rightarrow 1 < 1 + \sin\frac{\pi}{x} < 2 \Rightarrow f(x) = 1$]

17. Show that

(i) $-\frac{1}{2} < \int_0^1 \frac{x^3 \cos 5x}{2+x^2} dx < \frac{1}{2}$ (ii) $\frac{1}{3\sqrt{2}} < \int_0^1 \frac{x^2}{\sqrt{1+x}} dx < \frac{1}{3}$

(iii) $\frac{\pi^3}{24\sqrt{2}} < \int_0^{\pi/2} \frac{x^2}{\sin x + \cos x} dx < \frac{\pi^3}{24}$ (iv) $\frac{1}{3} < \int_0^1 \frac{dx}{1+x+x^2} < \frac{\pi}{4}$

(v) $\frac{\pi^3}{96} < \int_{-\pi/2}^{\pi/2} \frac{x^2}{5+3\sin x} dx < \frac{\pi^3}{24}$.

18. (i) If a function f is continuous on a closed interval $[a, b]$ and $\int_a^b fg = 0$ for every continuous function g on $[a, b]$, prove that $f(x) = 0$ for all $x \in [a, b]$.

[**Hint.** Take, in particular, $g = f$.]

(ii) A function f is integrable on $[a, b]$ and $\int_a^b f^2(x)dx = 0$. Prove that $f(x) = 0$ at every point of continuity in $[a, b]$.

19. A function f is continuous for all $x \geq 0$ and $f(x) \neq 0$ for all $x > 0$. If $\{f(x)\}^2 = 2\int_0^x f(t)dt$ prove that $f(x) = x$ for all $x \geq 0$.

[**Hint.** $f(0) = 0$ and $f'(x) = 1$ for all $x > 0$. Use Lagrange's mean value theorem to f on $[0, x]$.]

20. The functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are both continuous on $[a, b]$ and $\int_a^b | f - g | = 0$. Prove that $f = g$.

Give an example of functions f and g both integrable on $[a, b]$ such that $\int_a^b | f - g | = 0$, but $f \neq g$.

21. A function f is defined on $[0, 2]$ by
$$f(x) = 0, 0 \leq x \leq 1$$
$$= 1, 1 < x \leq 2$$
$$= 2, 2 < x \leq 3.$$

Let $F(x) = \int_0^x f(t)dt, x \in [0, 2]$. Find F. Show that F is continuous on $[0, 2]$.

22. A function f is defined on $[0, 3]$ by
$$f(x) = x, 0 \leq x \leq 1$$
$$= 1, 1 < x \leq 2$$
$$= x - 1, 2 < x \leq 3.$$

Show that f is integrable on $[0, 3]$.

Let $F(x) = \int_0^x f(t)dt, x \in [0, 3]$. Find F. Show that $F'(x) = f(x), x \in [0, 3]$.

23. A function f is defined on $[0, 3]$ by
$$f(x) = x, 0 \leq x < 1$$
$$= 1, 1 \leq x \leq 2$$
$$= x, 2 < x \leq 3.$$

Show that f is integrable on $[0, 3]$.

Let $F(x)=\int_0^x f(t)dt, 0\le x\le 3$. Find F. Find $F'(x)$ at all points where F is differentiable.

24. For $x\ge 0$, let $\phi(x)=\lim\limits_{n\to\infty}\frac{x^n+2}{x^n+1}$; and $f(x)=\int_0^x \phi(t)dt$.

Show that f is continuous at 1 but not differentiable at 1.

25. Find F' where F is defined on $[1,\infty)$ by

(i) $F(x)=\int_x^{e^x}\sqrt{1+t^2}dt$, (ii) $F(x)=\int_x^{x^2}\sin\sqrt{t}dt$,

26. Prove that

(i) $\lim\limits_{x\to 2}\dfrac{\int_2^x e^{\sqrt{1+t^2}}dt}{x-2}=e^{\sqrt{5}}$, (ii) $\lim\limits_{x\to 0}\dfrac{\int_0^{x^2}\sin\sqrt{t}dt}{x^3}=\frac{2}{3}$,

(iii) $\lim\limits_{x\to 0}\dfrac{\int_{-x}^{x} f(t)dt}{\int_0^{2x} f(t+1)dt}=\dfrac{f(0)}{f(1)}$, where f is continuous on $\mathbb{R}$.

27. A function f is continuous on $\mathbb{R}$ and $\int_{-x}^{x} f(t)dt=0$ for all $x\in\mathbb{R}$. Prove that f is an odd function on $\mathbb{R}$.

28. A function f is defined on $[-3,3]$ by
$$f(x)=2x\sin\tfrac{\pi}{x}-\pi\cos\tfrac{\pi}{x}, x\ne 0.$$
$$=0,\quad x=0.$$

Show that (i) f is not continuous on $[-3,3]$; (ii) f is integrable on $[-3,3]$;

(iii) f has an antiderivative ϕ on $[-3,3]$; (iv) $\int_{-3}^{3} f=\phi(3)-\phi(-3)$.

29. Let $f(x)=\operatorname{sgn} x, x\in[-1,3]$.

(i) Show that f is integrable on $[-1,3]$. (ii) Evaluate $\int_{-1}^{3} f$.

(iii) Show that the evaluation of $\int_{-1}^{3} f$ cannot be done by the fundamental theorem of Integral calculus.

30. Let $f(x)=x[x], x\in[0,3]$.

(i) Show that f is integrable on $[0,3]$. (ii) Evaluate $\int_0^3 f$.

(iii) Show that the evaluation of $\int_0^3 f$ cannot be done by the fundamental theorem of Integral calculus.

31. Let $f(x)=[x]$, $x\in[1,3]$; $\phi(x)=x$, $x\in[1,2]$
$$=2x-2,\ x\in(2,3].$$

(i) Show that f is integrable on $[1,3]$. (ii) Evaluate $\int_1^3 f$.

(iii) Without evaluating the integral show that $\int_1^3 f=\phi(3)-\phi(1)$.

32. A function f is defined on $[0,1]$ by $f(x) = 2x$, x is rational
$= 1 - x$, x is irrational.

Take the partition $P_n = (0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n})$ of $[0,1]$. Choose intermediate points α_r, β_r in $[x_{r-1}, x_r]$ to show that $\lim\limits_{n\to\infty} S(P_n, f, \alpha) \neq \lim\limits_{n\to\infty} S(P_n, f, \beta)$.

Deduce that f is not integrable on $[0,1]$.

33. Evaluate the limits

(i) $\lim\limits_{n\to\infty} [\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+3n}]$

(ii) $\lim\limits_{n\to\infty} [\sin \frac{\pi}{n} + \sin \frac{2\pi}{n} + \cdots + \sin \frac{n\pi}{n}]$

(iii) $\lim\limits_{n\to\infty} [\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+4n^2}]$

(iv) $\lim\limits_{n\to\infty} [(1+\frac{1}{n})(1+\frac{2}{n})\cdots(1+\frac{n}{n})]^{\frac{1}{n}}$

(v) $\lim\limits_{n\to\infty} [(1+\frac{1}{n^2})(1+\frac{2^2}{n^2})^2 \cdots (1+\frac{n^2}{n^2})^n]^{\frac{1}{n}}$.

34. Let $f : [a,b] \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ be both continuous on $[a,b]$ and $\int_a^b f = \int_a^b g$. Prove that there exists a point c in $[a,b]$ such that $f(c) = g(c)$.

35. A function $f : [0,1] \to \mathbb{R}$ is continuous on $[0,1]$ and $\int_0^x f(t)dt = \int_x^1 f(t)dt$ for all $x \in [0,1]$. Prove that $f(x) = 0$ for all $x \in [0,1]$.

36. (a) Discuss the applicability of the second Mean value theorem to the integral $\int_{-\pi/2}^{\pi/2} x^2 \cos x\, dx$.

(b) Verify second Mean value theorem (Weierstrass form) for the function f on the indicated intervals.

(i) $f(x) = x \sin x, x \in [-\pi/2, \pi/2]$, (ii) $f(x) = xe^x, x \in [-1,1]$, (iii) $f(x) = x \sin x, x \in [\pi, 2\pi]$.

37. Use Bonnet's form of second Mean value theorem to prove that $|\int_a^b \sin x^2 dx| \leq \frac{1}{a}$ if $0 < a < b < \infty$,

[**Hint.** Take $f(x) = \frac{1}{2x}$, $x \in [a,b]$, $\phi(x) = 2x \sin x^2$, $x \in [a,b]$ in the theorem 11.12.3.]

38. If f_0 is continuous on $[0,\infty)$ and for all $n \in \mathbb{N}$, $f_n(x) = \int_0^x f_{n-1}(t)dt$, $x \geq 0$. Prove that $f_n(x) = \frac{1}{(n-1)!}\int_0^x f_0(t)(x-t)^{n-1}dt$.

[**Hint.** f_n is continuous on $[0,\infty)$ for all $n \in \mathbb{N}$. On integration by parts,
$f_n(x) = xf_{n-1}(x) - \int_0^x tf_{n-2}(t)dt = \int_0^x (x-t)f_{n-2}(t)dt$.
Integrate by parts successively.]

39. A function f is continuous on $[0, \infty)$ and $\phi(x) = \frac{1}{3!}\int_0^x (x-t)^3 f(t)dt$, $x \geq 0$. Show that $\phi^{iv}(x) = f(x)$ for all $x \geq 0$.

40. Justifying each step evaluate the integrals by substitution

(i) $\int_0^2 t\sqrt{1+t^2}dt$, (ii) $\int_0^2 t^2\sqrt{1+t^3}dt$,

(iii) $\int_0^2 te^{t^2}dt$, (iv) $\int_0^{\frac{\pi}{2}} \sin^3 t \cos t dt$.

41. If $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and $g(x) = \int_a^x f(t)dt$ for all $x \in [a, b]$, prove that g is a function of bounded variation on $[a, b]$.

[**Hint.** Using Theorem 11.8.1, $|g(x_2) - g(x_1)| \leq k|x_2 - x_1|$ for any two points x_1, x_2 in $[a, b]$.]

42. If $n \in \mathbb{N}$ and $n \geq 2$, prove that $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \log n < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$.

[**Hint.** If $k \in \mathbb{N}$ and $k \geq 2$, $\frac{1}{k} < \int_{k-1}^k \frac{1}{t}dt < \frac{1}{k-1}$.]

43. If $f : [a, b] \to \mathbb{R}$ be bounded and integrable on $[a, b]$, prove that
(i) $\lim\limits_{n\to\infty} \int_a^b f(x) \sin nx\, dx = 0$ (ii) $\lim\limits_{n\to\infty} \int_a^b f(x) \cos nx\, dx = 0$.

[Riemann-Lebesgue theorem]

[**Hint.**(i) Let $\epsilon > 0$. Since f is integrable on $[a, b]$, there exists a partition $P = (a = x_0, x_1, ..., x_{p-1}, x_p = b)$ of $[a, b]$ such that $U(P, f) - L(P, f) < \frac{\epsilon}{2}$.

Let M_r, m_r be the supremum and the infimum of f on $[x_{r-1}, x_r]$, $r = 1, 2, ..., p$.

$\int_a^b f(x) \sin nx\, dx = \int_a^{x_1} f(x) \sin nx\, dx + \cdots + \int_{x_{p-1}}^b f(x) \sin nx\, dx$
$= \{\int_a^{x_1}[f(x) - f(x_1)] \sin nx\, dx + \cdots + \int_{x_{p-1}}^b [f(x) - f(b)] \sin nx\, dx\} + \{\int_a^{x_1} f(x_1) \sin nx\, dx + \cdots + \int_{x_{p-1}}^b f(b) \sin nx\, dx\} = s_1 + s_2$, say.

$|s_1| \leq \int_a^{x_1} |f(x) - f(x_1)|dx + \cdots + \int_{x_{p-1}}^b |f(x) - f(b)|dx \leq (M_1 - m_1)(x_1 - a) + \cdots + (M_p - m_p)(b - x_{p-1}) < \frac{\epsilon}{2}$.

$|s_2| \leq |f(x_1)| \int_a^{x_1} \sin nx\, dx + \cdots + |f(x_p)| \int_{x_{p-1}}^b \sin nx\, dx \leq \frac{2}{n}[|f(x_1)| + \cdots + |f(x_p)|]$, since $|\int_{x_{r-1}}^{x_r} \sin nx\, dx| \leq |\frac{\cos nx_{r-1} - \cos nx_r}{n}| \leq \frac{2}{n}$.

Since $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, there exists a natural number k such that $|s_2| < \frac{\epsilon}{2}$ for all $n > k$.

Therefore $|\int_a^b f(x) \sin nx\, dx| \leq |s_1| + |s_2| < \epsilon$ for all $n > k$.]

12. IMPROPER INTEGRALS

12.1. Introduction. In the preceding chapter on Riemann Integral the theory of integration was developed under two assumptions —

(i) the interval of integration was required to be a closed and bounded interval, and

(ii) the integrand was required to be bounded on the interval.

The scope of the theory of integration may be widened by relaxing these restrictions. If these restrictions are relaxed we have the following two types of integrals, called *improper integrals* or *infinite integrals* —

(a) *improper integrals on a finite interval where the integrand is unbounded*;

(b) *improper integrals on an unbounded interval.*

We define *convergence* of improper integrals and discuss the properties of each type separately.

A. Improper integrals on a closed and bounded interval, the integrand having infinite discontinuities.

12.2. Definitions.

I. Convergence of the improper integral $\int_a^b f(x)dx$ when a is the *only* point of infinite discontinuity of f in $[a, b]$.

Let the left end point a of the closed and bounded interval $[a, b]$ be the only point of infinite discontinuity of a function f which is bounded and integrable on $[a+\epsilon, b]$ for every ϵ satisfying $0 < \epsilon < b - a$.

Let $\phi(\epsilon) = \int_{a+\epsilon}^b f(x)dx, 0 < \epsilon < b - a$.

If $\lim\limits_{\epsilon \to 0+} \phi(\epsilon)$ exists (finitely) then the improper integral $\int_a^b f(x)dx$ is said to be *convergent*. If the limit be l, we write $\int_a^b f(x)dx = l$.

If $\lim\limits_{\epsilon \to 0+} \phi(\epsilon)$ does not exist (finitely) then the improper integral $\int_a^b f(x)dx$ is said to be *divergent*.

Note. If a be the *only* point of infinite discontinuity of a function f which is bounded and integrable on $[a+\epsilon, b]$ for every ϵ satisfying $0 < \epsilon < b-a$ and $\int_a^b f(x)dx$ is convergent, then $\int_a^c f(x)dx$ is also convergent for all $c \in (a,b)$.

Examples.

1. The integral $\int_0^1 \frac{1}{x}dx$ is improper, since 0 is a point of infinite discontinuity of the integrand.

The integrand is bounded and integrable on $[0+\epsilon, 1]$ for all ϵ satisfying $0 < \epsilon < 1$.

$\lim\limits_{\epsilon \to 0+} \int_{0+\epsilon}^1 \frac{1}{x}dx = \lim\limits_{\epsilon \to 0+} [-\log \epsilon] = \infty$.

Therefore the improper integral $\int_0^1 \frac{1}{x}dx$ is divergent.

2. The integral $\int_0^1 \frac{1}{\sqrt{x}}dx$ is improper, since 0 is a point of infinite discontinuity of the integrand.

The integrand is bounded and integrable on $[0+\epsilon, 1]$ for all ϵ satisfying $0 < \epsilon < 1$.

$\lim\limits_{\epsilon \to 0+} \int_{0+\epsilon}^1 \frac{1}{\sqrt{x}}dx = \lim\limits_{\epsilon \to 0+} [2 - 2\sqrt{\epsilon}] = 2$.

Therefore the improper integral $\int_0^1 \frac{1}{\sqrt{x}}dx$ is convergent and $\int_0^1 \frac{1}{\sqrt{x}}dx = 2$.

II. Convergence of the improper integral $\int_a^b f(x)dx$ when b is the *only* point of infinite discontinuity of f in $[a,b]$.

Let the right end point b of the closed and bounded interval $[a,b]$ be the only point of infinite discontinuity of a function f which is bounded and integrable on $[a, b-\epsilon]$ for every ϵ satisfying $0 < \epsilon < b-a$.

Let $\phi(\epsilon) = \int_a^{b-\epsilon} f(x)dx, 0 < \epsilon < b-a$.

If $\lim\limits_{\epsilon \to 0+} \phi(\epsilon)$ exists (finitely) then the improper integral $\int_a^b f(x)dx$ is said to be *convergent*. If the limit be l, we write $\int_a^b f(x)dx = l$.

If $\lim\limits_{\epsilon \to 0+} \phi(\epsilon)$ does not exist (finitely) then the improper integral $\int_a^b f(x)dx$ is said to be *divergent*.

Note. If b be the *only* point of infinite discontinuity of a function f which is bounded and integrable on $[a, b-\epsilon]$ for every ϵ satisfying $0 < \epsilon < b-a$ and $\int_a^b f(x)dx$ is convergent, then $\int_c^b f(x)dx$ is also convergent for all $c \in (a,b)$.

Examples (continued).

3. The integral $\int_0^1 \frac{1}{1-x}dx$ is improper, since 1 is a point of infinite discontinuity of the integrand.

The integrand is bounded and integrable on $[0, 1-\epsilon]$ for all ϵ satisfying $0 < \epsilon < 1$.

$\lim\limits_{\epsilon \to 0+} \int_0^{1-\epsilon} \frac{1}{1-x}dx = \lim\limits_{\epsilon \to 0+} [-\log \epsilon] = \infty.$

Therefore the improper integral $\int_0^1 \frac{1}{1-x}dx$ is divergent.

4. The integral $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$ is improper, since 1 is a point of infinite discontinuity of the integrand.

The integrand is bounded and integrable on $[0, 1-\epsilon]$ for all ϵ satisfying $0 < \epsilon < 1$.

$\lim\limits_{\epsilon \to 0+} \int_0^{1-\epsilon} \frac{1}{\sqrt{1-x^2}}dx = \lim\limits_{\epsilon \to 0+} [\sin^{-1}(1-\epsilon)] = \frac{\pi}{2}.$

Therefore the improper integral $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$ is convergent and $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \frac{\pi}{2}$.

III. Convergence of the improper integral $\int_a^b f(x)dx$ when a and b are the *only* points of infinite discontinuity of f in $[a, b]$.

Let the end points a, b of the closed and bounded interval $[a, b]$ be the only points of infinite discontinuity of a function f which is bounded and integrable on $[a+\epsilon, b-\epsilon']$ for every ϵ, ϵ' satisfying $0 < \epsilon < b-a$, $0 < \epsilon' < b-a$.

Let $c \in (a, b)$.

If the improper integrals $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ be both convergent according to the definitions given above, then the improper integral $\int_a^b f(x)dx$ is said to be *convergent* and we write

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$

Note. If the end points a and b of the closed and bounded interval $[a, b]$ be the *only* points of infinite discontinuity of a function f and the improper integral $\int_a^b f(x)dx$ be convergent, then for *any* point $d \in (a, b)$

$\int_a^b f(x)dx = \int_a^d f(x)dx + \int_d^b f(x)dx.$

Examples (continued).

5. The integral $\int_0^2 \frac{1}{\sqrt{x(2-x)}}dx$ is improper, since 0 and 2 are points of infinite discontinuity of the integrand.

The integrand is bounded and integrable on $[0+\epsilon, 2-\epsilon']$ for all ϵ, ϵ' satisfying $0 < \epsilon < 2$, $0 < \epsilon' < 2$.

Let us examine if $\lim\limits_{\epsilon \to 0+} \int_\epsilon^1 \frac{1}{\sqrt{x(2-x)}} dx$ and $\lim\limits_{\epsilon' \to 0+} \int_1^{2-\epsilon'} \frac{1}{\sqrt{x(2-x)}} dx$ exist.

$\lim\limits_{\epsilon \to 0+} \int_{0+\epsilon}^1 \frac{1}{\sqrt{x(2-x)}} dx = \lim\limits_{\epsilon \to 0+} [\sin^{-1}(x-1)]_\epsilon^1 = \frac{\pi}{2}$,

$\lim\limits_{\epsilon' \to 0+} \int_1^{2-\epsilon'} \frac{1}{\sqrt{x(2-x)}} dx = \lim\limits_{\epsilon' \to 0+} [\sin^{-1}(x-1)]_1^{2-\epsilon'} = \frac{\pi}{2}$.

Therefore $\int_0^2 \frac{1}{\sqrt{x(2-x)}} dx$ is convergent and $\int_0^2 \frac{1}{\sqrt{x(2-x)}} dx = \pi$.

IV. Convergence of the improper integral $\int_a^b f(x)dx$ when an interior point c is the *only* point of infinite discontinuity of f in $[a, b]$.

If the improper integrals $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ be both convergent according to the definitions given above, then the improper integral $\int_a^b f(x)dx$ is said to be *convergent* and we write

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

Therefore if both the limits $\lim\limits_{\epsilon \to 0+} \int_a^{c-\epsilon} f(x)dx$ and $\lim\limits_{\epsilon' \to 0+} \int_{c+\epsilon'}^b f(x)dx$ exist then $\int_a^b f(x)dx = \lim\limits_{\epsilon \to 0+} \int_a^{c-\epsilon} f(x)dx + \lim\limits_{\epsilon' \to 0+} \int_{c+\epsilon'}^b f(x)dx$.

If the improper integral $\int_a^b f(x)dx$ is convergent, its value is also equal to the symmetric limit $[\lim\limits_{\epsilon \to 0+} \int_a^{c-\epsilon} f(x)dx + \lim\limits_{\epsilon \to 0+} \int_{c+\epsilon}^b f(x)dx]$.

It may happen that the improper integral $\int_a^b f(x)dx$ is divergent but the limit $\lim\limits_{\epsilon \to 0+} [\int_a^{c-\epsilon} f(x)dx + \int_{c+\epsilon}^b f(x)dx]$ exists, then this symmetric limit is called the *Cauchy principal value* of the improper integral $\int_a^b f(x)dx$ and it is denoted by $P \int_a^b f(x)dx$.

For example, let us consider the improper integral $\int_{-1}^1 f(x)dx$, where

$f(x) = \frac{1}{x}, \ x \neq 0$
$\quad = 0, \ x = 0.$

Here $\lim\limits_{\epsilon \to 0+} \int_{-1}^{0-\epsilon} f(x)dx + \lim\limits_{\epsilon' \to 0+} \int_{0+\epsilon'}^1 f(x)dx$

$= \lim\limits_{\epsilon \to 0+} \log \epsilon + \lim\limits_{\epsilon' \to 0+} (-\log \epsilon')$

and this limit does not exist if $\epsilon \to 0+$, $\epsilon' \to 0+$ independently.

But $\lim\limits_{\epsilon \to 0+} [\int_{-1}^{0-\epsilon} f(x)dx + \int_{0+\epsilon}^1 f(x)dx] = \lim\limits_{\epsilon \to 0+} (\log \epsilon - \log \epsilon) = 0$.

Therefore the integral $\int_{-1}^1 f(x)dx$ is divergent but $P \int_{-1}^1 f(x)dx = 0$.

V. Convergence of the improper integral $\int_a^b f(x)dx$ when a finite number of points $c_1, c_2, \ldots, c_m$ are the only points of infinite discontinuity of f in $[a, b]$.

Case 1. Let $a < c_1 < c_2 < \cdots < c_m < b$.

If the improper integrals $\int_a^{c_1} f(x)dx$, $\int_{c_1}^{c_2} f(x)dx, \cdots, \int_{c_m}^b f(x)dx$ be all convergent according to the definitions given above, then the improper integral $\int_a^b f(x)dx$ is said to be *convergent* and we write

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \cdots + \int_{c_m}^b f(x)dx.$$

Case 2. Either $a = c_1$ or $b = c_m$ or both.

If $a = c_1$, then $\int_a^b f(x)dx = \int_a^{c_2} f(x)dx + \int_{c_2}^{c_3} f(x)dx + \cdots + \int_{c_m}^b f(x)dx$, provided each integral in the right hand side is convergent according to the definitions given.

If $b = c_m$, then $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \cdots + \int_{c_{m-1}}^b f(x)dx$, provided each integral in the right hand side is convergent according to the definitions given.

12.3. Tests for convergence, positive integrand.

Theorem 12.3.1. Let a be the *only* point of infinite discontinuity of a function f which is integrable on $[a+\epsilon, b]$ for all ϵ satisfying $0 < \epsilon < b-a$ and $f(x) > 0$ for all $x \in (a, b]$.

A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f(x)dx$ is that there exists a positive real number k such that

$$\int_{a+\epsilon}^b f(x)dx < k \text{ for all } \epsilon \text{ satisfying } 0 < \epsilon < b-a.$$

Proof. Let $\phi(\epsilon) = \int_{a+\epsilon}^b f(x)dx$, $0 < \epsilon < b-a$.

Let $0 < \epsilon_1 < \epsilon_2 < b-a$. Then $\phi(\epsilon_1) - \phi(\epsilon_2) = \int_{a+\epsilon_1}^b f(x)dx - \int_{a+\epsilon_2}^b f(x)dx = \int_{a+\epsilon_1}^{a+\epsilon_2} f(x)dx \geq 0$, since $f(x) > 0$ on $[a+\epsilon_1, a+\epsilon_2]$.

$0 < \epsilon_1 < \epsilon_2 \Rightarrow \phi(\epsilon_1) \geq \phi(\epsilon_2)$. This shows that ϕ is a monotone decreasing function on $(0, b-a)$. Therefore $\phi(\epsilon)$ will tend to a finite limit as $\epsilon \to 0+$ if and only if ϕ is bounded above.

In other words, the improper integral $\int_a^b f(x)dx$ is convergent if and only if there exists a positive real number k such that

$$\int_{a+\epsilon}^b f(x)dx < k \text{ for all } \epsilon \text{ satisfying } 0 < \epsilon < b-a.$$

Note. If ϕ be not bounded above, then $\phi(\epsilon)$ tends to ∞ as $\epsilon \to 0+$ and the improper integral $\int_a^b f(x)dx$ diverges to ∞.

Theorem 12.3.2. Let b be the *only* point of infinite discontinuity of a function f which is integrable on $[a, b-\epsilon]$ for all ϵ satisfying $0 < \epsilon < b-a$ and $f(x) > 0$ for all $x \in [a, b)$.

A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f(x)dx$ is that there exists a positive real number k such that

$$\int_a^{b-\epsilon} f(x)dx < k \text{ for all } \epsilon \text{ satisfying } 0 < \epsilon < b-a.$$

Similar proof.

Theorem 12.3.3. Comparison test.

Let a be the *only* point of infinite discontinuity of the functions f and g which are both integrable on $[a+\epsilon, b]$ for all ϵ satisfying $0 < \epsilon < b-a$ and $0 < f(x) \leq kg(x)$ for all $x \in (a, b]$, where $k > 0$. Then

(i) $\int_a^b g(x)dx$ is convergent $\Rightarrow \int_a^b f(x)dx$ is convergent;

(ii) $\int_a^b f(x)dx$ is divergent $\Rightarrow \int_a^b g(x)dx$ is divergent.

Proof. Since f and g are both integrable on $[a+\epsilon, b]$ and $0 < f(x) \leq kg(x)$ for all $x \in [a+\epsilon, b]$, we have $\int_{a+\epsilon}^b f(x)dx \leq k \int_{a+\epsilon}^b g(x)dx$.

This holds for all ϵ satisfying $0 < \epsilon < b-a$.

(i) If $\int_a^b g(x)dx$ be convergent then there exists a positive real number k' such that $\int_{a+\epsilon}^b g(x)dx < k'$ for all ϵ satisfying $0 < \epsilon < b-a$.

Then $\int_{a+\epsilon}^b f(x)dx < kk'$ for all ϵ satisfying $0 < \epsilon < b-a$ and this proves that $\int_a^b f(x)dx$ is convergent.

(ii) Let $\phi(\epsilon) = \int_{a+\epsilon}^b f(x)dx$, $\psi(\epsilon) = \int_{a+\epsilon}^b g(x)dx$ for $0 < \epsilon < b-a$. Then $\phi(\epsilon) \leq k\psi(\epsilon)$ for all $\epsilon \in (0, b-a)$.

If $\int_a^b f(x)dx$ be divergent, then ϕ is not bounded above on $(0, b-a)$ and therefore ψ is not bounded above on $(0, b-a)$.

This proves that $\int_a^b g(x)dx$ is divergent.

This completes the proof.

Theorem 12.3.4. Comparison test (limit form).

Let a be the *only* point of infinite discontinuity of the functions f and g which are both integrable on $[a+\epsilon, b]$ for all ϵ satisfying $0 < \epsilon < b-a$ and $f(x) > 0$, $g(x) > 0$ for all $x \in (a, b]$.

If $\lim\limits_{x \to a+} \frac{f(x)}{g(x)} = l$, where l is a *non-zero finite* number, then the two improper integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ converge or diverge together.

Proof. Since $\frac{f(x)}{g(x)} > 0$ for all $x \in (a, b]$, $l \geq 0$. Since l is non-zero, $l > 0$.

Let us choose a positive δ such that $l - \delta > 0$. Since $\lim_{x \to a+} \frac{f(x)}{g(x)} = l$, there exists a point c in (a, b) such that $l - \delta < \frac{f(x)}{g(x)} < l + \delta$ for all $x \in (a, c]$

or, $(l - \delta)g(x) < f(x) < (l + \delta)g(x)$ for all $x \in (a, c]$.

(i) Let $\int_a^b f(x)dx$ be convergent. Then $\int_a^c f(x)dx$ is convergent.

Since $(l - \delta)g(x) < f(x)$ for all $x \in (a, c]$ and $l - \delta > 0$, $\int_a^c g(x)dx$ is convergent by comparison test and therefore $\int_a^b g(x)dx$ is convergent.

(ii) Let $\int_a^b g(x)dx$ be convergent. Then $\int_a^c g(x)dx$ is convergent.

Since $f(x) < (l + \delta)g(x)$ for all $x \in (a, c]$ and $l + \delta > 0$, $\int_a^c f(x)dx$ is convergent by comparison test and therefore $\int_a^b f(x)dx$ is convergent

(iii) Let $\int_a^b f(x)dx$ be divergent. Then $\int_a^c f(x)dx$ is divergent.

Since $(l + \delta)g(x) > f(x)$ for all $x \in (a, c]$ and $l + \delta > 0$, $\int_a^c g(x)dx$ is divergent by comparison test and therefore $\int_a^b g(x)dx$ is divergent.

(iv) Let $\int_a^b g(x)dx$ be divergent. Then $\int_a^c g(x)dx$ is divergent.

Since $f(x) > (l - \delta)g(x)$ for all $x \in (a, c]$ and $l - \delta > 0$, $\int_a^c f(x)dx$ is divergent by comparison test and therefore $\int_a^b f(x)dx$ is divergent.

Thus the improper integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ converge or diverge together.

Note 1. If $\lim_{x \to a+} \frac{f(x)}{g(x)} = 0$ then for a pre-assigned positive ϵ, there exists a positive $\delta < b - a$ such that $f(x) < \epsilon g(x)$ for all x satisfying $a < x < a + \delta < b$. Then $\int_a^b g(x)dx$ converges $\Rightarrow \int_a^b f(x)dx$ converges.

2. If $\lim_{x \to a+} \frac{f(x)}{g(x)} = \infty$, then for a pre-assigned positive G, there exists a positive $\delta < b - a$ such that $f(x) > Gg(x)$ for all x satisfying $a < x < a + \delta < b$. Then $\int_a^b g(x)dx$ diverges $\Rightarrow \int_a^b f(x)dx$ diverges.

12.3.5. A useful comparison integral.

The integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is convergent if and only if $\mu < 1$.

Proof. The integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is proper if $\mu \leq 0$.

Let $\mu > 0$. Let $f(x) = \frac{1}{(x-a)^\mu}, a < x \leq b$. a is the only point of infinite discontinuity of f. f is integrable on $[a + \epsilon, b]$ for $0 < \epsilon < b - a$ and $f(x) > 0$ for all $x \in (a, b]$.

Let $\phi(\epsilon) = \int_{a+\epsilon}^{b} \frac{dx}{(x-a)^\mu}$, $0 < \epsilon < b - a$.

If $\mu \neq 1$, $\phi(\epsilon) = \int_{a+\epsilon}^{b} (x-a)^{-\mu} dx = \frac{1}{1-\mu}\left[\frac{1}{(b-a)^{\mu-1}} - \frac{1}{\epsilon^{\mu-1}}\right]$.

If $0 < \mu < 1$, $\lim\limits_{\epsilon \to 0+} \phi(\epsilon) = \dfrac{1}{1-\mu} \cdot \dfrac{1}{(b-a)^{\mu-1}}$ and if $\mu > 1$, $\lim\limits_{\epsilon \to 0+} \phi(\epsilon) = \infty$.

If $\mu = 1$, $\phi(\epsilon) = \int_{a+\epsilon}^{b} \frac{dx}{(x-a)} = \log|b-a| - \log|\epsilon|$ and $\lim\limits_{\epsilon \to 0+} \phi(\epsilon) = \infty$.

Since $\lim\limits_{\epsilon \to 0+} \phi(\epsilon)$ exists finitely when $0 < \mu < 1$ and $\lim\limits_{\epsilon \to 0+} \phi(\epsilon) = \infty$ when $\mu \geq 1$, the improper integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is convergent if and only if $0 < \mu < 1$.

Since the integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is proper if $\mu \leq 0$ and the improper integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is convergent if and only if $0 < \mu < 1$, it follows that the integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is convergent if and only if $\mu < 1$.

12.3.6. μ test. (A practical test)

Let a be the *only* point of infinite discontinuity of a function f which is integrable on $[a+\epsilon, b]$ for $0 < \epsilon < b - a$ and $f(x) > 0$ for all $x \in (a, b]$.

If $\lim\limits_{x \to a+} f(x)(x-a)^\mu = l$ where l is a *non-zero finite* number, then the integral $\int_a^b f(x)dx$ is convergent if and only if $\mu < 1$.

Proof. Since $f(x)(x-a)^\mu > 0$ for all $x \in (a, b]$, $l > 0$.

Let us choose a positive δ such that $l - \delta > 0$.

Since $\lim\limits_{x \to a+} f(x)(x-a)^\mu = l$, there exists a point c in (a, b) such that

$l - \delta < f(x)(x-a)^\mu < l + \delta$ for all $x \in (a, c]$

or, $\frac{l-\delta}{(x-a)^\mu} < f(x) < \frac{l+\delta}{(x-a)^\mu}$ for all $x \in (a, c]$.

If $\mu < 1$, the integral $\int_a^c \frac{1}{(x-a)^\mu} dx$ is convergent. Since $l + \delta > 0$, it follows from the right hand inequality that $\int_a^c f(x)dx$ is convergent.

If $\mu \geq 1$, the integral $\int_a^c \frac{1}{(x-a)^\mu} dx$ is divergent. Since $l - \delta > 0$, it follows from the left hand inequality that $\int_a^c f(x)dx$ is divergent.

Therefore $\int_a^c f(x)dx$ is convergent if and only if $\mu < 1$ and therefore $\int_a^b f(x)dx$ is convergent if and only if $\mu < 1$.

Note. If $\lim\limits_{x \to b-} f(x)(b-x)^\mu = l$ where l is a *non-zero finite* number, then the integral $\int_a^b f(x)dx$ is convergent if and only if $\mu < 1$.

Worked Examples.

1. Examine the convergence of $\int_0^1 \frac{x^{p-1}}{1+x} dx$.

The integral is a proper one if $p - 1 \geq 0$. If $p < 1$, 0 is the only point of infinite discontinuity of the integrand.

Let $f(x) = \frac{x^{p-1}}{1+x}$, $x \in (0,1]$, $g(x) = x^{p-1}$, $x \in (0,1]$. Then $f(x) > 0, g(x) > 0$ for all $x \in (0,1]$.

$\lim\limits_{x \to 0+} \frac{f(x)}{g(x)} = 1$ (a non-zero finite number) and $\int_0^1 g(x)dx$ is convergent if and only if $1 - p < 1$, i.e., if and only if $p > 0$.

By comparison test, $\int_0^1 f(x)dx$ is convergent if and only if $p > 0$.

Therefore $\int_0^1 \frac{x^{p-1}}{1+x} dx$ is convergent if and only if $p > 0$.

2. Show that $\int_0^1 \frac{1}{(x+1)(x+2)\sqrt{x(1-x)}} dx$ is convergent.

Let the given integral be $\int_0^1 f(x)dx$. 0 and 1 are the only points of infinite discontinuity of f. $f(x) > 0$ for all $x \in (0,1)$.

Let us examine the convergence of the improper integrals $\int_0^{\frac{1}{2}} f(x)dx$ and $\int_{\frac{1}{2}}^1 f(x)dx$.

Convergence of $\int_0^{\frac{1}{2}} f(x)dx$ at 0.

$\lim\limits_{x \to 0+} \sqrt{x} f(x) = \frac{1}{2}$. By μ test, $\int_0^{\frac{1}{2}} f(x)dx$ is convergent ... (i)

Convergence of $\int_{\frac{1}{2}}^1 f(x)dx$ at 1.

$\lim\limits_{x \to 1-} \sqrt{1-x} f(x) = \frac{1}{6}$. By μ test, $\int_{\frac{1}{2}}^1 f(x)dx$ is convergent... (ii)

From (i) and (ii) it follows that $\int_0^1 \frac{1}{(x+1)(x+2)\sqrt{x(1-x)}} dx$ is convergent.

3. Examine the convergence of $\int_0^1 \frac{x^{p-1}}{1-x} dx$.

1 is a point of infinite discontinuity of the integrand.

If $p < 1$, 0 is a point of infinite discontinuity of the integrand.

Let us examine the convergence of $\int_0^{\frac{1}{2}} \frac{x^{p-1}}{1-x} dx$ and of $\int_{\frac{1}{2}}^1 \frac{x^{p-1}}{1-x} dx$.

Convergence of $\int_0^{\frac{1}{2}} \frac{x^{p-1}}{1-x} dx$ at 0 when $p < 1$.

Let $f(x) = \frac{x^{p-1}}{1-x}$, $x \in (0, \frac{1}{2}]$; $g(x) = x^{p-1}$, $x \in (0, \frac{1}{2}]$. Then $f(x) > 0, g(x) > 0$ for all $x \in (0, \frac{1}{2}]$.

$\lim\limits_{x \to 0+} \frac{f(x)}{g(x)} = 1$ and $\int_0^{\frac{1}{2}} g(x)dx$ is convergent if and only if $p > 0$.

By comparison test, $\int_0^{\frac{1}{2}} f(x)dx$ is convergent if and only if $p > 0$, i.e., $\int_0^{\frac{1}{2}} \frac{x^{p-1}}{1-x}dx$ is convergent if $p > 0$ and divergent if $p \leq 0$ ...(i)

Convergence of $\int_{\frac{1}{2}}^1 \frac{x^{p-1}}{1-x}dx$ at 1.

Let $f(x) = \frac{x^{p-1}}{1-x}$, $g(x) = \frac{1}{1-x}$, $x \in [\frac{1}{2}, 1)$. Then $f(x) > 0, g(x) > 0$ for all $x \in [\frac{1}{2}, 1)$. $\lim\limits_{x \to 1-} \frac{f(x)}{g(x)} = 1$ and $\int_{\frac{1}{2}}^1 g(x)dx$ is divergent.

By comparison test, $\int_{\frac{1}{2}}^1 f(x)dx$ is divergent, i.e., $\int_{\frac{1}{2}}^1 \frac{x^{p-1}}{1-x}dx$ is divergent ...(ii)

From (i) and (ii) it follows that $\int_0^1 \frac{x^{p-1}}{1-x}dx$ is divergent.

4. Show that $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is convergent if and only if m, n are both positive.

Let the given integral be $\int_0^1 f(x)dx$. It is a proper integral if $m \geq 1$ and $n \geq 1$.

0 is the only point of infinite discontinuity of f if $m < 1$ and 1 is the only point of infinite discontinuity of f if $n < 1$.

Let us examine the convergence of $\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}dx$ when $m < 1$ and the convergence of $\int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}dx$ when $n < 1$.

Convergence of $\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}dx$ at 0 when $m < 1$.

$f(x) > 0$ for all $x \in (0, \frac{1}{2}]$. $\lim\limits_{x \to 0+} f(x)x^{1-m} = \lim\limits_{x \to 0+} (1-x)^{n-1} = 1$ (a non-zero finite number).

By μ test, $\int_0^{\frac{1}{2}} f(x)dx$ is convergent if and only if $1 - m < 1$, i.e., if and only if $m > 0$.

Convergence of $\int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}dx$ at 1 when $n < 1$.

$f(x) > 0$ for all $x \in [\frac{1}{2}, 1)$. $\lim\limits_{x \to 1-} f(x)(1-x)^{1-n} = \lim\limits_{x \to 1-} x^{m-1} = 1$ (a non-zero finite number).

By μ test, $\int_{\frac{1}{2}}^1 f(x)dx$ is convergent if and only if $1 - n < 1$, i.e., if and only if $n > 0$.

Therefore both the integrals $\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}dx$ and $\int_{\frac{1}{2}}^0 x^{m-1}(1-x)^{n-1}dx$ are convergent if and only if $m > 0$ and $n > 0$.

Hence $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is convergent if and only if $m > 0$ and $n > 0$.

Note. The integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$, $m > 0, n > 0$ is called the *Beta function* and it is denoted by $B(m, n)$.

5. Show that $\int_0^{\frac{\pi}{2}} \frac{x^m}{\sin^n x}dx$ is convergent if and only if $n < 1 + m$.

Let the given integral be $\int_0^{\frac{\pi}{2}} f(x)dx$.

If $m - n \geq 0$, it is a proper integral since $\lim\limits_{x\to 0+} (\frac{x}{\sin x})^n = 1$.

If $m - n < 0$, 0 is the only point of infinite discontinuity of f. $f(x) > 0$ for all $x \in (0, \frac{\pi}{2})$.

$f(x) > 0$ for all $x \in (0, \frac{\pi}{2}]$. $\lim\limits_{x\to 0+} x^{n-m} f(x) = \lim\limits_{x\to 0+} (\frac{x}{\sin x})^n = 1$ (a non-zero finite number).

By μ test, $\int_0^{\frac{\pi}{2}} f(x)dx$ is convergent if and only if $n - m < 1$, i.e., if and only if $n < 1 + m$.

Therefore the given integral is convergent if and only if $n < 1 + m$.

6. Examine the convergence of $\int_0^1 x^{n-1} \log x \, dx$.

0 is the only possible point of infinite discontinuity of the integrand.

Let us examine the convergence of $\int_0^{\frac{1}{2}} x^{n-1} \log \, x dx$. The integrand is negative in $(0, \frac{1}{2}]$.

Let $f(x) = -x^{n-1} \log \, x, \;\; x \in (0, \frac{1}{2}]$. Then $f(x) > 0$ for all $x \in (0, \frac{1}{2}]$.

If $n - 1 > 0$, the integral $\int_0^{\frac{1}{2}} f(x)dx$ is a proper one, since $\lim\limits_{x\to 0+} x^r \log \, x = 0$, for all $r > 0$.

If $n - 1 \leq 0$, 0 is the only point of infinite discontinuity of f.

Let m be a positive number such that $m + n - 1 > 0$. Then $\lim\limits_{x\to 0+} x^{m+n-1} \log \, x = 0$. Therefore $\lim\limits_{x\to 0+} x^m f(x) = 0$.

Let $g(x) = \frac{1}{x^m}$, $x \in (0, \frac{1}{2}]$. Then $g(x) > 0$ for all $x \in (0, \frac{1}{2}]$.

Since $\lim\limits_{x\to 0+} \frac{f(x)}{g(x)} = 0$ and $\int_0^{\frac{1}{2}} g(x)dx$ is convergent if $m < 1$, it follows that $\int_0^{\frac{1}{2}} f(x)dx$ is convergent if $m < 1$.

Therefore $\int_0^1 f(x)dx$ is convergent if $m < 1$ and $m + n - 1 > 0$, i.e., if $1 - n < m < 1$, i.e., if $n > 0$.

If $n = 0$, the integral reduces to $\int_0^1 \frac{\log x}{x}dx$.

$\int_\epsilon^1 \frac{\log x}{x}dx = -\frac{1}{2}(\log \epsilon)^2 \to -\infty$ as $\epsilon \to 0+$ and therefore $\int_0^1 f(x)dx$ is divergent if $n = 0$.

If $n < 0$, then $x^{n-1} \geq x^{-1}$ for all $x \in (0, 1]$. Since the integral $\int_0^1 \frac{\log x}{x}dx$ is divergent, it follows that $\int_0^1 f(x)dx$ is divergent.

Hence the given integral is convergent if and only if $n > 0$.

7. Examine the convergence of $\int_0^1 x^{m-1}(1-x)^{n-1} \log x \, dx$.

0 and 1 are the only possible points of infinite discontinuity of the integrand.

Let $f(x) = -x^{m-1}(1-x)^{n-1}\log\ x,\quad x \in (0, \frac{1}{2}]$. Then $f(x) > 0$ for all $x \in (0, \frac{1}{2}]$.

Convergence of $\int_0^{\frac{1}{2}} f(x)dx$ at 0.

If $m - 1 > 0$ then $\lim\limits_{x\to 0+} -x^{m-1}\log\ x = 0$ and therefore the integral $\int_0^{\frac{1}{2}} f(x)dx$ is proper.

When $m - 1 \le 0$, let p be a positive number such that $p + m - 1 > 0$. Then $\lim\limits_{x\to 0+} -x^{p+m-1}\log\ x = 0$ and therefore $\lim\limits_{x\to 0+} x^p f(x) = 0$.

Let $g(x) = \frac{1}{x^p}$, $x \in (0, \frac{1}{2}]$. Then $g(x) > 0$ for all $x \in (0, \frac{1}{2}]$ and $\lim\limits_{x\to 0+} \frac{f(x)}{g(x)} = 0$. Therefore the convergence of the integral $\int_0^{\frac{1}{2}} g(x)dx$ will imply convergence of the integral $\int_0^{\frac{1}{2}} f(x)dx$.

$\int_0^{\frac{1}{2}} g(x)dx$ is convergent if $p < 1$. Therefore $\int_0^{\frac{1}{2}} f(x)dx$ is convergent if $p < 1$ and $p + m - 1 > 0$, i.e., if $1 - m < p < 1$, i.e., if $m > 0$.

Let us examine the convergence of $\int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}\log\ xdx$. The integrand is negative in $[\frac{1}{2}, 1)$.

Let $f(x) = -x^{m-1}(1-x)^{n-1}\log\ x,\quad x \in [\frac{1}{2}, 1)$. Then $f(x) > 0$ for all $x \in [\frac{1}{2}, 1)$.

Convergence of $\int_{\frac{1}{2}}^1 f(x)dx$ at 1.

$\lim\limits_{x\to 1-} (1-x)^{n-1}\log\ x = \lim\limits_{x\to 1-} \frac{\log\ x}{(1-x)^{1-n}} = \lim\limits_{x\to 1-} \frac{(1-x)^n}{x(n-1)}$ is finite if $n \ge 0$.

Therefore the integral $\int_{\frac{1}{2}}^1 -x^{m-1}(1-x)^{n-1}\log\ xdx$ is proper if $n \ge 0$.

When $n < 0$, let q be a positive number such that $q + n > 0$. Then $\lim\limits_{x\to 1-} -(1-x)^{q+n-1}\log\ x = \lim\limits_{x\to 1-} \frac{-\log\ x}{(1-x)^{1-q-n}} = \lim\limits_{x\to 1-} \frac{-(1-x)^{q+n}}{x(1-q-n)} = 0.$ and therefore $\lim\limits_{x\to 1-} (1-x)^q f(x) = 0$.

Let $h(x) = \frac{1}{(1-x)^q}$, $x \in [\frac{1}{2}, 1)$. Then $h(x) > 0$ for all $x \in [\frac{1}{2}, 1)$ and $\lim\limits_{x\to 1-} \frac{f(x)}{h(x)} = 0$. Therefore the convergence of the integral $\int_{\frac{1}{2}}^1 h(x)dx$ will imply convergence of the integral $\int_{\frac{1}{2}}^1 f(x)dx$.

$\int_{\frac{1}{2}}^1 h(x)dx$ is convergent if $q < 1$. Therefore $\int_{\frac{1}{2}}^1 f(x)dx$ is convergent if $q < 1$ and $q + n > 0$, i.e., if $n > -q$ and $-q > -1$, i.e., if $n > -1$.

Hence the given integral is convergent if $m > 0$ and $n > -1$.

8. Examine the convergence of $\int_0^{\frac{\pi}{2}} \log \sin x \, dx$.

Let $f(x) = \log \sin x$, $x \in (0, \frac{\pi}{2}]$. 0 is a point of infinite discontinuity of f. $f(x) > 0$ for all $x \in (0, \frac{\pi}{2}]$.

We have $\lim\limits_{x \to 0+} \sqrt{x}(\log x) = 0$ and $\lim\limits_{x \to 0+} \sqrt{x} \log \frac{\sin x}{x} = 0$.

Therefore $\lim\limits_{x \to 0+} \sqrt{x}[\log x + \log \frac{\sin x}{x}] = 0$.

or, $\lim\limits_{x \to 0+} \sqrt{x} \log (\sin x) = 0$.

Let $g(x) = \frac{1}{\sqrt{x}}$, $x \in (0, \frac{\pi}{2}]$. Then $g(x) > 0$ for all $x \in (0, \frac{\pi}{2}]$.

$\lim\limits_{x \to 0+} \dfrac{f(x)}{g(x)} = 0$ and $\int_0^{\frac{\pi}{2}} g(x)dx$ is convergent. By comparison test, $\int_0^{\frac{\pi}{2}} f(x)dx$ is convergent, i.e., $\int_0^{\frac{\pi}{2}} \log \sin x \, dx$ is convergent.

9. Examine the convergence of $\int_0^{\frac{\pi}{2}} \sin^{m-1} x \cos^{n-1} x \, dx$.

The integral is a proper one if $m \geq 1$ and $n \geq 1$.

If $m < 1$, 0 is the only point of infinite discontinuity of the integrand and if $n < 1$, 1 is the only point of infinite discontinuity of the integrand.

Let us examine the convergence of $\int_0^{\frac{\pi}{4}} \sin^{m-1} x \cos^{n-1} x \, dx$ when $m < 1$ and the convergence of $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^{m-1} x \cos^{n-1} x \, dx$ when $n < 1$.

Convergence of $\int_0^{\frac{\pi}{4}} \sin^{m-1} x \cos^{n-1} x \, dx$ when $m < 1$.

Let $f(x) = \sin^{m-1} x \cos^{n-1} x$, $x \in (0, \frac{\pi}{4}]$. $f(x) > 0$ for all $x \in (0, \frac{\pi}{4}]$.

$\lim\limits_{x \to 0+} x^{1-m} f(x) = \lim\limits_{x \to 0+} (\frac{\sin x}{x})^{m-1} \cos^{n-1} x = 1$ (a non-zero finite number).

By μ test, $\int_0^{\frac{\pi}{4}} f(x)dx$ is convergent if and only if $1 - m < 1$, i.e., if and only if $m > 0$ ... (i)

Convergence of $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^{m-1} x \cos^{n-1} x \, dx$ when $n < 1$.

Let $f(x) = \sin^{m-1} x \cos^{n-1} x$, $x \in [\frac{\pi}{4}, \frac{\pi}{2})$. $f(x) > 0$ for all $x \in [\frac{\pi}{4}, \frac{\pi}{2})$.

$\lim\limits_{x \to \frac{\pi}{2}-} (\frac{\pi}{2} - x)^{1-n} f(x) = \lim\limits_{x \to \frac{\pi}{2}-} \sin^{m-1} x \, (\frac{\cos x}{\frac{\pi}{2} - x})^{n-1} = 1$ (a non-zero finite number).

By μ test, $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} f(x)dx$ is convergent if and only if $1 - n < 1$, i.e., if and only if $n > 0$ ... (ii)

From (i) and (ii) it follows that $\int_0^{\frac{\pi}{2}} \sin^{m-1} x \cos^{n-1} x \, dx$ is convergent if and only if $m > 0$ and $n > 0$.

12.4. Tests for convergence of an improper integral when the integrand does not necessarily keep the same sign on a bounded interval.

Theorem 12.4.1. (Cauchy)

Let a be the *only* point of infinite discontinuity of a function f which is integrable on $[a+\epsilon, b]$ for all ϵ satisfying $0 < \epsilon < b-a$ and $f(x)$ may not keep the same sign on $(a, b]$.

A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f(x)dx$ is that for a pre-assigned positive ϵ there exists a positive $\delta < b-a$ such that

$$\left| \int_{a+\lambda_1}^{a+\lambda_2} f(x)dx \right| < \epsilon \text{ for all } \lambda_1, \lambda_2 \text{ satisfying } 0 < \lambda_1 < \lambda_2 < \delta.$$

Proof. Let $F(\lambda) = \int_{a+\lambda}^b f(x)dx$, $0 < \lambda < b-a$.

The improper integral $\int_a^b f(x)dx$ is convergent if $\lim\limits_{\lambda \to 0+} F(\lambda)$ exists finitely. By Cauchy's criterion for the existence of finite limits, $\lim\limits_{\lambda \to 0+} F(\lambda)$ exists finitely if and only if for a pre-assigned positive ϵ there corresponds a positive $\delta < b-a$ such that

$|F(\lambda_1) - F(\lambda_2)| < \epsilon$ for all λ_1, λ_2 satisfying $0 < \lambda_1 < \lambda_2 < \delta$

or, $\left| \int_{a+\lambda_1}^{a+\lambda_2} f(x)dx \right| < \epsilon$ for all λ_1, λ_2 satisfying $0 < \lambda_1 < \lambda_2 < \delta$.

This completes the proof.

Definition.

The improper integral $\int_a^b f(x)dx$ is said to be *absolutely convergent* if $\int_a^b |f|(x)dx$ be convergent.

Theorem 12.4.2. An absolutely convergent improper integral $\int_a^b f(x)dx$ (where a is the only point of infinite discontinuity of f in $[a, b]$ and f is integrable on $[a+\epsilon, b]$ for all ϵ satisfying $0 < \epsilon < b-a$) is convergent.

Proof. Here the integral $\int_a^b |f|(x)dx$ is convergent and a is the only point of infinite discontinuity of f in $[a, b]$.

Let $\epsilon > 0$. Then there exists a positive $\delta < b-a$ such that

$$\left| \int_{a+\lambda_1}^{a+\lambda_2} |f|(x)dx \right| < \epsilon \text{ for all } \lambda_1, \lambda_2 \text{ satisfying } 0 < \lambda_1 < \lambda_2 < \delta.$$

We also have $\left| \int_{a+\lambda_1}^{a+\lambda_2} f(x)dx \right| \leq \int_{a+\lambda_1}^{a+\lambda_2} |f|(x)dx$.

Therefore $\left| \int_{a+\lambda_1}^{a+\lambda_2} f(x)dx \right| < \epsilon$ for all λ_1, λ_2 satisfying $0 < \lambda_1 < \lambda_2 < \delta$.

This proves that the integral $\int_a^b f(x)dx$ is convergent.

This completes the proof.

Note 1. The converse of the theorem is not true. We shall establish this by some examples.

Note 2. Since $|f(x)|$ is always positive, comparison tests can be applied to establish the convergence of the improper integral $\int_a^b |f|(x)dx$.

Worked Examples (continued).

10. Show that the improper integral $\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}dx$ is convergent.

Let $f(x) = \frac{\sin\frac{1}{x}}{\sqrt{x}}$, $x \in (0,1]$.

0 is the only point of infinite discontinuity of f. $f(x)$ does not keep the same sign in the interval $(0,1]$.

$|\frac{\sin\frac{1}{x}}{\sqrt{x}}| \leq \frac{1}{\sqrt{x}}$ for all $x \in (0,1]$ and $\int_0^1 \frac{1}{\sqrt{x}}dx$ is convergent.

Therefore $\int_0^1 |\frac{\sin\frac{1}{x}}{\sqrt{x}}|dx$ is convergent, i.e., $\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}dx$ is absolutely convergent and therefore $\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}dx$ is convergent.

11. A function f is defined on $[0,1]$ by $f(0)=0$,

$f(x) = (-1)^{n+1}(n+1)$, for $\frac{1}{n+1} < x \leq \frac{1}{n}$ $(n = 1,2,3,\ldots)$

Examine convergence of the integrals (i) $\int_0^1 f(x)dx$, (ii) $\int_0^1 |f|(x)dx$.

(i) f is bounded and integrable on $[\epsilon, 1]$ for every $\epsilon > 0$. 0 is the only point of infinite discontinuity of f in $[0,1]$

Let us choose $\epsilon > 0$. There exists a natural number p such that $\frac{1}{p+1} < \epsilon \leq \frac{1}{p}$.

$$\int_\epsilon^1 f(x)dx = \int_{\frac{1}{2}}^1 f(x)dx + \int_{\frac{1}{3}}^{\frac{1}{2}} f(x)dx + \cdots + \int_{\frac{1}{p}}^{\frac{1}{p-1}} f(x)dx + \int_\epsilon^{\frac{1}{p}} f(x)dx$$

$$= \int_{\frac{1}{2}}^1 2dx + \int_{\frac{1}{3}}^{\frac{1}{2}}(-3)dx + \cdots + \int_{\frac{1}{p}}^{\frac{1}{p-1}}(-1)^p p dx + \int_\epsilon^{\frac{1}{p}}(-1)^{p+1}(p+1)dx$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^p\frac{1}{p-1} + (-1)^{p+1}\int_\epsilon^{\frac{1}{p}} p+1dx.$$

$$|\int_\epsilon^1 f(x)dx - [1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^p\frac{1}{p-1}]| = |(p+1)\int_\epsilon^{\frac{1}{p}} dx|$$

$$< \frac{1}{p}, \text{ since } \int_\epsilon^{\frac{1}{p}} dx < \int_{\frac{1}{p+1}}^{\frac{1}{p}} dx.$$

As $\epsilon \to 0, p \to \infty$.

Therefore $\lim_{\epsilon\to 0}\int_\epsilon^1 f(x)dx = \lim_{p\to\infty}[1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^p\frac{1}{p-1}]$ (i)

Since the series $[1 - \frac{1}{2} + \frac{1}{3} - \cdots]$ is a convergent series, it follows from (i) that $\lim_{\epsilon\to 0}\int_\epsilon^1 f(x)dx$ is finite and therefore $\int_0^1 f(x)dx$ is convergent.

(ii) $|f|$ is bounded and integrable on $[\epsilon, 1]$ for every $\epsilon > 0$. 0 is the only point of infinite discontinuity of $|f|$ in $[0, 1]$.

Let us choose $\epsilon > 0$. There exists a natural number p such that $\frac{1}{p+1} < \epsilon \leq \frac{1}{p}$.

$\int_\epsilon^1 |f|(x)dx$

$= \int_{\frac{1}{2}}^1 |f|(x)dx + \int_{\frac{1}{3}}^{\frac{1}{2}} |f|(x)dx + \cdots + \int_{\frac{1}{p}}^{\frac{1}{p-1}} |f|(x)dx + \int_\epsilon^{\frac{1}{p}} |f|(x)dx$

$= \int_{\frac{1}{2}}^1 2dx + \int_{\frac{1}{3}}^{\frac{1}{2}} 3dx + \cdots + \int_{\frac{1}{p}}^{\frac{1}{p-1}} pdx + \int_\epsilon^{\frac{1}{p}} (p+1)dx$

$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1} + (p+1)(\frac{1}{p} - \epsilon)$

$> 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}]$, since $\frac{1}{p} > \epsilon$ (ii)

As $\epsilon \to 0, p \to \infty$.

Since the series $[1 + \frac{1}{2} + \frac{1}{3} + \cdots]$ is a divergent series, it follows from (ii) that $\lim\limits_{\epsilon \to 0} \int_\epsilon^1 f(x)dx$ is not finite and consequently, the integral $\int_0^1 |f|(x)dx$ is divergent.

Note. This example establishes that the converse of the theorem 15.4.2 is not true.

B. Improper integrals on an unbounded interval.

12.5. Definitions.

I. Convergence of the improper integral $\int_a^\infty f(x)dx$ where f is integrable on $[a, X]$ for all $X > a$.

Let $\phi(X) = \int_a^X f(x)dx, X > a$.

If $\lim\limits_{X \to \infty} \phi(X)$ exists (finitely) then the improper integral $\int_a^\infty f(x)dx$ is said to be *convergent*. If the limit be l, we write $\int_a^\infty f(x)dx = l$.

If $\lim\limits_{X \to \infty} \phi(X)$ does not exist (finitely) then the improper integral $\int_a^\infty f(x)dx$ is said to be *divergent*.

Examples.

1. Let us consider the integral $\int_0^\infty e^{-x}dx$. The integrand is integrable on any closed interval $[0, X]$, $X > 0$. The integral is improper.

Let $\phi(X) = \int_0^X e^{-x}dx, X > 0$. Then $\phi(X) = 1 - e^{-X}$. $\lim\limits_{X \to \infty} \phi(X) = 1$.

Therefore the integral $\int_0^\infty e^{-x}dx$ is convergent and $\int_0^\infty e^{-x}dx = 1$.

2. Let us consider the integral $\int_0^\infty \frac{1}{1+x}dx$. The integrand is integrable on any closed interval $[0, X]$, $X > 0$. The integral is improper.

Let $\phi(X) = \int_0^X \frac{1}{1+x}dx, X > 0$. Then $\phi(X) = \log(1 + X)$.

$\lim\limits_{X\to\infty} \phi(X) = \infty$. Therefore the integral $\int_0^\infty \frac{1}{1+x}dx$ is divergent.

II. Convergence of the improper integral $\int_{-\infty}^b f(x)dx$ where f is integrable on $[X, b]$ for all $X < b$.

Let $\phi(X) = \int_X^b f(x)dx, X < b$.

If $\lim\limits_{X\to-\infty} \phi(X)$ exists (finitely) then the improper integral $\int_{-\infty}^b f(x)dx$ is said to be *convergent*. If the limit be l, we write $\int_{-\infty}^b f(x)dx = l$.

III. Convergence of the improper integral $\int_{-\infty}^\infty f(x)dx$ where f is integrable on $[X_1, X_2]$ for all $X_1, X_2 \in \mathbb{R}$ satisfying $X_1 < X_2$.

