Important concepts, symbols, and equations

For a single joint with the joint velocity as the control:

- **Open-loop (feedforward) control**: $$\dot{\theta}(t) = \dot{\theta}_d(t)$$
- **Closed-loop (feedback) control**: $$\dot{\theta}(t) = f(\theta_d(t), \theta(t))$$
- **FF + Proportional-Integral (PI) FB control**: $\theta_e = \theta_d - \theta$

$$\underbrace{\dot{\theta}(t) = \dot{\theta}_d(t)}_{FF} + \underbrace{K_p\theta_e(t)}_{P} + \underbrace{K_i \int_0^t \theta_e(\mathrm{t})\, dt}_{I}\ , \quad K_p, K_i \geq 0$$

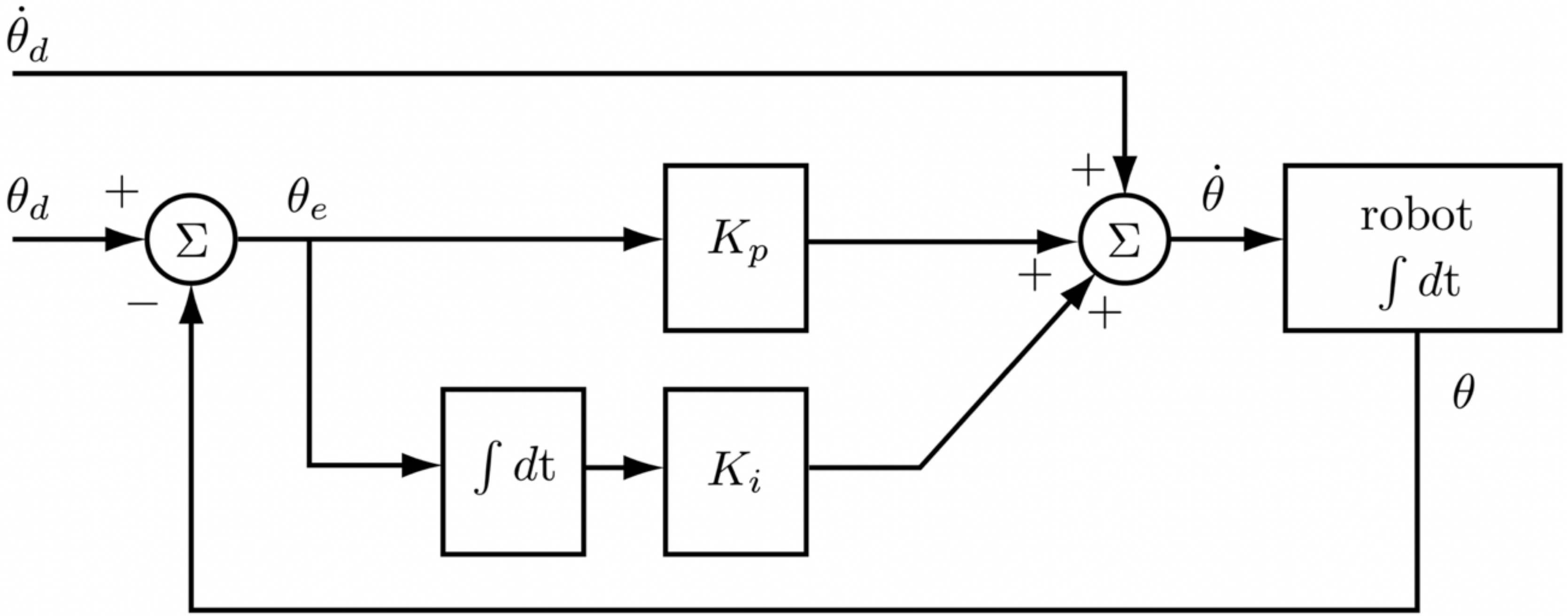
- *reduces to FF control if* $K_p, K_i = 0$
- *if no FF term:* **P control** *when* $K_i = 0$, **I control** *when* $K_p = 0$

What is the point of FF control in this control law?

Don't wait for error to start control. Better tracking.

Important concepts, symbols, and equations (cont.)

Block diagram



$$\dot{\theta}(t) = \dot{\theta}_d(t) + K_p\theta_e(t) + K_i \int_0^t \theta_e(\mathrm{t})\, dt$$

<u>Important concepts, symbols, and equations</u> (cont.)

