Introduction to Common Distributions

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Reading

BMLR: Chapter 3:
https://bookdown.org/roback/bookdown-BeyondMLR/

I would like to thank Wenxin Guo for helping correct some typos.

Agenda

- Bernoulli and Binomial Distribution
- Maximum Likelihood Estimation

Traditional inference

You are given **data** X and there is an **unknown parameter** you wish to estimate θ

How would you estimate θ?

- Find an unbiased estimator of θ.
- Find the maximum likelihood estimate (MLE) of θ by looking at the likelihood of the data.
- In later classes, STA 402, you will consider how to estimate θ when it's random

Bernoulli distribution

The Bernoulli distribution is very common due to binary outcomes.

- Consider flipping a coin (heads or tails).
- We can represent this a binary random variable where the probability of heads is θ and the probability of tails is $1 - \theta$.

Consider $X \sim \text{Bernoulli}(\theta)\mathbb{1}(0 < \theta < 1)$

The likelihood is

$$p(x \mid \theta) = \theta^x (1 - \theta)^{(1-x)} \mathbb{1}(0 < \theta < 1).$$

- Exercise: what is the mean and the variance of X?
- What is the connection with the Bernoulli and the Binomial distribution?

Bernoulli distribution

- Suppose that $X_1, \ldots, X_n \overset{iid}{\sim} \text{Bernoulli}(\theta)$. Then for $x_1, \ldots, x_n \in \{0, 1\}$ what is the likelihood?

Notation

- $x_{1:n}$ denotes $x_1, \ldots, x_n$

Bernoulli and Binomial Connection

$X_1, \ldots, X_n \overset{iid}{\sim} \text{Bernoulli}(\theta)$.[1]

Suppose $Y = \sum_{i=1}^{n} X_i$. Then $Y \sim \textit{Binomial}(n, \theta)$.[2]

[1] This represents n coin flips with success probability θ.

[2] This represents n Bernoulli trials with success probability θ.

Likelihood

$$X_1, \ldots, X_n \overset{iid}{\sim} \text{Bernoulli}(\theta).$$

$$\begin{aligned}
p(x_{1:n}|\theta) &= \mathbb{P}(X_1 = x_1, \ldots, X_n = x_n \mid \theta) \\
&= \prod_{i=1}^{n} \mathbb{P}(X_i = x_i \mid \theta) \\
&= \prod_{i=1}^{n} p(x_i|\theta) \\
&= \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} \\
&= \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}.
\end{aligned}$$

Traditional inference

You are given **data** X and there is an **unknown parameter** you wish to estimate θ

How would you estimate θ?

- Find an unbiased estimator of θ.
- Find the maximum likelihood estimate (MLE) of θ by looking at the likelihood of the data.
- Suppose that $\hat{\theta}$ estimates θ.

Note: $\hat{\theta}$ may depend on the data $x_{1:n} = x_1, \dots x_n$.

Unbiased Estimator

Recall that $\hat{\theta}$ is an **unbiased estimator** of θ if

$$E[\hat{\theta}] = \theta. \tag{1}$$

.

Maximum Likelihood Estimation

For each sample point $x_{1:n}$, let $\hat{\theta}$ be a parameter value at which $p(x_{1:n} \mid \theta)$ attains it's maximum as a function of θ, with $x_{1:n}$ held fixed.

A **maximum likelihood esimator** (MLE) of the parameter θ based on a sample $x_{1:n}$ is $\hat{\theta}$.

Finding the MLE

The solution to the MLE are the possible candidates (θ) that solve

$$\frac{\partial \log p(x_{1:n} \mid \theta)}{\partial \theta} = 0. \tag{2}$$

The solution to equation 2 are only **possible candidates** for the MLE since the first derivative being 0 is a **necessary condition** for a maximum but not a sufficient one.

Our job is to find a global maximum.

Thus, we need to ensure that we haven't found a local one. (Show that the second derivative is < 0.)

Exercise

$$X_1, \ldots, X_n \overset{iid}{\sim} \text{Bernoulli}(\theta). \tag{3}$$

Exercise: Derive (step by step) that the MLE for θ is $\frac{1}{n}\sum_i x_i = \bar{x}$.

Exercise Solution

$$L(\theta) = \theta^S (1-\theta)^{n-S}, \qquad S = \sum_{i=1}^{n} x_i,\ \ \theta \in [0,1].$$

The log-likelihood is

$$\ell(\theta) = S \log\theta + (n-S)\log(1-\theta).$$

For $\theta \in (0,1)$, differentiate and set to zero:

$$\ell'(\theta) = \frac{S}{\theta} - \frac{n-S}{1-\theta} = 0 \implies S(1-\theta) = (n-S)\theta$$

$$\implies \hat{\theta} = \frac{S}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

Exercise Solution (Continued)

- The log-likelihood is log-concave, which means the likelihood is concave (both functions are cap-shaped).
- This ensures that that solution is a global one (and we do not have to take the second derivative).

Approval ratings of Obama

What is the proportion of people that approve of President Obama in PA?

