

## Likelihoods (Part II)

Rebecca C. Steorts (slide adaption from Maria Tacket) and material from Chapter 2 of Roback and Legler text.

## Computing set up

```
knitr::opts_chunk$set(echo = T,  
                      results = "hide")  
  
library(tidyverse)  
library(tidymodels)  
library(GGally)  
library(knitr)  
library(patchwork)  
library(viridis)  
library(ggfortify)  
  
ggplot2::theme_set(ggplot2::theme_bw(base_size = 16))  
colors <- tibble::tibble(green = "#B5BA72")
```

# Using Likelihoods

- ▶ Describe the concept of a likelihood
- ▶ Construct the likelihood for a simple model
- ▶ Define the Maximum Likelihood Estimate (MLE) and use it to answer an analysis question
- ▶ Identify three ways to calculate or approximate the MLE and apply these methods to find the MLE for a simple model
- ▶ Use likelihoods to compare models

# What is the likelihood?

A **likelihood** is a function that tells us how likely we are to observe our data for a given parameter value (or values).

- ▶ Unlike Ordinary Least Squares (OLS), they do not require the responses be independent, identically distributed, and normal (iidN)
- ▶ They are not the same as probability functions

# Probability function vs. likelihood

- ▶ **Probability function:** Fixed parameter value(s) + input possible outcomes  $\Rightarrow$  probability of seeing the different outcomes given the parameter value(s)
- ▶ **Likelihood:** Fixed data + input possible parameter values  $\Rightarrow$  probability of seeing the fixed data for each parameter value

## Data: Fouls in college basketball games

The data set `04-refs.csv` includes 30 randomly selected NCAA men's basketball games played in the 2009 - 2010 season.<sup>1</sup>

We will focus on the variables `foul1`, `foul2`, and `foul3`, which indicate which team had a foul called them for the 1st, 2nd, and 3rd fouls, respectively.

- ▶ H: Foul was called on the home team
- ▶ V: Foul was called on the visiting team

We are focusing on the first three fouls for this analysis, but this could easily be extended to include all fouls in a game.

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<sup>1</sup>The data set was derived from `basektball0910.csv` used in BMLR Section 11.2

## Fouls in college basketball games

```
refs <- read_csv("data/04-refs.csv")  
refs |> slice(1:5) |> kable()
```

We will treat the games as independent in this analysis.

# Different likelihood models

## **Model 1 (Unconditional Model):**

- ▶ What is the probability the referees call a foul on the home team, assuming foul calls within a game are independent?



# Different likelihood models

## **Model 1 (Unconditional Model):**

- ▶ What is the probability the referees call a foul on the home team, assuming foul calls within a game are independent?

## **Model 2 (Conditional Model):**

- ▶ Is there a tendency for the referees to call more fouls on the visiting team or home team?
- ▶ Is there a tendency for referees to call a foul on the team that already has more fouls?

**Ultimately we want to decide which model is better.**

# Exploratory data analysis

```
refs |>  
count(foul1, foul2, foul3) |> kable()
```

There are

- ▶ 46 total fouls on the home team
- ▶ 44 total fouls on the visiting team

## Model 1: Unconditional model

What is the probability the referees call a foul on the home team, assuming foul calls within a game are independent?

# Likelihood

Let  $p_H$  be the probability the referees call a foul on the home team.

**The likelihood for a single observation**

$$Lik(p_H) = p_H^{y_i}(1 - p_H)^{n_i - y_i}$$

Where  $y_i$  is the number of fouls called on the home team. (In this example, we know  $n_i = 3$  for all observations.)

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## **Example**

For a single game where the first three fouls are  $H, H, V$ , then

$$Lik(p_H) = p_H^2(1 - p_H)^{3-2} = p_H^2(1 - p_H)$$

## Model 1: Likelihood contribution

Foul 1	Foul 2	Foul 3	n	Likelihood contribution
H	H	H	3	$p_H^3$
H	H	V	2	$p_H^2(1 - p_H)$
H	V	H	3	$p_H^2(1 - p_H)$
H	V	V	7	<b>A</b>
V	H	H	7	<b>B</b>
V	H	V	1	$p_H(1 - p_H)^2$
V	V	H	5	$p_H(1 - p_H)^2$
V	V	V	2	$(1 - p_H)^3$

Fill in **A** and **B**.