Let $c \in \mathbb{R}$. If both the integrals $\int_{-\infty}^c f(x)dx$ and $\int_c^\infty f(x)dx$ be convergent according to the definitions I and II above, then the improper integral $\int_{-\infty}^\infty f(x)dx$ is said to be *convergent* and we write

$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx.$

Example (continued).

3. Let us consider the integral $\int_{-\infty}^\infty \frac{dx}{1+x^2}$. The integrand is integrable on any closed interval $[X_1, X_2]$, $X_2 > X_1$. The integral is improper.

Let us consider the integrals $\int_{-\infty}^a \frac{dx}{1+x^2}$ and $\int_a^\infty \frac{dx}{1+x^2}$, where $a \in \mathbb{R}$.
Let $\phi(X) = \int_X^a \frac{dx}{1+x^2}dx$, $X < a$. Then $\phi(X) = \tan^{-1} a - \tan^{-1} X$.

$\lim\limits_{X\to-\infty} \phi(X) = \tan^{-1} a + \frac{\pi}{2}$.
Therefore the improper integral $\int_{-\infty}^a \frac{dx}{1+x^2}$ is convergent.

Let $\psi(X) = \int_a^X \frac{dx}{1+x^2}dx$, $X > a$. Then $\psi(x) = \tan^{-1} X - \tan^{-1} a$.

$\lim\limits_{X\to\infty} \psi(X) = \frac{\pi}{2} - \tan^{-1} a$.
Therefore the improper integral $\int_a^\infty \frac{dx}{1+x^2}$ is convergent.

Consequently, the integral $\int_{-\infty}^\infty \frac{dx}{1+x^2}$ is convergent and $\int_{-\infty}^\infty \frac{dx}{1+x^2} = (\tan^{-1} a + \frac{\pi}{2}) + (\frac{\pi}{2} - \tan^{-1} a) = \pi$.

IV. Convergence of the improper integral $\int_{-\infty}^\infty f(x)dx$ where f has a finite number of points of infinite discontinuity $c_1, c_2, \ldots, c_m$.

Let $c_1 < c_2 < \cdots < c_m$. If each of the integrals $\int_{-\infty}^{c_1} f(x)dx$, $\int_{c_1}^{c_2} f(x)dx, \ldots, \int_{c_{m-1}}^{c_m} f(x)dx$ and $\int_{c_m}^{\infty} f(x)dx$ be convergent according to the definitions given in 12.2 and 12.5, then the improper integral $\int_{-\infty}^{\infty} f(x)dx$ is said to be *convergent* and we write

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \cdots + \int_{c_{m-1}}^{c_m} f(x)dx + \int_{c_m}^{\infty} f(x)dx.$$

12.6. Tests for convergence, positive integrand.

Theorem 12.6.1. Let a function f be integrable on $[a, X]$ for all $X > a$ and $f(x) > 0$ for all $x \geq a$.

A necessary and sufficient condition for the convergence of the improper integral $\int_a^{\infty} f(x)dx$ is that there exists a positive real number k such that

$$\int_a^X f(x)dx < k \text{ for all } X > a.$$

Proof. Let $\phi(X) = \int_a^X f(x)dx$, $X > a$.

Let $a < X_1 < X_2$. Then $\phi(X_2) - \phi(X_1) = \int_a^{X_2} f(x)dx - \int_a^{X_1} f(x)dx = \int_{X_1}^{X_2} f(x)dx \geq 0$, since $f(x) > 0$ on $[X_1, X_2]$.

$a < X_1 < X_2 \Rightarrow \phi(X_1) < \phi(X_2)$. This shows that ϕ is a monotone increasing function on (a, ∞). Therefore $\phi(X)$ will tend to a finite limit as $X \to \infty$ if and only if ϕ is bounded above.

In other words, the improper integral $\int_a^{\infty} f(x)dx$ is convergent if and only if there exists a positive real number k such that

$$\int_a^X f(x)dx < k \text{ for all } X > a.$$

Note. If ϕ be not bounded above, then $\phi(X)$ tends to ∞ as $X \to \infty$ and the improper integral $\int_a^{\infty} f(x)dx$ diverges to ∞.

Theorem 12.6.2. Comparison test.

Let the functions f and g be both integrable on $[a, X]$ for all $X > a$ and $0 < f(x) \leq kg(x)$ for all $x \geq a$, where $k > 0$. Then

(i) $\int_a^{\infty} g(x)dx$ is convergent $\Rightarrow \int_a^{\infty} f(x)dx$ is convergent;

(ii) $\int_a^{\infty} f(x)dx$ is divergent $\Rightarrow \int_a^{\infty} g(x)dx$ is divergent.

Proof. Since f and g are both integrable on $[a, X]$ and $0 < f(x) \leq kg(x)$ for all $x \in [a, X]$, we have $\int_a^X f(x)dx \leq k \int_a^X g(x)dx$.

i) If $\int_a^{\infty} g(x)dx$ be convergent then there exists a positive real number k' such that $\int_a^X g(x)dx < k'$ for all $X > a$.

Then $\int_a^X f(x)dx < kk'$ for all $X > a$ and this proves that $\int_a^\infty f(x)d$ is convergent.

(ii) Let $\int_a^\infty f(x)dx$ be divergent. Then $\lim\limits_{X\to\infty} \int_a^X f(x)dx = \infty$.

Since $\int_a^X f(x)dx \leq k\int_a^X g(x)dx$ and $k > 0$ it follows tha $\lim\limits_{X\to\infty} \int_a^X g(x)dx = \infty$. Consequently, $\int_a^\infty g(x)dx$ is divergent.

This completes the proof.

Theorem 12.6.3. Comparison test (limit form).

Let the functions f and g be both integrable on $[a, X]$ for all $X >$ and $f(x) > 0, g(x) > 0$ for all $x \geq a$.

If $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = l$, where l is a *non-zero* finite number, then the two im proper integrals $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ converge or diverge togethe

Proof. Since $\frac{f(x)}{g(x)} > 0$ for all $x \geq a$, $l \geq 0$. Since l is non-zero, $l > 0$.

Let us choose a positive ϵ such that $l - \epsilon > 0$. Since $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} =$ there exists a positive real number $k > a$ such that $l - \epsilon < \frac{f(x)}{g(x)} < l +$ for all $x > k$

or, $(l - \epsilon)g(x) < f(x) < (l + \epsilon)g(x)$ for all $x > k > a$.

(i) Let $\int_a^\infty f(x)dx$ be convergent.

Since $(l - \epsilon)g(x) < f(x)$ for all $x > k > a$ and $l - \epsilon > 0$, $\int_a^\infty g(x)d$ is convergent, by comparison test.

(ii) Let $\int_a^b g(x)dx$ be convergent.

Since $f(x) < (l + \epsilon)g(x)$ for all $x > k > a$ and $l + \epsilon > 0$, $\int_a^\infty f(x)d$ is convergent, by comparison test.

(iii) Let $\int_a^\infty f(x)dx$ be divergent.

Since $(l + \epsilon)g(x) > f(x)$ for all $x > k > a$ and $l + \epsilon > 0$, $\int_a^\infty g(x)d$ is divergent, by comparison test.

(iv) Let $\int_a^\infty g(x)dx$ be divergent.

Since $(l - \epsilon)g(x) < f(x)$ for all $x > k > a$ and $l - \epsilon > 0$, $\int_a^\infty f(x)d$ is divergent, by comparison test.

Thus the improper integrals $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ converge o diverge together.

This completes the proof.

Note 1. If $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 0$, then for a pre-assigned positive ϵ there exists a positive real number $b > a$ such that $f(x) < \epsilon g(x)$ for all $x > b$.

Then $\int_a^\infty g(x)dx$ is convergent implies that $\int_a^\infty f(x)dx$ is convergent.

2. If $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ then for a pre-assigned positive G there exists a positive real number $b > a$ such that $f(x) > Gg(x)$ for all $x > b > a$.

Then $\int_a^\infty g(x)dx$ is divergent implies that $\int_a^\infty f(x)dx$ is divergent.

12.6.4. A useful comparison integral.

The improper integral $\int_a^\infty \frac{dx}{x^\mu}$, where $a > 0$, is convergent if and only if $\mu > 1$.

Proof. Let $\phi(X) = \int_a^X \frac{dx}{x^\mu}$, $X > a$.

If $\mu \neq 1$, we have $\phi(X) = \int_a^X \frac{dx}{x^\mu} = [\frac{x^{1-\mu}}{1-\mu}]_a^X = \frac{1}{1-\mu}[X^{1-\mu} - a^{1-\mu}]$;

and if $\mu = 1$, $\phi(X) = \int_a^X \frac{dx}{x} = \log X - \log a$.

If $\mu = 1$, $\lim\limits_{X\to\infty} \phi(X) = \lim\limits_{X\to\infty} \log X - \log a = \infty$.

If $\mu \neq 1$, $\lim\limits_{X\to\infty} \phi(X) = \lim\limits_{X\to\infty} \frac{1}{1-\mu}[X^{1-\mu} - a^{1-\mu}] = \infty$, if $\mu < 1$

$$= \frac{1}{(\mu-1)a^{\mu-1}}, \text{ if } \mu > 1.$$

Since $\lim\limits_{X\to\infty} \phi(X)$ exists finitely when $\mu > 1$ and $\lim\limits_{X\to\infty} \phi(X) = \infty$ when $\mu \leq 1$, the improper integral $\int_a^\infty \frac{dx}{x^\mu}$ is convergent if $\mu > 1$ and divergent if $\mu \leq 1$.

Hence the improper integral $\int_a^\infty \frac{dx}{x^\mu}$, where $a > 0$, is convergent if and only if $\mu > 1$. This completes the proof.

12.6.5. μ test. (A practical test)

Let $f(x) > 0$ for all $x \geq a$. If $\lim\limits_{x\to\infty} x^\mu f(x) = l$, where l is a *non-zero finite* number, the improper integral $\int_a^\infty f(x)dx$ is convergent if and only if $\mu > 1$.

Proof. Let $\lim\limits_{x\to\infty} x^\mu f(x) = l$. Then $l > 0$.

Let us choose a positive ϵ such that $l - \epsilon > 0$. Since $\lim\limits_{X\to\infty} x^\mu f(x) = l$, there exists a positive real number $k > a$ such that

$|x^\mu f(x) - l| < \epsilon$ for all $x \geq k$

or, $l - \epsilon < x^\mu f(x) < l + \epsilon$ for all $x \geq k$

or, $\frac{l-\epsilon}{x^\mu} < f(x) < \frac{l+\epsilon}{x^\mu}$ for all $x \geq k > a$.

$\int_a^\infty \frac{l}{x^\mu} dx$ is convergent if $\mu > 1$. Since $l + \epsilon > 0$, it follows from the last inequality that $\int_a^\infty f(x)dx$ is convergent if $\mu > 1$.

$\int_a^\infty \frac{l}{x^\mu} dx$ is divergent if $\mu \leq 1$. Since $l - \epsilon > 0$, it follows from the first inequality that $\int_a^\infty f(x)dx$ is divergent if $\mu \leq 1$.

Therefore $\int_a^\infty f(x)dx$ is convergent if and only if $\mu > 1$.

Worked Examples.

1. Examine the convergence of the improper integral $\int_1^\infty \frac{1}{x(1+x^2)} dx$.

Let the given integral be $\int_1^\infty f(x)dx$. Then $f(x) > 0$ for all $x \geq 1$.

Let $g(x) = \frac{1}{x^3}$. Then $\int_1^\infty g(x)dx$ is convergent and $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = 1$, a non-zero finite number.

By comparison test, $\int_1^\infty f(x)dx$ is convergent.

2. Examine the convergence of the improper integral $\int_0^\infty x^{-\frac{1}{2}} e^{-x} dx$.

Let the given integral be $\int_0^\infty f(x)dx$. Then 0 is a point of infinite discontinuity of f.

We are to examine the convergence at 0 as well as at ∞.

Convergence at 0.

Let us consider the integral $\int_0^1 f(x)dx$. $f(x) > 0$ for all x in $(0, 1]$.

Let $g(x) = \frac{1}{\sqrt{x}}$. Then $\lim\limits_{x \to 0} \frac{f(x)}{g(x)} = 1$, a non-zero finite number and $\int_0^1 g(x)dx$ is convergent.

By comparison test, $\int_0^1 f(x)dx$ is convergent (i)

Convergence at ∞.

Let us consider the integral $\int_1^\infty f(x)dx$. $f(x) > 0$ for all $x \geq 1$ and $e^x > x$ for all $x \geq 1$. Therefore $f(x) < \frac{1}{x^{\frac{3}{2}}}$ and $\int_1^\infty \frac{1}{x^{\frac{3}{2}}} dx$ is convergent.

By comparison test, $\int_1^\infty f(x)dx$ is convergent ... (ii)

From (i) and (ii) it follows that $\int_0^\infty f(x)dx$ is convergent.

3. Examine the convergence of the improper integral $\int_1^\infty \frac{1}{x^{\frac{1}{2}}(1+x)^{\frac{1}{4}}} dx$.

Let the given integral be $\int_1^\infty f(x)dx$. Then $f(x) > 0$ for all $x \geq 1$.

Let $g(x) = \frac{1}{x^{\frac{3}{4}}}$. Then $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = 1$, a non-zero finite number and $\int_1^\infty g(x)dx$ is divergent

By comparison test, $\int_1^\infty f(x)dx$ is divergent.

4. Prove that the integral $\int_0^\infty x^{m-1}e^{-x}dx$ is convergent if and only if $m > 0$.

Let the given integral be $\int_0^\infty f(x)dx$. If $m \geq 1$, 0 is not a point of infinite discontinuity of f. f has an infinite discontinuity at 0 if $m < 1$.

Convergence at 0. ($m < 1$)

$f(x) > 0$ for all $x \in (0,1]$. Let $g(x) = x^{m-1}$, $x \in (0,1]$. Then $g(x) > 0$ for all $x \in (0,1]$ and $\lim\limits_{x\to 0} \frac{f(x)}{g(x)} = 1$, a non-zero finite number.

$\int_0^1 g(x)dx$ is convergent if and only if $1-m < 1$, i.e., if and only if $m > 0$.

By comparison test, $\int_0^1 f(x)dx$ is convergent if and only if $m > 0$.

Convergence at ∞.

$f(x) > 0$ for all $x \geq 1$. Let $g(x) = \frac{1}{x^2}$, $x \geq 1$. Then $g(x) > 0$ for all $x \geq 1$ and $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \frac{x^{m+1}}{e^x} = 0$ for all m.

As the integral $\int_1^\infty g(x)dx$ is convergent, therefore the integral $\int_1^\infty x^{m-1}e^{-x}dx$ is convergent for all m.

Hence the given integral is convergent if and only if $m > 0$.

Note. The integral $\int_0^1 x^{m-1}e^{-x}dx$, $m > 0$ is called the *Gamma function* and is denoted by $\Gamma(m)$.

5. Prove that the integral $\int_0^\infty (\frac{1}{1+x} - \frac{1}{e^x})\frac{1}{x}dx$ is convergent.

Let the given integral be $\int_0^\infty f(x)dx$.

Since $e^x > 1+x$ for all $x > 0$, $f(x) > 0$ for all $x > 0$. Since $\lim\limits_{x\to 0}(\frac{1}{1+x} - \frac{1}{e^x})\frac{1}{x} = \lim\limits_{x\to 0}\frac{e^x-1-x}{x(1+x)e^x} = 1$, 0 is not a point of infinite discontinuity of f.

We are to examine the convergence of the integral at ∞.

Let $g(x) = \frac{1}{x^2}, x > 0$. Then $g(x) > 0$ for all $x > 0$ and $\int_0^\infty g(x)dx$ is convergent.

$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \frac{x(e^x-(1+x))}{(1+x)e^x} = 1$, a non-zero finite number.

By comparison test, $\int_0^\infty f(x)dx$ is convergent.

Hence the given integral is convergent.

6. Examine the convergence of the improper integral $\int_0^\infty \frac{x^{p-1}}{1+x}\,dx$.

Let us examine the convergence of the integrals $\int_0^1 \frac{x^{p-1}}{1+x}\,dx$ and $\int_1^\infty \frac{x^{p-1}}{1+x}\,dx$.

If $p \geq 1$ the integral $\int_0^1 \frac{x^{p-1}}{1+x}\,dx$ is a proper one. If $p < 1$, 0 is the only point of infinite discontinuity of the integrand.

Let $f(x) = \frac{x^{p-1}}{1+x}, 0 < x \le 1$. Let $g(x) = x^{p-1}, 0 < x \le 1$. Then $f(x) > 0$ and $g(x) > 0$ for all $x \in (0,1]$ and $\lim\limits_{x \to 0+} \frac{f(x)}{g(x)} = 1$.

$\int_0^1 g(x)dx$ is convergent if $p > 0$ and divergent if $p \le 0$.

By comparison test, $\int_0^1 \frac{x^{p-1}}{1+x}\, dx$ is convergent if $p > 0$ and divergent if $p \le 0$... (i)

Let us consider $\int_1^\infty \frac{x^{p-1}}{1+x}\, dx$.

Let $f(x) = \frac{x^{p-1}}{1+x}, x > 1$. Let $\phi(x) = x^{p-2}$, $x > 1$. Then $\lim\limits_{x \to \infty} \frac{f(x)}{\phi(x)} = \lim\limits_{x \to \infty} \frac{x}{1+x} = 1$.

$\int_1^\infty \phi(x)\, dx$ is convergent if $2 - p > 1$, i.e., if $p < 1$ and divergent if $p \ge 1$.

By comparison test, $\int_1^\infty \frac{x^{p-1}}{1+x}\, dx$ is convergent if $p < 1$ and divergent if $p \ge 1$... (ii)

It follows from (i) and (ii) that the improper integral $\int_0^\infty \frac{x^{p-1}}{1+x}\, dx$ is convergent if $0 < p < 1$ and divergent otherwise.

Theorem 12.6.6. (Cauchy-Maclaurin integral test)

If f is a monotone decreasing function on $[1,\infty)$ and $f(x) > 0$ for all $x \in [1,\infty)$, then the improper integral $\int_1^\infty f(x)dx$ and the infinite series $\sum\limits_1^\infty f(n)$ converge or diverge together.

Proof. Let $F(X) = \int_1^X f(x)dx$, $X \ge 1$.

Since $f(x) > 0$ for all $x \in [1,\infty)$, F is a monotone increasing function on $[1,\infty)$ and the improper integral $\int_1^\infty f(x)dx$ is convergent or divergent according as F is bounded above or unbounded above on $[1,\infty)$.

Let $s_n = f(1) + f(2) + \cdots + f(n)$.

Since $f(n) > 0$ for all $n \in \mathbb{N}$, the sequence $\{s_n\}$ is a monotone increasing sequence and the series $\sum\limits_1^\infty f(n)$ is convergent or divergent according as the sequence $\{s_n\}$ is bounded above or unbounded above.

Let $X > 1$. Then there exists a natural number n such that $n \le X < n+1$. Since $f(x) > 0$ for all $x \in [1,\infty)$, $\int_1^n f(x)dx \le \int_1^X f(x)dx \le \int_1^{n+1} f(x)dx$ ... (i)

Let r be a positive integer. Then for all $x \in [r, r+1]$; $f(r) \ge f(x) \ge f(r+1)$.

Therefore $\int_r^{r+1} f(r)dx \ge \int_r^{r+1} f(x)dx \ge \int_r^{r+1} f(r+1)dx$

or, $f(r) \geq \int_r^{r+1} f(x)dx \geq f(r+1)$ ... (ii)

From (ii) $f(1) + f(2) + \cdots + f(n) \geq \int_1^{n+1} f(x)dx \geq \int_1^X f(x)dx$
or, $s_n \geq F(X)$ ... (iii)

From (ii) $f(2) + f(3) + \cdots + f(n) \leq \int_1^n f(x)dx \leq \int_1^X F(X)dx$
or, $s_n - f(1) \leq F(X)$ (iv)

Let the series $\sum\limits_1^\infty f(n)$ be convergent. Then the sequence $\{s_n\}$ is bounded above and since $F(X) \leq s_n$, it follows that F is bounded above on $[1, \infty)$. Consequently, the improper integral $\int_1^\infty f(x)dx$ is convergent.

Let the series $\sum\limits_1^\infty f(n)$ be divergent. Then the sequence $\{s_n\}$ is unbounded above and since $F(X) \geq s_n - f(1)$, it follows that F is unbounded above on $[1, \infty)$. Consequently, the improper integral $\int_1^\infty f(x)dx$ is divergent.

Let the improper integral $\int_1^\infty f(x)dx$ be convergent. Then F is bounded above on $[1, \infty)$ and since $s_n \leq F(X) + f(1)$, it follows that the sequence $\{s_n\}$ is bounded above. Consequently, the series $\sum\limits_1^\infty f(n)$ is convergent.

Let the improper integral $\int_1^\infty f(x)dx$ be divergent. Then F is unbounded above on $[1, \infty)$ and since $s_n \geq F(X)$, it follows that the sequence $\{s_n\}$ is unbounded above. Consequently, the series $\sum\limits_1^\infty f(n)$ is divergent.

Therefore the integral $\int_1^\infty f(x)dx$ and the series $\sum\limits_1^\infty f(n)$ converge or diverge together. This completes the proof.

Worked Examples (continued).

7. Prove that for $s > 0$, the improper integral $\int_1^\infty \frac{x^{-s}}{1+x}dx$ is convergent.

Let $f(x) = \frac{x^{-s}}{1+x}$ for $x \geq 1$, where $s > 0$. Then $f(x) > 0$ for all $x \geq 1$.
For $1 \leq x_1 < x_2$, we have $f(x_2) - f(x_1) = \frac{1}{x_2^s(1+x_2)} - \frac{1}{x_1^s(1+x_1)} < 0$.
Therefore f is a monotone decreasing function on $[1, \infty)$.

By Cauchy-Maclaurin theorem, the improper integral $\int_1^\infty \frac{x^{-s}}{1+x}dx$ and the infinite series $\sum\limits_1^\infty f(n)$ converge or diverge together.

The series $\sum\limits_1^\infty f(n) = \sum\limits_1^\infty \frac{1}{n^s(1+n)}$ is convergent, since $s > 0$.

Consequently, $\int_1^\infty \frac{x^{-s}}{1+x}dx$ is convergent.

8. Use Cauchy-Maclaurin's theorem to test the convergence of the series $\sum\limits_{1}^{\infty} \frac{1}{n^p}$ for $p > 0$.

Let $f(x) = \frac{1}{x^p}, p > 0, x \in [1, \infty)$. $f(x) > 0$ for all $x \geq 1$ and f is a monotone decreasing function on $[1, \infty)$.

By Cauchy-Maclaurin theorem, the improper integral $\int_1^\infty \frac{1}{x^p} dx$ and the infinite series $\sum\limits_{1}^{\infty} f(n)$ converge or diverge together.

The integral $\int_1^\infty \frac{1}{x^p} dx$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Therefore the series $\sum\limits_{1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $0 < p \leq 1$.

12.7. Tests for convergence of the improper integral on an infinite range of integration, where the integrand may not keep the same sign.

Theorem 12.7.1. (Cauchy)

Let $a \in \mathbb{R}$ and a function f be integrable on $[a, X]$ for every $X > a$.

A necessary and sufficient condition for the convergence of the improper integral $\int_a^\infty f(x)dx$ is that for a pre-assigned positive ϵ there exists a positive number X_0 such that

$$\left| \int_{X_1}^{X_2} f(x)dx \right| < \epsilon \text{ for all } X_1, X_2 > X_0.$$

Proof. Let $F(X) = \int_a^X f(x)dx$, $X > a$.

The improper integral $\int_a^\infty f(x)dx$ is convergent if $\lim\limits_{X \to \infty} F(X)$ exists finitely.

By Cauchy's criterion for the existence of finite limits, $\lim\limits_{X \to \infty} F(X)$ exists finitely if and only if for a pre-assigned positive ϵ there corresponds a positive X_0 such that

$|F(X_1) - F(X_2)| < \epsilon$ for all X_1, X_2 satisfying $X_1, X_2 > X_0$,

i.e., $\left| \int_a^{X_1} f(x)dx - \int_a^{X_2} f(x)dx \right| < \epsilon$ for all X_1, X_2 satisfying $X_1, X_2 > X_0$

or, $\left| \int_{X_1}^{X_2} f(x)dx \right| < \epsilon$ for all X_1, X_2 satisfying $X_1, X_2 > X_0$.

This completes the proof.

Definitions.

The improper integral $\int_a^\infty f(x)dx$ is said to be *absolutely convergent* if the integral $\int_a^\infty |f|(x)dx$ be convergent.

The improper integral $\int_{-\infty}^\infty f(x)dx$ is said to be *absolutely convergent* if the integral $\int_{-\infty}^\infty |f|(x)dx$ be convergent.

Theorem 12.7.2. An absolutely convergent improper integral $\int_a^\infty f(x)dx$ (where f is bounded and integrable on $[a, X]$ for every $X > a$) is convergent.

Proof. Here f is integrable on $[a, X]$ for every $X > a$ and the improper integral $\int_a^\infty f(x)dx$ is absolutely convergent.

Then the integral $\int_a^\infty |f|(x)dx$ is convergent. Therefore for a pre-assigned positive ϵ there exists a positive X_0 such that

$$|\int_{X_1}^{X_2} |f|(x)dx| < \epsilon \text{ for all } X_1, X_2 \text{ satisfying } X_1, X_2 > X_0.$$

We also have $|\int_{X_1}^{X_2} f(x)dx| \le \int_{X_1}^{X_2} |f|(x)dx$.

Therefore for a pre-assigned positive ϵ there exists a positive X_0 such that $|\int_{X_1}^{X_2} f(x)dx| < \epsilon$ for all X_1, X_2 satisfying $X_1, X_2 > X_0$.

This implies that the integral $\int_a^\infty f(x)dx$ is convergent.

This completes the proof.

Note 1. The converse of the theorem is not true. We shall establish this by some examples.

Note 2. Since $|f(x)|$ is always positive, comparison tests can be applied to establish the convergence of the improper integral $\int_a^\infty |f|(x)dx$.

Worked Examples (continued).

9. Examine the convergence of the improper integral $\int_1^\infty f(x)dx$, where
$f(x) = \frac{1}{x^2}$, if x be rational ≥ 1
$= -\frac{1}{x^2}$, if x be irrational > 1.

$|f|(x) = \frac{1}{x^2}$, $x \ge 1$. $\int_1^\infty |f|(x)dx$ is convergent and therefore $\int_1^\infty f(x)dx$ is absolutely convergent.

Consequently, the integral $\int_1^\infty f(x)dx$ is convergent.

10. Examine the convergence of the improper integral $\int_0^\infty \frac{\cos mx}{x^2+a^2}dx$, $m > 0, a > 0$.

Let the given integral be $\int_0^\infty f(x)dx$. Let $g(x) = \frac{1}{x^2+a^2}$, $x \ge 0$. Then $|f|(x) \le g(x)$ for all $x \ge 0$.

$\lim\limits_{X\to\infty} \int_0^X g(x)dx = \lim\limits_{X\to\infty} \tan^{-1}\frac{X}{a} = \frac{\pi}{2}$.
Therefore $\int_0^\infty g(x)dx$ is convergent.

By comparison test, $\int_0^\infty |f|(x)dx$ is convergent and therefore $\int_0^\infty f(x)dx$ is convergent.

11. Examine the convergence of the improper integral $\int_0^\infty \frac{\cos x}{\sqrt{x^3+x}}dx$.

Let the given integral be $\int_0^\infty f(x)dx$. Then $|f|(x) \leq \frac{1}{\sqrt{x^3+x}}$. Let $g(x) = \frac{1}{\sqrt{x^3+x}}$, $x > 0$.

0 is a point of infinite discontinuity of g.

Convergence of $\int_0^1 g(x)dx$.

$g(x) > 0$ for all $x \in (0,1]$. Let $u(x) = \frac{1}{\sqrt{x}}$, $x \in (0,1]$. Then $u(x) > 0$ for all $x \in (0,1]$.

$\lim\limits_{x\to 0} \frac{g(x)}{u(x)} = 1$, a non-zero finite number and $\int_0^1 u(x)dx$ is convergent.

By comparison test, $\int_0^1 g(x)dx$ is convergent ... (i)

Convergence of $\int_1^\infty g(x)dx$.

$g(x) > 0$ for all $x > 1$. Let $v(x) = \frac{1}{x^{\frac{3}{2}}}$, $x > 1$. Then $v(x) > 0$ for all $x > 1$.

$\lim\limits_{x\to\infty} \frac{g(x)}{v(x)} = 1$, a non-zero finite number and $\int_1^\infty v(x)dx$ is convergent.

By comparison test, $\int_1^\infty g(x)dx$ is convergent ...(ii)

From (i) and (ii) it follows that $\int_0^\infty g(x)dx$ is convergent.

Since $|f|(x)$ and $g(x)$ are both positive for all $x > 0$ and $|f|(x) \leq g(x)$ for all $x > 0$, $\int_0^\infty |f|(x)dx$ is convergent, by comparison test.

Therefore $\int_0^\infty f(x)dx$ is absolutely convergent and hence the given integral is convergent.

12. A function f is defined on $[1,\infty)$ by

$f(x) = \frac{(-1)^{n-1}}{n}$, for $n \leq x < n+1$ $(n = 1, 2, 3, \ldots)$.

Examine convergence of the integrals (i) $\int_1^\infty f(x)dx$, (ii) $\int_1^\infty |f|(x)dx$.

(i) Let us choose $X > 1$. There exists a natural number n such that $n \leq X < n+1$.

Let $F(X) = \int_1^X f(x)dx$.

Then $F(X) = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-2}}{n-1} + \int_n^X \frac{(-1)^{n-1}}{n}dx$

$= 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-2}}{n-1} + \frac{(-1)^{n-1}}{n}(X-n)$.

So $|F(X) - [1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-2}}{n-1}]| < \frac{1}{n}$ ... (i)

As $X \to \infty, n \to \infty$.

From (i) it follows that $\lim\limits_{x\to\infty} F(X) = \lim\limits_{n\to\infty}[1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-2}}{n-1}]$.

Since the series $[1 - \frac{1}{2} + \frac{1}{3} - \cdots]$ is a convergent series, it follows that the integral $\int_1^\infty f(x)dx$ is convergent.

Note. Since the series $[1 - \frac{1}{2} + \frac{1}{3} - \cdots]$ converges to $\log 2$, $\int_1^\infty f(x)dx = \log 2$.

(ii) Let us choose $X > 1$. There exists a natural number n such that $n \leq X < n + 1$.

Let $F(X) = \int_1^X |f|(x)dx$.

Then $F(X) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \int_n^X \frac{1}{n} dx$

$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}(X - n)$.

$\geq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}$, since $X - n \geq 0$.

As $X \to \infty, n \to \infty$. As the series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is a divergent series, it follows that the integral $\int_1^\infty |f|(x)dx$ is divergent.

Note. This example establishes that the converse of the theorem 12.7.2 is not true.

12.8. Tests for convergence of the integral of a product.

Theorem 12.8.1. Let a function f be integrable on $[a, X]$ for every $X > a$ and the integral $\int_a^\infty f(x)dx$ is absolutely convergent and a function ϕ be bounded on $[a, \infty)$ and integrable on $[a, X]$ for every $X > a$.

Then the integral $\int_a^\infty f(x)\phi(x)dx$ is absolutely convergent.

Proof. Since the function ϕ is bounded on $[a, \infty)$, there exists a positive real number k such that $|\phi(x)| < k$ for all $x \geq a$.

Since $\int_a^\infty |f|(x)dx$ (with positive integrand) is convergent, there exists a positive real number k_1 such that $\int_a^X |f|(x)dx < k_1$ for all $X > a$.

$\int_a^X |f(x)\phi(x)|dx < k \int_a^X |f(x)|dx < kk_1$ for all $X > a$.

This implies $\int_a^X |f(x)\phi(x)|dx$ is bounded for all $X > a$ and therefore $\int_a^X |f(x)\phi(x)|dx$ is convergent.

Consequently, the integral $\int_a^\infty f(x)\phi(x)dx$ is absolutely convergent.

Theorem 12.8.2. (Abel's test)

Let (i) a function ϕ be monotonic and bounded on $[a, \infty)$ and

(ii) the integral $\int_a^\infty f(x)dx$ be convergent.

Then the integral $\int_a^\infty f(x)\phi(x)dx$ is convergent.

Proof. Since the function ϕ is monotonic on $[a, \infty)$, ϕ is integrable on $[a, X]$ for all $X > a$.

By the second Mean value theorem, $\int_{t_1}^{t_2} f(x)\phi(x)dx =$

$\phi(t_1) \int_{t_1}^{\xi} f(x)dx + \phi(t_2) \int_{\xi}^{t_2} f(x)dx$, where $a < t_1 \leq \xi \leq t_2$ ... ...(i)

Since ϕ is bounded on $[a,\infty)$, there exists a positive real number k such that $|\phi(x)| < k$ for all $x \geq a$. Therefore it follows that $|\phi(t_1)| < k$, $|\phi(t_2)| < k$.

Let us choose $\epsilon > 0$. Since $\int_a^\infty f(x)dx$ is convergent, there exists a positive real number X such that $|\int_{t_1}^{t_2} f(x)dx| < \frac{\epsilon}{2k}$ for all $t_1, t_2 > X$. Since $t_1 \leq \xi \leq t_2$, it follows that $|\int_{t_1}^{\xi} f(x)dx| < \frac{\epsilon}{2k}$, $|\int_{\xi}^{t_2} f(x)dx| < \frac{\epsilon}{2k}$.

From (i) we have

$|\int_{t_1}^{t_2} f(x)\phi(x)dx| < |\phi(t_1)||\int_{t_1}^{\xi} f(x)dx| + |\phi(t_2)||\int_{\xi}^{t_2} f(x)\phi(x)dx| < k.\frac{\epsilon}{2k} + k.\frac{\epsilon}{2k}$, i.e., $< \epsilon$ for all $t_1, t_2 > X$.

Therefore the integral $\int_a^\infty f(x)\phi(x)dx$ is convergent.

This completes the proof.

Theorem 12.8.3. (Dirichlet's test)

Let (i) a function ϕ be monotonic and bounded on $[a,\infty)$ and $\lim_{x\to\infty} \phi(x) = 0$ and

(ii) the integral $\int_a^X f(x)dx$ be bounded on $[a, X]$ for all $X > a$.

Then the integral $\int_a^\infty f(x)\phi(x)dx$ is convergent.

Proof. Since the function ϕ is monotonic on $[a,\infty)$, ϕ is integrable on $[a, X]$ for all $X > a$.

By the second Mean value theorem, $\int_{t_1}^{t_2} f(x)\phi(x)dx$

$= \phi(t_1)\int_{t_1}^{\xi} f(x)dx + \phi(t_2)\int_{\xi}^{t_2} f(x)dx$, where $a < t_1 \leq \xi \leq t_2$ ... ...(i)

Since the integral $\int_a^X f(x)dx$ is bounded on $[a, X]$ for all $X > a$, there exists a positive real number k such that $|\int_a^X f(x)dx| < k$ for all $X > a$.

Therefore $|\int_{t_1}^{\xi} f(x)dx| = |\int_a^{\xi} f(x)dx - \int_a^{t_1} f(x)dx|$

$\leq |\int_a^{\xi} f(x)dx| + |\int_a^{t_1} f(x)dx| < 2k$.

Similarly, $|\int_{\xi}^{t_2} f(x)dx| < 2k$.

Let us choose $\epsilon > 0$. Since $\lim_{x\to\infty} \phi(x) = 0$, there exists a positive real number X such that $|\phi(x)| < \frac{\epsilon}{4k}$ for all $x > X$.

Let $t_1, t_2 > X$. Then $|\phi(t_1)| < \frac{\epsilon}{4k}$ and $|\phi(t_2)| < \frac{\epsilon}{4k}$.

From (i) we have

$|\int_{t_1}^{t_2} f(x)\phi(x)dx| \leq |\phi(t_1)||\int_{t_1}^{\xi} f(x)dx| + |\phi(t_2)||\int_{\xi}^{t_2} f(x)dx| < \frac{\epsilon}{4k}.2k + \frac{\epsilon}{4k}.2k$, i.e., $< \epsilon$ for all $t_1, t_2 > X$.

Therefore the integral $\int_a^\infty f(x)\phi(x)dx$ is convergent.

This completes the proof.

Worked Examples (continued).

13. Show that the improper integral $\int_0^\infty \frac{\sin x}{x}\, dx$ is convergent.

Since $\lim\limits_{x \to 0+} \frac{\sin x}{x} = 1$, 0 is not a point of infinite discontinuity of the integrand. Therefore $\int_0^1 \frac{\sin x}{x}\, dx$ is convergent. ... (i)

Let us consider the improper integral $\int_1^\infty \frac{\sin x}{x}\, dx$.

Let $f(x) = \sin x,\ x \geq 1;\ g(x) = \frac{1}{x},\ x \geq 1$. Then g is a bounded and monotone decreasing function on $[1, \infty)$ and $\lim\limits_{x \to \infty} g(x) = 0$.

$|\int_1^X f(x)dx| = |-\cos X + \cos 1| < 2$. Therefore $\int_1^X f(x)dx$ is bounded on $[1, X]$ for all $X > 1$.

By Dirichlet's test, $\int_1^\infty \frac{\sin x}{x}\, dx$ is convergent. ... (ii)

From (i) and (ii) it follows that $\int_0^\infty \frac{\sin x}{x}\, dx$ is convergent.

14. Show that the improper integral $\int_0^\infty |\frac{\sin x}{x}|\, dx$ is not convergent.

Let $f(x) = |\frac{\sin x}{x}|,\ x > 0$
$\qquad\quad = 1,\ x = 0.$

Then f is continuous and hence integrable on $[0, X]$ for all $X > 0$.
Therefore $|\frac{\sin x}{x}|$ is integrable on $[0, X]$ for all $X > 0$.

Let us consider the integral $\int_0^{n\pi} |\frac{\sin x}{x}| dx$, where n is a positive integer.

$$\int_0^{n\pi} |\frac{\sin x}{x}| dx = \int_0^{n\pi} \frac{|\sin x|}{x}\, dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}\, dx.$$

Now $\int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}\, dx = \int_0^\pi \frac{|\sin u|}{(r-1)\pi + u}\, du, \quad [\, x = (r-1)\pi + u]$
$$= \int_0^\pi \frac{\sin u}{(r-1)\pi + u}\, du.$$

For all $u \in [0, \pi], (r-1)\pi + u \leq r\pi$.
Therefore $\int_0^\pi \frac{\sin u}{(r-1)\pi + u}\, du \geq \frac{1}{r\pi} \int_0^\pi \sin u\, du = \frac{2}{r\pi}$.

Hence $\int_0^{n\pi} |\frac{\sin x}{x}| dx \geq \frac{2}{\pi}(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n})$ ... (i)

As $n \to \infty$, the R.H.S. of (i) gives the series $\sum\limits_{n=1}^{\infty} \frac{2}{\pi n}$ which is a divergent series.

Hence $\lim\limits_{n \to \infty} \int_0^{n\pi} f(x)dx = \infty$. This implies that the improper integral $\int_0^\infty |\frac{\sin x}{x}| dx$ is divergent.

Note. These two examples establish that the converse of the theorem 12.7.2 is not true.

15. Show that the improper integral $\int_0^\infty e^{-ax}\frac{\sin x}{x}dx$ is convergent if $a \geq 0$.

If $a = 0$ the integral reduces to $\int_0^\infty \frac{\sin x}{x}dx$ and it is convergent. [Ex.1]

Let $a > 0$ and let $\phi(x) = e^{-ax}$, $x \geq 0$.

Then $\phi'(x) = -ae^{-ax} < 0$ for all $x \geq 0$.

Therefore ϕ is a bounded monotone function on $[0, \infty)$.
And $\int_0^\infty \frac{\sin x}{x}dx$ is convergent, by Dirichlet's test. [Ex.1]

By Abel's test, $\int_0^\infty \phi(x)\frac{\sin x}{x}dx$ is convergent.

Therefore the integral $\int_0^\infty e^{-ax}\frac{\sin x}{x}dx$ is convergent if $a \geq 0$.

16. Prove that $\int_0^\infty \frac{\sin mx}{x^n}dx$ $(m > 0, n > 0)$ is convergent if $0 < n < 2$ and absolutely convergent if $1 < n < 2$.

Let us choose a positive real number a such that $am < \pi$. Let us examine the convergence of the integrals $\int_0^a \frac{\sin mx}{x^n}dx$ and $\int_a^\infty \frac{\sin mx}{x^n}dx$.

Convergence of the integral $\int_0^a \frac{\sin mx}{x^n}dx$.

Let $f(x) = \frac{\sin mx}{x^n}$, $g(x) = \frac{1}{x^{n-1}}$, $x \in (0, a]$. Then $f(x) > 0, g(x) > 0$ for all $x \in (0, a]$.

$\lim\limits_{x\to 0}\frac{f(x)}{g(x)} = m$. $\int_0^a g(x)dx$ is convergent if $n - 1 < 1$, i.e., if $n < 2$.

By comparison test, $\int_0^a f(x)dx$ is convergent if $n < 2$ ...(i)

Since $f(x) > 0$ for all $x \in (0, a]$, it follows that $\int_0^a f(x)dx$ is absolutely convergent if $n < 2$ ...(ii)

Convergence of the integral $\int_a^\infty \frac{\sin mx}{x^n}dx$.

$\int_a^X \sin mx dx$ is bounded for all $X > a$; and for $n > 0$, $\frac{1}{x^n}$ is a monotone decreasing function, bounded below, on $[a, \infty)$ and $\lim\limits_{x\to\infty}\frac{1}{x^n} = 0$.

By Dirichlet's test, $\int_a^\infty \frac{\sin mx}{x^n}dx$ $(n > 0)$ is convergent ... (iii)

From (i) and (iii) it folloes that $\int_0^\infty \frac{\sin mx}{x^n}dx$ is convergent if $0 < n < 2$.

Absolute convergence of the integral $\int_a^\infty \frac{\sin mx}{x^n}dx$.

Let $f(x) = \frac{\sin mx}{x^n}$ and $g(x) = \frac{1}{x^n}$, $x \geq a$. Then $|f(x)| \leq g(x)$ for all $x \geq a$.

$\int_a^\infty g(x)dx$ is convergent if $n > 1$. By comparison test, $\int_a^\infty |f|(x)dx$ is convergent if $n > 1$, i.e., $\int_a^\infty f(x)dx$ is absolutely convergent if $n > 1$ ... (iv).

From (ii) and (iv) it follows that $\int_0^\infty \frac{\sin mx}{x^n}dx$ is absolutely convergent if $1 < n < 2$.

17. Show that the improper integral $\int_1^\infty \frac{x}{1+x^2} \sin x\, dx$ is convergent.

Let $f(x) = \sin x$, $\phi(x) = \frac{x}{1+x^2}$, $x \geq 1$. Then $\int_1^\infty f(x)dx$ is bounded.

$\phi'(x) = -\frac{1}{1+x^2} < 0$ for all $x \geq 1$. Therefore ϕ is monotone decreasing on $[1, \infty)$. $|\frac{x}{1+x^2}| \leq \frac{1}{2}$ for all $x \geq 1$. Therefore ϕ is bounded on $[1, \infty)$. $\lim\limits_{x \to \infty} \phi(x) = \lim\limits_{x \to \infty} \frac{x}{1+x^2} = 0$.

By Dirichlet's theorem, $\int_1^\infty f(x).\phi(x)dx$ is convergent, i.e., $\int_1^\infty \frac{x}{1+x^2} \sin x\, dx$ is convergent.

18. Show that the improper integral $\int_0^\infty \frac{1}{1+x^2 \sin^2 x} dx$ is divergent.

Let $f(x) = \frac{1}{1+x^2 \sin^2 x}$, $x \geq 0$. Then f is continuous and hence integrable on $[0, X]$ for all $X > 0$.

Let us consider the integral $\int_0^{n\pi} \frac{1}{1+x^2 \sin^2 x} dx$, $n \in \mathbb{N}$.

$\int_0^{n\pi} \frac{1}{1+x^2 \sin^2 x} dx = \sum\limits_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{1}{1+x^2 \sin^2 x} dx$.

For all $x \in [(r-1)\pi, r\pi]$, we have $\frac{1}{1+x^2 \sin^2 x} \geq \frac{1}{1+r^2\pi^2 \sin^2 x}$.

$\int_{(r-1)\pi}^{r\pi} \frac{1}{1+r^2\pi^2 \sin^2 x} dx = \int_0^\pi \frac{1}{1+r^2\pi^2 \sin^2 u} du \quad [x = (r-1)\pi + u]$

$= 2\int_0^{\frac{\pi}{2}} \frac{1}{1+r^2\pi^2 \sin^2 u} du = 2\int_0^{\frac{\pi}{2}} \frac{1}{1+k^2 \sin^2 u} du$, where $k^2 = r^2\pi^2$.

We have

$\int \frac{1}{1+k^2 \sin^2 u}\, du = \int \frac{\sec^2 u}{1+(k^2+1)\tan^2 u} du = \frac{1}{\sqrt{k^2+1}} \tan^{-1}(\sqrt{k^2+1} \tan\, u)$.

Therefore $2\int_0^{\frac{\pi}{2}} \frac{1}{1+k^2 \sin^2 u} du$

$= 2 \lim\limits_{\epsilon \to 0} [\frac{1}{\sqrt{k^2+1}} \tan^{-1}(\sqrt{k^2+1} \tan\, u)]_0^{\frac{\pi}{2}-\epsilon} = \frac{\pi}{\sqrt{1+k^2}}$.

Hence $\int_0^{n\pi} \frac{1}{1+x^2 \sin^2 x} dx \geq \sum\limits_{r=1}^{n} \frac{\pi}{\sqrt{1+r^2\pi^2}} \qquad \ldots \text{(i)}$

Let $t_n = \frac{\pi}{\sqrt{1+n^2\pi^2}}$. As $n \to \infty$, the R.H.S. of (i) becomes the infinite series $\sum\limits_{n=1}^{\infty} t_n$ which is a divergent series.

Hence $\lim\limits_{n \to \infty} \int_0^{n\pi} f(x)dx = \infty$. This implies that the improper integral $\int_0^\infty \frac{1}{1+x^2 \sin^2 x} dx$ is divergent.

19. Show that the improper integral $\int_0^\infty \frac{1}{1+x^4 \sin^2 x} dx$ is convergent.

Let $f(x) = \frac{1}{1+x^4 \sin^2 x}$, $x \geq 0$. Then f is continuous and hence integrable on $[0, X]$ for all $X > 0$.

Let us consider the integral $\int_0^{n\pi} \frac{1}{1+x^4 \sin^2 x} dx$, $n \in \mathbb{N}$.

$\int_0^{n\pi} \frac{1}{1+x^4\sin^2 x}dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{1}{1+x^4\sin^2 x}dx.$

For all $x \in [(r-1)\pi, r\pi]$, we have $\frac{1}{x^4 \sin^2 x} \leq \frac{1}{1+(r-1)^4\pi^4 \sin^2 x}$.

Now $\int_{(r-1)\pi}^{r\pi} \frac{1}{1+(r-1)^4\pi^4\sin^2 x}dx = \int_0^{\pi} \frac{1}{1+(r-1)^4\pi^4\sin^2 u}du$

[by the substitution $x = (r-1)\pi + u$]

$= 2\int_0^{\frac{\pi}{2}} \frac{1}{1+(r-1)^4\pi^4\sin^2 u}du$

$= 2\int_0^{\frac{\pi}{2}} \frac{1}{1+k^2\sin^2 u}du$, where $k^2 = (r-1)^4\pi^4$.

We have

$\int \frac{1}{1+k^2\sin^2 u}\, du = \int \frac{\sec^2 u}{1+(k^2+1)\tan^2 u}dx = \frac{1}{\sqrt{k^2+1}} \tan^{-1}(\sqrt{k^2+1}\tan\ u)$.

Therefore $2\int_0^{\frac{\pi}{2}} \frac{1}{1+k^2\sin^2 u}du$

$= 2\lim_{\epsilon \to 0}[\frac{1}{\sqrt{k^2+1}} \tan^{-1}(\sqrt{k^2+1}\tan\ u)]_0^{\frac{\pi}{2}-\epsilon} = \frac{\pi}{\sqrt{1+k^2}}$.

Hence $\int_0^{n\pi} \frac{1}{1+(n-1)^4\pi^4\sin^2 x}dx \leq \sum_{r=1}^{n} \frac{\pi}{\sqrt{1+(r-1)^4\pi^4}}$ ...(i)

Let $t_n = \frac{\pi}{\sqrt{1+\pi^4 n^4}}$. As $n \to \infty$, the R.H.S. of (i) becomes the infinite series $\sum_{n=0}^{\infty} t_n$. $\sum_{n=1}^{\infty} t_n$ is a convergent series, by comparison test. $[v_n = \frac{1}{n^2}]$

Hence $\lim_{n\to\infty} \int_0^{n\pi} f(x)dx$ is finite. This implies that the improper integral $\int_0^{\infty} \frac{1}{1+x^4\sin^2 x}dx$ is convergent.

12.9. Some theorems.

Theorem 12.9.1. Let a function f be bounded and integrable on the interval $[a, X]$ for every $X > a$ and the improper integral $\int_a^{\infty} f(x)dx$ be convergent. If $\lim_{x\to\infty} f(x) = l$, then $l = 0$.

Proof. Let $l > 0$. Let us choose a positive ϵ such that $l - \epsilon > 0$.

Since $\lim_{x\to\infty} f(x) = l$, there exists a positive real number B_1 such that $l - \epsilon < f(x) < l + \epsilon$ for all $x > B_1$.

Let $l - \epsilon = k$. Then $k > 0$ and $f(x) > k > 0$ for all $x > B_1$.

Since $\int_a^{\infty} f(x)dx$ is convergent, for the same chosen ϵ there exists a positive real number B_2 such that $|\int_{X_1}^{X_2} f(x)dx| < \epsilon$ for all X_1, X_2 satisfying $X_2 > X_1 > B_2$.

Let $B = \max\{B_1, B_2\}$. Then $f(x) > k > 0$ for all $x > B$ (i)

and $|\int_{X_1}^{X_2} f(x)dx| < \epsilon$ for all X_1, X_2 satisfying $X_2 > X_1 > B$ (ii)

Now $f(x) > k > 0$ for all $x > B \Rightarrow \int_{X_1}^{X_2} f(x)dx \geq k(X_2 - X_1) > k$ for all X_1, X_2 satisfying $X_2 > X_1 > B$.

This contradicts the condition (ii). Therefore $l \not> 0$ ... (iii)

Let $l < 0$. Let us choose a positive ϵ such that $l + \epsilon < 0$.

Since $\lim\limits_{x \to \infty} f(x) = l$, there exists a positive real number G_1 such that $l - \epsilon < f(x) < l + \epsilon$ for all $x > G_1$.

Let $l + \epsilon = k$. Then $k < 0$ and $f(x) < k < 0$ for all $x > G_1$.

Since $\int_a^\infty f(x)dx$ is convergent, for the same chosen ϵ there exists a positive real number G_2 such that $|\int_{X_1}^{X_2} f(x)dx| < \epsilon$ for all X_1, X_2 satisfying $X_2 > X_1 > G_2$.

Let $G = \max\{G_1, G_2\}$. Then $f(x) < k < 0$ for all $x > G$ (iv)
and $|\int_{X_1}^{X_2} f(x)dx| < \epsilon$ for all X_1, X_2 satisfying $X_2 > X_1 > G$ ... (v)

Now $f(x) < k < 0$ for all $x > G \Rightarrow \int_{X_1}^{X_2} f(x)dx \leq k(X_2 - X_1) < k$ for all X_1, X_2 satisfying $X_2 > X_1 > G$.
Therefore $|\int_{X_1}^{X_2} f(x)dx| > |k|$ for all X_1, X_2 satisfying $X_2 > X_1 > G$.

This contradicts the condition (v). Therefore $l \not< 0$ (vi)

From (iii) and (vi) it follows that $l = 0$.

Note. An important property of a convergent infinite series $\sum\limits_1^\infty u_n$ is that $\lim\limits_{n \to \infty} u_n = 0$. But this property does not hold in case of a convergent improper integral $\int_a^\infty f(x)dx$.

If f be bounded and integrable on $[a, X]$ for every $X > a$, then the convergence of the improper integral $\int_a^\infty f(x)dx$ does not necessarily imply that $\lim\limits_{x \to \infty} f(x) = 0$.

For example, let $f(x) = \cos x^2$, $x \in [1, \infty)$. f is bounded and integrable on $[1, X]$ for all $X > 1$ and $\int_1^\infty f(x)dx$ is convergent. But $\lim\limits_{x \to \infty} f(x)$ does not exist.

Theorem 12.9.2. Let a function ϕ be continuous on $(0, \infty)$ and $\lim\limits_{x \to 0+} \phi(x) = \phi_0$ (finite), $\lim\limits_{x \to \infty} \phi(x) = \phi_1$ (finite). Then

$$\int_0^\infty \frac{\phi(ax) - \phi(bx)}{x} dx = (\phi_0 - \phi_1) \log \frac{b}{a}, \text{ where } a > 0,\ b > 0 \text{ and } b > a.$$

Proof. Let $\epsilon > 0$ and let $X > \epsilon$.

Let us consider the integral $\int_\epsilon^X \frac{\phi(ax) - \phi(bx)}{x} dx$.

We have $\int_\epsilon^X \frac{\phi(ax) - \phi(bx)}{x} dx = \int_\epsilon^X \frac{\phi(ax)}{x} dx - \int_\epsilon^X \frac{\phi(bx)}{x} dx$.

Let $ax = u$, $bx = v$. Then we have $\int_\epsilon^X \frac{\phi(ax)}{x}dx = \int_{a\epsilon}^{aX} \frac{\phi(u)}{u}du$ and $\int_\epsilon^X \frac{\phi(bx)}{x}dx = \int_{b\epsilon}^{bX} \frac{\phi(v)}{v}dv = \int_{b\epsilon}^{bX} \frac{\phi(u)}{u}du$.

Therefore $\int_\epsilon^X \frac{\phi(ax)-\phi(bx)}{x}dx = \int_{a\epsilon}^{aX} \frac{\phi(u)}{u}du - \int_{b\epsilon}^{bX} \frac{\phi(u)}{u}du$

$= [\int_{a\epsilon}^{b\epsilon} \frac{\phi(u)}{u}du + \int_{b\epsilon}^{bX} \frac{\phi(u)}{u}du + \int_{bX}^{aX} \frac{\phi(u)}{u}du] - \int_{b\epsilon}^{bX} \frac{\phi(u)}{u}du$

$= \int_{a\epsilon}^{b\epsilon} \frac{\phi(u)}{u}du - \int_{aX}^{bX} \frac{\phi(u)}{u}du.$

By the first mean value theorem, there exists a point $\xi \in [a\epsilon, b\epsilon]$ and a point $\eta \in [aX, bX]$ such that

$\int_{a\epsilon}^{b\epsilon} \frac{\phi(u)}{u}du = \phi(\xi) \int_{a\epsilon}^{b\epsilon} \frac{1}{u}du = \phi(\xi) \log \frac{b}{a}$ and

$\int_{aX}^{bX} \frac{\phi(u)}{u}du = \phi(\eta) \int_{aX}^{bX} \frac{1}{u}du = \phi(\eta) \log \frac{b}{a}$.

Therefore $\int_\epsilon^X \frac{\phi(ax)-\phi(bx)}{x}dx = [\phi(\xi) - \phi(\eta)] \log \frac{b}{a}$.

Let $\epsilon \to 0+$. Then $\phi(\xi) \to \phi_0$

Let $X \to \infty$. Then $\phi(\eta) \to \phi_1$.

Therefore $\int_0^\infty \frac{\phi(ax)-\phi(bx)}{x}dx$ is convergent and equals to $[\phi_0 - \phi_1] \log \frac{b}{a}$.

Note. If $\int_0^1 \frac{\phi(x)}{x}dx$ be convergent at 0, then by the general principle of convergence, $\lim_{\epsilon\to 0}\int_{a\epsilon}^{b\epsilon} \frac{\phi(x)}{x}dx = 0$.

If $\int_1^\infty \frac{\phi(x)}{x}dx$ be convergent at ∞, then by the general principle of convergence, $\lim_{X\to\infty}\int_{aX}^{bX} \frac{\phi(x)}{x}dx = 0$.

Therefore (i) if $\int_0^1 \frac{\phi(x)}{x}dx$ be convergent at 0 and $\lim_{x\to\infty} \phi(x) = \phi_1$, then $\int_0^\infty \frac{\phi(ax)-\phi(bx)}{x}dx = -\phi_1 \log \frac{b}{a}$, where $a > 0$, $b > 0$ and $b > a$;

(ii) if $\int_1^\infty \frac{\phi(x)}{x}dx$ be convergent at ∞ and $\lim_{x\to 0+} \phi(x) = \phi_0$, then $\int_0^\infty \frac{\phi(ax)-\phi(bx)}{x}dx = \phi_0 \log \frac{b}{a}$, where $a > 0$, $b > 0$ and $b > a$;

(iii) if $\int_0^1 \frac{\phi(x)}{x}dx$ be convergent at 0 and $\int_1^\infty \frac{\phi(x)}{x}dx$ be convergent at ∞, then $\int_0^\infty \frac{\phi(ax)-\phi(bx)}{x}dx = 0$, where $a > 0$, $b > 0$ and $b > a$.

Worked Examples (continued).

20. Show that $\int_0^\infty \frac{\tan^{-1}(ax)-\tan^{-1}(bx)}{x} = \frac{\pi}{2}\log(a/b)$, $0 < b < a$.

Let $\phi(x) = \tan^{-1} x$, $x \geq 0$. Then ϕ is continuous on $[0, \infty)$.

$\lim_{x\to 0+} \phi(x) = \phi(0) = 0$, $\lim_{x\to\infty} \phi(x) = \frac{\pi}{2}$.

Therefore $\int_0^\infty \frac{\phi(ax)-\phi(bx)}{x}dx = [0-\frac{\pi}{2}].\log(b/a)$

or, $\int_0^\infty \frac{\tan^{-1}(ax)-\tan^{-1}(bx)}{x} = \frac{\pi}{2}\log(a/b)$.

21. Show that $\int_0^\infty \frac{\cos\, ax-\cos\, bx}{x}dx = \log\frac{b}{a}$, $0<b<a$.

Let $\phi(x) = \cos\, x$, $x \geq 0$. Then ϕ is continuous on $[0,\infty)$.

Let $\epsilon > 0$ and let $X > \epsilon$.

$\int_\epsilon^X \frac{\cos\, ax-\cos\, bx}{x}dx = \int_\epsilon^X \frac{\cos\, ax}{x}dx - \int_\epsilon^X \frac{\cos\, bx}{x}dx$

$= \int_{a\epsilon}^{aX} \frac{\cos\, x}{x}dx - \int_{b\epsilon}^{bX} \frac{\cos\, x}{x} = \int_{a\epsilon}^{b\epsilon} \frac{\cos\, x}{x}dx - \int_{aX}^{bX} \frac{\cos\, x}{x}dx.$

By the first Mean value theorem, there exists a real number $\xi \in [a\epsilon, b\epsilon]$ such that $\int_{a\epsilon}^{b\epsilon} \frac{\cos\, x}{x}dx = \cos\,\xi \int_{a\epsilon}^{b\epsilon} \frac{1}{x}dx = \cos\,\xi \log\frac{b}{a}$.

Therefore $\lim\limits_{\epsilon\to 0}\int_{a\epsilon}^{b\epsilon} \frac{\cos\, x}{x}dx = \log\frac{b}{a}$, since $\xi \to 0$ as $\epsilon \to 0$.

Since $\int_1^\infty \frac{\cos\, x}{x}dx$ is convergent at ∞, $\lim\limits_{X\to\infty}\int_{aX}^{bX} \frac{\cos\, x}{x} = 0$.

Therefore $\int_0^\infty \frac{\cos\, ax-\cos\, bx}{x}dx = \log\frac{b}{a}$.

Deduction.

Prove that $\int_0^\infty \frac{\sin\, ax \sin\, bx}{x}\, dx = \frac{1}{2}\log\frac{a+b}{a-b}$, $0<b<a$.

We have $\int_0^\infty \frac{\cos\, px-\cos\, qx}{x}dx = \log\frac{q}{p}$, $0<q<p$.

or, $\int_0^\infty \frac{2\sin\,\frac{1}{2}(p+q)x \sin\,\frac{1}{2}(q-p)x}{x}dx = \log\frac{q}{p}$.

Let $p+q = 2a, q-p = 2b$. Then the result follows.

12.10. Evaluation of some improper integrals.

1. Evaluate $\int_0^\infty \frac{\sin x}{x}dx$.

$\int_0^\infty \frac{\sin x}{x}\, dx$ is a convergent improper integral. [page 28]

Let us consider the integral $\int_0^{\frac{\pi}{2}} \phi(x)\sin(2n+1)x\, dx$, where $\phi(x) = \frac{1}{x} - \frac{1}{\sin x}$, $x>0$.

ϕ is continuous, and therefore integrable, on $[\epsilon, \frac{\pi}{2}]$, where $0<\epsilon<\frac{\pi}{2}$ and $\lim\limits_{x\to 0}\phi(x) = 0$. Therefore ϕ is integrable on $[0, \frac{\pi}{2}]$.

By Riemann-Lebesgue theorem, $\lim\limits_{n\to\infty}\int_0^{\frac{\pi}{2}} \phi(x)\sin(2n+1)x\, dx = 0$. [page 484]

Let $I_n = \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{x}dx$, $J_n = \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x}dx$.

Then $J_n - J_{n-1} = \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin x}dx = \int_0^{\frac{\pi}{2}} 2\cos 2nx\, dx = 0$.

Therefore $J_n = J_{n-1} = \cdots = J_1 = \frac{\pi}{2}$.

As $\lim\limits_{n\to\infty}(I_n - J_n) = 0$ and $J_n = \frac{\pi}{2}$, it follows that $\lim\limits_{n\to\infty} I_n = \frac{\pi}{2}$, i.e., $\lim\limits_{n\to\infty}\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{x}\,dx = \frac{\pi}{2}$.

Now $\lim\limits_{n\to\infty}\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{x}\,dx = \lim\limits_{n\to\infty}\int_0^{\frac{(2n+1)\pi}{2}} \frac{\sin u}{u}\,du.$ $\quad [u = (2n+1)x]$

$= \int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx$, since $\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx$ is convergent.

