Unifying Theory of GLMs

Rebecca C. Steorts

Exponential Families

Consider the following:

$$p(y \mid \eta) = h(y)\exp\{\eta^T t(y) - a(\eta)\},$$

where

- η is the natural parameter
- $h(y)$ is the underlying measure
- $a(\eta)$ is the log normalizer (ensuring the distribution integrates to one).

It follows that

$$a(\eta) = log \int h(y)\exp\{\eta^T t(y) - a(\eta)\}\, dy.$$

Exponential Families

We can alternatively write the exponential family as follows:

$$p(y \mid \eta) = \frac{1}{Z(\eta)} h(y) \exp\{\eta^T u(y)\},$$

where

- η are the natural parameters
- $u(y)$ are the sufficient statistics
- $Z(\eta)$ is a partition function that ensures the density is normalized.

$$Z(\eta) = \int h(y) \exp\{\eta^T u(y)\} \, dy.$$

Bernouli

$$p(y \mid \eta) = \frac{1}{Z(\eta)} h(y) \exp\{\eta^T u(y)\}$$

Consider

$$\begin{aligned}
\text{Bern}(y \mid \mu) &= \mu^y (1-\mu)^{1-y} && (1)\\
&= (1-\mu)(\frac{\mu}{1-\mu})^y && (2)\\
&= (1-\mu) \times 1 \times \exp\{\log(\frac{\mu}{1-\mu}) \times y\} && (3)
\end{aligned}$$

- $Z(\eta) = \frac{1}{1-\mu}$
- $h(y) = 1$
- $\eta = \log(\frac{\mu}{1-\mu})$
- $u(y) = y$

Normal (μ)

$$p(y \mid \eta) = \frac{1}{Z(\eta)} h(y) \exp\{\eta^T u(y)\}$$

$$\begin{align}
p(y \mid \mu, \sigma^2) &= \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\{\frac{-1}{2\sigma^2}(y-\mu)^2\} \tag{4} \\
&= \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\{\frac{-1}{2\sigma^2}y^2 + \frac{\mu}{\sigma^2}y - \frac{\mu^2}{2\sigma^2}\} \tag{5} \\
&= \underbrace{\frac{1}{(2\pi\sigma^2)^{1/2}} \exp\{\frac{-1}{2\sigma^2}y^2\}}_{h(y)} \underbrace{\exp\{-\frac{\mu^2}{2\sigma^2}\}}_{\frac{1}{Z(\eta)}} \exp\{\underbrace{\frac{\mu}{\sigma^2}}_{\eta^T} \underbrace{y}_{u(y)}\} \tag{6}
\end{align}$$