## Model 1: Solution Likelihood contribution

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H	H	H	3	$p_H^3$
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H	V	H	3	$p_H^2(1 - p_H)$
H	V	V	7	$p_H(1 - p_H)^2$
V	H	H	7	$p_H^2(1 - p_H)$
V	H	V	1	$p_H(1 - p_H)^2$
V	V	H	5	$p_H(1 - p_H)^2$
V	V	V	2	$(1 - p_H)^3$

Observe that  $n$  does not matter here. The order of the fouls does not matter either.

## Model 1: Likelihood function

Because the observations (the games) are independent, the **likelihood** is

$$Lik(p_H) = \prod_{i=1}^n p_H^{y_i} (1 - p_H)^{3-y_i}$$

We will use this function to find the **maximum likelihood estimate (MLE)**. The MLE is the value between 0 and 1 where we are most likely to see the observed data.

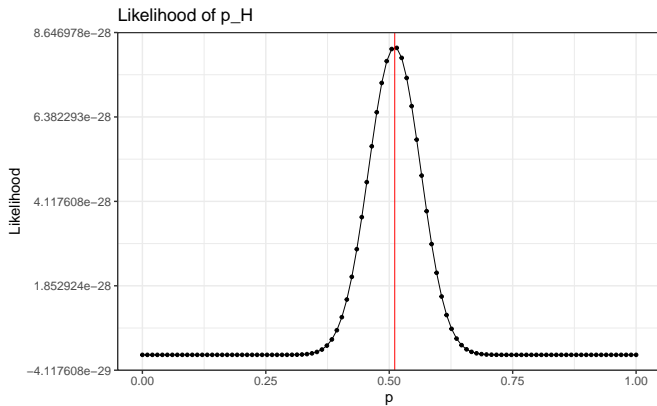


# Finding the maximum likelihood estimate

There are three primary ways to find the MLE:

- ▶ Approximate using a graph
- ▶ Numerical approximation
- ▶ Using calculus

# Approximate MLE from a graph



## Visualizing the likelihood

What is your best guess for the MLE,  $\hat{p}_H$ ?

- a. 0.489
- b. 0.500
- c. 0.511
- d. 0.556

## Find the MLE using numerical approximation

Specify a finite set of possible values the for  $p_H$  and calculate the likelihood for each value

```
# write an R function for the likelihood
```

```
ref_lik <- function(ph) {  
  ph^46 *(1 - ph)^44  
}
```

```
# search possible values for p and return max
```

```
nGrid = 1000  
ph <- seq(0, 1, length = nGrid)  
lik <- ref_lik(ph)  
ph[lik == max(lik)]
```

```
# use the optimize function to find the MLE
```

```
optimize(ref_lik, interval = c(0,1), maximum = TRUE)
```

# Find the MLE using numerical approximation

Exercise:

1. Re-write the code to find the MLE using the log-likelihood.
2. The MLE should be the same.
3. Remember why we work with the log-likelihood (convenience and numerical stability).

Video on connections between likelihood and log-likelihood:

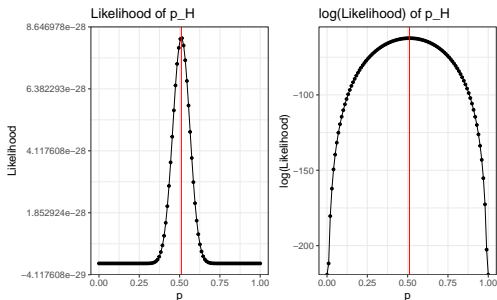
<https://www.youtube.com/watch?v=8nogLkirA3I>

## Find MLE using calculus

- ▶ Find the MLE by taking the first derivative of the likelihood function.
- ▶ This can be tricky because of the Product Rule, so we can maximize the **log(Likelihood)** instead. The same value maximizes the likelihood and  $\log(\text{Likelihood})$

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Find MLE using calculus

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## Find MLE using calculus

$$Lik(p_H) = \prod_{i=1}^n p_H^{y_i} (1 - p_H)^{3-y_i}$$

$$\log(Lik(p_H)) = \sum_{i=1}^n y_i \log(p_H) + (3 - y_i) \log(1 - p_H)$$

$$= 46 \log(p_H) + 44 \log(1 - p_H)$$

Find MLE using calculus

$$\frac{d}{dp_H} \log(Lik(p_H)) = \frac{46}{p_H} - \frac{44}{1 - p_H} = 0$$

## Find MLE using calculus

$$\frac{d}{dp_H} \log(\text{Lik}(p_H)) = \frac{46}{p_H} - \frac{44}{1 - p_H} = 0$$

$$\Rightarrow \frac{46}{p_H} = \frac{44}{1 - p_H}$$

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$$\Rightarrow 46(1 - p_H) = 44p_H$$

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$$\Rightarrow 46 = 90p_H$$

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$$\Rightarrow 46 = 90p_H$$

$$\hat{p}_H = \frac{46}{90} = 0.511$$

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$$\Rightarrow \frac{46}{p_H} = \frac{44}{1 - p_H}$$

$$\Rightarrow 46(1 - p_H) = 44p_H$$

$$\Rightarrow 46 = 90p_H$$

$$\hat{p}_H = \frac{46}{90} = 0.511$$

## Model 2: Conditional model

Is there a tendency for referees to call more fouls on the visiting team or home team?

