ORIE 4580/5580/5581 Assignment 3

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Github link: Github link

Instructions

- Due Thursday September 25, at 11.59pm on Gradescope.
- Assignment .ipynb files available for download on Canvas.
- Do all your work in provided notebook (text answers typeset in markdown; show all required code and generate plots inline), and then generate and submit a pdf.
- Ideally do assignments in groups of 2, and submit a single pdf with both names
- Please show your work and clearly mark your answers.
- You can use any code fragments given in class, found online (for example, on StackOverflow), or generated via Gemini/Claude/ChatGPT (you are encouraged to use these for first drafts) **with proper referencing**.
- You can also discuss with others (again, please reference them if you do so); but you must write your final answers on your own as a team.

Suggested reading

Chapters 7 (you can skim through this), and chapters 8 and 9 of Introduction to Probability by Grinstead and Snell.

Chapter 3 and chapter 4 (up to section 4.5) of Simulation by Ross.

In [9]:
```
#importing necessary packages
import numpy as np
import matplotlib.pyplot as plt
import scipy.stats
%matplotlib inline
# Make sure we have the packages we need
import numpy as np
from scipy import stats
import math

# Configuring matplotlib
plt.rcParams["figure.figsize"] = (10,10)
plt.rcParams['axes.labelsize'] = 14
```

```
plt.rcParams['xtick.labelsize'] = 12
plt.rcParams['ytick.labelsize'] = 12
plt.style.use('dark_background')
plt.rcParams["image.cmap"] = 'Set3'

# Choosing a colormap for the plot colors
cmap=plt.get_cmap('Set3')
```

Question 1: Combining LCGs (20 points)

In order to avoid biases, simulations should not use anywhere near the full period of an LCG (otherwise, the random sequence repeats \dots). For example, a typical traffic simulator may have $10,000$ vehicles, each experiencing thousands of random disturbances, thus needing around 10^7 random samples per replication -- for this, an LCG using $m = 2^{31} - 1 \approx 2 \times 10^9$ is insufficient, as after 100 replications the sequences get correlated.

One method to combine multiple LCGs to obtain a generator with a longer period is to add a smaller period LCG to it. For example, suppose we have two generators $X_{n+1} = (a_1 X_n) \bmod m_1$ and $Y_{n+1} = (a_2 Y_n) \bmod m_2$, with $m_1 > m_2$. We can derive a combined generator by setting $Z_n = (X_n + Y_n) \bmod m_1$. If properly designed, the resulting period can be on the order of $m_1 m_2$. We will now study a small example to see how this works.

(a) Consider two LCGs, $x_{n+1} = (5x_n) \mod 16$ and $y_{n+1} = (2y_n) \mod 7$. Starting both with seed $x_0 = y_0 = 1$, plot the sequences x_n, y_n using the clock visualization introduced in class (separate plot for each sequence; you can use and modify the code in Demo-PRNGs.ipynb on Canvas).

In [10]:
```
# Functions to visualize LCG sequence on clock (see demo notebook)
def plot_clock_face(m, fig, annotate=False):
  """
  Plot points on a unit circle representing the LCG sequence on a cloc

  Parameters:
  m (int): The modulus value for the LCG sequence.
  fig (matplotlib.figure.Figure): The figure object to draw on.
  annotate (bool): Whether to annotate points with their index.

  Returns:
  None
```

```
    """
    # Plot m points on the unit circle
    for i in range(m):
        theta = 2.0 * np.pi * i / m
        plt.plot(np.sin(theta), np.cos(theta), 'rs', markersize = 10)
        if annotate:
            plt.annotate(str(i), (np.pi/2 - theta, 1.05), xycoords='pola

def plot_clock_path(m, x, fig, color='y'):
    """
    Plot the path of an LCG sequence on a clock face.

    Parameters:
    m (int): The modulus value for the LCG sequence.
    x (numpy.ndarray): The LCG sequence.
    fig (matplotlib.figure.Figure): The figure object to draw on.
    color (str): The color for the path.

    Returns:
    None
    """
    # Plot the seed node
    theta_0 = 2.0 * np.pi * (x[0] * (m + 1) - 1) / m
    plt.plot(np.sin(theta_0), np.cos(theta_0), 'gs', markersize = 10)

    # Plot the path of the LCG sequence
    for i in range(len(x) - 1):
        theta_start = 2.0 * np.pi * (x[i] * (m + 1) - 1) / m
        theta_end = 2.0 * np.pi * (x[i + 1] * (m + 1) - 1) / m
        x_start = np.sin(theta_start)
        y_start = np.cos(theta_start)
        del_x = np.sin(theta_end) - np.sin(theta_start)
        del_y = np.cos(theta_end) - np.cos(theta_start)
        if abs(del_x) > 0 or abs(del_y) > 0:
            plt.arrow(x_start, y_start, del_x, del_y,
                      length_includes_head=True, head_width=0.05, head_l
```

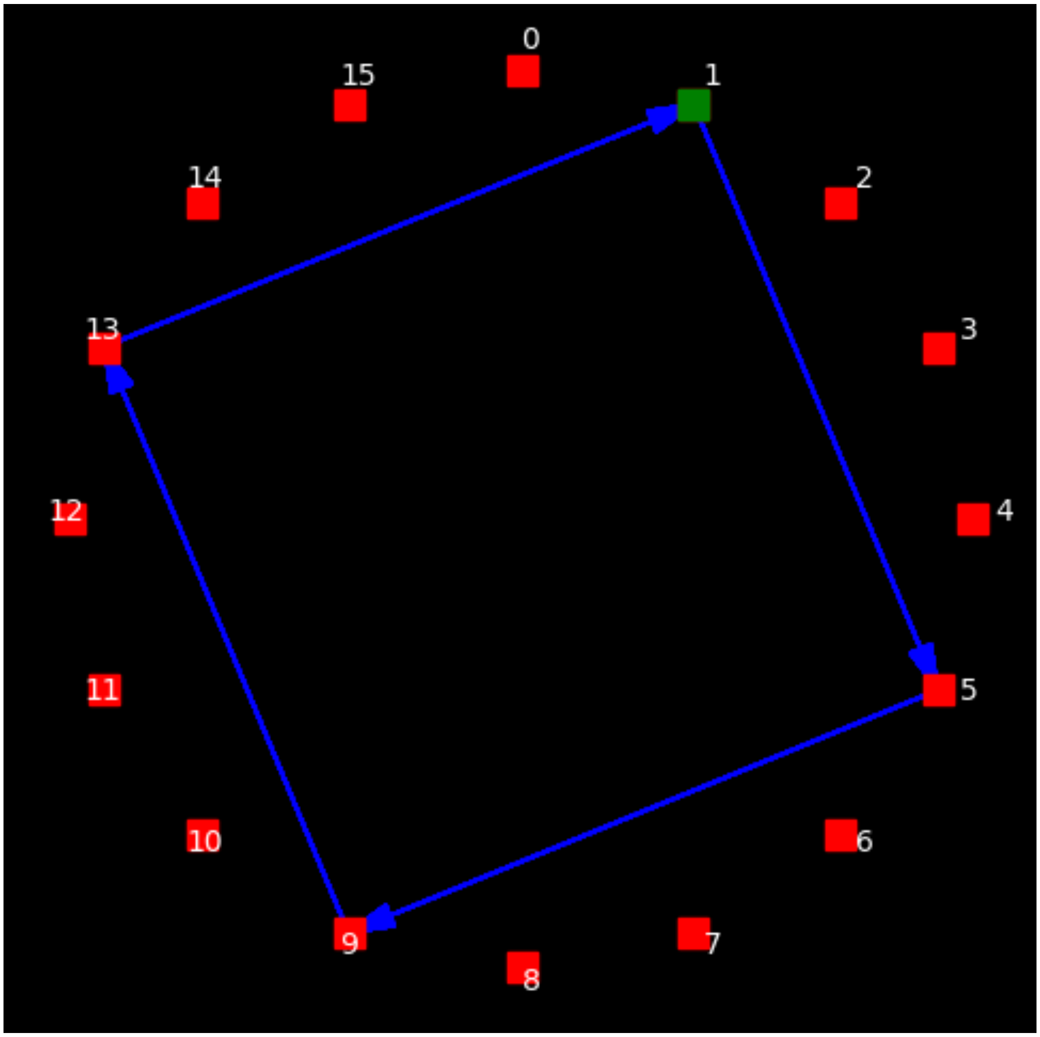
In [11]:

```
# Function to generate pseudorandom sequence using LCG
# Set default parameters to glibc specifications (see demo notebook)
def LCG(n, m=2**31-1, a=1103515245, c=12345, seed=1):
    """
    Generate a pseudorandom sequence using a Linear Congruential Gener

    Parameters:
    n (int): The number of pseudorandom numbers to generate.
    m (int): The modulus value (default is 2^31-1, following glibc spe
    a (int): The multiplier value (default is 1103515245, following gl
    c (int): The increment value (default is 12345, following glibc sp
    seed (int): The initial seed value (default is 1).

    Returns:
    numpy.ndarray: An array of pseudorandom numbers in the range [0, 1
```

