ORIGAMI IN MODERN GEOMETRY: THE BELOCH FOLD

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1. Introduction

1.1. **Historical Context and Discovery.** Margharita P. Beloch was the first to demonstrate that a simple paper fold could perform complex geometric constructions equivalent to solving cubic equations. This method, known as the Beloch Fold, involves aligning a single crease to simultaneously place two points onto two distinct lines.

1.2. **Mathematical Description.** The Beloch Fold, named after Margharita P. Beloch, represents a significant advancement in origami geometry, introducing an origami axiom that allows for solving cubic equations. This capability transcends the traditional bounds set by straightedge and compass constructions, which are limited to quadratic equations.

- **Origami Axiom:** The axiom enables the simultaneous alignment of two distinct points onto two distinct lines through a single fold. This unique capability is what sets the Beloch Fold apart from traditional geometric constructions and previous origami axioms.
- **Geometric Implications:** By performing this fold, the created crease acts as a common tangent to two parabolas defined by each point-line pair. This tangency is not merely a geometric curiosity but a powerful tool for solving cubic equations, as it implies the intersection of complex geometric constructs not possible with simpler tools.
- **Application and Utility:** The fold does not just solve theoretical problems but also practical ones such as trisecting angles and duplicating cubes, thus expanding the scope of problems approachable through geometric constructions.

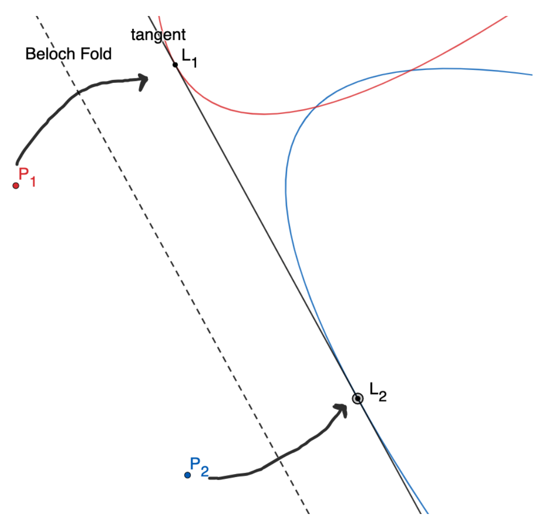

Figure 1. Beloch Fold being performed on two parabolas

2. Bisecting an Angle Using Origami

Having explored the theoretical and historical contexts of the Beloch Fold and its application in solving complex geometric problems such as cubic equations, we now transition to a more commonly encountered geometric task: bisecting an angle. This next section demonstrates the practical utility of origami in classical geometric constructions, further showcasing the flexibility and power of paper folding in geometry.

2.1. Construction Steps.

(1) Begin with angle $\angle ABC$ where B is our vertex.
(2) Fold the paper such that point A aligns directly over point C. This manipulation creates a fold BD that intersects $\angle ABC$ at point B and another point D on $\overline{AC}$.
(3) Unfold the paper to reveal fold BD, which now acts as the angle bisector of $\angle ABC$.

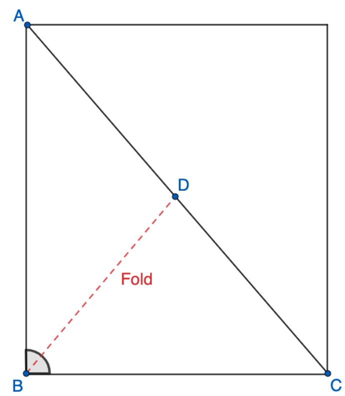

Figure 2. Bisecting $\angle ABC$ with a fold

2.2. Geometric Justification.

- During the construction, when point B is folded onto point C, crease BD is formed, intersecting $\overline{AC}$ at point D. This folding operation ensures that D becomes the midpoint of $\overline{AC}$, hence $\overline{AD}$ is congruent to $\overline{DC}$.
- The fold creates two angles, $\angle ADB$ and $\angle BDC$, which are congruent due to the symmetric nature of the fold. This symmetric fold effectively makes $\overline{BD}$ the angle bisector, thus ensuring that $\angle ADB$ is congruent to $\angle DBC$.
- By I.15, if two straight lines cut one another, they make the vertical angles equal. Thus, $\angle ADB$ and $\angle DBC$ are not only congruent by construction but also by the properties of vertical angles, reinforcing their equality.
- With $\overline{BD}$ shared between triangles $\triangle ADB$ and $\triangle DBC$, and having established that $\overline{AD}$ is congruent to $\overline{DC}$ and $\angle ADB$ is congruent to $\angle DBC$, by I.4 (SAS Congruence), $\triangle ADB$ is congruent to $\triangle DBC$.
- The congruence of these triangles solidifies that BD is indeed the angle bisector of $\angle ABC$. This follows Euclid's method of showing that if two triangles are congruent (SAS), their corresponding parts are equal, thereby justifying that the line dividing them bisects the angle formed at their common vertex.

3. Understanding the Beloch Fold

The Beloch Fold is a fascinating concept in origami that goes beyond traditional paper folding, crossing into the realm of advanced geometry. Here's a simple way to understand how this fold works and what it accomplishes geometrically.