Consequently, $\int_0^{\infty}\frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

2. Prove that $\int_0^{\infty}\frac{\sin mx}{x}\,dx = \frac{\pi}{2}$ or $-\frac{\pi}{2}$ according as $m > 0$ or $m < 0$.

Let $m > 0$. Let us choose $\epsilon > 0$ and $X > \epsilon$.

$\int_\epsilon^X \frac{\sin mx}{x}\,dx = \int_{m\epsilon}^{mX}\frac{\sin u}{u}\,du$ $\quad$ [by the substitution $mx = u$].

$\lim\limits_{X\to\infty,\epsilon\to 0}\int_{m\epsilon}^{mX}\frac{\sin u}{u}\,du = \int_0^{\infty}\frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

Therefore $\lim\limits_{X\to\infty,\epsilon\to 0}\int_\epsilon^X \frac{\sin mx}{x}\,dx = \frac{\pi}{2}$ and therefore $\int_0^{\infty}\frac{\sin mx}{x}\,dx = \frac{\pi}{2}$.

Similar proof when $m < 0$.

3. Show that $\int_0^{\infty}\frac{\cos\, ax - \cos\, bx}{x^2}\,dx = \frac{\pi}{2}(b-a),\ a > 0, b > 0$.

$\int_\epsilon^X \frac{\cos\, ax-\cos\, bx}{x^2}\,dx = \left[\frac{\cos\, ax - \cos\, bx}{-x}\right]_\epsilon^X + \int_\epsilon^X \frac{-a\sin\, ax + b\sin\, bx}{x}\,dx$

$= -\frac{\cos\, aX - \cos\, bX}{X} + \frac{\cos\, a\epsilon - \cos\, b\epsilon}{\epsilon} + \int_\epsilon^X \frac{-a\sin\, ax}{x}\,dx + \int_\epsilon^X \frac{b\sin\, bx}{x}\,dx.$

$\lim\limits_{X\to\infty}\frac{\cos\, aX - \cos\, bX}{-X} = 0,\ \lim\limits_{\epsilon\to 0}\frac{\cos\, a\epsilon - \cos\, b\epsilon}{\epsilon} = 0,$

$\lim\limits_{X\to\infty,\epsilon\to 0}\int_\epsilon^X \frac{-a\sin ax}{x}\,dx = -a\int_0^{\infty}\frac{\sin ax}{x} = -a\frac{\pi}{2},$

$\lim\limits_{X\to\infty,\epsilon\to 0}\int_\epsilon^X \frac{b\sin bx}{x}\,dx = b\int_0^{\infty}\frac{\sin bx}{x} = b\frac{\pi}{2}.$

Therefore $\int_0^{\infty}\frac{\cos\, ax - \cos\, bx}{x^2}\,dx = \frac{\pi}{2}(b-a),\ a > 0, b > 0.$

4. Evaluate $\int_0^{\infty}(\frac{\sin x}{x})^2\,dx$.

Let us choose $\epsilon > 0$ and $X > \epsilon$.

$\int_\epsilon^X \frac{\sin^2 x}{x^2}\,dx = \frac{\sin^2\epsilon}{\epsilon} - \frac{\sin^2 X}{X} + \int_\epsilon^X \frac{\sin 2x}{x}\,dx.$

As $\lim\limits_{\epsilon\to 0}\frac{\sin^2\epsilon}{\epsilon} = 0$ and $\lim\limits_{X\to\infty}\frac{\sin^2 X}{X} = 0$,

$\lim\limits_{X\to\infty\epsilon\to 0}\int_\epsilon^X \frac{\sin^2 x}{x^2}\,dx = \lim\limits_{X\to\infty,\epsilon\to 0}\int_\epsilon^X \frac{\sin 2x}{x}\,dx$

$= \int_0^{\infty}\frac{\sin 2x}{x}\,dx$, since $\int_0^{\infty}\frac{\sin 2x}{x}\,dx$ is convergent

$= \frac{\pi}{2}$.

Therefore $\int_0^{\infty}(\frac{\sin x}{x})^2\,dx = \frac{\pi}{2}$.

Evaluate $\int_0^{\frac{\pi}{2}} \log\sin x\, dx$.

$\int_0^{\frac{\pi}{2}} \log\sin x\, dx$ is a convergent improper integral. [worked out Ex.8,

Let $I = \int_0^{\frac{\pi}{2}} \log\sin x\, dx$. Let $\phi(\epsilon) = \int_\epsilon^{\frac{\pi}{2}} \log\sin x\, dx$, $0 < \epsilon < \frac{\pi}{2}$.

Then $I = \lim\limits_{\epsilon\to 0} \phi(\epsilon)$.

Now $\phi(\epsilon) = \int_\epsilon^{\frac{\pi}{2}} \log\sin x\, dx = \int_0^{\frac{\pi}{2}-\epsilon} \log\cos y\, dy$ $\quad [x = \frac{\pi}{2} - y]$

$= \int_0^{\frac{\pi}{2}-\epsilon} \log\cos x\, dx.$

Then $2\phi(\epsilon) = \int_\epsilon^{\frac{\pi}{2}} \log\sin x\, dx + \int_0^{\frac{\pi}{2}-\epsilon} \log\cos x\, dx$

$= \int_\epsilon^{\frac{\pi}{2}-\epsilon} [\log\sin x + \log\cos x]\, dx + \int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} \log\sin x\, dx + \int_0^\epsilon \log\cos x\, dx$

$= \int_\epsilon^{\frac{\pi}{2}-\epsilon} \log\frac{\sin 2x}{2} dx + 2\int_0^\epsilon \log\cos x dx$, since $\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} \log\sin x\, dx =$
$\int_0^\epsilon \log\cos x\, dx$, by the substitution $y = \frac{\pi}{2} - x$.

Therefore $2I = \lim\limits_{\epsilon\to 0} 2\phi(\epsilon)$

$= \lim\limits_{\epsilon\to 0} [\int_\epsilon^{\frac{\pi}{2}-\epsilon} \log\frac{\sin 2x}{2}\, dx + 2\int_0^\epsilon \log\cos x\, dx]$

$= \lim\limits_{\epsilon\to 0} [\int_\epsilon^{\frac{\pi}{2}-\epsilon} \log\sin 2x\, dx - \int_\epsilon^{\frac{\pi}{2}-\epsilon} \log 2\, dx + 2\int_0^\epsilon \log\cos x\, dx]$

$= \lim\limits_{\epsilon\to 0} [\frac{1}{2}\int_{2\epsilon}^{\pi-2\epsilon} \log\sin u\, du - (\frac{\pi}{2} - 2\epsilon)\log 2 + 2\int_0^\epsilon \log\cos x\, dx]$ $\quad [u = 2x]$

$= \lim\limits_{\epsilon\to 0} [\frac{1}{2}\int_{2\epsilon}^{\frac{\pi}{2}} \log\sin u\, du + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi-2\epsilon} \log\sin u\, du - (\frac{\pi}{2} - 2\epsilon)\log 2 +$
$2\int_0^\epsilon \log\cos x\, dx]$

$= \lim\limits_{\epsilon\to 0} [\frac{1}{2}\int_{2\epsilon}^{\frac{\pi}{2}} \log\sin u\, du + \frac{1}{2}\int_{2\epsilon}^{\frac{\pi}{2}} \log\sin t\, dt - (\frac{\pi}{2} - 2\epsilon)\log 2$
$+2\int_0^\epsilon \log\cos x\, dx]$ $\quad [t = \pi - u]$

$= \lim\limits_{\epsilon\to 0} [\int_{2\epsilon}^{\frac{\pi}{2}} \log\sin x\, dx - (\frac{\pi}{2} - 2\epsilon)\log 2 + 2\int_0^\epsilon \log\cos x\, dx]$.

$= \lim\limits_{\epsilon\to 0} [\phi(2\epsilon) - (\frac{\pi}{2} - 2\epsilon)\log 2 + 2\int_0^\epsilon \log\cos x\, dx]$ ... (i)

Let $f(\epsilon) = \int_0^\epsilon \log\cos x\, dx$, $0 \le \epsilon < \frac{\pi}{2}$. Then f is a continuous function on $[0, \frac{\pi}{4}]$, since $\log\cos x$ is integrable on $[0, \frac{\pi}{4}]$. Therefore $\lim\limits_{\epsilon\to 0} f(\epsilon) = f(0) = 0$.

$\lim\limits_{\epsilon\to 0} \phi(2\epsilon) = \lim\limits_{\epsilon\to 0} \phi(\epsilon) = I$, and $\lim\limits_{\epsilon\to 0} [\frac{\pi}{2} - 2\epsilon] = \frac{\pi}{2}$.

From (i) it follows that $2I = I - \frac{\pi}{2}\log 2$

or, $I = -\frac{\pi}{2}\log 2$, i.e., $\int_0^{\frac{\pi}{2}} \log\sin x\, dx = \frac{\pi}{2}\log\frac{1}{2}$.

6. Assuming that the integral $\int_0^{\frac{\pi}{2}} \log\sin x\ dx$ is convergent, prove that $\int_0^{\frac{\pi}{2}} \log\cos x\ dx$ is convergent and evaluate $\int_0^{\frac{\pi}{2}} \log\cos x\ dx$.

$\int_0^{\frac{\pi}{2}} \log\cos x\ dx$ is an improper integral. The integrand is continuous, and therefore integrable, on $[0, \frac{\pi}{2} - \epsilon]$ for all ϵ satisfying $0 < \epsilon < \frac{\pi}{2}$.

Let $\psi(\epsilon) = \int_0^{\frac{\pi}{2}-\epsilon} \log\cos x\ dx$.
Then $\psi(\epsilon) = \int_\epsilon^{\frac{\pi}{2}} \log\sin y\ dy \quad [x = \frac{\pi}{2} - y]$.
$\qquad = \int_\epsilon^{\frac{\pi}{2}} \log\sin x\ dx$.

Since $\int_0^{\frac{\pi}{2}} \log\sin x\ dx$ is convergent, $\lim\limits_{\epsilon\to 0}\int_\epsilon^{\frac{\pi}{2}} \log\sin x\ dx$ is finite, i.e., $\lim\limits_{\epsilon\to 0} \psi(\epsilon)$ is finite and this proves that $\int_0^{\frac{\pi}{2}} \log\cos x\ dx$ is convergent.

Let $I = \int_0^{\frac{\pi}{2}} \log\cos x\ dx$. Then $I = \lim\limits_{\epsilon\to 0} \psi(\epsilon) = \lim\limits_{\epsilon\to 0}\int_\epsilon^{\frac{\pi}{2}} \log\sin x\ dx$
$= \int_0^{\frac{\pi}{2}} \log\sin x\ dx = \frac{\pi}{2}\log\frac{1}{2}$.

7. Prove that $\int_0^{\frac{\pi}{2}} \cos 2nx\ \log\sin x\ dx$ is convergent. Evaluate the integral when n is a positive integer.

Let $f(x) = \cos 2nx\ \log\sin x, 0 < x \le \frac{\pi}{2}$. Then f is bounded and integrable on $[\epsilon, \frac{\pi}{2}]$ for all ϵ satisfying $0 < \epsilon < \frac{\pi}{2}$.
We have $\lim\limits_{x\to 0+} \frac{\log\sin x}{1/\sqrt{x}} = 0$. [using L'Hospital's rule]

Let $g(x) = \frac{1}{\sqrt{x}}, 0 < x \le \frac{\pi}{2}$. Then $g(x) > 0$ for all $x \in (0, \frac{\pi}{2}]$ and $\lim\limits_{x\to 0+} \frac{|f(x)|}{g(x)} = 0$. Therefore $\int_0^{\frac{\pi}{2}} |f(x)|\ dx$ is convergent, since $\int_0^{\frac{\pi}{2}} g(x)\ dx$ is convergent.

Consequently, $\int_0^{\frac{\pi}{2}} f(x)\ dx$ is convergent.

Second part.
$\int_0^{\frac{\pi}{2}} f(x)\ dx = \lim\limits_{\epsilon\to 0+}\int_\epsilon^{\frac{\pi}{2}} f(x)\ dx$, since $\int_0^{\frac{\pi}{2}} f(x)\ dx$ is convergent.

Integrating by parts, we have $\lim\limits_{\epsilon\to 0+} \int_\epsilon^{\frac{\pi}{2}} \cos 2nx \log\sin x\ dx$
$= \lim\limits_{\epsilon\to 0+}[\frac{\sin n\pi}{2n}\log\sin\frac{\pi}{2} - \frac{\sin 2n\epsilon}{2n}\log\sin\epsilon]$- $\lim\limits_{\epsilon\to 0+}\frac{1}{2n}\int_\epsilon^{\frac{\pi}{2}} \frac{\sin 2nx\cos x}{\sin x}\ dx$

$=-\lim\limits_{\epsilon\to 0+}\frac{1}{n}\int_\epsilon^{\frac{\pi}{2}} \cos x[\cos x + \cos 3x + \cdots + \cos(2n-1)x]\ dx$, since n is a positive integer and $\frac{\sin 2nx}{2\sin x} = \cos x + \cos 3x + \cdots + \cos(2n-1)x$ and $\lim\limits_{\epsilon\to 0+} -\frac{\sin 2n\epsilon}{2n}\log\sin\epsilon = 0$

$= -\lim\limits_{\epsilon\to 0+}\frac{1}{n}\int_\epsilon^{\frac{\pi}{2}} (\frac{1}{2} + 2\cos 2x + 2\cos 4x + \cdots + 2\cos(2n-1)x + \cos 2nx)\ dx$
$= -\frac{\pi}{4n}$.

8. Assuming that $\int_0^{\frac{\pi}{2}} \cos 2nx \ \log \sin x \ dx = -\frac{\pi}{4n}$, when n is a positive integer, show that

$\int_0^{\pi} \cos nx \ \log 2(1-\cos x) \ dx = -\frac{\pi}{n}$, when n is a positive integer.

Let $f(x) = \cos nx \log 2(1-\cos x), 0 < x \leq \pi$. Then f is bounded and integrable on $[\epsilon, \pi]$ for every ϵ satisfying $0 < \epsilon < \pi$.

Let us evaluate $\lim\limits_{\epsilon \to 0+} \int_{\epsilon}^{\pi} \cos nx \log 2(1-\cos x) \ dx$.

$$\lim_{\epsilon \to 0+} \int_{\epsilon}^{\pi} \cos nx \log 2(1-\cos x) \ dx$$

$$= \lim_{\epsilon \to 0+} \int_{\epsilon}^{\pi} \cos nx \ 2\log(2\sin \tfrac{x}{2}) \ dx$$

$$= \lim_{\epsilon \to 0+} 2\int_{\frac{\epsilon}{2}}^{\frac{\pi}{2}} \cos 2nu \ 2\log(2\sin u) \ du, \quad \text{by the substitution } x = 2u.$$

$$= \lim_{\epsilon \to 0+} 4\int_{\frac{\epsilon}{2}}^{\frac{\pi}{2}} \cos 2nx[\log 2 + \log \sin x] \ dx.$$

$$= \lim_{\epsilon \to 0+} [4\log 2\int_{\frac{\epsilon}{2}}^{\frac{\pi}{2}} \cos 2nx \ dx + 4\int_{\frac{\epsilon}{2}}^{\frac{\pi}{2}} \cos 2nx \ \log \sin x \ dx]$$

$$= \lim_{\epsilon \to 0+} [4\log 2.\tfrac{1}{2n}(\sin n\pi - \sin n\epsilon)] - 4.\tfrac{\pi}{4n}, \text{ since}$$

$$\lim_{\epsilon \to 0+} \int_{\frac{\epsilon}{2}}^{\frac{\pi}{2}} \cos 2nx \log \sin x \ dx = \int_0^{\frac{\pi}{2}} \cos 2nx \ \log \sin x \ dx = -\frac{\pi}{4n}$$

$$= -\frac{\pi}{n}.$$

9. Evaluate $\int_0^{\infty} \frac{x^{p-1}}{1+x} \ dx, \quad 0 < p < 1.$

The improper integral $\int_0^{\infty} \frac{x^{p-1}}{1+x} \ dx$ is convergent if $0 < p < 1$. [worked out Ex.6, page 506]

Let $I = \int_0^{\infty} \frac{x^{p-1}}{1+x} \ dx$.

Then $I = \int_0^1 \frac{x^{p-1}}{1+x} \ dx + \int_1^{\infty} \frac{x^{p-1}}{1+x} \ dx$

$$= \int_0^1 \frac{x^{p-1}}{1+x} \ dx - \int_1^0 \frac{u^{-p+1}}{u(1+u)} \ du \qquad [u = \tfrac{1}{x}]$$

$$= \int_0^1 \frac{x^{p-1}}{1+x} \ dx + \int_0^1 \frac{u^{-p}}{(1+u)} \ du$$

$$= \int_0^1 \frac{x^{p-1}+x^{-p}}{1+x} dx.$$

We have $\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^n x^n + \frac{(-1)^{n+1}x^{n+1}}{1+x}$ for all real $x \neq -1$ and for all $n \in \mathbb{N}$.

Therefore $\int_0^1 \frac{x^{p-1}+x^{-p}}{1+x} \ dx$

$$= \int_0^1 (x^{p-1}+x^{-p})[\sum_{r=0}^{n}(-1)^r x^r]dx + \int_0^1 (-1)^{n+1}(x^{p-1}+x^{-p})(\tfrac{x^{n+1}}{1+x}) \ dx$$

$= \int_0^1 [\sum_{r=0}^{n}(-1)^r(x^{p+r-1} + x^{-p+r})]\, dx + R_n$, where $R_n =$
$$(-1)^{n+1} \int_0^1 \frac{x^{n+p}+x^{n-p+1}}{1+x} dx$$

$= \sum_{r=0}^{n}(-1)^r[\frac{1}{p+r} + \frac{1}{-p+r+1}] + R_n$

$= (\frac{1}{p} + \frac{1}{-p+1}) - (\frac{1}{p+1} + \frac{1}{-p+2}) + (\frac{1}{p+2} + \frac{1}{-p+3}) - \cdots + (-1)^n(\frac{1}{p+n} + \frac{1}{-p+n+1}) + R_n$.

$|R_n| = |\int_0^1 \frac{x^{n+p}+x^{n-p+1}}{1+x}\, dx| \leq \int_0^1 |\frac{x^{n+p}+x^{n-p+1}}{1+x}|\, dx \leq 2\int_0^1 x^n\, dx \leq \frac{2}{n+1}$, since $0 < p < 1 \Rightarrow 0 < x^p < 1$ and $0 < x^{1-p} < 1$ and $\frac{1}{1+x} < 1$ for all $x \in [0,1]$.

Therefore $\lim |R_n| = 0$ and this implies $\lim R_n = 0$.

Therefore $\int_0^1 \frac{x^{p-1}+x^{-p}}{1+x}\, dx = \frac{1}{p} - (\frac{1}{p-1} + \frac{1}{p+1}) + (\frac{1}{p-2} + \frac{1}{p+2}) - \cdots$

Let us recall $\operatorname{cosec}\theta = \frac{1}{\theta} - \frac{1}{\theta-\pi} - \frac{1}{\theta+\pi} + \frac{1}{\theta-2\pi} + \frac{1}{\theta+2\pi} - \frac{1}{\theta-3\pi} - \frac{1}{\theta+3\pi} + \cdots$, if $\theta \neq n\pi$, n being an integer.

Therefore $\pi\operatorname{cosec} p\pi = \frac{1}{p} - \frac{1}{p-1} - \frac{1}{p+1} + \frac{1}{p-2} + \frac{1}{p+2} - \frac{1}{p-3} - \frac{1}{p+3} + \cdots$

Hence $\int_0^1 \frac{x^{p-1}+x^{-p}}{1+x}\, dx = \frac{\pi}{\sin p\pi}$, $0 < p < 1$.

That is, $\int_0^\infty \frac{x^{p-1}}{1+x}\, dx = \frac{\pi}{\sin p\pi}$, $0 < p < 1$.

Exercises 22

1. Examine the convergence of the improper integrals:

(i) $\int_0^1 \log x dx$, (ii) $\int_0^1 \frac{\log x}{\sqrt{1-x}} dx$, (iii) $\int_0^1 \frac{\log(1-x)}{\sqrt{x}} dx$,

(iv) $\int_0^1 \frac{1}{\sqrt{x(1-x)}} dx$, (v) $\int_0^{\frac{\pi}{2}} \frac{1}{e^x - \cos x} dx$, (vi) $\int_0^\pi \frac{\tan x}{x}\, dx$,

(vii) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{x^n} dx$, (viii) $\int_0^1 \frac{x^{p-1}+x^{-p}}{1+x} dx$, (ix) $\int_0^1 \frac{x^p \log x}{1+x^2} dx$,

(x) $\int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{\tan x}} dx$, (xi) $\int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{\tan x}} dx$, (xii) $\int_0^\pi \frac{dx}{\cos\alpha - \cos x}$, $0 \leq \alpha \leq \pi$.

[**Hint.** (xii) Consider the cases $\alpha = 0$, $\alpha = \pi$, $0 < \alpha < \pi$.]

2. Examine the convergence of the improper integrals:

(i) $\int_0^\infty \frac{1}{(1+x)\sqrt{x}} dx$, (ii) $\int_0^\infty \frac{1}{x\log x} dx$, (iii) $\int_0^\infty \frac{1}{(x+\sin^2 x)\log x} dx$,

(iv) $\int_0^\infty (\frac{1}{x^2} - \frac{1}{x\sinh x})dx$, (v) $\int_0^\infty \frac{\cosh 2x}{\cosh 3x}dx$, (vi) $\int_0^\infty \log(1+\text{sech}\, x)dx$.

[**Hint.** (iv) 0 is not a point of infinite discontinuity. $\frac{1}{x\sinh x} > 0$ for $x > 0$. (v) $\frac{\cosh 2x}{\cosh 3x} = \frac{e^{2x}+e^{-2x}}{e^{3x}+e^{-3x}} < \frac{2e^{2x}}{e^{3x}}$. (vi) For $x > 0, \log(1+x) < x$.]

3. Show that the following improper integrals are absolutely convergent.

(i) $\int_0^\infty \frac{\sin x}{1+x^2}dx$, (ii) $\int_0^\infty \frac{\cos x}{1+x^2}dx$, (iii) $\int_0^\infty e^{-ax}\cos bx dx, (a > 0)$,

(iv) $\int_0^\infty \frac{\sin x}{\sqrt{x+x^3}}dx$, (v) $\int_0^\infty \frac{x\sin x}{1+x^3}dx$, (vi) $\int_0^\infty e^{-a^2x^2}\cos bx dx$.

4. Assuming the result $\int_0^\infty \frac{\sin mx}{x}dx = \frac{\pi}{2}$ if $m > 0$, prove that

(i) $\int_0^\infty \frac{\sin^3 x}{x^3}dx = \frac{3\pi}{8}$, (ii) $\int_0^\infty \frac{\sin^3 x}{x}dx = \frac{\pi}{4}$, (iii) $\int_0^\infty \frac{\sin^4 x}{x^2}dx = \frac{\pi}{4}$.

[**Hint.** (iii) Integrate by parts and note that $4\sin^3 x\cos x = \sin 2x - \frac{1}{2}\sin 4x$.]

5. Prove that

(i) $\int_0^\infty \frac{x^m(1+x^n)}{1+x^p}dx$ $(m > 0, n > 0)$ is convergent if $p > 1+m+n$;

(ii) $\int_0^\infty \frac{x^{2m}}{1+x^{2n}}dx$ $(m > 0, n > 0)$ is convergent if $n - m > \frac{1}{2}$;

(iii) $\int_0^\infty \frac{x^{m-1}-x^{n-1}}{1-x}dx$ is convergent if $0 < m < 1$ and $0 < n < 1$;

(iv) $\int_0^\infty x^m(\log x)^n\, dx$ is convergent if $m < -1$;

(v) $\int_0^\infty \frac{\sin x(1-\cos x)}{x^n}dx$ is convergent if $0 < n < 4$, and absolutely convergent if $1 < n < 4$.

6. Prove that

(i) $\int_0^\infty \frac{1}{1+x^4\cos^2 x}dx$ is convergent; (ii) $\int_0^\infty \frac{1}{1+x^2\cos^2 x}dx$ is divergent;

(iii) $\int_0^\infty \frac{x}{1+x^4\sin^2 x}dx$ is divergent; (iv) $\int_0^\infty \frac{x}{1+x^6\sin^2 x}dx$ is convergent.

7. Assuming convergence of the integral $\int_0^{\frac{\pi}{2}} \log\sin x\ dx$ to $-\frac{\pi}{2}\log 2$, prove that

(i) $\int_0^\pi \log(1+\cos x)\ dx$ converges to $-\pi\log 2$,

(ii) $\int_0^\pi \log(1-\cos x)\ dx$ converges to $-\pi\log 2$,

(iii) $\int_0^\infty \log(x+\frac{1}{x})\frac{1}{1+x^2}\ dx$ converges to $-\pi\log 2$,

(iv) $\int_0^\infty \frac{\log(1+x^2)}{1+x^2} dx$ converges to $\pi \log 2$.

8. Assuming convergence of the integral $\int_0^{\frac{\pi}{2}} \cos 2nx \log \sin x \, dx$ to $-\frac{\pi}{4n}$, when n is a positive integer, prove that

(i) $\int_0^{\frac{\pi}{2}} \cos 2nx \log \cos x \, dx$ converges to $(-1)^{n+1} \frac{\pi}{4n}$, when n is a positive integer;

(ii) $\int_0^{\pi} \cos nx \log 2(1 + \cos x) \, dx$ converges to $(-1)^{n+1} \frac{\pi}{n}$, when n is a positive integer.

12.11. Beta function and Gamma function.

The improper integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is convergent if $m > 0, n > 0$. The integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$, $m > 0, n > 0$ is called the *Beta function* and it is denoted by $B(m, n)$.

Thus $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$, $m > 0, n > 0$.

The improper integral $\int_0^\infty e^{-x}x^{n-1}dx$ is convergent if $n > 0$. The integral $\int_0^\infty e^{-x}x^{n-1}dx$, $n > 0$ is called the *Gamma function* and it is denoted by $\Gamma(n)$.

Thus $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$, $n > 0$.

Properties.

1. $B(1,1) = 1$.

Proof. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$, $m > 0, n > 0$.

Therefore $B(1,1) = \int_0^1 dx = 1$.

2. $B(m,n) = B(n,m)$.

Proof. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$, $m > 0, n > 0$.

$$= \lim_{\epsilon\to 0\delta\to 0} \int_\epsilon^{1-\delta} x^{m-1}(1-x)^{n-1} dx.$$

Let $x = 1-y$. Then $dx = -dy$.

$$\lim_{\epsilon\to 0\delta\to 0} \int_\epsilon^{1-\delta} x^{m-1}(1-x)^{n-1} dx = \lim_{\epsilon\to 0\delta\to 0} \int_\delta^{1-\epsilon} (1-y)^{m-1}y^{n-1} dy.$$

$$= \lim_{\delta\to 0\epsilon\to 0} \int_\delta^{1-\epsilon} y^{n-1}(1-y)^{m-1} dy = \int_0^1 y^{n-1}(1-y)^{m-1}dy = B(n,m).$$

Therefore $B(m,n) = B(n,m)$.

3. $B(m+1,n) = \frac{m}{m+n}B(m,n)$, $m > 0, n > 0$.

Proof. $B(m+1,n) = \int_0^1 x^m(1-x)^{n-1}dx$

$$= \left[\frac{x^m(1-x)^n}{-n}\right]_0^1 + \frac{m}{n}\int_0^1 x^{m-1}(1-x)^n dx$$
$$= \frac{m}{n}\int_0^1 (1-x)x^{m-1}(1-x)^{n-1}dx$$
$$= \frac{m}{n}\int_0^1 x^{m-1}(1-x)^{n-1}dx - \frac{m}{n}\int_0^1 x^m(1-x)^{n-1}dx$$
$$= \frac{m}{n}B(m,n) - \frac{m}{n}B(m+1,n).$$

Therefore $(1+\frac{m}{n})B(m+1,n) = \frac{m}{n}B(m,n)$

or, $B(m+1,n) = \frac{m}{m+n}B(m,n)$.

4. $B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta, m > 0, n > 0.$

Proof. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,\ m > 0, n > 0.$

Let $x = \sin^2\theta$. Then $dx = 2\sin\theta\cos\theta\, d\theta$.

As $x \to 0+, \theta \to 0+$; as $x \to 1-, \theta \to \frac{\pi}{2}-$.

Therefore $B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta\cos^{2n-2}\theta\sin\theta\cos\theta\, d\theta$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta, m > 0, n > 0.$$

Deductions.

(i) $\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\, d\theta = \frac{1}{2}B(\frac{m+1}{2}, \frac{n+1}{2}), m > -1, n > -1.$

(ii) $\int_0^{\frac{\pi}{2}} \sin^n\theta d\theta = \int_0^{\frac{\pi}{2}} \cos^n\theta\, d\theta = \frac{1}{2}B(\frac{n+1}{2}, \frac{1}{2}), n > --1.$

(iii) $B(\frac{1}{2}, \frac{1}{2}) = 2\int_0^{\frac{\pi}{2}} d\theta = \pi.$

5. $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx,\ m > 0, n > 0.$

Proof. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,\ m > 0, n > 0.$

Let $x = \frac{t}{1+t}$. Then $dx = \frac{1}{(1+t)^2}$.

As $x \to 0+, t \to 0+$; as $x \to 1-, t \to \infty$.

Therefore $B(m,n) = \int_0^\infty (\frac{t}{1+t})^{m-1}(\frac{1}{1+t})^{n-1}\frac{1}{(1+t)^2}dt$

$$= \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}dt$$
$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx.$$

6. $B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx.$

Proof. We have $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$
$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx.$

Let $x = \frac{1}{t}$ in the second integral. Then $dx = -\frac{1}{t^2}$.
As $x \to 1+, t \to 1-$; as $x \to \infty, t \to 0+$.

$$\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^1 \frac{1}{t^{m-1}} \frac{t^{m+n}}{(1+t)^{m+n}} \frac{1}{t^2} dt$$
$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}} dt = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx.$$

Therefore $B(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$
$= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} dx.$

7. $\Gamma(1) = 1$.

Proof. $\Gamma(1) = \int_0^\infty e^{-x} dx = \lim_{X\to\infty} \int_0^X e^{-x} dx = \lim_{X\to\infty} [1 - e^{-X}]_0^X = 1.$

8. $\Gamma(n+1) = n\Gamma(n)$, $n > 0$.

Proof. $\int_\epsilon^X x^n e^{-x} dx = [\frac{x^n e^{-x}}{-1}]_\epsilon^X + n \int_\epsilon^X x^{n-1} e^{-x} dx$
$= -X^n e^{-X} + \epsilon^n e^\epsilon + n \int_\epsilon^X x^{n-1} e^{-x} dx.$

Proceeding to limit as $X \to \infty$ and $\epsilon \to 0$, we have
$\int_0^\infty e^{-x} x^n dx = n \int_0^\infty e^{-x} x^{n-1} dx$
or, $\Gamma(n+1) = n\Gamma(n)$, $n > 0$.

Corollary. If n be a positive integer then $\Gamma(n+1) = n!$.
$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n(n-1)...2.1\Gamma(1) = n!$.

9. $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, $m > 0, n > 0$.

The proof of the property is beyond the scope of this book.

Deductions.

(i). $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

$B(\frac{1}{2}, \frac{1}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} = \Gamma(\frac{1}{2})\Gamma(\frac{1}{2})$.
Therefore $(\Gamma(\frac{1}{2}))^2 = B(\frac{1}{2}, \frac{1}{2}) = \pi$ and this gives $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

(ii). If m, n be positive integers, $B(m+1, n+1) = \frac{m!n!}{(m+n+1)!}$.

$B(m+1, n+1) = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}$, $m > -1, n > -1$.

If m, n are postive integers, $\Gamma(m+1) = m!$, $\Gamma(n+1) = n!$ and therefore $B(m+1, n+1) = \frac{m!n!}{(m+n+1)!}$.

10. Legender's Duplication formula.

$\sqrt{\pi}\Gamma(2n) = 2^{2n-1}\Gamma(n)\Gamma(n+\frac{1}{2})$, $n > 0$.

Proof. $\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m,n)$
$= 2\int_0^{\pi} 2\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$, $m > 0, n > 0$ ... (i)

Taking $m = n$, we have $\frac{(\Gamma(n))^2}{\Gamma(2n)} = 2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta\cos^{2n-1}\theta\, d\theta$
$= \frac{1}{2^{2n-2}}\int_0^{\frac{\pi}{2}} \sin^{2n-1} 2\theta\, d\theta$
$= \frac{1}{2^{2n-1}}\int_0^{\pi} \sin^{2n-1}\phi\, d\phi$ $[2\theta = \phi]$
$= \frac{1}{2^{2n-1}}.2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\phi\, d\phi$... (ii)

Taking $m = \frac{1}{2}$ in (i), we have $\frac{\Gamma(n)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})} = 2\int_0^{\frac{\pi}{2}} \cos^{2n-1}\theta\, d\theta = 2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta\, d\theta$ ... (iii)

From (ii) and (iii) we have $\frac{(\Gamma(n))^2}{\Gamma(2n)} = \frac{1}{2^{2n-1}}\frac{\Gamma(n)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})}$, $n > 0$

or, $\sqrt{\pi}\Gamma(2n) = 2^{2n-1}\Gamma(n)\Gamma(n+\frac{1}{2})$, since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

11. $\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi}$, $0 < m < 1$.

Proof. We have $B(m, 1-m) = \frac{\Gamma(m)\Gamma(1-m)}{\Gamma(1)} = \Gamma(m)\Gamma(1-m)$.

Since $B(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}dx$, $m > 0, n > 0$, $B(m, 1-m) = \int_0^{\infty} \frac{x^{m-1}}{1+x}dx$, $0 < m < 1$.

Therefore $\Gamma(m)\Gamma(1-m) = \int_0^{\infty} \frac{x^{m-1}}{1+x}dx = \frac{\pi}{\sin m\pi}$, $0 < m < 1$. [worked Ex.9, page 524]

12. (i) $\int_0^{\infty} e^{-kt}t^{n-1}dt = \frac{\Gamma(n)}{k^n}$, $k > 0, n > 0$;

(ii) $\int_1^{\infty} \frac{(\log y)^{n-1}}{y^{k+1}} = \frac{\Gamma(n)}{k^n}$, $k > 0, n > 0$.

Proof. (i) $\int_0^{\infty} e^{-kt}t^{n-1}dt$

$= \int_0^{\infty} e^{-y}(\frac{y}{k})^{n-1}\frac{1}{k}dy$ [Let $y = kt$. As $t \to \infty, t \to \infty$ since $k > 0$.]

$= \frac{1}{k^n}\int_0^{\infty} e^{-y}y^{n-1}dy$, $n > 0$

$= \frac{\Gamma(n)}{k^n}$.

(ii) $\int_1^{\infty} \frac{(\log y)^{n-1}}{y^{k+1}}dy$

$= \int_0^\infty t^{n-1}e^{-kt}dt$ [Let $\log y = t$. Then $y = e^t$. $y = 1 \Rightarrow t = 0$]
$= \frac{\Gamma(n)}{k^n}$, since $k > 0, n > 0$. [using (i)]

Worked Examples.

1. Prove that (i) $\int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}$; (ii) $\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}$;

(i) Let $x^2 = t$. Then $dx = \frac{1}{2\sqrt{t}}dt$. As $x \to \infty, t \to \infty$.

$\int_0^\infty e^{-x^2}dx = \frac{1}{2}\int_0^\infty e^{-t}t^{-\frac{1}{2}}dt = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$.

(ii) Let $f(x) = e^{-x^2}, x \in \mathbb{R}$. Then f is an even function on $\mathbb{R}$.

Therefore $\int_{-\infty}^\infty e^{-x^2}dx = 2\int_0^\infty e^{-x^2}dx$, assuming convergence of the integral on the right

$= \sqrt{\pi}$.

2. Prove that $\int_0^{\frac{\pi}{2}} \sin^p x\, dx \times \int_0^{\frac{\pi}{2}} \sin^{p+1} x\, dx = \frac{\pi}{2(p+1)}$, $p > -1$.

We have $\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\, d\theta = \frac{1}{2}B(\frac{m+1}{2}, \frac{n+1}{2}), m > -1, n > -1$.

Therefore $\int_0^{\frac{\pi}{2}} \sin^p x\, dx = \frac{1}{2}B(\frac{p+1}{2}, \frac{1}{2}) = \frac{1}{2}\frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{p+2}{2})}$, $p > -1$

and $\int_0^{\frac{\pi}{2}} \sin^{p+1} x\, dx = \frac{1}{2}B(\frac{p+2}{2}, \frac{1}{2}) = \frac{1}{2}\frac{\Gamma(\frac{p+2}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{p+3}{2})}$, $p > -2$.

$$\int_0^{\frac{\pi}{2}} \sin^p x\, dx \times \int_0^{\frac{\pi}{2}} \sin^{p+1} x\, dx = \frac{1}{4}\ \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{p+2}{2})}.\ \frac{\Gamma(\frac{p+2}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{p+3}{2})},\ p > -1$$

$$= \frac{1}{4}\frac{\pi\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p+3}{2})}, \text{ since } \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$= \frac{1}{4}\frac{2\pi}{p+1}, \text{ since } \Gamma(\tfrac{p+3}{2}) = \tfrac{p+1}{2}\Gamma(\tfrac{p+1}{2})$$

$$= \frac{\pi}{2(p+1)}.$$

3. Prove that $\int_a^b (x-a)^{m-1}(b-x)^{n-1}dx = (b-a)^{m+n-1}B(m,n)$, $m > 0, n > 0$.

We have $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$, $m > 0, n > 0$.

$(x-a) + (b-x) = b - a \Rightarrow \frac{x-a}{b-a} + \frac{b-x}{b-a} = 1$.

Let $\frac{x-a}{b-a} = y$. Then $\frac{b-x}{b-a} = 1 - y$, $dx = (b-a)dy$.
As $x \to a, y \to 0$; as $x \to b, y \to 1$.

$$\int_a^b (x-a)^{m-1}(b-x)^{n-1}dx = \int_0^1 (b-a)^{m+n-1}y^{m-1}(1-y)^{n-1}dy$$
$$= (b-a)^{m+n-1}B(m,n).$$

4. Prove that $\int_0^1 \frac{1}{(1-x^n)^{\frac{1}{n}}}dx = \frac{\pi}{n}\text{cosec } \frac{\pi}{n}$, $n > 1$.

Let $x^n = t$. Then $dx = \frac{1}{nt^{\frac{n-1}{n}}}dt$.

$\int_0^1 \frac{1}{(1-x^n)^{\frac{1}{n}}}\, dx = \int_0^1 (1-t)^{-\frac{1}{n}} . \frac{1}{nt^{\frac{n-1}{n}}} dt$

$= \frac{1}{n}\int_0^1 t^{\frac{1}{n}-1}(1-t)^{(1-\frac{1}{n})-1}dt$

$= \frac{1}{n}B(\frac{1}{n}, 1-\frac{1}{n})$, since $0 < \frac{1}{n} < 1$

$= \frac{1}{n}\frac{\Gamma(\frac{1}{n})\Gamma(1-\frac{1}{n})}{\Gamma(1)}$

$= \frac{1}{n}\frac{\pi}{\sin\frac{\pi}{n}} = \frac{\pi}{n}\text{cosec } \frac{\pi}{n}$.

5. If n be a positive integer, prove that $\Gamma(\frac{1}{n})\Gamma(\frac{2}{n})\Gamma(\frac{3}{n})\ldots\Gamma(\frac{n-1}{n}) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}$.

Let $P = \Gamma(\frac{1}{n})\Gamma(\frac{2}{n})\Gamma(\frac{3}{n})\ldots\Gamma(\frac{n-1}{n})$.

Then $P = \Gamma(1-\frac{1}{n})\Gamma(1-\frac{2}{n})\ldots\Gamma(\frac{2}{n})\Gamma(\frac{1}{n})$. [taking the factors in the reverse order]

$P^2 = [\Gamma(\frac{1}{n})\Gamma(1-\frac{1}{n})][\Gamma(\frac{2}{n})\Gamma(1-\frac{2}{n}]\ldots[\Gamma(\frac{n-1}{n})\Gamma(\frac{1}{n})]$

$= \frac{\pi}{\sin\frac{\pi}{n}}.\frac{\pi}{\sin\frac{2\pi}{n}}\cdots\frac{\pi}{\sin\frac{(n-1)\pi}{n}} = \frac{\pi^{n-1}}{\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{n}}\ldots$(i)

We prove the following lemma.

Lemma. $\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$.

Proof. $x^{2n} - 2x^n\cos\, 2n\theta + 1 = 0$ gives $x^n = \cos\, 2n\theta + i\sin\, 2n\theta$, i.e., $x = \cos(2\theta + \frac{2k\pi}{n}) + i\sin(2\theta + \frac{2k\pi}{n})$, where $k = 0, 1, \ldots, n-1$.

Therefore $x^{2n} - 2x^n\cos\, 2n\theta + 1 = \prod_{k=0}^{n-1}[x^2 - 2x\cos(2\theta + \dfrac{2k\pi}{n}) + 1]$.

Taking $x = 1$, we have $4\sin^2\, n\theta = \prod_{k=0}^{n-1} 4\sin^2(\theta + \dfrac{k\pi}{n})$.

$\sin^2\, n\theta = 4^{n-1}\sin^2\theta\sin^2(\theta + \frac{\pi}{n})\ldots\sin^2(\theta + \frac{(n-1)\pi}{n})$

or, $\sin\, n\theta = 2^{n-1}\sin\,\theta\sin\,(\theta + \frac{\pi}{n})\sin\,(\theta + \frac{2\pi}{n})\ldots\sin(\theta + \frac{(n-1)\pi}{n})$

or, $\frac{\sin\, n\theta}{\sin\,\theta} = 2^{n-1}\sin\,(\theta + \frac{\pi}{n})\sin\,(\theta + \frac{2\pi}{n})\ldots\sin(\theta + \frac{(n-1)\pi}{n})$.

Proceeding to limit as $\theta \to 0$, we have

$n = 2^{n-1}\sin\,\frac{\pi}{n}\sin\,\frac{2\pi}{n}\ldots\sin\,\frac{(n-1)\pi}{n}$

or, $\sin\,\frac{\pi}{n}\sin\,\frac{2\pi}{n}\ldots\sin\,\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$.

This proves the lemma.

Using the lemma, we have from (i) $P^2 = \frac{\pi^{n-1}2^{n-1}}{n}$.

Therefore $P = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}$.

6. Show that $\int_0^1 \log\Gamma(x)dx$ is convergent and evaluate it.

$\Gamma(x) = \int_0^\infty e^{t}t^{x-1}dt,\ x > 0$.

Let $f(t) = e^{t}t^{x-1},\ t > 0$. If $x > 0$ then f is a continuous function of t on $(0, \infty)$ and for $x > 0$, $f(t) > 0$ for all $t > 0$.

For $x > 0$, $\int_0^\infty f(t)dt$ is a convergent integral and for $x > 0$, $\int_0^\infty f(t)dt > 0$, i.e., $\Gamma(x) > 0$ for $x > 0$. So $\log\Gamma(x)$ is defined for $x > 0$.

We have $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$.

Therefore $\log\Gamma(x+1) = \log x + \log\Gamma(x)$ for $x > 0$

or, $\log\Gamma(x) = \log\Gamma(x+1) - \log x$ for $x > 0$ ... (i)

$\Gamma(x+1)$ is defined for all $x > -1$ and $\Gamma(x+1) > 0$ for all $x > -1$. Therefore the integral $\int_0^1 \log\Gamma(x+1)dx$ is a proper one.

The integral $\int_0^1 \log x dx$ is an improper integral, since 0 is a point of infinite discontinuity.

$\lim_{\epsilon\to 0+}\int_\epsilon^1 \log x dx = \lim_{\epsilon\to 0+}[x\log x - x]_\epsilon^1 = \lim_{\epsilon\to 0+}[\epsilon - \epsilon\log\epsilon - 1] = -1$.

Therefore $\int_0^1 \log x dx$ is convergent.

From (i) it follows that $\int_0^1 \log\Gamma(x)dx$ is convergent.

Let $\phi(\epsilon) = \int_\epsilon^1 \log\Gamma(x)dx$.

Let $x = 1-y$. Then $\phi(\epsilon) = \int_0^{1-\epsilon}\log\Gamma(1-y)dy = \int_0^{1-\epsilon}\log\Gamma(1-x)dx$.

Since $\int_0^1 \log\Gamma(x)dx$ is convergent, $\lim_{\epsilon\to 0+}\phi(\epsilon)$ is finite.

Let $I = \int_0^1 \log\Gamma(x)dx = \int_0^1 \log\Gamma(1-x)dx$.

$2I = \int_0^1 \log\Gamma(x)dx$. Then

$= \int_0^1 \log\Gamma(x)dx + \int_0^1 \log\Gamma(1-x)dx$

$= \int_0^1 \log(\Gamma(x).\Gamma(1-x))\,dx = \int_0^1 \log\frac{\pi}{\sin\pi x}dx$

$= \log\pi - \int_0^1 \log\sin\pi x\,dx = \log\pi - \frac{1}{\pi}\int_0^\pi \log\sin$ ...

$= \log\pi - \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\log\sin y\,dy = \log\pi - \frac{2}{\pi}\cdot\frac{\pi}{2}$ ... $\log 2$ ...

$= \log\pi + \log 2 = \log 2$ ...

Therefore $I = \int_0^1 \log\Gamma$ ... $= \log 2$ ...

Exercises 23

1. Prove that (i) $\int_0^{\frac{\pi}{2}} \tan^p \theta \, d\theta = \frac{\pi}{2} \sec \frac{p\pi}{2}$, if $-1 < p < 1$;

(ii) $\int_0^{\frac{\pi}{2}} \cot^p \theta \, d\theta = \frac{\pi}{2} \sec \frac{p\pi}{2}$, if $-1 < p < 1$.

[**Hint.** (i) $\tan^p \theta = \sin^p \theta \cos^{-p} \theta$.]

2. Prove that (i) $\int_0^1 x^{m-1}(1-x^p)^{n-1} = \frac{1}{p} B(\frac{m}{p}, n)$, if $m > 0, n > 0, p > 0$.

(ii) $\int_0^1 x^{m-1}(\log \frac{1}{x})^{n-1} \, dx = \frac{\Gamma(n)}{m^n}$, if $m > 0, n > 0$.

(iii) $\int_{-1}^1 (1+x)^{p-1}(1-x)^{q-1} dx = 2^{p+q-1} B(p, q)$, if $p > 0, q > 0$.

[**Hint.** (iii) Let $1 + x = 2y$.]

3. Prove that (i) $\int_0^1 \frac{1}{(1-x^6)^{\frac{1}{6}}} \, dx = \frac{\pi}{3}$; (ii) $\int_0^1 \frac{1}{(1-x^3)^{\frac{1}{3}}} \, dx = \frac{2\pi}{3\sqrt{3}}$;

4. Prove that (i) $\int_0^1 \frac{x^2}{\sqrt{1-x^4}} dx \times \int_0^1 \frac{1}{\sqrt{1+x^4}} dx = \frac{\pi}{4\sqrt{2}}$;

(ii) $\int_0^\infty x^2 e^{-x^4} dx \times \int_0^\infty e^{-x^4} dx = \frac{\pi}{8\sqrt{2}}$.

5. Prove that (i) $\Gamma(\frac{1}{9})\Gamma(\frac{2}{9})...\Gamma(\frac{8}{9}) = \frac{16\pi^4}{3}$;

(ii) $\Gamma(\frac{1+n}{2})\Gamma(\frac{1-n}{2}) = \pi \sec \frac{n\pi}{2}$, $-1 < n < 1$;

(iii) $2^n \Gamma(\frac{n+1}{2})\Gamma(\frac{n+2}{2}) = \sqrt{\pi}\Gamma(n+1)$, $n > -1$.

[**Hint.** (iii) Use Duplication formula.]

6. Prove that (i) $B(m, m) = 2^{1-2m} B(m, \frac{1}{2})$, $m > 0$;

(ii) $B(m, m).B(m + \frac{1}{2}, m + \frac{1}{2}) = \frac{\pi}{m}.2^{1-4m}$, $m > 0$.

[**Hint.** (i) $B(m, m) = 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2m-1} \theta d\theta$, $m > 0$.]

7.(a) Prove that $\int_0^\pi \frac{\sin^{m-1} x}{(2+\cos x)^m} dx = \frac{2^{m-1}}{3^{\frac{m}{2}}} B(\frac{m}{2}, \frac{m}{2})$, $m > 0$.

(b) Prove that $\int_0^\infty x^{2n} e^{-ax^2} dx = \frac{1.3.5...(2n-1)\sqrt{\pi}}{2^{n+1} a^{n+\frac{1}{2}}}$, $a > 0$.

8.(a) Show that $\int_0^1 x^{m-1}(\log x)^{n-1} dx$ is convergent if $m > 0, n > 0$.

(b) Show that (i) $\int_0^1 \sqrt{x}(\log x)^2 dx = \frac{16}{27}$, (ii) $\int_0^1 \frac{(\log x)^2}{\sqrt{x}} dx = 16$.

[**Hint.** (a) Let $x = e^{-y}$.]

9. Show that (i) $\int_0^{\frac{\pi}{2}} \sqrt{\cot x} \, dx = \frac{\pi}{\sqrt{2}}$, (ii) $\int_0^{\frac{\pi}{2}} \sqrt[3]{\cot x} \, dx = \frac{\pi}{\sqrt{3}}$,

(iii) $\int_0^\infty \frac{x^2(x^3-1)}{(1+x)^9} dx = 0$, (iv) $\int_0^\infty e^{-3x^2} x^3 dx = \frac{1}{9}$,

(v) $\int_0^\infty e^{-x^2} x^3 dx = 1$, (vi) $\int_{-1}^1 \frac{1}{(1+x)^{\frac{1}{3}}(1-x)^{\frac{2}{3}}} dx = \frac{2\pi}{\sqrt{3}}$.

13. SEQUENCE OF FUNCTIONS

13.1. Sequence of functions.

Let D be a subset of $\mathbb{R}$ and for each $n \in \mathbb{N}$, let $f_n : D \to \mathbb{R}$ be a function. Then $\{f_n\}$ is a sequence of functions on D to $\mathbb{R}$. D is said to be the domain of the sequence of functions $\{f_n\}$.

In particular, D may be a closed interval $[a, b]$ (or $[a, \infty)$), or an open interval (a, b) (or (a, ∞)).

To each $x_0 \in D$ the sequence $\{f_n\}$ gives rise to a sequence of real numbers $\{f_n(x_0)\}$, which is obtained by evaluating each f_n at x_0.

For some $x \in D$, the sequence $\{f_n(x)\}$ may converge to a limit and for some other $x \in D$, the sequence $\{f_n(x)\}$ may not converge.

13.2. Pointwise convergence.

Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}$, let $f_n : D \to \mathbb{R}$ be a function.

The sequence $\{f_n\}$ is said to be *pointwise convergent* on D if for each $x \in D$, the sequence $\{f_n(x)\}$ converges.

Let the sequence $\{f_n\}$ be pointwise convergent on D and let $c \in D$.

Then the sequence $\{f_n(c)\}$ is convergent. Let $\lim f_n(c) = l_c$. Since for all $x \in D$, $\{f_n(x)\}$ converges to a limit, l_x exists for all $x \in D$.

Let us define a function $f : D \to \mathbb{R}$ by $f(x) = l_x, x \in D$. Then f is said to be the *limit function* of the sequence $\{f_n\}$ on D.

In this case we also say that the sequence $\{f_n\}$ converges to f on D and we write $f = \lim f_n$ on D, or $f_n \to f$ on D.

Examples.

1. For each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = x^n, x \in \mathbb{R}$.

Then $\{f_n\}$ is a sequence of functions on $\mathbb{R}$. For each $x \in (-1, 1)$ the sequence $\{f_n(x)\}$ converges to 0 and for $x = 1$, the sequence $\{f_n(x)\}$ converges to 1. For all other $x \in \mathbb{R}$, the sequence $\{f_n(x)\}$ is divergent.

Therefore the sequence $\{f_n\}$ is pointwise convergent on $(-1, 1]$ and the limit function f is defined by

$$\begin{aligned} f(x) &= 0, -1 < x < 1 \\ &= 1, x = 1. \end{aligned}$$

Note. Here we observe that although the domain of the sequence $\{f_n\}$ is $\mathbb{R}$, the domain of pointwise convergence of the sequence is a proper subset of $\mathbb{R}$.

2. For each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{x}{n}, x \in \mathbb{R}$. Then $\{f_n\}$ is a sequence of functions on $\mathbb{R}$. For each $x \in \mathbb{R}$, the sequence $\{f_n(x)\}$ converges to 0.

Therefore the sequence $\{f_n\}$ is pointwise convergent on $\mathbb{R}$ and the limit function f is defined by $f(x) = 0, x \in \mathbb{R}$.

3. Let $D = \{x \in \mathbb{R} : x \geq 0\}$ and for each $n \in \mathbb{N}$, let $f_n : D \to \mathbb{R}$ be defined by $f_n(x) = \frac{n}{x+n}, x \geq 0$. Then $\{f_n\}$ is a sequence of functions on D. For each $x \in D$, $\lim\limits_{n \to \infty} f_n(x) = 1$.

Therefore the sequence is pointwise convergent on D to the function f defined by $f(x) = 1, x \geq 0$.

4. For each natural number n, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{nx}{1+n^2x^2}, x \in \mathbb{R}$. Then $\{f_n\}$ is a sequence of functions on $\mathbb{R}$.

For $x = 0$, the sequence is $\{0, 0, 0, \ldots\}$. This converges to 0.
For $x \neq 0$, $\lim\limits_{n \to \infty} f_n(x) = \lim\limits_{n \to \infty} \frac{nx}{1+n^2x^2} = 0$.

Therefore the sequence $\{f_n\}$ is pointwise convergent on $\mathbb{R}$ to the function f defined by $f(x) = 0, x \in \mathbb{R}$.

5. Let $D = \{x \in \mathbb{R} : x \geq 0\}$ and for each natural number n, let $f_n : D \to \mathbb{R}$ be defined by $f_n(x) = \frac{nx}{1+nx}, x \geq 0$. Then $\{f_n\}$ is a sequence of functions on D.

For $x = 0$, the sequence is $\{0, 0, 0, \ldots\}$. This converges to 0.
For $x > 0$, $\lim\limits_{n \to \infty} f_n(x) = \lim\limits_{n \to \infty} \frac{nx}{1+nx} = 1$.

Therefore the sequence $\{f_n\}$ is pointwise convergent on D to the function f defined by
$$f(x) = 0, x = 0$$
$$= 1, x > 0.$$

6. Let $D = \{x \in \mathbb{R} : x \geq 0\}$ and for each natural number n, let $f_n : D \to \mathbb{R}$ be defined by $f_n(x) = \frac{x}{1+nx}, x \geq 0$. Then $\{f_n\}$ is a sequence of functions on D.

For $x = 0$, the sequence is $\{0, 0, 0, \cdots\}$. This converges to 0.
For $x > 0$, $\lim\limits_{n \to \infty} f_n(x) = \lim\limits_{n \to \infty} \frac{x}{1+nx} = 0$.

Therefore the sequence $\{f_n\}$ is pointwise convergent on D to the function f defined by $f(x) = 0, x > 0$.

7. Let $f_n(x) = \tan^{-1} nx, x \in \mathbb{R}$.

$$\text{Then } \lim_{n\to\infty} f_n(x) \begin{array}{ll} = & \frac{\pi}{2}, \text{ if } x > 0 \\ = & 0 \text{ if } x = 0 \\ = & -\frac{\pi}{2} \text{ if } x < 0. \end{array}$$

Therefore the sequence $\{f_n\}$ is pointwise convergent on $\mathbb{R}$ to the function f where $f(x) = \frac{\pi}{2}\text{sgn } x, x \in \mathbb{R}$.

8. For each natural number n let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{\sin nx}{n}, x \in \mathbb{R}$. Then $\{f_n\}$ is a sequence of functions on $\mathbb{R}$.

For each $x \in \mathbb{R}$, $\lim\limits_{n\to\infty} f_n(x) = 0$, since $\lim \frac{1}{n} = 0$ and $|\sin nx| \leq 1$.

Therefore the sequence $\{f_n\}$ is pointwise convergent on $\mathbb{R}$ to the function f defined by $f(x) = 0, x \in \mathbb{R}$.

9. Let $f_n(x) = xe^{-nx}, x \geq 0$.

For all $x \geq 0$, $0 \leq xe^{-nx} < \frac{1}{n}$, since $e^{nx} > nx$ for all $x > 0$.
By Sandwich theorem 5.5.5, $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \geq 0$.

Therefore the sequence $\{f_n\}$ is pointwise convergent on $[0, \infty)$ to the function f defined by $f(x) = 0, x \geq 0$.

10. Let $f_n(x) = x^2e^{-nx}, x \geq 0$.

For all $x \geq 0$, $0 \leq x^2e^{-nx} < \frac{2}{n^2}$, since $e^{nx} > \frac{n^2x^2}{2}$ for all $x \geq 0$.
By Sandwich theorem 5.5.5, $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \geq 0$.

Therefore the sequence $\{f_n\}$ is pointwise convergent on $[0, \infty)$ to the function f defined by $f(x) = 0, x \geq 0$.

Let $\{f_n\}$ be a sequence of functions that converges pointwise on a domain $D \subset \mathbb{R}$ to the function f.

Let $x' \in D$. Let us choose $\epsilon > 0$.

Since the sequence $\{f_n(x')\}$ converges to $f(x')$, there exists a natural number k' such that $|f_n(x') - f(x')| < \epsilon$ for all $n \geq k'$.

This k' depends on ϵ as well as on x'.

Let $x'' \in D$. Since the sequence $\{f_n(x'')\}$ converges to $f(x'')$, there exists a natural number k'' such that $|f_n(x'') - f(x'')| < \epsilon$ for all $n \geq k''$.

It is quite natural that k'' is different from k'.

If it is possible that for a pre-assigned positive ϵ, there exists a natural number k such that

for all $x \in D$, $|f_n(x) - f(x)| < \epsilon$ for all $n \geq k$

then we say that the convergence of the sequence $\{f_n\}$ is uniform on D.

13.3. Uniform convergence.

Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}$, let $f_n : D \to \mathbb{R}$ be a function. The sequence $\{f_n\}$ is said to be *uniformly convergent* on D to a function f if corresponding to a pre-assigned positive ϵ there exists a natural number $k(\epsilon)$ (depending on ϵ but not on $x \in D$) such that
for all $x \in D$, $| f_n(x) - f(x) | < \epsilon$ for all $n \geq k$.

In this case we write $\lim f_n = f$ uniformly on D, or $f_n \to f$ uniformly on D.

f is said to be the *uniform limit* of the sequence $\{f_n\}$ on D.

It follows that if the sequence $\{f_n\}$ is uniformly convergent on D to the function f then the sequence $\{f_n\}$ also converges pointwise on D to f. But that the converse is not true is discussed in the following examples.

Examples.

1. In Example 1 of 13.2, the sequence $\{f_n\}$ converges on $(-1, 1]$ to the function f where $f(x) = 0, -1 < x < 1$.
$= 1, x = 1$.

Let us examine if the convergence of the sequence $\{f_n\}$ is uniform on $(0, 1)$.

Let $c \in (0, 1)$. Then $| f_n(c) - f(c) | = c^n$.

Let $0 < \epsilon < 1$. Then $| f_n(c) - f(c) | < \epsilon$ whenever $c^n < \epsilon$,
i.e., whenever $n \log(1/c) > \log(1/\epsilon)$,
i.e., whenever $n > \log(1/\epsilon)/\log(1/c)$.

Let $k = [\log(1/\epsilon)/\log(1/c)] + 1$. Then k is a natural number and $| f_n(c) - f(c) | < \epsilon$ for all $n \geq k$.

Therefore for all $x \in (0, 1)$, $| f_n(x) - f(x) | < \epsilon$ for all $n \geq k$, where $k = [\log(1/\epsilon)/\log(1/x)] + 1$.
This k depends on ϵ as well as on x. As $x \to 1-, k \to \infty$.

It follows that there does not exist a natural number k such that for all $x \in (0, 1)$, $| f_n(x) - f(x) | < \epsilon$ holds for all $n \geq k$.

Consequently, $\{f_n\}$ is not uniformly convergent on $(0, 1)$.

Let $a \in \mathbb{R}$ such that $0 < a < 1$.
In $[0, a]$, the greatest value of $\log(1/\epsilon)/\log(1/x)$ is $\log(1/\epsilon)/\log(1/a)$.

Let $k = [\log(1/\epsilon)/\log(1/a)] + 1$. Then k is a natural number and for all $x \in [0, a]$, $| f_n(x) - f(x) | < \epsilon$ for all $n \geq k$.

This proves that the sequence $\{f_n\}$ is uniformly convergent on $[0, a]$.

2. In Example 2 of 13.2, the sequence $\{f_n\}$ converges on $\mathbb{R}$ to the function f where $f(x) = 0, x \in \mathbb{R}$.

Let us examine if the convergence of the sequence is uniform on $[0, \infty)$.
For all $x \geq 0, \mid f_n(x) - f(x) \mid = \frac{x}{n}$.
Let $\epsilon > 0$. If $k = [\frac{x}{\epsilon}] + 1$, then k is a natural number and for all $x \geq 0, \mid f_n(x) - f(x) \mid < \epsilon$ for all $n \geq k$.
This k depends on ϵ as well as on x. As $x \to \infty, k \to \infty$.

It follows that there does not exist a natural number k (depending only on the chosen ϵ) such that
for all $x \geq 0, \mid f_n(x) - f(x) \mid < \epsilon$ holds for all $n \geq k$.

This proves that the convergence of the sequence $\{f_n\}$ is not uniform on $[0, \infty)$.

Let $a \in \mathbb{R}$ such that $a > 0$. In $[0, a]$ the greatest value of $(\frac{x}{\epsilon})$ is $(\frac{a}{\epsilon})$.
Let $k = [\frac{a}{\epsilon}] + 1$. Then k is a natural number and for all $x \in [0, a]$, $\mid f_n(x) - f(x) \mid < \epsilon$ for all $n \geq k$.