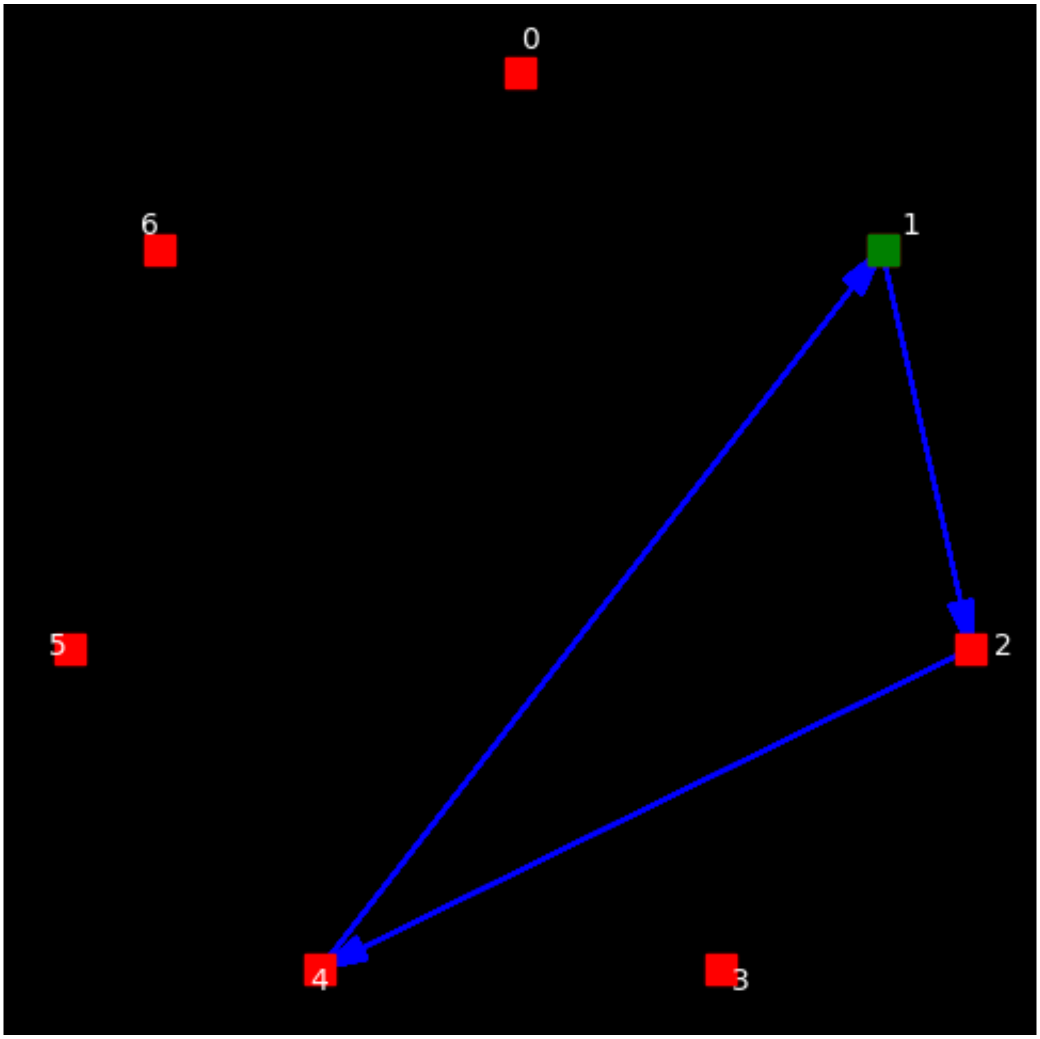
```
    """
    # Initialize an array to store the generated pseudorandom numbers
    output = np.zeros(n)

    x = seed
    for i in range(n):
        # Calculate the pseudorandom number and normalize it to [0, 1)
        output[i] = (x + 1.0) / (m + 1.0)

        # Update the LCG state using the specified parameters
        x = (a * x + c) % m

    return output
```

In [12]:

```
#LCG 1
m1 = 16
m2 = 7
a1 = 5
c1 = 0
seed = 1

fig = plt.figure(figsize=(6,6))

plot_clock_face(m1,fig,annotate = True)

x = LCG(n=m1*m2,m=m1,a=a1,c=c1,seed=seed)
plot_clock_path(m1,x,fig,color='b')

plt.axis('off')
plt.show()
```



In [13]:
```
#LCG 2
m2 = 7
a2 = 2
c2 = 0
seed = 1

fig = plt.figure(figsize=(6,6))

plot_clock_face(m2,fig,annotate = True)

y = LCG(n=m2*m1,m=m2,a=a2,c=c2,seed=seed)
plot_clock_path(m2,y,fig,color='b')

plt.axis('off')
plt.show()
```