3.1. **The Basic Idea.** Imagine you have a piece of paper. On this paper, you mark two points, let's call them Point A and Point B. You also draw two lines, which we'll refer to as Line 1 and Line 2. The goal of the Beloch Fold is to fold the paper so that Point A touches Line 1 and Point B touches Line 2 with just one fold.

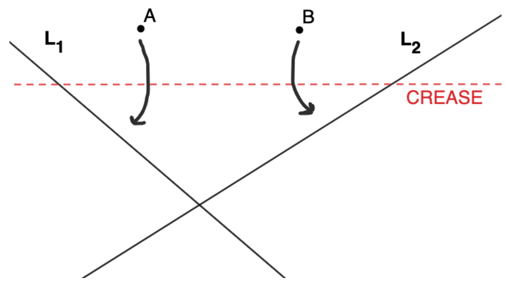

Figure 3. The Beloch Origami Fold

3.2. **Connecting Points and Lines.** When you make this fold, you're not just randomly folding the paper. This fold is precise. It creates a crease that has special properties:

- It touches Point A and Line 1 in such a way that the crease is exactly equidistant at every point along it from Point A and the nearest point on Line 1.
- Similarly, it touches Point B and Line 2, maintaining the same precise equidistance.

3.3. **The Role of Parabolas.** Each of these actions—touching a point and a line—creates a tangent to a parabola. A parabola in geometry is defined by its focus (a fixed point) and its directrix (a fixed line). The paper's crease becomes a tangent line to the parabola where Point A is the focus and Line 1 is the directrix, and similarly for Point B and Line 2.

3.4. **Producing Parabolas Through Repeated Folding.** When repeatedly folding a point onto a line, we create multiple creases. These creases accumulate in such a way that their envelope forms a parabola. This is evident because each crease acts as a tangent to the parabola formed with the folded point as the focus and the line as the directrix. This parabolic envelope can be visualized if a significant number of such creases are made. By aligning a point with a line multiple times and in diverse locations along the line, we gradually sketch out the shape of the parabola.

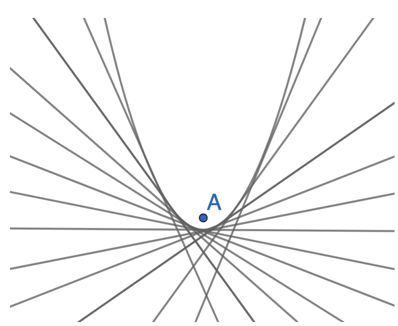

Figure 4. Creation of a Parabola from Repeated Creases

3.5. **Geometric Implications and the Solving of Cubic Equations.** The Beloch Fold reveals the potential of origami to solve cubic equations by constructing a single crease that acts as a common tangent to two distinct parabolas. In the setup, Point A and Point B are used as foci for these parabolas, with Lines L_1 and L_2 serving as their respective directrices. The fold brings Point A onto Line L_1 and Point B onto Line L_2, with the crease maintaining equidistance from each point to its corresponding line.

This precise alignment results in the crease line being tangent to the parabola formed around Point A and Line L_1, and similarly tangent to the parabola formed around Point B and Line L_2. Therefore, the Beloch Fold not only demonstrates the geometric process of connecting points to lines but also the advanced concept of tangency to multiple curves simultaneously, a problem equivalent to solving cubic equations.

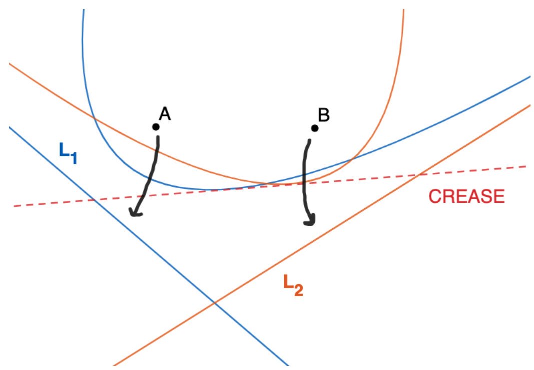

FIGURE 5. The Beloch Origami Fold with Parabolas

4. The Scope of Euclidean Geometry and Beyond

4.1. **Capabilities of Euclidean Geometry.** Euclidean geometry, grounded in the operations of a straightedge and compass, is adept at constructing a variety of figures and performing numerous arithmetic operations. Starting from fundamental points like 0 and 1 on the complex plane, which can be thought of as a two-dimensional space, this traditional approach can create any rational length. It allows for the addition, subtraction, multiplication, and division of lengths, as well as the construction of square roots for any established length.

For instance, using basic Euclidean techniques, one can construct numbers such as $\frac{7}{5}$, $\sqrt{3}$, or even complex expressions like $\frac{1+i}{35}$ and composite numbers like $2 + \frac{7}{5}$. However, the operations possible with Euclidean methods are restricted to intersecting lines and circles. This means the types of equations we can solve are limited to quadratic ones, which involve the square of a variable.

Thus, while numbers like $\sqrt{2}$, $\sqrt{3}$, or rational numbers are within reach, transcendent numbers like e and π, or even roots of cubic equations such as $\sqrt[3]{2}$ and $\sqrt[3]{5}$ remain beyond the capabilities of straightedge and compass. This limitation is intrinsic to the nature of the tools and methods prescribed in classical Euclidean constructions.

