Likelihoods (Part II)

Rebecca C. Steorts (slide adaption from Maria Tacket) and material from Chapter 2 of Roback and Legler text.

Computing set up

```r
knitr::opts_chunk$set(echo = T,
                      results = "hide")
library(tidyverse)
library(tidymodels)
library(GGally)
library(knitr)
library(patchwork)
library(viridis)
library(ggfortify)

ggplot2::theme_set(ggplot2::theme_bw(base_size = 16))
colors <- tibble::tibble(green = "#B5BA72")
```

Using Likelihoods

- Describe the concept of a likelihood
- Construct the likelihood for a simple model
- Define the Maximum Likelihood Estimate (MLE) and use it to answer an analysis question
- Identify three ways to calculate or approximate the MLE and apply these methods to find the MLE for a simple model
- Use likelihoods to compare models

What is the likelihood?

A **likelihood** is a function that tells us how likely we are to observe our data for a given parameter value (or values).

- Unlike Ordinary Least Squares (OLS), they do not require the responses be independent, identically distributed, and normal (iidN)
- They are not the same as probability functions

Probability function vs. likelihood

- **Probability function:** Fixed parameter value(s) + input possible outcomes $\Rightarrow$ probability of seeing the different outcomes given the parameter value(s)
- **Likelihood:** Fixed data + input possible parameter values $\Rightarrow$ probability of seeing the fixed data for each parameter value

Data: Fouls in college basketball games

The data set `04-refs.csv` includes 30 randomly selected NCAA men's basketball games played in the 2009 - 2010 season.[1]

We will focus on the variables `foul1`, `foul2`, and `foul3`, which indicate which team had a foul called them for the 1st, 2nd, and 3rd fouls, respectively.

- `H`: Foul was called on the home team
- `V`: Foul was called on the visiting team

We are focusing on the first three fouls for this analysis, but this could easily be extended to include all fouls in a game.

[1] The data set was derived from `basektball0910.csv` used in BMLR Section 11.2

Fouls in college basketball games

```r
refs <- read_csv("data/04-refs.csv")
refs |> slice(1:5) |> kable()
```

We will treat the games as independent in this analysis.

Different likelihood models

Model 1 (Unconditional Model):

- What is the probability the referees call a foul on the home team, assuming foul calls within a game are independent?

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Model 2 (Conditional Model):

- Is there a tendency for the referees to call more fouls on the visiting team or home team?
- Is there a tendency for referees to call a foul on the team that already has more fouls?

Ultimately we want to decide which model is better.

Exploratory data analysis

```
refs |>
count(foul1, foul2, foul3) |> kable()
```

There are

- 46 total fouls on the home team
- 44 total fouls on the visiting team

Model 1: Unconditional model

What is the probability the referees call a foul on the home team, assuming foul calls within a game are independent?

Likelihood

Let p_H be the probability the referees call a foul on the home team.
The likelihood for a single observation

$$Lik(p_H) = p_H^{y_i}(1 - p_H)^{n_i - y_i}$$

Where y_i is the number of fouls called on the home team. (In this example, we know $n_i = 3$ for all observations.)

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Example

For a single game where the first three fouls are H, H, V, then

$$Lik(p_H) = p_H^2(1 - p_H)^{3-2} = p_H^2(1 - p_H)$$

Model 1: Likelihood contribution

Foul 1	Foul 2	Foul 3	n	Likelihood contribution
H	H	H	3	p_H^3
H	H	V	2	$p_H^2(1-p_H)$
H	V	H	3	$p_H^2(1-p_H)$
H	V	V	7	**A**
V	H	H	7	**B**
V	H	V	1	$p_H(1-p_H)^2$
V	V	H	5	$p_H(1-p_H)^2$
V	V	V	2	$(1-p_H)^3$

Fill in **A** and **B**.

Model 1: Solution Likelihood contribution

Foul 1	Foul 2	Foul 3	n	Likelihood contribution
H	H	H	3	p_H^3
H	H	V	2	$p_H^2(1-p_H)$
H	V	H	3	$p_H^2(1-p_H)$
H	V	V	7	$p_H(1-p_H)^2$
V	H	H	7	$p_H^2(1-p_H)$
V	H	V	1	$p_H(1-p_H)^2$
V	V	H	5	$p_H(1-p_H)^2$
V	V	V	2	$(1-p_H)^3$

Observe that n does not matter here. The order of the fouls does not matter either.