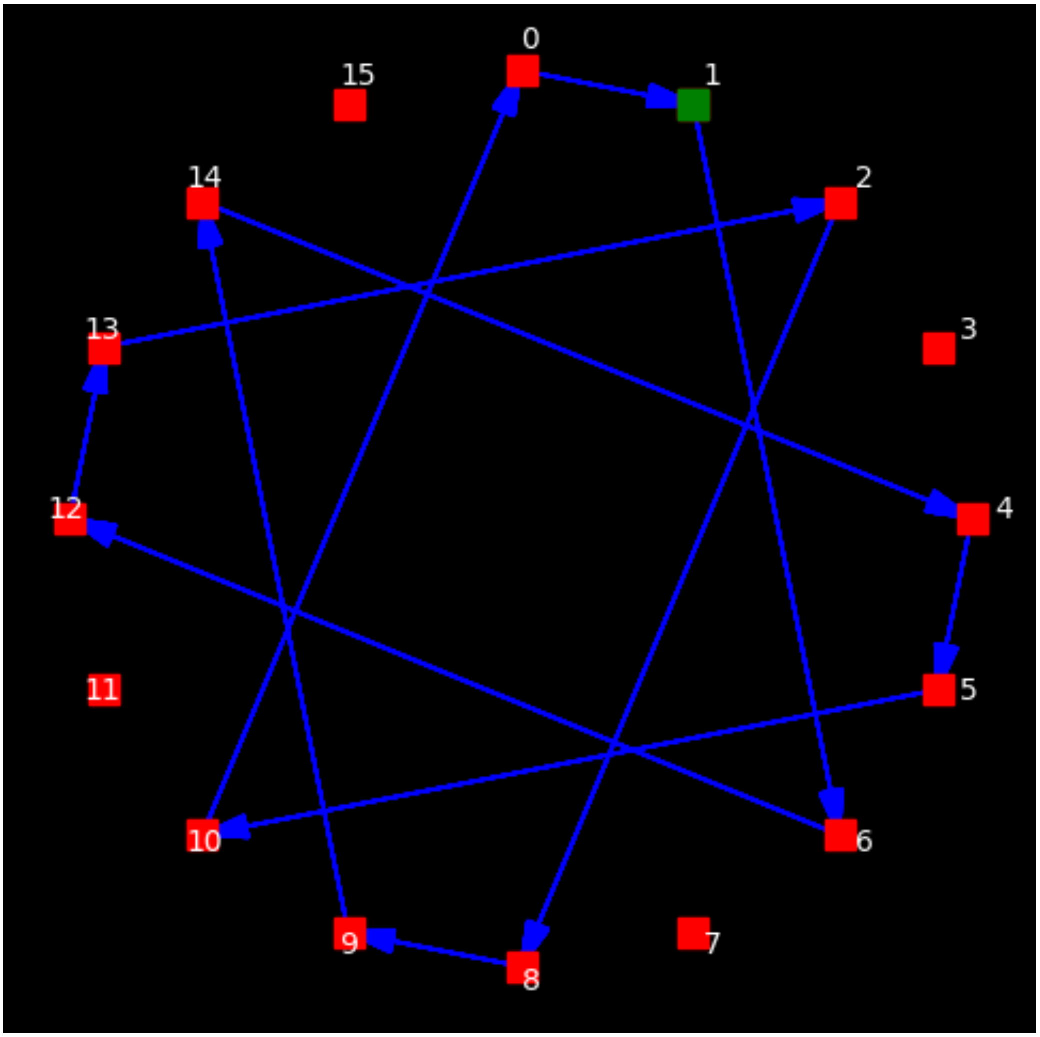


(b) Next, define a combined LCG as $z_n = (x_n + y_n) \mod 16$. Starting both the base LCGs with seed $x_0 = y_0 = 1$, plot the sequence z_n using the clock visualization given in class.

In [14]:

```
#LCG Z
m3 = 16
c2 = 0
seed = 1

fig = plt.figure(figsize=(6,6))

plot_clock_face(m3,fig,annotate = True)

# Revert the U back to x and y
xx = (x * (m1 + 1.0) - 1).astype(int)
yy = (y * (m2 + 1.0) - 1).astype(int)
z = (xx+yy)%16

plot_clock_path(m3,z,fig,color='b')
```

```
plt.axis('off')
plt.show()
```



(c) What are the periods of the pseudo-random sequences x_n, y_n and z_n?

Ans.

Assume the starting seed is 1,

In [15]:
```
print(f"The pseudo-random sequences for Xn is {len(set(x))}")
print(f"The pseudo-random sequences for yn is {len(set(y))}")
print(f"The pseudo-random sequences for zn is {len(set(z))}")
```

```
The pseudo-random sequences for Xn is 4
The pseudo-random sequences for yn is 3
The pseudo-random sequences for zn is 12
```

Question 2: inverting cdfs (25 pts)

In class, we defined $F^{-1}(y)$ for a continuous increasing cdf $F(x)$ as the unique x such that $F(x) = y$ (for $y \in [0, 1]$). More generally, for any cdf F we can use the inversion method based on its generalized inverse or *pseudoinverse*:

$$F^{-1}(y) = \inf\{x|F(x) \geq y\}$$

(where $\inf$ denotes the \href{https://en.wikipedia.org/wiki/Infimum_and_supremum}{infimum}; if you have not seen this before, treat it as minimum).

(a) Find the pseudoinverse $F^{-1}(y)$ for the following mixed (discrete/continuous) cdf

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \leq x < \frac{1}{2}, \\ \frac{1}{2} & \text{for } \frac{1}{2} \leq x < 1, \\ 1 & \text{for } x \geq 1 \end{cases}$$

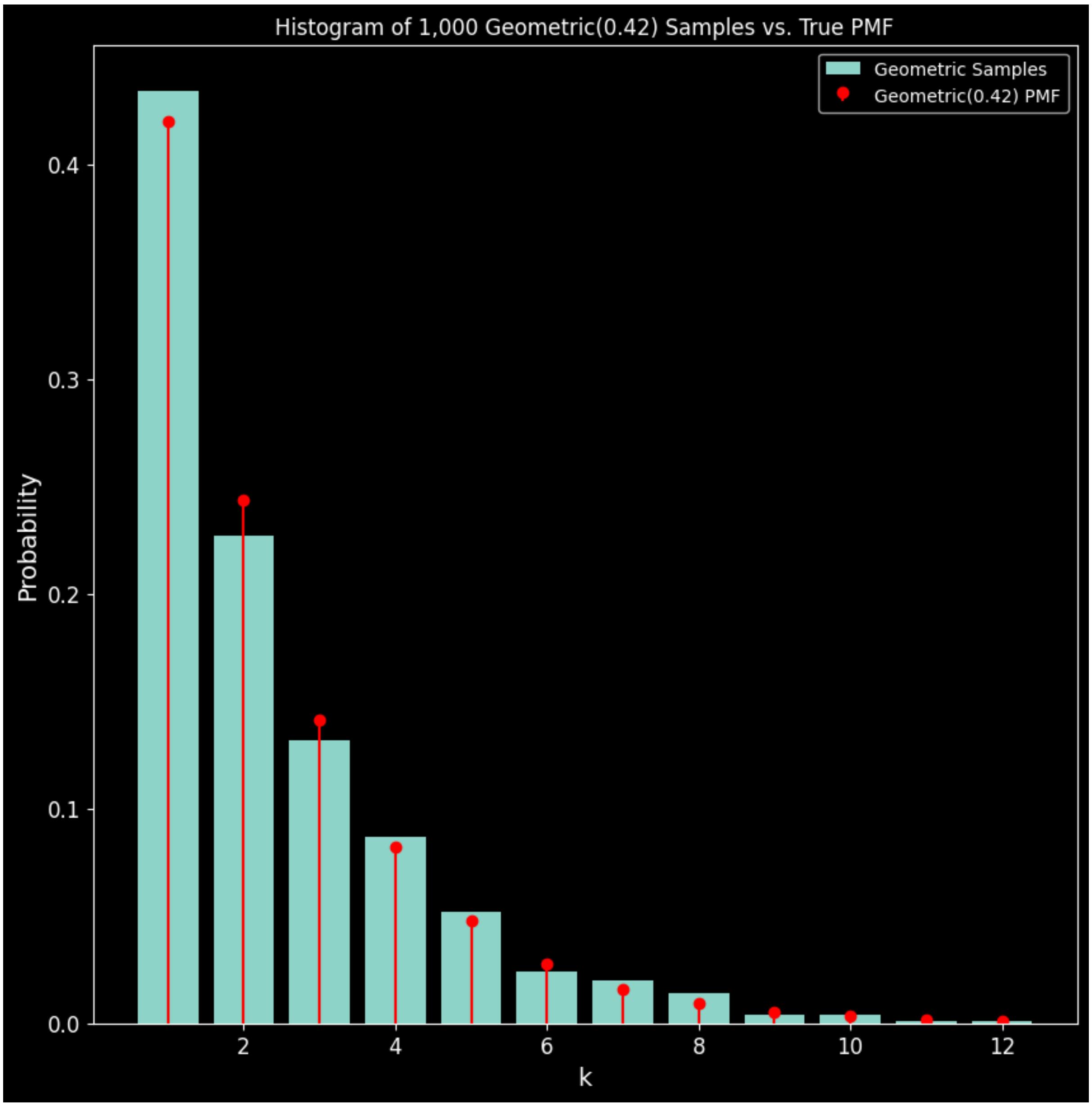
Ans.

