

## Poisson Regression

Rebecca C. Steorts (slide adaption from Maria Tacket) and material from Chapter 4 of Roback and Legler text.

# Computing set up

```
library(tidyverse)
library(tidymodels)
library(knitr)
library(patchwork)
library(viridis)
library(gridExtra)
library(dplyr)

ggplot2:::theme_set(ggplot2:::theme_bw(base_size = 16))

colors <- tibble::tibble(green = "#B5BA72")
```

## Topics

- ▶ Describe properties of the Poisson random variable
- ▶ Write the mathematical equation of the Poisson regression model
- ▶ Describe how the Poisson regression differs from least-squares regression
- ▶ Interpret the coefficients for the Poisson regression model
- ▶ Compare two Poisson regression models

Notes based on Section 4.4 - 4.5, and 4.9 of Roback and Legler (2021) unless noted otherwise.

## Scenarios to use Poisson regression

- ▶ Does the number of employers conducting on-campus interviews during a year differ for public and private colleges?
- ▶ Does the daily number of asthma-related visits to an Emergency Room differ depending on air pollution indices?
- ▶ Does the number of paint defects per square foot of wall differ based on the years of experience of the painter?

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Each response variable is a **count per a unit of time or space.**

## Poisson distribution

Let  $Y$  be the number of events in a given unit of time or space.  
Then  $Y$  can be modeled using a **Poisson distribution**

$$P(Y = y) = \frac{e^{-\lambda} \lambda^y}{y!} \quad y = 0, 1, 2, \dots, \infty$$

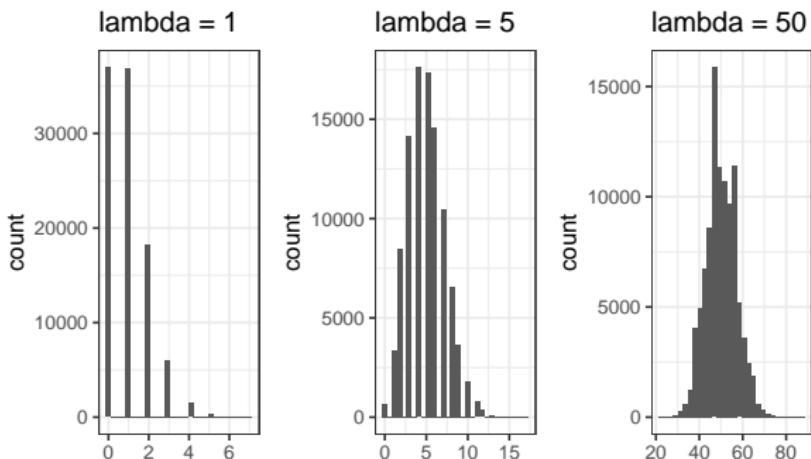
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- ▶  $E(Y) = Var(Y) = \lambda$
- ▶ The distribution is typically skewed right, particularly if  $\lambda$  is small
- ▶ The distribution becomes more symmetric as  $\lambda$  increases
  - ▶ If  $\lambda$  is sufficiently large, it can be approximated using a normal distribution ([Click here for an example.](#))

# Simulation



---

|             | Mean     | Variance   |
|-------------|----------|------------|
| lambda = 1  | 0.99351  | 0.9902178  |
| lambda = 5  | 4.99367  | 4.9865798  |
| lambda = 50 | 49.99288 | 49.8962683 |

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## Earthquakes

The annual number of earthquakes registering at least 2.5 on the Richter Scale and having an epicenter within 40 miles of downtown Memphis follows a Poisson distribution with mean 6.5.<sup>1</sup>

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<sup>1</sup>Example adapted from [Introduction to Probability Theory Example 28-2](<https://online.stat.psu.edu/stat414/lesson/28/28.2>).

## Earthquakes

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$$= \frac{e^{-6.5} 6.5^0}{0!} + \frac{e^{-6.5} 6.5^1}{1!} + \frac{e^{-6.5} 6.5^2}{2!} + \frac{e^{-6.5} 6.5^3}{3!}$$

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$$= 0.112$$

```
ppois(3, 6.5)
```

```
## [1] 0.1118496
```

## Poisson regression

# Poisson regression: Household size in the Philippines

The data fHH1.csv come from the 2015 Family Income and Expenditure Survey conducted by the Philippine Statistics Authority.