This proves that the sequence $\{f_n\}$ is uniformly convergent on $[0, a]$.

3. In Example 4 of 13.2, the sequence $\{f_n\}$ converges on $\mathbb{R}$ to the function f where $f(x) = 0, x \in \mathbb{R}$.

Let us examine if the convergence of the sequence is uniform on $[0, \infty)$.
For all $x \geq 0, \mid f_n(x) - f(x) \mid = \frac{nx}{1+n^2x^2}$.

Let $u(x) = \frac{nx}{1+n^2x^2}$ for $x > 0$. Then $u'(x) = \frac{n(1-n^2x^2)}{(1+n^2x^2)^2}$.
$u'(x) > 0$ for $x < \frac{1}{n}$, $u'(x) = 0$ for $x = \frac{1}{n}$, $u'(x) < 0$ for $x > \frac{1}{n}$.
u is a maximum at $x = \frac{1}{n}$ and $u(\frac{1}{n}) = \frac{1}{2}$, i.e., $|f_n(\frac{1}{n}) - f(\frac{1}{n})| = \frac{1}{2}$.

Let $\epsilon = \frac{1}{4}$. If the sequence $\{f_n\}$ be uniformly convergent on $[0, \infty)$ to the function f, then for the chosen ϵ there must exist a natural number k such that for all $x \geq 0, \mid f_n(x) - f(x) \mid < \frac{1}{4}$ holds for all $n \geq k$.

But for every natural number k, $|f_k(\frac{1}{k}) - f(\frac{1}{k})| = \frac{1}{2} \not< \frac{1}{4}$.

This shows that no natural number k can be found so that for all $x \in [0, \infty), \mid f_n(x) - f(x) \mid < \frac{1}{4}$ holds for all $n \geq k$.

Therefore the sequence $\{f_n\}$ is not uniformly convergent on $[0, \infty)$.

Let $a \in \mathbb{R}$ such that $a > 0$.
For all $x > 0, \mid f_n(x) - f(x) \mid = \frac{nx}{1+n^2x^2} < \frac{1}{nx}$.
Then for all $x \geq a, \mid f_n(x) - f(x) \mid < \frac{1}{na}$.
Let us choose $\epsilon > 0$. If $k = [\frac{1}{a\epsilon}] + 1$, then k is a natural number and for all $x \geq a, \mid f_n(x) - f(x) \mid < \epsilon$ for all $n \geq k$.

This proves that the sequence $\{f_n\}$ is uniformly convergent on $[a, \infty)$.

4. In Example 6 of 13.2, the sequence $\{f_n\}$ converges to the function f where $f(x) = 0, x \geq 0$.

For all $x \geq 0, \mid f_n(x) - f(x) \mid = \frac{x}{1+nx} < \frac{1}{n}$.

Let $\epsilon > 0$. Then $\mid f_n(x) - f(x) \mid < \epsilon$ for all $n > \frac{1}{\epsilon}$.

Let $k = [\frac{1}{\epsilon}] + 1$. Then k is a natural number and for all $x \geq 0$, $\mid f_n(x) - f(x) \mid < \epsilon$ for all $n \geq k$.

This proves that the sequence $\{f_n\}$ is uniformly convergent on $[0, \infty)$.

5. In example 8 of 13.2, the sequence $\{f_n\}$ converges on $\mathbb{R}$ to the function f where $f(x) = 0, x \in \mathbb{R}$.

For all $x \in \mathbb{R}, \mid f_n(x) - f(x) \mid = \mid \frac{\sin nx}{n} \mid \leq \frac{1}{n}$.

Let $\epsilon > 0$. Then $\mid f_n(x) - f(x) \mid < \epsilon$ for all $n > \frac{1}{\epsilon}$.

Let $k = [\frac{1}{\epsilon}] + 1$. Then k is a natural number and for all $x \in \mathbb{R}$, $\mid f_n(x) - f(x) \mid < \epsilon$ for all $n \geq k$.

This proves that the sequence $\{f_n\}$ is uniformly convergent on $\mathbb{R}$.

6. In Example 9 of 13.2, the sequence $\{f_n\}$ converges to the function f where $f(x) = 0, x \geq 0$.

For all $x \geq 0, \mid f_n(x) - f(x) \mid = xe^{-nx} < \frac{1}{n}$, since $e^{nx} > nx$ for all $x \geq 0$.

Let $\epsilon > 0$. Then $\mid f_n(x) - f(x) \mid < \epsilon$ for all $n > \frac{1}{\epsilon}$.

Let $k = [\frac{1}{\epsilon}] + 1$. Then k is a natural number and for all $x \geq 0$, $\mid f_n(x) - f(x) \mid < \epsilon$ for all $n \geq k$.

This proves that the sequence $\{f_n\}$ is uniformly convergent on $[0, \infty)$.

Theorem 13.3.1. (Cauchy criterion)

Let $D \subset \mathbb{R}$ and let $\{f_n\}$ be a sequence of functions on D to $\mathbb{R}$.

A necessary and sufficient condition for uniform convergence of the sequence $\{f_n\}$ on D is that for a pre-assigned positive ϵ there exists a natural number k such that for all $x \in D$,
$\mid f_{n+p}(x) - f_n(x) \mid < \epsilon$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

Proof. Let the sequence $\{f_n\}$ be uniformly convergent on D and let the limit function be f. Then for a pre-assigned positive ϵ there exists a natural number k (depending only on ϵ) such that

for all $x \in D, \mid f_n(x) - f(x) \mid < \frac{\epsilon}{2}$ for all $n \geq k$.

Therefore if $p = 1, 2, 3, \ldots$ then for all $x \in D$,
$\mid f_{n+p}(x) - f(x) \mid < \frac{\epsilon}{2}$ holds for all $n \geq k$.

Thus for all $x \in D$,

$$\begin{aligned} | f_{n+p}(x) - f_n(x) | &\le | f_{n+p}(x) - f(x) | + | f_n(x) - f(x) | \\ &< \epsilon \quad \text{for all } n \ge k \text{ and } p = 1, 2, 3, \ldots. \end{aligned}$$

Conversely, let the condition be satisfied. Then for a chosen $\epsilon > 0$ there exists a natural number k such that for all $x \in D$,
$| f_{n+p}(x) - f_n(x) | < \epsilon$ for all $n \ge k$ and $p = 1, 2, 3, \ldots$

Let $x_0 \in D$. Then $| f_{n+p}(x_0) - f_n(x_0) | < \epsilon$ for all $n \ge k$ and $p = 1, 2, 3, \ldots$

It follows that the sequence $\{f_n(x_0)\}$ is Cauchy sequence in $\mathbb{R}$ and therefore it is convergent. Consequently, the sequence $\{f_n\}$ is pointwise convergent on D. Let the limit function be f.

Let us choose $\epsilon > 0$. Then by the condition, there exists a natural number k (depending only on ϵ) such that for all $x \in D$,
$| f_{n+p}(x) - f_n(x) | < \frac{\epsilon}{2}$ for all $n \ge k$ and $p = 1, 2, 3, \ldots$

Therefore for all $x \in D$, $f_k(x) - \frac{\epsilon}{2} < f_{k+p}(x) < f_k(x) + \frac{\epsilon}{2}$ for $p = 1, 2, 3, \ldots$

Since $\lim\limits_{p \to \infty} f_{k+p}(x) = f(x)$, taking limit as $p \to \infty$ we have
for all $x \in D$, $f_k(x) - \frac{\epsilon}{2} \le f(x) \le f_k(x) + \frac{\epsilon}{2}$
or, $| f_k(x) - f(x) | \le \frac{\epsilon}{2} < \epsilon$ for all $x \in D$.
Similar inequalities hold for $k+1, k+2, \ldots$

Therefore for all $x \in D$, $| f_n(x) - f(x) | < \epsilon$ for all $n \ge k$.

This proves that the sequence $\{f_n\}$ is uniformly convergent on D.

Equivalent statement for Cauchy criterion.

A necessary and sufficient condition for uniform convergence of a sequence $\{f_n\}$ on D is that for a pre-assigned positive ϵ there exists a natural number k such that
for all $x \in D$, $| f_m(x) - f_n(x) | < \epsilon$ for all $m, n \ge k$.

Worked Examples.

1. A sequence of functions $\{f_n\}$ is defined on $[0, a]$, $0 < a < 1$, by $f_n(x) = x^n$, $x \in [0, a]$. Show that the sequence $\{f_n\}$ converges uniformly on $[0, a]$.

Let us choose $\epsilon > 0$ such that $0 < \epsilon < 2$.
For all $x \in [0, a]$ and for all $m, n \in \mathbb{N}$,

$$\begin{aligned} | f_m(x) - f_n(x) | &= | x^m - x^n | \\ &\le | x |^m + | x |^n \\ &\le a^m + a^n \\ &\le 2a^m \text{ if } m \le n. \end{aligned}$$

Now $| f_m(x) - f_n(x) | < \epsilon$ holds if $a^m < \frac{\epsilon}{2}$

i.e., if $m \log a < \log \frac{\epsilon}{2}$

i.e., if $m > \frac{\log \frac{\epsilon}{2}}{\log a}$, since $\log a < 0$.

Let $k = [\frac{\log \frac{\epsilon}{2}}{\log a}] + 1$. Then k is a natural number and for all $x \in [0, a]$, $| f_m(x) - f_n(x) | < \epsilon$ for all natural numbers m, n satisfying $n \geq m \geq k$.

By Cauchy's criterion, the sequence $\{f_n\}$ is uniformly convergent on $[0, a], 0 < a < 1$.

2. Let $r_1, r_2, r_3, \ldots$ be an enumeration of the set of all rational points in $[0, 1]$ and a sequence of functions $\{f_n\}$ is defined on $[0, 1]$ by

$$\begin{aligned} f_n(x) &= 0, x = r_1, r_2, \ldots, r_n \\ &= 1, x \in [0, 1] - \{r_1, r_2, \ldots, r_n\}. \end{aligned}$$

Show that the sequence $\{f_n\}$ is not uniformly convergent on $[0, 1]$.

Let us take $\epsilon = \frac{1}{2}$.

It is sufficient to establish that for all $k \in \mathbb{N}$ there exist natural numbers m and n such that $m, n \geq k$ and

$| f_m(x_0) - f_n(x_0) | \not< \frac{1}{2}$ for some $x_o \in [0, 1]$.

$$\begin{aligned} \text{For all } k \in \mathbb{N}, f_k(x) &= 0 \text{ if } x \in \{r_1, r_2, \ldots, r_k\} \\ &= 1 \text{ if } x \in [0, 1] - \{r_1, r_2, \ldots, r_k\}. \end{aligned}$$

For every natural number k, there exists a point $r_{k+1} \in [0, 1]$ such that $| f_k(r_{k+1}) - f_{k+1}(r_{k+1}) | = | 1 - 0 | = 1$.

Therefore no natural number k can be found such that for all $x \in [0, 1], | f_m(x) - f_n(x) | < \frac{1}{2}$ holds for all $m, n \geq k$.

By Cauchy criterion for uniform convergence of a sequence of functions, the sequence $\{f_n\}$ is not uniformly convergent on $[0, 1]$.

Theorem 13.3.2. Let $D \subset \mathbb{R}$ and let $\{f_n\}$ be a sequence of functions pointwise convergent on D to a function f. Let $M_n = \sup_{x \in D} | f_n(x) - f(x) |$.

Then $\{f_n\}$ is uniformly convergent on D to f if and only if $\lim M_n = 0$.

Proof. Let the sequence $\{f_n\}$ be uniformly convergent on D to f.

Let $\epsilon > 0$. Then there exists a natural number k (depending only on ϵ) such that for all $x \in D$, $| f_n(x) - f(x) | < \frac{\epsilon}{2}$ for all $n \geq k$.

This implies $\sup_{x \in D} | f_n(x) - f(x) | \leq \dfrac{\epsilon}{2} < \epsilon$ for all $n \geq k$

or, $| M_n | < \epsilon$ for all $n \geq k$. This proves that $\lim M_n = 0$.

Conversely, let $\lim M_n = 0$.

Let $\epsilon > 0$. Then there exists a natural number k such that $| M_n | < \epsilon$ for all $n \geq k$.

or, $\sup_{x \in D} | f_n(x) - f(x) | < \epsilon$ for all $n \geq k$.

$$\text{Therefore for all } x \in D, |f_n(x) - f(x)| \le \sup_{x \in D} |f_n(x) - f(x)| < \epsilon \text{ for all } n \ge k.$$

This proves that the sequence $\{f_n\}$ is uniformly convergent to f on D.

Worked Examples (continued).

3. A sequence of functions $\{f_n\}$ is defined by $f_n(x) = \frac{nx}{1+n^2x^2}, 0 \le x \le 1$. Show that the sequence $\{f_n\}$ is not uniformly convergent on $[0, 1]$.

For $x = 0$, the sequence is $\{0, 0, 0, \ldots\}$. This converges to 0.
For $0 < x \le 1$, $\lim_{n\to\infty} f_n(x) = 0$.

Thus the sequence $\{f_n\}$ is convergent on $[0, 1]$ and the limit function f is defined by $f(x) = 0, 0 \le x \le 1$.

Let $M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)|$. Then $M_n = \sup_{x\in[0,1]} \dfrac{nx}{1 + n^2x^2}$.

For $x > 0$, $\frac{\frac{1}{nx}+nx}{2} \ge \sqrt{\frac{1}{nx}.nx}$, the equality occurs when $x = \frac{1}{n}$.
That is, for $x > 0$, $\frac{nx}{1+n^2x^2} \le \frac{1}{2}$ and $\frac{nx}{1+n^2x^2} = \frac{1}{2}$ at $x = \frac{1}{n}$.
For $x = 0$, $\frac{nx}{1+n^2x^2} = 0$.
Therefore for $0 < x \le 1$, $\frac{nx}{1+n^2x^2} \le \frac{1}{2}$ and $\frac{nx}{1+n^2x^2} = \frac{1}{2}$ at $x = \frac{1}{n}$.
Clearly, $\sup_{x\in[0,1]} \frac{nx}{1+n^2x^2} = \frac{1}{2}$. Therefore $M_n = \frac{1}{2}$ for all $n \in \mathbb{N}$.

Since $\lim M_n = \frac{1}{2} \ne 0$, the sequence $\{f_n\}$ is not uniformly convergent on $[0, 1]$, by Theorem 13.3.2.

4. For each natural number n, let $f_n(x) = 1 - \frac{x^n}{n}, x \in [0, 1]$. Show that the sequence $\{f_n\}$ is uniformly convergent on $[0, 1]$.

For $0 \le x \le 1$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} (1 - \frac{x^n}{n}) = 1$.

Hence the sequence $\{f_n\}$ converges pointwise on $[0, 1]$ to the function f where $f(x) = 1, x \in [0, 1]$.

Let $M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)|$.

Then $M_n = \sup_{x\in[0,1]} \frac{|x|^n}{n} = \frac{1}{n}$ and $\lim_{n\to\infty} M_n = 0$.

Hence the sequence $\{f_n\}$ converges uniformly on $[0, 1]$.

5. For each natural number n let $f_n(x) = \frac{x}{1+nx^2}, x \in [0, 1]$. Show that the sequence $\{f_n\}$ converges uniformly on $[0, 1]$.

For $x = 0$, the sequence is $\{0, 0, 0, \ldots\}$. This converges to 0.
For $0 < x \le 1$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{1+nx^2} = 0$.

Therefore the sequence $\{f_n\}$ converges pointwise on $[0, 1]$ to the function f where $f(x) = 0, x \in [0, 1]$.

Let $M_n = \sup\limits_{x\in[0,1]} \mid f_n(x) - f(x) \mid$. Then $M_n = \sup\limits_{x\in[0,1]} \frac{x}{1+nx^2}$.

For $x > 0$, $\frac{\frac{1}{x}+nx}{2} \geq \sqrt{\frac{1}{x}.nx}$, the equality occurs when $x = \frac{1}{\sqrt{n}}$.

That is, for $x > 0$, $\frac{x}{1+nx^2} \leq \frac{1}{2\sqrt{n}}$ and $\frac{x}{1+nx^2} = \frac{1}{2\sqrt{n}}$ at $x = \frac{1}{\sqrt{n}}$.

For $x = 0$, $\frac{x}{1+nx^2} = 0$.

Clearly, $\sup\limits_{x\in[0,1]} \frac{x}{1+nx^2} = \frac{1}{2\sqrt{n}}$. Therefore $M_n = \frac{1}{2\sqrt{n}}$ and $\lim M_n = 0$.

Hence the sequence $\{f_n\}$ is uniformly convergent on $[0, 1]$.

6. Let $f_n(x) = x^n, x \in [0, 1]$. Show that the sequence of functions $\{f_n\}$ is not uniformly convergent on $[0, 1]$.

For all $x \in [0, 1)$, $\lim\limits_{n\to\infty} f_n(x) = 0$.

For $x = 1$, the sequence is $\{1, 1, 1, \ldots\}$. This converges to 1.

Therefore the sequence $\{f_n\}$ converges to the function f where

$$\begin{aligned} f(x) &= 0, 0 \leq x < 1 \\ &= 1, x = 1. \end{aligned}$$

Let $M_n = \sup\limits_{x\in[0,1]} \mid f_n(x) - f(x) \mid$.

Then $M_n = 1$ for all $n \in \mathbb{N}$ and therefore $\lim\limits_{n\to\infty} M_n = 1$.

Hence the convergence of the sequence $\{f_n\}$ is not uniform on $[0, 1]$.

7. Let $f_n(x) = nx(1 - x)^n, x \in [0, 1]$. Show that the sequence $\{f_n\}$ is not uniformly convergent on $[0, 1]$.

At $x = 0$, the sequence is $\{0, 0, 0, \ldots\}$. This converges to 0.

At $x = 1$, the sequence is $\{0, 0, 0, \ldots\}$. This converges to θ.

When $0 < x < 1$, $0 < 1 - x < 1$.

Let $1 - x = \frac{1}{y}, y > 1$. Then $y = 1 + a, a > 0$.

$f_n(x) = \frac{nx}{(1+a)^n} < \frac{2nx}{n(n-1)a^2}$ since $(1 + a)^n > \frac{n(n-1)}{2}a^2$.

Therefore $0 < f_n(x) < \frac{2x}{(n-1)a^2}$ for all $x \in (0, 1)$.

By Sandwich theorem 5.5.5, $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \in (0, 1)$.

Thus the sequence $\{f_n\}$ converges to the function f on $[0, 1]$ where $f(x) = 0, x \in [0, 1]$.

Let $M_n = \sup\limits_{x\in[0,1]} \mid f_n(x) - f(x) \mid$.

$$\begin{aligned} \text{Then } M_n &= \sup_{x\in[0,1]} nx(1 - x)^n \\ &\geq n.\tfrac{1}{n}(1 - \tfrac{1}{n})^n, \text{ since } \tfrac{1}{n} \in [0, 1]. \end{aligned}$$

$\lim\limits_{n\to\infty} M_n \geq \lim\limits_{n\to\infty} (1 - \frac{1}{n})^n = e^{-1}$.

As $\lim_{n\to\infty} M_n \neq 0$, the convergence of the sequence is not uniform on $[0,1]$.

8. Let $f_n(x) = x^2e^{-nx}, x \in [0,\infty)$. Show that the sequence $\{f_n\}$ is uniformly convergent on $[0,1]$.

The sequence $\{f_n\}$ converges to the function f where $f(x) = 0, x \in [0,1]$. [worked Ex.10, 13.2.]

Let $M_n = \sup_{x\in[0,\infty)} | f_n(x) - f(x) |$. Then $M_n = \sup_{x\in[0,\infty)} x^2e^{-nx}$.

Let $u(x) = x^2e^{-nx}, x \geq 0$. Then $u'(x) = \frac{x(2-nx)}{e^{nx}}$.

$u'(x) = 0$ at $x = \frac{2}{n}$. $u'(x) > 0$ for $0 < x < \frac{2}{n}$. $u'(x) < 0$ for $x > \frac{2}{n}$.

Therefore u is an increasing function for $0 < x < \frac{2}{n}$, u is a maximum at $x = \frac{2}{n}$, u is a decreasing function for $x > \frac{2}{n}$ and $\lim_{x\to\infty} u(x) = 0$.

Therefore $M_n = \sup_{x\in[0,\infty)} u(x) = u(\frac{2}{n}) = \frac{4}{e^2n^2}$. $\lim_{n\to\infty} M_n = 0$.

Hence the sequence $\{f_n\}$ is uniformly convergent on $[0,\infty)$.

9. Let $f_n(x) = \frac{x}{n+x^2}, x \in [0,1]$. Show that the sequence $\{f_n\}$ is uniformly convergent on $[0,1]$.

The sequence $\{f_n\}$ converges to the function f where $f(x) = 0, x \in [0,1]$.

Let $M_n = \sup_{x\in[0,1]} | f_n(x) - f(x) |$. Then $M_n = \sup_{x\in[0,1]} \frac{x}{n+x^2}$.

Let $u_n(x) = \frac{x}{n+x^2}, x \in [0,1]$. Then $u_n'(x) = \frac{n-x^2}{(n+x^2)^2} > 0$ for all $x \in [0,1]$ and for all $n > 1$.

Therefore for all $n > 1, u_n$ is a strictly increasing function of x on $[0,1]$ and therefore $\sup_{x\in[0,1]} u_n(x) = \frac{1}{n+1}$.

That is, $M_n = \frac{1}{n+1}$ for all $n > 1$ and therefore $\lim M_n = 0$.

Hence the sequence $\{f_n\}$ is uniformly convergent on $[0,1]$.

13.4. Consequences of uniform convergence.

Theorem 13.4.1. Let D be a subset of $\mathbb{R}$ and a sequence of functions $\{f_n\}$ be uniformly convergent on D to a function f. Let $x_0 \in D'$ (the derived set of D) and $\lim_{x\to x_0} f_n(x) = a_n$. Then

(i) the sequence $\{a_n\}$ is convergent, and

(ii) $\lim_{x\to x_0} f(x)$ exists and equals $\lim_{n\to\infty} a_n$.

Proof. Let us choose $\epsilon > 0$. Since the sequence $\{f_n\}$ is uniformly convergent, there exists a natural number k such that

for all $x \in D$, $| f_m(x) - f_n(x) |< \frac{\epsilon}{2}$ for all $m, n \geq k$ ... ... (i)

As $\lim\limits_{x \to x_0} f_n(x) = a_n$ and $\lim\limits_{x \to x_0} f_m(x) = a_m$, it follows that $\lim\limits_{x \to x_0} \{f_m(x) - f_n(x)\} = a_m - a_n$ and therefore $\lim\limits_{x \to x_0} |f_m(x) - f_n(x)| = |a_m - a_n|$.

It follows from (i) that $| a_m - a_n |\leq \frac{\epsilon}{2} < \epsilon$ for all $m, n \geq k$.

This shows that $\{a_n\}$ is a Cauchy sequence in $\mathbb{R}$ and is therefore convergent.

Let $\lim a_n = l$. Let us choose $\epsilon > 0$.

Since the sequence $\{f_n\}$ converges uniformly on D, there exists a natural number k_1 such that

for all $x \in D$, $| f_n(x) - f(x) |< \frac{\epsilon}{3}$ for all $n \geq k_1$.

Since $\lim a_n = l$, there exists a natural number k_2 such that

$| a_n - l |< \frac{\epsilon}{3}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $| f_k(x) - f(x) |< \frac{\epsilon}{3}$ for all $x \in D$ and $| a_k - l |< \frac{\epsilon}{3}$.

Since $\lim\limits_{x \to x_0} f_k(x) = a_k$, there exists a positive δ such that

$| f_k(x) - a_k |< \frac{\epsilon}{3}$ for all $x \in N'(x_0, \delta) \cap D$.

By triangle inequality,

$$\begin{aligned} | f(x) - l | &\leq | f(x) - f_k(x) | + | f_k(x) - a_k | + | a_k - l | \\ &< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} (= \epsilon) \text{ for all } x \in N'(x_0, \delta) \cap D. \end{aligned}$$

This proves $\lim\limits_{x \to x_0} f(x) = l$. Therefore $\lim\limits_{x \to x_0} f(x) = \lim\limits_{n \to \infty} a_n$.

Note. In consequence of uniform convergence of the sequence $\{f_n\}$, $\lim\limits_{x \to x_0} \lim\limits_{n \to \infty} f_n(x) = \lim\limits_{n \to \infty} \lim\limits_{x \to x_0} f_n(x)$. This indicates that the interchange of limits is permissible.

Corollary. Let I be an interval and a sequence of functions $\{f_n\}$ be uniformly convergent on I to a function f. Let $c \in I$ and each f_n be continuous at c. Then f is continuous at c.

Proof. Since each f_n is continuous at c, $\lim\limits_{x \to c} f_n(x) = f_n(c)$, for all $n \in \mathbb{N}$. Since the sequence $\{f_n\}$ converges on I to the function f, the sequence $\{f_n(c)\}$ converges to $f(c)$.

By the theorem, $\lim\limits_{x \to c} f(x)$ exists and equals $\lim\limits_{n \to \infty} f_n(c)$, i.e., $\lim\limits_{x \to c} f(x) = f(c)$.

This proves that f is continuous at c.

Example 1. Let $f_n(x) = x^n, x \in [0, 1)$.

The sequence $\{f_n\}$ is pointwise convergent on $[0, 1)$ and the limit function f is given by $f(x) = 0, x \in [0, 1)$. 1 is a limit point of $[0, 1)$.

Let $a_n = \lim\limits_{x \to 1} f_n(x)$. Then $a_n = \lim\limits_{x \to 1} x^n = 1$.

Since $a_n = 1$ for all $n \in \mathbb{N}$, $\lim\limits_{n \to \infty} a_n = 1$. $\lim\limits_{x \to 1} f(x) = 0 \neq \lim a_n$.

This proves that $\{f_n\}$ is not uniformly convergent on $[0, 1)$.

Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}, f_n : D \to \mathbb{R}$ is bounded on D. If $\{f_n\}$ be pointwise convergent on D then the limit function f may not be bounded on D.

For example, let $f_n(x) = 1 + x + x^2 + \cdots + x^{n-1}, x \in [0, 1)$.

Then $\lim\limits_{n \to \infty} f_n(x) = \frac{1}{1-x}, x \in [0, 1)$.

The sequence $\{f_n\}$ converges on $[0, 1)$ to the function f is given by $f(x) = \frac{1}{1-x}, x \in [0, 1)$.

$$| f_n(x) | = | 1 + x + x^2 + \cdots + x^{n-1} | \leq 1 + | x | + | x^2 | + \cdots + |x^{n-1}|$$
$$< n \text{ for all } x \in [0, 1).$$

Each f_n is bounded on $[0, 1)$. But f is unbounded on $[0, 1)$.

Theorem 13.4.2. Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}, f_n : D \to \mathbb{R}$ is bounded on D. If the sequence $\{f_n\}$ be uniformly convergent on D, then the limit function f is bounded on D.

Proof. Let us choose $\epsilon > 0$. Since $\{f_n\}$ is uniformly convergent on D to f, there exists a natural number k such that

for all $x \in D, | f_n(x) - f(x) | < \epsilon$ for all $n \geq k$.

Let $\epsilon = 1$. There exists a natural number m such that for all $x \in D$, $| f_n(x) - f(x) | < 1$ for all $n \geq m$.

Therefore for all $x \in D$, $| f_m(x) - f(x) | < 1$.

Since $| \, | f(x) | - | f_m(x) | \, | \leq | f(x) - f_m(x) |$, it follows that

$| f(x) | \leq | f_m(x) | + | f(x) - f_m(x) | < | f_m(x) | + 1$.

Since f_m is bounded on D, there exists a positive number B such that $| f_m(x) | < B$ for all $x \in D$.

Therefore for all $x \in D$, $|f(x)| < B + 1$ and this proves that f is bounded on D.

Note. If each f_n be bounded on D, the uniform convergence of the sequence $\{f_n\}$ on D is a sufficient but not a necessary condition for boundedness of the limit function f on D.

For example, let $f_n(x) = \frac{nx}{1+n^2x^2}, x \in [0, 1]$. Then the limit function f is defined by $f(x) = 0, x \in [0, 1]$.

Since $\sup_{x \in [0,1]} | f_n(x) | = \frac{1}{2}$, each f_n is bounded on $[0, 1]$. Also the limit function f is bounded in $[0, 1]$. But the convergence of the sequence $\{f_n\}$ is not uniform on $[0, 1]$.

Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}, f_n : D \to \mathbb{R}$ is continuous on D. If the sequence $\{f_n\}$ be pointwise convergent on D then the limit function f may not be continuous on D.

For example, let $f_n(x) = x^{n-1}, x \in [0, 1]$.

Then each f_n is continuous on $[0, 1]$.

The sequence $\{f_n\}$ converges on $[0, 1]$ to the function f where

$$\begin{aligned} f(x) &= 0, 0 \leq x < 1 \\ &= 1, x = 1. \end{aligned}$$

The limit function f is not continuous on $[0, 1]$.

Theorem 13.4.3. Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}, f_n : D \to \mathbb{R}$ is continuous on D. If the sequence $\{f_n\}$ be uniformly convergent on D to a function f, then f is continuous on D.

Proof. Let $c \in D$. Let us choose $\epsilon > 0$.

Since $\{f_n\}$ is uniformly convergent on D to the function f, there exists a natural number k such that

for all $x \in D, | f_n(x) - f(x) | < \frac{\epsilon}{3}$ for all $n \geq k$.

Therefore $| f_k(x) - f(x) | < \frac{\epsilon}{3}$ for all $x \in D$ and $| f_k(c) - f(c) | < \frac{\epsilon}{3}$.

Since f_k is continuous on at c, there exists a positive δ such that $| f_k(x) - f_k(c) | < \frac{\epsilon}{3}$ for all $x \in N(c, \delta) \cap D$.

By triangle inequality,

$$\begin{aligned} | f(x) - f(c) | &\leq | f(x) - f_k(x) | + | f_k(x) - f_k(c) | + \\ &\quad | f_k(c) - f(c) | \\ &< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} \text{ for all } x \in N(c, \delta) \cap D. \end{aligned}$$

That is, $| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$.

This proves that f is continuous at c. Since c is arbitrary, f is continuous on D.

Note 1. If each f_n be continuous on D, the uniform convergence of the sequence $\{f_n\}$ on D is a sufficient but not a necessary condition for continuity of the limit function f on D.

For example, let $f_n(x) = \frac{nx}{1+n^2x^2}, x \in [0, 1]$.

Each f_n is continuous on $[0, 1]$. The sequence $\{f_n\}$ converges on $[0, 1]$ to the function f where $f(x) = 0, x \in [0, 1]$.

The limit function f is continuous on $[0, 1]$.

But the convergence of the sequence $\{f_n\}$ is not uniform on $[0, 1]$, as established in worked Example 3, page 543.

Note 2. If each f_n be continuous on D and the sequence $\{f_n\}$ converges pointwise on D to a function f not continuous on D, then it follows from the theorem that the convergence is not uniform on D.

Worked Examples .

1. Let $f_n(x) = \tan^{-1} nx, x \in [0,1]$. Prove that the sequence $\{f_n\}$ is not uniformly convergent on $[0, 1]$.

$$\begin{aligned} \lim_{n\to\infty} f_n(x) &= \tfrac{\pi}{2}, \text{ if } x \in (0,1] \\ &= 0, \text{ if } x = 0. \end{aligned}$$

The sequence $\{f_n\}$ is convergent on $[0,1]$ to the function f where

$$\begin{aligned} f(x) &= 0, x = 0 \\ &= \tfrac{\pi}{2}, 0 < x \leq 1. \end{aligned}$$

Each f_n is continuous on $[0,1]$ but the limit function f is not continuous on $[0,1]$.

This proves that the convergence of the sequence is not uniform on $[0, 1]$.

2. Prove that the sequence $\{f_n\}$ where $f_n(x) = \frac{x^n}{1+x^n}, x \in [0,2]$ is not uniformly convergent on $[0,2]$.

When $0 \leq x < 1$, $\lim_{n\to\infty} f_n(x) = 0$.

When $x = 1$, $\lim_{n\to\infty} f_n(x) = \frac{1}{2}$.

When $1 < x \leq 2$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \dfrac{1}{(\frac{1}{x})^n + 1} = 1$.

The sequence $\{f_n\}$ converges pointwise to the function f where

$$\begin{aligned} f(x) &= 0, 0 \leq x < 1 \\ &= \tfrac{1}{2}, x = 1 \\ &= 1, 1 < x \leq 2. \end{aligned}$$

Each f_n is continuous on $[0,2]$. The function f is not continuous on $[0,2]$. Therefore the convergence of the sequence is not uniform on $[0,2]$.

3. For each $n \in \mathbb{N}$, let
$$\begin{aligned} f_n(x) &= 1 - nx, 0 \leq x \leq \tfrac{1}{n} \\ &= 0, \quad \tfrac{1}{n} < x \leq 1. \end{aligned}$$

Show that the sequence $\{f_n\}$ converges on $[0,1]$ to a function f but the convergence of the sequence is not uniform on $[0,1]$.

At $x = 0$, the sequence is $\{1, 1, 1, \ldots\}$. This converges to 1.

At $x = 1$, the sequence is $\{0, 0, 0, \ldots\}$. This converges to 0.

Let $c \in (0,1)$. By Archimedean property of $\mathbb{R}$ there exists a natural number m such that $0 < \frac{1}{m} < c$ and therefore $0 < \frac{1}{n} < c$ for all $n \geq m$. $f_m(c) = 0$ and $f_n(c) = 0$ for all $n \geq m$. This proves $\lim\limits_{n\to\infty} f_n(c) = 0$.

Therefore the sequence $\{f_n\}$ converges to the function f on $[0,1]$ given by

$$\begin{aligned} f(x) &= 1, x = 0 \\ &= 0, 0 < x \leq 1. \end{aligned}$$

Each f_n is continuous on $[0,1]$. The limit function f is not continuous on $[0,1]$.

Therefore the convergence of the sequence $\{f_n\}$ is not uniform on $[0,1]$, since uniform convergence of the sequence $\{f_n\}$ of continuous functions on $[0,1]$ implies continuity of f on $[0,1]$.

4. Prove that the sequence $\{f_n\}$ defined by $f_n(x) = \frac{nx}{1+nx}, x \geq 0$ is not uniformly convergent on $[0,\infty)$, but the convergence is uniform on $[a,\infty)$ if $a > 0$.

Each f_n is continuous on $[0,\infty)$ but the sequence $\{f_n\}$ converges to the function f which is not continuous on $[0,\infty)$. [Ex.5, 13.2]

Therefore the convergence of the sequence $\{f_n\}$ is not uniform on $[0,\infty)$.

For all $x > 0$, $| f_n(x) - f(x) | = \frac{1}{1+nx} < \frac{1}{nx}$.

Then for all $x \geq a$, $| f_n(x) - f(x) | < \frac{1}{na}$.

Let $\epsilon > 0$. Then for all $x \geq a$, $| f_n(x) - f(x) | < \epsilon$ holds if $n > \frac{1}{a\epsilon}$.

Let $k = [1/(a\epsilon)] + 1$. Then k is a natural number and for all $x \geq a$, $| f_n(x) - f(x) | < \epsilon$ for all $n \geq k$.

This proves that $\{f_n\}$ is uniformly convergent on $[a,\infty)$.

The following theorem due to U.Dini gives a partial converse of the theorem 13.4.3.

Theorem 13.4.4. (Dini)

Let D be a compact subset of $\mathbb{R}$ and $f_n : D \to \mathbb{R}$ be a sequence of continuous functions on D that converges pointwise to a continuous function f. If the sequence $\{f_n\}$ be a monotone sequence on D, i.e., either $f_{n+1}(x) \geq f_n(x)$ for each $n \in \mathbb{N}$ and each $x \in D$, or $f_{n+1}(x) \leq f_n(x)$ for each $n \in \mathbb{N}$ and each $x \in D$, then the convergence of the sequence $\{f_n\}$ is uniform on D.

Proof. If $\{f_n\}$ be monotone increasing, let $g_n = f - f_n$. If $\{f_n\}$ be monotone decreasing, let $g_n = f_n - f$.

Then $g_{n+1} - g_n \leq 0$ for all $n \in \mathbb{N}$ and for all $x \in D$.

So $\{g_n\}$ is a monotone decreasing sequence of continuous functions with $\lim g_n(x) = 0$ for every $x \in D$. Also $g_n(x) \geq 0$ for all $n \in \mathbb{N}$ and all $x \in D$.

Let $M_n = \sup_{x \in D} g_n(x)$. Then $M_{n+1} \leq M_n$ for all $n \in \mathbb{N}$.

Since g_n is continuous on D, g_n attains the supremum M_n at a point, say $x_n, \in D$, i.e., $g_n(x_n) = M_n$ for all $n \in \mathbb{N}$.

The sequence $\{x_n\}$ is a sequence in a compact set D. Therefore there exists a subsequence $\{x_{r_n}\}$ of $\{x_n\}$ such that $\{x_{r_n}\}$ converges to a point x^* in D.

Since $\lim g_n(x^*) = 0$, for a pre-assigned positive ϵ, there exists a natural number m such that $g_n(x^*) < \frac{\epsilon}{2}$ for all $n \geq m$.

Since g_m is continuous at x^*, there exists a neighbourhood U of x^* such that $| g_m(x) - g_m(x^*) | < \frac{\epsilon}{2}$ for all $x \in U \cap D$.

It follows that $g_m(x) < \epsilon$ for all $x \in U \cap D$.

Since $\lim x_{r_n} = x^*$, there exists a natural number k_1 such that $x_{r_n} \in U \cap D$ (neighbourhood of x^*) for all $n \geq k_1$.

Also there exists a natural number k_2 such that $r_n > m$ for all $n \geq k_2$.

Let $k = max\{k_1, k_2\}$. Then $g_{r_n}(x_{r_n}) < g_m(x_{r_n}) < \epsilon$ for all $n \geq k$.

This proves $M_{r_n} < \epsilon$ for all $n \geq k$, i.e., $\lim M_{r_n} = 0$.

Since $\{M_n\}$ is a monotone decreasing sequence having a convergent subsequence $\{M_{r_n}\}$ with limit 0, the sequence $\{M_n\}$ converges to 0.

Therefore the sequence $\{g_n\}$ converges uniformly to 0 on D.

Consequently, the sequence $\{f_n\}$ converges uniformly to f on D.

This completes the proof.

Another proof.

Without loss of generality, let us assume that the sequence $\{f_n\}$ is monotone increasing.

Let $a \in D$. Since the sequence $\{f_n(a)\}$ is a monotone increasing sequence converging to $f(a)$, for a pre-assigned positive ϵ there exists a natural number k such that

$$f(a) - \epsilon < f_k(a) \leq f(a), \text{ i.e., } 0 \leq f(a) - f_k(a) < \epsilon.$$

Since f and f_k are continuous at a, there exists a positive δ such that $|f(x) - f(a)| < \epsilon$ and $|f_k(x) - f_k(a)| < \epsilon$ for all $x \in N(a, \delta) \cap D$.

Therefore $|f(x)-f_k(x)| \leq |f(x)-f(a)|+|f(a)-f_k(a)|+|f_k(a)-f_k(x)|$
$< 3\epsilon$ for all $x \in N(a, \delta) \cap D$,

i.e., $0 \leq f(x) - f_k(x) < 3\epsilon$ for all $x \in N(a, \delta) \cap D$.

Thus corresponding to a chosen positive ϵ, for each point $a \in D$ there exist a positive number δ_a and a natural number k_a such that

$$0 \leq f(x) - f_{k_a}(x) < 3\epsilon \text{ for all } x \in N(a, \delta_a) \cap D.$$

The set of neighbourhoods $\{N(a, \delta_a) : a \in D\}$ form an open cover of D and since D is compact, there exist a finite number of points $a_1, a_2, \ldots, a_m$ in D such that the union of the corresponding neighbourhoods of $a_1, a_2, \ldots, a_m$ determined by ϵ form an open cover of D.

Let the neighbourhood $N(a_i, \delta_i)$ and the natural number k_i correspond to the point a_i, for $i = 1, 2, \ldots, m$.

Then $0 \leq f(x) - f_{k_i}(x) < 3\epsilon$ for all $x \in N(a_i, \delta_i) \cap D$.

Let $k_0 = \max\{k_1, k_2, \ldots, k_m\}$. Then

$$0 \leq f(x) - f_{k_0}(x) < 3\epsilon \text{ for all } x \in D.$$

Since $\{f_n\}$ is a monotone increasing sequence converging to f, for all $x \in D$, $|f(x) - f_n(x)| < 3\epsilon$ for all $n \geq k_0$.

This proves that convergence of the sequence $\{f_n\}$ is uniform on D. This completes the proof.

Note. The compactness of D in the theorem is essential.

Let us consider the following examples.

Let $I = [0, 1)$ and let $f_n : I \to \mathbb{R}$ be defined by $f_n(x) = x^n, x \in I$. The sequence $\{f_n\}$ is a monotone decreasing sequence of continuous functions on I. It converges to the 0-function on I, which is a continuous function. But the convergence is not uniform on I. [worked Example 1, 13.4]

This does not violate Dini's theorem, since I is not a compact subset of $\mathbb{R}$.

Let $I = [0, \infty)$ and let $f_n : I \to \mathbb{R}$ be defined by $f_n(x) = \frac{x}{n}, x \in I$. The sequence is a monotone decreasing sequence of continuous functions on I and it converges to the 0-function on I, which is a continuous function. But the convergence is not uniform on I. [worked Example 2,13.3.]

This does not violate Dini's theorem, since I is not a compact subset of $\mathbb{R}$.

Worked Examples (continued).

5. For each $n \in \mathbb{N}$, let $f_n(x) = x^{n-1} - x^n, x \in [0, 1]$. Use Dini's theorem to prove that the sequence $\{f_n\}$ is uniformly convergent on $[0, 1]$.

$\lim\limits_{n \to \infty} f_n(x) = \lim\limits_{n \to \infty} x^{n-1}(1 - x) = 0$ for $x \in [0, 1]$.

The sequence $\{f_n\}$ is convergent on $[0, 1]$ to the function f where $f(x) = 0, x \in [0, 1]$.

Each f_n is continuous on $[0, 1]$; f is also continuous on $[0, 1]$.

$$\begin{aligned}\text{For each } x \in [0,1], f_{n+1}(x) - f_n(x) &= (x^n - x^{n+1}) - (x^{n-1} - x^n)\\ &= -x^{n-1}(x-1)^2 \leq 0.\end{aligned}$$

Therefore the sequence $\{f_n\}$ is a monotone decreasing sequence on $[0, 1]$, a compact subset of $\mathbb{R}$.

By Dini's theorem, the convergence of the sequence is uniform on $[0, 1]$.

6. A sequence of functions $\{f_n\}$ is defined by $f_1(x) = \sqrt{x}, f_{n+1}(x) = \sqrt{xf_n(x)}$ for all $n \geq 1$.

Use Dini's theorem to prove that the sequence $\{f_n\}$ is uniformly convergent on $[0, 1]$.

$f_1(x) = x^{\frac{1}{2}}, f_2(x) = x^{\frac{1}{2}+\frac{1}{2^2}}, \ldots, f_n(x) = x^{\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}}, \ldots$

At $x = 0$, the sequence converges to 0.

When $x > 0$, $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} x^{1-\frac{1}{2^n}} = x$.

Therefore the sequence $\{f_n\}$ converges to f on $[0, 1]$ where $f(x) = x, x \in [0, 1]$.

Each f_n is continuous on $[0, 1]$ and the limit function f is also continuous on $[0, 1]$.

$f_{n+1}(x) - f_n(x) = x^{\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}}[x^{\frac{1}{2^{n+1}}} - 1] \leq 0$ for each $x \in [0, 1]$.

Therefore the sequence $\{f_n\}$ is a monotone decreasing sequence on $[0, 1]$, a compact subset of $\mathbb{R}$.

Thus the sequence $\{f_n\}$ is a sequence of continuous functions on the compact set $[0, 1]$ and converges to a function f continuous on $[0, 1]$. Also $\{f_n\}$ is a monotone decreasing sequence on $[0, 1]$.

By Dini's theorem the convergence of the sequence is uniform on $[0, 1]$.

Let $I = [a, b]$ be closed and bounded interval and for each natural number n, let $f_n : I \to \mathbb{R}$ be $\mathcal{R}$-integrable on I. If the sequence $\{f_n\}$ be pointwise convergent on I to a function f then f may not be $\mathcal{R}$-integrable on I.

For Example, let $I = [0, 1]$. Let $r_1, r_2, r_3, \ldots$ be an enumeration of the set of all rational points in I. For each $n \in \mathbb{N}$, let $f_n : I \to \mathbb{R}$ be defined by

$$\begin{aligned} f_n(x) &= 0,\ x = r_1, r_2, \ldots, r_n\\ &= 1,\ x \in [0,1] - \{r_1, r_2, \ldots, r_n\}.\end{aligned}$$

Then each f_n is $\mathcal{R}$-integrable on $[0, 1]$, since f_n is continuous on $[0, 1]$ except only at n points.

$$\begin{aligned}\lim_{n\to\infty} f_n(x) &= 0,\ \text{if } x \in [0,1] \cap \mathbb{Q}\\ &= 1,\ \text{if } x \in [0,1] - \mathbb{Q}.\end{aligned}$$

Therefore the sequence $\{f_n\}$ converges to f on $[0,1]$ where

$$f(x) = 0, \text{ if } x \in [0,1] \cap \mathbb{Q}$$
$$= 1, \text{ if } x \in [0,1] - \mathbb{Q}.$$

f is discontinuous at every point in $[0,1]$. So f is not $\mathcal{R}$- integrable on $[0,1]$.

Let $I = [a,b]$ and for each $n \in \mathbb{N}$, let $f_n : I \to \mathbb{R}$ be integrable on I and the sequence $\{f_n\}$ converges pointwise to a function f which is also integrable on I.

We now ask if it is true that the sequence $\{\int_a^b f_n\}$ converges to $\int_a^b f$.

That is , if it is true that $\lim_{n\to\infty} (\int_a^b f_n) = \int_a^b (\lim_{n\to\infty} f_n)$.

The answer is 'no'.

For example, let $f_n(x) = nxe^{-nx^2}, x \in [0,1]$.

When $x = 0$, the sequence is $\{0,0,0,\ldots\}$. This converges to 0.

When $0 < x \leq 1, e^{nx^2} > \frac{n^2x^4}{2}$.

For all $x \in (0,1]$, we have $0 < nxe^{-nx^2} < \frac{2}{nx^3}$.

By Sandwich theorem, $\lim_{n\to\infty} nxe^{-nx^2} = 0$, for $x \in (0,1]$.

Therefore the sequence $\{f_n\}$ converges on $[0,1]$ to the function f where $f(x) = 0, x \in [0,1]$.

Each f_n is integrable on $[0,1]$ and f is also integrable on $[0,1]$.

$\int_0^1 f_n(x)dx = [-\frac{1}{2}e^{-nx^2}]_0^1 = \frac{1}{2}(1 - e^{-n})$.

$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \lim_{n\to\infty} \frac{1}{2}(1 - e^{-n}) = \frac{1}{2}$.

Hence the sequence $\{\int_0^1 f_n\}$ converges to $\frac{1}{2}$ but $\int_0^1 f(x)dx = 0$.

Therefore $\lim_{n\to\infty} (\int_0^1 f_n) \neq \int_0^1 (\lim_{n\to\infty} f_n)$.

Theorem 13.4.5. Let $I = [a,b]$ be a closed and bounded interval and for each $n \in \mathbb{N}, f_n : I \to \mathbb{R}$ be $\mathcal{R}$-integrable on I. If the sequence $\{f_n\}$ converges uniformly to a function f on I then f is $\mathcal{R}$-integrable on I and moreover, the sequence $\{\int_a^b f_n\}$ converges to $\int_a^b f$.

Proof. Let us choose $\epsilon > 0$. Since $\{f_n\}$ is uniformly convergent on $[a,b]$ to the function f there exists a natural number k such that

for all $x \in [a,b], \mid f_n(x) - f(x) \mid < \frac{\epsilon}{4(b-a)}$ for all $n \geq k$.

Therefore for all $x \in [a,b], \mid f_k(x) - f(x) \mid < \frac{\epsilon}{4(b-a)}$.

or, $f_k(x) - \frac{\epsilon}{4(b-a)} < f(x) < f_k(x) + \frac{\epsilon}{4(b-a)}$ for all $x \in [a,b]$ ... (i)

Since f_k is integrable on $[a,b]$, there exists a partition $P = (x_0, x_1, x_2, \ldots, x_n)$ of $[a,b]$ such that $U(P, f_k) - L(P, f_k) < \frac{\epsilon}{2}$ ... (ii)

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x), m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$

$$M'_r = \sup_{x \in [x_{r-1}, x_r]} f_k(x), m'_r = \inf_{x \in [x_{r-1}, x_r]} f_k(x),\ r = 1, 2, \ldots, n.$$

From (i) it follows that $m_r \geq m'_r - \frac{\epsilon}{4(b-a)}$; $M_r \leq M'_r + \frac{\epsilon}{4(b-a)}$.

$$\begin{aligned} U(P, f) &= M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1}) \\ &\leq M'_1(x_1 - x_0) + M'_2(x_2 - x_1) + \cdots + M'_n(x_n - x_{n-1}) + \tfrac{\epsilon}{4}. \end{aligned}$$

$$\begin{aligned} L(P, f) &= m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1}) \\ &\geq m'_1(x_1 - x_0) + m'_2(x_2 - x_1) + \cdots + m'_n(x_n - x_{n-1}) - \tfrac{\epsilon}{4}. \end{aligned}$$

Therefore $U(P, f) - L(P, f) \leq U(P, f_k) - L(P, f_k) + \frac{\epsilon}{2}$

$< \epsilon$, by using (ii).

This proves that f is $\mathcal{R}$-integrable on $[a,b]$.

Second part.

Let us choose $\epsilon > 0$.

Since the sequence $\{f_n\}$ converges uniformly to f on $[a,b]$, there exists a natural number k such that for all $x \in [a,b]$,
$|f_n(x) - f(x)| < \frac{\epsilon}{2(b-a)}$ for all $n \geq k$.

We have $| \int_a^b [f_n(x) - f(x)]dx | \leq \int_a^b | f_n(x) - f(x) | \, dx \leq \frac{\epsilon}{2(b-a)} \cdot (b-a)$, i.e., $< \epsilon$ for all $n \geq k$

or $| \int_a^b f_n(x)dx - \int_a^b f(x)dx | < \epsilon$ for all $n \geq k$.

This implies $\lim\limits_{n \to \infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.

In other words, the sequence $\{\int_a^b f_n\}$ converges to $\int_a^b f$.

This completes the proof.

Remarks. In symbols, $\lim\limits_{n \to \infty} (\int_a^b f_n) = \int_a^b (\lim\limits_{n \to \infty} f_n)$.

This says that if the convergence of the sequence $\{f_n\}$ be uniform on $[a,b]$, it is permissible to interchange $\lim\limits_{n \to \infty}$ and $\int_a^b$.

Corollary. For each $x \in [a,b]$, the sequence $\{\int_a^x f_n\}$ converges to $\int_a^x f$.

Note 1. If each f_n be integrable on $[a,b]$ and the sequence $\{f_n\}$ converges pointwise to a function f which is also integrable on $[a,b]$, the uniform convergence of the sequence $\{f_n\}$ is a sufficient but not a necessary condition for the convergence of the sequence $\{\int_a^b f_n\}$ to $\int_a^b f$.

For example, let $f_n(x) = \frac{nx}{1+n^2x^2}, x \in [0,1]$.

This sequence $\{f_n\}$ converges on $[0,1]$ to the function f where $f(x) = 0, x \in [0,1]$.

Each f_n is integrable on $[0,1]$ and also f is integrable on $[0,1]$.

$\int_0^1 f_n(x)dx = [\frac{1}{2n}\log(1+n^2x^2)]_0^1 = \frac{1}{2n}\log(1+n^2)$.

$\lim\limits_{x\to\infty} \frac{\log(1+x^2)}{2x} = 0$. By sequential criterion for limits, $\lim\limits_{n\to\infty} \frac{\log(1+n^2)}{2n} = 0$.

$\lim\limits_{n\to\infty} \int_0^1 f_n(x)dx = 0$ and $\int_0^1 f(x)dx = 0$.

Thus the sequence $\{\int_0^1 f_n\}$ converges to $\int_0^1 f$ but the convergence of the sequence $\{f_n\}$ is not uniform on $[0,1]$.

Note 2. If each f_n be integrable on $[a,b]$ and the sequence $\{f_n\}$ converges pointwise to a function f which is not integrable on $[a,b]$, then it follows from the theorem that the convergence of the sequence $\{f_n\}$ is not uniform on $[a,b]$.

If each f_n be integrable on $[a,b]$ and the sequence $\{f_n\}$ converges pointwise to a function f which is also integrable on $[a,b]$ but the sequence $\{\int_a^b f_n\}$ does not converge to $\int_a^b f$, then it follows from the theorem that the convergence of the sequence $\{f_n\}$ is not uniform on $[a,b]$.

Worked Examples (continued).

7. For each $n \in \mathbb{N}$, let $f_n(x) = nxe^{-nx^2}, x \in [0,1]$. Show that the sequence $\{f_n\}$ is not uniformly convergent on $[0,1]$.

For all $x \in (0,1], nxe^{-nx^2} > 0$. Let $u_n = nxe^{-nx^2}, x \in (0,1]$.

Then $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty}(1+\frac{1}{n})e^{-x^2} = e^{-x^2} > 0$.

By theorem 5.8.1, $\lim\limits_{n\to\infty} u_n = 0$, i.e., $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \in (0,1]$.

Also for $x = 0$, the sequence converges to 0.

Thus the sequence $\{f_n\}$ converges pointwise on $[0,1]$ to the function f where $f(x) = 0, x \in [0,1]$.

Each f_n is integrable on $[0,1]$. $\int_0^1 f_n(x)dx = [-\frac{1}{2}e^{-nx^2}]_0^1 = \frac{1}{2}(1-e^{-n})$.

f is integrable on $[0,1]$ and $\int_0^1 f(x)dx = 0$.

$\lim\limits_{n\to\infty} \int_0^n f_n(x)dx = \lim\limits_{n\to\infty} \frac{1}{2}(1-e^{-n}) = \frac{1}{2}$.

Hence the sequence $\{\int_0^1 f_n\}$ converges to $\frac{1}{2}$ which is not equal to $\int_0^1 f$.

This proves that the convergence of the sequence $\{f_n\}$ is not uniform on $[0,1]$.

8. Let $f_n(x) = nx(1 - x^2)^n, x \in [0, 1]$. Show that the sequence of functions $\{f_n\}$ converges to a function f integrable on $[0, 1]$ but the convergence is not uniform on $[0, 1]$.

When $x = 0$, the sequence is $\{0, 0, 0, \ldots\}$. This converges to 0.
When $x = 1$, the sequence is $\{0, 0, 0, \ldots\}$. This converges to 0.
$0 < x <$ implies $0 < 1 - x^2 < 1$.

Let $1 - x^2 = \frac{1}{y}, y > 1$. Then $y = 1 + a, a > 0$.

Then $f_n(x) = \frac{nx}{(1+a)^n} < \frac{2nx}{n(n-1)a^2}$, since $(1 + a)^n > \frac{n(n-1)}{2}a^2$.
We have, $0 < f_n(x) < \frac{2x}{(n-1)a^2}$, when $0 < x < 1$.

By Sandwich theorem 5.5.5, $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \in (0, 1)$.

Therefore the sequence $\{f_n\}$ is convergent on $[0, 1]$ to the function f where $f(x) = 0, x \in [0, 1]$.

Each f_n is integrable on $[0, 1]$ and $\int_0^1 f_n(x)dx = \frac{n}{2(n+1)}$.
f is integrable on $[0, 1]$ and $\int_0^1 f(x)dx = 0$. $\lim\limits_{n\to\infty} \int_0^1 f_n = \frac{1}{2} \neq \int_0^1 f$.

This proves that the sequence $\{f_n\}$ is not uniformly convergent on $[0, 1]$.

Let $\{f_n\}$ be a sequence of functions on $[a, b]$ such that for each $n \in \mathbb{N}$, $f_n'(x)$ exists for all $x \in [a, b]$. Let $\{f_n\}$ be pointwise convergent on $[a, b]$ to a function f. We ask if $f'(x)$ exists for all $x \in [a, b]$. If it exists, we ask again if the sequence $\{f_n'\}$ converges to f' on $[a, b]$.

The answer to both the questions is 'no'.

For example, let $f_n(x) = x^{n-1}, x \in [0, 1]$.
Then the limit function f is given by
$$\begin{aligned} f(x) &= 0, x \in [0, 1) \\ &= 1, x = 1. \end{aligned}$$
For each $n \in \mathbb{N}, f_n'(x)$ exists for all $x \in [0, 1]$.
But $f'(x) = 0, x \in [0, 1]$ and $f'(1)$ does not exist.

To discuss the second question, let us consider the sequence $\{f_n\}$ where $f_n(x) = x - \frac{x^n}{n}, x \in [0, 1]$. $\lim\limits_{n\to\infty} f_n(x) = x, x \in [0, 1]$.

Therefore the sequence converges to the function f on $[0, 1]$ where $f(x) = x, x \in [0, 1]$.

$f_n'(x) = 1 - x^{n-1}$ and $f'(x) = 1, x \in [0, 1]$.

For each $n \in \mathbb{N}, f_n'(x)$ exists for all $x \in [0, 1]$. Also $f'(x)$ exists for all $x \in [0, 1]$.
$$\begin{aligned} \lim_{n\to\infty} f_n'(x) &= 1, x \in [0, 1) \\ &= 0, x = 1. \end{aligned}$$

This shows that the sequence $\{f_n'\}$ does not converge to f' on $[0, 1]$.

Let $\{f_n\}$ be a sequence of functions on $[a, b]$ such that for each $n \in \mathbb{N}$, $f_n'(x)$ exists for all $x \in [a, b]$. Let the sequence $\{f_n\}$ converge uniformly to a function f on $[a, b]$.

We ask if the sequence $\{f_n'\}$ converges to f' on $[a, b]$.

The answer is 'no'.

For example, let $f_n(x) = \frac{\sin nx}{n}, x \in [0, 1]$.

The sequence $\{f_n\}$ converges uniformly to the function f where $f(x) = 0, x \in [0, 1]$. [Example 5, 13.3.]

$f'(x) = 0, x \in [0, 1]$. $f_n'(x) = \cos nx$, $x \in [0, 1]$.

The sequence $\{f_n'\}$ converges at $x = 0$ but does not converge for $x \in (0, 1]$.

Theorem 13.4.6. Let $\{f_n\}$ be a sequence of functions on $[a, b]$ such that for each $n \in \mathbb{N}$, $f_n'(x)$ exists for all $x \in [a, b]$. If the sequence of derivatives $\{f_n'\}$ converges uniformly on $[a, b]$ to a function g and the sqeuence $\{f_n\}$ converges *at least at one point* $x_0 \in [a, b]$, then the sequence $\{f_n\}$ is uniformly convergent on $[a, b]$ and if the limit function be f then $f'(x) = g(x)$ for all $x \in [a, b]$.

Proof. Let us choose $\epsilon > 0$.

Since the sequence $\{f_n'\}$ is uniformly convergent on $[a, b]$, there exists a natural number k_1 such that

for all $x \in [a, b], | f_{n+p}'(x) - f_n'(x) | < \frac{\epsilon}{2(b-a)}$ for all $n \geq k_1$ and $p = 1, 2, 3, \ldots$

Also since $\{f_n(x_0)\}$ is convergent, there exists a natural number k_2 such that $| f_{n+p}(x_0) - f_n(x_0) | < \frac{\epsilon}{2}$ for all $n \geq k_2$ and $p = 1, 2, 3, \ldots$

Let $k = \max\{k_1, k_2\}$.

Then for all $x \in [a, b], | f_{n+p}'(x) - f_n'(x) | < \frac{\epsilon}{2(b-a)}$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$ and $| f_{n+p}(x_0) - f_n(x_0) | < \frac{\epsilon}{2}$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

Applying Lagrange's Mean value theorem to the function $f_{n+p} - f_n$ on $[x_0, x]$ or $[x, x_0]$ where $x \in [a, b]$,

$| f_{n+p}(x) - f_n(x) - f_{n+p}(x_0) + f_n(x_0) | = | x - x_0 || f_{n+p}'(\xi) - f_n'(\xi) |$ where $x_0 < \xi < x$ or $x < \xi < x_0$, as the case may be.

Now $| f_{n+p}'(\xi) - f_n'(\xi) | < \frac{\epsilon}{2(b-a)}$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$ and $| x - x_0 | < b - a$ for $x \in [a, b]$.

It follows that for all $x \in [a, b]$,

$| f_{n+p}(x) - f_n(x) - f_{n+p}(x_0) + f_n(x_0) | < \frac{\epsilon}{2}$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

Using triangle inequality,
for all $x \in [a,b]$, $| f_{n+p}(x) - f_n(x) | \leq | f_{n+p}(x) - f_n(x) - f_{n+p}(x_0) + f_n(x_0) | + | f_{n+p}(x_0) - f_n(x_0) | < \epsilon$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

This proves that the sequence $\{f_n\}$ is uniformly convergent on $[a, b]$.

Second part.

Let f be the uniform limit of the sequence $\{f_n\}$ on $[a, b]$.

We now prove that $f'(x) = g(x), x \in [a, b]$.

Let $c \in [a, b]$.

Let us define a sequence of functions $\{\phi_n\}$ on $D = [a, b] - \{c\}$ by

$$\phi_n(x) = \frac{f_n(x) - f_n(c)}{x - c},\ x \in [a, b] - \{c\}.$$

Then for $x \in [a, b] - \{c\}$, $\lim\limits_{n \to \infty} \phi_n(x) = \lim\limits_{n \to \infty} \frac{f_n(x) - f_n(c)}{x - c} = \frac{f(x) - f(c)}{x - c}$.

Applying Mean value theorem to the function $f_{n+p} - f_n$ on $[c, x]$ or $[x, c]$, we have $| f_{n+p}(x) - f_n(x) - f_{n+p}(c) + f_n(c) | = | x - c || f'_{n+p}(\eta) - f'_n(\eta) |$ where $c < \eta < x$ or $x < \eta < c$.