4.2. **Extending the Geometric Reach with Origami.** Enter origami: this ancient art introduces an axiom that vastly extends the reach of geometric construction. With origami, not only can we solve quadratic equations, but we can also address cubic equations, allowing for the construction of cube roots of any length previously constructed. This capability stems from the precise and complex folds permissible in origami, which enable the creation of angles and intersections that are impossible with the more rigid tools of Euclidean geometry.

Though the full mathematical proof of origami's capabilities in solving higher-order equations involves complex algebra and geometry, as detailed in works by Cox and Martin, the fundamental concept remains accessible. Origami not only enhances our geometric toolkit but also bridges a gap between theoretical mathematics and practical construction, opening up new realms of exploration and understanding in both educational and professional settings.

4.3. **The Classical Problem of Doubling the Cube.** "Duplicating the cube" is a classical problem from ancient Greek mathematics, often referred to as the Delian problem. The challenge was posed to solve an oracle's demand to double the size of an altar, which translated mathematically into doubling the volume of a cube. Specifically, if we start with a cube of volume 1 cubic unit, the task is to construct a cube with a volume of 2 cubic units.

This problem is not just a puzzle but a significant mathematical challenge. The solution requires finding a cube whose side length is the cube root of 2, as the volume of a cube is given by $V = s^3$, where s is the side length. Thus, to double the volume from 1 to 2 cubic units, the side length of the new cube needs to be $\sqrt[3]{2}$.

4.4. **Why Can't Euclidean Tools Solve It?** Using the tools stipulated by Euclidean geometry—a straightedge and a compass—one can perform many constructions, including drawing straight lines, perfect circles, and marking intersections. These methods are inherently capable of solving quadratic equations, which involve the square of a variable (e.g., x^2).

However, the task of doubling a cube requires solving a cubic equation, specifically calculating $\sqrt[3]{2}$ to determine the new side length of the cube. This operation is outside the capabilities of traditional Euclidean constructions because it involves a third-degree equation, which cannot be solved by merely intersecting lines and circles.

5. Doubling a Cube

5.1. Constructing the Beloch Square Using Origami. To tackle this ancient problem using origami, we employ the Beloch Square construction:

(1) Begin with two points, A and B, and two lines, r and s. These will form the basis for the Beloch Square.
(2) Compute the perpendicular distances from A to r and from B to s. Construct new lines r' and s' parallel to r and s at these distances, respectively.
(3) Perform the Beloch fold: fold A onto r' and B onto s'. This folding creates a crease that acts as the perpendicular bisector of segments AA' and BB', where A' and B' are the new positions of A and B on r' and s'.
(4) Define points X and Y as the midpoints of AA' and BB' respectively. This setup ensures that X lies on r and Y on s, forming one side of the Beloch Square.

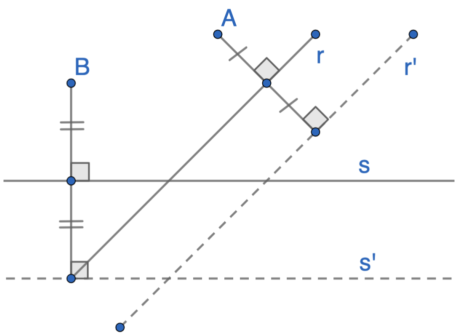

Figure 6. Step (1) and (2) construction

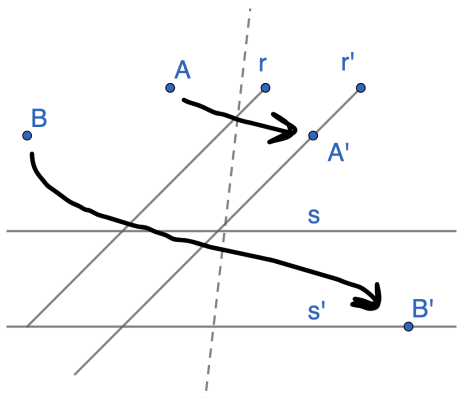

Figure 7. (3) Beloch fold

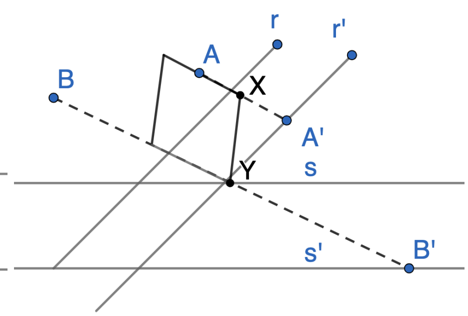

Figure 8. Final Beloch Square

5.2. **Computing the Cube Root of 2.** To solve the classical problem of doubling a cube using origami, we employ a construction based on the Beloch Fold, which geometrically calculates the cube root of two. This method utilizes origami's capability to solve cubic equations through precise folding techniques, exceeding the capabilities of traditional straightedge and compass methods.

Consider points $A = (-1, 0)$ and $B = (0, -2)$ positioned with respect to coordinate axes where we set r as the y-axis and s as the x-axis. Construct lines $r' = x = 1$ and $s' = y = 2$ parallel to r and s, respectively. Performing the Beloch fold aligns A onto r' and B onto s', resulting in a crease that intersects the axes at points X and Y.

This setup creates several right triangles: OAX, OXY, and OBY, where O is the origin. These triangles are similar due to the perpendicular nature of XY to segments AA' and BB', where A' and B' are the new locations of A and B after the fold.