**Goal:** Understand the association between household size and various characteristics of the household

**Response:**

- ▶ total: Number of people in the household other than the head

**Predictors:**

- ▶ location: Where the house is located
- ▶ age: Age of the head of household
- ▶ roof: Type of roof on the residence (proxy for wealth)

**Other variables:**

- ▶ numLT5: Number in the household under 5 years old

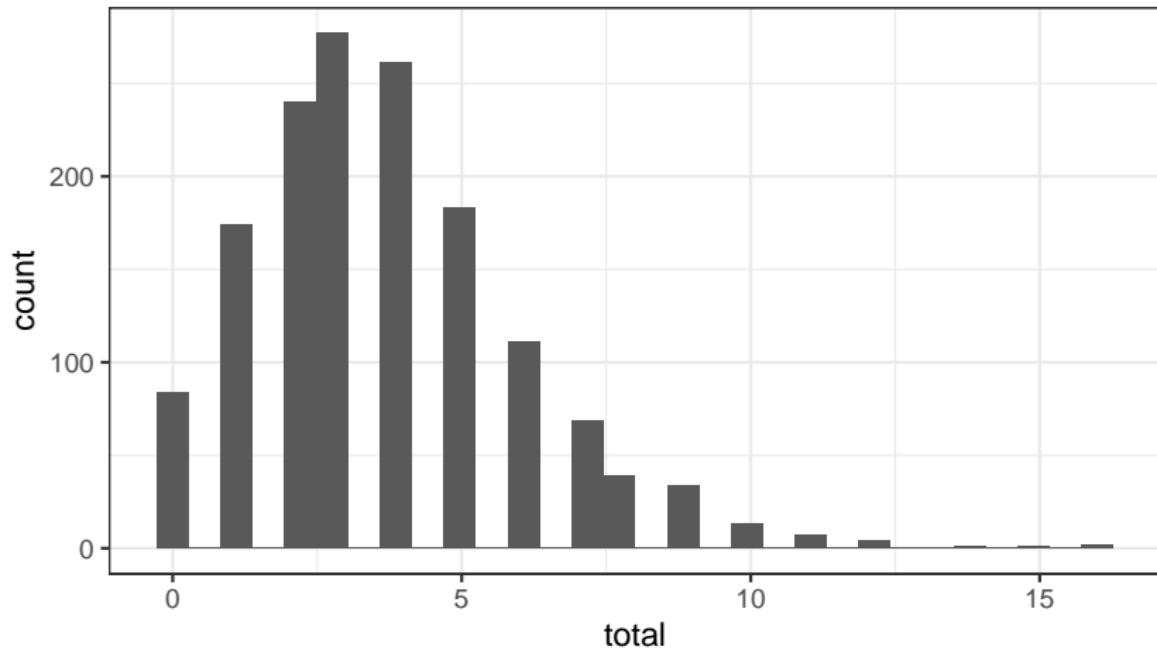
## The data

```
hh_data <- read_csv("data/fHH1.csv")
hh_data |> slice(1:5) |> kable()
```

| location     | age | total | numLT5 | roof                          |
|--------------|-----|-------|--------|-------------------------------|
| CentralLuzon | 65  | 0     | 0      | Predominantly Strong Material |
| MetroManila  | 75  | 3     | 0      | Predominantly Strong Material |
| DavaoRegion  | 54  | 4     | 0      | Predominantly Strong Material |
| Visayas      | 49  | 3     | 0      | Predominantly Strong Material |
| MetroManila  | 74  | 3     | 0      | Predominantly Strong Material |

## Response variable

Total number in household other than the head



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| mean  | var   |
|-------|-------|
| 3.685 | 5.534 |

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The goal is to model  $\lambda$ , the expected number of people in the household (other than the head), as a function of the predictors (covariates)

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$$\lambda_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}$$

This model won't work because...

- ▶ It could produce negative values of  $\lambda$  for certain values of the predictors
- ▶ The equal variance assumption required to conduct inference for linear regression is violated.