$$F^-1(u) = \begin{cases} u & \text{for } 0 \leq u \leq \frac{1}{2}, \\ 1 & \text{for } \frac{1}{2} < u \leq 1 \end{cases}$$

(b) Use the above definition to get an inversion algorithm for the $Geometric(p)$ distribution (with pmf $p(k) = p(1-p)^{k-1} \, \forall \, k \in \{1, 2, 3, \ldots\}$). Implement this, and generate and plot the histogram of 1000 samples from a $Geometric(0.42)$ distribution. (For this, it may be useful for you to first understand how the scipy.stats library works, and in particular, how it provides methods to compute various statistics for many different random variables, including the geometric r.v.)

In [16]:
```
import matplotlib.pyplot as plt
import numpy as np
import scipy.stats as sc
```

```
n = 1000
u = np.random.uniform(0,1,n)

geo = scipy.stats.geom.isf(1-u, 0.42)
plt.hist(geo, bins=range(1, int(max(geo))+2), density=True, label='Geo
k = np.arange(1, int(max(geo)) + 1)
geo_pmf = sc.geom.pmf(k, 0.42)
plt.stem(k, geo_pmf, 'r-', label='Geometric(0.42) PMF', basefmt=" ")
plt.title('Histogram of 1,000 Geometric(0.42) Samples vs. True PMF')
plt.xlabel('k')
plt.ylabel('Probability')
plt.legend()
plt.show()
```



(c) The p.d.f. of the random variable X is given by

$$f(x) = \begin{cases} e^{x-2} & \text{for } 0 \le x \le 2, \\ e^{-x} & \text{for } x > 2, \\ 0 & \text{otherwise}, \end{cases}$$

Describe and implement an inversion algorithm to generate samples of X. Generate 1,000 samples and plot a histogram. Compare the histogram and the p.d.f.

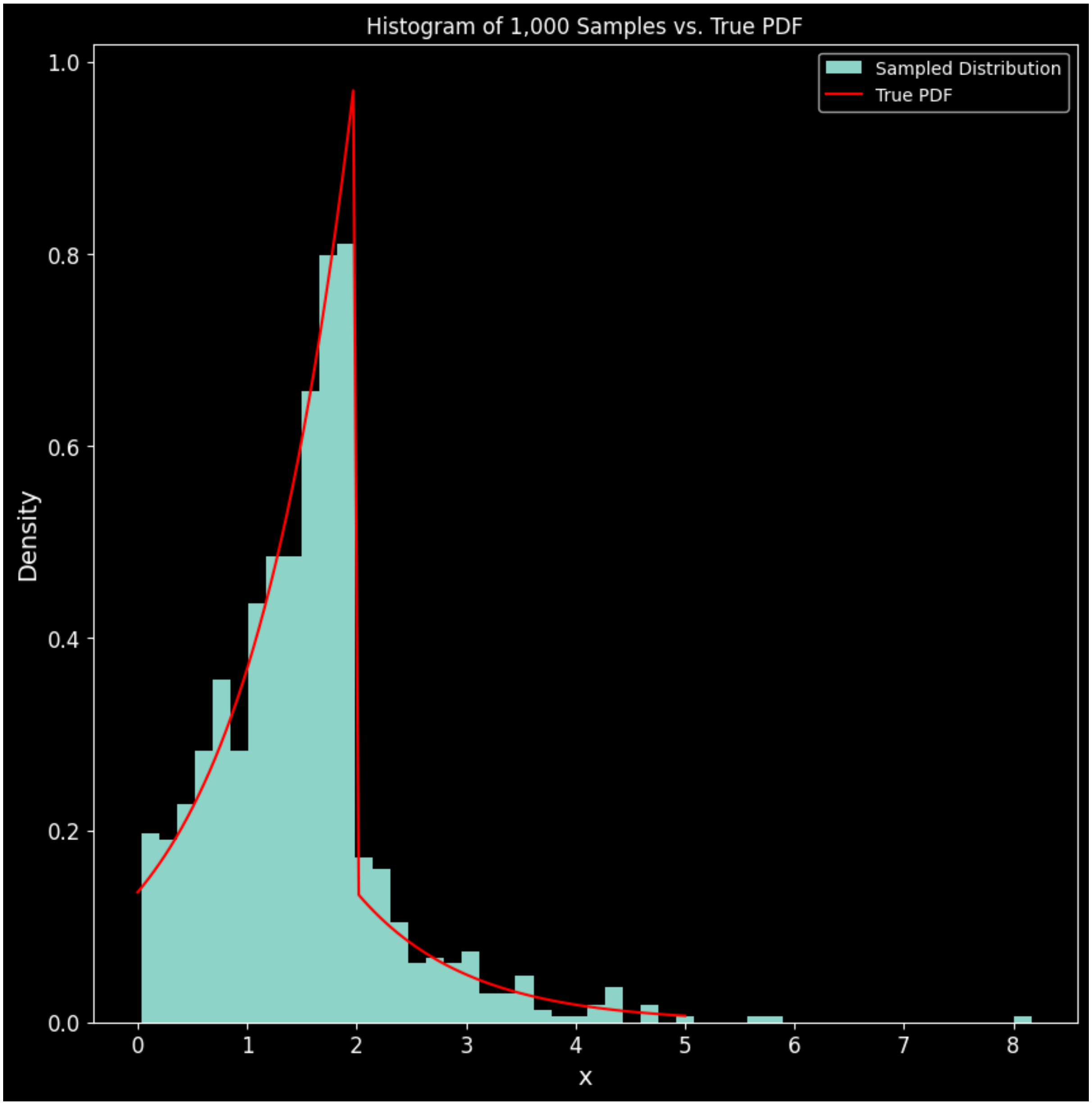
Ans.

$$F(x) = \begin{cases} 0 & x < 0 \\ e^{x-2} - e^{-2} & 0 \le x \le 2 \\ -e^{-x} + 1 & x > 2 \end{cases}$$

$$F^{-1}(u) = \begin{cases} ln(ue^2 + 1) & 0 \le u \le 1 - e^{-2} \\ -ln(1-u) & u > 1 - e^{-2} \end{cases}$$

In [17]:

```
import matplotlib.pyplot as plt
import numpy as np
import scipy.stats as sc
import math

np.random.seed(42)
n = 1000
u = np.random.uniform(0,1,n)

plt.hist(np.where(u<1-math.e**(-2),np.log(u*np.e**2+1), -np.log(1-u)),
x_range = np.linspace(0, 5, 100)
pdf = np.where(x_range <= 2, np.exp(x_range - 2), np.exp(-x_range))
plt.plot(x_range, pdf, 'r-', label='True PDF')
plt.title('Histogram of 1,000 Samples vs. True PDF')
plt.xlabel('x')
plt.ylabel('Density')
plt.legend()
plt.show()
```



Question 3: Acceptance-Rejection (25 pts)

Let the random variable X have density

$$f(x) = \begin{cases} (5x^4 + 4x^3 + 3x^2 + 1)/4 & \text{for } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Give an acceptance-rejection algorithm to generate samples of X.