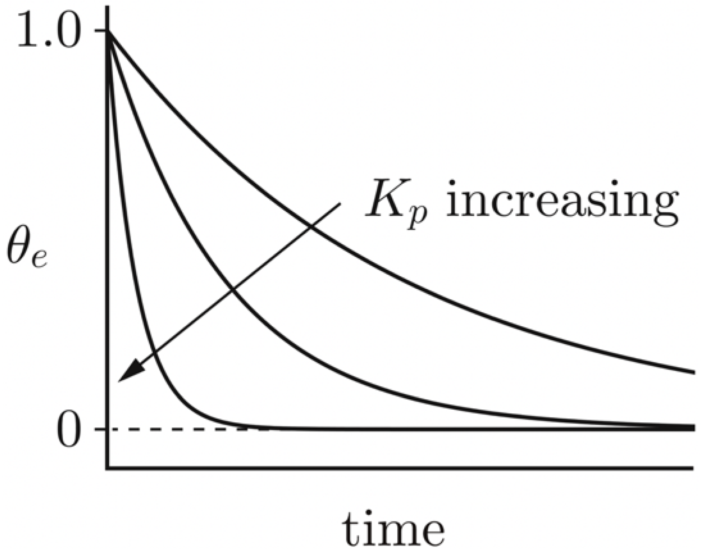
Setpoint control, $\theta_d(t) = c$, with a P controller

$$\dot{\theta}_e(t) = \cancelto{0}{\dot{\theta}_d(t)} - \dot{\theta}(t)$$

$$\dot{\theta}_e(t) = -K_p\theta_e(t) \quad \rightarrow \quad \dot{\theta}_e(t) + K_p\theta_e(t) = 0$$

$$\mathfrak{t} = 1/K_p$$

$$\theta_e(t) = e^{-t/\mathfrak{t}}\theta_e(0)$$



Important concepts, symbols, and equations (cont.)

Constant velocity control, $\theta_d(t) = ct + a$

P control

$$\dot{\theta}_e(t) = \dot{\theta}_d(t) - \dot{\theta}(t) = c - K_p\theta_e(t)$$

$$\dot{\theta}_e(t) + K_p\theta_e(t) = c$$

$$\theta_e(t) = \underbrace{\boxed{\frac{c}{K_p}}}_{e_{ss}} + \underbrace{\left(\theta_e(0) - \frac{c}{K_p}\right) e^{-K_p t}}_{\to 0 \text{ as } t \to \infty}$$

PI control

$$\dot{\theta}(t) = K_p\theta_e(t) + K_i \int_0^t \theta_e(\mathrm{t})\, d\mathrm{t}$$

$$\dot{\theta}_e(t) + \underset{2\zeta\omega_n}{K_p}\theta_e(t) + \underset{\omega_n^2}{K_i} \int_0^t \theta_e(\mathrm{t})\, d\mathrm{t} = c$$

$$\ddot{\theta}_e(t) + K_p\dot{\theta}_e(t) + K_i\theta_e(t) = 0$$

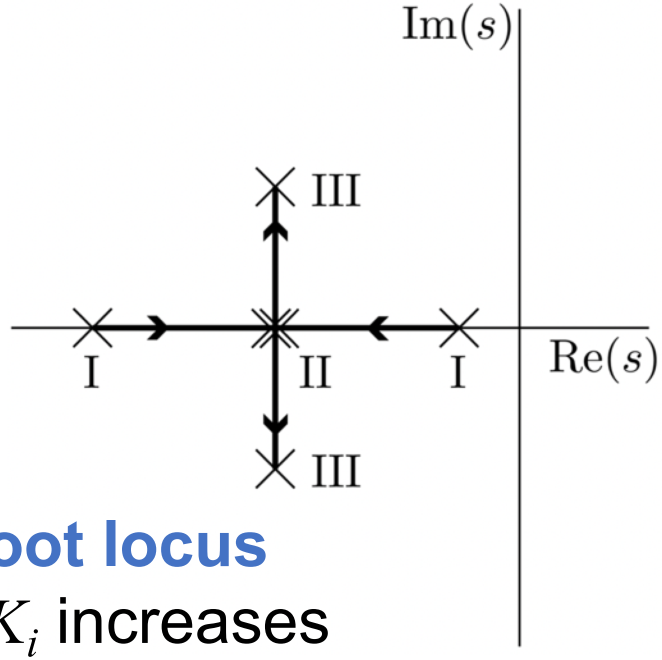
$$\zeta = K_p/(2\sqrt{K_i})$$

$$\omega_n = \sqrt{K_i}$$

Important concepts, symbols, and equations (cont.)