## Poisson regression model

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If  $Y_i \sim Poisson$  with  $\lambda = \lambda_i$  for the given values  $x_{i1}, \dots, x_{ip}$ , then

$$\log(\lambda_i) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \dots + \beta_p x_{ip}$$

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$$\log(\lambda_i) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \dots + \beta_p x_{ip}$$

- ▶ Each observation can have a different value of  $\lambda$  based on its value of the predictors  $x_1, \dots, x_p$
- ▶  $\lambda$  determines the mean and variance, so we don't need to estimate a separate error term

## Poisson vs. multiple linear regression

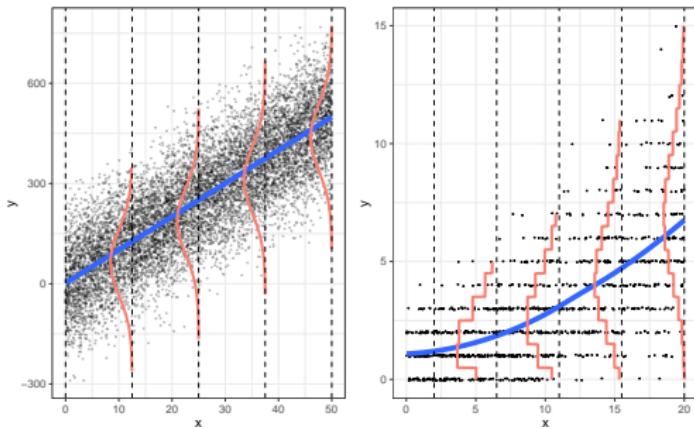


Figure 1: Regression models: Linear regression (left) and Poisson regression (right).

Figures recreated from BMLR Figure 4.1

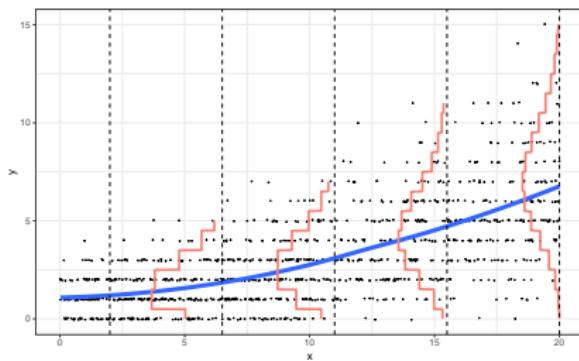
# Assumptions for Poisson regression

**Poisson response:** The response variable is a count per unit of time or space, described by a Poisson distribution, at each level of the predictor(s)

**Independence:** The observations must be independent of one another

**Mean = Variance:** The mean must equal the variance

**Linearity:** The log of the mean rate,  $\log(\lambda)$ , must be a linear function of the predictor(s)



## Model 1: Household vs. Age

```
model1 <- glm(total ~ age,  
               data = hh_data, family = poisson)  
  
tidy(model1) |>  
  kable(digits = 4)
```

| term        | estimate | std.error | statistic | p.value |
|-------------|----------|-----------|-----------|---------|
| (Intercept) | 1.5499   | 0.0503    | 30.8290   | 0       |
| age         | -0.0047  | 0.0009    | -5.0258   | 0       |

$$\log(\hat{\lambda}) = 1.5499 - 0.0047 \text{ age}$$

## Interpretation of coefficient estimates

Consider a comparison of two models – one for a given age ( $x$ ) and another for age ( $x + 1$ ).

$$\begin{aligned} \log(\lambda_x) &= \beta_0 + \beta_1 X \\ \log(\lambda_{x+1}) &= \beta_0 + \beta_1(X + 1) \\ \log(\lambda_{x+1}) - \log(\lambda_x) &= \beta_1 \\ \log\left(\frac{\lambda_{x+1}}{\lambda_x}\right) &= \beta_1 \\ \frac{\lambda_{x+1}}{\lambda_x} &= e^{\beta_1} \end{aligned} \tag{1}$$

Exponentiating the coefficient on age provides the multiplicative factor by which the mean count changes.

## Interpretation of coefficient estimates

The mean household size is predicted to decrease by 0.47% (or multiply by a factor of  $e^{-0.0047}$ ) for each year older the head of the household is.

## Interpretation of coefficient estimates

1. The mean number in the house changes by a factor of  $e^{-0.0047} = 0.995$  with each additional year older the household head is.
2. The mean number in the houses decreases by 0.5 percent with each additional year older the household head is. (Because  $1 - 0.995 = 0.005$ )
3. We predict a 0.47 percent increase in mean household size for a 1-year decrease in age of the household head (because  $1/0.995 = 1.0047$ ).
4. We predict a 0.47 percent decrease in mean household size for a 1-year increase in age of the household head (because  $1/0.995 = 1.0047$ ).

