Nicolás Bernal
NYU Abu Dhabi
NeMO-C 2024

I.9 Solving the Boltzmann Equation

Solve numerically the Boltzmann equation for the dark matter (DM) number density n

$$\frac{dn}{dt} + 3\,H\,n = -\langle\sigma v\rangle \left(n^2 - n_{\rm eq}^2\right), \tag{I.15}$$

corresponding to the case of 2-to-2 DM annihilations into standard model (SM) states. Assume *i*) a Universe dominated by SM radiation, *ii*) conservation of the SM entropy, and *iii*) an s-wave (i.e. constant) thermally-averaged annihilation cross section $\langle\sigma v\rangle$.

Hints:

1. *i*) implies that the Hubble expansion rate as a function of the SM temperature T is

$$H(T) = \sqrt{\frac{\rho_R(T)}{3\,M_P^2}}, \tag{I.16}$$

with the SM energy density is given by

$$\rho_R(T) = \frac{\pi^2}{30}\,g_\star(T)\,T^4, \tag{I.17}$$

$M_P \simeq 2.4\times 10^{18}$ GeV is the reduced Planck mass and $g_\star(T)$ is the number of relativistic degrees of freedom that contributes to the energy density of the SM. As a first approximation, take, e.g. $g_\star = 106.75$. It is a good number for temperatures higher than the top mass.

2. For the DM number density in equilibrium (without chemical potential), use the Maxwell-Boltzmann approximation

$$n_{\rm eq}(T) = \frac{g}{2\pi^2}\,m^2\,T\,K_2\left(\frac{m}{T}\right), \tag{I.18}$$

where m is the mass of the DM, g is the number of internal degrees of freedom of the DM particle, and K_i the modified Bessel function of the second order.

3. Given *i*) and *ii*), Eq. (I.15) can be conveniently rewritten as

$$\frac{dY}{dx} = -\frac{\langle\sigma v\rangle\,s}{x\,H}\left[Y^2 - (Y^{\rm eq})^2\right], \tag{I.19}$$

where $x \equiv m/T$, $Y(T) \equiv n(T)/s(T)$, and $Y_{\rm eq}(T) \equiv n_{\rm eq}(T)/s(T)$, with

$$s(T) = \frac{2\pi^2}{45}\,g_{\star s}(T)\,T^3 \tag{I.20}$$

is the SM entropy density and $g_{\star s}(T)$ the number of relativistic degrees of freedom contributing to the DM entropy. Take $g_{\star s}(T) = g_\star(T)$, which is a good approximation for temperatures higher than a few MeVs.

4. For wisely chosen values of m and $\langle\sigma v\rangle$,[16] solve Eq. (I.19). Recall that observations of the DM relic density require that at low temperatures (i.e. $x \gg 100$)

$$Y \times m \simeq 4.3 \times 10^{-10}\ \text{GeV}. \tag{I.21}$$

For the initial conditions (yes, a first-order differential equation requires an initial condition!), take $x_{\rm ini} = 15$ and $Y(x_{\rm ini}) = Y_{\rm eq}(x_{\rm ini})$. For a given DM mass, what is the required value of $\langle\sigma v\rangle$? Calculate $\langle\sigma v\rangle$ in GeV^{-2}, picobarns, and cm^3/s.

5. How does the relic abundance (i.e. Y) evolves with an increase / decrease of $\langle\sigma v\rangle$?

6. Play wildly with the initial conditions $x_{\rm ini}$ and $Y(x_{\rm ini})$.

7. Repeat point 4 for other values of m, and plot $\langle\sigma v\rangle$ versus m.

8. Considering the simple parametrization

$$\langle\sigma v\rangle = \frac{\alpha^2}{m^2}\,, \tag{I.22}$$

with α a dimensionless coupling, plot α versus m. What is the maximal WIMP DM mass compatible with perturbativity?