Let us choose $\epsilon > 0$.

Since the sequence $\{f'_n\}$ converges uniformly to g on $[a, b]$, there exists a natural number k_1 such that

for all $x \in [a, b]$, $| f'_{n+p}(x) - f'_n(x) | < \epsilon$ for all $n \geq k$ and $p = 1, 2, \ldots$

Therefore $| f'_{n+p}(\eta) - f'_n(\eta) | < \epsilon$ for all $n \geq k$ and $p = 1, 2, \ldots$

For all $x \in [a, b] - \{c\}$, $|\phi_{n+p}(x) - \phi_n(x)|$

$$= \left| \frac{f_{n+p}(x) - f_n(x) - f_{n+p}(c) + f_n(c)}{x - c} \right|$$

$< \epsilon$ for all $n \geq k$ and $p = 1, 2, \ldots$

This proves that the sequence $\{\phi_n\}$ is uniformly convergent on D.

Now c is a limit point of D.

Since the the sequence $\{\phi_n\}$ is uniformly convergent on D, it follows that $\lim\limits_{n \to \infty} \lim\limits_{x \to c} \phi_n(x) = \lim\limits_{x \to c} \lim\limits_{n \to \infty} \phi_n(x)$, by Theorem 12.4.1.

$\lim\limits_{n \to \infty} \lim\limits_{x \to c} \phi_n(x) = \lim\limits_{n \to \infty} \lim\limits_{x \to c} \frac{f_n(x) - f_n(c)}{x - c} = \lim\limits_{n \to \infty} f'_n(c) = g(c)$ and

$\lim\limits_{x \to c} \lim\limits_{n \to \infty} \phi_n(x) = \lim\limits_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$.

Therefore $g(c) = f'(c)$.

Since c is arbitrary, $f'(x) = g(x)$ for all $x \in [a, b]$.

This completes the proof.

Note 1. For a sequence of functions $\{f_n\}$ where each f_n is differentiable on $[a, b]$, mere uniform convergence of the sequence $\{f'_n\}$ on $[a, b]$ is not enough to ensure uniform convergence of the sequence $\{f_n\}$ on $[a, b]$.

For example, let $f_n(x) = \log(n + x^2), x \in [0, 1]$.

Then $f'_n(x) = \frac{2x}{n+x^2}, x \in [0, 1]$. The sequence $\{f'_n\}$ converges to the function g where $g(x) = 0, x \in [0, 1]$.

Let $M_n = \sup\limits_{x\in[0,1]} | f'_n(x) - g(x) |$. Then $M_n = \sup\limits_{x\in[0,1]} \frac{2x}{n+x^2}$.

Let $u_n(x) = \frac{2x}{n+x^2}, x \in [0, 1]$. Then $u'_n(x) = \frac{2n-2x^2}{(n+x^2)^2} > 0$, for all $x \in [0, 1]$ and for all $n > 1$.

Therefore for all $n > 1, u_n$ is a strictly increasing function of x on $[0, 1]$ and therefore $\sup\limits_{x\in[0,1]} u_n(x) = \frac{2}{n+1}$.

That is, $M_n = \frac{2}{n+1}$ for all $n > 1$ and therefore $\lim M_n = 0$.

Hence the sequence $\{f'_n\}$ is uniformly convergent on $[0, 1]$.

But the sequence $\{f_n\}$ is not even pointwise convergent on $[0, 1]$.

Note 2. For a sequence of functions $\{f_n\}$ where each f_n is differentiable on $[a, b]$ and the sequence $\{f_n\}$ is pointwise convergent on $[a, b]$, the uniform convergence of the sequence $\{f'_n\}$ on $[a, b]$ is only a sufficient but not a necessary condition for the uniform convergence of $\{f_n\}$ on $[a, b]$.

For example, let $f_n(x) = x - \frac{x^n}{n}, x \in [0, 1]$. $\lim\limits_{n\to\infty} f_n(x) = x, x \in [0, 1]$.

The sequence $\{f_n\}$ converges to the function f where $f(x) = x, x \in [0, 1]$.

Let $M_n = \sup\limits_{x\in[0,1]} | f_n(x) - f(x) |$. Then $M_n = \frac{1}{n}$ and $\lim\limits_{n\to\infty} M_n = 0$.

This establishes uniform convergence of the sequence $\{f_n\}$ on $[0, 1]$.

$$\begin{aligned} f'_n(x) = 1 - x^{n-1}. \lim_{n\to\infty} f'_n(x) &= 1, 0 \le x < 1 \\ &= 0, x = 1. \end{aligned}$$

The limit function of the sequence $\{f'_n\}$ is not continuous on $[0, 1]$.

As each f'_n is continuous on $[0, 1]$ and the limit function is not continuous on $[0, 1]$, the convergence of the sequence $\{f'_n\}$ is not uniform on $[0, 1]$.

Thus the sequence $\{f_n\}$ is uniformly convergent on $[0, 1]$ inspite of non-uniform convergence of the sequence $\{f'_n\}$ on $[0, 1]$ and our assertion is established.

Worked Examples (continued).

9. Show that the sequence $\{f_n\}$ where $f_n(x) = \frac{x}{1+nx^2}, 0 \leq x \leq 1$ converges uniformly to a function f but $\lim\limits_{n\to\infty} f_n'(x) = f'(x)$ is true if $x \neq 0$.

When $x = 0$, the sequence converges to 0.
When $0 < x \leq 1$, $\lim\limits_{n\to\infty} f_n(x) = 0$.

Therefore the sequence $\{f_n\}$ converges to the function f on $[0,1]$ where $f(x) = 0, 0 \leq x \leq 1$.

Let $M_n = \sup\limits_{x\in[0,1]} \mid f_n(x) - f(x) \mid$. Then $M_n = \sup\limits_{x\in[0,1]} \dfrac{x}{1+nx^2}$.
$M_n = \frac{1}{2\sqrt{n}}$ [worked Ex.5, 13.3.] and therefore $\lim\limits_{n\to\infty} M_n = 0$.
This proves that the sequence $\{f_n\}$ is uniformly convergent on $[0,1]$.

$f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}$.
When $x = 0$, $\lim\limits_{n\to\infty} f_n'(x) = 1$. When $0 < x \leq 1$, $\lim\limits_{n\to\infty} f_n'(x) = 0$.
But $f'(x) = 0, 0 \leq x \leq 1$.
Therefore $\lim\limits_{n\to\infty} f_n'(x) = 0 = f'(x)$, when $0 < x \leq 1$
and $\lim\limits_{n\to\infty} f_n'(x) = 1 \neq f'(x)$ when $x = 0$.

Miscellaneous Examples.

1. Let a sequence of functions $\{f_n\}$ be uniformly convergent on an interval I and each f_n be bounded on I. Prove that the sequence f_n is uniformly bounded on I.

[A sequence of functions $f_n(x)$ is said to be uniformly bounded on an interval I if there exists a constant B such $|f_n(x)| < B$ for all $x \in I$ and for all $n \in \mathbb{N}$.]

Let the sequence $\{f_n\}$ be uniformly convergent on I to the function f. Since each f_n is bounded on I and the the sequence $\{f_n\}$ is uniformly convergent on I to the function f, f is bounded on I. Therefore there exists positive real number k_1 such that $|f(x)| < k_1$ for all $x \in I$.

Let $M_n = \sup\limits_{x\in I} \mid f_n(x) - f(x) \mid$.

Since $\{f_n\}$ is uniformly convergent on I to the function f, $\lim M_n = 0$. Consequently, the sequence $\{M_n\}$ is a bounded sequence and therefore there exists positive real number k_2 such that $|M_n| < k_2$ for all $n \in \mathbb{N}$.
That is, $\sup\limits_{x\in I} |f_n(x) - f(x)| < k_2$ for all $n \in \mathbb{N}$.
Therefore for all $x \in I$, $|f_n(x)| < k_1 + k_2$ for all $n \in \mathbb{N}$.

This proves that the sequence f_n is uniformly bounded on I.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $\mathbb{R}$. For each natural number n, let $f_n(x) = f(x + \frac{1}{n}), x \in \mathbb{R}$.

Prove that the sequence $\{f_n\}$ is uniformly convergent on $\mathbb{R}$.

For all $x \in \mathbb{R}$, $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} f(x + \frac{1}{n})$
$= f(x)$, since f is continuous at x.

Therefore the sequence $\{f_n\}$ converges to the function f on $\mathbb{R}$.

Let $\epsilon > 0$. Since f is uniformly continuous on $\mathbb{R}$ there exists a positive δ such that

for all $x, u \in \mathbb{R}, |\ x - u\ | < \delta \Rightarrow |\ f(x) - f(u)\ | < \epsilon$ ... ... (i)

There exists a natural number k such that $0 < \frac{1}{n} < \delta$ for all $n \geq k$.

It follows from (i) that

for all $x \in \mathbb{R}, |\ f(x + \frac{1}{n}) - f(x)\ | < \epsilon$ for all $n \geq k$.

That is, for all $x \in \mathbb{R}, |\ f_n(x) - f(x)\ | < \epsilon$ for all $n > k$.

This proves that the sequence $\{f_n\}$ is uniformly convergent on $\mathbb{R}$ to the function f.

3. Let $\{f_n\}$ be a sequence of functions on an interval I that converges uniformly on I to a continuous function f. Let $c \in I$ and $\{x_n\}$ is any sequence in I converging to c. Prove that $\lim\limits_{n\to\infty} f_n(x_n) = f(c)$.

Let $\epsilon > 0$. Since the sequence $\{f_n\}$ is uniformly convergent on I, there exists a natural number k_1 such that

for all $x \in I$, $|\ f_n(x) - f(x)\ | < \frac{\epsilon}{2}$ for all $n \geq k_1$.

Since $x_n \in I$, $|\ f_n(x_n) - f(x_n)\ | < \frac{\epsilon}{2}$ for all $n \geq k_1$.

Since f is continuous at c and $\lim x_n = c$, $\lim f(x_n) = f(c)$. Therefore there exists a natural number k_2 such that

$|\ f(x_n) - f(c)\ | < \frac{\epsilon}{2}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $|\ f_n(x_n) - f(x_n)\ | < \frac{\epsilon}{2}$ and $|\ f(x_n) - f(c)\ | < \frac{\epsilon}{2}$ for all $n \geq k$.

By triangle inequality,

$|\ f_n(x_n) - f(c)\ | \leq |\ f_n(x_n) - f(x_n)\ | + |\ f(x_n) - f(c)\ | < \epsilon$ for all $n \geq k$.

This implies $\lim\limits_{n\to\infty} f_n(x_n) = f(c)$.

4. Let $\{f_n\}$ be a sequence of differentiable functions on $[a, b]$ that converges on $[a, b]$ to f. If $\{f'_n\}$ be a sequence of continuous functions on $[a, b]$ that converges uniformly to g on $[a, b]$ then show that $g(x) = f'(x)$ for all $x \in [a, b]$.

Since $\{f'_n\}$ is a uniformly convergent sequence of continuous functions on $[a, b]$, the limit function g is continuous on $[a, b]$.

Since each f'_n is continuous on $[a, b]$, each f'_n is integrable on $[a, b]$.

Since $\{f'_n\}$ is uniformly convergent on $[a, b]$ to the function g, by the corollary of the theorem 13.4.5,

for all $x \in [a, b]$, $\lim\limits_{n\to\infty} \int_a^x f'_n(t)dt = \int_a^x g(t)dt$.

But $\int_a^x f'_n(t)dt = f_n(x) - f_n(a)$, by the fundamental theorem.

$\lim\limits_{n\to\infty} \int_a^x f'_n(t)dt = \lim\limits_{n\to\infty} \{f_n(x) - f_n(a)\} = f(x) - f(a)$.

Therefore for all $x \in [a, b]$, $f(x) - f(a) = \int_a^x g(t)dt$.

Since g is continuous on $[a, b]$, by the corollary of the theorem 11.8.2, $f'(x) = g(x)$ for all $x \in [a, b]$.

Exercises 24

1. Let $\{u_n\}$ and $\{v_n\}$ be sequences of functions uniformly convergent on $[a, b]$ to the limit functions u and v respectively. Prove that the sequence $\{u_n + v_n\}$ converges uniformly on $[a, b]$ to the limit function $u + v$.

2. For each $n \in \mathbb{N}$, let $f_n(x) = x - \frac{1}{n}, g_n(x) = x + \frac{2}{n}$ on $[0, \infty)$.

Show that the sequences $\{f_n\}$ and $\{g_n\}$ are uniformly convergent on $[0, \infty)$ but the sequence $\{f_n g_n\}$ is not so.

3. Define uniform convergence of a sequence of functions $\{f_n\}$ on an interval I. Use the definition to examine uniform convergence of the sequence $\{f_n\}$ on $[0, \infty)$, where

(i) $f_n(x) = \frac{x}{x+n}, x \in [0, \infty)$;

(ii) $f_n(x) = xe^{-nx}, x \in [0, \infty)$;

(iii) $f_n(x) = x^2e^{-nx}, x \in [0, \infty)$;

(iv) $f_n(x) = n^2x^2e^{-nx}, x \in [0, \infty)$.

4. Let $a < c < b$. Let $\{f_n\}$ be a sequence of functions converging uniformly on $[a, c]$ and $[c, b]$. Prove that $\{f_n\}$ converges uniformly on $[a, b]$.

5. Prove that a sequence of functions $\{f_n\}$ is uniformly convergent on $[a, b]$ to a function f if and only if $\lim\limits_{n\to\infty} M_n = 0$, where

$$M_n = \sup_{x\in[a,b]} | f_n(x) - f(x) | .$$

Utilise this to examine uniform convergence of the sequence $\{f_n\}$ on $[0, 1]$:

(i) for each $n \in \mathbb{N}, f_n(x) = \frac{x}{1+nx^2}, x \in [0, 1]$;

(ii) for each $n \in \mathbb{N}, f_n(x) = \frac{nx}{1+n^3x^2}, x \in [0, 1]$;

(iii) for each $n \in \mathbb{N}$, $f_n(x) = \frac{n^2x}{1+n^4x^2}$, $x \in [0, 1]$;

(iv) for each $n \in \mathbb{N}$, $f_n(x) = nx(1-x)^n$, $x \in [0, 1]$;

(v) For each $n \in \mathbb{N}$, $f_n(x) = \frac{x}{1+nx}$, $x \in [0, 1]$;

(vi) For each $n \in \mathbb{N}$, $f_n(x) = xe^{-nx}$, $x \in [0, \infty)$.

6. Let $D = \{x \in \mathbb{R} : x \geq 0\}$ and $f_n(x) = e^{-nx}$, $x \in D$.

(i) Show that the sequence $\{f_n\}$ converges to a function f on D.

(ii) Show that f is not continuous on D. Deduce that the convergence of the sequence $\{f_n\}$ is not uniform on D.

(iii) Show that the convergence of the sequence $\{f_n\}$ is uniform on $[a, \infty)$, if $a > 0$.

7. Let $f_n(x) = \frac{nx}{1+nx}$, $x \in [0, 1]$.

(i) Show that the sequence $\{f_n\}$ converges to a function f on $[0, 1]$.

(ii) Show that f is not continuous on $[0, 1]$. Deduce that the convergence of the sequence is not uniform on $[0, 1]$.

8. For each $n \in \mathbb{N}$, let
$$f_n(x) = nx, 0 \leq x \leq 1/n.$$
$$= 1, \tfrac{1}{n} < x \leq 1.$$

(i) Show that the sequence $\{f_n\}$ converges to a function f on $[0, 1]$.

(ii) Show that f is not continuous on $[0, 1]$. Deduce that the convergence of the sequence $\{f_n\}$ is not uniform on $[0, 1]$.

9. For each $n \in \mathbb{N}$, let
$$f_n(x) = nx^2, 0 \leq x \leq 1/n$$
$$= x, \tfrac{1}{n} < x \leq 1.$$

(i) Show that the sequence $\{f_n\}$ converges to a function f on $[0, 1]$.

(ii) Find M_n, where $M_n = \sup\limits_{x \in [0,1]} | f_n(x) - f(x) |$. Show that the sequence $\{f_n\}$ is uniformly convergent on $[0, 1]$.

10. For each $n \geq 2$, let
$$f_n(x) = n^2x, \ 0 \leq x \leq \tfrac{1}{n}$$
$$= -n^2x + 2n, \ \tfrac{1}{n} < x < \tfrac{2}{n}$$
$$= 0, \ \tfrac{2}{n} \leq x \leq 1.$$

(i) Show that the sequence $\{f_n\}_2^\infty$ converges to a function f on $[0, 1]$;

(ii) Show that the convergence of the sequence is not uniform on $[0, 1]$ by establishing that $\lim\limits_{n \to \infty} \int_0^1 f_n \neq \int_0^1 f$.

[**Hint.** If $c \in (0, 1)$ there exists a natural number $p > 2$ such that $0 < \frac{2}{p} < c$ and therefore $\frac{2}{n} < c < 1$ for all $n \geq p$. $f_p(c) = 0$ and $f_n(c) = 0$ for all $n \geq p$. Therefore $\lim\limits_{n \to \infty} f_n(x) = 0$ for all $x \in (0, 1)$ and for all $n > 2$.]

11. Show that the sequence of functions f_n defined on $[0, 1]$ by

$f_n(x) = n(1-nx),\ 0 \le x < \frac{1}{n}$
$\quad = 0,\ \frac{1}{n} \le x \le 1$
converges to the function f given by $f(x) = 0,\ x \in [0,1]$.

Show that $\lim\limits_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 f(x)dx$. Is the convergence of the sequence uniform ?

[**Hint.** If $c \in (0,1)$ there exists a natural number m such that $0 < \frac{1}{m} < c$ and therefore $\frac{1}{n} < c < 1$ for all $n \ge m$. $f_m(c) = 0$ and $f_n(c) = 0$ for all $n \ge m$. Therefore $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \in (0,1)$.]

12. For each $n \in \mathbb{N}$, let $f_n(x) = \lim\limits_{m\to\infty} (\cos n!\pi x)^{2m}$.

(i) Show that the sequence $\{f_n\}$ converges on $\mathbb{R}$ to the function f defined by $f(x) = 1, x \in \mathbb{Q}$
$\quad = 0, x \in \mathbb{R} - \mathbb{Q}$.

(ii) If $[a,b]$ be a closed and bounded interval, show that each f_n is integrable on $[a,b]$.

Deduce that the sequence $\{f_n\}$ is not uniformly convergent on $[a,b]$.

[**Hint.** If $x \in \mathbb{R} - \mathbb{Q}, 0 < \cos^2 n!\pi x < 1$ and $\lim\limits_{m\to\infty} (\cos n!\pi x)^{2m} = 0$.

If $x = \frac{p}{q}$, where p, q are integers and $q \ge 1$ then $(\cos n!\pi x)^{2m} = 1$ if $n \ge q$.]

13. Let $f_n(x) = n^2 x(1-x^2)^n, 0 \le x \le 1$. Show that

(i) the sequence $\{f_n\}$ converges to a function f on $[0,1]$;

(ii) the sequence is not uniformly convergent on $[0,1]$ by establishing that $\lim\limits_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f$.

14. Let $f_n(x) = \frac{nx}{1+nx}, x \in [0,1]$. Show that

(i) the sequence $\{f_n\}$ converges to a function f on $[0,1]$;

(ii) f is integrable on $[0,1]$ and $\lim\limits_{n\to\infty} \int_0^1 f_n = \int_0^1 f$, but still the convergence of the sequence is not uniform on $[0,1]$.

15. Let $f_n(x) = nx(1-x)^n, x \in [0,1]$. Show that

(i) the sequence $\{f_n\}$ converges to a function f on $[0,1]$;

(ii) f is integrable on $[0,1]$ and $\lim\limits_{n\to\infty} \int_0^1 f_n = \int_0^1 f$, but still the convergence of the sequence is not uniform on $[0,1]$.

16. Let $\{f_n\}$ be a sequence of continuous functions on $[a,b]$ and the sequence is uniformly convergent on $[a,b]$. Let $g_n(x) = \int_a^x f_n(x)dx, a \le x \le b$.

Prove that the sequence $\{g_n\}$ is uniformly convergent on $[a,b]$.

[**Hint.** $g_n'(x) = f_n(x), x \in [a,b]$. $\{g_n'\}$ is uniformly convergent on $[a,b]$ and the sequence $\{g_n\}$ is convergent at a.]

17. Let $f_n(x) = \frac{x}{1+nx}, 0 \leq x \leq 1$. Show that

(i) the sequence $\{f_n\}$ converges uniformly to a function on $[0, 1]$;

(ii) the sequence $\{f'_n\}$ converges to a function g on $[0, 1]$ and $f'(0) \neq g(0)$.

18. Let $f_n(x) = \frac{x^n}{n}, 0 \leq x \leq 1$. Show that

(i) the sequence $\{f_n\}$ converges uniformly to a function f on $[0, 1]$;

(ii) the sequence $\{f'_n\}$ converges to a function g on $[0, 1]$ and $f'(x) = g(x), x \in (0, 1)$, $f'(1) \neq g(1)$.

19. Show that the sequence of functions f_n defined on $[-1, 1]$ by $f_n(x) = |x|^{1+\frac{1}{n}}$, $x \in [-1, 1]$ converges uniformly to the function given by $f(x) = |x|$, $x \in [-1, 1]$ but the convergence of the sequence $\{f'_n\}$ is not uniform on $[-1, 1]$.

20. Let $f_n(x) = \log(n^2 + x^2), x \in \mathbb{R}$. Show that

(i) the sequence $\{f'_n\}$ is uniformly convergent on $\mathbb{R}$;

(ii) the sequence $\{f_n\}$ is not uniformly convergent on $\mathbb{R}$.

21. Let $f_n(x) = \frac{\log(1+n^2x^2)}{n^2}, x \in [0, 1]$. Show that

(i) the sequence $\{f'_n\}$ is uniformly convergent on $[0, 1]$;

(ii) the sequence $\{f_n\}$ is uniformly convergent on $[0, 1]$.

22. Let $f_n(x) = n + \frac{x}{n}, x \in \mathbb{R}$. Show that

(i) the sequence $\{f'_n\}$ is uniformly convergent on $\mathbb{R}$;

(ii) the sequence $\{f_n\}$ is not uniformly convergent on $\mathbb{R}$.

Explain why the sequence $\{f_n\}$ is not uniformly convergent on $\mathbb{R}$ although (i) is satisfied.

14. SERIES OF FUNCTIONS

14.1. Uniform convergence.

Let $D \subset \mathbb{R}$. Let $\{f_n\}$ be a sequence of functions on D to $\mathbb{R}$. Then $f_1 + f_2 + f_3 + \cdots$ is said to be *a series of functions* on D. The infinite series is denoted by Σf_n (or by $\sum\limits_1^\infty f_n$).

Let the sequence of functions $\{s_n\}$ be defined for $x \in D$ by

$$\begin{aligned} s_1(x) &= f_1(x), \\ s_2(x) &= f_1(x) + f_2(x), \\ s_3(x) &= f_1(x) + f_2(x) + f_3(x), \\ &\ldots \quad \ldots \\ s_n(x) &= f_1(x) + f_2(x) + \cdots + f_n(x), \\ &\ldots \quad \ldots \end{aligned}$$

The sequence $\{s_n\}$ is said to be the sequence of *partial sums* of the infinite series Σf_n.

If the sequence $\{s_n\}$ be pointwise convergent on D to a function s then the series Σf_n is said to be *pointwise convergent* on D and s is said to be the *sum function* of the series Σf_n on D.

If the sequence $\{s_n\}$ be uniformly convergent on D to a function s then the series Σf_n is said to be *uniformly convergent* on D to the sum function s.

If the series $\Sigma \mid f_n(x) \mid$ converges for each $x \in D$, then the series Σf_n is said to be *absolutely convergent* on D.

Note. We shall use the symbol Σf_n (or $\sum\limits_1^\infty f_n$) to denote either the series of functions $f_1 + f_2 + f_3 + \cdots \cdots$, or the sum of the series, when it exists.

Worked Examples.

1. Prove that the series of functions $1 + x + x^2 + \cdots, 0 \le x < 1$ is convergent on $0 \le x < 1$, but the convergence is not uniform on $[0, 1)$.

Let $s_n(x) = 1 + x + x^2 + \cdots + x^{n-1}$, $0 \le x < 1$.
Then $s_n(x) = \frac{1-x^n}{1-x}, x \in [0, 1)$. $\lim\limits_{n\to\infty} s_n(x) = \frac{1}{1-x}, x \in [0, 1)$.

The sequence $\{s_n\}$ converges on $[0,1)$ to the function s where $s(x) = \frac{1}{1-x}, x \in [0,1)$.

The series Σf_n is, therefore, pointwise convergent on $[0,1)$ to the sum function s.

Each s_n is bounded on $[0,1)$ but the limit function s is not bounded on $[0,1)$.

Hence the convergence of the sequence $\{s_n\}$ is not uniform on $[0,1)$ and, by definition, Σf_n is not uniformly convergent on $[0,1)$.

2. Prove that the series of functions

$\frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \frac{x}{(2x+1)(3x+1)} + \cdots, x \geq 0$

is convergent on $[0,\infty)$ but the convergence is not uniform on $[0,\infty)$.

Let $s_n(x) = \frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \cdots + \frac{x}{[(n-1)x+1](nx+1)]}$.

$$\begin{aligned}
\text{Then } s_n(x) &= (1 - \tfrac{1}{x+1}) + (\tfrac{1}{x+1} - \tfrac{1}{2x+1}) + \cdots \\
&\quad + (\tfrac{1}{(n-1)x+1} - \tfrac{1}{nx+1}) \\
&= 1 - \tfrac{1}{nx+1} = \tfrac{nx}{nx+1}, x \in [0,\infty).
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} s_n(x) &= 0, x = 0 \\
&= 1, x > 0.
\end{aligned}$$

Therefore the sequence $\{s_n\}$ converges on $[0,\infty)$ to the function s where $s(x) = 0, x = 0,$
$= 1, x > 0.$

The function s is not continuous on $[0,\infty)$ but each s_n is continuous on $[0,\infty)$. This implies that the convergence of the sequence $\{s_n\}$ is not uniform on $[0,\infty)$.

By definition, the convergence of the given series is not uniform on $[0,\infty)$.

3. Prove that the series $x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \cdots$, $x \in [0,1]$ is not uniformly convergent on $[0,1]$.

Let the series be $f_1(x) + f_2(x) + f_3(x) + \cdots$

Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$.

Then $s_n(x) = x^4[1 + \frac{1}{(1+x^4)} + \cdots + \frac{1}{(1+x^4)^{n-1}}]$.

When $x = 0, s_n(x) = 0$.

When $0 < x \leq 1, s_n(x) = (1 + x^4).[1 - \frac{1}{(1+x^4)^n}]$.

$$\begin{aligned}
\lim_{n\to\infty} s_n(x) &= 0, \ x = 0 \\
&= 1 + x^4, \ 0 < x \leq 1.
\end{aligned}$$

The sequence $\{s_n\}$ converges to the function s where

$$\begin{aligned}
s(x) &= 0, \quad x = 0 \\
&= 1 + x^4, 0 < x \leq 1.
\end{aligned}$$

s is not continuous on $[0,1]$, the point of discontinuity being 0. Each s_n is continuous on $[0,1]$. Therefore the convergence of the sequence $\{s_n\}$ is not uniform on $[0,1]$.

By definition, the convergence of the series is not uniform on $[0,1]$.

Theorem 14.1.1. (Cauchy's principle of convergence)

Let $D \subset \mathbb{R}$ and Σf_n be a series of functions on D to $\mathbb{R}$. The series Σf_n is uniformly convergent on D if and only if for a pre-assigned positive ϵ there exists a natural number k such that for all $x \in D$,
$|f_{n+1}(x)+f_{n+2}(x)+\cdots+f_{n+p}(x)| < \epsilon$ for all $n \geq k$ and for $p = 1,2,3,\ldots$

Proof. Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x), x \in D$.

Let the series Σf_n be uniformly convergent on D. Then the sequence of functions $\{s_n\}$ is uniformly convergent on D.

By Cauchy's principle for the sequence, for a pre-assigned positive ϵ there exists a natural number k such that for all $x \in D$,
$| s_{n+p}(x) - s_n(x) | < \epsilon$ for all $n \geq k$ and for $p = 1,2,3,\ldots$

That is, for all $x \in D$, $| f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x) | < \epsilon$ for all $n \geq k$ and for $p = 1,2,3,\ldots$ showing that the condition is necessary.

Conversely, let the condition hold.

Then for all $x \in D$, $| s_{n+p}(x) - s_n(x) | < \epsilon$ for all $n \geq k$ and for $p = 1,2,3,\ldots$

By Cauchy's principle of convergence for the sequence, the sequence $\{s_n\}$ is uniformly convergent on D and by definition, the series of functions Σf_n is uniformly convergent on D.

Worked Examples (continued).

4. Let a series of functions Σf_n be uniformly convergent on the intervals $[a,c]$ and $[c,b]$. Show that the series is uniformly convergent on $[a,b]$.

Let us choose $\epsilon > 0$.

Since Σf_n is uniformly convergent on $[a,c]$, there exists a natural number k_1 such that for all $x \in [a,c]$,
$| f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x) | < \epsilon$ for all $n \geq k_1, p = 1,2,3,\ldots$.

Since Σf_n is uniformly convergent on $[c,b]$, there exists a natural number k_2 such that for all $x \in [c,b]$,
$| f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x) | < \epsilon$ for all $n \geq k_2, p = 1,2,3,\ldots$

Let $k = \max\{k_1, k_2\}$. Then for all $x \in [a,b]$,
$| f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x) | < \epsilon$ for all $n \geq k, p = 1,2,3,\ldots$

This proves that Σf_n is uniformly convergent on $[a,b]$.

Theorem 14.1.2. (Weierstrass' M-test)

Let $D \subset \mathbb{R}$ and Σf_n be a series of functions on D to $\mathbb{R}$.

Let $\{M_n\}$ be a sequence of positive real numbers such that for all $x \in D, \mid f_n(x) \mid \leq M_n$ for all $n \in \mathbb{N}$. If the series ΣM_n be convergent then the series Σf_n is uniformly and absolutely convergent on D.

Proof. Let us choose $\epsilon > 0$.

Since ΣM_n is convergent, there exists a natural number k (by Cauchy's principle) such that

$\mid M_{n+1} + M_{n+2} + \cdots + M_{n+p} \mid < \epsilon$ for all $n \geq k, p = 1, 2, 3, \ldots$

For all $x \in D$,
$$\begin{aligned} & \mid f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x) \mid \\ \leq & \mid f_{n+1}(x) \mid + \mid f_{n+2}(x) \mid + \cdots + \mid f_{n+p}(x) \mid \\ \leq & \; M_{n+1} + M_{n+2} + \cdots + M_{n+p} \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Hence for all $x \in D, \mid f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x) \mid < \epsilon$ for all $n \geq k, p = 1, 2, 3, \ldots$.

By Cauchy's principle, the series Σf_n is uniformly convergent on D.

Again, for all $x \in D$,
$$\begin{aligned} & \mid \; |f_{n+1}(x)| + |f_{n+2}(x)| + \cdots + |f_{n+p}(x)| \; \mid \\ & = \mid f_{n+1}(x) \mid + \mid f_{n+2}(x) \mid + \cdots + \mid f_{n+p}(x) \mid \\ & \leq M_{n+1} + M_{n+2} + \cdots + M_{n+p}. \end{aligned}$$

Therefore for all $x \in D$, $\mid \; |f_{n+1}(x)| + \cdots + |f_{n+p}(x)| \; \mid$
$< \epsilon$ for all $n \geq k, p = 1, 2, 3, \ldots$

By Cauchy's principle, $\Sigma \mid f_n \mid$ is convergent on D.

This implies that the series Σf_n is absolutely convergent on D.

Worked Examples (continued).

5. Prove that the series $\Sigma \frac{1}{n^3+n^4x^2}$ is uniformly convergent for all real x.

Let $f_n(x) = \frac{1}{n^3+n^4x^2}, x \in \mathbb{R}$.

For all $x \in \mathbb{R}, f_n(x) \leq \frac{1}{n^3}$. This holds for all $n \in \mathbb{N}$.

Let $M_n = \frac{1}{n^3}$. Then for all $x \in \mathbb{R}, \mid f_n(x) \mid \leq M_n$ for all $n \in \mathbb{N}$.

The series ΣM_n is a convergent series of positive real numbers.

By Weierstrass' M-test, the series Σf_n is uniformly convergent on $\mathbb{R}$.

6. Prove that the series $\Sigma \frac{x}{n+n^2x^2}$ is uniformly convergent for all real x.

Let $f_n(x) = \frac{x}{n+n^2x^2}$.

When $x = 0, f_n(x) = 0$.

When $x \neq 0, \frac{n}{|x|} + n^2|x| \geq 2n^{3/2}$, the equality occurs when $\mid x \mid = \frac{1}{\sqrt{n}}$

or, $\mid f_n(x) \mid \leq \frac{1}{2n^{3/2}}$, the equality occurs when $\mid x \mid = \frac{1}{\sqrt{n}}$.

It follows that $\mid f_n(x) \mid \leq \frac{1}{2n^{3/2}}$ for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$.

Let $M_n = \frac{1}{2n^{3/2}}$. Then for all real x, $| f_n(x) | \leq M_n$ for all $n \in \mathbb{N}$. The series ΣM_n is a convergent series of positive real numbers.

By Weierstrass' M-test, Σf_n is uniformly convergent on $\mathbb{R}$.

7. Show that the series
$1 - \frac{e^{-2x}}{2^2-1} + \frac{e^{-4x}}{4^2-1} - \frac{e^{-6x}}{6^2-1} + \cdots$ converges uniformly for all $x \geq 0$.

Let $\sum\limits_{n=0}^{\infty} f_n$ be the given series.

Then $f_0(x) = 1, f_n(x) = (-1)^n \frac{e^{-2nx}}{4n^2-1}, n \geq 1$.

$$\begin{aligned} \text{For all } x \geq 0, | f_n(x) | &= \frac{e^{-2nx}}{4n^2-1}, \text{ for all } n \geq 1 \\ &\leq \frac{1}{4n^2-1}, \text{since } e^{2nx} \geq 1 \text{ for all } x \geq 0. \end{aligned}$$

Let $M_n = \frac{1}{4n^2-1}$. Then ΣM_n is a convergent series of positive real numbers and for all $x \geq 0$, $| f_n(x) | \leq M_n$ for all $n \geq 1$.

By Weierstrass' M-test, Σf_n is uniformly convergent for all $x \geq 0$.

14.2. Consequences of uniform convergence.

Theorem 14.2.1. Let D be a subset of $\mathbb{R}$ and a series of functions Σf_n be uniformly convergent on D to a function f. Let $x_0 \in D'$ (the derived set of D) and $\lim\limits_{x \to x_0} f_n(x) = a_n$. Then

(i) the series Σa_n is convergent, and

(ii) $\lim\limits_{x \to x_0} f(x)$ exists and equals Σa_n.

Proof. (i) Let us choose $\epsilon > 0$.

Since the series Σf_n is uniformly convergent on D, there exists a natural number k such that

for all $x \in D$, $|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x)| < \frac{\epsilon}{2}$ for all $n \geq k$ and for $p = 1, 2, \ldots$ ... ... (i)

As $\lim\limits_{x \to x_0} f_n(x) = a_n$, it follows that

$\lim\limits_{x \to x_0} \{f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x)\} = a_{n+1} + a_{n+2} + \cdots + a_{n+p}$

and therefore

$\lim\limits_{x \to x_0} |f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x)| = |a_{n+1} + a_{n+2} + \cdots + a_{n+p}|$.

It follows from (i) that $|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| \leq \frac{\epsilon}{2} < \epsilon$ for all $n \geq k$ and for $p = 1, 2, \ldots$.

This shows that the series Σa_n is convergent, by Cauchy's principle.

(ii) Let $\Sigma a_n = s$. Let $b_n = a_1 + a_2 + \cdots + a_n$. Then $\lim b_n = s$.

Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$, $x \in D$.

Let us choose $\epsilon > 0$.

Since the series Σf_n converges uniformly on D to the function f, the sequence $\{s_n\}$ is uniformly convergent on D to the function f and therefore there exists a natural number k_1 such that for all $x \in D$, $|s_n(x) - f(x)| < \frac{\epsilon}{3}$ for all $n \geq k_1$.

Since $\lim b_n = s$, there exists a natural number k_2 such that $|b_n - s| < \frac{\epsilon}{3}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $|s_k(x) - f(x)| < \frac{\epsilon}{3}$ for all $x \in D$ and $|b_k - s| < \frac{\epsilon}{3}$.

Since $\lim\limits_{x \to x_0} f_n(x) = a_n$ for $n = 1, 2, \ldots$, $\lim\limits_{x \to x_0} s_k(x) = b_k$ and therefore there exists a positive δ such that

$| s_k(x) - b_k | < \frac{\epsilon}{3}$ for all $x \in N'(x_0, \delta) \cap D$.

By triangle inequality,

$$\begin{aligned} |f(x) - s| &\leq |f(x) - s_k(x)| + |s_k(x) - b_k| + |b_k - s| \\ &< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} (= \epsilon) \text{ for all } x \in N'(x_0, \delta) \cap D. \end{aligned}$$

This proves $\lim\limits_{x \to x_0} f(x) = s$. That is, $\lim\limits_{x \to x_0} f(x) = \Sigma a_n$.

Note. In consequence of uniform convergence of the series Σf_n, $\lim\limits_{x \to x_0} \Sigma f_n(x) = \Sigma \lim\limits_{x \to x_0} f_n(x)$. This means that the interchange of the symbols Σ and $\lim\limits_{x \to x_0}$ is permissible.

Corollary. Let I be an interval and a series of functions Σf_n be uniformly convergent on I to a function f. Let $c \in I$ and each f_n be continuous at c. Then f is continuous at c.

Worked Example.

1. Find $\lim\limits_{x \to 0} \sum\limits_{n=1}^{\infty} \frac{\cos nx}{n(n+1)}$.

Let $f_n(x) = \frac{\cos nx}{n(n+1)}$ for $n = 1, 2, \ldots$. Then $|f_n(x)| \leq \frac{1}{n(n+1)}$ for $n = 1, 2, \ldots$ and for all $x \in \mathbb{R}$.

Let $M_n = \frac{1}{n(n+1)}$ for $n = 1, 2, \ldots$. Then $|f_n(x)| \leq M_n$ for $n = 1, 2, \ldots$.

ΣM_n is a convergent series of positive real numbers and therefore by Weierstrass' M-test, $\sum\limits_{n=1}^{\infty} f_n(x)$ is uniformly convergent for all real x.

Since the series is uniformly convergent for all real x, $\lim\limits_{x \to 0} \sum\limits_{n=1}^{\infty} f_n(x) = \sum\limits_{n=1}^{\infty} \lim\limits_{x \to 0} f_n(x)$. That is, $\lim\limits_{x \to 0} f(x) = \sum\limits_{n=1}^{\infty} \lim\limits_{x \to 0} \frac{\cos nx}{n(n+1)} = \sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Let $t_n = \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{n(n+1)}$. Then $t_n = 1 - \frac{1}{n+1}$ and $\lim\limits_{n \to \infty} t_n = 1$.

This implies $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ and therefore $\lim\limits_{x \to 0} \sum\limits_{n=1}^{\infty} \frac{\cos nx}{n(n+1)} = 1$.

Theorem 14.2.2. Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}, f_n : D \to \mathbb{R}$ is a continuous function on D. If the series Σf_n be uniformly convergent on D then sum function s is continuous on D.

Proof. Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x), x \in D$.

Then each s_n is continuous on D.

Since the series Σf_n is uniformly convergent on D, the sequence $\{s_n\}$ is uniformly convergent on D to the function s.

Let us choose $\epsilon > 0$. Then there exists a natural number k such that for all $x \in D, |s_n(x) - s(x)| < \frac{\epsilon}{3}$ for all $n \geq k$.

It follows that for all $x \in D, |s_k(x) - s(x)| < \frac{\epsilon}{3}$ ... ... (i)

Let $c \in D$. Then (i) gives $| s_k(c) - s(c) | < \frac{\epsilon}{3}$.

Since s_k is continuous at c, there exists a positive δ such that $| s_k(x) - s_k(c) | < \frac{\epsilon}{3}$ for all $x \in \mathbb{N}(c, \delta) \cap D$.

By triangle inequality,

$$\begin{aligned} | s(x) - s(c) | &\leq |s(x) - s_k(x)| + |s_k(x) - s_k(c)| + |s_k(c) - s(c)| \\ &< \epsilon \text{ for all } x \in N(c, \delta) \cap D. \end{aligned}$$

This proves that s is continuous at c.

Since c is arbitrary, s is continuous on D.

Note 1. If for each $n \in \mathbb{N}, f_n$ is continuous on D and the sum function s of the series Σf_n is not continuous on D then it follows from the theorem that the convergence of the series Σf_n is not uniform on D.

Note 2. If each f_n be continuous on D, the uniform convergence of the series Σf_n on D is a sufficient but not a necessary condition for continuity of the sum function s on D.

For example, let $f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}, x \in \mathbb{R}$.

Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x), x \in \mathbb{R}$.

Then $s_n(x) = \frac{nx}{1+n^2x^2}$. $\lim\limits_{n \to \infty} s_n(x) = 0$ for all $x \in \mathbb{R}$.

The sequence $\{s_n\}$ converges to the function s where $s(x) = 0$, for all $x \in \mathbb{R}$.

The convergence of the sequence $\{s_n\}$ is not uniform on $\mathbb{R}$ by the Example 5 of 13.3 and therefore the convergence of the series Σf_n is not uniform on $\mathbb{R}$.

But each f_n is continuous on $\mathbb{R}$ and also the sum function s is continuous on $\mathbb{R}$.

This proves that for a convergent series of continuous functions, the uniform convergence of the series is not necessary for continuity of the sum function.

Worked Example (continued).

2. Show that the series $(1-x)+x(1-x)+x^2(1-x)+\cdots$ is not uniformly convergent on $[0,1]$.

Let the series be $\sum\limits_1^\infty f_n(x)$.

Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x), x \in [0,1]$.

Then $s_n(x) = (1-x)[1+x+x^2+\cdots x^{n-1}]$.

$$\begin{aligned} \lim s_n(x) &= 1, \text{ for } 0 \le x < 1 \\ &= 0, \text{ for } x = 1. \end{aligned}$$

The sequence $\{s_n\}$ converges to the function s

$$\begin{aligned} \text{where } s(x) &= 1, 0 \le x < 1 \\ &= 0, x = 1. \end{aligned}$$

Therefore the series Σf_n converges to the function s on $[0,1]$.

Each f_n is continuous on $[0,1]$ but s is not continuous on $[0,1]$.

This proves that the convergence of the series Σf_n is not uniform on $[0,1]$.

Theorem 14.2.3. Let $I = [a,b]$ be a closed and bounded interval and for each $n \in \mathbb{N}, f_n : I \to \mathbb{R}$ be intergrable on I. If the series Σf_n be uniformly convergent on I to the function s then

(i) s is integrable on I,

(ii) $\Sigma \int_a^b f_n(x)dx = \int_a^b s(x)dx$.

Proof. Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x), x \in [a,b]$.

Then each s_n is integrable on $[a,b]$.

Since the series Σf_n is uniformly convergent on $[a,b]$ to the function s, the sequence $\{s_n\}$ is uniformly convergent on $[a,b]$ to s.

By the Theorem 13.4.5, s is integrable on $[a,b]$ and $\lim\limits_{n\to\infty} \int_a^b s_n(x)dx = \int_a^b s(x)dx$.

But $\int_a^b s_n(x)dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx + \cdots + \int_a^b f_n(x)dx$.

Since $\lim\limits_{n\to\infty} \int_a^b s_n(x)dx = \int_a^b s(x)dx$, the series $\Sigma \int_a^b f_n(x)dx$ converges to $\int_a^b s(x)dx$. This completes the proof.

Note 1. The equality $\Sigma \int_a^b f_n(x) = \int_a^b s(x)dx$ can be expressed as

$$\sum_{k=1}^\infty \int_a^b f_k(x)dx = \int_a^b \sum_{k=1}^\infty f_k(x)dx.$$

Thus it is permissible to interchange the symbols $\sum\limits_{k=1}^\infty$ and $\int_a^b$ if each f_n be integrable on $[a,b]$ and the convergence of the series Σf_n be uniform on $[a,b]$.

Equivalently, this can be expressed as
$\int_a^b [f_1(x) + f_2(x) + \cdots]dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx + \cdots$.
That is, the series of functions can be integrated term-by-term on $[a, b]$ if the convergence of the series is uniform on $[a, b]$.

Note 2. If each f_n be integrable on $[a, b]$ and the series Σf_n converges to a function s which is not integrable on $[a, b]$, then it follows from the theorem that the convergence of the series is not uniform on $[a, b]$.

If each f_n be integrable on $[a, b]$ and the series Σf_n converges to a function s which is also integrable on $[a, b]$ but the series $\Sigma \int_a^b f_n(x)dx$ does not converge to $\int_a^b s(x)dx$, then it follows from the theorem that the convergence of the series is not uniform on $[a, b]$.

Note 3. If each f_n be integrable on $[a, b]$, the uniform convergence of the series is only a sufficient but not a necessary condition for the integrability of the sum function.

For example, let $f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}, x \in [0, 1]$.
Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x), x \in [0, 1]$.
Then $s_n(x) = \frac{nx}{1+n^2x^2}, x \in [0, 1]$.
The sequence $\{s_n\}$ converges to the function s where $s(x) = 0$.
But the convergence is not uniform on $[0, 1]$ by Example 5, 13.3.

Thus the series Σf_n is such that each f_n is integrable on $[0, 1]$ and it converges to the function s which is also integrable on $[0, 1]$ but the convergence of the series is not uniform on $[0, 1]$.

This proves that for a series of integrable functions on $[0, 1]$ the uniform convergence of the series is not necessary for integrability of the sum function.

Note 4. If each f_n be integrable on $[a, b]$, and the series Σf_n converges to a function s which is also integrable on $[a, b]$, the uniform convergence of the series is only a sufficient but not a necessary condition for term-by-term integration of the series on $[a, b]$.

For example, let $f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}, x \in [0, 1]$.
Each f_n is integrable on $[0, 1]$.
The series converges to the function s on $[0, 1]$ where $s(x) = 0, x \in [0, 1]$.
$\int_0^1 s(x)dx = 0$.
$\int_0^1 f_1(x)dx = \int_0^1 \frac{x}{1+x^2}dx = \frac{1}{2}\log 2$, and
for $n \geq 2$, $\int_0^1 f_n(x)dx = \frac{1}{2n}\log(1 + n^2) - \frac{1}{2(n-1)}\log(1 + (n-1)^2)$.

Let $t_n = \int_0^1 f_1(x)dx + \int_0^1 f_2(x)dx + \cdots + \int_0^1 f_n(x)dx$. Then $t_n = \frac{\log(1+n^2)}{2n}$ and $\lim\limits_{n\to\infty} t_n = 0$. Therefore $\int_0^1 s(x)dx = \lim\limits_{n\to\infty} t_n$.

or, $\int_0^1 \{f_1(x) + f_2(x) + \cdots\}dx = \int_0^1 f_1(x)dx + \int_0^1 f_2(x)dx + \cdots$

But the convergence of the sequence $\{s_n\}$ and hence the convergence of the series Σf_n is not uniform on $[0, 1]$.

Thus the series can be integrated term-by-term on $[0, 1]$, although the convergence of the series is not uniform on $[0, 1]$.

This establishes that uniform convergence of a series on $[a, b]$ is a sufficient but not a necessary condition for term-by-term integration of the series on $[a, b]$.

Worked Examples (continued).

3. For the series $\sum\limits_1^\infty f_n(x)$ where

$f_n(x) = n^2xe^{-n^2x^2} - (n-1)^2xe^{-(n-1)^2x^2}, x \in [0, 1]$ show that

$\sum\limits_1^\infty \int_0^1 f_n(x)dx \neq \int_0^1 (\sum\limits_1^\infty f_n(x))dx$.

Is the series $\sum\limits_1^\infty f_n(x)$ uniformly convergent on $[0, 1]$?

Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$. Then $s_n(x) = n^2xe^{-n^2x^2}$.

For all $x \in (0, 1]$, $e^{n^2x^2} > \frac{n^4x^4}{2} > 0$.

Therefore $0 < s_n(x) < \frac{2}{n^2x^3}$ for all $x \in (0, 1]$.

By Sandwich theorem, $\lim\limits_{n\to\infty} s_n(x) = 0$, for all $x \in (0, 1]$.

And for $x = 0$, the sequence $\{s_n\}$ converges to 0.

Hence the series $\sum\limits_1^\infty f_n(x)$ is convergent on $[0, 1]$ and the sum function f is given by $f(x) = 0, x \in [0, 1]$.

Therefore $\int_0^1 (\sum\limits_1^\infty f_n(x))dx = 0$.

$\int_0^1 f_n(x)dx = \frac{1}{2}[-e^{-n^2x^2} + e^{-(n-1)^2x^2}]_0^1 = \frac{1}{2}[e^{-(n-1)^2} - e^{-n^2}]$.

Let $t_n = \int_0^1 f_1(x)dx + \int_0^1 f_2(x)dx + \cdots + \int_0^1 f_n(x)dx$.

Then $t_n = \frac{1}{2}[1 - e^{-n^2}]$ and $\lim\limits_{n\to\infty} t_n = \frac{1}{2}$.

Therefore $\sum\limits_1^\infty \int_0^1 f_n(x)dx = \frac{1}{2} \neq \int_0^1 (\sum\limits_1^\infty f_n(x))dx$.

Note. It follows that the series $\sum\limits_1^\infty f_n(x)$ is not uniformly convergent on $[0, 1]$, since uniform convergence of the series implies the equality of $\sum\limits_1^\infty \int_0^1 f_n(x)dx$ and $\int_0^1 (\sum\limits_1^\infty f_n(x))dx$.

4. If $f(x)$ be the sum of the series $e^{-x}+2e^{-2x}+3e^{-3x}+\cdots, x>0$ show that f is continuous for all $x>0$. Evaluate $\int_{\log 2}^{\log 3} f(x)dx$.

Let the series be $\sum\limits_{n=1}^{\infty} f_n(x)$. Then $f_n(x)=ne^{-nx}$.

$|f_n(x)| = \frac{n}{e^{nx}} < \frac{6n}{n^3x^3}$ for all $x>0$.

Let $[a,b]$ be a closed and bounded interval $\subset (0,\infty)$.

For all $x\in[a,b]$, $|f_n(x)| < \frac{6}{a^3n^3}$ for all $n\in\mathbb{N}$.

Let $M_n = \frac{6}{a^3n^3}$. Then ΣM_n is a convergent series of positive real numbers.

By Weierstrass' M-test, Σf_n is uniformly convergent on $[a,b]$.

Let $c>0$. Let us choose a positive δ such that $c-\delta>0$.

Then Σf_n is uniformly convergent on $[c-\delta, c+\delta]$.

Since each f_n is continuous on $[c-\delta,c+\delta]$, the sum function f is continuous on $[c-\delta,c+\delta]$. Hence f is continuous at c. It follows that f is continuous for all $x>0$.

Let $a=\log 2, b=\log 3$. Then $[a,b]$ is a closed and bounded interval $\subset (0,\infty)$.

Each f_n is integrable on $[a,b]$. Since the series is uniformly convergent on $[a,b]$,

$\int_a^b f(x)dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx + \int_a^b f_3(x)dx + \cdots$

$$\begin{aligned} \text{That is, } \int_{\log 2}^{\log 3} f(x)dx &= \int_{\log 2}^{\log 3} e^{-x}dx + \int_{\log 2}^{\log 3} 2e^{-2x}dx + \cdots \\ &= (\tfrac{1}{2}-\tfrac{1}{3}) + (\tfrac{1}{2^2}-\tfrac{1}{3^2}) + (\tfrac{1}{2^3}-\tfrac{1}{3^3}) + \cdots \\ &= (1+\tfrac{1}{2}+\tfrac{1}{2^2}+\cdots) - (1+\tfrac{1}{3}+\tfrac{1}{3^2}+\cdots) \\ &= 2-\tfrac{3}{2} = \tfrac{1}{2}. \end{aligned}$$

Theorem 14.2.4. Let $[a,b]$ be a closed and bounded interval and for each $n\in\mathbb{N}$, let f_n be differentiable on $[a,b]$. If the series of functions $f_1'+f_2'+f_3'+\cdots$ converges uniformly on $[a,b]$ to a function g and the series $f_1+f_2+f_3+\cdots$ converges *at least at one point* $x_0\in[a,b]$, then the series $f_1+f_2+f_3+\cdots$ converges uniformly on $[a,b]$ to a function s such that $s'(x)=g(x)$ for all $x\in[a,b]$.

Proof. Let $s_n(x)=f_1(x)+f_2(x)+\cdots+f_n(x), x\in[a,b]$.

Each s_n is differentiable on $[a,b]$ and $s_n'(x)=f_1'(x)+f_2'(x)+\cdots+f_n'(x)$.

Since $\Sigma f_n'$ converges uniformly on $[a,b]$ to g, the sequence $\{s_n'\}$ converges uniformly to g on $[a,b]$.

Since Σf_n converges at x_0, the sequence $\{s_n\}$ converges at x_0.

Hence by the Theorem 13.4.6, the sequence $\{s_n\}$ is uniformly convergent on $[a,b]$ to a function s such that $s'(x)=g(x)$ for all $x\in[a,b]$.

This implies that the series Σf_n is uniformly convergent on $[a, b]$ to a function s such that $s'(x) = g(x)$ for all $x \in [a, b]$.

Note 1. The theorem says that under the conditions stated, $f_1(x) + f_2(x) + f_3(x + \cdots = s(x)$ for all $x \in [a, b]$ and

$\frac{d}{dx} f_1(x) + \frac{d}{dx} f_2(x) + \frac{d}{dx} f_3(x) + \cdots = \frac{d}{dx} s(x)$ for all $x \in [a, b]$.

That is, $\frac{d}{dx}[f_1(x)+f_2(x)+f_3(x)+\cdots] = \frac{d}{dx} f_1(x)+\frac{d}{dx} f_2(x)+\frac{d}{dx} f_3(x)+ \cdots$ for all $x \in [a, b]$.

In other words, term-by-term differentiation of the series of functions is valid under the conditions.

Note 2. Only the uniform convergence of the series of functions $f_1(x) + f_2(x) + f_3(x) + \cdots$ on $[a, b]$ is not sufficient to ensure validity of term-by-term differentiation of the series on $[a, b]$.

For example, let the series be $f_1(x) + f_2(x) + f_3(x) + \cdots, x \in [0, 1]$ such that $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x) = \frac{x}{1+nx^2}, x \in [0, 1]$.

$$\lim_{n\to\infty} s_n(x) = \lim_{n\to\infty} \frac{x}{1+nx^2}$$
$$= 0 \text{ for all } x \in [0, 1].$$

The sequence $\{s_n\}$ converges to the function s where $s(x) = 0$.

Let $M_n = \sup\limits_{x\in[0,1]} | s_n(x) - s(x) |$. Then $M_n = \sup\limits_{x\in[0,1]} \frac{x}{1+nx^2}$.

$M_n = \frac{1}{2\sqrt{n}}$ [by Example 5, 13.3.2.] and therefore $\lim\limits_{n\to\infty} M_n = 0$.

This implies that the convergence of the sequence $\{s_n\}$ is uniform on $[0, 1]$. Hence the series $f_1(x) + f_2(x) + \cdots$ converges uniformly to the function s on $[0, 1]$.

$\frac{d}{dx} s_n(x) = \frac{1-nx^2}{(1+nx^2)^2}$. $\lim\limits_{n\to\infty} s'_n(x) = 0, 0 < x \leq 1$
$= 1, x = 0.$

Therefore the series $f'_1(x) + f'_2(x) + \cdots$ converges to the function g where $g(x) = 0, 0 < x \leq 1$
$= 1, x = 0.$

Hence $\frac{d}{dx} f_1(x) + \frac{d}{dx} f_2(x) + \cdots = 0 = \frac{d}{dx}[f_1(x) + f_2(x) + \cdots]$ for $0 < x \leq 1$; and $\frac{d}{dx} f_1(x) + \frac{d}{dx} f_2(x) + \cdots = 1 \neq \frac{d}{dx}[f_1(x) + f_2(x) + \cdots]$ for $x = 0$.

Note 3. If the series $f_1(x) + f_2(x) + f_3(x) + \cdots$ be convergent, then the uniform convergence of the series $f'_1(x) + f'_2(x) + f'_3(x) + \cdots$ is only a sufficient but not a necessary condition for the validity of term-by-term differentiation of the series $f_1(x) + f_2(x) + f_3(x) + \cdots$

For example, let the series be

$f_1(x) + f_2(x) + f_3(x) + \cdots, x \in [0,1]$ such that
$s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x) = \frac{\log(1+n^2x^2)}{2n}, x \in [0,1]$.
$\lim_{n\to\infty} s_n(x) = \lim_{n\to\infty} \frac{\log(1+n^2x^2)}{2n} = 0$, for all $x \in [0,1]$.

The sequence $\{s_n\}$ converges to the function s where $s(x) = 0, x \in [0,1]$. Hence the series $f_1(x) + f_2(x) + \cdots$ converges to the function s on $[0,1]$.

$s_n'(x) = f_1'(x) + f_2'(x) + \cdots + f_n'(x) = \frac{nx}{1+n^2x^2}, x \in [0,1]$.
$\lim_{n\to\infty} s_n'(x) = 0$, for all $x \in [0,1]$.

The sequence $\{s_n'\}$ converges to the function g where $g(x) = 0, x \in [0,1]$. Hence the series $f_1'(x) + f_2'(x) + \cdots$ converges to the function $g(x)$ on $[0,1]$.

Now $\frac{d}{dx}s(x) = 0, x \in [0,1]$. Therefore $\frac{d}{dx}s(x) = g(x), x \in [0,1]$.
Hence $\frac{d}{dx}f_1(x) + \frac{d}{dx}f_2(x) + \cdots = \frac{d}{dx}[f_1(x) + f_2(x) + \cdots]$.

This shows that term-by-term differentiation of the series Σf_n is valid.

But the convergence of the series $\Sigma f_n'$ is not uniform on $[0,1]$ since the convergence of the sequence $\{s_n'\}$ is not uniform on $[0,1]$ by worked Example 3, 13.3.2.

Theorem 14.2.5. Let $[a,b]$ be a closed and bounded interval and for each $n \in \mathbb{N}$, let f_n be differentiable on $[a,b]$. If each f_n' be continuous on $[a,b]$ and the series of functions $f_1' + f_2' + f_3' + \cdots$ converges uniformly on $[a,b]$ to a function g and the series $f_1 + f_2 + f_3 + \cdots$ converges to s on $[a,b]$, then $s'(x) = g(x)$ for all $x \in [a,b]$.

Proof. Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x), x \in [a,b]$.

Each s_n is differentiable on $[a,b]$ and $s_n'(x) = f_1'(x) + f_2'(x) + \cdots + f_n'(x)$.

Since each $f_n'(x)$ is continuous on $[a,b]$, each s_n' is continuous on $[a,b]$.

Since the series $\Sigma f_n'$ is uniformly convergent to g on $[a,b]$, the sequence $\{s_n'\}$ converges uniformly to g on $[a,b]$.

Since s_n' is continuous on $[a,b]$, g is continuous on $[a,b]$.

Therefore each s_n' is integrable on $[a,b]$ and g is also integrable on $[a,b]$.

By the corollary of the Theorem 13.4.5, for each $x \in [a,b]$
$\lim_{n\to\infty} \int_a^x s_n'(x)dx = \int_a^x g(x)dx$ ... ... (i)
But $\int_a^x s_n'(x)dx = s_n(x) - s_n(a)$ by the fundamental theorem.
Therefore $\lim_{n\to\infty} \int_a^x s_n'(x)dx = s(x) - s(a)$.
From (i) $s(x) - s(a) = \int_a^x g(x)dx$.
Since g is continuous on $[a,b]$, $s'(x) = g(x)$ for all $x \in [a,b]$.

Worked Examples (continued).

5. Let $f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}, x \in [0,1]$.
Show that at $x=0$, $\frac{d}{dx}\Sigma f_n(x) \neq \Sigma \frac{d}{dx} f_n(x)$.

Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$.
Then $s_n(x) = \frac{nx}{1+n^2x^2}$. $\lim_{n\to\infty} s_n(x) = 0$ for all $x \in [0,1]$.

The sequence $\{s_n\}$ converges to the function s where $s(x) = 0, x \in [0,1]$. Therefore the series $\Sigma f_n(x)$ converges to $s(x)$ for all $x \in [0,1]$.

$\frac{d}{dx}(\Sigma f_n(x)) = \frac{d}{dx}s(x) = 0$ for all $x \in [0,1]$.

$\frac{d}{dx} f_n(x) = \frac{n-n^2x^2}{(1+n^2x^2)} - \frac{(n-1)-(n-1)^2x^2}{[1+(n-1)^2x^2]^2}$.
At $x=0$, $\frac{d}{dx}f_n(x) = n-(n-1) = 1$.
At $x=0, \Sigma \frac{d}{dx} f_n(x) = 1+1+1+\cdots$ This is divergent.

Hence $\frac{d}{dx}\Sigma f_n(x) \neq \Sigma \frac{d}{dx} f_n(x)$ at $x=0$.

6. Show that the series $\Sigma \frac{1}{n^3+n^4x^2}$ is uniformly convergent for all real x. If $s(x)$ be the sum function verify that $s'(x)$ is obtained by term-by-term differentiation.

Let $f_n(x) = \frac{1}{n^3+n^4x^2}, x \in \mathbb{R}$.
For all $x \in \mathbb{R}, \mid f_n(x) \mid \leq \frac{1}{n^3}$ for all $n \in \mathbb{N}$.

Let $M_n = \frac{1}{n^3}$. Then ΣM_n is a convergent series of positive terms.

By Weierstrass' M-test, Σf_n is uniformly convergent for all real x.

$f_n'(x) = \frac{-2x}{n^2(1+nx^2)^2} = u(x)$, say. Then $u'(x) = \frac{2(3nx^2-1)}{n^2(1+nx^2)^3}$.

$u'(x) = 0$ at $x = \pm\frac{1}{\sqrt{3n}}, u'(x) < 0$ for $\mid x \mid < \frac{1}{\sqrt{3n}}, u'(x) > 0$ for $\mid x \mid > \frac{1}{\sqrt{3n}}$.