## Is the coefficient of age statistically significant?

| term        | estimate | std.error | statistic | p.value | conf.low | conf.high |
|-------------|----------|-----------|-----------|---------|----------|-----------|
| (Intercept) | 1.5499   | 0.0503    | 30.8290   | 0       | 1.4512   | 1.6482    |
| age         | -0.0047  | 0.0009    | -5.0258   | 0       | -0.0065  | -0.0029   |

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- $H_0 : \beta_1 = 0$  vs.  $H_a : \beta_1 \neq 0$
- $Z = \frac{\hat{\beta}_1 - 0}{SE(\hat{\beta}_1)} = \frac{-0.0047 - 0}{0.0009} = -5.026$  (using exact values)
- $P(|Z| > |-5.026|) = 5.01 \times 10^{-7} \approx 0.$
- Yes, it is statistically significant.

## What are plausible values for the coefficient of age?

| term        | estimate | std.error | statistic | p.value | conf.low | conf.high |
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**95% confidence interval for the coefficient of age**

$$\hat{\beta}_1 \pm z^* \times SE(\hat{\beta}_1)$$

where  $z^* \sim N(0, 1)$

$$-0.0047 \pm 1.96 \times 0.0009 = (-0.0065, -0.0029)$$

Interpret the interval in terms of the change in mean household size.

## What are plausible values for the coefficient of age?

Interpret the interval  $(-.0065, -0.0029)$  in terms of the change in mean household size.

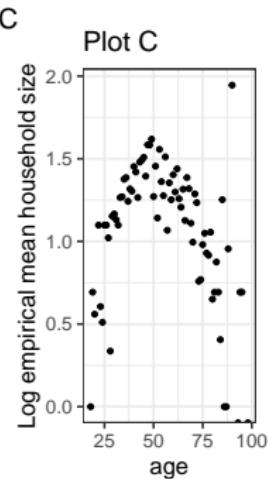
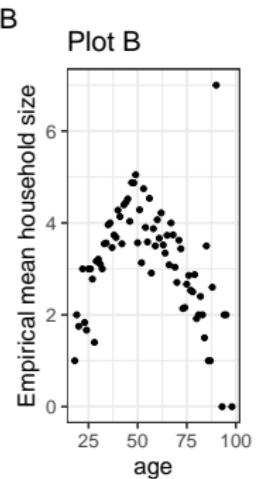
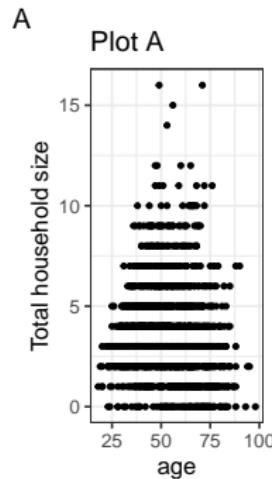
Recall: Exponentiating the endpoints yields a confidence interval for the relative risk; i.e., the percent change in household size for each additional year older.

Thus,

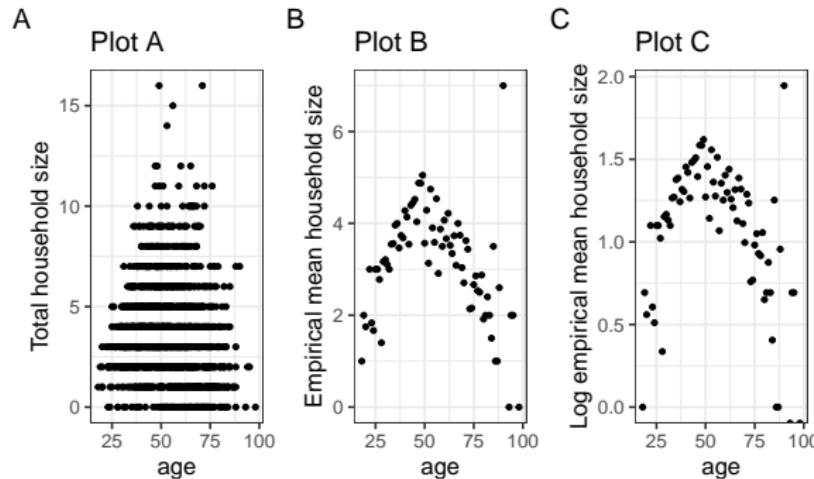
$$(e^{-0.0065}, e^{-0.0029}) = (0.993, 0.997).$$

suggests that we are 95% confident that the mean number in the house decreases between 0.7% and 0.3% for each additional year older the head of household is.

