Beta mapper proof

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Lemma: the combination of expected value and c-statistic uniqly identifies a beta distirbution

Proof

Let μ and c be, respectively, the expected value and the c-statistic of a random variable (RV) $Beta(\alpha,\beta)$ distribution, with $\mu = \mathbf{E}\pi = \frac{\alpha}{\alpha+\beta}$ and $c = P(\pi_1 > \pi_0 | Y_1 > Y_0)$ where π_0, π_1 are iid RVs all with $Beta(\alpha,\beta)$ distribution, and Y_0, Y_1 are Bernoulli RVs with oribabilities π_0 and π_1, respectively. We shall prove a unique α, β solution exists for a given μ and c.

Noting that $m = \frac{\alpha}{\alpha+\beta}$, we remove β as $\beta = \alpha\frac{\mu}{1-\mu}$.

Next, we apply Bayes' theorem to the definition of c:

$c = P(Y_1 > Y_0 | \pi_1 > \pi_0)\frac{P(\pi_1>\pi_0)}{P(Y_1>Y_0)} = P(Y_1 = 1, Y_0 = 0 | \pi_1 > \pi_0)\frac{0.5}{P(Y_1=1,Y_0=0)} = \frac{\mathbf{E}\pi_M(1-\mathbf{E}\pi_m)}{2\mu(1-\mu)}$,

where π_M and π_m are, respectively, the maximum and minimum of a pair of independent RVs from $Beta(\alpha,\beta)$.

The goal is achieved by showing that $\mathbf{E}\pi_M$ is monotonically decreasing as a function of α. By symmetry, the opposite can be proven for π_m.

$\mathbf{E}\pi_M = \int_0^1 2xf(x;\alpha)F(x;\alpha).dx$, with f() and F(), respectively, the PDF and CDF of the $Beta(\alpha, \alpha\frac{\mu}{1-\mu})$. Integrating by parts (with $u' = 2xf(x;\alpha)F(x;\alpha)$ and $v = x$), we have

$\mathbf{E}\pi_M = xF^2(x;\alpha)|_0^1 - \int_0^1 F^2(x;\alpha).dx = 1 - \int_0^1 F^2(x;\alpha).dx$

So the task in front of us can be formulated as showing that $\int_0^1 F^2(x;\alpha).dx$ is monotonic with respect to α.

To proceed, we take advantage of a particular property of the beta distribution: that the integral of its CDF is equal to one minus its expected value:

Proposition: $\int_0^1 F(x).dx = xF(x) + \int_0^1 xf(x).dx = [1F(1) - 0F(0)] - \mu = 1 - \mu$.

Let f_0 and f_1 be PDF of two beta distributions with mean μ but with two different α_0 and α_1 parameters, and $\alpha_1 > \alpha_0$. We note that around $x = 0$, the CDF F_0 rises faster than F_1 (proof?). Given the proposition, the two CDFs, being smooth and monotonical, must cross. However, they can only cross once, given the identifiability of beta distribution by a pair of quantiles proven by Shih et al (if the two CDFs crosses 2 or more times, any pairs of quantiles defined by crossing points will fail to identify uniquely a beta distribution).

Let z be the unique crossing point of the two CDFs (where $F_0(z) = F_1(z)$) be the value of these CDFs at this point.

$\int_0^1 (F_0^2(x) - F_1^2(x)).dx = \int_0^z (F_0^2(x) - F_1^2(x)).dx + \int_z^1 (F_0^2(x) - F_1^2(x)).dx = \int_0^z (F_0(x) - F_1(x))(F_0(x) + F_1(x)).dx + \int_z^1 (F_0(x) - F_1(x))(F_0(x) + F_1(x)).dx$.

In the left region ($x \in [0,z]$), $F_0(x) - F_1(x) \geq 0$, and $0 < F_0(x) + F_1(x) \leq F_0(z) + F_1(z)$. As such, replacing $F_0(x) + F_1(x)$ by the larger positive quantity $F_0(z) + F_1(z)$ will increase this term. As well, in the right region ($x \in [z,1]$), $F_0(x) - F_1(x) \leq 0$, and $0 < F_0(x) + F_1(x) \leq F_0(z) + F_1(z)$. As such, replacing $F_0(x) + F_1(x)$ by the smaller positive quantity $F_0(z) + F_1(z)$ will also increase this term. Therefore we have

$\int_0^1 (F_0^2(x) - F_1^2(x)).dx \leq \int_0^z (F_0(x) - F_1(x))(F_0(z) + F_1(z)).dx + \int_z^1 (F_0(x) - F_1(x))(F_0(z) + F_1(z)).dx$, and the term on the right hand side is zero because of the proposition. Therefore, $\int_0^1 (F_0^2(x) - F_1^2(x)).dx \leq 0$, establishing the results.