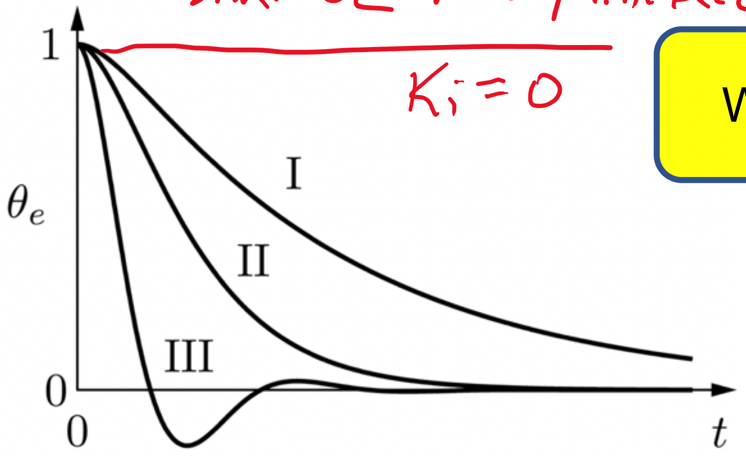
characteristic equation of PI error dynamics: $s^2 + K_p s + K_i = 0$

$$s_{1,2} = -\frac{K_p}{2} \pm \sqrt{\frac{K_p^2}{4} - K_i}$$



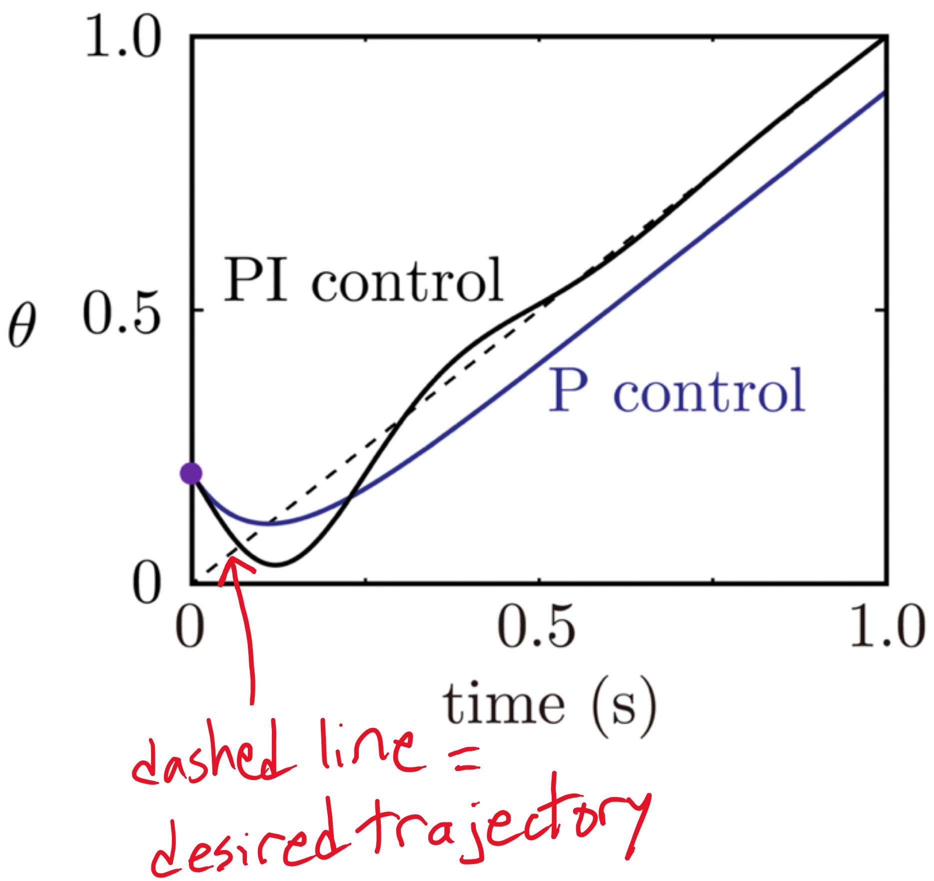
root locus
as K_i increases

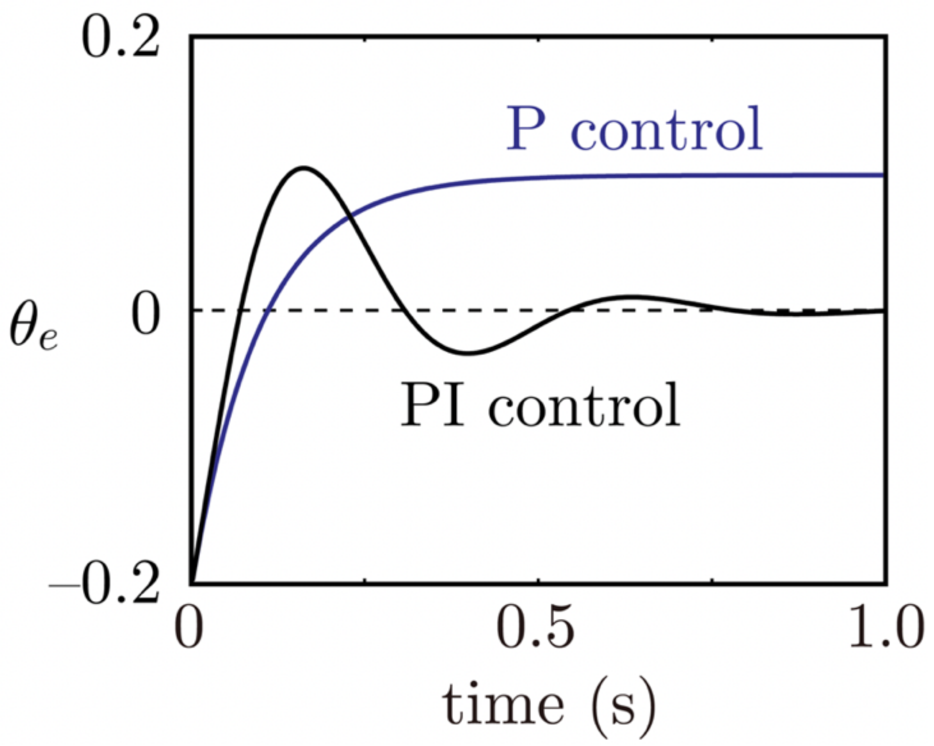