# Which plot can best help us determine whether Model 1 is a good fit?



# Which plot can best help us determine whether Model 1 is a good fit?



Solution: Plot C. Observe a curvi-linear relationship between age and the log of the mean household size, implying that adding a quadratic term should be considered.

## Model 2: Add a quadratic effect for age

```
hh_data <- hh_data |>  
  mutate(age2 = age*age)  
  
model2 <- glm(total ~ age + age2,  
               data = hh_data, family = poisson)  
tidy(model2, conf.int = T) |>  
  kable(digits = 4)
```

| term        | estimate | std.error | statistic | p.value | conf.low | conf.high |
|-------------|----------|-----------|-----------|---------|----------|-----------|
| (Intercept) | -0.3325  | 0.1788    | -1.8594   | 0.063   | -0.6863  | 0.0148    |
| age         | 0.0709   | 0.0069    | 10.2877   | 0.000   | 0.0575   | 0.0845    |
| age2        | -0.0007  | 0.0001    | -11.0578  | 0.000   | -0.0008  | -0.0006   |

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We can determine whether to keep  $age^2$  in the model in two ways:

1. Use the p-value (or confidence interval) for the coefficient (since we are adding a single term to the model). This is known as a Wald-type statistic.
2. Conduct a drop-in-deviance test

## Wald-type statistic

Observe that  $Z = -11.058$  with p-value approximately 0.

This supports the alternative hypothesis that the quadratic term is statistically significant in the model.

## Deviance

A **deviance** is a way to measure how the observed data differs (deviates) from the model predictions.

- ▶ It's a measure unexplained variability in the response variable (similar to SSE in linear regression )
- ▶ Lower deviance means the model is a better fit to the data

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We can calculate the “deviance residual” for each observation in the data (more on the formula later). Let  $(\text{deviance residual})_i$  be the deviance residual for the  $i^{\text{th}}$  observation, then

$$\text{deviance} = \sum (\text{deviance residual})_i^2$$

*Note: Deviance is also known as the “residual deviance”*

## Drop-in-Deviance Test

We can use a **drop-in-deviance test** to compare two models. To conduct the test

1. Compute the deviance for each model
2. Calculate the drop in deviance

$$\text{drop-in-deviance} = \text{Deviance}(\text{reduced model}) - \text{Deviance}(\text{larger model})$$

3. Given the reduced model is the true model ( $H_0$  true), then

$$\text{drop-in-deviance} \sim \chi_d^2$$

where  $d$  is the difference in degrees of freedom between the two models (i.e., the difference in number of terms)

## Summary of the Drop-in-Deviance

- ▶ To use the drop-in-deviance test, the models must be nested
- ▶ This means the terms in the smaller model must appear in the larger model
- ▶ When the reduced (or smaller model) is true, the drop-in-deviance  $\approx \chi_d^2$
- ▶ A large drop-in-deviance favors the larger model

Refer to Section 4.4.4 for more details.

## Drop-in-deviance to compare Model1 and Model2

```
anova(model1, model2, test = "Chisq") |>  
  kable(digits = 3)
```

| Resid. Df | Resid. Dev | Df | Deviance | Pr(>Chi) |
|-----------|------------|----|----------|----------|
| 1498      | 2337.089   | NA | NA       | NA       |
| 1497      | 2200.944   | 1  | 136.145  | 0        |

- Write the null and alternative hypotheses.
- What does the value 2337.089 tell you?
- What does the value 1 tell you?
- What is your conclusion?

## Drop-in-deviance to compare Model1 and Model2

a.

$$\text{Null (reduced) Model : } \log(\lambda) = \beta_0 + \beta_1 \text{age}$$

$$\text{Larger (full) Model : } \log(\lambda) = \beta_0 + \beta_1 \text{age} + \beta_2 \text{age}^2$$

b. What does the value 2337.089 tell you?

The value 2337.1 (with 1498 df.) provides the residual deviance for the null model.

c. There is only 1 degree of freedom difference between the two models.

## Drop-in-deviance to compare Model1 and Model2 (continued)

- d. The drop-in-deviance is  $2337.089 - 2200.94 = 136.15 \approx \chi_1^2$ .

The p-value is 0, indicating there is statistically significant evidence that average household size decreases as age of the head of household increases. This provides significant support for rejecting the null hypothesis (in favor of the alternative) and including the quadratic term.