Ans.

1. Generate (U,V) with $U \sim Unif(0,1)$ and $V \sim Unif(0,M)$ for some M that is the an upper bound of the maximum of f(x)
2. Accept X = U if $V \le f(U)$. Reject and repeat step 1 otherwise

In [18]:
```
rand_u = np.random.uniform(0,1)
M = 13.0/4 # the maximum value from f(x) in interval [0, 1]
rand_v = np.random.uniform(0,M)
prob_accept = (5*(rand_u**4)+4*(rand_u**3)+3*(rand_u**2)+1)/4
if (rand_v <= prob_accept):
  acc = True
else:
  acc = False
print(acc)
```

```
False
```

(b) On average, how many samples from the uniform distribution over $[0, 1]$ would your acceptance-rejection algorithm need in order to generate one sample of X?

Ans.

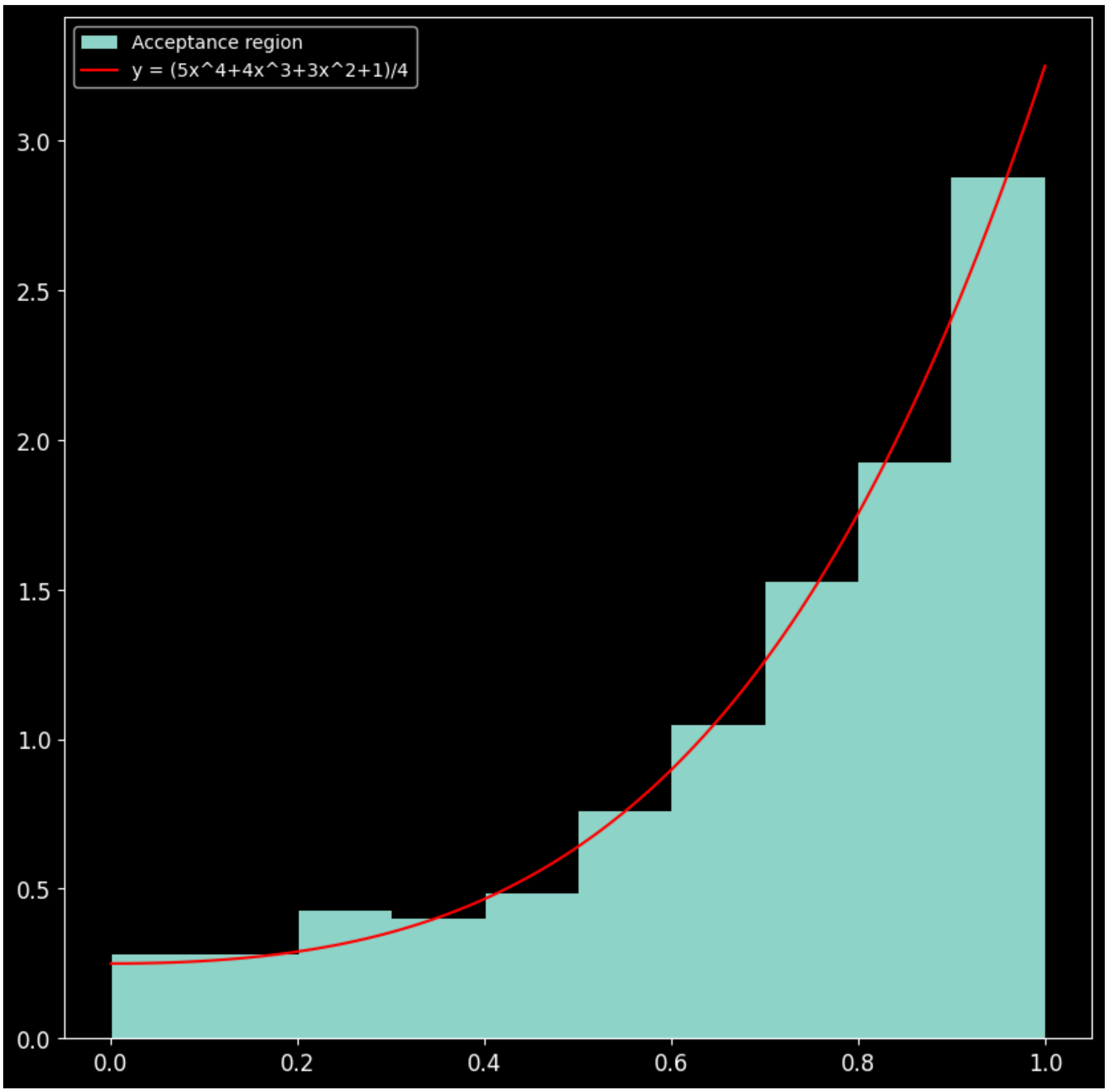
By calculus, we know the maximum of f(x) is 13/4 for $0 \le x \le 1$, thus we set M = 13/4. The distribution of sample needed to get one sample is geo(1/M), thus the average is $M = 3.25$ samples.

(c) Use your algorithm in (a) to generate 2,500 samples of X. Note that this will require more than 2500 uniform random variables.

Plot a histogram of your sample and compare it against the true pdf.

In [19]:
```
import numpy as np
import matplotlib.pyplot as plt
import math

n = 2500
x = np.zeros(n)
i = 0

def f(x):
```

```
    return (5*x**4+4*x**3+3*x**2+1)/4

while i != n:
    u = np.random.uniform(0,1)
    v = np.random.uniform(0,3.25)
    if v <= f(u):
        x[i] = u
        i +=1

x_pt = np.linspace(0,1,100)
y_pt = f(x_pt)

plt.hist(x, density=True, label="Acceptance region")
plt.plot(x_pt, y_pt, "r-", label='y = (5x^4+4x^3+3x^2+1)/4')
plt.legend()
plt.show()
```



Question 4: Generalized Acceptance-Rejection (30 pts)

We want to generate a $\mathcal{N}(0,1)$ rv X, with pdf $f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$, using generalized acceptance-rejection.

