**Section #1**

***SOLUTIONS***

**Topic covered:**

- Properties of summations

**Problem 1**

Prove the following properties of summations:

**a.** $$\sum_{i=1}^{n} \frac{X_i}{Y_i} \neq \frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} Y_i}$$

**Solution:**

We can prove this by providing a counterexample.

Let N=2
$X_1 = 2$, $X_2 = 4$
$Y_1 = 4$, and $Y_2 = 8$.

Then

$$\frac{\sum_{i=1}^{2} X_i}{\sum_{i=1}^{2} Y_i} = \frac{X_1 + X_2}{Y_1 + Y_2} = \frac{2+4}{4+8} = \frac{1}{2}$$

$$\sum_{i=1}^{2} \frac{X_i}{Y_i} = \frac{2}{4} + \frac{4}{8} = 1$$

Clearly $\frac{1}{2} \neq 1$, so the claim is false.

**b.** Prove that given data on two variables X and Y, and the sample means $\bar{X}$ and $\bar{Y}$ the following holds:

$$\sum_{i=1}^{N} (X_i - \bar{X})(Y_i - \bar{Y}) = \sum_{i=1}^{N} X_i Y_i - N\,\bar{X}\bar{Y}$$

**Solution:**

$$\sum_{i=1}^{N}(X_i - \bar{X})\,(Y_i - \bar{Y}) =$$

$$\sum_{i=1}^{N}(X_iY_i - X_i\bar{Y} - Y_i\bar{X} + \bar{X}\bar{Y}\ ) = (multiplying\ the\ terms)$$

$$\sum_{i=1}^{N} X_iY_i - \sum_{i=1}^{N} X_i\bar{Y} - \sum_{i=1}^{N} Y_i\bar{X} + \sum_{i=1}^{N} \bar{X}\bar{Y} = (regrouping\ the\ summation\ terms)$$

$$\sum_{i=1}^{N} X_iY_i - \bar{Y}\sum_{i=1}^{N} X_i - \bar{X}\sum_{i=1}^{N} Y_i + \ N\bar{X}\bar{Y} = (using\ the\ fact\ that\ \bar{X}\ \ and\ \bar{Y}\ are\ constants)$$

$$\sum_{i=1}^{N} X_iY_i - \bar{Y}(N\bar{X}) - \bar{X}(N\bar{Y}) + \ N\bar{X}\bar{Y} =$$

$$= \ (using\ the\ definitions\ of\ \bar{X}\ \ and\ \bar{Y}, and\ expressing\ \sum_{i=1}^{N} X_i\ \ and\ \sum_{i=1}^{N} Y_i)$$

$$\sum_{i=1}^{N} X_iY_i \ -\ N\ \bar{X}\bar{Y}\ (\text{as} - \bar{X}(N\bar{Y})\ \text{and}\ \ N\bar{X}\bar{Y}\ \text{cancel})$$

This completes the proof.

**Note:** from here it follows that when $X$=$Y$, then:

$$\sum_{i=1}^{N}(X_i - \bar{X})^2 = \ \sum_{i=1}^{N} X_i^2 - \ N\bar{X}^2$$

We are going to see these in econometrics part the course when we start talking about regression analysis.

**c.** Prove that given a sample of size $N$ with data on a variable X the following holds:

$$\sum_{i=1}^{N}[X_i\,(X_i - \bar{X})] = \sum_{i=1}^{N}(X_i - \bar{X})^2$$

**Solution:**

**Method I**

**Start with the left hand side:**

$$\sum_{i=1}^{N}[X_i\,(X_i-\bar{X})] =$$

$$= \sum_{i=1}^{N}(X_i^2 - X_i\bar{X}) \qquad (multiplying\ the\ terms)$$

$$= \sum_{i=1}^{N} X_i^2 - \sum_{i=1}^{N} X_i\bar{X} \qquad (regrouping\ the\ summation\ terms)$$

$$= \sum_{i=1}^{N} X_i^2 - \bar{X}\sum_{i=1}^{N} X_i \qquad (using\ the\ fact\ that\ \bar{X}\ is\ a\ constant)$$

$$= \sum_{i=1}^{N} X_i^2 - \bar{X}(N\bar{X}) \qquad (using\ the\ definition\ of\ \bar{X}\ \ , and\ expressing\ \sum_{i=1}^{N} X_i\ )$$

$$= \sum_{i=1}^{N} X_i^2 - \ N\bar{X}^2$$

**And from part b. the right hand side:**

$$\sum_{i=1}^{N}(X_i-\bar{X})^2 = \ \sum_{i=1}^{N} X_i^2 - \ N\bar{X}^2$$

**So, the left hand side and the right hand side both equal $\sum_{i=1}^{N} \boldsymbol{X_i^2} - \boldsymbol{N\bar{X}^2}$. Hence, they are equal to each other. This completes the proof.**

**Method II**

**A considerably more elegant but harder proof is by noticing the following:**

$$\boldsymbol{RHS} = \sum_{i=1}^{N}(X_i-\bar{X})^2 = \ \sum_{i=1}^{N}[X_i\,(X_i-\bar{X}) - \bar{X}(X_i-\bar{X})]$$

$$= \sum_{i=1}^{N} X_i\,(X_i-\bar{X}) - \ \sum_{i=1}^{N}\bar{X}(X_i-\bar{X}) = LHS - \ \sum_{i=1}^{N}\bar{X}(X_i-\bar{X})$$

$$But\ \sum_{i=1}^{N}\bar{X}(X_i-\bar{X}) = \bar{X}\sum_{i=1}^{N}(X_i-\bar{X}) = \ \bar{X}(\sum_{i=1}^{N} X_i - N\bar{X}) = \ 0\,, \textbf{which completes the proof}.$$