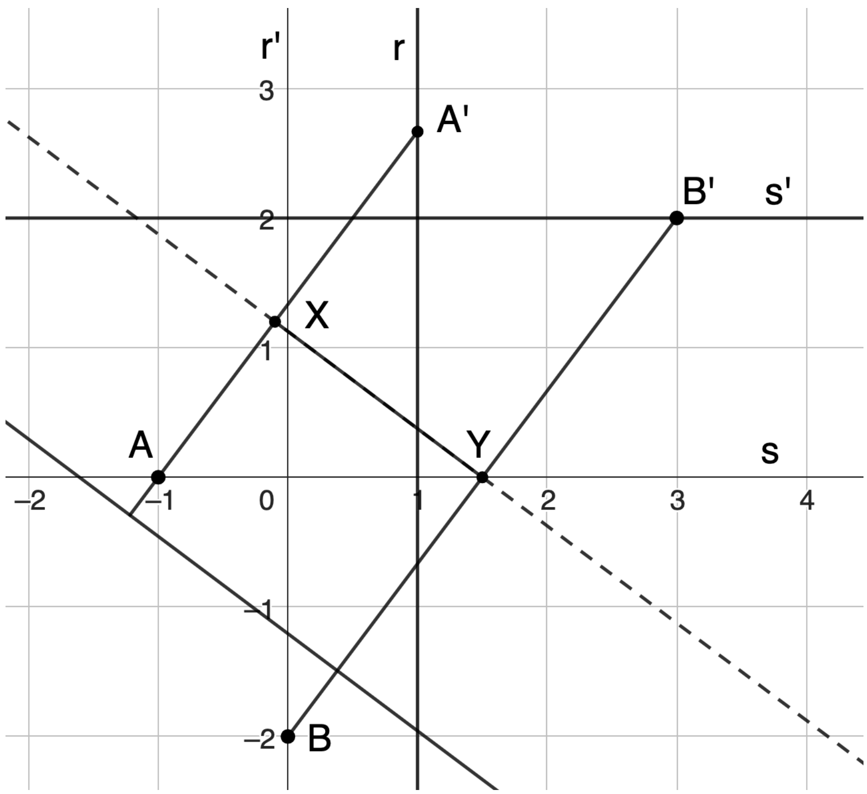

FIGURE 9. Beloch's origami construction of the cube root of two

From the properties of these triangles, the segment ratios are established as follows:

$$\frac{|OX|}{|OA|} = \frac{|OY|}{|OX|} = \frac{|OB|}{|OY|}$$

Given $|OA| = 1$ and $|OB| = 2$, solving these proportions gives $|OX| = \sqrt[3]{2}$, which is the side length needed to construct a cube with double the volume of a cube with unit side length.

6. Trisecting an Angle

6.1. **Origami Construction.**

(1) Begin with a rectangular sheet of paper. Let the angle θ be located at the bottom corner of the sheet, defined between the bottom edge (line l_2) and the left side of the paper.
(2) Fold the sheet twice to create two new creases that are parallel to the bottom edge of the paper such that these creases are equidistant from the bottom and pass through points P_1 and P_2 respectively, located along the initial angle rays.
(3) Fold the sheet so that point P_1 is brought to a new point Q_1 on line l_2 and P_2 moves along its path.
(4) After making the fold and creasing the paper sharply, unfold it. The crease made will intersect the initial angle's arms such that it creates three segments along the bottom edge, effectively trisecting the original angle θ.

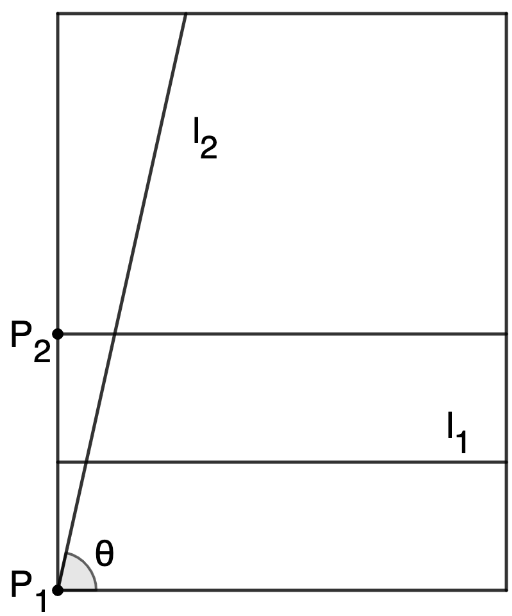

Figure 10. Steps (1) and (2)

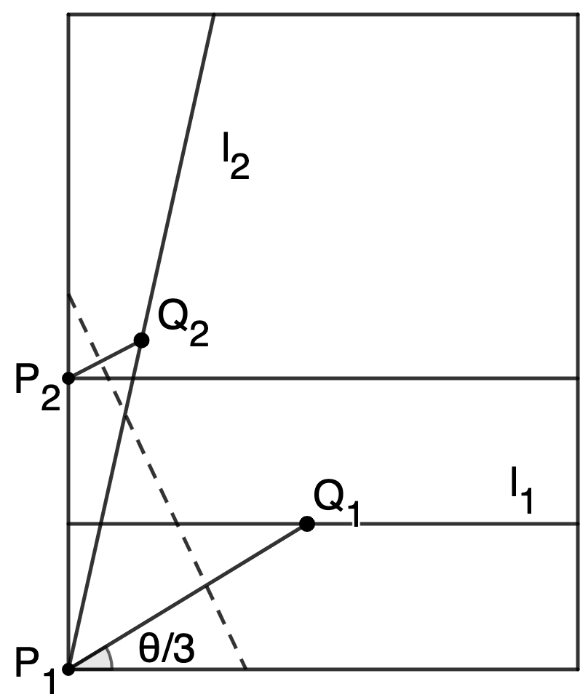

Figure 11. Steps (3) and (4)