Therefore u is a minimum at $\frac{1}{\sqrt{3n}}$ and maximum at $= \frac{-1}{\sqrt{3n}}$.

$u_{\max} = \frac{9}{8\sqrt{3}}\cdot\frac{1}{n^{5/2}}, u_{\min} = \frac{-9}{8\sqrt{3}}\cdot\frac{1}{n^{5/2}}$

$u(0) = 0, u$ is decreasing on $(0, \frac{1}{\sqrt{3n}}), u$ is a minimum at $\frac{1}{\sqrt{3n}}, u$ is increasing for $x > \frac{1}{\sqrt{3n}}$ and $\lim_{x\to\infty} u(x) = 0$.

Since u is an odd function, it follows that for all real x,

$$\mid f_n'(x) \mid \leq \frac{3\sqrt{3}}{8}\cdot\frac{1}{n^{5/2}}.$$

By Weierstrass' M-test, $\Sigma f_n'$ is uniformly convergent for all real x.

This ensures validity of term-by-term differentiation of the series $\Sigma f_n(x)$. Therefore $f_1'(x) + f_2'(x) + \cdots = s'(x)$.

14.3. Abel's and Dirichlet's tests.

Definition. A sequence of functions $\{f_n\}$ is said to be *uniformly bounded* on an interval I if there exists a positive real number k such that for all $x \in I$, $| f_n(x) | < k$ for all $n \in \mathbb{N}$.

For example, the sequence $\{f_n\}$ where $f_n(x) = \sin nx, x \in \mathbb{R}$ is uniformly bounded on $\mathbb{R}$.

The sequence $\{f_n\}$ where $f_n = \frac{nx}{1+x}, x \in [0, \infty)$ is not uniformly bounded on $[0, \infty)$ although each f_n is bounded on $[0, \infty)$.

Theorem 14.3.1. (Abel's test)

Let (i) the series of functions Σu_n be uniformly convergent on $[a, b]$

and (ii) the sequence of functions $\{v_n\}$ be monotonic for every $x \in [a, b]$ and uniformly bounded on $[a, b]$.

Then the series $u_1(x)v_1(x)+u_2(x)v_2(x)+u_3(x)v_3(x)+\cdots$ is uniformly convergent on $[a, b]$.

Proof. Since the sequence $\{v_n\}$ is uniformly bounded on $[a, b]$, there is a positive real number B such that $| v_n(x) | < B$ for all $[a, b]$ and for all $n \in \mathbb{N}$.

Let us choose $\epsilon > 0$. Since the series Σu_n is uniformly convergent on $[a, b]$, there exists a natural number k such that for all $x \in [a, b]$,

$| u_{n+1}(x) + u_{n+2}(x) + \cdots + u_{n+p}(x) | < \frac{\epsilon}{3B}$ for all $n \geq k, p = 1, 2, 3, \ldots$

Let $R_{n,p}(x) = u_{n+1}(x) + u_{n+2}(x) + \cdots + u_{n+p}(x)$.

Then $| u_{n+1}(x)v_{n+1}(x) + u_{n+2}(x)v_{n+2}(x) + \cdots + u_{n+p}(x)v_{n+p}(x) |$

$= | R_{n,1}(x)v_{n+1}(x) + \{R_{n,2}(x) - R_{n,1}(x)\}v_{n+2}(x) + \cdots + \{R_{n,p}(x) - R_{n,p-1}(x)\}v_{n+p}(x) |$

$= | R_{n,1}(x)\{v_{n+1}(x) - v_{n+2}(x)\} + R_{n,2}(x)\{v_{n+2}(x) - v_{n+3}(x)\} + \cdots + R_{n,p-1}(x)\{v_{n+p-1}(x) - v_{n+p}(x)\} + R_{n,p}(x).v_{n+p}(x) |$

$\leq | R_{n,1}(x) || v_{n+1}(x) - v_{n+2}(x) | + | R_{n,2}(x) || v_{n+2}(x) - v_{n+3}(x) | + \cdots + | R_{n,p-1}(x) || v_{n+p-1}(x) - v_{n+p}(x) | + | R_{n,p}(x) || v_{n+p}(x) |$

$< \frac{\epsilon}{3B}[| v_{n+1}(x) - v_{n+2}(x) | + | v_{n+2}(x) - v_{n+3}(x) | + \cdots | v_{n+p-1}(x) - v_{n+p}(x) | + | v_{n+p}(x) |]$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

Since $\{v_n\}$ is monotonic for every $x \in [a, b]$,

$| v_{n+1}(x) - v_{n+2}(x) | + \cdots + | v_{n+p-1}(x) - v_{n+p}(x) |$

$= | v_{n+1}(x) - v_{n+p}(x) | \leq | v_{n+1}(x) | + | v_{n+p}(x) |$.

Therefore $| u_{n+1}(x)v_{n+1}(x) + \cdots + u_{n+p}(x)v_{n+p}(x) | < \frac{\epsilon}{3B}.2B + \frac{\epsilon}{3B}.B$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

Hence for all $x \in [a, b]$, $| u_{n+1}(x)v_{n+1}(x) + \cdots + u_{n+p}(x)v_{n+p}(x) | < \epsilon$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

This proves that the series $u_1(x)v_1(x) + u_2(x)v_2(x) + \cdots$ is uniformly convergent on $[a, b]$.

Worked Examples.

1. If $a_0 + a_1 + a_2 + \cdots$ be a convergent series of real numbers prove that the series $a_0 + a_1x + a_2x^2 + \cdots$ is uniformly convergent on $[0, 1]$.

Let $v_n(x) = x^n, x \in [0, 1]$.

Then $| v_n(x) | \leq 1$ for all $x \in [0, 1]$ and all $n \in \mathbb{N}$.

The sequence $\{v_n\}$ is monotonic for every fixed $x \in [0, 1]$.

The series $a_0 + a_1 + a_2 \cdots$ being a convergent series of real numbers is uniformly convergent on $[0, 1]$.

By Abel's test, the series $a_0 + a_1x + a_2x^2 + \cdots$ is uniformly convergent on $[0, 1]$.

2. If $a_1 + a_2 + a_3 + \cdots$ be a convergent series of real numbers prove that the series $a_1 + \frac{a_2}{2^x} + \frac{a_3}{3^x} + \cdots$ is uniformly convergent on $[0, \infty)$.

Let $v_n(x) = \frac{1}{n^x}, x \geq 0$.

For a fixed $x \in [0, \infty)$ the sequence $\{v_n\}$ is monotonic and for all $x \geq 0, | v_n(x) | = \frac{1}{n^x} \leq 1$ for all $n \in \mathbb{N}$.

The series $a_1 + a_2 + a_3 + \cdots$ being a convergent series of real numbers is uniformly convergent on $[0, \infty)$.

By Abel's test, the series $a_1 + \frac{a_2}{2^x} + \frac{a_3}{3^x} + \cdots$ is uniformly convergent on $[0, \infty)$.

3. Prove that the series $e^{-x} - \frac{e^{-2x}}{2} + \frac{e^{-3x}}{3} - \frac{e^{-4x}}{4} + \cdots$ is uniformly convergent on $[0, 1]$.

Let $v_n(x) = e^{-nx}, x \in [0, 1]$.

For each $x \in [0, 1]$, the sequence $\{v_n\}$ is monotonic, and for all $x \in [0, 1], | v_n(x) | = \frac{1}{e^{nx}} \leq 1$ for all $n \in \mathbb{N}$.

The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is a convergent series of real numbers and therefore it is uniformly convergent on $[0, 1]$.

By Abel's test, the series $1 - \frac{1}{2}e^{-2x} + \frac{1}{3}e^{-3x} - \frac{1}{4}e^{-4x} + \cdots$ is uniformly convergent on $[0, 1]$.

4. Prove that the series $\Sigma \frac{(-1)^{n-1}x^n}{n(1+x^n)}$ is uniformly convergent on $[0, 1]$.

Let $v_n(x) = \frac{x^n}{1+x^n}, x \in [0, 1]$.

Then $v_{n+1} - v_n = \frac{x^n(x-1)}{(1+x^n)(1+x^{n+1})} \leq 0$ for all $x \in [0, 1]$.

For each $x \in [0,1]$, the sequence $\{v_n\}$ is monotonic and for all $x \in [0,1], | v_n(x) |< 1$ for all $n \in \mathbb{N}$.

The series $\Sigma \frac{(-)^{n-1}}{n}$ is a convergent series of real numbers and therefore it is uniformly convergent on $[0,1]$.

By Abel's test, the series $\Sigma \frac{(-1)^{n-1} x^n}{n(1+x^n)}$ is uniformly convergent on $[0,1]$.

Theorem 14.3.2. (Dirichlet's test)

Let (i) the sequence of partial sums $\{s_n\}$ of the series of functions $u_1(x)+u_2(x)+u_3(x)+\cdots$ be uniformly bounded on $[a,b]$

(ii) the sequence of functions $\{v_n\}$ be monotonic for every $x \in [a,b]$,

(iii) the sequence $\{v_n\}$ be uniformly convergent to 0 on $[a,b]$.

Then the series of functions $u_1(x)v_1(x)+u_2(x)v_2(x)+u_3(x)v_3(x)+\cdots$ is uniformly convergent on $[a,b]$.

Proof. Since the sequence $\{s_n\}$ is uniformly bounded on $[a,b]$, there exists a positive number B such that for all $x \in [a,b], | s_n(x) |< B$ for all $n \in \mathbb{N}$.

Let us choose $\epsilon > 0$.

Since the sequence $\{v_n\}$ converges uniformly to 0 on $[a,b]$, there exists a natural number k such that for all $x \in [a,b], | v_n(x) |< \frac{\epsilon}{4B}$ for all $n \geq k$.

Now $u_{n+1}(x)v_{n+1}(x)+u_{n+2}(x)v_{n+2}(x)+\cdots+u_{n+p}(x)v_{n+p}(x)$

$= [s_{n+1}(x)-s_n(x)]v_{n+1}(x)+[s_{n+2}(x)-s_{n+1}(x)]v_{n+2}(x)+\cdots+[s_{n+p}(x)-s_{n+p-1}(x)].v_{n+p}(x)$

$= s_{n+1}(x)[v_{n+1}(x)-v_{n+2}(x)]+\cdots+s_{n+p-1}(x)[v_{n+p-1}(x)-v_{n+p}(x)]+s_{n+p}(x)v_{n+p}(x)-s_n(x)v_{n+1}(x)$.

For all $x \in [a,b], | u_{n+1}(x)v_{n+1}(x)+\cdots+u_{n+p}(x)v_{n+p}(x) |$

$\leq | s_{n+1}(x) | \, | v_{n+1}(x)-v_{n+2}(x) | +\cdots+ | s_{n+p-1}(x) | \, | v_{n+p-1}(x)-v_{n+p}(x) | + | s_{n+p}(n) | \, | v_{n+p}(x) | + | s_n(x) | v_{n+1}(x) |$

$< B[| v_{n+1}(x)-v_{n+2}(x) | +\cdots+ | v_{n+p-1}(x)-v_{n+p}(x) | + | v_{n+p}(x) | + | v_{n+1}(x) |]$.

Since $\{v_n\}$ is monotonic for every $x \in [a,b]$,
$| v_{n+1}(x)-v_{n+2}(x) | +\cdots+ | v_{n+p-1}(x)-v_{n+p}(x) |$
$=| v_{n+1}(x)-v_{n+p}(x) |< \frac{\epsilon}{2B}$ for all $n \geq k$ and $p=1,2,3,\ldots$

Therefore for all $x \in [a,b]$,
$| u_{n+1}(x)v_{n+1}(x)+\cdots+u_{n+p}(x)v_{n+p}(x) |< B.\frac{\epsilon}{2B}+2B.\frac{\epsilon}{4B}(=\epsilon)$
for all $n \geq k$ and $p=1,2,3,\ldots$

This proves that the series $u_1(x)v_1(x) + u_2(x)v_2(x) + \cdots$ is uniformly convergent on $[a, b]$.

Worked Examples (continued).

5. Prove that the series $\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots$ is uniformly convergent on any closed interval $[a, b]$ contained in the open interval $(0, 2\pi)$.

Let $u_n(x) = \sin nx, x \in [a, b] \subset (0, 2\pi)$; and $v_n = \frac{1}{n}$.
Then $\{v_n\}$ is a monotone decreasing sequence converging to 0.
Let $s_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x)$.

Then $s_n(x) = \sin x + \sin 2x + \cdots + \sin nx$
$= \dfrac{\sin \frac{n\pi}{2} \sin \frac{(n+1)\pi}{2}}{\sin \frac{x}{2}}$.

For each $n \in \mathbb{N}$, $| s_n(x) | \leq | \frac{1}{\sin x/2} |$.

$\sin \frac{x}{2} \neq 0$ for all $x \in [a, b] \subset (0, 2\pi)$. Therefore the function f defined by $f(x) = \frac{1}{\sin \frac{x}{2}}, x \in [a, b]$ is continuous on $[a, b]$ and therefore it is bounded on $[a, b]$. Then there exists a positive real number k such that $| \frac{1}{\sin \frac{x}{2}} | \leq k$ for all $x \in [a, b] \subset (0, 2\pi)$.

This shows that the sequence $\{s_n\}$ is uniformly bounded on $[a, b]$.

By Dirichlet's test, the series $\Sigma u_n v_n$, i.e., the series $\Sigma \frac{\sin nx}{n}$ is uniformly convergent on $[a, b] \subset (0, 2\pi)$.

6. Prove that the series $\Sigma(-1)^n x^n(1 - x)$ converges uniformly on $[0, 1]$, but the series $\Sigma x^n(1 - x)$ is not uniformly convergent on $[0, 1]$.

Let $u_n = (-1)^n, v_n = x^n(1 - x), x \in [0, 1]$.
Let $s_n = u_1 + u_2 + \cdots + u_n$. Then the sequence $\{s_n\}$ is bounded.

$v_{n+1} - v_n = x^{n+1}(1 - x) - x^n(1 - x) = -x^n(1 - x)^2 \leq 0$ for all $x \in [0, 1]$.

This implies $v_{n+1}(x) \leq v_n(x)$ for each x in $[0, 1]$.

$\lim_{n \to \infty} v_n(x) = 0$ for all $x \in [0, 1]$.

Then the sequence of functions $\{v_n\}$ is such that each v_n is continuous on $[0, 1]$, the sequence converges to a continuous function on $[0, 1]$ and $\{v_n\}$ is a monotone decreasing sequence on $[0, 1]$.

By Dini's theorem, the sequence $\{v_n\}$ is uniformly convergent on $[0, 1]$.

Since (i) the sequence $\{s_n\}$ is bounded and (ii) the sequence $\{v_n\}$ is a monotone decreasing sequence on $[0, 1]$ converging uniformly to 0, the series $\Sigma(-1)^n v_n$ is uniformly convergent on $[0, 1]$, by Dirichlet's test.

Second part. Let $v_n(x) = x^n(1-x), x \in [0,1]$.

Then $v_1(x) + v_2(x) + \cdots + v_n(x) = (x + x^2 + \cdots + x^n)(1-x)$.

Let $s_n(x) = v_1(x) + \cdots + v_n(x)$. Then $s_n(x) = x(1-x^n)$ if $x \neq 1$
$= 0$ if $x = 1$.

$$\lim_{n\to\infty} s_n(x) = x \text{ if } x \neq 1$$
$$= 0 \text{ if } x = 1.$$

The sequence $\{s_n\}$ converges to a function discontinuous on $[0,1]$, while each s_n is continuous on $[0,1]$.

This implies that the sequence $\{s_n\}$ not uniformly convergent on $[0,1]$ and consequently the series Σv_n is not uniformly convergent on $[0,1]$.

Note. This example shows that the uniform convergence of a series $\Sigma f_n(x)$ does not necessarily imply uniform convergence of the series $\Sigma \mid f_n(x) \mid$.

7. Prove that the series $\Sigma(-1)^n \frac{x^2+n}{n^2}$ converges uniformly in any closed and bounded interval $[a,b]$, but does not converge absolutely for any real x.

Let $u_n = (-1)^n, v_n = \frac{x^2+n}{n^2}, x \in [a,b]$.

Let $s_n = u_1 + u_2 + \cdots + u_n$. Then the sequence $\{s_n\}$ is bounded.

$v_{n+1} - v_n = \frac{x^2+n+1}{(n+1)^2} - \frac{x^2+n}{n^2} = x^2[\frac{1}{(n+1)^2} - \frac{1}{n^2}] + [\frac{1}{n+1} - \frac{1}{n}] < 0$ for all $x \in [a,b]$.

This shows that $\{v_n\}$ is a monotone decreasing sequence for each x in $[a,b]$.

$\lim_{n\to\infty} v_n(x) = 0$ for all $x \in [a,b]$.

Thus the sequence of functions $\{v_n\}$ is such that each v_n is continuous on $[a,b]$, the sequence converges to a continuous function on $[a,b]$ and $\{v_n\}$ is a monotone decreasing sequence on $[a,b]$.

By Dini's theorem, the sequence $\{v_n\}$ is uniformly convergent on $[a,b]$.

Since (i) the sequence $\{s_n\}$ is uniformly bounded on $[a,b]$ and (ii) the sequence $\{v_n\}$ is a monotone decreasing sequence on $[a,b]$ and converges uniformly to 0, the series $\Sigma(-1)^n v_n$ is uniformly convergent on $[a,b]$, by Dirichlet's test.

Second part. Let the series be $(-1)^n v_n(x)$.

For each real x, the series $\Sigma v_n(x)$ is a series of positive terms.

Let $w_n = \frac{1}{n}$. Then $\lim_{n\to\infty} \frac{v_n}{u_n} = 1$.

By comparison test, the series $\Sigma v_n(x)$ is divergent for each real x.

Since $\{v_n\}$ is a monotone decreasing sequence for each $x \in \mathbb{R}$ and $\lim\limits_{n\to\infty} v_n(x) = 0$ for each $x \in \mathbb{R}$, the series $\Sigma(-1)^{n-1}v_n(x)$ is convergent for each real x.

Therefore the series $\Sigma(-1)^{n-1}\frac{x^2+n}{n^2}$ does not converge absolutely for any real x.

Exercises 25

1. If $\Sigma u_n(x)$ be uniformly convergent on $[a, b]$ and $v : [a, b] \to \mathbb{R}$ be a bounded function on $[a, b]$, prove that $\Sigma u_n(x)v(x)$ is uniformly convergent on $[a, b]$.

2. Prove that the series $\sum\limits_1^\infty f_n(x)$ where $f_n(x) = n^2x^2e^{-n^2x^2} - (n-1)^2x^2 e^{-(n-1)^2x^2}, x \in [0, 1]$ is not uniformly convergent on $[0, 1]$.

3. Prove that the series $(1-x)^2 + x(1-x)^2 + x^2(1-x)^2 + \cdots$ is uniformly convergent on $[0, 1]$.

4. Show that the series $x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots$ is not uniformly convergent on $[0, 1]$.

5. Show that the following series are uniformly convergent for all real x.

   (i) $\sum\limits_1^\infty \dfrac{1}{(n+x^2)^2}$, (ii) $\sum\limits_1^\infty r^n \cos nx, 0 < r < 1$,

   (iii) $\sum\limits_1^\infty r^n \sin nx, 0 < r < 1$, (iv) $\sum\limits_1^\infty \dfrac{(-1)^{n-1}x^{2n}}{n^2(1+x^{2n})}$,

   (v) $\sum\limits_1^\infty \dfrac{x}{n(1+nx^2)}$.

6. If $p \neq 0, \neq \pm 1, \neq \pm 2, \cdots$ prove that the series $\frac{1}{p^2} - \frac{\cos x}{p^2-1^2} + \frac{\cos 2x}{p^2-2^2} - \frac{\cos 3x}{p^2-3^2} + \cdots$ is uniformly convergent on any closed and bounded interval $[a, b]$.

[**Hint.** For a fixed p, there exists a natural number k such that $n^2 > 2p^2$ for all $n \geq k$. Then for all $n \geq k$, $|\frac{(-1)^n \cos nx}{p^2-n^2}| \leq \frac{1}{n^2-p^2} < \frac{2}{n^2}$.]

7. Show that the series $\sum\limits_1^\infty \dfrac{1}{n^2 + [f(x)]^2}$ is uniformly convergent on any set $D \subset \mathbb{R}$ on which f is defined.

8. Show that the series $\sum\limits_1^\infty \dfrac{nx^2}{n^3+x^3}$ is uniformly convergent on $[0, k]$ for any $k > 0$.

9. Show that both the series $\sum\limits_0^\infty \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$ and $\sum\limits_1^\infty \dfrac{(-1)^n x^{2n}}{(2n)!}$ are uniformly convergent on any closed and bounded interval $[a, b]$.

10. Show that the series $\sum\limits_1^\infty \dfrac{n^5+1}{n^7+3}(\dfrac{x}{3})^n$ is absolutely and uniformly convergent on $[-3,3]$.

11. If Σa_n be an absolutely convergent series of real numbers prove that the series

(i) $\Sigma \frac{a_n x^n}{1+x^{2n}}$ is absolutely and uniformly convergent for all real x;

(ii) $\Sigma a_n \sin nx$ is absolutely and uniformly convergent for all real x;

(iii) $\Sigma a_n x^n$ is absolutely and uniformly convergent on $[-1,1]$.

12. Let $\sum\limits_1^\infty f_n(x)$ be uniformly convergent to $f(x)$ on $[a,b]$ where each f_n is continuous on $[a,b]$. If $g:[a,b]\to\mathbb{R}$ be integrable on $[a,b]$, prove that

$$\int_a^b f(x)g(x)dx = \sum_1^\infty \int_a^b f_n(x)g(x)dx.$$

13. If $f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}, x\in[0,1]$, show that $\int_0^1(\sum\limits_1^\infty f_n(x))dx = \sum\limits_1^\infty (\int_0^1 f_n(x)dx)$, although the series $\sum\limits_1^\infty f_n(x)$ is not uniformly convergent on $[0,1]$.

14. Prove that the series is uniformly convergent for all real x. Show that the derivative of the sum function can be obtained by differentiating the series term-by-term.

(i) $\sum\limits_1^\infty \frac{1}{n^2+nx^2}$, (ii) $\sum\limits_1^\infty \frac{1}{n^2+n^3x^2}$.

15. Show that the series $\sum\limits_1^\infty \frac{\sin nx}{n^3}$ is uniformly convergent for all real x and the derivative of the sum function $s(x)$ can be obtained by differentiating the series term-by-term, i.e., $\sum\limits_1^\infty \frac{\cos nx}{n^2} = s'(x)$.

16. Prove that the series $\sum\limits_{n=1}^\infty \frac{\sin nx}{n^2}$ is uniformly convergent for every real x. If $f(x)$ be the sum of the series prove that f is continuous on $\mathbb{R}$.

Also prove that $\int_0^\pi f(x)dx = 2(1+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\cdots\cdots)$.

17. Prove that the series $\sum\limits_1^\infty \frac{x}{n(x+n)}$ is uniformly convergent on $[0,1]$. If $f(x)$ be the sum of the series prove that f is continuous on $[0,1]$.

Also prove that $\int_0^1 f(x)dx = \gamma$, where $\gamma = \lim\limits_{n\to\infty}(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n)$.

18. If Σa_n is a convergent series of real numbers prove that the series

(i) $\Sigma a_n e^{-nx}$ is uniformly convergent on $[0,\infty)$;

(ii) $\Sigma \frac{a_n}{n^x}$ is uniformly convergent on $[0,1]$.

19. Prove that the series $\sum\limits_{1}^{\infty} \frac{\cos nx}{n^p}$ and $\sum\limits_{1}^{\infty} \frac{\sin nx}{n^p}$ both converge uniformly for all p satisfying $0 < p \leq 1$ on any closed interval $[a, b]$ contained in $[0, 2\pi)$.

[**Hint.** Use Dirichlet's test.]

20. Prove that the series $\sum\limits_{1}^{\infty} \frac{(-1)^{n-1} x^n}{n^p(1+x^n)}$ converges uniformly for all p satisfying $p > 0$ on $[0, 1]$.

[**Hint.** If $p > 1$, use Weierstrass' M-test. If $0 < p \leq 1$, use Abel's test.]

21. Prove that the series $\sum\limits_{1}^{\infty} \frac{\cos nx}{\{\log(n+1)\}^x}$ is uniformly convergent on any closed interval $[a, b]$ lying within $(0, 2\pi)$.

15. POWER SERIES

15.1. Introduction.

We now study an important class of series of functions, called a power series.

A series of the form $a_0 + a_1x + a_2x^2 + \cdots$ where $a_0, a_1, a_2, \ldots$ are real numbers, is called a *power series*. The general form of a power series is
$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$ where $a_0, a_1, a_2, \ldots, x_0 \in \mathbb{R}$.
This is called a power series about the point x_0.

The general form reduces to the form $a_0 + a_1x + a_2x^2 + \cdots$ (which is a power series about 0) by the substitution $x' = x - x_0$.

To study the nature and properties of a power series we shall consider the power series about 0, i.e., a series of the form

$$a_0 + a_1x + a_2x^2 + \cdots$$

This is denoted by $\sum\limits_{n=0}^{\infty} a_nx^n$. It is a series of functions $\sum\limits_{n=0}^{\infty} f_n(x)$ where, for $n = 0, 1, 2, \ldots, f_n(x) = a_nx^n, x \in \mathbb{R}$.

Although each f_n is defined for all real x, it is not expected that the series $\sum\limits_{n=0}^{\infty} a_nx^n$ will converge for all real x.

For example,

the series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ converges for all real x; while
the series $1 + x + x^2 + \cdots$ converges only for all $x \in (-1, 1)$; and
the series $1 + x + 2!x^2 + 3!x^3 + \cdots$ converges only for $x = 0$.

It appears that some power series converge for all $x \in \mathbb{R}$. They are called *everywhere convergent* power series. Some power series converge *only* for $x = 0$. They are called *nowhere convergent* power series. Some power series converge for *some real* x and diverge for the others.

We shall see, however, that an arbitrary subset of $\mathbb{R}$ cannot be the precise set of points of convergence of a power series.

Note. We shall use the symbol $\sum\limits_{n=0}^{\infty} a_nx^n$ to denote the power series $a_0 + a_1x + a_2x^2 + \cdots$ and also to denote the sum of the series, when the sum exists.

Theorem 15.1.1. If a power series $a_0 + a_1x + a_2x^2 + \cdots$ converges for $x = x_1$, then the series converges absolutely for all real x satisfying $|x| < |x_1|$.

Proof. Since the series converges for $x = x_1$, $\sum\limits_{n=0}^{\infty} a_n x_1^n$ is convergent. It follows that $\lim a_n x_1^n = 0$. Again the convergence of the sequence $\{a_n x_1^n\}$ implies that the sequence $\{a_n x_1^n\}$ is bounded.

Therefore there exists a positive real number k such that $| a_n x_1^n | \leq k$ for all $n \in \mathbb{N}$.

$|a_n x^n| = |a_n x_1^n|.|\frac{x}{x_1}|^n \leq k|\frac{x}{x_1}|^n$.

For all x satisfying $|\frac{x}{x_1}| < 1$, $\sum\limits_{n=0}^{\infty} |\frac{x}{x_1}|^n$ is a convergent series of positive real numbers.

By Comparison test, $\sum\limits_{n=0}^{\infty} | a_n x^n |$ is convergent if $|x| < |x_1|$.

Therefore $\sum\limits_{n=0}^{\infty} a_n x^n$ is absolutely convergent if $|x| < |x_1|$.

This completes the proof.

Theorem 15.1.2. If a power series $a_0 + a_1x + a_2x^2 + \cdots$ diverges for $x = x_1$, then the series diverges for all real x satisfying $|x| > |x_1|$.

Proof. Let the power series be convergent for $x = c$ such that $|c| > |x_1|$. Since the series converges for $x = c$ and $|x_1| < |c|$, by the previous theorem, the series would be convergent for $x = x_1$, a contradiction to the hypothesis.

This proves that the series is divergent for all real x satisfying $|x| > |x_1|$.

Theorem 15.1.3. If a power series $\sum\limits_{n=0}^{\infty} a_n x^n$ be neither nowhere convergent nor everywhere convergent, then there exists a positive real number R such that the series converges absolutely for all real x satisfying $| x |< R$ and diverges for all x satisfying $| x |> R$.

Proof. Since the series is neither nowhere convergent nor everywhere convergent, there exists at least one non-zero point of convergence, say $x = c$ and there exists at least one point of divergence, say $x = d$.

Let $c_1 > 0$ be such that $c_1 <| c |$ and $d_1 > 0$ be such that $d_1 >| d |$. Then c_1 is a point of convergence and d_1 is a point of divergence of the series, by Theorems 15.1.1 and 15.1.2.

We assert that $c_1 < d_1$. Because if $c_1 > d_1$ then c_1 being a point of convergence of the series, d_1 will also be a point of convergence by the Theorem 15.1.1 and a contradiction will arise.

Let I_1 be the closed and bounded interval $[c_1, d_1]$. Then the series converges at c_1 and diverges at d_1.

Let $c_1' = \frac{1}{2}(c_1 + d_1)$. If c_1' be a point of convergence of the series we select the closed subinterval $[c_1', d_1]$ and call it $[c_2, d_2]$.

If c_1' be a point of divergence of the series we select the closed subinterval $[c_1, c_1']$ and call it $[c_2, d_2]$.

Thus the closed interval $I_2 = [c_2, d_2]$ is such that

(i) c_2 is a point of convergence and d_2 is a point of divergence of the series,

(ii) $I_2 \subset I_1$, and

(iii) $| I_2 |= \frac{1}{2}(d_1 - c_1)$.

Let $c_2' = \frac{1}{2}(c_2 + d_2)$. If c_2' be a point of convergence of the series we select the closed subinterval $[c_2', d_2]$ and call it $[c_3, d_3]$.

If c_2' be a point of divergence of the series we select the closed subinterval $[c_2, c_2']$ and call it $[c_3, d_3]$.

Thus the closed interval $I_3 = [c_3, d_3]$ is such that

(i) c_3 is a point of convergence and d_3 is a point of divergence of the series,

(ii) $I_3 \subset I_2$, and

(iii) $| I_3 |= \frac{1}{2^2}(d_1 - c_1)$.

Let $c_3' = \frac{1}{2}(c_3 + d_3)$. Proceeding in a similar manner we obtain a sequence of closed and bounded intervals $\{I_n\}$ such that for every $n \in \mathbb{N}$,

(i) c_n is a point of convergence and d_n is a point of divergence of the series,

(ii) $I_{n+1} \subset I_n$, and

(iii) $| I_n |= \frac{1}{2^{n-1}}(d_1 - c_1)$.

The sequence $\{I_n\}$ is a sequence of nested intervals and $\lim | I_n |= 0$.

By Cantor's theorem, there exists one and only one point α such that $c_n \leq \alpha \leq d_n$ for all $n \in \mathbb{N}$ and $\sup\{c_n\} = \alpha = \inf\{d_n\}$.

Let x_0 be such that $0 < x_0 < \alpha$.

Since $\alpha = \sup\{c_n\}$, there exists a natural number m such that

$x_0 < c_m \leq \alpha$.

Since the power series converges at c_m and $0 < x_0 < c_m$, the power series converges for $x = x_0$.

By Theorem 15.1.1, the power series converges absolutely for all x such that $| x |< x_0$. Since x_0 is arbitrary, the power series converges for all x satisfying $| x |< \alpha$.

Let x_1 be such that $x_1 > \alpha$.

Since $\alpha = \inf\{d_n\}$, there exists a natural number k such that $\alpha \leq d_k < x_1$.

Since the power series is divergent for $x = d_k$ and $0 < d_k < x_1$, the power series diverges for $x = x_1$.

By Theorem 15.1.2, the power series diverges for all x satisfying $|x| > x_1$. Since x_1 is arbitrary, the power series diverges for all x satisfying $|x| > \alpha$.

Hence $\alpha = R$ and the theorem is proved.

Definition. R is called the *radius of convergence* of the power series $\sum\limits_{n=0}^{\infty} a_n x^n$. The open interval $(-R, R)$ is called the *interval of convergence* of the series.

Note 1. We define $R = 0$ for a power series which is nowhere convergent; and $R = \infty$ for a power series which is everywhere convergent.

Note 2. The convergence of the power series at $x = R, x = -R$ depends on the nature of the sequence $\{a_n\}$. There are power series for which both R and $-R$ are points of convergence, or both R and $-R$ are points of divergence, or one of R and $-R$ is a point of convergence and the other is a point of divergence.

15.2. Determination of the radius of convergence.

Theorem 15.2.1. (Cauchy-Hadamard)

Let $\sum\limits_{n=0}^{\infty} a_n x^n$ be a power series and let $\overline{\lim} \, |a_n|^{1/n} = \mu$. Then

(i) if $\mu = 0$, the series is everywhere convergent;

(ii) if $0 < \mu < \infty$, the series is absolutely convergent for all x satisfying $|x| < 1/\mu$ and is divergent for all x satisfying $|x| > 1/\mu$;

(iii) if $\mu = \infty$, the series is nowhere convergent.

Proof. (i) Let $x_0 \neq 0$ and $\epsilon = \frac{1}{2|x_0|}$.

Since $\overline{\lim} \, |a_n|^{1/n} = 0$, there exists a natural number k such that $|a_n|^{1/n} < \epsilon$ for all $n \geq k$. or, $|a_n x_0^n| < \frac{1}{2^n}$ for all $n \geq k$.

Since $\sum\limits_{n=0}^{\infty} \dfrac{1}{2^n}$ is a convergent series of positive terms, $\sum\limits_{n=0}^{\infty} |a_n x_0^n|$ is a convergent series, by Comparison test.

It follows that $\sum\limits_{n=0}^{\infty} a_n x_0^n$ is absolutely convergent and is therefore convergent.

As x_0 is arbitrary, the series $\sum\limits_{n=0}^{\infty} a_n x^n$ is everywhere convergent.

(ii) $\overline{\lim} \sqrt[n]{|a_n x^n|} = \overline{\lim}(\sqrt[n]{|a_n|}\,|x|) = |x|\,\mu$.

By Cauchy's root test, the series $\sum\limits_{n=0}^{\infty} |a_n x^n|$ is convergent if $|x|\mu < 1$.

Therefore if $|x| < 1/\mu$, the series $\sum\limits_{n=0}^{\infty} |a_n x^n|$ is convergent i.e., the series $\sum\limits_{n=0}^{\infty} a_n x^n$ is absolutely convergent.

If $|x|\,\mu > 1, \overline{\lim} \sqrt[n]{|a_n x^n|} = \overline{\lim} \sqrt[n]{|a_n|}.|x| > 1$.

Let $u_n = a_n x^n$. Then $\overline{\lim}\,|u_n| > 1$ and this implies $\lim |u_n| \neq 0$. Consequently, $\lim u_n \neq 0$ and it follows that $\sum\limits_{n=0}^{\infty} u_n$ is divergent.

(iii) If possible, let the series $\sum\limits_{n=0}^{\infty} a_n x^n$ is convergent for $x = x_0, (x_0 \neq 0)$. Then $\lim a_n x_0^n = 0$.

The sequence $\{a_n x_0^n\}$ being a bounded sequence, there exists a positive real number B such that $|a_n x_0^n| < B$ for all $n \in \mathbb{N}$.

This shows that the sequence $\{|a_n|^{1/n}\}$ is a bounded sequence and this contradicts that $\overline{\lim}\,|a_n|^{1/n} = \infty$.

Thus the series $\Sigma a_n x^n$ is not convergent for $x = x_0$. As x_0 is an arbitrary non-zero real number, the series $\Sigma a_n x^n$ is nowhere convergent.

This completes the proof.

Note. The radius of convergence of the series is $\frac{1}{\overline{\lim}|a_n|^{1/n}}$.

When $0 < \mu < \infty, R = \frac{1}{\mu}$; when $\mu = 0, R = \infty$; when $\mu = \infty, R = 0$.

Theorem 15.2.2. (Ratio test)

Let $\sum\limits_{n=0}^{\infty} a_n x^n$ be a power series and let $\lim \left|\frac{a_{n+1}}{a_n}\right| = \mu$. Then

(i) if $\mu = 0$ the series is everywhere convergent;

(ii) if $0 < \mu < \infty$ the series is absolutely convergent for all x satisfying $|x| < \frac{1}{\mu}$ and the series is divergent for all x satisfying $|x| > \frac{1}{\mu}$;

(iii) if $\mu = \infty$, the series is nowhere converegnt.

Proof. (i) Let $x \neq 0$ and let $u_n = a_n x^n$.

Then $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |x| = 0 < 1$.

By D'Alembert's Ratio test, the series $\Sigma|u_n|$ is convergent.

Therefore $\sum\limits_{n=0}^{\infty} a_n x^n$ is absolutely convergent for all non-zero real x.

Consequently, $\sum_{n=0}^{\infty} a_n x^n$ is convergent for all non-zero real x, i.e., the series $\sum_{n=0}^{\infty} a_n x^n$ is everywhere convergent.

(ii) Let $x \neq 0$. Then $\lim \mid \frac{a_{n+1}x^{n+1}}{a_n x^n} \mid = \mu|x|$.

By D'Alembert's ratio test, the series $\sum_{n=0}^{\infty} \mid a_n x^n \mid$ is convergent if $|x| < \frac{1}{\mu}$.

The series is convergent for $x = 0$ also.

Hence the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent for all real x satisfying $\mid x \mid < \frac{1}{\mu}$.

When $\mid x \mid > \frac{1}{\mu}$, $\lim \mid \frac{a_{n+1}x^{n+1}}{a_n x^n} \mid > 1$. Let $u_n = a_n x^n$.

Then $\lim \mid \frac{u_{n+1}}{u_n} \mid > 1$. Let $\lim \mid \frac{u_{n+1}}{u_n} \mid = l$. Then $l > 1$.

Let us choose $\epsilon > 0$ such that $l - \epsilon > 1$. There exists a natural number m such that $l - \epsilon < \mid \frac{u_{n+1}}{u_n} \mid < l + \epsilon$ for all $n \geq m$.

Therefore $\mid \frac{u_{n+1}}{u_n} \mid > 1$ for all $n \geq m$
or, $\mid u_{n+1} \mid > \mid u_n \mid$ for all $n \geq m$.

This shows that the sequence $\{\mid u_n \mid\}$ is ultimately a monotone increasing sequence of positive real numbers and therefore $\lim \mid u_n \mid$ cannot be 0.

It follows that $\lim u_n \neq 0$ and consequently $\sum_{n=0}^{\infty} a_n x^n$ is divergent for all real x satisfying $\mid x \mid > \frac{1}{\mu}$.

(iii) Let $u_n = a_n x^n$.
$\lim \mid \frac{u_{n+1}}{u_n} \mid = \lim \mid \frac{a_{n+1}}{a_n} \mid \mid x \mid = \infty$. (not considering the case, $x = 0$.)
Let us choose $G > 1$. There exists a natural number m such that $\mid \frac{u_{n+1}}{u_n} \mid > G$ for all $n \geq m$.
Therefore $\mid u_{n+1} \mid > \mid u_n \mid$ for all $n \geq m$.

Proceeding with similar arguments as in the last part of case (ii) it can be proved that $\sum_{n=0}^{\infty} a_n x^n$ is divergent for all real $x (\neq 0)$.

Thus the series $\sum_{n=0}^{\infty} a_n x^n$ is nowhere convergent.

This completes the proof.

Note 1. The radius of convergence of the power series is $\frac{1}{\lim \mid \frac{a_{n+1}}{a_n} \mid}$.

Note 2. We have $\underline{\lim} \mid \frac{a_{n+1}}{a_n} \mid \leq \underline{\lim} \mid a_n \mid^{1/n}$
$\leq \overline{\lim} \mid a_n \mid^{1/n} \leq \overline{\lim} \mid \frac{a_{n+1}}{a_n} \mid$. [Theorem 5.16.4.]

Therefore if $\lim |\frac{a_{n+1}}{a_n}|$ exists then $\lim | a_n |^{1/n}$ exists, but the converse is not true.

It follows that if the ratio test be applicable to a series then Cauchy-Hadamard test is also applicable to the series. But there are cases where Cauchy-Hadamard test is applicable while the ratio test fails to be applicable. Therefore Cauchy-Hadamard test is more powerful than the ratio test for the determination of the nature of a power series.

Worked Examples.

1. Determine the radius of convergence of the power series
$x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \cdots$
Let $\sum\limits_{n=0}^{\infty} a_nx^n$ be the given series. Then $a_0 = 0, a_n = \frac{n^n}{n!}$ for all $n \in \mathbb{N}$.
$| \frac{a_{n+1}}{a_n} |= \frac{(n+1)^{n+1}}{(n+1)!}.\frac{n!}{n^n} = (1 + \frac{1}{n})^n$ for $n \geq 1$. $\lim\limits_{n\to\infty} | \dfrac{a_{n+1}}{a_n} |= e$.
The radius of convergence of the power series is $1/e$.

2. Determine the radius of convergence of the power series
$\frac{1}{3} - x + \frac{x^2}{3^2} - x^3 + \frac{x^4}{3^4} - x^5 + \cdots$
Let $\sum\limits_{n=0}^{\infty} a_nx^n$ be the given series.
Then $a_0 = \frac{1}{3}, a_1 = -1, a_2 = \frac{1}{3^2}, a_3 = -1, a_4 = \frac{1}{3^4}, \cdots$
$\overline{\lim} | a_n |^{1/n}= 1$.
The radius of convergence of the series is 1.

3. Find the radius of convergence of the power series
$x + \frac{(2!)^2}{4!}x^2 + \frac{(3!)^2}{6!}x^3 + \cdots + \frac{(n!)^2}{(2n)!}x^n + \cdots$
Let $\sum\limits_{n=0}^{\infty} a_nx^n$ be the given series.
Then $a_0 = 0, a_1 = 1, a_n = \frac{(n!)^2}{(2n)!}$ for all $n \geq 2$.
$\lim | \frac{a_{n+1}}{a_n} |= \lim \frac{n+1}{2(2n+1)} = \frac{1}{4}$.
The radius of convergence of the series is 4.

15.3. Properties of a power series.

Theorem 15.3.1. Let $\sum\limits_{n=0}^{\infty} a_nx^n$ be a power series with radius of convergence $R(> 0)$. Then the series is uniformly convergent on $[-s, s]$, where $0 < s < R$.

Proof. Let $f_n(x) = a_nx^n, n \geq 0$.

Since R is the radius of convergence of the power series, the series is absolutely convergent for all real x satisfying $| x |< R$.

Since $0 < s < R$, the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent for all x satisfying $|x| \leq s < R$.

Therefore the series $\sum_{n=0}^{\infty} |a_n s^n|$ is convergent.

Now $|f_n(x)| = |a_n x^n| \leq |a_n| s^n$ for all real x satisfying $|x| \leq s$.

Let $M_n = |a_n| s^n$ for all $n \in \mathbb{N}$.

Then $\sum_{n=1}^{\infty} M_n$ is a convergent series of positive real numbers and for all $n \in \mathbb{N}$, $|f_n(x)| \leq M_n$ for all $x \in [-s, s]$.

By Weierstrass' M-test, the series $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on $[-s, s]$. Consequently, the series $\sum_{n=0}^{\infty} f_n(x)$, i.e., the power series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[-s, s]$.

Corollary 1. Let $R(> 0)$ be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$. Then the series is uniformly convergent on $[-R+\epsilon, R-\epsilon]$ where ϵ is an arbitrarily small positive number satisfying $R - \epsilon > 0$.

Proof. $R - \epsilon > 0$. Let $s = R - \epsilon$. Then $0 < s < R$ and therefore the power series is uniformly convergent on $[-s, s]$, i.e., on $[-R + \epsilon, R - \epsilon]$.

Corollary 2. Let $R(> 0)$ be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$. If $[a, b]$ be any closed interval contained in $(-R, R)$, then the series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[a, b]$.

Proof. Let us choose a positive ϵ such that $R - \epsilon > 0$ and $-R < -R + \epsilon < a < b < R - \epsilon < R$.

Let $R - \epsilon = s$. Then $0 < s < R$ and $-R < -s < a < b < s < R$.

Since the power series is uniformly convergent on $[-s, s]$ and $[a, b] \subset [-s, s]$, the power series is uniformly convergent on $[a, b]$.

Theorem 15.3.2. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R(> 0)$. Let $f(x)$ be the sum of the series on $(-R, R)$. Then f is continuous on $(-R, R)$.

Proof. Since R is the radius of convergence of the power series, the series is uniformly convergent on $[-R + \delta, R - \delta]$ where δ is an arbitrarily small positive number satisfying $R - \delta > 0$.

Let $f_n(x) = a_n x^n, n \geq 0$.

Let $s_n(x) = f_0(x) + f_1(x) + \cdots + f_n(x), n \geq 1$.

Since the series is uniformly convergent on $[-R+\delta, R-\delta]$ to the function f, the sequence $\{s_n\}$ is uniformly convergent to f on $[-R+\delta, R-\delta]$. Let $c \in [-R+\delta, R-\delta]$.

Let us choose $\epsilon > 0$. There exists a natural number k such that for all $x \in [-R+\delta, R-\delta]$, $| s_n(x) - f(x) | < \frac{\epsilon}{3}$ for all $n \geq k$.
Hence for all $x \in [-R+\delta, R-\delta]$, $| s_k(x) - f(x) | < \frac{\epsilon}{3}$
Therefore $| s_k(c) - f(c) | < \frac{\epsilon}{3}$.
Since each f_n is continuous at c, $s_n(x)$ is continuous at c for all $n \geq 1$.
Therefore there exists a positive δ' such that
$| s_k(x) - s_k(c) | < \frac{\epsilon}{3}$ for all $x \in N(c, \delta') \cap [-R+\delta, R-\delta]$. We have

$$\begin{aligned} | f(x) - f(c) | &= | f(x) - s_k(x) + s_k(x) - s_k(c) + s_k(c) - f(c) | \\ &\leq | f(x) - s_k(x) | + | s_k(x) - s_k(c) | \\ &\quad + | s_k(c) - f(c) | \\ &< \epsilon \text{ for all } x \in N(c, \delta') \cap [-R+\delta, R-\delta] \end{aligned}$$

or, $| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta') \cap [-R+\delta, R-\delta]$.
This shows that f is continuous at c.

Since c is arbitrary, f is continuous on $[-R+\delta, R-\delta]$.

Since δ is arbitrary, f is continuous on $(-R, R)$ and this completes the proof.

Note. A power series with radius of convergence $R(> 0)$ has a continuous sum function on the interval of convergence $(-R, R)$.

Theorem 15.3.3. A power series can be integrated term-by-term on any closed and bounded interval contained within the interval of convergence.

Let $\sum\limits_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R and let $f(x)$ be the sum of the series on $(-R, R)$.

The theorem states that for any closed interval $[a, b] \subset (-R, R)$,
$\int_a^b a_0 dx + \int_a^b a_1 x dx + \int_a^b a_2 x^2 dx + \cdots = \int_a^b f(x) dx$.

Proof. Since R is the radius of convergence of the series and the closed and bounded interval $[a, b] \subset (-R, R)$, the series is uniformly convergent on $[a, b]$ to the sum function f.

Since each term of the series is integrable on $[a, b]$, f is also intergrable on $[a, b]$ and
$\int_a^b a_0 dx + \int_a^b a_1 x dx + \int_a^b a_2 x^2 dx + \cdots = \int_a^b f(x) dx$.

Note. For any x satisfying $| x | < R$, the series is uniformly convergent on $[0, x]$ or $[x, 0]$ and
$\int_0^x a_0 dx + \int_0^x a_1 x dx + \int_0^x a_2 x^2 dx + \cdots = \int_0^x f(x) dx$
or, $a_0 x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} + \cdots + \frac{a_n x^{n+1}}{n+1} + \cdots = \int_0^x f(x) dx$.

The convergence of the left hand series (obtained by term-by-term integration) to $\int_0^x f(x)dx$ is established by the theorem.

Since the left hand series is a power series we now determine the radius of convergence of the series.

Lemma 15.3.4. Let $\{u_n\}$ be a bounded sequence where $u_n \geq 0$ for all $n \in \mathbb{N}$ and let $v_n \geq 0$ for all $n \in \mathbb{N}$ and $\lim v_n = v$. Then $\overline{\lim}\ (u_n v_n) = v.\overline{\lim}\ u_n$.

Proof. **Case 1.** $v \neq 0$.

Since the sequence $\{v_n\}$ is convergent, it is a bounded sequence. So the sequence $\{u_n v_n\}$ is bounded and therefore $\overline{\lim}\ (u_n v_n)$ exists.

Let $\overline{\lim}\ (u_n v_n) = p$. Then there exists a subsequence $\{u_{r_n} v_{r_n}\}$ of $\{u_n v_n\}$ such that $\lim u_{r_n} v_{r_n} = p$. Again $\lim v_n = v \Rightarrow \lim v_{r_n} = v$.

Since $\lim u_{r_n} v_{r_n} = p$ and $\lim v_{r_n} = v \neq 0$, $\lim u_{r_n} = \frac{p}{v}$.

Clearly, $\frac{p}{v}$ is a subsequential limit of the sequence $\{u_n\}$ and so $\frac{p}{v} \leq \overline{\lim}\ u_n (= u$, say). Therefore $p \leq uv$ since $v > 0$.

Since $\overline{\lim}\ u_n = u$, there exists a subsequence $\{u_{k_n}\}$ of $\{u_n\}$ such that $\lim u_{k_n} = u$. Also $\lim v_n = v \Rightarrow \lim v_{k_n} = v$.

Therefore $\lim u_{k_n} v_{k_n} = uv$.

Clearly, uv is a subsequential limit of $\{u_n v_n\}$ and therefore $uv \leq p$.

It follows that $p = uv$. That is, $\overline{\lim}(u_n v_n) = v.\overline{\lim}\ u_n$.

Case 2. $v = 0$.

Let $\overline{\lim}\ u_n = u$. Then there exists a natural number k_1 such that $u_n < u + 1$ for all $n \geq k_1$. Clearly $u + 1 > 0$.

Let $\epsilon > 0$. Since $\lim v_n = 0$, there exists a natural number k_2 such that $v_n < \frac{\epsilon}{u+1}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $u_n v_n < \epsilon$ for all $n \geq k$. This proves that $\lim u_n v_n = 0$. Therefore $\overline{\lim}(u_n v_n) = \lim u_n v_n = 0 = v\overline{\lim}\ u_n$.

This completes the proof.

Lemma 15.3.5. Let $u_n > 0$ for all $n \in \mathbb{N}$ and $\{u_n\}$ be a bounded sequence such that $\overline{\lim}\ u_n > 0$. Let $v_n > 0$ for all $n \in \mathbb{N}$ and $\lim v_n = v > 0$. Then $\overline{\lim}\ (u_n)^{v_n} = (\overline{\lim}\ u_n)^v$.

Proof. Since $\{u_n\}$, $\{v_n\}$ are bounded sequences and $u_n > 0$ for all $n \in \mathbb{N}$, the sequence $\{(u_n)^{v_n}\}$ is a bounded sequence. So $\overline{\lim}\ (u_n)^{v_n}$ exists.

Let $\overline{\lim}\ (u_n)^{v_n} = p$. Then $p > 0$.

Let $\overline{\lim}\ u_n = u$. Since $\overline{\lim}\ (u_n)^{v_n} = p$, there exists a subsequence $\{(u_{r_n})^{v_{r_n}}\}$ of the sequence $\{(u_n)^{v_n}\}$ such that $\lim(u_{r_n})^{v_{r_n}} = p$. Also

$\lim v_n = v \Rightarrow \lim v_{r_n} = v$.

$$\begin{aligned}\log p &= \log \lim (u_{r_n})^{v_{r_n}} \\ &= \lim [v_{r_n} \log(u_{r_n})].\end{aligned}$$

$$\begin{aligned}\text{Since } \lim v_{r_n} = v \neq 0, \lim \log(u_{r_n}) &= \tfrac{\log p}{v}. \\ \text{or, } \log \lim u_{r_n} &= \tfrac{1}{v} \log p \\ \text{or, } \lim u_{r_n} &= e^{\frac{1}{v}\log p} = p^{\frac{1}{v}}.\end{aligned}$$

Clearly, $p^{\frac{1}{v}}$ is a subsequential limit of the sequence $\{u_n\}$ and so $p^{\frac{1}{v}} \leq u$. Therefore $p \leq u^v$, since $p > 0, v > 0$.

Since $\overline{\lim}\ u_n = u$, there exists a subsequence $\{u_{k_n}\}$ of the sequence $\{u_n\}$ such that $\lim u_{k_n} = u$. Also $\lim v_n = v \Rightarrow \lim v_{k_n} = v$.
Since $\lim u_{k_n} = u > 0$ and $\lim v_{k_n} = v$, $\lim (u_{k_n})^{v_{k_n}} = u^v$.

Clearly, u^v is a subsequential limit of the sequence $\{(u_n)^{v_n}\}$. Therefore $u^v \leq p$. It follows that $p = u^v$. That is, $\overline{\lim}(u_n)^{v_n} = (\overline{\lim} u_n)^v$.
This completes the proof.

Theorem 15.3.6. Let $R(> 0)$ be the radius of convergence of the power series $a_0 + a_1x + a_2x^2 + \cdots$ Then the radius of convergence of the power series$a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots + \frac{a_{n-1}}{n}x^n + \cdots$, obtained by term-by-term intergration, is also R.

Proof. $\frac{1}{R} = \overline{\lim}\ \sqrt[n]{|\ a_n\ |}$. Let R' be the radius of convergence of the power series $a_0x + \frac{a_1}{2}x^2 + \cdots + \frac{a_{n-1}}{n}x^n + \cdots$ Then $\frac{1}{R'} = \overline{\lim}\ \sqrt[n]{\frac{|a_{n-1}|}{n}}$.

$\frac{1}{R'} = \overline{\lim}\ \sqrt[n]{\frac{|a_{n-1}|}{n}} = \overline{\lim} \frac{\{|a_{n-1}|^{\frac{1}{n-1}}\}^{\frac{n-1}{n}}}{\sqrt[n]{n}} = \overline{\lim}\ (u_n.v_n)$, where $u_n = \frac{1}{\sqrt[n]{n}}$ and $v_n = \{|\ a_{n-1}\ |^{\frac{1}{n-1}}\}^{\frac{n-1}{n}}$.

As $\overline{\lim}\ \sqrt[n]{|\ a_n\ |} = \frac{1}{R}$, we have $\overline{\lim}\ |\ a_{n-1}\ |^{\frac{1}{n-1}} = \frac{1}{R}$.
Since $\overline{\lim}\ |\ a_{n-1}\ |^{\frac{1}{n-1}} = \frac{1}{R}$ and $\lim \frac{n-1}{n} = 1$, it follows that $\overline{\lim}\ v_n = \frac{1}{R}$.
As $\lim \sqrt[n]{n} = 1$, we have $\lim u_n = 1$.
Since $\lim u_n = 1$ and $\overline{\lim}\ v_n = \frac{1}{R}$, we have $\overline{\lim}\ (u_nv_n) = \lim u_n . \overline{\lim}\ v_n = \overline{\lim}\ v_n = \frac{1}{R}$.
Therefore $\frac{1}{R'} = \frac{1}{R}$, i.e., $R' = R$.

Note. It follows that the series obtained by integrating the power series Σa_nx^n term-by-term is also uniformly convergent on any closed and bounded sub-interval contained in the interval of convergence.

Worked Example.

1. A function f is defined on $(-\frac{1}{3}, \frac{1}{3})$ by
$f(x) = 1 + 2.3x + 3.3^2x^2 + \cdots + n.3^{n-1}x^{n-1} + \cdots$
Show that f is continuous on $(-\frac{1}{3}, \frac{1}{3})$. Evaluate $\int_0^{\frac{1}{4}} f$.

Let $\sum_{n=0}^{\infty} a_n x^n$ be the given series.

Then $a_0 = 1, a_n = (n+1)3^n$ for $n \geq 1$.

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{3(n+2)}{n+1} = 3$.

The radius of convergence of the power series is $\frac{1}{3}$.

Therefore f is continuous on $(-\frac{1}{3}, \frac{1}{3})$.

The series can be integrated term-by-term on any closed interval contained within $(-\frac{1}{3}, \frac{1}{3})$. $[0, \frac{1}{4}] \subset (-\frac{1}{3}, \frac{1}{3})$.

Therefore $\int_0^{1/4} f(x)dx$

$= \int_0^{1/4} dx + \int_0^{1/4} 2.3x dx + \cdots + \int_0^{1/4} n.3^{n-1}x^{n-1}dx + \cdots$

$= \frac{1}{4} + \frac{1}{4}(\frac{3}{4}) + \frac{1}{4}(\frac{3}{4})^2 + \cdots$

$= \frac{1}{4} \cdot \frac{1}{1-3/4} = 1$.

Theorem 15.3.7. Let $R(>0)$ be the radius of convergence of the power series $a_0 + a_1 x + a_2 x^2 + \cdots$ Then the radius of convergence of the power series $a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (n+1)a_{n+1}x^n + \cdots$ obtained by term-by-term differentiaton, is also R.

Proof. $\frac{1}{R} = \overline{\lim} \sqrt[n]{| a_n |}$. Let R' be the radius of convergence of the series $a_1 + 2a_2 + 3a_3 x^2 + \cdots$ Then $\frac{1}{R'} = \overline{\lim} \sqrt[n]{(n+1) \mid a_{n+1} \mid}$.

$\frac{1}{R'} = \overline{\lim} \sqrt[n]{(n+1) \mid a_{n+1} \mid}$

$= \overline{\lim}[\{(n+1)^{\frac{1}{n+1}}\}^{\frac{n+1}{n}}.\{\mid a_{n+1} \mid^{\frac{1}{n+1}}\}^{\frac{n+1}{n}}] = \overline{\lim}(u_n v_n)$ where $u_n = \{(n+1)^{\frac{1}{n+1}}\}^{\frac{n+1}{n}}$, and $v_n = \{\mid a_{n+1} \mid^{\frac{1}{n+1}}\}^{\frac{n+1}{n}}$.

As $\overline{\lim} \sqrt[n]{| a_n |} = \frac{1}{R}$, we have $\overline{\lim} \mid a_{n+1} \mid^{\frac{1}{n+1}} = \frac{1}{R}$.

Since $\overline{\lim} \mid a_{n+1} \mid^{\frac{1}{n+1}} = \frac{1}{R}$ and $\lim \frac{n+1}{n} = 1$, it follows that $\overline{\lim}\, v_n = \frac{1}{R}$.

As $\lim \sqrt[n]{n} = 1$, we have $\lim (n+1)^{\frac{1}{n+1}} = 1$.

Since $\lim (n+1)^{\frac{1}{n+1}} = 1$ and $\lim \frac{n+1}{n} = 1$, it follows that $\lim u_n = 1$.

Since $\lim u_n = 1$ and $\overline{\lim} v_n = \frac{1}{R}$, we have $\overline{\lim}(u_n v_n) = \lim u_n . \overline{\lim} v_n = \overline{\lim}\, v_n = \frac{1}{R}$.

Therefore $\frac{1}{R'} = \frac{1}{R}$, i.e., $R' = R$.

Theorem 15.3.8. A power series can be differentiated term-by-term within the interval of convergence.

Proof. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R(>0)$. Let $f(x)$ be the sum of the series on $(-R, R)$. The theorem states that $\frac{d}{dx}(a_0) + \frac{d}{dx}(a_1 x) + \frac{d}{dx}(a_2 x^2) + \cdots = \frac{d}{dx} f(x)$ on $(-R, R)$.

Differentiating the series $\sum_{n=0}^{\infty} a_n x^n$ term-by-term, we obtain the series $a_1 + 2a_2x + 3a_3x^2 + \cdots$

Let R' be the radius of convergence of this power series.

Then $R' = R$, by Theorem 15.3.6.

Since R is the radius of convergence of both the series, both of these are uniformly convergent on $[-R+\epsilon, R-\epsilon]$ for any positive ϵ satisfying $R - \epsilon > 0$.

Let $f(x)$ be the sum of the series $a_0 + a_1x + a_2x^2 + \cdots$ on $[-R+\epsilon, R-\epsilon]$. Then

$\frac{d}{dx}(a_0) + \frac{d}{dx}(a_1x) + \frac{d}{dx}(a_2x^2) + \cdots = \frac{d}{dx}f(x)$ on $[-R+\epsilon, R-\epsilon]$, by Theorem 14.2.3.

Since ϵ is arbitrary, it follows that f is differentiable at each point of $(-R, R)$ and $\frac{d}{dx}(a_0) + \frac{d}{dx}(a_1x) + \frac{d}{dx}(a_2x^2) + \cdots = \frac{d}{dx}f(x)$ on $(-R, R)$.

Theorem 15.3.9. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R(> 0)$ and $f(x)$ be the sum of the series on $(-R, R)$. Then $f^k(0) = k!a_k$ $(k = 0, 1, 2, \ldots)$.

Proof. $a_0 + a_1x + a_2x^2 + \cdots = f(x)$ on $(-R, R)$ ... ... (i)

Therefore $a_0 = f(0)$.

Differentiating the series (i) term-by-term, we have

$a_1 + 2a_2x + 3a_3x^2 + \cdots = f'(x)$ on $(-R, R)$ ... ... (ii)

Therefore $a_1 = f'(0)$.

Differentiating the series (ii) term-by-term, we have

$1.2a_2 + 2.a_3x + 3.4a_4x^2 + \cdots = f''(x)$ on $(-R, R)$ ... ... (iii)

Therefore $2!a_2 = f''(0)$.

Differentiating the series (iii) term-by-term, we have

$1.2.3a_3 + 2.3.4a_4x + \cdots = f'''(x)$ on $(-R, R)$.

Therefore $3!a_3 = f'''(0)$.

Proceeding similarly, $k!a_k = f^k(0)$ for $k = 0, 1, 2, \ldots$

Note. The power series takes the form $\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$, the coefficients depending on the values at the origin of the sum function f and its successive derivatives.

Definition. If a function f defined on some neighbourhood $N(0)$ of 0, has derivatives of all orders on $N(0)$, then the series

$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$ is called the *Taylor's series* of f about the point 0.

The Theorem 15.3.9 shows that every power series $\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence $R(>0)$ is the Taylor's series about 0 of its sum function f.