in the mbk analogy, where θ_e = position, there is no spring to pull θ_e to 0, and the damper eventually gives constant θ_e for any initial condition



What if $K_i = 0$?

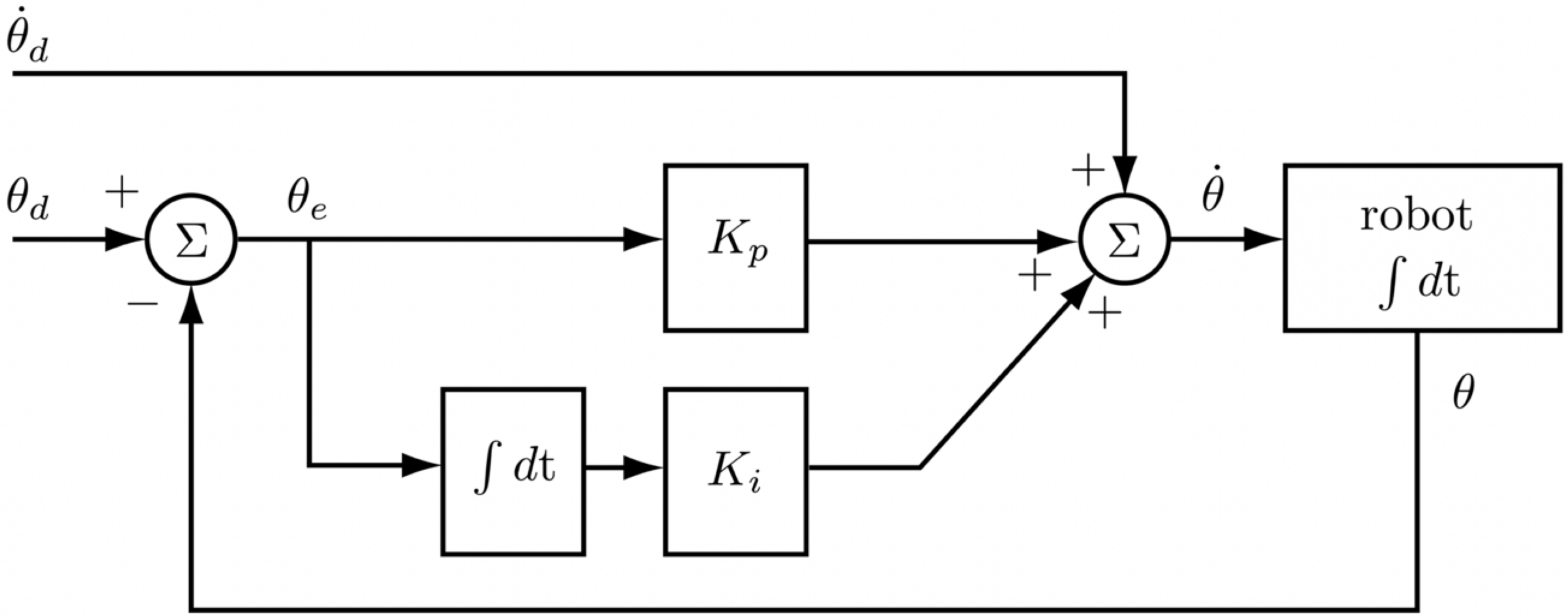
Important concepts, symbols, and equations (cont.)