## Add location to the model?

Suppose we want to add location to the model, so we compare the following models:

**Model A:**  $\lambda_i = \beta_0 + \beta_1 \text{ age}_i + \beta_2 \text{ age}_i^2$

**Model B:**  $\lambda_i =$

$$\beta_0 + \beta_1 \text{ age}_i + \beta_2 \text{ age}_i^2 + \beta_3 \text{ Loc1}_i + \beta_4 \text{ Loc2}_i + \beta_5 \text{ Loc3}_i + \beta_6 \text{ Loc4}_i$$

Which of the following are reliable ways to determine if location should be added to the model? (See Section 4.5, regarding comparison of linear versus Poisson models)

1. Drop-in-deviance test
2. Use the p-value for each coefficient
3. Likelihood ratio test
4. Nested F Test
5. BIC
6. AIC

## Add location to the model?

- ▶ See Section 4.4.7 regarding adding location to the model.  
(Pages 109 – 110).

## Supplementary Material

Let's consider the connection between the Drop in deviance test and LRT for Poisson regression.

## Drop in deviance and LRT

In Poisson regression, the deviance is defined as  $-2$  times the log-likelihood ratio of a fitted model relative to the saturated model.

The saturated model is the model that has one free parameter per observation, so it fits the data perfectly.

When comparing two nested models, the saturated model term cancels in the difference of deviances, leaving exactly the likelihood ratio test statistic.

## Proof

Proof: Let  $M_0 \subset M_1$  be two nested Poisson regression models.

The deviance of a model  $M$  is defined as

$$D(M) = 2 \left\{ \ell(\hat{\lambda}^{\text{sat}}) - \ell(\hat{\lambda}^{(M)}) \right\},$$

where the saturated model satisfies  $\hat{\lambda}_i^{\text{sat}} = y_i$ .

Because both  $M_0$  and  $M_1$  are compared to the same saturated model,

$$\begin{aligned} D(M_0) - D(M_1) &= 2 \left\{ \ell(\hat{\lambda}^{(1)}) - \ell(\hat{\lambda}^{(0)}) \right\} \\ &= -2 \log \frac{L(\hat{\lambda}^{(0)})}{L(\hat{\lambda}^{(1)})}, \end{aligned}$$

which is exactly the likelihood ratio test statistic for testing  $M_0$  against  $M_1$ .

Since  $M_0$  is nested in  $M_1$ , the statistic is asymptotically  $\chi^2$  with degrees of freedom equal to the number of additional parameters in  $M_1$ .

## Formal treatment

Is the LRT a special case of the Drop-in-deviance test for GLMs?

- ▶ Yes! Please see

<http://users.stat.umn.edu/~helwig/notes/generalized-linear-models.html#example-2-poisson-regression>

## Interpretation of GLM regression coefficients

A generalized linear model (GLM) is defined by

$$g(\mathbb{E}[Y | X]) = \eta = X\beta,$$

where  $g(\cdot)$  is the link function and  $\eta$  is the linear predictor.

*GLM coefficients describe how predictors affect the **mean of the response** through the link function.*

## Coefficient interpretation for GLMs

For a predictor  $X_j$ ,

$$\beta_j = \frac{\partial}{\partial X_j} [g(\mathbb{E}[Y | X])],$$

that is,  $\beta_j$  is the additive change in the *link-transformed mean (or median)* associated with a one-unit increase in  $X_j$ , holding other variables fixed.

To interpret  $\beta_j$  on the response scale, apply the inverse link:

$$\mathbb{E}[Y | X] = g^{-1}(X\beta).$$

## Linear versus Poisson regression

- ▶ **Identity link (linear regression):**

$$\mathbb{E}[Y | X] = X\beta,$$

so  $\beta_j$  is the additive change in the mean/median of  $Y$ .

- ▶ **Log link (Poisson):**

$$\log \mathbb{E}[Y | X] = X\beta,$$

so

$$\mathbb{E}[Y | X] = \exp(X\beta), \quad e^{\beta_j} = \text{multiplicative change in the mean.}$$

## Poisson regression

## Looking ahead

- ▶ For next time - Chapter 4 - Poisson Regression
  - ▶ Sections 4.6, 4.10

## References

Roback, Paul, and Julie Legler. 2021. *Beyond multiple linear regression: applied generalized linear models and multilevel models in R*. CRC Press.