Is there a tendency for referees to call a foul on the team that already has more fouls?



## Model 2: Conditional model

Now let's assume fouls are not independent within each game. We will specify this dependence using conditional probabilities.

- ▶ **Conditional probability:**  $P(A|B)$  = Probability of  $A$  given  $B$  has occurred

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Define new parameters:

- ▶  $p_{H|N}$ : Probability referees call foul on home team given there are equal numbers of fouls on the home and visiting teams

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Define new parameters:

- ▶  $p_{H|N}$ : Probability referees call foul on home team given there are equal numbers of fouls on the home and visiting teams
- ▶  $p_{H|HBias}$ : Probability referees call foul on home team given there are more prior fouls on the home team

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Define new parameters:

- ▶  $p_{H|N}$ : Probability referees call foul on home team given there are equal numbers of fouls on the home and visiting teams
- ▶  $p_{H|HBias}$ : Probability referees call foul on home team given there are more prior fouls on the home team
- ▶  $p_{H|VBias}$ : Probability referees call foul on home team given there are more prior fouls on the visiting team

## Model 2: Likelihood contributions

Foul 1	Foul 2	Foul 3	n	Likelihood contribution
H	H	H	3	$(p_{H N})(p_{H HBias})(p_{H HBias}) =$ $(p_{H N})(p_{H HBias})^2$
H	H	V	2	$(p_{H N})(p_{H HBias})(1 - p_{H HBias})$
H	V	H	3	$(p_{H N})(1 - p_{H HBias})(p_{H N})$
H	V	V	7	<b>A</b>
V	H	H	7	<b>B</b>
V	H	V	1	$(1 - p_{H N})(p_{H VBias})(1 - p_{H N}) =$ $(1 - p_{H N})^2(p_{H VBias})$
V	V	H	5	$(1 - p_{H N})(1 - p_{H VBias})(p_{H VBias})$
V	V	V	2	$(1 - p_{H N})(1 - p_{H VBias})(1 - p_{H VBias})$ $= (1 - p_{H N})(1 - p_{H VBias})^2$

Fill in **A** and **B**.

## Likelihood function

$$Lik(p_{H|N}, p_{H|HBias}, p_{H|VBias}) = [(p_{H|N})^{25}(1 - p_{H|N})^{23}(p_{H|HBias})^8 \\ (1 - p_{H|HBias})^{12}(p_{H|VBias})^{13}(1 - p_{H|VBias})^9]$$

**(Note: The exponents sum to 90, the total number of fouls in the data)**

## Likelihood function

$$Lik(p_{H|N}, p_{H|HBias}, p_{H|VBias}) = [(p_{H|N})^{25}(1 - p_{H|N})^{23}(p_{H|HBias})^8 \\ (1 - p_{H|HBias})^{12}(p_{H|VBias})^{13}(1 - p_{H|VBias})^9]$$

**(Note: The exponents sum to 90, the total number of fouls in the data)**

$$\log(Lik(p_{H|N}, p_{H|HBias}, p_{H|VBias})) = 25 \log(p_{H|N}) + 23 \log(1 - p_{H|N}) \\ + 8 \log(p_{H|HBias}) + 12 \log(1 - p_{H|HBias}) \\ + 13 \log(p_{H|VBias}) + 9 \log(1 - p_{H|VBias})$$

## Exercise

On your own, fill in the table and verify that the likelihood is indeed correct.