6.2. **Proof Explanation.** To prove that our construction is correct, we will rely on a proof format set out by Cox. The construction is as follows:

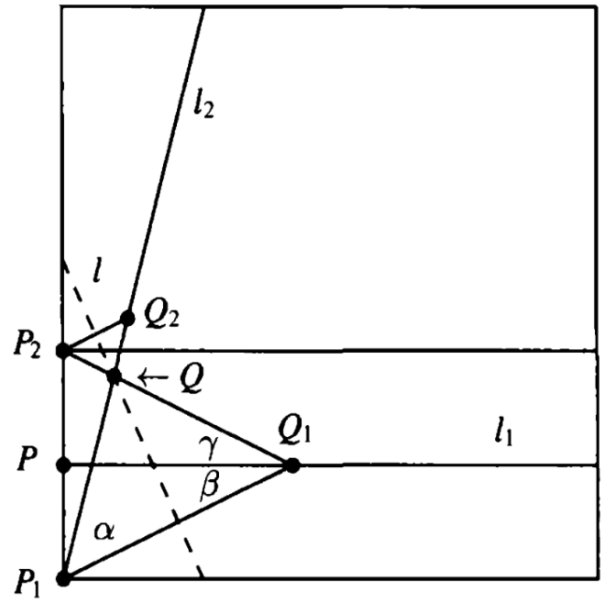

FIGURE 12. Trisecting Angle Proof Construction from Cox, adapted from *Galois Theory*, Second Edition by David A. Cox, John Wiley & Sons, 2012.

6.2.1. *Prove that Q lies on the dashed line l.* Given that the dashed line l is the fold (reflection line) that maps P_1 to Q_1 and P_2 to Q_2, we start by noting that l must be the perpendicular bisector of segments $\overline{P_1Q_1}$ and $\overline{P_2Q_2}$. This follows from the basic properties of reflections, where the reflection line is equidistant from each point to its image.

Let s be the reflection with respect to line l. Then $s(P_2) = Q_2$ and $s(Q_1) = P_1$, implying s maps the line $\overline{P_2Q_1}$ onto $\overline{P_1Q_2}$. Let Q be the intersection of these lines. Since reflections preserve intersection points, $s(Q)$ must also lie on $\overline{P_1Q_2}$ and $\overline{P_2Q_1}$, specifically at their intersection, which is Q itself. Hence, $s(Q) = Q$.

This condition $s(Q) = Q$ signifies that Q is invariant under the reflection s, which means Q lies on the line l itself, as this is the defining property of points on a reflection line.

Therefore, $Q \in l$, completing the proof.

6.2.2. *Prove that θ is congruent to $\alpha + \beta$.* Let D be the point at the bottom right corner, and define δ as the measure of angle $(\widehat{\overrightarrow{P_1D}, \overrightarrow{P_1Q_1}})$.

According to I.13, if two angles are adjacent and their sum is equivalent to two right angles, they form a straight line. Therefore:

$$(\widehat{\overrightarrow{P_1D}, \overrightarrow{P_1Q_1}}) + (\widehat{\overrightarrow{P_1Q_1}, \overrightarrow{P_1Q_2}}) = (\widehat{\overrightarrow{P_1D}, \overrightarrow{P_1Q_2}})$$

This confirms that $\theta = \delta + \alpha$.

Given the parallel lines P_1Q_1 and P_1D with a transversal through Q_1 and I.29, the alternate interior angles are equal. This makes $\beta = \delta$.

Combining the equalities, we conclude:

$$\theta = \delta + \alpha = \beta + \alpha$$

Therefore, θ is congruent to $\alpha + \beta$, as required.

6.2.3. *Use triangles $\triangle P_1PQ_1$ and $\triangle P_2PQ_1$ to prove that β and γ are congruent.* Given the reflection r about the line l_1 which maps P_1 to P_2 and fixes Q_1, we have $Q_1P_1 = Q_1P_2$, making $\triangle Q_1P_1P_2$ isosceles.

The definition of an isosceles triangle (I.5) states that the angles opposite the equal sides are congruent. Hence, in $\triangle Q_1P_1P_2$, the angles at P_1 and P_2 are equal.

By the properties of reflections and isosceles triangles:

$$\beta = (\widehat{\overrightarrow{Q_1P}, \overrightarrow{Q_1P_2}}) = -(\widehat{\overrightarrow{Q_1P}, \overrightarrow{Q_1P_1}}) = \gamma$$

These angles, β and γ, are the base angles of $\triangle Q_1P_1P_2$.

Therefore, β and γ are congruent, as they are both base angles of the isosceles triangle $\triangle Q_1P_1P_2$ formed by the reflection.

6.2.4. *Use triangle $\triangle P_1QQ_1$ to prove that α is congruent to $\beta+\gamma$.* Reflection s maps P_1 to Q_1 and fixes Q, making QP_1 equal to QQ_1. Thus, $\triangle P_1QQ_1$ is isosceles.