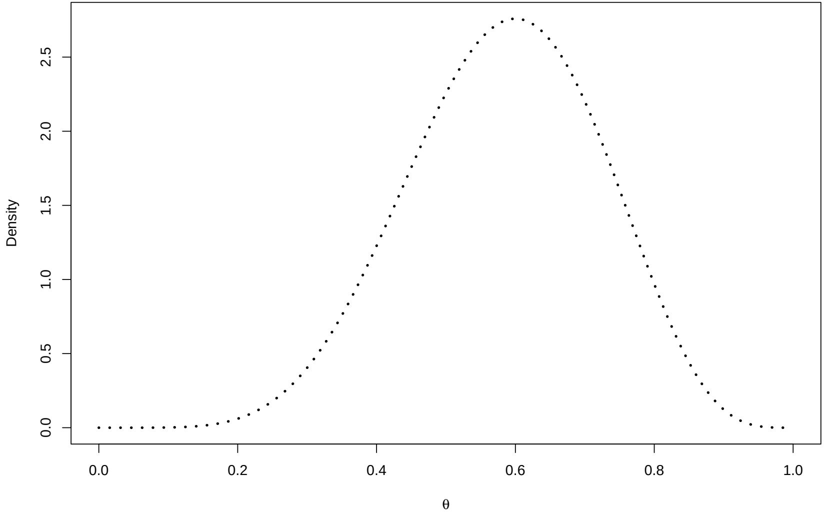
- We take a random sample of 10 people in PA and find that 6 approve of President Obama.

Obama Example

```
n <- 10
th <- seq(0, 1, length = 500)
x <- 6
like <- dbeta(th, x + 1 , n - x + 1)
```

Likelihood

```
plot(th, like, type = "l", ylab = "Density",
     lty = 3, lwd = 3, xlab = expression(theta))
```



Supplemental Material

- Continuous Random Variables
- Discrete Random Variables

Continuous Random Variables

A continuous random variable (r.v.) can take on an uncountably infinite number of values.

Given a probability density function (pdf), f(y), allows us to compute

$$P(a \leq Y \leq b) = \int_a^b f(y)\,dy.$$

Properties of continuous random r.v.'s:

- $\int f(y)\,dy = 1$.
- For any value y,

$$P(Y = y) = \int_y^y f(y)\,dy = 0 \implies$$

$$P(y < Y) = P(y \leq Y).$$

Discrete Random Variables

A discrete random variable has a countable number of possible values, where the associated probabilities are calculated for each possible value using a probabilitiy mass function (pmf).

A pmf is a function that calculates $P(Y = y)$, given each variable's parameters.

Common Discrete distributions

- Bernoulli/Binomial (already covered)
- Poisson
- Geometric
- Negative Binomial
- Hypergeometric

Common Continuous distributions

- Exponential
- Beta
- Gamma
- Normal (Gaussian)

Beta distribution

The Beta distribution is frequently used in situations where the data are constrained to the interval $[0, 1]$. It often used to model proportions, rates, and probabilities.

Examples:

- In manufacturing, the proportion of defective items in a batch is a common quantity that can be modeled using the Beta distribution.
- In finance, the proportion of a portfolio invested in risky assets (such as stocks) is typically between 0 and 1.
- The Beta distribution is often used as a prior for the probability of success in Bernoulli or Binomial experiments in Bayesian statistics (STA 402).

Beta distribution

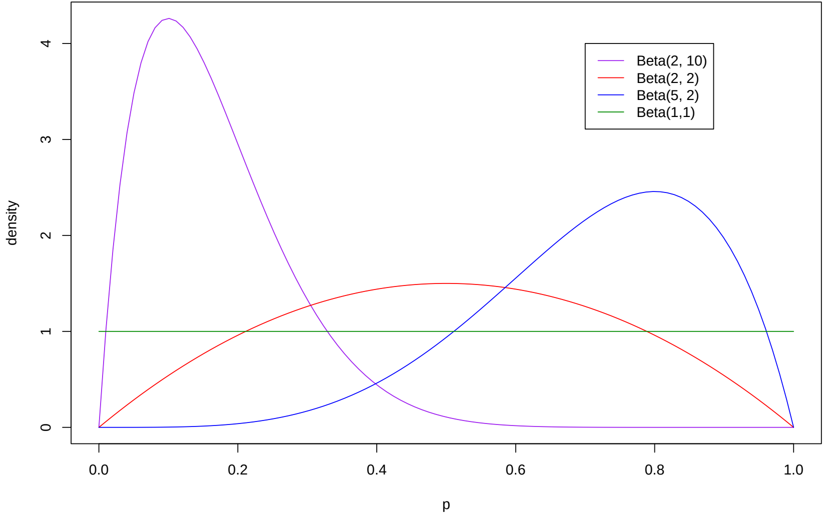
Given $a, b > 0$, we write $\theta \sim \text{Beta}(a, b)$ to mean that θ has pdf

$$p(y) = \text{Beta}(y|a, b) = \frac{1}{B(a, b)} y^{a-1}(1-y)^{b-1}\mathbb{1}(0 < y < 1),$$

i.e., $p(y) \propto y^{a-1}(1-y)^{b-1}$ on the interval from 0 to 1.