Important concepts, symbols, and equations (cont.)

Multi-joint control: θ, θ_d, and θ_e are vectors and $K_p = k_p I$, $K_i = k_i I$



$$\dot{\theta}(t) = \dot{\theta}_d(t) + K_p \theta_e(t) + K_i \int_0^t \theta_e(\mathrm{t})\, dt$$

Important concepts, symbols, and equations (cont.)

$$\dot{\theta}(t) = \dot{\theta}_d(t) + K_p\theta_e(t) + K_i \int_0^t \theta_e(\mathrm{t})\ dt$$

Task-space motion control

desired motion

$$X_d \in SE(3)$$

$$[\mathcal{V}_d] = X_d^{-1}\dot{X}_d$$

actual motion

$$X \in SE(3)$$

$$[\mathcal{V}_b] = X^{-1}\dot{X}$$

error: $[X_e] = \log X_{bd} = \log(X^{-1}X_d)$

$X_{sb}^{-1} X_{sd}$

$X_{b\cancel{s}} X_{\cancel{s}d}$

recall

$\mathcal{V}_b = J_b \dot{\theta}$

$$\boxed{\begin{aligned} \mathcal{V}_b(t) &= [\mathrm{Ad}_{X_{bd}}]\mathcal{V}_d(t) + K_p X_e(t) + K_i \int_0^t X_e(\mathrm{t})dt \\ \dot{\theta} &= J_b^{\dagger}(\theta)\mathcal{V}_b \end{aligned}}$$

Important concepts, symbols, and equations (cont.)

Decoupled task-space motion control

$$X_e(t) = \begin{bmatrix} \omega_e(t) \\ p_d(t) - p(t) \end{bmatrix} \qquad [\omega_e] = \log(R^{\mathrm{T}} R_d)$$