9. One can go one step forward and take into account the temperature dependence of $g_\star$ and $g_{\star s}$. Use the tabulated values of $g_\star$ and $g_{\star s}$ in the file `Data/hgEff/DHS.thg` of MicrOMEGAs. In their notation heff $= g_{\star s}$ while geff $= g_\star$. Plot again $\langle\sigma v\rangle$ versus m taking now into account the variation of $g_\star$ and $g_{\star s}$.

10. Decrease the (constant) value of $\langle\sigma v\rangle$ without fear. Use the initial conditions $x_{\rm ini} = 10^{-1}$ and $Y(x_{\rm ini}) = 0$. Discover a second solution satisfying Eq. (I.21) and corresponding to ultraviolet freeze-in (UV FIMP) paradigm. Why is this freeze-in mechanism called *ultraviolet*?

11. How does the relic abundance (i.e. Y) evolves with an increase / decrease of $\langle\sigma v\rangle$?

12. Play wildly with the initial conditions $x_{\rm ini}$ and $Y(x_{\rm ini})$.

13. Plot $\langle\sigma v\rangle$ versus m for UV FIMPs.

14. Repeat the WIMP analysis, now for DM semi-annihialtion (i.e. reactions DM + DM → DM + SM)

$$\frac{dY}{dx} = -\frac{\langle\sigma v\rangle\, s}{x\,H}\left[Y^2 - Y\,Y^{\rm eq}\right]. \tag{I.23}$$

15. Repeat the WIMP analysis, now for 3-to-2 SIMP DM self-annihilation (i.e. reactions DM + DM + DM → DM + DM)

$$\frac{dY}{dx} = -\frac{\langle\sigma v^2\rangle_{\text{3-to-2}}\, s^2}{x\,H}\left[Y^3 - Y^2\,Y^{\rm eq}\right]. \tag{I.24}$$

[16] Wisely means masses in the GeV to TeV ballpark and cross sections at the picobarn scale!

16. Considering the simple parametrization

$$\langle \sigma v^2 \rangle_{\text{3-to-2}} = \frac{\alpha^3}{m^5}\,, \tag{I.25}$$

with α a dimensionless coupling, plot α versus m. What is the maximal SIMP DM mass compatible with perturbativity?

17. Forget assumptions *i)* and *ii)*. Assume instead the existence of a long-lived non-relativistic particle ϕ of mass m_ϕ, that only decays into SM particles with a total decay width Γ_ϕ corresponding to a lifetime $\tau_\phi = 1/\Gamma_\phi \sim 1$ s. The evolution of the number density n_ϕ of ϕ and the SM radiation energy density ρ_R can be track by the system of coupled Boltzmann equations

$$\frac{dn_\phi}{dt} + 3\,H\,n = -\Gamma_\phi\,n_\phi\,, \tag{I.26}$$

$$\frac{d\rho_R}{dt} + 4\,H\,\rho_R = +\Gamma_\phi\,n_\phi\,m_\phi\,, \tag{I.27}$$

and the Friedmann equation

$$H = \sqrt{\frac{\rho_R + m_\phi\,n_\phi}{3\,M_P}}\,, \tag{I.28}$$

where $\rho_\phi = m_\phi\,n_\phi$ is the energy density of ϕ. Rewrite Eqs. (I.26), (I.27) and (I.28) in terms of the comoving quantities $\Phi \equiv n_\phi \times a^3$, $R \equiv \rho_R \times a^4$, and the cosmic scale factor a. Solve numerically the new equations, looking for periods where ϕ dominates the total energy density of the Universe. Extract the evolution of the SM temperature using Eq. (I.17). Use the initial condition $a_{\text{ini}} = 1$ with $\rho_R(a_{\text{ini}}) = 0$ and $\rho_\phi(a_{\text{ini}}) \neq 0$ for a reheating-like scenario or $\rho_R(a_{\text{ini}}) \gg \rho_\phi(a_{\text{ini}}) > 0$ and for an early matter domination era. Recall that $H(T) < 2 \times 10^{-5}\ M_P$.

18. Rewrite Eq. (I.15) using $N \equiv n \times a^3$ and a. Solve it in the previously discussed background. Understand WIMPs during low-temperature reheating, and WIMPs during an early matter dominated era.