Now it is natural to ask if a function f, having derivatives of all orders on some neighbourhood $N(0)$ of 0, be chosen first and the Taylor's series $\sum\limits_{n=0}^{\infty} \dfrac{f^n(0)}{n!} x^n$ be constructed, does this power series will have f as its sum function on $N(0)$?

The answer is 'no'.

Let us consider the function f (due to Cauchy) defined on some neighbourhood $N(0)$ of 0 by $f(x) = e^{-1/x^2}$, $x \neq 0$
$= 0$, $x = 0$.

We have $f^n(0) = 0$ for $n = 0, 1, 2, \ldots$

The Taylor's series of f about 0 is $0 + 0 + 0 + \cdots$ and this converges obviously to 0, and not to f, on $N(0)$.

Theorem 15.3.10. (Abel)

Let $\sum\limits_{0}^{\infty} a_n x^n$ be a power series with radius of convergence $R(>0)$. If the series converges at the end point R of the interval of convergence $(-R, R)$, then the series is uniformly convergent on the closed interval $[0, R]$.

[i.e., the range of uniform convergence extends upto and includes R].

If the series converges at the end point $-R$ of the interval of convergence $(-R, R)$, then the series is uniformly convergent on the closed interval $[-R, 0]$.

[i.e., the range of uniform convergence of the series extends upto and includes $-R$].

Proof. First part.

The series $\sum\limits_{n=0}^{\infty} a_n R^n$ is convergent.

Let us choose $\epsilon > 0$. Then there exists a natural number k such that $|a_{n+1}R^{n+1} + a_{n+2}R^{n+2} + \cdots + a_{n+p}R^{n+p}| < \epsilon$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

$$\begin{aligned}
\text{Let } s_{n,1} &= a_{n+1}R^{n+1}, \\
s_{n,2} &= a_{n+1}R^{n+1} + a_{n+2}R^{n+2}, \\
&\ldots \quad \ldots \\
s_{n,p} &= a_{n+1}R^{n+1} + a_{n+2}R^{n+2} + \cdots + a_{n+p}R^{n+p}, \\
&\ldots \quad \ldots
\end{aligned}$$

Then $|s_{n,p}| < \epsilon$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

$| a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots + a_{n+p}x^{n+p} |$

$= | a_{n+1}R^{n+1}(\frac{x}{R})^{n+1} + a_{n+2}R^{n+2}(\frac{x}{R})^{n+2} + \cdots + a_{n+p}R^{n+p}(\frac{x}{R})^{n+p} |$

$= | s_{n,1}(\frac{x}{R})^{n+1} + (s_{n,2} - s_{n,1})(\frac{x}{R})^{n+2} + (s_{n,3} - s_{n,2})(\frac{x}{R})^{n+3} + \cdots + (s_{n,p} - s_{n,p-1})(\frac{x}{R})^{n+p} |$

$= | s_{n,1}\{(\frac{x}{R})^{n+1} - (\frac{x}{R})^{n+2}\} + s_{n,2}\{(\frac{x}{R})^{n+2} - (\frac{x}{R})^{n+3}\} + \cdots + s_{n,p-1}\{(\frac{x}{R})^{n+p-1} - (\frac{x}{R})^{n+p}\} + s_{n,p}(\frac{x}{R})^{n+p} |$

$\leq | s_{n,1} || (\frac{x}{R})^{n+1} - (\frac{x}{R})^{n+2} | + | s_{n,2} || (\frac{x}{R})^{n+2} - (\frac{x}{R})^{n+3} | + \cdots + | s_{n,p-1} || (\frac{x}{R})^{n+p-1} - (\frac{x}{R})^{n+p} | + | s_{n,p} || \frac{x}{R} |^{n+p}$

$= | s_{n,1} | \{(\frac{x}{R})^{n+1} - (\frac{x}{R})^{n+2}\} + | s_{n,2} | \{(\frac{x}{R})^{n+2} - (\frac{x}{R})^{n+3}\} + \cdots + | s_{n,p} | (\frac{x}{R})^{n+p}$,

since for all $x \in [0, R], 0 \leq (\frac{x}{R})^{n+p} \leq (\frac{x}{R})^{n+p-1} \leq \cdots \leq (\frac{x}{R})^{n+1} \leq 1$

$< \epsilon[\{(\frac{x}{R})^{n+1} - (\frac{x}{R})^{n+2}\} + \{(\frac{x}{R})^{n+2} - (\frac{x}{R})^{n+3}\} + \cdots + \{(\frac{x}{r})^{n+p-1} - (\frac{x}{R})^{n+p}\} + (\frac{x}{R})^{n+p}]$ for all $n \geq k, p = 1, 2, 3, \ldots$

$= \epsilon.(\frac{x}{R})^{n+1}$.

Therefore for all $x \in [0, R]$, $| a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots + a_{n+p}x^{n+p} | < \epsilon$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

This proves that $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[0, R]$.

Second part. Similar proof.

Theorem 15.3.11. Abel's theorem (Another form)

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence 1. If $\sum_{n=0}^{\infty} a_n$ be convergent then the series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[0, 1]$.

Proof. The series $\sum_{n=0}^{\infty} a_n$ is convergent.

Let us choose $\epsilon > 0$. Then there exists a natural number k such that $| a_{n+1} + a_{n+2} + \cdots + a_{n+p} | < \epsilon$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

$$\begin{aligned} \text{Let } s_{n,1} &= a_{n+1}, \\ s_{n,2} &= a_{n+1} + a_{n+2}, \\ &\cdots \quad \cdots \\ s_{n,p} &= a_{n+1} + a_{n+2} + \cdots + a_{n+p}, \\ &\cdots \quad \cdots \end{aligned}$$

Then $| s_{n,p} | < \epsilon$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

$| a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots + a_{n+p}x^{n+p} |$

$= | s_{n,1}x^{n+1} + (s_{n,2} - s_{n,1})x^{n+2} + \cdots + (s_{n,p} - s_{n,p-1})x^{n+p} |$

$= | s_{n,1}\{x^{n+1} - x^{n+2}\} + s_{n,2}\{x^{n+2} - x^{n+3}\} + \cdots + s_{n,p-1}\{x^{n+p-1} - x^{n+p}\} + s_{n,p}x^{n+p} |$

$\leq | s_{n,1} || x^{n+1} - x^{n+2} | + | s_{n,2} || x^{n+2} - x^{n+3} | + \cdots + | s_{n,p-1} || x^{n+p-1} - x^{n+p} | + | s_{n,p} || x^{n+p} |$

$= | s_{n,1} | \{x^{n+1} - x^{n+2}\} + | s_{n,2} | \{x^{n+2} - x^{n+3}\} + \cdots + | s_{n,p-1} | \{x^{n+p-1} - x^{n+p}\} + | s_{n,p} | x^{n+p}$, for all $x \in [0,1]$

$< \epsilon . x^{n+1}$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

Therefore for all $x \in [0,1]$, $| a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots + a_{n+p}x^{n+p} | < \epsilon$ for all $n \geq k$ and $p = 1, 2, 3, \ldots$

This proves that $\sum\limits_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[0,1]$.

Theorem 15.3.12. Abel's theorem (Limit form)

Let $\sum\limits_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R and let the sum of the series be $f(x)$ on $(-R, R)$. If the series $\sum\limits_{n=0}^{\infty} a_n R^n$ be convergent then $\sum\limits_{n=0}^{\infty} a_n R^n = \lim\limits_{x \to R-} f(x)$.

Proof. Since R is the radius of convergence of the series and $\sum\limits_{n=0}^{\infty} a_n R^n$ is convergent, the series is uniformly convergent on $[0, R]$. Let $\phi(x)$ be the sum of the series on $[0, R]$.

Since each term of the series is continuous on $[0, R]$, the sum function ϕ is also continuous on $[0, R]$. Also $\phi(x) = f(x)$ on $[0, R)$.

Since ϕ is continuous at $R, \phi(R) = \lim\limits_{x \to R-} \phi(x) = \lim\limits_{x \to R-} f(x)$.

Therefore $\Sigma a_n R^n = \lim\limits_{x \to R-} f(x)$.

Corollary. Let $\sum\limits_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence 1 and let the sum of the series be $f(x)$ on $(-1, 1)$. Then

(i) if the series Σa_n be convergent, then $\Sigma a_n = \lim\limits_{x \to 1-} f(x)$;

(ii) if the series $\Sigma(-1)^n a_n$ be convergent, then $\Sigma(-1)^n a_n = \lim\limits_{x \to -1+0} f(x)$.

Note. The converse of Abel's theorem is not true. For a power series $\sum\limits_{n=0}^{\infty} a_n x^n$ with radius of convergence R, $\lim\limits_{x \to R-} f(x)$ may exist, yet the series $\sum\limits_{n=0}^{\infty} a_n x^n$ may not converge at R.

For example, the sum of the series $1 - x + x^2 - x^3 + \cdots$ is $\frac{1}{1+x}$ on $(-1, 1)$, 1 being the radius of convergence of the series. $\lim\limits_{x \to 1-} \dfrac{1}{1+x} = 2$, but the series $1 - x + x^2 - x^3 + \cdots$ is not convergent at $x = 1$.

Theorem 15.3.13. (Uniqueness theorem)

If two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ converge on the same interval $(-R, R), R > 0$, to the same function f, then $a_n = b_n$ for $n = 0, 1, 2, \ldots$

Proof. By the given condition,

$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ on $(-R, R)$

and $f(x) = b_0 + b_1 x + b_2 x^2 + \cdots$ on $(-R, R)$.

At $x = 0$, $f(0) = a_0 = b_0$.

Differentiating both the series term-by-term, we have

$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$ on $(-R, R)$

and $f'(x) = b_1 + 2b_2 x + 3b_3 x^2 + \cdots$ on $(-R, R)$.

At $x = 0$, $f'(0) = a_1 = b_1$.

Differentiating again, we have by similar arguments

$f''(0) = a_2 = b_2$.

Proceeding similarly, we have $a_n = b_n$ for $n = 0, 1, 2, \ldots$

Theorem 15.3.14. If R_1, R_2 be the radii of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ respectively and $\sum_{n=0}^{\infty} a_n x^n = f(x)$ for $|x| < R_1$, $\sum_{n=0}^{\infty} b_n x^n = g(x)$ for $|x| < R_2$, then the radius of convergence of the series $\sum_{n=0}^{\infty} (a_n + b_n) x^n$ is $R = \min\{R_1, R_2\}$ and the sum of the series is $f(x) + g(x)$ on $(-R, R)$.

Proof left to the reader.

An alternative proof of Theorem 6.6.4

Abel's theorem.

If the series $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$, $\sum_{n=0}^{\infty} c_n$ converge to A, B, C respectively and if $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$, then $C = AB$.

Proof. Let $a_0 + a_1 x + a_2 x^2 + \cdots$, $b_0 + b_1 x + b_2 x^2 + \cdots$, $c_0 + c_1 x + c_2 x^2 + \cdots$ be three power series with radius of convergence 1 having the sums $f(x), g(x), h(x)$ respectively.

Each of the series is absolutely convergent for $0 \le x < 1$. Therefore $f(x).g(x) = h(x)$ for all x satisfying $0 \le x < 1$, by theorem 6.6.2.

Since the series $a_0 + a_1 x + a_2 x^2 + \cdots$ is convergent for $x = 1$, f is continuous at 1 and $\lim_{x \to 1-} f(x) = f(1) = A$.

Similarly, $\lim_{x \to 1-} g(x) = g(1) = B$ and $\lim_{x \to 1-} h(x) = h(1) = C$.

Since $f(x).g(x) = h(x)$ for $0 \le x < 1$ and the functions f,g,h are continuous at 1, $\lim_{x\to 1-} [f(x).g(x)] = \lim_{x\to 1-} h(x)$, i.e., $AB = C$.

Worked Examples (continued).

2. Let $f(x)$ be the sum of the power series $\sum_{n=0}^{\infty} a_n x^n$ on $(-R, R)$ for some $R > 0$. If $f(x) = f(-x)$ for all $x \in (-R, R)$, show that $a_n = 0$ for all odd n.

$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = f(x)$ for all $x \in (-R, R)$
Therefore $a_0 - a_1x + a_2x^2 - a_3x^3 + \cdots = f(-x)$ for all $x \in (-R, R)$.

As $f(x) = f(-x)$, both the power series
$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ and $a_0 - a_1x + a_2x^2 - a_3x^3 + \cdots$
have the same sum $f(x)$ on $(-R, R)$.

By uniqueness theorem, $a_1 = -a_1, a_3 = -a_3, a_5 = -a_5, \cdots$
Hence $a_n = 0$ for all odd n.

3. Assuming the power series expansion for $\frac{1}{\sqrt{1-x^2}}$ as
$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}.x^4 + \frac{1.3.5}{2.4.6}x^6 + \cdots$
obtain the power series expansion for $\sin^{-1} x$.
Deduce that $1 + \frac{1}{2.3} + \frac{1.3}{2.4.5} + \frac{1.3.5}{2.4.6.7} + \cdots = \frac{\pi}{2}$.

Let $x^2 = y$. The series becomes $1 + \frac{1}{2}y + \frac{1.3}{2.4}y^2 + \frac{1.3.5}{2.4.6}y^3 + \cdots$
Let $\sum_{n=0}^{\infty} a_n y^n$ be the series. Then $a_0 = 1, a_n = \frac{1.3.5\cdots(2n-1)}{2.4.6\cdots 2n}$ for $n \ge 1$.
$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$.

Hence the interval of convergence of the series is $\{y \in \mathbb{R} : -1 < y < 1\}$. It follows that the interval of convergence of the given series is $\{x \in \mathbb{R} : -1 < x < 1\}$.

Integrating term-by-term on $[0, x]$ where $|x| < 1$, we have
$x + \frac{1}{2}.\frac{x^3}{3} + \frac{1.3}{2.4}.\frac{x^5}{5} + \frac{1.3.5}{2.4.6}.\frac{x^7}{7} + \cdots = \sin^{-1} x$ for $-1 < x < 1$.
At $x = 1$, the series becomes $1 + \frac{1}{2}.\frac{1}{3} + \frac{1.3}{2.4}.\frac{1}{5} + \cdots$
The series is convergent by Raabe's test. (Ex. 1, Theorem 6.3.10)

By Abel's theorem, the sum of the series at $x = 1$ is $\sin^{-1} 1$.

At $x = -1$, the series becomes $-1 - \frac{1}{2}.\frac{1}{3} - \frac{1.3}{2.4}.\frac{1}{5} - \cdots$
This is also convergent.
By Abel's theorem the sum of the series at $x = -1$ is $\sin^{-1}(-1)$.

Hence $\sin^{-1} x = x + \frac{1}{2}.\frac{x^3}{3} + \frac{1.3}{2.4}.\frac{x^5}{5} + \frac{1.3.5}{2.4.6}.\frac{x^7}{7} + \cdots$ for $-1 \le x \le 1$

and $1+\frac{1}{2.3}+\frac{1.3}{2.4.5}+\frac{1.3.5}{2.4.6.7}+\cdots=\sin^{-1}1=\frac{\pi}{2}$.

4. Assuming the expansion

$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$ for $-1<x\leq 1$

prove that $\int_0^1 \frac{\log(1+x)}{x}dx=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$

Let us consider the series $1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\cdots$ ... ... (i)

The radius of convergence of the series is 1. Let $\phi(x)$ be the sum of the series on $-1<x<1$.

$$\begin{aligned}\text{Then } \phi(x) &= \frac{\log(1+x)}{x}, \text{ for } 0<|x|<1\\ &= 1, \text{ for } x=0.\end{aligned}$$

At $x=1$, the series is convergent. By Abel's theorem, the sum of the series at $x=1$ is $\lim_{x\to 1-}\phi(x)=\log 2$.

At $x=-1$, the series is divergent.

Hence $1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\cdots=f(x)$ for $-1<x\leq 1$,

$$\begin{aligned}\text{where } f(x) &= \frac{\log(1+x)}{x}, \text{ for } -1<x\leq 1, x\neq 0\\ &= 1, x=0.\end{aligned}$$

The series (i) is uniformly convergent on $[0,1]$. Integrating term-by-term on $[0,1]$, we have

$\int_0^1 f(x)dx=\int_0^1 dx-\frac{1}{2}\int_0^1 xdx+\frac{1}{3}\int_0^1 x^2dx-\frac{1}{4}\int_0^1 x^3dx+\cdots$

or, $\int_0^1 \frac{\log(1+x)}{x}dx=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$

5. Assuming the expansion

$\log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots$ for $-1\leq x<1$

prove that $\int_0^1 \log(1-x)dx=-1$.

The radius of convergence of the series $-x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots$ is 1.

Integrating term-by-term on $[0,x]$ where $|x|<1$, we have

$-\frac{x^2}{2}-\frac{x^3}{2.3}-\frac{x^4}{3.4}-\cdots=\int_0^x \log(1-x)dx$ for $-1<x<1$.

At $x=1$, the series becomes $-\frac{1}{1.2}-\frac{1}{2.3}-\frac{1}{3.4}-\cdots$

Let $s_n=-\frac{1}{1.2}-\frac{1}{2.3}-\frac{1}{3.4}-\cdots-\frac{1}{n(n+1)}$.

Then $s_n=-[1-\frac{1}{n+1}]$ and $\lim s_n=-1$.

Therefore the series converges to -1 at $x=1$.

By Abel's theorem, the sum of the series at $x=1$ is $\lim_{x\to 1-}\int_0^x \log(1-x)dx$.

Therefore $\lim_{x\to 1-}\int_0^x \log(1-x)dx=-1$.

Since $\lim_{x\to 1-}\int_0^x \log(1-x)dx$ exists, this limit is $\int_0^1 \log(1-x)dx$. Therefore $\int_0^1 \log(1-x)dx=-1$.

6. Find the sum of the series $\sum_{n=0}^{\infty}(2^n + 3^n)x^n$, indicating the range of validity.

Let the series be $\sum_{n=0}^{\infty} a_n x^n$. Then $a_n = 2^n + 3^n$.

$\sum_{n=0}^{\infty} 2^n x^n$ is a power series whose radius of convergence is $\frac{1}{2}$ and the sum of the series is $\frac{1}{1-2x}$ for $|x| < \frac{1}{2}$.

$\sum_{n=0}^{\infty} 3^n x^n$ is a power series whose radius of convergence is $\frac{1}{3}$ and the sum of the series is $\frac{1}{1-3x}$ for $|x| < \frac{1}{3}$.

Hence the radius of convergence of the series $\sum_{n=0}^{\infty} a_n x^n$ is $\frac{1}{3}$ and $\sum_{n=0}^{\infty} a_n x^n = \frac{1}{1-2x} + \frac{1}{1-3x}$ for $|x| < \frac{1}{3}$.

At $x = \frac{1}{3}$ the series becomes $\sum_{n=0}^{\infty} \frac{2^n+3^n}{3^n}$.

As $\lim[(\frac{2}{3})^n + 1] \neq 0$, the series is divergent at $x = \frac{1}{3}$.
By similar arguments, the series is divergent at $x = -\frac{1}{3}$.

Hence the sum of the series is $\frac{1}{1-2x} + \frac{1}{1-3x}$ for $-\frac{1}{3} < x < \frac{1}{3}$.

7. Assuming that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ for $-1 \leq x \leq 1$ and $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$ for $-1 < x < 1$, deduce that

$\frac{1}{2}(\tan^{-1} x)^2 = \frac{1}{2}x^2 - \frac{1}{4}(1+\frac{1}{3})x^4 + \frac{1}{6}(1+\frac{1}{3}+\frac{1}{5})x^6 - \cdots$ for $-1 \leq x \leq 1$.

The radius of convergence of each of the series $x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ and $1 - x^2 + x^4 - x^6 + \cdots$ is 1 and therefore both the series are absolutely convergent for $-1 < x < 1$.

So their Cauchy product will converge absolutely to the product of their sums for $-1 < x < 1$.

Let the Cauchy product be $c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$
Then $c_0 = 0, c_1 = 1, c_2 = 0, c_3 = -(1+\frac{1}{3}), c_4 = 0, c_5 = (1+\frac{1}{3}+\frac{1}{5}), \ldots$

Therefore $\frac{\tan^{-1} x}{1+x^2} = x - (1+\frac{1}{3})x^3 + (1+\frac{1}{3}+\frac{1}{5})x^5 + \cdots$ for $-1 < x < 1$.

Integrating the series term-by-term on $[0, x]$ where $|x| < 1$, we have
$\frac{1}{2}(\tan^{-1} x)^2 = \frac{1}{2}x^2 - \frac{1}{4}(1+\frac{1}{3})x^4 + \frac{1}{6}(1+\frac{1}{3}+\frac{1}{5})x^6 - \cdots$ for $-1 < x < 1$.

At $x = \pm 1$ the series becomes $\frac{1}{2} - \frac{1}{4}(1+\frac{1}{3}) + \frac{1}{6}(1+\frac{1}{3}+\frac{1}{5}) - \cdots$ This is an alternating series and it is convergent by Leibnitz's test.

By Abel's theorem, $\frac{1}{2}(\tan^{-1} x)^2 = \frac{1}{2}x^2 - \frac{1}{4}(1+\frac{1}{3})x^4 + \frac{1}{6}(1+\frac{1}{3}+\frac{1}{5})x^6 - \cdots$ for $-1 \leq x \leq 1$.

Exercises 26

1. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ where

(i) $a_n = \frac{(-1)^n n^n}{n! 2^n}, n = 1, 2, \ldots, a_0 = 0$; (ii) $a_n = \frac{2^n}{n^2}, n = 1, 2, \ldots, a_0 = 0$;

(iii) $a_n = (1/3)^n$ if n be odd
$= (1/2)^n$ if n be even ;

(iv) $a_n = 2^n + 3^n,\ n \geq 1$;

(v) $a_0 = 1, a_n = (\sqrt[n]{n} + 1)^n, n \geq 1$; (vi) $a_0 = 1, 2 \leq |a_n| \leq 3$ for $n \geq 1$.

2. Find the radius of convergence of the power series

(i) $1 + 3x + \frac{3^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \cdots$

(ii) $1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \cdots$

(iii) $1 - \frac{2^2}{3^2}x + \frac{2^2.4^2}{3^2.5^2}x^2 - \frac{2^2.4^2.6^2}{3^2.5^2.7^2}x^3 + \cdots$

(iv) $1 - \frac{x}{1.2} + x^2 - \frac{x^3}{2.4} + x^4 - \frac{x^5}{4.8} + \cdots$

(v) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}(x+1)^n$

(vi) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)(n+2)}(x-2)^n$.

3. $\sum_{n=0}^{\infty} a_n x^n$ is a power series with radius of convergence $R(> 0)$. Construct a power series $\sum_{n=0}^{\infty} b_n x^n$, other than $\sum_{n=0}^{\infty} x^n$, such that the radius of convergence of the series $\sum_{n=0}^{\infty} a_n b_n x^n$ is also R.

[**Hint.** Take $b_n = \frac{1}{n+1}$.]

4. $\sum_{n=0}^{\infty} a_n x^n$ is a power series with radius of convergence $R(> 0)$. Construct a power series $\sum_{n=0}^{\infty} b_n x^n$, other than $\sum_{n=0}^{\infty} (x/2)^n$, such that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n b_n x^n$ is $2R$.

[**Hint.** Take $b_n = \frac{1}{2^n}$ if n be even
$= \frac{1}{3^n}$ if n be odd .]

5. Find the sum of the power series $1 + x + x^2 + \cdots$ on its interval of convergence.

Deduce the power series expansion of $\log(1 - x)$ and use Abel's theorem to prove that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2$.

6. Prove that $\frac{1}{2}[\log(1 - x)]^2 = \frac{1}{2}x^2 + (1 + \frac{1}{2})\frac{x^3}{3} + (1 + \frac{1}{2} + \frac{1}{3})\frac{x^4}{4} + \cdots$ for $-1 < x \leq 1$.

7. Find the sum of the power series $1+x+x^2+\cdots$ on its interval of convergence. By repeated differentiation prove that

$$(1-x)^{-3} = 1 + 3x + \tfrac{3.4}{1.2}x^2 + \tfrac{3.4.5}{1.2.3}x^3 + \cdots \text{ on } (-1,1).$$

8. Assuming the power series expansion for $(1+x)^{-1}$ as

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

obtain the power series expansion for $\log(1+x)$. Deduce that

(i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2$.

(ii) $\frac{1}{2}[\log(1+x)]^2 = \frac{1}{2}x^2 - \frac{1}{3}(1+\frac{1}{2})x^3 + \frac{1}{4}(1+\frac{1}{2}+\frac{1}{3})x^4 - \cdots$ for $-1 < x \le 1$.

9. Assuming the power series expansion for $(1+x^2)^{-1}$ as

$$(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \cdots$$

obtain the power series expansion for $\tan^{-1} x$.

Deduce that (i) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$;

(ii) $\int_0^1 \frac{\tan^{-1} x}{x} dx = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \cdots$

10. Assuming that the sum of the power series $x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ on its interval of convergence is $\log(1+x)$, deduce that

$$\frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \cdots = 2\log 2 - 1.$$

11. Assuming that $\sin^{-1} x = x + \frac{1}{2}.\frac{x^3}{3} + \frac{1.3}{2.4}.\frac{x^5}{5} + \cdots$ for $-1 \le x \le 1$ prove that

$$\int_0^1 \frac{\sin^{-1} x}{x} dx = 1 + \frac{1}{2}.\frac{1}{3^2} + \frac{1.3}{2.4}.\frac{1}{5^2} + \cdots$$

12. Assuming the expansion $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ for $-1 < x < 1$ prove that

(i) $\int_0^1 \frac{x}{1+x} dx = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$

(ii) $\int_0^1 \frac{x^2}{1+x} dx = \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

13. Find the sum of the series indicating the range of validity.

(i) $\sum_{n=0}^{\infty} (1+2^n)x^n$, (ii) $\sum_{n=0}^{\infty} (n+3)x^n$.

14. Let $f(x)$ be the sum of the power series $\sum_{n=0}^{\infty} a_n x^n$ on $(-R,R)$ for $R > 0$. If $f(x) + f(-x) = 0$ for all $x \in (-R,R)$ prove that $a_n = 0$ for all even n.

15. Let $f(x)$ be the sum of the power series $a_0 + a_1 x + a_2 x^2 + \cdots$ on R. If $f'(x) = f(x)$ for all $x \in R$ and $f(0) = 1$, prove that $a_n = \frac{1}{n!}$ for all $n \in \mathbb{N}$.

16. If a function f be defined for $|x| < R$ and if there exists a constant k such that for all $x \in (-R,R)$, $|f^n(x)| \le k$ for all $n \in \mathbb{N}$, prove that the Taylor's series $\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$ converges to $f(x)$ for all $x \in (-R,R)$.

A1.1. Introduction.

$\mathbb{R}^2$ is the set of all ordered pairs $\{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$. An ordered pair $\{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$ is also called a *point* in $\mathbb{R}^2$, denoted by $\mathbf{x}$.

Definition. Let $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$. We define

(i) $\mathbf{x} = \mathbf{y}$ if and only if $x_1 = x_2, y_1 = y_2$;

(ii) $\mathbf{x} + \mathbf{y} = (x_1 + x_2, y_1 + y_2)$;

(iii) $c\mathbf{x} = (cx_1, cy_1) \quad (c \in \mathbb{R})$.

Norm. Let $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. The norm of $\mathbf{x}$ is denoted by $\| \mathbf{x} \|$ and is defined by $\| \mathbf{x} \| = \sqrt{x_1^2 + x_2^2}$.

Note. Norm is a mapping from $\mathbb{R}^2$ to $\mathbb{R}$.

Properties of the norm. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. Then

(i) $\| \mathbf{x} \| \geq 0$, and $\| \mathbf{x} \| = 0$ if and only if $\mathbf{x} = \mathbf{0}$;

(ii) $\| c\mathbf{x} \| = | c | \,.\, \| \mathbf{x} \|$ for all $c \in \mathbb{R}$;

(iii) $\| \mathbf{x} + \mathbf{y} \| \leq \| \mathbf{x} \| + \| \mathbf{y} \|$ (Triangle inequality).

Note. The triangle inequality is also expressed in the form –

$\| \mathbf{x} - \mathbf{y} \| + \| \mathbf{y} - \mathbf{z} \| \leq \| \mathbf{x} - \mathbf{z} \|$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$.

A1.2. Cell in $\mathbb{R}^2$. Disc in $\mathbb{R}^2$.

Let $a_1, b_1 \in \mathbb{R}$ and $a_1 < b_1 : a_2, b_2 \in \mathbb{R}$ and $a_2 < b_2$.

Let $I_1 = \{x_1 \in \mathbb{R} : a_1 < x_1 < b_1\}$ and $I_2 = \{x_2 \in \mathbb{R} : a_2 < x_2 < b_2\}$ be open intervals in $\mathbb{R}$. Then the Cartesian product $I_1 \times I_2 = \{(x_1, x_2) \in \mathbb{R}^2 : a_1 < x_1 < b_1, a_2 < x_2 < b_2\}$ is said to be an *open cell* in $\mathbb{R}^2$.

Let $J_1 = \{x_1 \in \mathbb{R} : a_1 \leq x_1 \leq b_1\}$ and $J_2 = \{x_2 \in \mathbb{R} : a_2 \leq x_2 \leq b_2\}$ be closed intervals in $\mathbb{R}$. Then the Cartesian product $J_1 \times J_2 = \{(x_1, x_2) \in \mathbb{R}^2 : a_1 \leq x_1 \leq b_1, a_2 \leq x_2 \leq b_2\}$ is said to be a *closed cell* in $\mathbb{R}^2$.

If I_1 and I_2 be open (closed) bounded intervals in $\mathbb{R}$ then the Cartesian product $I_1 \times I_2$ is said to be an *open (closed) bounded cell* in $\mathbb{R}^2$.

An open cell in $\mathbb{R}^2$ is also called an *open rectangle* in $\mathbb{R}^2$ and a closed cell in $\mathbb{R}^2$ is also called a *closed rectangle* in $\mathbb{R}^2$.

Let $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$ and $\delta_1 > 0, \delta_2 > 0$. The set $S = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - a_1| < \delta_1, |x_2 - a_2| < \delta_2\}$ is said to be an *open cell about* $\mathbf{a}$ (or an *open rectangle about* $\mathbf{a}$).

This is a rectangular region in $\mathbb{R}^2$ with the centre at $\mathbf{a}$.

Let $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$ and $\delta > 0$. The set of all points $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ satisfying the condition $\| \mathbf{x} - \mathbf{a} \| < \delta$, i.e., $\sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2} < \delta$ is said to be an *open δ-disc* about $\mathbf{a}$.

This is a circular region in $\mathbb{R}^2$ with the centre at $\mathbf{a}$ and radius δ.

It is a matter of simple verification that an open cell about $\mathbf{a}$ contains an open disc about $\mathbf{a}$ and vice-versa.

Note. Let $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and $\delta, \delta_1, \delta_2, \ldots, \delta_n > 0$.

An open n-cell about $\mathbf{a}$ is the set of all points $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ satisfying the condition $|x_1 - a_1| < \delta_1, |x_2 - a_2| < \delta_2, \ldots, |x_n - a_n| < \delta_n$.

An open n-ball about $\mathbf{a}$ is the set of all points $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ satisfying the condition $\| \mathbf{x} - \mathbf{a} \| < \delta$, i.e., $\sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2} < \delta$ and it is denoted by $B(\mathbf{a}, \delta)$.

In particular, if $n = 1$, the open 1-ball is the open interval $(a - \delta, a + \delta)$; if $n = 2$, the open 2-ball is the open disc about $\mathbf{a}$.

A1.3. Neighbourhood of a point in $\mathbb{R}^2$.

Let $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$.

A set $S \subset \mathbb{R}^2$ is said to be a neighbourhood of $\mathbf{a}$ if an open cell about $\mathbf{a}$ (or an open disc about $\mathbf{a}$) is contained in S. A neighbourhood of $\mathbf{a}$ is denoted by $N(\mathbf{a})$.

Clearly, an open disc about $\mathbf{a}$ is also a neighbourhood of the point $\mathbf{a}$. This is a *circular neighbourhood* of the point $\mathbf{a}$. It is also denoted by $N(\mathbf{a}, \delta)$, if δ be the radius.

An open cell about $\mathbf{a}$ is also a neighbourhood of the point $\mathbf{a}$. This is a *rectangular neighbourhood* of the point $\mathbf{a}$ (and also a *square neighbourhood*, in particular) of the point $\mathbf{a}$.

It can be observed that a rectangular neighbourhood of a point $\mathbf{a}$ contains a circular neighbourhood of the point and vice-versa.

Any type of neighbourhood of the point $\mathbf{a}$ is denoted by $N(\mathbf{a})$.

A1.4. Interior point.

Let S be a subset of $\mathbb{R}^2$. A point $\mathbf{x}$ in S is said to be an *interior point* of S if there exists a neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x})$ is contained in S.

The set of all interior points of a set $S \subset \mathbb{R}^2$ is called the *interior* of S and is denoted by int S (or by S^o).

Examples.

1. Let $S = \{(x, y) \in \mathbb{R}^2 : 1 < x < 3, 1 < y < 5\}$. The set S is an open cell.

Let $(p, q) \in S$. Then $1 < p < 3, 1 < q < 5$. There exists a positive δ_1 such that $(p - \delta_1, p + \delta_1) \subset (1, 3)$ and there exists a positive δ_2 such that $(q - \delta_2, q + \delta_2) \subset (1, 5)$. $N = \{(x, y) \in \mathbb{R}^2 : p - \delta_1 < x < p + \delta_1, q - \delta_2 < y < q + \delta_2\}$ is a neighbourhood of the point (p, q) and $N \subset S$. Therefore (p, q) is an interior point of S. Thus each point of S is an interior point of S.

2. Let $S = \{(x, y) \in \mathbb{R}^2 : x + y < 1\}$. The set S is the open half plane containing the origin bounded by the line $x + y = 1$.

Let $(a, b) \in S$. Then $a + b < 1$. Let p be the length of perpendicular from the point (a, b) upon the line $x + y = 1$. Then $p > 0$. The set $N = \{(x, y) \in \mathbb{R}^2 : \sqrt{(x - a)^2 + (y - b)^2} < p$ is a neighbourhood of (a, b) and $N \subset S$. Therefore (a, b) is an interior point of S. Thus each point of S is an interior point of S.

3. Let $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. The set S is the interior of the circle of unit radius with centre at $(0, 0)$.

Let $(a, b) \in S$. Then $a^2 + b^2 < 1$. Let $a^2 + b^2 = r^2$. Then $0 \leq r < 1$. Let $p = \frac{1}{2}(1 - r)$. The set $N = \{(x, y) \in \mathbb{R}^2 : \sqrt{(x - a)^2 + (y - b)^2} < p$ is a neighbourhood of (a, b) and $N \subset S$. Therefore (a, b) is an interior point of S. Thus each point of S is an interior point of S.

4. Let $S = \{(x, y) \in \mathbb{R}^2 : x \leq 1\}$. The set S is the half plane containing the origin bounded by the line $x = 1$.

No point on the line $x = 1$ is an interior point of S,because every neighbourhood of a point on the line $x = 1$ contains points not in S. Every point in S not on the line $x = 1$ is an interior point of S.

5. Let $S = \mathbb{R}^2$. Each point of S is an interior point of S.

6. Let $S = \phi$. Here int $S = \phi$, i.e., int $S = S$.

A1.5. Open set.

Let S be a subset of $\mathbb{R}^2$. S is said to be an *open set* in $\mathbb{R}^2$ if each point of S is an interior point of S.

Clearly, S is an open set in $\mathbb{R}^2$ if int $S = S$.

Examples.

1. Let $S = \{(x, y) \in \mathbb{R}^2 : 1 < x < 3, 1 < y < 5\}$. Since each point of S is an interior point of S, S is an open set.

2. Let $S = \{(x, y) \in \mathbb{R}^2 : x + y < 1\}$. Since each point of S is an interior point of S, S is an open set.

3. Let $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Since each point of S is an interior point of S, S is an open set.

4. Let $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. No point of S is an interior point of S. S is not an open set.

5. Let $S = \mathbb{R}^2$. Each point of S is an interior point of S. Therefore S is an open set.

6. Let $S = \phi$. Here int $S = S$. Therefore S is an open set.

Theorem A1.5.1. The union of a finite number of open sets in $\mathbb{R}^2$ is an open set in $\mathbb{R}^2$.

Proof. Let $G_1, G_2 \ldots, G_m$ be m open sets in $\mathbb{R}^2$.

Let $G = G_1 \cup G_2 \cup \cdots \cup G_m$.

Let $\mathbf{x} \in G$. Then $\mathbf{x}$ belongs to at least one of the sets $G_i, i = 1, 2, \ldots, m$.

Let $\mathbf{x} \in G_k$. Since G_k is an open set in $\mathbb{R}^2$, $\mathbf{x}$ is an interior point of G_k. Therefore there exists a neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \subset G_k$. It follows that $N(\mathbf{x}) \subset G$ and this shows that $\mathbf{x}$ is an interior point of G.

Thus every point of G is an interior point of G. Hence G is an open set. This completes the proof.

Theorem A1.5.2. The intersection of a finite number of open sets in $\mathbb{R}^2$ is an open set in $\mathbb{R}^2$.

Proof. Let $G_1, G_2 \ldots, G_m$ be m open sets in $\mathbb{R}^2$.

Let $G = G_1 \cap G_2 \cap \cdots \cap G_m$.

Case 1. Let $G = \phi$. Then G is an open set in $\mathbb{R}^2$, since ϕ is an open set in $\mathbb{R}^2$.

Case 2. Let $G \neq \phi$.

Let $\mathbf{x} \in G$. Then $\mathbf{x} \in G_i$ for $i = 1, 2, \cdots, m$.

Since G_1 is an open set in $\mathbb{R}^2$, $\mathbf{x}$ is an interior point of G_1. Therefore there exists a neighbourhood $N(\mathbf{x}, \delta_1)$ of $\mathbf{x}$ such that $N(\mathbf{x}, \delta_1) \subset G_1$.

Since G_2 is an open set in $\mathbb{R}^2$, $\mathbf{x}$ is an interior point of G_2. Therefore there exists a neighbourhood $N(\mathbf{x}, \delta_2)$ of $\mathbf{x}$ such that $N(\mathbf{x}, \delta_2) \subset G_2$.

... ...

Since G_m is an open set in $\mathbb{R}^2$, $\mathbf{x}$ is an interior point of G_m. Therefore there exists a neighbourhood $N(\mathbf{x}, \delta_m)$ of $\mathbf{x}$ such that $N(\mathbf{x}, \delta_m) \subset G_m$.

Let $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_m\}$. Then $\delta > 0$ and the neighbourhood $N(\mathbf{x}, \delta)$ of $\mathbf{x}$ is such that $N(\mathbf{x}, \delta) \subset G$. This shows that $\mathbf{x}$ is an interior point of G.

Thus every point of G is an interior point of G. Hence G is an open set. This completes the proof.

Theorem A1.5.3. The union of an arbitrary collection of open sets in $\mathbb{R}^2$ is an open set in $\mathbb{R}^2$.

Proof. Let $\{G_\lambda : \lambda \in \Lambda\}$, Λ being the index set, be a collection of open sets in $\mathbb{R}^2$.

Let $G = \cup G_\lambda$. Let $\mathbf{x} \in G$. Then $\mathbf{x}$ belongs to at least one set G_α of the collection, where $\alpha \in \Lambda$.

Since G_α is an open set in $\mathbb{R}^2$ and $\mathbf{x} \in G_\alpha$, $\mathbf{x}$ is an interior point of G_α. Therefore there exists a neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \subset G_\alpha$. This implies $N(\mathbf{x}) \subset G$. This shows that $\mathbf{x}$ is an interior point of G.

Thus every point of G is an interior point of G hence G is an open set. This completes the proof.

Note. The intersection of an infinite collection of open sets in $\mathbb{R}^2$ is not necessarily an open set in $\mathbb{R}^2$.

Let us consider the sets
$$\begin{aligned} G_1 &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \\ G_2 &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < \tfrac{1}{2}\} \\ &\cdots \qquad \cdots \qquad \cdots \\ G_n &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < \tfrac{1}{n}\} \\ &\cdots \qquad \cdots \qquad \cdots \end{aligned}$$

Each G_i is an open set in $\mathbb{R}^2$. $\bigcap\limits_{i=1}^{\infty} G_i = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 0\}$, i.e., $\{(0,0)\}$. This is not an open set in $\mathbb{R}^2$.

Let us consider the sets
$$\begin{aligned} G_1 &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \\ G_2 &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 2\} \\ &\cdots \qquad \cdots \qquad \cdots \\ G_n &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < n\} \\ &\cdots \qquad \cdots \qquad \cdots \end{aligned}$$

Each G_i is an open set in $\mathbb{R}^2$. $\bigcap_{i=1}^{\infty} G_i = G_1$ and it is an open set in $\mathbb{R}^2$.

These two examples establish that the intersection of an infinite number of open sets in $\mathbb{R}^2$ is not necessarily an open set in $\mathbb{R}^2$.

Worked Examples.

1. Let G_1, G_2 be open sets in $\mathbb{R}$. Prove that $G_1 \times G_2$ is an open set in $\mathbb{R}^2$.

If one or both of G_1 and G_2 be the null set then $G_1 \times G_2 = \phi$, an open set in $\mathbb{R}^2$.

Let $(c, d) \in G_1 \times G_2$. Then $c \in G_1$ and $d \in G_2$.

Since G_1 is an open set in $\mathbb{R}$ and $c \in G_1, c$ is an interior point of G_1. Therefore there exists a positive δ_1 such that the set $N_1 = \{x \in \mathbb{R} : c - \delta_1 < x < c + \delta_1\}$ is entirely contained in G_1.

Since G_2 is an open set in $\mathbb{R}$ and $d \in G_2, d$ is an interior point of G_2. Therefore there exists a positive δ_2 such that the set $N_2 = \{y \in \mathbb{R} : d - \delta_2 < y < d + \delta_2\}$ is entirely contained in G_2.

Clearly, the set $N_1 \times N_2$ is a neighbourhood of (c, d) and $N_1 \times N_2$ is entirely contained in $G_1 \times G_2$. This shows that (c, d) is an interior point of $G_1 \times G_2$. Hence $G_1 \times G_2$ is an open set in $\mathbb{R}^2$.

Corollary. An open cell in $\mathbb{R}^2$, being the Cartesian product of two open intervals in $\mathbb{R}$, is an open set in $\mathbb{R}^2$.

2. Prove that the set $S = \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1\}$ is an open set in $\mathbb{R}^2$.

The set S can be expressed as $S = A \cap B \cap C \cap D$ where $A = \{(x, y) \in \mathbb{R}^2 : x + y < 1\}$, $B = \{(x, y) \in \mathbb{R}^2 : x - y < 1\}$, $C = \{(x, y) \in \mathbb{R}^2 : -x + y < 1\}$ and $D = \{(x, y) \in \mathbb{R}^2 : -x - y < 1\}$.

Each of A, B, C, D is an open set in $\mathbb{R}^2$. S being the intersection of a finite number of open sets in $\mathbb{R}^2$, is an open set in $\mathbb{R}^2$.

3. Prove that an open bounded interval in $\mathbb{R}$ is not an open set in $\mathbb{R}^2$.

Let $a, b \in \mathbb{R}$ and $a < b$. Then $S = \{x \in \mathbb{R} : a < x < b\}$ is an open bounded interval in $\mathbb{R}$. In $\mathbb{R}^2$, S can be considered as the set $T = \{(x, y) \in \mathbb{R}^2 : a < x < b, y = 0\}$.

Let $(c, 0) \in T$. Then $a < c < b$. Any neighbourhood of $(c, 0)$ contains points of T and also points not in T. Therefore $(c, 0)$ is not an interior point of T. Hence T is not an open set in $\mathbb{R}^2$.

A1.6. Closed set.

Let S be a subset of $\mathbb{R}^2.S$ is said to be a *closed set* in $\mathbb{R}^2$ if the complement of S in $\mathbb{R}^2$ is an open set in $\mathbb{R}^2$.

Examples.

1. Let $S = \{(x, y) \in \mathbb{R}^2 : x + y \leq 1\}$. S is the complement of the set A in $\mathbb{R}^2$ where $A = \{(x, y) \in \mathbb{R}^2 : x + y > 1\}$.

A is an open sets in $\mathbb{R}^2$. S being the complement of an open set in $\mathbb{R}^2$, is a closed set in $\mathbb{R}^2$.

2. Let $S = \{(x, y) \in \mathbb{R}^2 : x+y = 1\}$. S is the complement of the set $A \cup B$ in $\mathbb{R}^2$ where $A = \{(x, y) \in \mathbb{R}^2 : x+y < 1\}$, $B = \{(x, y) \in \mathbb{R}^2 : x+y > 1\}$.

Since A and B are open sets in $\mathbb{R}^2$, $A \cup B$ is an open set in $\mathbb{R}^2.S$ being the complement of an open set in $\mathbb{R}^2$, is an closed set in $\mathbb{R}^2$.

3. Let $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. S is the complement of the set $A \cup B$ in $\mathbb{R}^2$ where $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$.

Since A and B are open sets in $\mathbb{R}^2$, $A \cup B$ is an open set in $\mathbb{R}^2.S$ being the complement of an open set in $\mathbb{R}^2$, is a closed set in $\mathbb{R}^2$.

4. Let S be the closed cell $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

S is the complement of the set $A \cup B \cup C \cup D$, in $\mathbb{R}^2$ where $A = \{(x, y) \in \mathbb{R}^2 : x < 0\}$, $B = \{(x, y) \in \mathbb{R}^2 : x > 1\}$, $C = \{(x, y) \in \mathbb{R}^2 : y < 0\}$, $D = \{(x, y) \in \mathbb{R}^2 : y > 1\}$.

A, B, C, D being open sets in $\mathbb{R}^2$, $A \cup B \cup C \cup D$ is an open set in $\mathbb{R}^2$. S being the complement of an open set in $\mathbb{R}^2$, is a closed set in $\mathbb{R}^2$.

Note. A closed cell in $\mathbb{R}^2$ is a closed set in $\mathbb{R}^2$.

5. Let $S = \mathbb{R}^2$. The complement of S in $\mathbb{R}^2$ is ϕ and it is an open set in $\mathbb{R}^2$. Therefore S is a closed set.

6. Let $S = \phi$. The complement of S in $\mathbb{R}^2$ is $\mathbb{R}^2$ and it is an open set. Therefore S is a closed set.

The following theorems are immediate consequences of the definition of a closed set.

Theorem A1.6.1. The union of a finite number of closed sets in $\mathbb{R}^2$ is a closed set in $\mathbb{R}^2$.

Theorem A1.6.2. The intersection of a finite number of closed sets in $\mathbb{R}^2$ is a closed set in $\mathbb{R}^2$.

Theorem A1.6.3. The intersection of an arbitrary collection of closed sets in $\mathbb{R}^2$ is a closed set in $\mathbb{R}^2$.

Note. The union of an infinite collection of closed sets in $\mathbb{R}^2$ is not necessarily a closed set in $\mathbb{R}^2$.

Let us consider the sets
$$\begin{aligned} F_1 &= \{(x,y) \in \mathbb{R}^2 : -1 \le x \le 1\} \\ F_2 &= \{(x,y) \in \mathbb{R}^2 : -2+\tfrac{1}{2} \le x \le 2-\tfrac{1}{2}\} \\ &\cdots \quad \cdots \quad \cdots \\ F_n &= \{(x,y) \in \mathbb{R}^2 : -2+\tfrac{1}{n} \le x \le 2-\tfrac{1}{n}\} \\ &\cdots \quad \cdots \quad \cdots \end{aligned}$$

Each F_i is a closed set in $\mathbb{R}^2$.

$\bigcup_{i=1}^{\infty} F_i = \{(x,y) \in \mathbb{R}^2 : -2 < x < 2\}$. This is not a closed set in $\mathbb{R}^2$.

Let us consider the sets
$$\begin{aligned} F_1 &= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\} \\ F_2 &= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le \tfrac{1}{2}\} \\ &\cdots \quad \cdots \quad \cdots \\ F_n &= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le \tfrac{1}{n}\} \\ &\cdots \quad \cdots \quad \cdots \end{aligned}$$

Each F_i is a closed set in $\mathbb{R}^2$. $\bigcup_{i=1}^{\infty} F_i = F_1$ and it is a closed set in $\mathbb{R}^2$.

These two examples establish that the union of an infinite number of closed sets in $\mathbb{R}^2$ is not necessarily a closed set in $\mathbb{R}^2$.

Worked Examples.

1. Prove that a closed and bounded interval in $\mathbb{R}$ is a closed set in $\mathbb{R}^2$.

Let $a, b \in \mathbb{R}$ and $a < b$. Then $S = \{x \in \mathbb{R} : a \le x \le b\}$ is a closed and bounded interval in $\mathbb{R}$.

In $\mathbb{R}^2$, S can be considered as the set $T = \{(x,y) \in \mathbb{R}^2 : a \le x \le b, y = 0\}$.

The set T^c (the complement of T in $\mathbb{R}^2$) can be expressed as $T^c = A \cup B \cup C \cup D$ where $A = \{(x,y) \in \mathbb{R}^2 : x < a\}$, $B = \{(x,y) \in \mathbb{R}^2 : x > b\}$, $C = \{(x,y) \in \mathbb{R}^2 : y < 0\}$, $D = \{(x,y) \in \mathbb{R}^2 : y > 0\}$.

Since A, B, C, D are open sets in $\mathbb{R}^2$, $A \cup B \cup C \cup D$ is an open set in $\mathbb{R}^2$. T being the complement of an open set in $\mathbb{R}^2$, is a closed set in $\mathbb{R}^2$.

Note. An infinite closed interval in $\mathbb{R}$ is a closed set in $\mathbb{R}^2$.

2. Prove that a one-element set $\{(a,b)\}$ $[a \in \mathbb{R}, b \in \mathbb{R}]$ is a closed set in $\mathbb{R}^2$.

Let $S = \{(a,b)\}$. S can be considered as the complement of the union $A \cup B \cup C \cup D$ where $A = \{(x,y) \in \mathbb{R}^2 : x < a\}$, $B = \{(x,y) \in \mathbb{R}^2 : x > a\}$, $C = \{(x,y) \in \mathbb{R}^2 : y < b\}$, $D = \{(x,y) \in \mathbb{R}^2 : y > b\}$.

A, B, C, D are open sets in $\mathbb{R}^2$. Therefore $A \cup B \cup C \cup D$ is an open set in $\mathbb{R}^2$. S being the complement of an open set in $\mathbb{R}^2$, is a closed set in $\mathbb{R}^2$.

Note. A finite subset in $\mathbb{R}^2$ is a closed set in $\mathbb{R}^2$.

A1.7. Limit point.

Let S be a subset of $\mathbb{R}^2$. A point $\mathbf{x}$ in $\mathbb{R}^2$ is said to be a *limit point* (or, an *accumulation point*) of S if every neighbourhood of $\mathbf{x}$ contains a point of S other than $\mathbf{x}$. That is, every deleted neighbourhood of $\mathbf{x}$ contains a point of S.

If $N(\mathbf{x})$ be a neighbourhood $\mathbf{x}$ then $N'(\mathbf{x}) = N(\mathbf{x}) - \{\mathbf{x}\}$ is called a *deleted neighbourhood* of the point $\mathbf{x}$.

This is to note that a limit point of a set S may not belong to S.

The set of all limit points of a set $S \subset \mathbb{R}^2$ is called the *derived set* of S. The derived set of S is denoted by S'.

Theorem A1.7.1. If $\mathbf{x}$ be a limit point of a set $S \subset \mathbb{R}^2$ then every neighbourhood of $\mathbf{x}$ contains infinitely many points of S.

Proof. Let us assume the contrary. Let a neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ contains only a finite number of points of S distinct from $\mathbf{x}$, say, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$.

Let $r = \min\{\| \mathbf{x} - \mathbf{x}_1 \|, \| \mathbf{x} - \mathbf{x}_2 \|, \cdots, \| \mathbf{x} - \mathbf{x}_m \|\}$. Then the neighbourhood $N(\mathbf{x}, \frac{r}{2})$ of $\mathbf{x}$ contains no point of S, contradicting that $\mathbf{x}$ is a limit point of S. This proves the theorem.

Corollary. A finite subset of $\mathbb{R}^2$ has no limit point.

Theorem A1.7.2. Let $S \subset \mathbb{R}^2$. S is a closed set if and only if $S' \subset S$.

Proof. Let S be a closed set in $\mathbb{R}^2$. Then S^c, the complement of S in $\mathbb{R}^2$, is an open set in $\mathbb{R}^2$.

If $S^c = \phi$, then $S = \mathbb{R}^2$ and $S' \subset S$.

If $S^c \neq \phi$, let $\mathbf{x} \in S^c$. Then there exists a neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \subset S^c$.

So $N(\mathbf{x}) \cap S = \phi$. This shows that $\mathbf{x}$ is not a limit point of S. That is, $\mathbf{x} \notin S'$.

$\mathbf{x} \in S^c \Rightarrow \mathbf{x} \notin S'$. Contrapositively, $\mathbf{x} \in S' \Rightarrow \mathbf{x} \notin S^c$, i.e., $\mathbf{x} \in S' \Rightarrow \mathbf{x} \in S$. Therefore $S' \subset S$.

Conversely, let S be a subset of $\mathbb{R}^2$ such that $S' \subset S$.

Let $\mathbf{x} \in S^c$. Then $\mathbf{x} \notin S$ and therefore $\mathbf{x} \notin S'$. Since $\mathbf{x}$ is not a limit point of S, there exists a neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \cap S = \phi$, i.e., $N(\mathbf{x}) \subset S^c$.

$\mathbf{x} \in S^c \Rightarrow N(\mathbf{x}) \subset S^c$. Therefore $\mathbf{x}$ is an interior point of S^c.

Thus every point of S^c is an interior point of S^c. Therefore S^c is an open set in $\mathbb{R}^2$. S being the complement of an open set in $\mathbb{R}^2$, is a closed set in $\mathbb{R}^2$. This completes the proof.

A1.8. Isolated point.

Let S be a subset of $\mathbb{R}^2$. A point $\mathbf{x}$ in S is said to be an *isolated point* of S if there exists a neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \cap \{\mathbf{x}\} = \{\mathbf{x}\}$.

Clearly, an isolated point of S is not a limit point of S.

Worked Example.

1. Let $S = \{(1,1), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{3}, \frac{1}{3}), \ldots\}$.

Prove that (i) each point of S is an isolated point of S,

(ii) $(0,0)$ is a limit point of S.

(i) Let $p \in \mathbb{N}$. Then $(\frac{1}{p}, \frac{1}{p}) \in S$. Let $\mathbf{a} = (\frac{1}{p}, \frac{1}{p})$ and $\epsilon = \frac{1}{p(p+1)}$.

Then the neighbourhood $N(\mathbf{a}, \epsilon)$ defined by $\{(x, y) \in \mathbb{R}^2 : \mid x - \frac{1}{p} \mid < \epsilon, \mid y - \frac{1}{p} \mid < \epsilon\}$ contains no point of S. This proves that $(\frac{1}{p}, \frac{1}{p})$ is an isolated point of S.

Therefore every point of S is an isolated point of S.

(ii) Let $\mathbf{0} = (0,0)$.

Let $\epsilon > 0$. by Archimedean property of $\mathbb{R}$, there exists a natural number p such that $0 < \frac{1}{p} < \frac{\epsilon}{\sqrt{2}}$. Therefore $0 < \sqrt{\frac{1}{p^2} + \frac{1}{p^2}} < \epsilon$.

Then the neighbourhood $N(\mathbf{0}, \epsilon)$ defined by $\{(x, y) \in \mathbb{R}^2 : \sqrt{(x-0)^2 + (y-0)^2} < \epsilon\}$ contains a point $(\frac{1}{p}, \frac{1}{p})$ of S other than $(0,0)$.

This proves that $(0,0)$ is a limit point of S.

Note. Since the point $(0,0)$ does not belong to S, it follows that S is not a closed set in $\mathbb{R}^2$.

A1.9. Adherent point.

Let S be a subset of $\mathbb{R}^2$. A point $\mathbf{x}$ in $\mathbb{R}^2$ is said to be an *adherent point* of S if every neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ contains a point of S. [That is, if $N(\mathbf{x}) \cap S \neq \phi$.]

The set of all adherent points of a set $S \subset \mathbb{R}^2$ is called the *closure* of S. The closure of S is denoted by $\bar{S}$.

Theorem A1.9.1. $\bar{S} = S \cup S'$.

From the definition it follows that (i) if $\mathbf{x} \in S$ then $\mathbf{x} \in \bar{S}$ and (ii) if $\mathbf{x} \in S'$ then $\mathbf{x} \in \bar{S}$. Therefore $S \cup S' \subset \bar{S}$ ... (i)

Let $\mathbf{x} \notin S \cup S'$. Then $\mathbf{x} \notin S$ and $\mathbf{x} \notin S'$.

Therefore there exists a neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \cap S = \phi$. This shows that $\mathbf{x}$ is not an adherent point of S, i.e., $\mathbf{x} \notin \bar{S}$.

$\mathbf{x} \notin S \cup S' \Rightarrow \mathbf{x} \notin \bar{S}$. Contrapositively, $\mathbf{x} \in \bar{S} \Rightarrow \mathbf{x} \in S \cup S'$ and therefore $\bar{S} \subset S \cup S'$ ... (ii)

From (i) and (ii) $S \cup S' = \bar{S}$.

Theorem A1.9.2. A set S in $\mathbb{R}^2$ is a closed set in $\mathbb{R}^2$ if and only if $S = \bar{S}$.

Proof. Let S be a closed set in $\mathbb{R}^2$. Then $\mathbb{R}^2 - S$ is an open set in $\mathbb{R}^2$.

Let $\mathbf{x} \in \mathbb{R}^2 - S$. Then there exists a neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \subset \mathbb{R}^2 - S$.

Therefore $N(\mathbf{x}) \cap S = \phi$. This implies $\mathbf{x} \notin \bar{S}$.

$\mathbf{x} \in \mathbb{R}^2 - S \Rightarrow \mathbf{x} \notin \bar{S}$. Contrapositively, $\mathbf{x} \in \bar{S} \Rightarrow \mathbf{x} \notin \mathbb{R}^2 - S$, i.e., $\mathbf{x} \in S$.

Therefore $\bar{S} \subset S$. Also by definition, $S \subset \bar{S}$. Combining, we have $S = \bar{S}$.

Conversely, let $S = \bar{S}$.

Let $\mathbf{x} \in \mathbb{R}^2 - S$. Then $\mathbf{x} \notin S$ and therefore $\mathbf{x} \notin \bar{S}$, since $S = \bar{S}$.

Therefore there exists a neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \cap S = \phi$. That is, $N(\mathbf{x}) \subset \mathbb{R}^2 - S$.

Thus $\mathbf{x} \in \mathbb{R}^2 - S \Rightarrow N(\mathbf{x}) \subset \mathbb{R}^2 - S$, showing that $\mathbf{x}$ is an interior point of $\mathbb{R}^2 - S$. This proves that $\mathbb{R}^2 - S$ is an open set. Therefore S is a closed set in $\mathbb{R}^2$.

This completes the proof.

Theorem A1.9.3. Let $S \subset \mathbb{R}^2$. Then $\bar{S}$ is a closed set in $\mathbb{R}^2$.

Proof left to the reader.

A1.10. Nested cells in $\mathbb{R}^2$.

Let $\{I_n : n \in \mathbb{N}\}$ be a family of cells in $\mathbb{R}^2$ such that $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$, then the family $\{I_n : n \in \mathbb{N}\}$ is said to be a family of *nested cells* in $\mathbb{R}^2$.

Theorem A1.10.1. (Theorem on nested cells in $\mathbb{R}^2$)

Let $\{I_1, I_2, I_3 \ldots\}$ be a family of non-empty closed and bounded cells such that $I_1 \supset I_2 \supset I_3 \supset \cdots$ Then $\bigcap_{n=1}^{\infty} I_n$ is non-empty.

Proof. Let $I_1 = \{(x_1, x_2) \in \mathbb{R}^2 : a_{11} \leq x_1 \leq b_{11}, a_{21} \leq x_2 \leq b_{21}\}$,

$$I_2 = \{(x_1, x_2) \in \mathbb{R}^2 : a_{12} \leq x_1 \leq b_{12}, a_{22} \leq x_2 \leq b_{22}\},$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$I_n = \{(x_1, x_2) \in \mathbb{R}^2 : a_{1n} \leq x_1 \leq b_{1n}, a_{2n} \leq x_2 \leq b_{2n}\},$$

$$\ldots \qquad \ldots \qquad \ldots$$

Then $I_1 = [a_{11}, b_{11}] \times [a_{21}, b_{21}], I_2 = [a_{12}, b_{12}] \times [a_{22}, b_{22}], \quad \cdots, I_n = [a_{1n}, b_{1n}] \times [a_{2n}, b_{2n}], \cdots$

Since $I_1 \supset I_2 \supset I_3 \supset \cdots$ it follows that $[a_{11}, b_{11}] \supset [a_{12}, b_{12}] \supset [a_{13}, b_{13}] \supset \cdots$ and $[a_{21}, b_{21}] \supset [a_{22}, b_{22}] \supset [a_{23}, b_{23}] \supset \cdots$

$\{[a_{1n}, b_{1n}] : n \in \mathbb{N}\}$ is a family of nested closed and bounded intervals in $\mathbb{R}$. By the theorem on nested intervals, there exists a point $x_1 \in \mathbb{R}$ such that $x_1 \in \bigcap_{i=1}^{\infty} [a_{1i}, b_{1i}]$.

$\{[a_{2n}, b_{2n}] : n \in \mathbb{N}\}$ is a family of nested closed and bounded intervals in $\mathbb{R}$. By the theorem on nested intervals, there exists a point $x_2 \in \mathbb{R}$ such that $x_2 \in \bigcap_{i=1}^{\infty} [a_{2i}, b_{2i}]$.

Therefore $(x_1, x_2) \in \bigcap_{n=1}^{\infty} I_n$, showing that $\bigcap_{n=1}^{\infty} I_n$ is non-empty.

This completes the proof.

A1.11. Bounded set in $\mathbb{R}^2$.