# Model Comparisons

- ▶ Nested Models
- ▶ Non-nested Models

## Comparing Nested Models

# Nested Models

Nested models: Models such that the parameters of the reduced model are a subset of the parameters for a larger model

Example:

$$\text{Model A: } y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \epsilon$$

$$\text{Model B: } y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \epsilon$$

Model A is nested in Model B. We could use likelihoods to test whether it is useful to add  $x_3$  and  $x_4$  to the model.

$$H_0 : \beta_3 = \beta_4 = 0$$

$$H_a : \text{at least one } \beta_j \text{ is not equal to } 0$$

## Nested Models

Another way to think about nested models: Parameters in larger model can be equated to get the simpler model or if some parameters can be set to constants

Example:

Model 1:  $p_H$

Model 2:  $p_{H|N}, p_{H|HBias}, p_{H|VBias}$

Model 1 is nested in Model 2. The parameters  $p_{H|N}$ ,  $p_{H|HBias}$ , and  $p_{H|VBias}$  can be set equal to  $p_H$  to get Model 1.

$$H_0 : p_{H|N} = p_{H|HBias} = p_{H|VBias} = p_H$$

$H_a$  : At least one of  $p_{H|N}, p_{H|HBias}, p_{H|VBias}$  differs from the others

# Steps to compare models

1. Find the MLEs for each model.
2. Plug the MLEs into the log-likelihood function for each model to get the maximum value of the log-likelihood for each model.
3. Find the difference in the maximum log-likelihoods
4. Use the Likelihood Ratio Test to determine if the difference is statistically significant

## Step 1

Model 1:  $\hat{p}_H = 46/90 = 0.511$

```
loglik1 <- function(ph){  
  log(ph^46 * (1 - ph)^44)  
}  
loglik1(46/90)
```

Solution = -62.36

## Step 2

Model 2:

- ▶  $\hat{p}_{H|N} = 25/48 = 0.521$
- ▶  $\hat{p}_{H|HBias} = 8/20 = 0.4$
- ▶  $\hat{p}_{H|VBias} = 13/22 = 0.591$

```
loglik2 <- function(phn, phh, phv) {  
  log(phn^25 * (1 - phn)^23 * phh^8 *  
      (1 - phh)^12 * phv^13 * (1 - phv)^9)  
}  
(loglik2(25/48, 8/20, 13/22))
```

Solution = -61.57

## Step 3

Find the difference in the log-likelihoods

```
(diff <- loglik2(25/48, 8/20, 13/22) - loglik1(46/90))
```

Diff = 0.7878

Is the difference in the maximum log-likelihoods statistically significant?



# Likelihood Ratio Test

Test statistic

$$\begin{aligned}LRT &= 2[\max\{\log(Lik(\text{larger model}))\} - \max\{\log(Lik(\text{reduced model}))\}] \\&= 2\log\left(\frac{\max\{(Lik(\text{larger model}))\}}{\max\{(Lik(\text{reduced model}))\}}\right)\end{aligned}$$

LRT follows a  $\chi^2$  distribution where the degrees of freedom equal the difference in the number of parameters between the two models

## Step 4

```
(LRT <- 2 * (loglik2(25/48, 8/20, 13/22) - loglik1(46/90)))
```

$LRT = 1.576$

The test statistic follows a  $\chi^2$  distribution with 2 degrees of freedom. Therefore, the p-value is  $P(\chi^2 > LRT)$ .

```
pchisq(LRT, 2, lower.tail = FALSE)
```

p-value = 0.4548

The p-value is very large, so we fail to reject  $H_0$ . We do not have convincing evidence that the conditional model is an improvement over the unconditional model.

Therefore, we can stick with the **unconditional** model.

## Comparing non-nested models

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We can also consider AIC and BIC. We cannot consider a LRT for a non-nested model.

## Comparing non-nested models

$$\text{AIC} = -2(\text{max log-likelihood}) + 2p$$

```
(Model1_AIC <- 2 * loglik1(46/90) + 2 * 1)
```

$$\text{AIC}_1 = -122.72$$

```
(Model2_AIC <- 2 * loglik2(25/48, 8/20, 13/22) + 2 * 3)
```

$$\text{AIC}_2 = -117.15$$

## Comparing non-nested models

```
(Model1_BIC <- 2 * loglik1(46/90) + 1 * log(30))
```

BIC\_1 = -121.32

```
(Model2_BIC <- 2 * loglik2(25/48, 8/20, 13/22) + 3 * log(30))
```

BIC\_2 = -112.94

Which model would you choose and why based upon the AIC and BIC values?

## Looking ahead

- ▶ Likelihoods help us answer the question of how likely we are to observe the data given different parameters
- ▶ In this example, we did not consider covariates, so in practice the parameters we want to estimate will look more similar to this

$$p_H = \frac{e^{\beta_0 + \beta_1 x_1 + \dots + \beta_p x_p}}{1 + e^{\beta_0 + \beta_1 x_1 + \dots + \beta_p x_p}}$$

- ▶ Finding the MLE becomes much more complex (in practice) and numerical methods may be required.

# References