PageRank Algorithm

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PageRank algorithm focuses on *how important* each webpage is.

The “Random Surfer” Model

The PageRank algorithm

- iterates on a **graph** where webpages are **nodes**, and links are **directed edges**,
- calculates the probability of a “random surfer” visiting each webpage,
- outputs a probability distribution vector.

$$\boldsymbol{p} = \begin{bmatrix} \Pr(1) & \Pr(2) & \cdots & \Pr(n) \end{bmatrix}^{\mathsf{T}}$$

Algorithm

Notations and Conventions

- Let n denote the number of nodes (i.e. total number of webpages).
- Let $\deg^+(u)$ denote the **out degree** of node u (i.e. number of outreaching links on webpage u).

Definition

Google PageRank[a] **iteration**: Initially, $\Pr(i) := 1/n$ for each webpage $1 \le i \le n$. Then iterate by

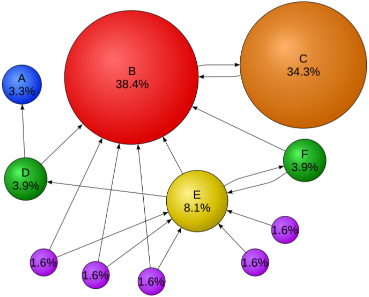
$$\Pr(v) := (1-d)\frac{1}{n} + d \sum_{\text{edge } u \to v} \frac{1}{\deg^+(u)} \Pr(u)$$

where d is the **damping factor** $(0 < d < 1)$.

[a] Trademark of Google; U.S. patent 6,285,999.

Computing PageRank

- A node is important if important nodes point to it.
- A node contributes part of its importance to the nodes it points to.



Matrix Algebra I

A Simplified Model

Let **transition matrix**

$$\mathscr{M}_{ij} := \begin{cases} \dfrac{1}{\deg^{+}(j)} & \text{edge } j \to i \\ 0 & \text{otherwise} \end{cases}$$

i.e., given **adjacent matrix** $A_{ij} = [\text{edge } i \to j]$ and diagonal matrix K with the outdegrees in the diagonal,

$$\mathscr{M} := (K^{-1}A)^{\mathsf{T}}$$

Matrix Algebra II

Definition

Let **probability distribution vector** of the k-th iteration be

$$\boldsymbol{p}(k) := \begin{bmatrix} \Pr(1) & \Pr(2) & \cdots & \Pr(n) \end{bmatrix}^{\mathsf{T}}$$

which is initially set to

$$\boldsymbol{p}(0) := \begin{bmatrix} 1/n & 1/n & \cdots & 1/n \end{bmatrix}^{\mathsf{T}}$$

Matrix Algebra III

Definition

The **Google matrix** $\widehat{\mathscr{M}}$ is defined by

$$\widehat{\mathscr{M}} := d\mathscr{M} + \frac{1-d}{n}E$$

where E is $n \times n$ matrix of all ones (so that $E\boldsymbol{p} = \mathbf{1}$), and $0 < d < 1$ is the damping factor.

The Power Method

Google PageRank can be computed by

$$\boldsymbol{p}(k+1) = \widehat{\mathscr{M}}\boldsymbol{p}(k)$$

One may assume PageRank converges after $|\boldsymbol{p}(t) - \boldsymbol{p}(t-1)| < \epsilon$.

PageRank and Eigenvectors

Recall the equation

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In fact, $\boldsymbol{p}$ is the **principal eigenvector** of the Google matrix $\widehat{\mathscr{M}}$, and 1 is the corresponding eigenvalue (i.e. the eigenvalue with the largest magnitude).

Properties

Fact

- *Both $\mathscr{M}$ and $\widehat{\mathscr{M}}$ are **column-stochastic** (sums of each column are 1), positive.*

$$\sum_i \widehat{M}_{ij} = 1 \ (\forall j) \Longleftrightarrow \mathbf{1}^\top \widehat{M} = \mathbf{1}^\top$$

($\mathbf{1}^\top$ is a left eigenvector with eigenvalue $1 \Rightarrow 1$ is an eigenvalue of $\widehat{\mathscr{M}}$)

- *Both $\mathscr{M}$ and $\widehat{\mathscr{M}}$ are positive matrices.*

By the **Perron–Frobenius theorem**,

- all other eigenvalues λ_i satisfy $|\lambda_i| < 1$,
- the corresponding eigenvector r has all positive entries.

A View on Probability

Matrix multiplication → updating probability distribution vector.

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- With a probability of d, “random surfing” continues;
  - Each u has equal chance $1/\deg^{+}(u)$ to redirect to v
- with a probability of $1 - d$, surfer redirects to another random page.
  - Each of the n pages has equal chance to redirect to the current page.

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$$\mathbb{P}(\text{arriving at } v \mid \text{arriving at } u) := (1-d)\frac{1}{n} + d \sum_{\text{edge } u \to v} \frac{1}{\deg^{+}(u)} \mathbb{P}(\text{arriving at } u)$$

Applications

Generalizing *importance*.

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- Evaluating academic papers based on citations.
- Determine species that are essential to the continuing health of ecosystems.
- Ranking performance of sports teams.
- ...

Further Reading I

Suggested topics for further reading:

Topic

PageRank is actually a variation of **the Markov Chains**.

Fact

Random walk which is

- ***irreducible**: with probability of $1/d$, surfer can reach any node with one step $\Rightarrow$ all nodes are reachable $(\Pr > 0)$ to the surfer.*
- ***aperiodic**: with probability of d, surfer stops on their current page $\Rightarrow$ no fixed-length cycles in transition*
- ***positive recurrent**: expected return time to any node is finite*

converges to a stationary distribution.

Question

Will the simplified model $\boldsymbol{p}(k+1) = \mathscr{M}\,\boldsymbol{p}(k)$ we defined before always converge?

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Will the simplified model $\boldsymbol{p}(k+1) = \mathscr{M}\,\boldsymbol{p}(k)$ we defined before always converge?
If not, what will happen and why? Construct test-cases to prove.

Further Reading II

Topic

Some Easy Ways to Briefly Analyze “Relevance”:

- Word frequency (TF / TF-IDF)
- Proximity scoring (word vectors)
- Matching phrases (bags-of-words)
- ...

Ending

Thanks!