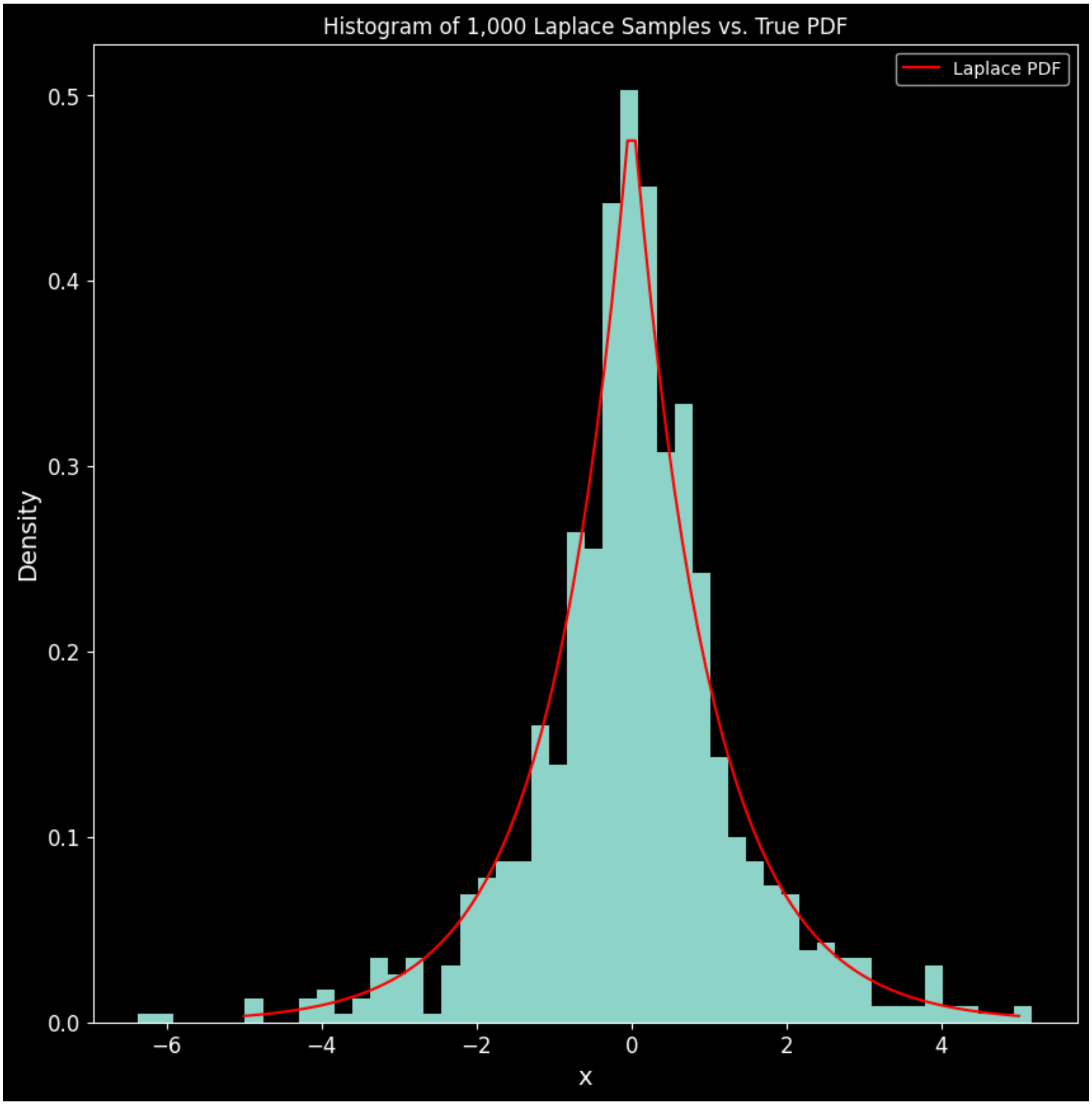
(a) First, suppose we choose the proposal distribution to be a $\emph{Laplace}$ (i.e., two-sided Exponential) distribution, which has pdf $g(x) = e^{-|x|}/2$. Describe (and implement) an inversion algorithm to get samples from this distribution.

We can sample U from unif(0,1) and plug into $G^-1(u)$ which has the same distribution as g, where G is the CDF

$$G(x) = \begin{cases} e^x/2 & x < 0 \\ 1 - e^{-x}/2 & x \geq 0 \end{cases}$$

$$G^-1(u) = \begin{cases} ln(2u) & u \in [0, 1/2) \\ -ln(2-2u) & u \in [1/2, 1] \end{cases}$$

In [20]:
```
import numpy as np
import math
import matplotlib.pyplot as plt

u = np.random.uniform(0,1, 1000)
G = np.where(u<0.5, np.log(2*u), -np.log(2-2*u))

plt.hist(G, density=True, bins=50)
x = np.linspace(-5, 5, 100)
laplace_pdf = 0.5 * np.exp(-np.abs(x))
plt.plot(x, laplace_pdf, 'r-', label='Laplace PDF')
plt.title('Histogram of 1,000 Laplace Samples vs. True PDF')
plt.xlabel('x')
plt.ylabel('Density')
plt.legend()
plt.show()
```



(b) Determine the smallest k such that $kg(x) \geq f(x) \, \forall \, x \in \mathbb{R}$. Using this, propose (and implement) an acceptance-rejection algorithm for sampling $X \sim \mathcal{N}(0,1)$, and compute the expected number of samples needed for generating each sample.

Ans.

Define $h(x) = f(x)/g(x) = e^{-x^2/2+x}(2/\sqrt{2\pi})$.

Maximum of $h(x)$ occurs at $x = 1$ with $h(1) = \sqrt{2e/\pi} = k$. The acceptance-rejection algorithm is:

1. Generate $U \sim G$
2. Accept U as a sample from f(U) with probability $\frac{f(U)}{k*g(U)}$. Else, reject U and return to step 1

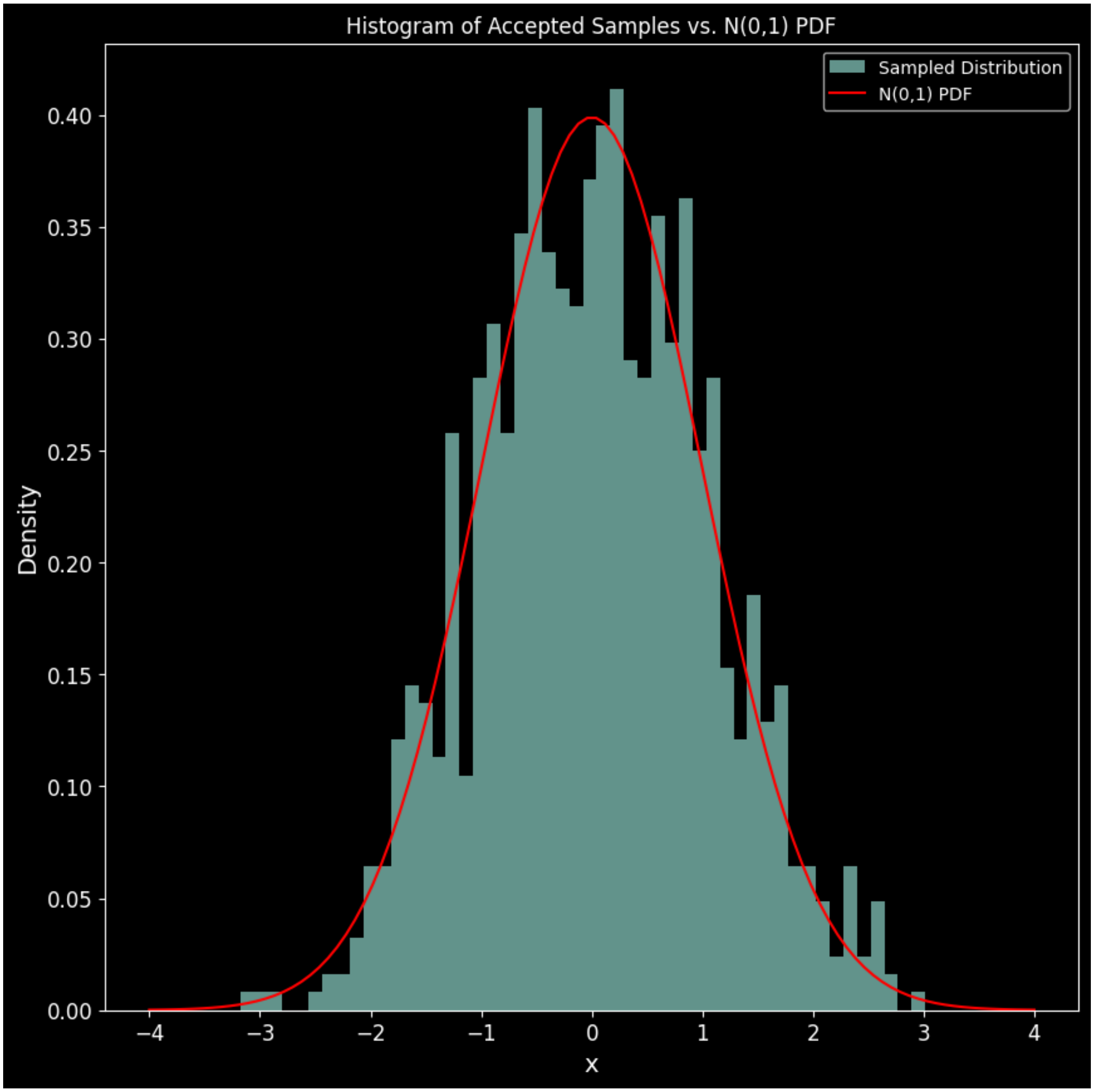
In [21]:
```
k = np.sqrt(2*np.e/np.pi)
print(f"The expected sample needed is {k:.3f}")
```

```
The expected sample needed is 1.315
```

(c) Generate 1000 samples from your method in part (b), and plot the histogram of the samples. Also report the average and 95% CI for the number of $U[0, 1]$ samples needed to generate the 1000 samples.