Model 1: Likelihood function

Because the observations (the games) are independent, the **likelihood** is

$$Lik(p_H) = \prod_{i=1}^{n} p_H^{y_i}(1 - p_H)^{3 - y_i}$$

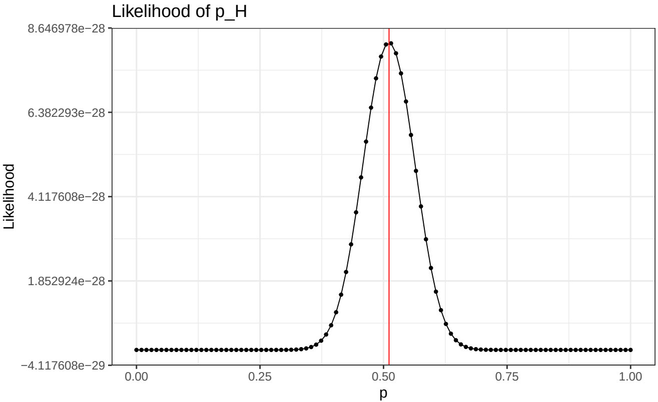
We will use this function to find the **maximum likelihood estimate (MLE)**. The MLE is the value between 0 and 1 where we are most likely to see the observed data.

Finding the maximum likelihood estimate

There are three primary ways to find the MLE:

- Approximate using a graph
- Numerical approximation
- Using calculus

Approximate MLE from a graph



Visualizing the likelihood

What is your best guess for the MLE, $\hat{p}_H$?

a. 0.489
b. 0.500
c. 0.511
d. 0.556

Find the MLE using numerical approximation

Specify a finite set of possible values the for p_H and calculate the likelihood for each value

```r
# write an R function for the likelihood
ref_lik <- function(ph) {
  ph^46 *(1 - ph)^44
}

# search possible values for p and return max
nGrid = 1000
ph <- seq(0, 1, length = nGrid)
lik <- ref_lik(ph)
ph[lik == max(lik)]
```

```r
# use the optimize function to find the MLE
optimize(ref_lik, interval = c(0,1), maximum = TRUE)
```

Find the MLE using numerical approximation

Exercise:

1. Re-write the code to find the MLE using the log-likelihood.
2. The MLE should be the same.
3. Remember why we work with the log-likelihood (convenience and numerical stability).

Video on connections between likelihood and log-likelihood:
https://www.youtube.com/watch?v=8nogLkirA3I

Find MLE using calculus

- Find the MLE by taking the first derivative of the likelihood function.
- This can be tricky because of the Product Rule, so we can maximize the **log(Likelihood)** instead. The same value maximizes the likelihood and log(Likelihood)

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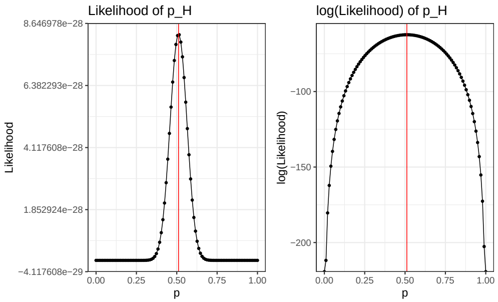

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Find MLE using calculus

$$Lik(p_H) = \prod_{i=1}^{n} p_H^{y_i}(1-p_H)^{3-y_i}$$

$$\log(Lik(p_H)) = \sum_{i=1}^{n} y_i \log(p_H) + (3-y_i)\log(1-p_H)$$

$$= 46\log(p_H) + 44\log(1-p_H)$$

Find MLE using calculus

$$\frac{d}{dp_H}\log(Lik(p_H)) = \frac{46}{p_H} - \frac{44}{1-p_H} = 0$$

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$$\Rightarrow 46 = 90p_H$$

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$$\Rightarrow 46 = 90p_H$$

$$\hat{p}_H = \frac{46}{90} = 0.511$$

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$$\Rightarrow 46 = 90p_H$$

$$\hat{p}_H = \frac{46}{90} = 0.511$$

Model 2: Conditional model

Is there a tendency for referees to call more fouls on the visiting team or home team?

Is there a tendency for referees to call a foul on the team that already has more fouls?

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Now let's assume fouls are not independent within each game. We will specify this dependence using conditional probabilities.