In $\triangle P_1QQ_1$, $QP_1 = QQ_1$, and by I.5, the angles opposite these sides are equal, denoted as β at P_1 and γ at Q_1.

The full angle at Q is formed by β and γ, adjacent angles along segment QQ_1. The principle that the sum of parts equals the whole implies:

$$\alpha = \beta + \gamma$$

by considering the angles around point Q.

Therefore, $\alpha = \beta + \gamma$, confirming the angle sum.

6.2.5. *Conclude that α is congruent to $2\theta/3$ and that the angle formed by $\overline{P_1Q_1}$ and the bottom of the square is $\theta/3$.* From previous results and the equalities:

$$\theta = \alpha + \beta, \quad \beta = \gamma, \quad \alpha = \beta + \gamma,$$

we use substitution to refine these relationships:

$$\alpha = 2\beta, \quad \theta = 3\beta,$$

and therefore:

$$\alpha = 2\left(\frac{\theta}{3}\right) = \frac{2\theta}{3},$$

confirming $\alpha = 2\theta/3$.

Additionally, since $\beta = \delta$ and $\theta = 3\beta$:

$$\delta = \frac{\theta}{3},$$

establishing the measure of the angle formed by $\overline{P_1Q_1}$ and the bottom of the square as $\theta/3$.

7. Constructing a Heptagon

7.1. **Overview of Constructible Polygons.** In Euclidean geometry, not all regular polygons can be constructed using just a straightedge and a compass. Constructible polygons must have a number of sides, n, that is a product of distinct Fermat primes and a power of 2, such as $n = 2^k p_1 p_2 \dots p_m$. For example, while pentagons (5 sides) and 15-sided polygons are constructible, a regular heptagon (7-sided polygon) is not, because 7 does not meet these criteria.

7.2. **Exploring the 7th Roots of Unity.** The corners of a heptagon can be perfectly placed as points on a circle, represented by something called the 7th roots of unity. These roots solve the equation $z^7 - 1 = 0$, where each solution z_k represents a point on the circle, corresponding to an angle $\frac{2\pi k}{7}$ radians, with k ranging from 1 to 7.

To put it simply, the 7th roots of unity are complex numbers (think of them as points on a circular path) that, when multiplied by themselves seven times, loop back to 1. These points help lay out a heptagon by marking seven spots evenly around the circle. They are represented as $e^{\left(\frac{2\pi i \cdot k}{7}\right)}$ for each $k = 1, 2, \ldots, 7$, arranging each vertex of the heptagon equally around the circle. Here's a breakdown of the roots:

- $z_1 = e^{\left(\frac{2\pi i \cdot 1}{7}\right)}$ is the first point after $(1, 0)$ going counterclockwise,
- $z_2 = e^{\left(\frac{2\pi i \cdot 2}{7}\right)}$, and so on until
- $z_7 = e^{\left(\frac{2\pi i \cdot 7}{7}\right)} = 1$.

Each of these corresponds to a coordinate on the circle where the angle from the horizontal x-axis is $\frac{2\pi k}{7}$ of a full turn. This matches up with Euler's formula, which connects complex exponentials to cosines and sines: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. Thus, each root translates into a coordinate $(\cos(\frac{2\pi k}{7}), \sin(\frac{2\pi k}{7}))$.

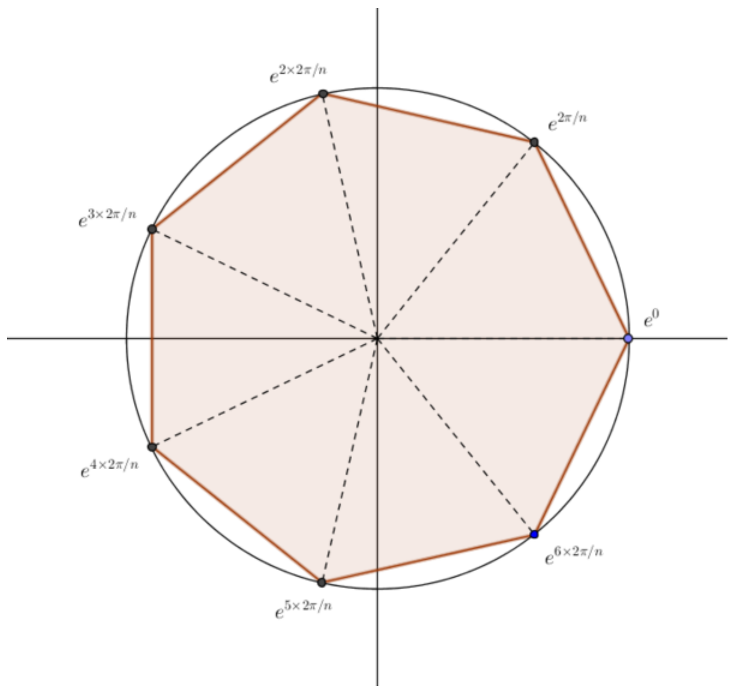

FIGURE 13. 7 Roots of Unity, adapted from an image on StackExchange Mathematics. Retrieved from `https://math.stackexchange.com/questions/658662/how-pi-3-1415-and-180o-are-adaptive-together`, accessed 5/12/24.