In [32]:
```
S = 500
n = 1000

Total_sample_needed = []
for s in range(S):
  reject = []
  xa = []
  i = 0
  while i<1001:
    k = np.sqrt(2*np.e/np.pi)
    u = np.random.uniform(0,1)
    U = np.where(u<0.5, np.log(2*u), -np.log(2-2*u))
    g = np.exp(-abs(U))/2
    f = np.exp(-U**2/2)/np.sqrt(2*np.pi)
    p = f/(k*g)
    u = np.random.uniform(0,1)
    if u <= p:
      xa.append(U)
      i+=1
    elif u > p:
      reject.append(0)
  Total_sample_needed.append(2*(len(xa)+len(reject)))

# Plot histogram of accepted samples
plt.hist(xa, bins=50, density=True, alpha=0.7, label='Sampled Distribu
x = np.linspace(-4, 4, 100)
plt.plot(x, np.exp(-x**2 / 2) / np.sqrt(2 * np.pi), 'r-', label='N(0,1
plt.title('Histogram of Accepted Samples vs. N(0,1) PDF')
plt.xlabel('x')
plt.ylabel('Density')
plt.legend()
```

Out[32]: <matplotlib.legend.Legend at 0x7cb74a385640>



In [35]:

```
# Construct the 95% CI
mean_sample = np.mean(Total_sample_needed)
se = np.std(Total_sample_needed, ddof=1)
print(f"The mean is {mean_sample}")
print(f"The 95% CI is {mean_sample-1.96*se:.3f},{mean_sample+1.96*se:.
print(f"Theoretical expected total proposals of uniform generation: {2
```

```
The mean is 2635.132
The 95% CI is 2559.114,2711.150
Theoretical expected total proposals of uniform generation: 2631.0
```

Here I used 500 sample to simulate the generation of normal random variable for reaching 1000 accepted.

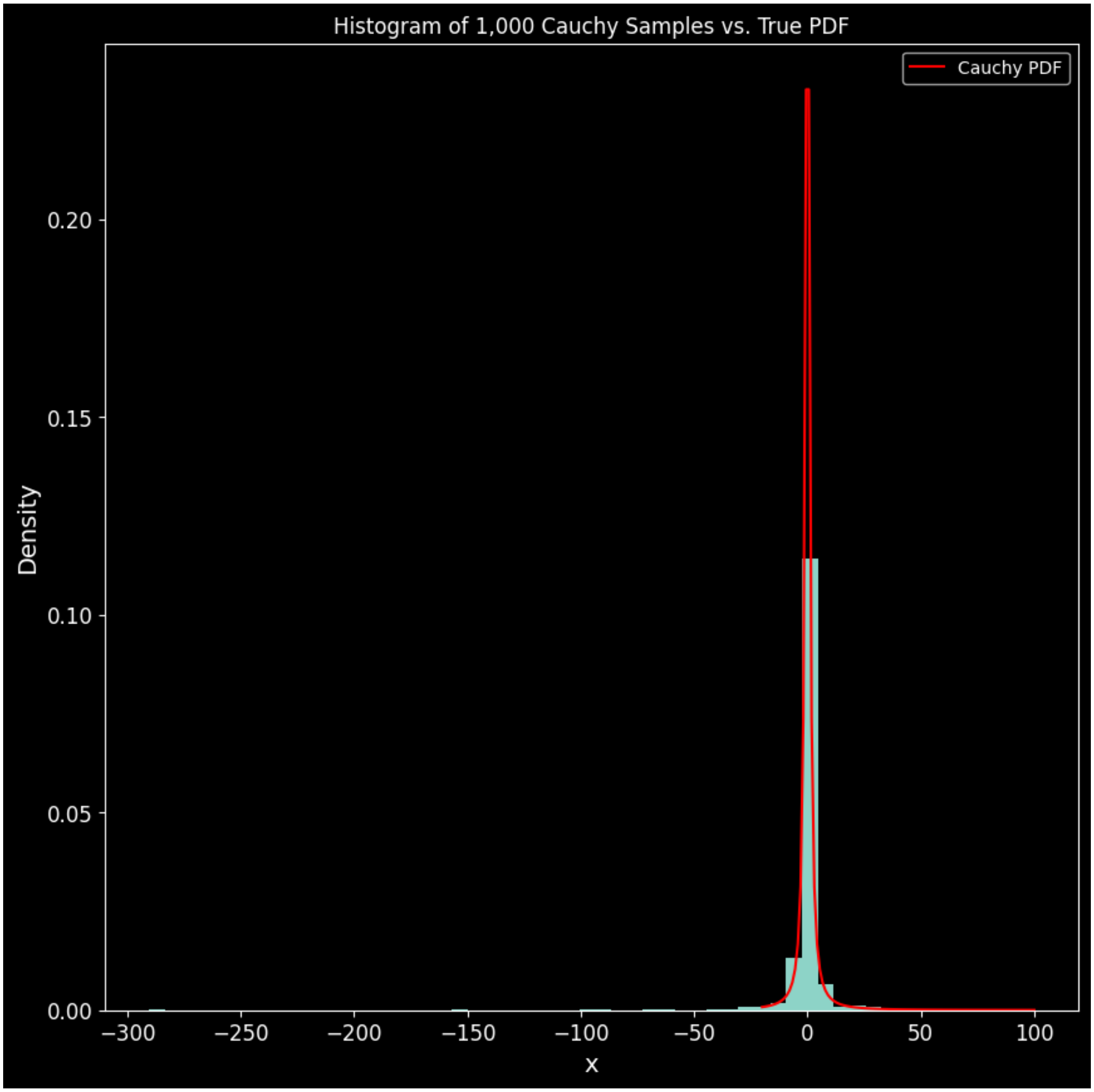
(d) Now, suppose instead we choose the proposal distribution to be a Cauchy distribution with pdf $g(x) = \frac{1}{\pi(1+x^2)}$. Describe and implement an inversion algorithm to get samples from this distribution, and plot the histogram of 1000 samples from this distribution.

We can sample U from unif(0,1) and plug into $G^-1(u)$ which has the same distribution as g, where G is the CDF

$$G(x) = \arctan(x)/\pi + 1/2, x \in \mathbf{R}$$
$$G^-1(u) = tan(\pi u - \pi/2), u \in [0, 1]$$

In [24]:

```
import numpy as np
import matplotlib.pyplot as plt

n=1000
u = np.random.uniform(0,1, n)
U = np.tan(np.pi*(u-0.5))

plt.hist(U, bins=50, density=True)
x = np.linspace(-20, 100, 100)
cauchy_pdf = 1 / (np.pi * (1 + x**2))
plt.plot(x, cauchy_pdf, 'r-', label='Cauchy PDF')
plt.title('Histogram of 1,000 Cauchy Samples vs. True PDF')
plt.xlabel('x')
plt.ylabel('Density')
plt.legend()
plt.show()
```



(e) Repeat parts (b) and (c) for this proposal distribution.