- **Conditional probability**: $P(A|B) =$ Probability of A given B has occurred

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Define new parameters:

- $p_{H|N}$: Probability referees call foul on home team given there are equal numbers of fouls on the home and visiting teams

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- $p_{H|HBias}$: Probability referees call foul on home team given there are more prior fouls on the home team

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- **Conditional probability**: $P(A|B) =$ Probability of A given B has occurred

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- $p_{H|HBias}$: Probability referees call foul on home team given there are more prior fouls on the home team
- $p_{H|VBias}$: Probability referees call foul on home team given there are more prior fouls on the visiting team

Model 2: Likelihood contributions

Foul 1	Foul 2	Foul 3	n	Likelihood contribution
H	H	H	3	$(p_{H\|N})(p_{H\|HBias})(p_{H\|HBias}) = (p_{H\|N})(p_{H\|HBias})^2$
H	H	V	2	$(p_{H\|N})(p_{H\|HBias})(1 - p_{H\|HBias})$
H	V	H	3	$(p_{H\|N})(1 - p_{H\|HBias})(p_{H\|N})$
H	V	V	7	**A**
V	H	H	7	**B**
V	H	V	1	$(1 - p_{H\|N})(p_{H\|VBias})(1 - p_{H\|N}) = (1 - p_{H\|N})^2(p_{H\|VBias})$
V	V	H	5	$(1 - p_{H\|N})(1 - p_{H\|VBias})(p_{H\|VBias})$
V	V	V	2	$(1 - p_{H\|N})(1 - p_{H\|VBias})(1 - p_{H\|VBias}) = (1 - p_{H\|N})(1 - p_{H\|VBias})^2$

Fill in **A** and **B.**

Model 2: Solution Likelihood contributions

Foul 1	Foul 2	Foul 3	n	Likelihood contribution
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H	H	V	2	$(p_{H\|N})(p_{H\|HBias})(p_{H\|HBias}) = (p_{H\|N})(p_{H\|HBias})^2$
H	V	H	3	$(p_{H\|N})(p_{H\|HBias})(1 - p_{H\|HBias})$
H	V	V	7	**A**
V	H	H	7	**B**
V	H	V	1	$(1 - p_{H\|N})(p_{H\|VBias})(1 - p_{H\|N}) = (1 - p_{H\|N})^2(p_{H\|VBias})$
V	V	H	5	$(1 - p_{H\|N})(1 - p_{H\|VBias})(p_{H\|VBias})$
V	V	V	2	$(1 - p_{H\|N})(1 - p_{H\|VBias})(1 - p_{H\|VBias}) = (1 - p_{H\|N})(1 - p_{H\|VBias})^2$

Fill in **A** and **B.**

Likelihood function

$$Lik(p_{H|N}, p_{H|HBias}, p_{H|VBias}) = [(p_{H|N})^{25}(1 - p_{H|N})^{23}(p_{H|HBias})^{8} (1 - p_{H|HBias})^{12}(p_{H|VBias})^{13}(1 - p_{H|VBias})^{9}]$$

(Note: The exponents sum to 90, the total number of fouls in the data)

Likelihood function

$$
\begin{aligned}
Lik(p_{H|N}, p_{H|HBias}, p_{H|VBias}) = [&(p_{H|N})^{25}(1 - p_{H|N})^{23}(p_{H|HBias})^{8} \\
&(1 - p_{H|HBias})^{12}(p_{H|VBias})^{13}(1 - p_{H|VBias})^{9}]
\end{aligned}
$$

(Note: The exponents sum to 90, the total number of fouls in the data)

$$
\begin{aligned}
\log(Lik(p_{H|N}, p_{H|HBias}, p_{H|VBias})) &= 25\log(p_{H|N}) + 23\log(1 - p_{H|N}) \\
&+ 8\log(p_{H|HBias}) + 12\log(1 - p_{H|HBias}) \\
&+ 13\log(p_{H|VBias}) + 9\log(1 - p_{H|VBias})
\end{aligned}
$$

Model Comparisons

- Nested Models
- Non-nested Models

Comparing Nested Models

Nested Models

Nested models: Models such that the parameters of the reduced model are a subset of the parameters for a larger model

Example:

$$\text{Model A: } y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \epsilon$$
$$\text{Model B: } y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \epsilon$$

Model A is nested in Model B. We could use likelihoods to test whether it is useful to add x_3 and x_4 to the model.

$$H_0 : \beta_3 = \beta_4 = 0$$
$$H_a : \text{ at least one } \beta_j \text{ is not equal to } 0$$

Nested Models

Another way to think about nested models: Parameters in larger model can be equated to get the simpler model or if some parameters can be set to constants

Example:

Model 1: p_H

Model 2: $p_{H|N}, p_{H|HBias}, p_{H|VBias}$

Model 1 is nested in Model 2. The parameters $p_{H|N}$, $p_{H|HBias}$, and $p_{H|VBias}$ can be set equal to p_H to get Model 1.