- Here,
$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$
.
- Parameters a, b control the shape of the distribution.
- This distribution models random behavior of percentages/proportions.

Beta distribution



Beta distribution example

Suppose that a college models probabilities of student accepting admission via the Beta(a, b) distribution, where a,b > 0 are fixed and known.

What is the probability that a randomly selected student has prob of accepting an offer larger than 80 percent, where a$=4/3$ and b$=2$.

```
pbeta(0.8, shape1 = 4/3, shape2 = 2, lower.tail = FALSE)
```

```
## [1] 0.05930466
```

Exponential distribution

Data that follows an Exponential distribution typically represents the time between events in a Poisson process, where events happen at a constant average rate and are independent of each other.

The Exponential distribution is widely used in various fields to model waiting times, lifetimes, and inter-arrival times.

Examples: time until a device fails, time between arrivals in a line, time between arrivals/departures, among others.

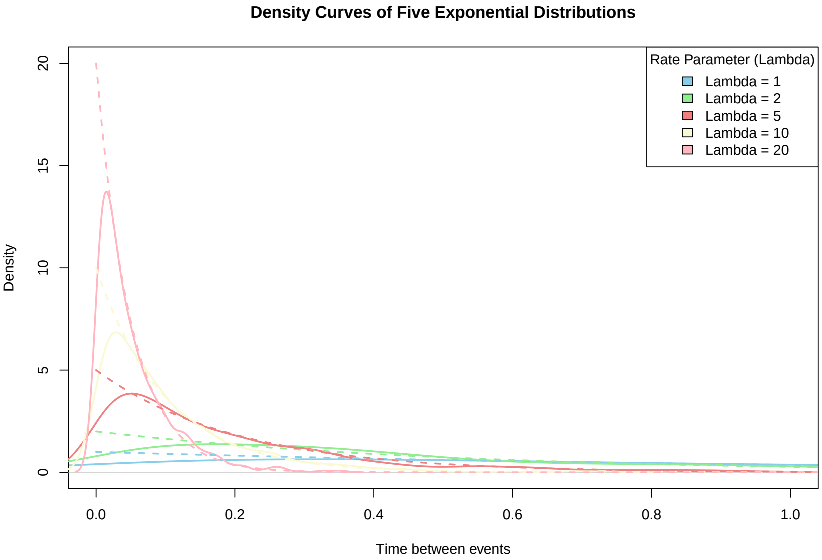
Exponential distribution

Assume $\lambda > 0$, which is the called rate parameter (the rate at which some event occurs).

The density function is given by

$$p(y) = \mathrm{Exp}(y \mid \lambda) = \lambda \exp^{-\lambda y} I(y > 0).$$

Exponential distribution



Gamma distribution

The Gamma distribution is a continuous probability distribution that is often used to model waiting times, lifetimes, and other phenomena where the events are continuous and the process involves a sum of exponentially distributed random variables.

The Gamma distribution is commonly used in reliability theory, queueing theory, Bayesian statistics, and life data analysis.

Rainfall example

The Gamma distribution is used to model the accumulated rainfall over a given period, particularly in areas where precipitation events occur at a constant rate.

The total accumulated rainfall over a month could be modeled as a Gamma distribution, where the shape parameter k reflects the number of significant rainfall events, and the rate λ represents the intensity of the rainfall.

For example, the accumulated rainfall in a region that experiences 10 or more rainstorms per year, with an average rainfall rate of 0.5 inches per storm, could be modeled as a Gamma(10, 0.5) distribution.

Queueing Systems (Time Until k Customers Arrive)

The Gamma distribution is used to model the waiting time for the occurrence of k events, such as the arrival of k customers at a service station.

In a service system where customers arrive at an average rate of λ per minute, the time it takes for the system to serve k customers is modeled as a Gamma distribution with shape parameter k and rate parameter λ.

For example, the time to serve 4 customers in a queue, where customers arrive at a rate of 2 per minute, could be modeled with a Gamma(4, 2) distribution.

Gamma distribution (shape, rate)

Assuming shape parameter k and rate parameter λ, the density function is

$$f(y \mid k, \lambda) = \text{Gamma}(y \mid k = \text{shape}, \lambda = \text{rate}) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{\Gamma(k)}, \quad y \geq 0$$

This parameterization tends to be more common in Bayesian statistics and other applied fields. However, there exists another parameterization for other contexts.

Gamma distribution (shape, scale)

Assuming shape parameter k and scale parameter $\theta = 1/\lambda$, the density function is

$$f(y \mid k, \theta) = \text{Gamma}(y \mid k = \text{shape}, \theta = \text{scale}) = \frac{y^{k-1} e^{-y/\theta}}{\Gamma(k)\theta^k}, \quad y \geq 0$$

Summary of the Gamma distribution:
https://en.wikipedia.org/wiki/Gamma_distribution