7.3. **Why Origami?** With origami, we can create these angles and thus the vertices of a heptagon much more accurately than with basic geometry tools. This advantage comes from origami's ability to solve complex equations, like dividing a circle into thirds—something impossible with just a compass and straightedge. This is crucial because the cosine of important angles, such as $\cos(\frac{2\pi}{7})$, fits a type of equation that traditional tools can't handle.

Let's consider the vertex of a heptagon closest to the point (1,0) on a unit circle, moving counterclockwise. This point can be expressed in terms of complex numbers as $e^{(2\pi i/7)}$. By applying Euler's formula, we can simplify this expression to $\cos(2\pi/7) + i\sin(2\pi/7)$. We'll denote this complex number as $A = e^{(2\pi i/7)}$. The complex conjugate of A, noted as $\overline{A}$, is key to finding its reciprocal, which gives us:

$$\frac{1}{A} = \frac{\overline{A}}{A\overline{A}} = \overline{A}$$

This relationship helps us establish:

$$A + \frac{1}{A} = A + \overline{A} = 2\cos(2\pi/7)$$

Exploring further, raising A to the sixth power, denoted as A^6, calculates to $e^{(2\pi i \cdot 6/7)}$, which resolves to $\cos(12\pi/7) + i\sin(12\pi/7)$. On the unit circle, $12\pi/7$ simplifies to $2\pi/7$, implying that:

$$A^6 = \cos(2\pi/7) + i\sin(2\pi/7) = \overline{A} = \frac{1}{A}$$

Additionally, analyzing A^2 and its reciprocal gives:

$$\frac{1}{A^2} = \overline{A}^2$$

But, interestingly, the negative angle $-4\pi/7$ corresponds to $10\pi/7$ when mapped on the circle, leading to:

$$\frac{1}{A^2} = \cos(10\pi/7) + i\sin(10\pi/7) = A^5$$

This pattern confirms that A^4 equals $\frac{1}{A^3}$. These findings demonstrate that the seventh roots of unity—ranging from A to A^6—are solutions to the polynomial equation:

$$z^7 - 1 = 0 \tag{1}$$

Incorporating these roots into the equation, we get:

$$A^6 + A^5 + A^4 + A^3 + A^2 + A + 1 = 0 \tag{2}$$

Expanding the expressions for $A + \frac{1}{A}$, where A is the complex number representing a vertex of a heptagon on the unit circle. When squared and cubed, these expressions reveal deeper relationships:

- The square of $A + \frac{1}{A}$ breaks down as follows:

$$(A + \frac{1}{A})^2 = A^2 + 2\frac{A}{A} + \frac{1}{A^2} \to A^2 + \frac{1}{A^2} = (A + \frac{1}{A})^2 - 2$$

- The cube of $A + \frac{1}{A}$ expands to:

$$(A + \frac{1}{A})^3 = A^3 + 3A^2\frac{1}{A} + 3A\frac{1}{A^2} + \frac{1}{A^3} \to A^3 + \frac{1}{A^3} = (A + \frac{1}{A})^3 - 3(A + \frac{1}{A})$$

Inserting these into Equation 2 gives us:

$$(A + \frac{1}{A})^3 + 3(A + \frac{1}{A})^2 + (A + \frac{1}{A})^2 + 2 + (A + \frac{1}{A}) + 1 = 0 \tag{3}$$

This simplifies further to:

$$(A + \frac{1}{A})^3 + (A + \frac{1}{A})^2 + 2(A + \frac{1}{A}) + 1 = 0 \tag{4}$$

From here, we recognize that $A + \frac{1}{A} = 2\cos\left(\frac{2\pi}{7}\right)$ fits as a solution to Equation 4. Defining $z = A + \frac{1}{A}$, we can simplify Equation 4 further to:

$$z^3 + z^2 + 2z + 1 = 0$$

This equation is what we aim to solve using origami.

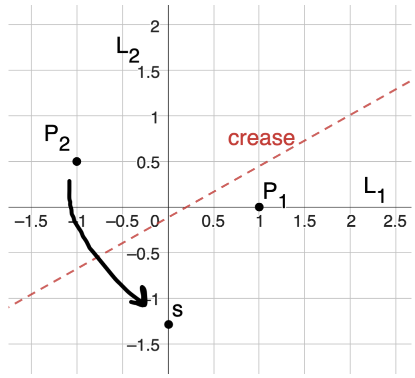

FIGURE 14. Beloch Fold in our Construction

7.4. **Origami in the Construction.** To demonstrate this in a construction, we start with the point at $(1, 0)$ and aim to construct the next nearest counterclockwise vertex. By determining $2\cos\left(\frac{2\pi}{7}\right)$ using the Beloch fold we set up two points for the fold: $P_1 = (0, 1)$ and $P_2 = (-1, \frac{1}{2})$. We then place P_1 and P_2 onto the x-axis and y-axis, respectively, labeling the new locations $P_1' = (t, 0)$ and $P_2' = (0, s)$.