$H_0 : p_{H|N} = p_{H|HBias} = p_{H|VBias} = p_H$

H_a : At least one of $p_{H|N}, p_{H|HBias}, p_{H|VBias}$ differs from the others

Steps to compare models

1. Find the MLEs for each model.
2. Plug the MLEs into the log-likelihood function for each model to get the maximum value of the log-likelihood for each model.
3. Find the difference in the maximum log-likelihoods
4. Use the Likelihood Ratio Test to determine if the difference is statistically significant

Step 1

Model 1: $\hat{p}_H = 46/90 = 0.511$

```
loglik1 <- function(ph){
 log(ph^46 * (1 - ph)^44)
}
loglik1(46/90)
```

Solution = -62.36

Step 2

Model 2:

- $\hat{p}_{H|N} = 25/48 = 0.521$
- $\hat{p}_{H|HBias} = 8/20 = 0.4$
- $\hat{p}_{H|VBias} = 13/22 = 0.591$

```
loglik2 <- function(phn, phh, phv) {
  log(phn^25 * (1 - phn)^23 * phh^8 *
        (1 - phh)^12 * phv^13 * (1 - phv)^9)
}
(loglik2(25/48, 8/20, 13/22))
```

Solution = -61.57

Step 3

Find the difference in the log-likelihoods

```
(diff <- loglik2(25/48, 8/20, 13/22) - loglik1(46/90))
```

Diff $= 0.7878$

Is the difference in the maximum log-likelihoods statistically significant?

Likelihood Ratio Test

Test statistic

$$
\begin{aligned}
LRT &= 2[\max\{\log(Lik(\text{larger model}))\} - \max\{\log(Lik(\text{reduced model}))\}] \\
&= 2\log\left(\frac{\max\{(Lik(\text{larger model})\}}{\max\{(Lik(\text{reduced model})\}}\right)
\end{aligned}
$$

LRT follows a χ^2 distribution where the degrees of freedom equal the difference in the number of parameters between the two models

Step 4

```
(LRT <- 2 * (loglik2(25/48, 8/20, 13/22) - loglik1(46/90)))
```

LRT $= 1.576$

The test statistic follows a χ^2 distribution with 2 degrees of freedom. Therefore, the p-value is $P(\chi^2 > LRT)$.

```
pchisq(LRT, 2, lower.tail = FALSE)
```

p-value $= 0.4548$

The p-value is very large, so we fail to reject H_0. We do not have convincing evidence that the conditional model is an improvement over the unconditional model.

Therefore, we can stick with the **unconditional** model.

Comparing non-nested models

Comparing non-nested models

We can also consider AIC and BIC. We cannot consider a LRT for a non-nested model.

Comparing non-nested models

AIC = -2(max log-likelihood) + 2p

```
(Model1_AIC <- 2 * loglik1(46/90) + 2 * 1)
```

AIC_1 = -122.72

```
(Model2_AIC <-2 * loglik2(25/48, 8/20, 13/22) + 2 * 3)
```

AIC_2 = -117.15

Comparing non-nested models

```
(Model1_BIC <- 2 * loglik1(46/90) + 1 * log(30))
```

BIC_1 = -121.32

```
(Model2_BIC <-2 * loglik2(25/48, 8/20, 13/22) + 3 * log(30)
```

BIC_2 = -112.94

Which model would you choose and why based upon the AIC and BIC values?

Looking ahead

- Likelihoods help us answer the question of how likely we are to observe the data given different parameters
- In this example, we did not consider covariates, so in practice the parameters we want to estimate will look more similar to this

$$p_H = \frac{e^{\beta_0+\beta_1 x_1+\cdots+\beta_p x_p}}{1+e^{\beta_0+\beta_1 x_1+\cdots+\beta_p x_p}}$$

- Finding the MLE becomes much more complex (in practice) and numerical methods may be required.

References