Definition. Let S be a subset of $\mathbb{R}^2$. S is said to be a *bounded set* in $\mathbb{R}^2$ if there exists a positive real number b such that $\| \mathbf{x} \| \leq b$ for all $\mathbf{x} \in S$.

If $\mathbf{x} = (x_1, x_2)$ then $| x_1 | \leq \| \mathbf{x} \|$ and $| x_2 | \leq \| \mathbf{x} \|$. It follows that if S is a bounded set then S is contained in the closed cell $[-b, b] \times [-b, b]$.

Diameter of a set. Let S be a subset of $\mathbb{R}^2$. The *diameter* of S is the supremum of the set $\{\| \mathbf{x} - \mathbf{y} \| : \mathbf{x} \in S, \mathbf{y} \in S\}$. It is denoted by $d(S)$.

Definition. Let S be a subset of $\mathbb{R}^2$. S is said to be a *bounded set* if $d(S)$ is finite.

Note. These two definitions are equivalent.

Theorem A1.11.1. Bolzano-Weierstrass theorem.

Every bounded infinite subset of $\mathbb{R}^2$ has at least one limit point.

Proof. Let S be a bounded subset of $\mathbb{R}^2$ containing infinite number of points. Since S is a bounded set, there exists a closed cell I_1 in $\mathbb{R}^2$ such that $S \subset I_1$.

Let $I_1 = \{(x_1, x_2) \in \mathbb{R}^2 : a_{11} \le x_1 \le b_{11}, a_{21} \le x_2 \le b_{21}\}$.

Let $l(I_1) = \max\,\{b_{11} - a_{11}, b_{21} - a_{21}\}$. Then $d(I_1)=$ the diameter of $I_1 \le \sqrt{2}l(I_1)$.

Let us divide I_1 into four closed subcells by bisecting each side of the rectangle. At least one of these subcells must contain infinitely many elements of S. We call one such I_2.

Let $I_2 = \{(x_1, x_2) \in \mathbb{R}^2 : a_{12} \le x_1 \le b_{12}, a_{22} \le x_2 \le b_{22}\}$.

Then $I_2 \subset I_1$ and $d(I_2) = \frac{1}{2}d(I_1)$. I_2 contains infinite number of points of S.

Let us divide I_2 into four closed subcells by bisecting each side of the rectangle. At least one of these subcells must contain infinitely many elements of S. We call one such I_3.

Then $I_3 \subset I_2 \subset I_1$ and $d(I_3) = \frac{1}{4}d(I_1)$.$I_3$ contains infinite number of points of S.

Continuing thus we obtain a family of closed cells $\{I_1, I_2, I_3, \ldots\}$ in $\mathbb{R}^2$ such that for all $n \in \mathbb{N}$,
(i) $I_{n+1} \subset I_n$, (ii) I_n contains infinite number of points of S and (iii) $d(I_n) = \frac{1}{2^{n-1}}d(I_1)$.

Since each I_n is a closed and bounded cell, by the theorem on nested cells in $\mathbb{R}^2$ there exists a point $\mathbf{x} = (x_1, x_2)$ in $\mathbb{R}^2$ such that $\mathbf{x} \in \bigcap\limits_{n=1}^{\infty} I_n$ (i)

We prove that $\mathbf{x}$ is a limit point of S.

Since $d(I_1) \in \mathbb{R}$ and $d(I_n) = \frac{1}{2^{n-1}}d(I_1)$ for all $n \in \mathbb{N}$, the sequence $\{d(I_n)\}$ is a null sequence in $\mathbb{R}$.

Let us choose a positive ϵ. Then there exists a natural number m such that $d(I_n) < \epsilon$ for all $n \ge m$ ... (ii)

By (i), $\mathbf{x} \in I_m$ and by (ii), $d(I_m) < \epsilon$. Therefore $I_m \subset N(\mathbf{x}, \epsilon)$.

Since I_m contains infinite number of points of S, the neighbourhood $N(\mathbf{x}, \epsilon)$ of $\mathbf{x}$ contains infinite number of points of S

Since ϵ is arbitrary, it follows that $\mathbf{x}$ is a limit point of S.

This proves the existence of a limit point of S.

A1.12. Cover of a set in $\mathbb{R}^2$.

Let S be a subset of $\mathbb{R}^2$ and C be a collection of sets in $\mathbb{R}^2$ given by $\{A_\alpha : \alpha \in \Lambda\}, \Lambda$ being the index set. C is said to be a *cover* of S if

$S \subset \bigcup_{\alpha\in\Lambda} A_\alpha$.

Let $\mathcal{G}$ be a collection of open sets in $\mathbb{R}^2$ given by $\{A_\alpha : \alpha \in \Lambda\}, \Lambda$ being the index set. $\mathcal{G}$ is said to be an *open cover* of S if $S \subset \bigcup_{\alpha\in\Lambda} A_\alpha$.

Let $\mathcal{G}$ be a collection of sets in $\mathbb{R}^2$ such that $\mathcal{G}$ covers S. If $\mathcal{G}'$ be a subcollection of $\mathcal{G}$ such that G' also covers S, then $\mathcal{G}'$ is said to be a *subcover* of the cover $\mathcal{G}$.

If the subcollection $\mathcal{G}'$ contains a finite number of sets of $\mathcal{G}$ and $\mathcal{G}'$ covers S, then $\mathcal{G}'$ is said to be a *finite subcover* of the cover $\mathcal{G}$.

Worked Example.

1. Let $I_n = \{(x, y) \in \mathbb{R}^2 : -n < x < n, -n < y < n\}, n = 1, 2, 3, \ldots$ and $\mathcal{G} = \{I_n : n \in \mathbb{N}\}$. Prove that the family $\mathcal{G}$ is an open cover of the set $\mathbb{R}^2$. Show that there is no finite subfamily $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers the set $\mathbb{R}^2$.

Let $(p, q) \in \mathbb{R}^2$. Then $p, q \in \mathbb{R}$ and $|\, p \,| \geq 0, |\, q \,| \geq 0$. There exist natural numbers u, v such that $u - 1 \leq |\, p \,| < u, v - 1 \leq |\, q \,| < v$. Let $w=$ max $\{u, v\}$. Then $w \in \mathbb{N}$ and $(p, q) \in I_w$.

$(p, q) \in \mathbb{R}^2 \Rightarrow (p, q) \in \bigcup_{n\in\mathbb{N}} I_n$ and this implies $\mathbb{R}^2 \in \bigcup_{n=1}^{\infty} I_n$. Therefore $\mathcal{G}$ is an open cover of the set $\mathbb{R}^2$.

Let $\mathcal{G}' = \{I_{r_1}, I_{r_2}, \ldots, I_{r_m}\}$ be a finite subcollection of $\mathcal{G}$ such that $\mathcal{G}'$ covers the set $\mathbb{R}^2$.

Then $\mathbb{R}^2 \subset I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m}$ ... (i)

Let $p =$ max $\{r_1, r_2, \ldots, r_m\}$. Then $I_{r_1} \subset I_p, I_{r_2} \subset I_p, \cdots, I_{r_m} \subset I_p$ and therefore $I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m} \subset I_p$.

From (i) it follows that $\mathbb{R}^2 \subset I_p$ but this cannot be, since $(p, p) \in \mathbb{R}^2$ but $(p, p) \notin I_p$.

Therefore $\mathcal{G}'$ cannot cover the set $\mathbb{R}^2$. So no finite subfamily of $\mathcal{G}$ can be a cover of the set $\mathbb{R}^2$.

Compact set. Let S be a subset of $\mathbb{R}^2$. S is said to be a *compact set* in $\mathbb{R}^2$ if every open cover $\mathcal{G}$ of S has a finite subcover. That is, if $\mathcal{G}$ be a collection of open sets in $\mathbb{R}^2$ that covers S then there exists a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers S.

Heine-Borel theorem. Let S be a closed and bounded set in $\mathbb{R}^2$. Then every open cover of S has a finite subcover.

Heine-Borel theorem states that a closed and bounded set in $\mathbb{R}^2$ is a compact set in $\mathbb{R}^2$.

Converse of Heine-Borel theorem. A compact set in $\mathbb{R}^2$ is a closed and bounded set in $\mathbb{R}^2$.

The Heine-Borel theorem and its converse characterise the compact sets in $\mathbb{R}^2$. The closed and bounded sets in $\mathbb{R}^2$ are the only compact sets in $\mathbb{R}^2$.

Worked Examples.

1. Let T be a finite subset of $\mathbb{R}^2$. Using the definition of a compact set, show that T is a compact set in $\mathbb{R}^2$.

Let $T = \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m\}$ be a finite set in $\mathbb{R}^2$. Let $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$, Λ being the index set, be an open cover of T.

Each $\mathbf{x}_i$ is contained in some open set G_{α_i} of the collection $\mathcal{G}$. Then the union of the finite collection $\{G_{\alpha_1}, G_{\alpha_2}, \cdots, G_{\alpha_m}\}$ is a cover of T.

So $\mathcal{G}$ has a finite subcover. Since $\mathcal{G}$ is arbitrary, every open cover of T has a finite subcover. Consequently, T is compact in $\mathbb{R}^2$.

2. Show that the set $S = \{(1,1), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{3}, \frac{1}{3}), \ldots\}$ is not a compact set in $\mathbb{R}^2$.

$(0,0)$ is a limit point of S and $(0,0)$ does not belong to S. Therefore the set S is not a closed set in $\mathbb{R}^2$. [Worked Ex.1, page 620.]

S is not a compact set in $\mathbb{R}^2$, since a compact set in $\mathbb{R}^2$ is a closed and bounded set in $\mathbb{R}^2$.

Exercises

1. Show that the set S is an open set in $\mathbb{R}^2$.

(i) $S = \{(x,y) \in \mathbb{R}^2 : |x| < 1\}$,

(ii) $S = \{(x,y) \in \mathbb{R}^2 : |x| > 1\}$,

(iii) $S = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0, x + y < 1\}$,

(iv) $S = \{(x,y) \in \mathbb{R}^2 : x > 0, x^2 + y^2 < 1\}$.

2. Show that the set S is a closed set in $\mathbb{R}^2$.

(i) $S = \{(x,y) \in \mathbb{R}^2 : |x| \leq 1\}$,

(ii) $S = \{(x,y) \in \mathbb{R}^2 : |x| = 1\}$,

(iii) $S = \{(x,y) \in \mathbb{R}^2 : 2x + 3y = 1\}$,

(iv) $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$,

3. Show that the set S is not a compact set in $\mathbb{R}^2$.

(i) $S = \{(x,y) \in \mathbb{R}^2 : |x| \geq 1\}$,

(ii) $S = \{(x,y) \in \mathbb{R}^2 : x = y\}$,

(iii) $S = \{(1,1), (1, \frac{1}{2}), (1, \frac{1}{3}), (1, \frac{1}{4}), \cdots\}$,

(iv) $S = \{(\frac{1}{m}, \frac{1}{n}) : m \in \mathbb{N}, n \in \mathbb{N}\}$,

(v) $S = \{(1,1), (2,2), (3,3), (4,4), \cdots\}$.

4. Use the definition of a compact set in $\mathbb{R}^2$ to prove that

(i) the union of two compact sets in $\mathbb{R}^2$ is a compact set in $\mathbb{R}^2$;

(ii) the intersection of two compact sets in $\mathbb{R}^2$ is a compact set in $\mathbb{R}^2$.

5. Define a compact set in $\mathbb{R}^2$. Use the definition to prove that

(i) the set $\mathbb{Z} \times \mathbb{Z}$ is not a compact set in $\mathbb{R}^2$;

(ii) the set $\mathbb{N} \times \mathbb{N}$ is not a compact set in $\mathbb{R}^2$.

[**Hint.** Let $C_n = \{(x,y) \in \mathbb{R}^2 : |x| < n, |y| < n\}$, $n = 1, 2, 3, \ldots$ and $\mathcal{G} = \{C_n : n \in \mathbb{N}\}$. Show that $\mathcal{G}$ is an open cover of the set having no finite subcover.]

6. Let $C_n = \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} < n\}, n = 1, 2, 3, \ldots$ and $\mathcal{G} = \{C_n : n \in \mathbb{N}\}$. Show that

(i) $\mathcal{G}$ is an open cover of the set $\mathbb{R}^2$;

(ii) $\mathcal{G}$ has no finite subcover.

7. Let A and B be subsets of $\mathbb{R}^2$ of which A is closed in $\mathbb{R}^2$ and B is compact in $\mathbb{R}^2$. Prove that $A \cap B$ is a compact set in $\mathbb{R}^2$.

A2.1. Introduction.

$\mathbb{R}^2$ is the set of all ordered pairs $\{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$.

A sequence in $\mathbb{R}^2$ is a mapping $X : \mathbb{N} \to \mathbb{R}^2$. To each natural number n the X-image, generally denoted by X_n, is an element in $\mathbb{R}^2$ and it is called the *nth element* of the sequence X. Let $X_n = (x_{1n}, x_{2n})$. Then the elements of the sequence are $(x_{11}, x_{21}), (x_{12}, x_{22}), (x_{13}, x_{23}), \quad \ldots \quad \ldots$

The sequence X is also denoted by the symbol $\{X_n\}$. The symbol $\{(x_{11}, x_{21}), (x_{12}, x_{22}), (x_{13}, x_{23}), \quad \ldots \quad \ldots\}$ is also used to describe the sequence X. The elements of the sequence have an order induced by the order of the natural numbers. The image set of the sequence is $\{X_n : n \in \mathbb{N}\}$.

Examples.

1. Let $X : \mathbb{N} \to \mathbb{R}^2$ be defined by $X_n = (n, \frac{1}{n}), n \in \mathbb{N}$. Then $X_1 = (1,1), X_2 = (2, \frac{1}{2}), X_3 = (3, \frac{1}{3}), \quad \ldots \quad \ldots$ The sequence is $\{(1,1), (2, \frac{1}{2}), (3, \frac{1}{3}), \quad \ldots \quad \ldots\}$.

2. Let $X : \mathbb{N} \to \mathbb{R}^2$ be defined by $X_n = ((-1)^n, (-1)^{n+1}), n \in \mathbb{N}$. The sequence is $\{(-1, 1), (1, -1), (-1, 1), \quad \ldots \quad \ldots\}$. The image set of the sequence is $\{(-1, 1), (1, -1)\}$, a set containing only two elements.

3. Let $X : \mathbb{N} \to \mathbb{R}^2$ be defined by $X_n = (\frac{1}{n}, 1 - \frac{1}{n}), n \in \mathbb{N}$. The sequence is $\{(1, 0), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{3}, \frac{2}{3}), (\frac{1}{4}, \frac{3}{4}), \quad \ldots \quad \ldots\}$.

4. Let $X : \mathbb{N} \to \mathbb{R}^2$ be defined by $X_n = (1, 1)$ for all $n \in \mathbb{N}$. The sequence is $\{(1,1), (1,1), (1,1), \quad \ldots \quad \ldots\}$. This is called a *constant sequence.*

A2.2. Bounded sequence.

A sequence $\{X_n\}$ in $\mathbb{R}^2$ is said to be a *bounded sequence* if there exists a positive real number b such that $\| X_n \| \leq b$ for all $n \in \mathbb{N}$, where $\| X_n \| = \sqrt{{x_{1n}}^2 + {x_{2n}}^2}$.

Examples.

1. Let $X_n = (\frac{1}{2^n}, \frac{1}{3^n})$. Then the sequence $\{X_n\}$ is a bounded sequence, because $\| X_n \| = \sqrt{(\frac{1}{2^n})^2 + (\frac{1}{3^n})^2} < \frac{1}{\sqrt{2}}$ for all $n \in \mathbb{N}$.

2. Let $X_n = ((-1)^n, (-1)^{n+1})$. Then the sequence $\{X_n\}$ is a bounded sequence because $\| X_n \| = \sqrt{(-1)^{2n} + (-1)^{2n+2}} = \sqrt{2}$ for all $n \in \mathbb{N}$.

3. Let $X_n = (n, \frac{1}{n})$. Then the sequence $\{X_n\}$ is not a bounded sequence.

Theorem A2.2.1. A sequence $\{X_n\}$ in $\mathbb{R}^2$ where $X_n = (x_{1n}, x_{2n})$ for all $n \in \mathbb{N}$, is a bounded sequence if and only if both the sequences $\{x_{1n}\}$ and $\{x_{2n}\}$ are bounded.

[The sequences $\{x_{1n}\}$ and $\{x_{2n}\}$ are called the *co-ordinate sequences* of the sequence $\{X_n\}$.]

Proof. Let $\{X_n\}$ be a bounded sequence. Then there exists a positive real number b such that $\| X_n \| \leq b$ for all $n \in \mathbb{N}$,

i.e., $\sqrt{x_{1n}{}^2 + x_{2n}{}^2} \leq b$ for all $n \in \mathbb{N}$.

But $\sqrt{x_{1n}{}^2 + x_{2n}{}^2} \geq |\, x_{1n} \,|$ and also $\sqrt{x_{1n}{}^2 + x_{2n}{}^2} \geq |\, x_{2n} \,|$ for all $n \in \mathbb{N}$.

Hence $|\, x_{1n} \,| \leq b$ as well as $|\, x_{2n} \,| \leq b$ for all $n \in \mathbb{N}$.

This implies that $\{x_{1n}\}$ and $\{x_{2n}\}$ are both bounded sequences.

Conversely, let $\{x_{1n}\}$ and $\{x_{2n}\}$ be both bounded sequences.

Then there exist positive real numbers b_1, b_2 such that

$$|\, x_{1n} \,| \leq b_1, |\, x_{2n} \,| \leq b_2 \text{ for all } n \in \mathbb{N}.$$

Let $b = \max \{b_1, b_2\}$.

Then $\| X_n \| = \sqrt{x_{1n}^2 + x_{2n}^2} \leq \sqrt{2}b$ for all $n \in \mathbb{N}$.

This proves that $\{X_n\}$ is a bounded sequence.

A2.3. Limit of a sequence in $\mathbb{R}^2$.

Let $\{X_n\}$ be a sequence in $\mathbb{R}^2$. An element $\mathbf{x}$ in $\mathbb{R}^2$ is said to be a *limit* of $\{X_n\}$ if for a pre-assigned positive ϵ there exists a natural number k such that

$$\| X_n - \mathbf{x} \| < \epsilon \text{ for all } n \geq k.$$

Theorem A2.3.1. A sequence in $\mathbb{R}^2$ can have at most one limit.

Proof. If possible, let a sequence $\{X_n\}$ in $\mathbb{R}^2$ have two distinct limits $\mathbf{x}'$ and $\mathbf{x}''$.

Let us choose $\epsilon = \frac{1}{2} \| \mathbf{x}' - \mathbf{x}'' \|$. Then the ϵ -balls $B(\mathbf{x}', \epsilon)$ and $B(\mathbf{x}'', \epsilon)$ are disjoint.

[Note that $B(\mathbf{x}', \epsilon) = \{\mathbf{x} \in \mathbb{R}^2 : \| \mathbf{x}' - \mathbf{x} \| < \epsilon\}, B(\mathbf{x}'', \epsilon) = \{\mathbf{x} \in \mathbb{R}^2 : \| \mathbf{x}'' - \mathbf{x} \| < \epsilon\}$.]

Since $\mathbf{x}'$ is a limit of the sequence, there exists a natural number k_1 such that $\| X_n - \mathbf{x}' \| < \epsilon$ for all $n \geq k_1$.

Since $\mathbf{x}''$ is a limit of the sequence, there exists a natural number k_2 such that $\| X_n - \mathbf{x}'' \| < \epsilon$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $\| X_k - \mathbf{x}' \| < \epsilon$ as well as $\| X_k - \mathbf{x}'' \| < \epsilon$.

Consequently, $X_k \in B(\mathbf{x}', \epsilon)$ and $X_k \in B(\mathbf{x}'', \epsilon)$ i.e., $X_k \in B(\mathbf{x}', \epsilon) \cap B(\mathbf{x}'', \epsilon)$.

But $B(\mathbf{x}', \epsilon) \cap B(\mathbf{x}'', \epsilon) = \phi$ and we arrive at a contradiction.

Therefore our assumption is wrong and this proves the theorem.

A2.4. Convergent sequence.

A sequence in $\mathbb{R}^2$ is said to be a *convergent sequence* if it has a limit $\mathbf{x}$ in $\mathbb{R}^2$. In this case we also say that the sequence $\{X_n\}$ *converges to* $\mathbf{x}$. We write $\lim X_n = \mathbf{x}$.

If a sequence $\{X_n\}$ has no limit then we say that the sequence $\{X_n\}$ is *divergent*.

The following theorem establishes a connection between the convergence of $\{X_n\}$ and the convergence of the co-ordinate sequences.

Theorem A2.4.1. A sequence $\{X_n\}$ in $\mathbb{R}^2$ where $X_n = (x_{1n}, x_{2n}), n \in \mathbb{N}$ converges to an element $\mathbf{x} = (x_1, x_2)$ in $\mathbb{R}^2$ if and only if the real sequences $\{x_{1n}\}$ and $\{x_{2n}\}$ converge to x_1 and x_2 respectively.

Proof. Since $\lim X_n = \mathbf{x}$, for a pre-assigned positive ϵ, there exists a natural number k such that

$$\| X_n - \mathbf{x} \| < \epsilon \text{ for all } n \geq k.$$

$$\begin{aligned} \text{But } \| X_n - \mathbf{x} \| &= \sqrt{(x_{1n} - x_1)^2 + (x_{2n} - x_2)^2} \\ &\geq | x_{1n} - x_1 |. \\ \text{Also } \| X_n - \mathbf{x} \| &\geq | x_{2n} - x_2 |. \end{aligned}$$

Therefore $| x_{1n} - x_1 | < \epsilon$ and $| x_{2n} - x_2 | < \epsilon$ for all $n \geq k$.

This shows that the real sequences $\{x_{1n}\}$ and $\{x_{2n}\}$ are convergent with the limits x_1 and x_2 respectively.

Conversely, let us suppose that the sequences $\{x_{1n}\}$ and $\{x_{2n}\}$ converge to x_1 and x_2 respectively.

Let $\epsilon > 0$. There exist natural numbers k_1 and k_2 such that

$| x_{1n} - x_1 | < \frac{\epsilon}{\sqrt{2}}$ for all $n \geq k_1$ and $| x_{2n} - x_2 | < \frac{\epsilon}{\sqrt{2}}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$.

Then $| x_{1n} - x_1 | < \frac{\epsilon}{\sqrt{2}}$ and $| x_{2n} - x_2 | < \frac{\epsilon}{\sqrt{2}}$ for all $n \geq k$.

Now $\| (x_{1n}, x_{2n}) - (x_1, x_2) \| = \sqrt{(x_{1n} - x_1)^2 + (x_{2n} - x_2)^2} < \epsilon$ for all $n \geq k$.

That is, $\| X_n - \mathbf{x} \| < \epsilon$ for all $n \geq k$. This proves that the sequence $\{X_n\}$ converges to $\mathbf{x}$.

Examples.

1. Let $X_n = (\frac{1}{n}, \frac{1}{n+1}), n \geq 1$. Prove that the sequence $\{X_n\}$ is convergent.

Let $X_n = (x_{1n}, x_{2n})$. Then $x_{1n} = \frac{1}{n}$ for $n \geq 1; x_{2n} = \frac{1}{n+1}$ for $n \geq 1$.
Now $\lim x_{1n} = 0, \lim x_{2n} = 0$. Therefore $\lim X_n = (0, 0)$.

2. Let $X_n = ((-1)^n, \frac{1}{n}), n \geq 1$. Prove that the sequence $\{X_n\}$ is divergent.

Let $X_n = (x_{1n}, x_{2n})$. Then $x_{1n} = (-1)^n$ and $x_{2n} = \frac{1}{n}$ for $n \geq 1$.
$\{x_{1n}\}$ is a divergent sequence.

Therefore $\{X_n\}$ is a divergent sequence, because the convergence of the sequence $\{X_n\}$ implies the convergence of both the real sequences $\{x_{1n}\}$ and $\{x_{2n}\}$.

Theorem A2.4.2. A convergent sequence in $\mathbb{R}^2$ is bounded.

Proof. Let $\{X_n\}$ be a convergent sequence in $\mathbb{R}^2$. Let $\lim X_n = \mathbf{x}$.

Let $\epsilon = 1$. Then there exists a natural number k such that

$$\| X_n - \mathbf{x} \| < 1 \text{ for all } n \geq k.$$

$\| X_n \| = \| X_n - \mathbf{x} + \mathbf{x} \| \leq \| X_n - \mathbf{x} \| + \| \mathbf{x} \|$.

Therefore $\| X_n \| \leq 1 + \| \mathbf{x} \|$ for all $n \geq k$.

Let $b = \max\{\| X_1 \|, \| X_2 \|, \ldots, \| X_{k-1} \|, \| \mathbf{x} \| + 1\}$. Then $\| X_n \| \leq b$ for all $n \in \mathbb{N}$. This proves that the sequence $\{X_n\}$ is bounded.

Note. The converse of the theorem is not true.

The sequence $\{((-1)^n, \frac{1}{n})\}$ is a bounded sequence but it not a convergent sequence.

Theorem A2.4.3. Let $\{X_n\}$ and $\{Y_n\}$ be two convergent sequences in $\mathbb{R}^2$ and $\lim X_n = \mathbf{x}, \lim Y_n = \mathbf{y}$. Then

(i) $\lim(X_n + Y_n) = \mathbf{x} + \mathbf{y}$,

(ii) $\lim cX_n = c\mathbf{x}$, c being a real number.

Proof. (i) To show that $\lim(X_n + Y_n) = \mathbf{x} + \mathbf{y}$, we need to establish that for a pre-assigned positive ϵ there exists a natural number k such that

$$\| (X_n + Y_n) - (\mathbf{x} + \mathbf{y}) \| < \epsilon \text{ for all } n \geq k.$$

$\| (X_n + Y_n) - (\mathbf{x} + \mathbf{y}) \| \leq \| X_n - \mathbf{x} \| + \| Y_n - \mathbf{y} \|$, by triangle inequality.

Let $\epsilon > 0$. Since $\lim X_n = \mathbf{x}$ and $\lim Y_n = \mathbf{y}$, there exist natural numbers k_1 and k_2 such that $\| X_n - \mathbf{x} \| < \frac{\epsilon}{2}$ for all $n \geq k_1$ and $\| Y_n - \mathbf{y} \| < \frac{\epsilon}{2}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$.

Then $\| X_n - \mathbf{x} \| < \frac{\epsilon}{2}$ and $\| Y_n - \mathbf{y} \| < \frac{\epsilon}{2}$ for all $n \geq k$.

Therefore $\| (X_n + Y_n) - (\mathbf{x} + \mathbf{y}) \| < \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, $\lim(X_n + Y_n) = \mathbf{x} + \mathbf{y}$.

(ii) Proof left to the reader.

Theorem A2.4.4. Let $\{X_n\}$ be a convergent sequence in $\mathbb{R}^2$ and let $\{c_n\}$ be a sequence in $\mathbb{R}$ that converges to $c \in \mathbb{R}$. Then the sequence $\{c_n X_n\}$ in $\mathbb{R}^2$ converges to $c\mathbf{x}$.

Proof.
$$\begin{aligned} \| c_n X_n - c\mathbf{x} \| &= \| c_n X_n - cX_n + cX_n - c\mathbf{x} \| \\ &\leq \| c_n X_n - cX_n \| + | c | \| X_n - \mathbf{x} \| \\ &= | c_n - c | \| X_n \| + | c | \| X_n - \mathbf{x} \| . \end{aligned}$$

Since $\{X_n\}$ is a convergent sequence, it is bounded. Therefore there exists a positive number b_1 such that $\| X_n \| \leq b_1$ for all $n \in \mathbb{N}$.

Let $b = \max\{b_1, | c |\}$. Then $b > 0$ and
$$\| c_n X_n - c\mathbf{x} \| \leq b \, | c_n - c | + b \, \| X_n - \mathbf{x} \| .$$

Let $\epsilon > 0$. Since $\lim c_n = c$ and $\lim X_n = \mathbf{x}$ there exist natural numbers k_1, k_2 such that $| c_n - c | < \frac{\epsilon}{2b}$ for all $n \geq k_1$ and $\| X_n - \mathbf{x} \| < \frac{\epsilon}{2b}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$.

Then $\| c_n X_n - c\mathbf{x} \| < \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, $\lim c_n X_n = c\mathbf{x}$.

Theorem A2.4.5. Let $\{X_n\}, \{Y_n\}$ be two convergent sequences in $\mathbb{R}^2$ and $\lim X_n = \mathbf{x}, \lim Y_n = \mathbf{y}$.

Let the sequence $\{X_n.Y_n\}$ be defined by
$$\begin{aligned} X_n.Y_n &= (x_{1n}, x_{2n}).(y_{1n}, y_{2n}) \\ &= x_{1n}y_{1n} + x_{2n}y_{2n} \quad \text{(the inner product of } X_n \text{ and } Y_n). \end{aligned}$$

Then the real sequence $\{X_n.Y_n\}$ converges to $\mathbf{x}.\mathbf{y}$.

Proof. $| X_n.Y_n - \mathbf{x}.\mathbf{y} | = | X_n.Y_n - X_n.\mathbf{y} + X_n.\mathbf{y} - \mathbf{x}.\mathbf{y} |$
$\leq | X_n.(Y_n - \mathbf{y}) + (X_n - \mathbf{x}).\mathbf{y} |$
$\leq | X_n.(Y_n - \mathbf{y}) | + | (X_n - \mathbf{x}).\mathbf{y} |$.

By Schwarz's inequality,
$| X_n.(Y_n - \mathbf{y}) | \leq \| X_n \| \| Y_n - \mathbf{y} \|$, $\quad | (X_n - \mathbf{x}).\mathbf{y} | \leq \| X_n - \mathbf{x} \| \| \mathbf{y} \|$.

Therefore $| X_n.Y_n - \mathbf{x}.\mathbf{y} | \leq \| X_n \| \| Y_n - \mathbf{y} \| + \| X_n - \mathbf{x} \| \| \mathbf{y} \|$.

Since $\{X_n\}$ is a convergent sequence, it is bounded.

Therefore There exists a positive number b_1, such that $\| X_n \| \leq b_1$ for all $n \in \mathbb{N}$. Let $b = \max\{b_1, \| \mathbf{y} \|\}$.

Then $| X_n.Y_n - \mathbf{x}.\mathbf{y} | \leq b\{\| Y_n - \mathbf{y} \| + \| X_n - \mathbf{x} \|\}$.

Let $\epsilon > 0$. There exist natural numbers k_1 and k_2 such that

$\| X_n - \mathbf{x} \| < \frac{\epsilon}{2b}$ for all $n \geq k_1$ and $\| Y_n - \mathbf{y} \| < \frac{\epsilon}{2b}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$.

Then $\| X_n - \mathbf{x} \| < \frac{\epsilon}{2b}$ and $\| Y_n - \mathbf{y} \| < \frac{\epsilon}{2b}$ for all $n \geq k$.

Therefore $| X_n.Y_n - \mathbf{x}.\mathbf{y} | < \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, $\lim X_n.Y_n = \mathbf{x}.\mathbf{y}$.

This completes the proof.

A2.5. Subsequence.

Let $\{X_n\}$ be a sequence in $\mathbb{R}^2$ and $\{r_n\}$ be a strictly increasing sequence of natural numbers. Then $\{X_{r_n}\}$ is said to be a *subsequence* of the sequence $\{X_n\}$.

Examples.

1. Let $X_n = (\frac{1}{n}, (-1)^n), n \geq 1$ and $r_n = 2n, n \geq 1$.

Then $\{X_{r_n}\} = \{X_{2n}\} = \{(\frac{1}{2}, 1), (\frac{1}{4}, 1), (\frac{1}{6}, 1), \quad \ldots\}$ is a subsequence of the sequence $\{X_n\}$.

2. Let $X_n = (\frac{1}{n}, (-1)^n), n \geq 1$ and $r_n = 2n - 1, n \geq 1$.

Then $\{X_{r_n}\} = \{X_{2n-1}\} = \{(1, -1), (\frac{1}{3}, -1), (\frac{1}{5}, -1), \quad \ldots \quad \ldots\}$ is a subsequence of the sequence $\{X_n\}$.

Theorem A2.5.1. If a sequence $\{X_n\}$ in $\mathbb{R}^2$ converges to a limit $\mathbf{x}$ in $\mathbb{R}^2$ then every subsequence of $\{X_n\}$ converges to $\mathbf{x}$.

Proof. Let $X_n = (u_n, v_n), n \geq 1$ and let $\mathbf{x} = (u, v)$.

$\lim X_n = \mathbf{x}$ implies $\lim u_n = u$ and $\lim v_n = v$.

Let $\{r_n\}$ be a strictly increasing sequence of natural numbers.

Then $\{X_{r_n}\}$ is a subsequence of $\{X_n\}$. $\quad X_{r_n} = (u_{r_n}, v_{r_n})$.

$\{u_{r_n}\}$ is a subsequence of the sequence $\{u_n\}$ and $\{v_{r_n}\}$ is a subsequence of the sequence $\{v_n\}$.

$\lim u_n = u$ implies $\lim u_{r_n} = u$ and $\lim v_n = v$ implies $\lim v_{r_n} = v$.

Hence $\lim X_{r_n} = (u, v) = \mathbf{x}$. This proves the theorem.

Note. If there exist two different subsequences of $\{X_n\}$ that converge to two distinct limits then the sequence $\{X_n\}$ is divergent.

Worked Examples.

1. Prove that the sequence $\{(-1,1),(1,\frac{1}{2}),(-1,\frac{1}{3}),(1,\frac{1}{4}),\quad \ldots \quad\}$ is not convergent.

Let $\{X_n\}$ be the given sequence. Then $X_n = ((-1)^n, \frac{1}{n}), n \geq 1$.
$X_{2n-1} = (-1, \frac{1}{2n-1})$. The subsequence $\{X_{2n-1}\}$ converges to $(-1,0)$.
$X_{2n} = (1, \frac{1}{2n})$. The subsequence $\{X_{2n}\}$ converges to $(1,0)$.

As the subsequences $\{X_{2n-1}\}$ and $\{X_{2n}\}$ converge to two different limits, the sequence $\{X_n\}$ is divergent.

2. Prove that the sequence $\{(1,1),(1,-1),(\frac{1}{2},1),(\frac{1}{2},-1),(\frac{1}{3},1),(\frac{1}{3},-1),\quad \ldots \quad\}$ is not convergent.

Let $\{X_n\}$ be the given sequence. Let $X_n = (x_n, y_n)$.

$$\text{Then } x_n = \frac{2}{n+1} \text{ if } n \text{ be odd} \qquad y_n = 1 \text{ if } n \text{ be odd}$$
$$= \frac{2}{n} \text{ if } n \text{ be even}; \qquad = -1 \text{ if } n \text{ be even}.$$

$X_{2n-1} = (\frac{1}{n}, 1)$. The sequence $\{X_{2n-1}\}$ converges to $(0,1)$.
$X_{2n} = (\frac{1}{n}, -1)$. The sequence $\{X_{2n}\}$ converges to $(0,-1)$.

As the subsequences $\{X_{2n-1}\}$ and $\{X_{2n}\}$ converge to two different limits, the sequence $\{X_n\}$ is divergent.

Theorem A2.5.2. Let $\{I_n\}$ be a sequence of non-empty closed and bounded cells in $\mathbb{R}^2$ such that $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} I_n$ is non-empty.

Proof. Let $I_1 = [a_{11}, b_{11}] \times [a_{21}, b_{21}], I_2 = [a_{12}, b_{12}] \times [a_{22}, b_{22}], \quad \cdots$
$\cdots \quad , I_n = [a_{1n}, b_{1n}] \times [a_{2n}, b_{2n}], \quad \cdots \quad \cdots$

Since $I_{n+1} \subset I_n, [a_{1\,n+1}, b_{1\,n+1}] \times [a_{2\,n+1}, b_{2\,n+1}] \subset [a_{1n}, b_{1n}] \times [a_{2n}, b_{2n}]$.

It follows that $[a_{1\,n+1}, b_{1\,n+1}] \subset [a_{1n}, b_{1n}]$ and $[a_{2\,n+1}, b_{2\,n+1}] \subset [a_{2n}, b_{2n}]$ for all $n \in \mathbb{N}$.

Therefore the sequence $\{[a_{1n}, b_{1n}]\}$ is a sequence of closed and bounded intervals in $\mathbb{R}$ such that $[a_{1\,n+1}, b_{1\,n+1}] \subset [a_{1n}, b_{1n}]$ for all $n \in \mathbb{N}$.
By the nested intervals theorem in $\mathbb{R}$, there exists a real number ξ such that $\xi \in [a_{1n}, b_{1n}]$ for all $n \in \mathbb{N}$.

By similar arguments there exists a real number η such that
$\eta \in [a_{2n}, b_{2n}]$ for all $n \in \mathbb{N}$. Hence $(\xi, \eta) \in I_n$ for all $n \in \mathbb{N}$ and this

proves that $\bigcap_{n=1}^{\infty} I_n$ is non-empty.

This completes the proof.

Theorem A2.5.3. (Bolzano)

Every bounded sequence in $\mathbb{R}^2$ has a convergent subsequence.

Proof. Let $\{X_n\}$ be a bounded sequence in $\mathbb{R}^2$. Then there exists a closed and bounded cell in $\mathbb{R}^2$, say $I = [a_1, b_1] \times [a_2, b_2]$ such that $X_n \in I$ for all $n \in \mathbb{N}$.

Let $l(I) = \max\{b_1 - a_1, b_2 - a_2\}$. Then $d(I)$ = the diameter of $I \leq \sqrt{2}l(I)$.

Let $c_1 = \frac{a_1+b_1}{2}, c_2 = \frac{a_2+b_2}{2}$. Then I is divided into 4 closed subcells $[a_1, c_1] \times [a_2, c_2], [a_1, c_1] \times [c_2, b_2], [c_1, b_1] \times [a_2, c_2], [c_1, b_1] \times [c_2, b_2]$ in $\mathbb{R}^2$.

At least one of these subcells contains infinite number of elements of $\{X_n\}$. We call it I_1 and let $I_1 = [a_{11}, b_{11}] \times [a_{21}, b_{21}]$.

Then $I_1 \subset I$ and $d(I_1) = \frac{1}{2}d(I)$.

Let $c_{11} = \frac{a_{11}+b_{11}}{2}, c_{21} = \frac{a_{21}+b_{21}}{2}$. Then I_1 is divided into 4 subcells

$[a_{11}, c_{11}] \times [a_{21}, c_{21}], [a_{11}, c_{11}] \times [c_{21}, b_{21}], [c_{11}, b_{11}] \times [a_{21}, c_{21}], [c_{11}, b_{11}] \times [c_{21}, b_{21}]$ in $\mathbb{R}^2$.

At least one of these subcells contains infinite number of elements of $\{X_n\}$. We call it I_2 and let $I_2 = [a_{12}, b_{12}] \times [a_{22}, b_{22}]$.

Then $I_2 \subset I_1$ and $d(I_2) = \frac{1}{2}d(I_1) = \frac{1}{2^2}d(I)$.

By induction, we obtain a sequence of closed cells $\{I_n\}$ in $\mathbb{R}^2$ such that each I_n contains infinite number of elements of $\{X_n\}$ and $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$. Therefore $\bigcap_{n=1}^{\infty} I_n$ is non-empty.

Let $\mathbf{x} = (\xi, \eta) \in \bigcap_{n=1}^{\infty} I_n$. I_1 contains infinitely many elements of the sequence $\{X_n\}$.

Therefore the set $S_1 = \{n : X_n \in I_1\}$ is an infinite subset of $\mathbb{N}$.

By the well ordering property of the set $\mathbb{N}$,S_1 has a least element, say r_1. Then $X_{r_1} \in I_1$.

Since I_2 contains infinitely many elements of the sequence $\{X_n\}$ the set $S_2 = \{n : X_n \in I_2\}$ is an infinite subset of $\mathbb{N}$. Therefore there exists a natural number $r_2 > r_1$ such that $X_{r_2} \in I_2$.

By induction, we obtain a strictly increasing sequence $\{r_1, r_2, r_3, \ldots\}$ of natural numbers such that $X_{r_k} \in I_k$ for all $k \in \mathbb{N}$.

We now prove that the subsequence $\{X_{r_n}\}$ converges to $\mathbf{x}$.

Let $\epsilon > 0$. There exists a natural number m such that $\frac{d(I)}{2^m} < \epsilon$.

Since $X_{r_m} \in I_m$ and $\mathbf{x} \in I_m$, $\| X_{r_m} - \mathbf{x} \| \leq d(I_m) = \frac{1}{2^m}d(I) < \epsilon$.

$n \geq m \Rightarrow X_{r_n} \in I_m$. Hence $\| X_{r_n} - X \| \leq d(I_m) < \epsilon$ for all $n \geq m$.

Since ϵ is arbitrary, $\lim X_{r_n} = \mathbf{x}$. This shows that the subsequence $\{X_{r_n}\}$ is a convergent subsequence of $\{X_n\}$.

A2.6. Cauchy sequence.

A sequence $\{X_n\}$ in $\mathbb{R}^2$ is said to be a *Cauchy sequence* if for a pre-assigned positive ϵ there exists a natural number k such that

$$\| X_m - X_n \| < \epsilon \text{ for all } m, n \geq k.$$

Replacing m by $n+p$, where $p = 1, 2, 3, \ldots$ the condition for a Cauchy sequence can be equivalently stated as –

$$\| X_{n+p} - X_n \| < \epsilon \text{ for all } n \geq k \text{ and } p = 1, 2, 3, \ldots$$

Theorem A2.6.1. A convergent sequence in $\mathbb{R}^2$ is a Cauchy sequence.

Proof. Let $\{X_n\}$ be a convergent sequence in $\mathbb{R}^2$ and let $\lim X_n = \mathbf{x}$.

Let $\epsilon > 0$. There exists a natural number k such that

$$\| X_n - \mathbf{x} \| < \tfrac{\epsilon}{2} \text{ for all } n \geq k.$$

Hence if $m, n \geq k$, $\| X_n - \mathbf{x} \| < \frac{\epsilon}{2}$ and $\| X_m - \mathbf{x} \| < \frac{\epsilon}{2}$.

$$\begin{aligned} \| X_m - X_n \| &= \| X_m - \mathbf{x} + \mathbf{x} - X_n \| \\ &\leq \| X_m - \mathbf{x} \| + \| X_n - \mathbf{x} \| \\ &< \epsilon \text{ for all } m, n \geq k. \end{aligned}$$

This proves that $\{X_n\}$ is a Cauchy sequence.

Theorem A2.6.2. A Cauchy sequence in $\mathbb{R}^2$ is a bounded sequence.

Proof left to the reader.

Theorem A2.6.3. If a subsequence $\{X_{r_n}\}$ of a Cauchy sequence $\{X_n\}$ converges to a limit $\mathbf{x}$ then the sequence $\{X_n\}$ also converges to $\mathbf{x}$.

Proof. Let $\epsilon > 0$. Since $\{X_n\}$ is a Cauchy sequence, there exists a natural number k such that

$$\| X_m - X_n \| < \tfrac{\epsilon}{2} \text{ for all } m, n \geq k \quad \cdots \quad \text{(i)}$$

Since $\{X_{r_n}\}$ converges to $\mathbf{x}$, there is a natural number $p > k$ belonging to the set $\{r_1, r_2, \ldots\}$ such that $\| X_p - \mathbf{x} \| < \frac{\epsilon}{2}$.

Also from (i) $\| X_n - X_p \| < \frac{\epsilon}{2}$ for all $n \geq k$.

$$\begin{aligned} \| X_n - \mathbf{x} \| &\leq \| X_n - X_p \| + \| X_p - \mathbf{x} \| \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon \text{ for all } n \geq k. \end{aligned}$$

This proves that $\lim X_n = \mathbf{x}$.

Theorem A2.6.4. A Cauchy sequence in $\mathbb{R}^2$ is convergent.

Proof. Let $\{X_n\}$ be a Cauchy sequence in $\mathbb{R}^2$. Then $\{X_n\}$ is a bounded sequence.

By Bolzano's theorem, $\{X_n\}$ has a convergent subsequence, say $\{X_{r_n}\}$. Let $\lim X_{r_n} = \mathbf{x}$.

Then by the previous theorem, $\lim X_n = \mathbf{x}$ and therefore the sequence $\{X_n\}$ is convergent.

Exercises

1. Let $\{X_n\}$ be a sequence in $\mathbb{R}^2$ that converges to $\mathbf{x}$. Prove that the real sequence $\{\| X_n \|\}$ converges to $\| \mathbf{x} \|$.

2. Let $X_n = ((-1)^n, (-1)^{n+1})$ for $n \geq 1$. Show that $\{X_n\}$ is a divergent sequence but the real sequence $\{\| X_n \|\}$ is convergent.

3. Let $X_n = ((-1)^n, \frac{1}{n})$ and $Y_n = ((-1)^{n+1}, \frac{1}{n})$ for $n \geq 1$.

Show that the sequence $\{X_n + Y_n\}$ is convergent but none of the sequences $\{X_n\}$ and $\{Y_n\}$ is convergent.

4. Let $X_n = (\frac{1}{n}, n)$ and $Y_n = (n, \frac{1}{n})$ for $n \geq 1$.

Show that the sequence $\{X_n.Y_n\}$ (where $X_n.Y_n$ denotes the inner product of X_n and Y_n) is convergent but none of the sequences $\{X_n\}$ and $\{Y_n\}$ is convergent.

5. Let $u_n = \frac{n+1}{n}$ and $X_n = (\frac{1}{n}, \frac{n}{2n+1})$ for $n \geq 1$.

Verify that $\lim u_n X_n = \lim u_n . \lim X_n$.

6. If the subsequences $\{X_{2n-1}\}$ and $\{X_{2n}\}$ of a sequence $\{X_n\}$ converge to the same limit $\mathbf{x}$ prove that the sequence $\{X_n\}$ converges to $\mathbf{x}$.

Show that the sequence $\{(1,1), (1,-1), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}), \quad \cdots \quad\}$ is convergent.

7. Let $X_1 = (1,1)$ and $X_2 = (2,2)$ and $X_{n+2} = \frac{1}{2}(X_{n+1} + X_n)$ for $n \geq 1$. Prove that the sequence $\{X_n\}$ converges to $(\frac{5}{3}, \frac{5}{3})$.

8. If $\{X_n\}$ and $\{Y_n\}$ are Cauchy sequences in $\mathbb{R}^2$, prove directly that the sequence $\{X_n + Y_n\}$ is a Cauchy sequence in $\mathbb{R}^2$.

ANSWERS TO EXERCISES

Exercises 1. (Page 12)

2. (i) $(-1,1),\{0\}$; (ii) $(-1,2),[0,1]$.

5. (i) $g(x) = \cos x - \sin x, x \in [0, \frac{\pi}{4})$
$= 0, x = \frac{\pi}{4}$
$= \sin x - \cos x, x \in (\frac{\pi}{4}, \frac{\pi}{2}]$
Therefore $f \neq g$.

(ii) $2\tan^{-1} x = \tan^{-1}\frac{2x}{1-x^2}$ for $|x| < 1$
$= \pi + \tan^{-1}\frac{2x}{1-x^2}$ for $x > 1$
$= -\pi + \tan^{-1}\frac{2x}{1-x^2}$ for $x < -1$.
Therefore $f \neq g$.

Exercises 2. (Page 41)

8. (i) $(-\infty, \frac{1}{2}) \cup (1, \infty)$, (ii) $(-\infty, 1) \cup (2, \infty)$, (iii) $(-\frac{3}{2}, -\frac{3}{14}) \cup (\frac{3}{2}, \infty)$,
(iv) $(-\infty, \frac{9}{11}) \cup (\frac{5}{3}, \infty)$, (v) $(-\infty, \frac{23}{5}) \cup (13, \infty)$.

10. (i) $1, -1$; (ii) $\frac{1}{3}, -3$; (iii) $2, 0$; (iv) $\frac{3}{2}, 0$.

Exercises 3. (Page 69)

1. (iii) $\{\cos\frac{n\pi}{2} + \frac{1}{n} : n \in \mathbb{N}\}$ (iv) $\{\cos\frac{n\pi}{3} + \frac{1}{n} : n \in \mathbb{N}\}$. **2.** $\mathbb{N}$. **3.**(ii) $\{-1, 1\}$.
6. (iii) $0, \frac{1}{2}, \frac{1}{2^2}, \cdots$ **9.** (i) yes, (ii) – (v) no. **17.** (iv) no.

Exercises 4. (Page 83)

1. $I_n = \{x \in \mathbb{R} : x \geq n\}$. **2.** $I_n = \{x \in \mathbb{R} : 0 < x < \frac{1}{n}\}$.

Exercises 5. (Page 97)

4. $I_n = \{x \in \mathbb{R} : \frac{1}{n+1} < x < \frac{n+1}{n}\}$; $\mathcal{G} = \{I_n : n \in \mathbb{N}\}$.

5. $I_n = \{x \in \mathbb{R} : -\frac{1}{n} < x < n\}$; $\mathcal{G} = \{I_n : n \in \mathbb{N}\}$.

6. Let $K_n = \{x \in \mathbb{R} : \frac{1}{n+1} \leq x \leq 1\}$. Then $\bigcup\limits_{n=1}^{\infty} K_n$ is not compact.

Exercises 6. (Page 119)

1. (i) $(-2, 2)$, (ii) $[-\frac{1}{3}, 1]$, (iii) $[-1, 2]$, (iv) $[-2, 0]$,
(v) $\bigcup\limits_{m \in z} (2m\pi, \overline{2m+1}\pi)$, (vi) $(-\infty, -1) \cup [0, \infty)$.

3. (i) odd, (ii) odd, (iii) even, (iv) odd.

4. (i)$[\frac{1}{2}(\sqrt{1+x} + \sqrt{1-x})] + [\frac{1}{2}(\sqrt{1+x} - \sqrt{1-x})]$,
(ii) $[\frac{1}{2}(x + \sqrt{1+x^2}) + \frac{1}{2}(-x + \sqrt{1-x^2})] + [\frac{1}{2}(x + \sqrt{1+x^2}) - \frac{1}{2}(-x + \sqrt{1-x^2})]$.

5. (i) $\frac{2\pi}{3}$, (ii) π, (iii) π.

Exercises 7. (Page 145)

2. (i) $2, -1 - \frac{1}{\sqrt{2}}$; (ii) $1, -1 - \frac{1}{3}$. **12.** 3.

Exercises 8. (Page 169)

1. (i) $e^{\frac{1}{3}}$; (ii) e; (iii) $e^{\frac{1}{3}}$; (iv) e.

7. (i) $1, -1$; (ii) ∞, ∞; (iii) $\infty, 0$; (iv) $\sqrt{2}, -\sqrt{2}$.

Exercises 9. (Page 198)

8. (i) convergent if $p > 2$, divergent if $p \leq 2$; (ii) convergent, (iii) divergent, (iv) divergent, (v) divergent, (vi) convergent.

9. (i)–(vi) convergent, (vii)–(ix) divergent.

10. (i) conv, (ii) conv, (iii) div, (iv) conv, (v) conv, (vi) conv, (vii) conv, (viii) conv, (ix) div, (x) conv, (xi) div, (xii) conv, (xiii) div.

(xiv)–(xv) conv if $0 < x < 1$, div if $x \geq 1$; (xvi) conv if $0 < x < e$, div if $x \geq e$; (xvii) convergent if $0 < x \leq 1$, divergent if $x > 1$; (xviii) convergent if $0 < x < \frac{1}{e}$, divergent if $x \geq \frac{1}{e}$; (xix) convergent if $0 < x < 1$, divergent if $x \geq 1$; (xx) convergent if $0 < x < 4$, divergent if $x \geq 4$.

Exercises 10. (Page 215)

9. (i) abs, (ii) abs, (iii) cond, (iv) abs, (v) div, (vi) abs, (vii)-(viii) cond.

Exercises 11. (Page 243)

4. (i) 0, 1; (ii) 0, 1; (iii) 0, 0; (iv) 1, 1; (v) 0, 0; (vl) 1, 1.

5. (i) 1, (ii) 0, (iii) 1, (iv) 0, (v) e^2, (vi) 0, (vii) 0, (viii) 0.

Exercises 12. (Page 270)

1. $f(x) = 1, x \in \mathbb{Q}$ $\quad g(x) = -1, x \in \mathbb{Q}$
$ = -1, x \in \mathbb{R} - \mathbb{Q}, \quad = 1, x \in \mathbb{R} - \mathbb{Q}$

Therefore $(f + g)(x) = 0, x \in \mathbb{R}$.

2. $f(x) = 1, x \in \mathbb{Q}$ $\quad g(x) = -1, x \in \mathbb{Q}$
$ = -1, x \in \mathbb{R} - \mathbb{Q}, \quad = 1, x \in \mathbb{R} - \mathbb{Q}$.

Therefore $fg(x) = -1, x \in \mathbb{R}$.

12. (i) $n\pi, (4n+1)\frac{\pi}{2}$ where n is an integer; (ii) $0, \pm 1, \pm 2, \cdots\cdots$
(iii) $0, \pm 1, \pm 2, \cdots\cdots$ (iv) $1, \frac{1}{2}, \frac{1}{3}, \cdots\cdots$ (v) $1, -1$.

13. (i) infinite, (ii) removable, (iii) infinite, (iv) removable, (v) oscillatory, (vi) infinite, (vii) jump, (viii) infinite.

Exercises 13. (Page 303)

1. (i) for $x \in [-1, 1]$, $f(x) = \sin\frac{1}{x}, x \neq 0$ $\quad$ (ii) $f(x) = \operatorname{sgn} x, x \in [-1, 1]$.
$ = 0, x = 0$.

4. (i) $f(x) = x \sin\frac{1}{x}, x \neq 0$ $\quad$ (ii) $f(x) = \sin x, x \in \mathbb{R}$.
$ = 0, x = 0$.

11. (i) $I = [0, \infty), f(x) = x^2, x \in I$. (ii) $I = [0, \infty), f(x) = \frac{1}{1+x^2}, x \in I$.

Exercises 14. (Page 317)

2. $Lf'(0) = 1$, $Rf'(0) = 0$. **4.** $(-\infty, 0) \cup (0, 1) \cup (1, 2) \cup (2, \infty)$.

5. (i) $f'(x) = \frac{2}{\sqrt{1-x^2}}, x \in (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
$= -\frac{2}{\sqrt{1-x^2}}, x \in (-1, -\frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, 1)$

$f'(x)$ does not exist at $x = \pm 1, \pm\frac{1}{\sqrt{2}}$.

(ii) $f'(x) = \frac{3}{\sqrt{1-x^2}}, x \in (-\frac{1}{2}, \frac{1}{2})$
$= -\frac{3}{\sqrt{1-x^2}}, x \in (-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1)$

$f'(x)$ does not exist at $x = \pm 1, \pm\frac{1}{2}$.

(iii) $f'(x) = \frac{4}{\sqrt{1-x^2}}, x \in (-1, -\frac{1}{\sqrt{2}}) \cup (0, \frac{1}{\sqrt{2}})$
$= -\frac{4}{\sqrt{1-x^2}}, x \in (-\frac{1}{\sqrt{2}}, 0) \cup (\frac{1}{\sqrt{2}}, 1)$

$f'(x)$ does not exist at $x = \pm 1, \pm\frac{1}{\sqrt{2}}, 0$.

12. (ii) $\frac{1}{4}, \frac{1}{9}$.

Exercises 15. (Page 336)

6. (i) $\frac{\pi}{2}$, (ii) $\sqrt{3}$, (iii) $\frac{e}{e-1}$, (iv) $\frac{\sqrt{6}-1}{\sqrt{6}}$.

Exercises 18. (Page 364)

1. (i) a minimum, (ii) neither a maximum nor a minimum, (iii) neither a maximum nor a minimum, (iv) a maximum, (v) a minimum (vi) a maximum, (vii) a minimum, (viii) a minimum.

2. (i) max at 1, min at 2; (ii) max at -2, min at 2, global max 7 at -2, global min $\frac{1}{7}$ at 2; (iii) max at -1, min at 0; (iv) max at 1, min at -1, global max 3 at 1, global min $\frac{1}{3}$ at -1; (v) max at $\frac{1}{2}$, min at 2; (vi) max at 1, min at -1, global max $\frac{1}{2}$ at 1, global min $-\frac{1}{2}$ at -1; (vii) max at 1, max at $\frac{5}{3}$, min at 2; (viii) max at 2, min at $\frac{9}{14}$; (ix) max at $\frac{1}{\sqrt{2}}$, min at $-\frac{1}{\sqrt{2}}$; (x) max at $\frac{1}{2}$, min at $-\frac{1}{2}$.

3. (i) $\frac{3\sqrt{3}}{4}, \frac{-3\sqrt{3}}{4}$; (ii) max 2, 0; min $-\frac{9}{8}$;

(iii) max $\frac{4\sqrt{3}\pm 3}{6}$, min $\frac{\sqrt{3}}{4}$; (iv) max $\frac{11}{6}, -\frac{5}{12}$; min $-\frac{1}{2}, -\frac{5}{6}$.

4. (i) $(\frac{1}{e})^{1/e}$, (ii) $e^{1/e}$, (iii) $\frac{1}{e}$, (iv) $\frac{e\log 2}{2}$. 6. (i) 5, 5; (ii) 6, 6.

7. (i) $\frac{3s}{4}, \frac{3s}{4}, \frac{s}{2}$; (ii) $\frac{3s}{5}, \frac{3s}{5}, \frac{4s}{5}$. 8. (i) $\frac{2\sqrt{3}}{3}r$, (ii) $\frac{4r}{3}$.

Exercises 19 (Page 377)

2. (i) 0, (ii) 1, (iii) $\frac{1}{3}$, (iv) 1, (v) 0, (vi) 0, (vii) 2, (viii) $\frac{1}{2}$.

5. (i) $a = 2$, (ii) $a = 1, b = -1$; (iii) $a = -\frac{1}{4}, b = -\frac{3}{8}$; (iv) $a = 2, b = 1, c = -3$.

Exercises 20. (Page 400)

5. (i) $V(x) = \sin 2x,\ 0 \le x < \frac{\pi}{4}; V(x) = 2 - \sin 2x,\ \frac{\pi}{4} \le x \le \frac{\pi}{2}$.

(ii) $V(x) = \sin x + \cos x - 1,\ 0 \le x < \frac{\pi}{4}; V(x) = -\sin x - \cos x + 2\sqrt{2} - 1\ \frac{\pi}{4} \le x \le \frac{\pi}{2}$. (iii) $V(x) = 0,\ 0 \le x < \frac{\pi}{4}; V(x) = 1,\ x = \frac{\pi}{4}; V(x) = 2,\ \frac{\pi}{4} < x \le \frac{\pi}{2}$.

6. (i) $V(x) = -x^2 + 2x,\ 0 \le x < 1; V(x) = x^2 - 2x + 2,\ 1 \le x \le 2$.

(ii) $V(x) = x,\ 0 \le x < 1; V(x) = 1 + x,\ 1 \le x < 2; V(x) = 4,\ x = 2$.

(iii) $V(x) = x,\ 0 \le x \le 2$.

7. (i) $V_f[0,3] = 5, p_f[0,3] = 1, n_f[0,3] = 4$. (ii) $V_f[0,3] = 2, p_f[0,3] = 2, n_f[0,3] = 0$.
(iii) $V_f[0,3] = 3, p_f[0,3] = 1, n_f[0,3] = 2$.

8. (i) $p(x) = 0,\ 0 \le x < 1; p(x) = x^2 - 2x + 1,\ 1 \le x \le 2$.
$n(x) = -x^2 + 2x,\ 0 \le x < 1; n(x) = 1,\ 1 \le x \le 2$.

(ii) $p(x) = 0.\ 0 \le x < 1; p(x) = 1,\ x = 1; p(x) = 2,\ 1 < x \le 2$.
$n(x) = 0,\ 0 \le x \le 2$.

(iii) $p(x) = 0,\ 0 \le x < 1; p(x) = x - 1,\ 1 \le x \le 2$.
$n(x) = x,\ 0 \le x < 1; n(x) = 1,\ 1 \le x \le 2$.

Exercises 21. (Page 479)

5. $\frac{1}{4}, \frac{1}{3}$. **6.** $\frac{7}{12}, \frac{5}{6}$. **7.** $1, \frac{\pi^2}{8}$. **11.** $\frac{13}{12}$. **12.** $\frac{3}{2}$. **13.** 10.

21. $F(x) = 0,\quad 0 \le x \le 1$
$\quad = x - 1, 1 < x \le 2$
$\quad = 2x - 3, 2 < x \le 3$.

22. $F(x) = \frac{1}{2}x^2,\quad 0 \le x \le 1$
$\quad = x - \frac{1}{2}, 1 < x \le 2$
$\quad = \frac{1}{2}(x^2 - 2x + 3), 2 < x \le 3$.

25. (i) $e^x\sqrt{1 + e^{2x}} - \sqrt{1 + x^2}$, (ii) $2x \sin x - \sin\sqrt{x}$. **29.** (ii) 2. **30.** (ii) $\frac{13}{2}$.

33. (i) $2\log 2$, (ii) $\frac{2}{\pi}$, (iii) $\tan^{-1} 2$, (iv) $\frac{4}{e}$, (v) $\frac{4}{e}$.

40. (i) $\frac{1}{3}(5^{3/2} - 1)$, (ii) $\frac{52}{9}$, (iii) $\frac{1}{2}(3e^4 - 1)$, (iv) $\frac{1}{4}$.

Exercises 22. (Page 525)

1. (i) convergent, (ii) convergent, (iii) convergent, (iv) convergent, (v) divergent, (vi) divergent, (vii) convergent if $n < 1$, (viii) convergent if $0 < p < 1$, (ix) convergent if $p > -1$, (x) convergent, (xi) convergent, (xii) divergent.

2. (i) convergent, (ii) convergent, (iii) convergent, (iv) convergent, (v) divergent, (vi) convergent.

Exercises 24. (Page 563)

3. (i) no, (ii) yes, (iii) yes, (iv) no.

5. (i)–(ii) uniform, (iii)–(iv) non-uniform, (v)–(vi) uniform.

Exercises 26. (Page 609)

1. (i) $\frac{2}{e}$, (ii) $\frac{1}{2}$, (iii) 2, (iv) 1, (v) $\frac{1}{2}$, (vi) 1. **2.** (i) $\frac{1}{3}$, (ii)–(vi) 1.

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