Ans.

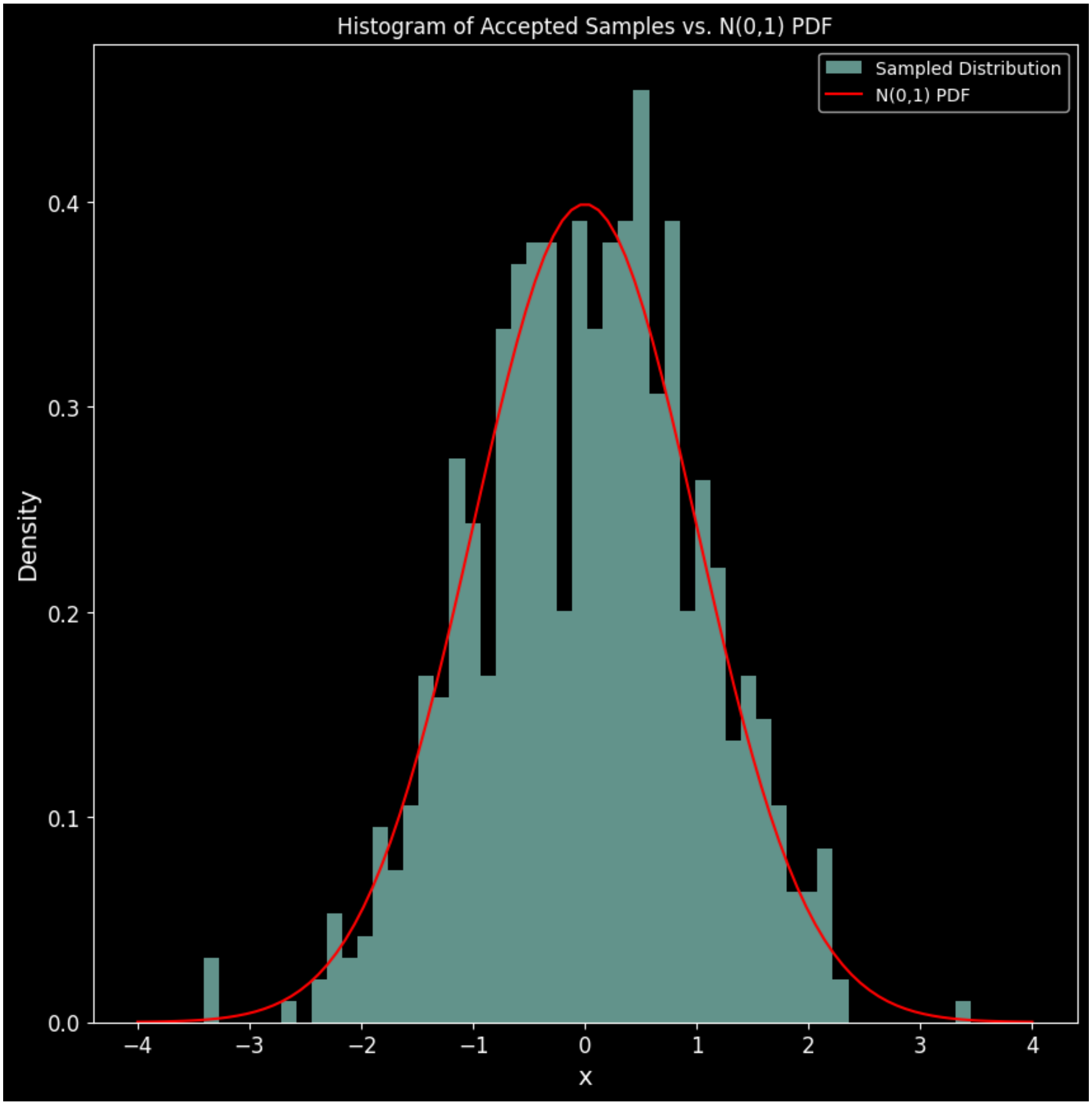
Define $h(x) = f(x)/g(x) = e^{-x^2/2}(g(x)/\sqrt{2\pi})$.

Maximum of $h(x)$ occurs at $x = 1$ with $h(1) = \sqrt{2\pi/e} = k$. The acceptance-rejection algorithm is:

1. Generate $U \sim G$
2. Accept U as a sample from f(U) with probability $\frac{f(U)}{k*g(U)}$. Else, reject U and return to step 1

In [25]:
```
n = 1000
k = np.sqrt(2*np.pi/np.e)
u = np.random.uniform(0,1, n)
U = np.tan(np.pi*(u-0.5))
g = 1/(np.pi*(1+U**2))
f = np.exp(-U**2/2)/np.sqrt(2*np.pi)
p = f/(k*g)
u = np.random.uniform(0,1, n)
xa = U[u<=p]

# Plot histogram of accepted samples
plt.hist(xa, bins=50, density=True, alpha=0.7, label='Sampled Distribu
x = np.linspace(-4, 4, 100)
plt.plot(x, np.exp(-x**2 / 2) / np.sqrt(2 * np.pi), 'r-', label='N(0,1
plt.title('Histogram of Accepted Samples vs. N(0,1) PDF')
plt.xlabel('x')
plt.ylabel('Density')
plt.legend()
```

Out[25]: <matplotlib.legend.Legend at 0x7cb74a1ef980>



In [29]:
```
import numpy as np
import matplotlib.pyplot as plt

np.random.seed(42)

n_target = 1000  # Target number of accepted samples per run
k = np.sqrt(2 * np.pi / np.e)  # Correct k for Cauchy proposal ≈ 1.520
S = 100  # Number of simulation runs to estimate CI

total_proposals_per_run = []

# Run S simulations to estimate variability in total proposals needed
for s in range(S):
    accepted_samples = []
    proposals = 0
    while len(accepted_samples) < n_target:
        u = np.random.uniform(0, 1)
        U = np.tan(np.pi * (u - 0.5))
```

```
        proposals += 1
        g = 1 / (np.pi * (1 + U**2))
        f = np.exp(-U**2 / 2) / np.sqrt(2 * np.pi)
        p = f / (k * g)
        if np.random.uniform(0, 1) <= p:
            accepted_samples.append(U)
    total_proposals_per_run.append(proposals*2)
    # Store the last run's samples for histogram
    if s == S - 1:
        xa = np.array(accepted_samples)

# Compute mean and 95% CI for total proposals needed
mean_proposals = np.mean(total_proposals_per_run)
std_proposals = np.std(total_proposals_per_run, ddof=1)
se_proposals = std_proposals / np.sqrt(S)
ci_lower = mean_proposals - 1.96 * se_proposals
ci_upper = mean_proposals + 1.96 * se_proposals

print(f"Mean total proposals of uniform generation for {n_target} acce
print(f"95% CI for total proposals of uniform generation needed: ({ci_
print(f"Theoretical expected total proposals of uniform generation: {2
print(f"Empirical acceptance rate (average): {n_target / (mean_proposa
print(f"Theoretical acceptance rate (1/k): {1 / k:.3f}")
```

```
Mean total proposals of uniform generation for 1000 accepted samples (o
ver 100 runs): 3042.1
95% CI for total proposals of uniform generation needed: (3031.8, 3052.
3)
Theoretical expected total proposals of uniform generation: 3040.7
Empirical acceptance rate (average): 0.657
Theoretical acceptance rate (1/k): 0.658
```

Clarification

The homework was finished with the help of LLM AI assistant for skeleton code and topic explation.