The next step is to determine the equation for the fold line based on the properties of the segments formed:

The segment P_1P_1' has a slope calculated by:

$$\frac{1-0}{0-t} = -\frac{1}{t}$$

Due to the unique properties of the Beloch fold, we know that the segment connecting P_1 to its projected point P_1' on the fold line is perpendicular to the fold itself. This means the fold has a slope of t and passes through the midpoint of P_1P_1', located at $\left(\frac{t}{2}, \frac{1}{2}\right)$. Using the point-slope form of a line, the equation for the fold becomes:

$$y - \frac{1}{2} = t\left(x - \frac{t}{2}\right) \to y = tx - \frac{t^2}{2} + \frac{1}{2}$$

Next, let's apply the same process to P_2 and its projection P_2'. The slope calculation here, considering the points' placement and movement due to the fold, is a bit more complex:

$$\text{Slope of } P_2P_2' = \frac{s + \left(-\frac{1}{2}\right)}{0 + (-1)} = s + \frac{1}{2} = \frac{2s+1}{2}$$

The midpoint of P_2P_2' is at $\left(-\frac{1}{2}, \frac{2s-1}{4}\right)$, and using the point-slope form with the negative reciprocal of the slope, we derive the equation:

$$y + \frac{2s-1}{4} = -\frac{2}{2s+1}\left(x + \frac{1}{2}\right) \to y = -\frac{2}{2s+1}x + \frac{1}{2s+1} + \frac{2s+1}{4}$$

For the origami fold to be correct, the slope and y-intercept of the fold line must be consistent across both sets of points. Ensuring that the y-intercepts from both fold line equations are equal gives us a system of equations:

$$\frac{t^2}{2} + \frac{1}{2} = \frac{1}{2s+1} + \frac{2s-1}{2t}$$

Further adjustments and simplifications lead to a cubic equation in t:

$$-t^2 + 1 = t - \frac{1}{t} \to t^3 + t^2 - 2t - 1 = 0$$

We've discovered that one of the solutions to our earlier equation is $2\cos\left(\frac{2\pi}{7}\right)$. This is key because it shows we've successfully constructed this value using origami. Even better, since the rules of geometry allow us to divide numbers, we can halve this result to get $\cos\left(\frac{2\pi}{7}\right)$, which is a crucial angle for building our heptagon. Let's call this point t'.

Now, let's use t' to help us place the next point in our heptagon construction. Starting from t', we draw a line straight up from the x-axis—this is perpendicular to it, meaning it forms a right angle. All integers can be constructed using origami, including the number 1, so we can make a line exactly one unit long from the center point (the origin) to meet this vertical line. We'll label where they meet as point a.

The angle between the x-axis and this line is exactly $\frac{2\pi}{7}$ radians. This precision comes from the right triangle formed by the x-axis, the vertical line, and our one-unit line. By applying the Pythagorean theorem here, we find:

$$\cos^2\left(\frac{2\pi}{7}\right) + w^2 = 1$$

where w is the length of the vertical segment, leading us to:

$$w^2 = 1 - \cos^2\left(\frac{2\pi}{7}\right) = \sin^2\left(\frac{2\pi}{7}\right)$$

This confirms that point a is perfectly placed on the unit circle, with the hypotenuse of our triangle exactly one unit in length.

7.5. **Additional Vertices.** With point a now set as the next vertex of the heptagon, constructing the remaining vertices becomes much simpler. If we shift our starting point to a, treating it as the new (1,0), we can repeat the same steps to methodically build out the rest of the heptagon.

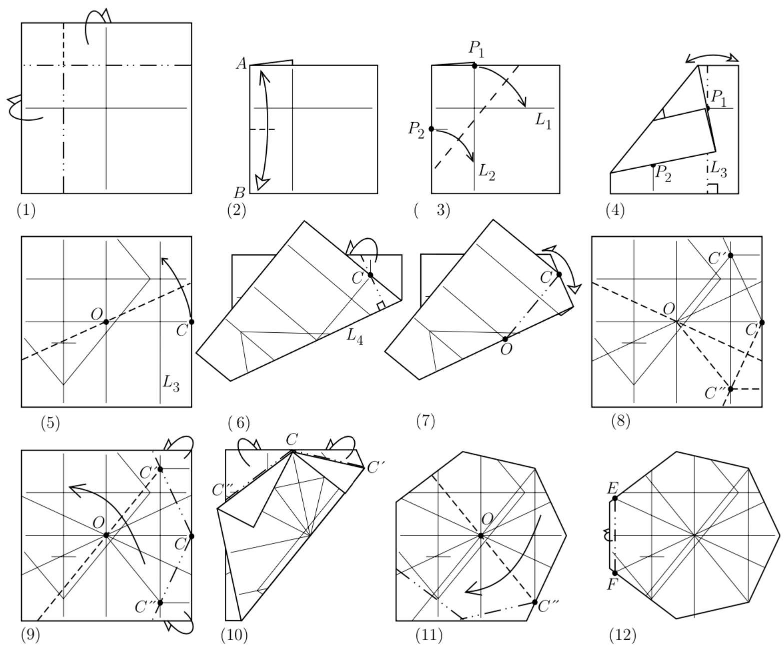

Figure 15. Folding a regular heptagon from a square piece of paper, adapted from Thomas C. Hull's demonstration in *Folding Regular Heptagons* (Merrimack College).

8. Conclusion

This project has allowed us to delve deeply into the intricate relationship between origami and Euclidean geometry. By incorporating origami into classical geometric constructions, we've expanded our toolkit beyond the traditional straightedge and compass, enabling the resolution of problems previously deemed impossible within the strictures of Euclidean postulates.

As the mathematician Euclid once stated, "What is asserted without proof can be dismissed without proof."

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