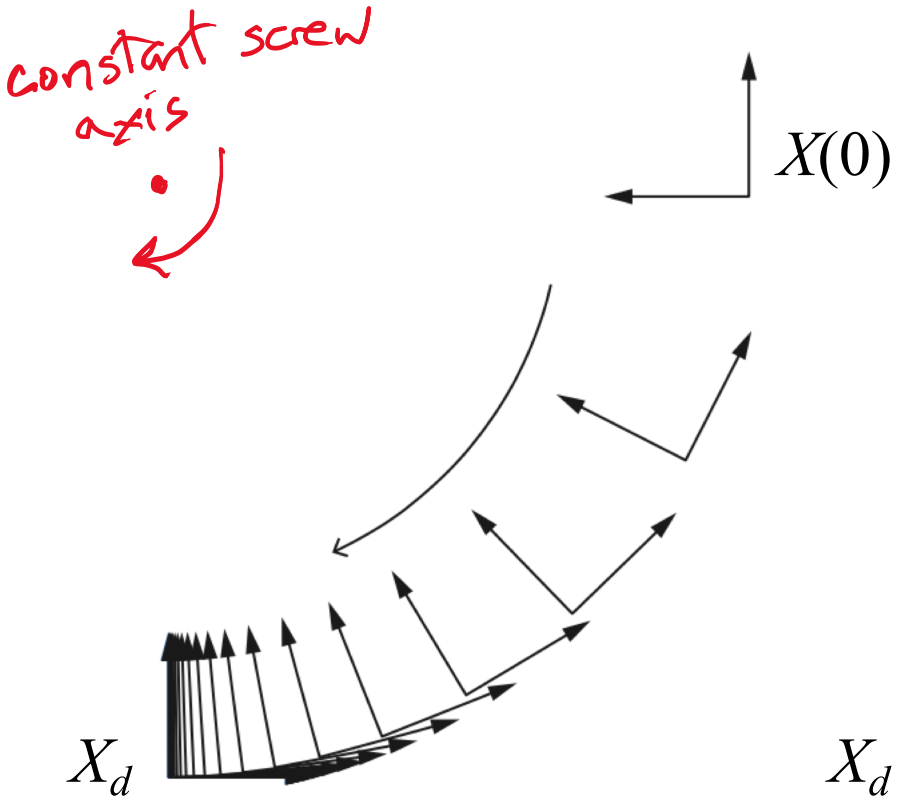
$$\begin{bmatrix} \omega_b(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} R^{\mathrm{T}}(t)R_d(t) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \omega_d(t) \\ \dot{p}_d(t) \end{bmatrix} + K_p X_e(t) + K_i \int_0^t X_e(\mathrm{t})\, dt$$

$$\dot{\theta} = J^{\dagger}(\theta) \begin{bmatrix} \omega_b(t) \\ \dot{p}(t) \end{bmatrix}$$

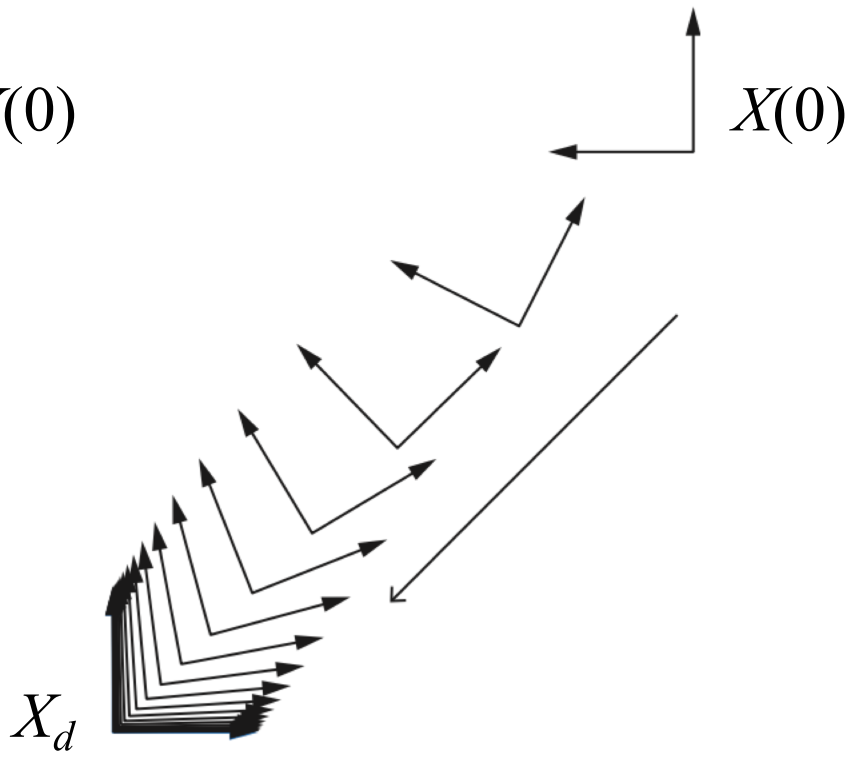
What is this Jacobian?

top three rows are same as J_b
bottom three rows are neither from J_b nor J_s

Important concepts, symbols, and equations (cont.)



task-space P controller



decoupled task-space P controller

You are designing a P controller to track a joint reference trajectory that is moving at a constant rate of 3 radians/s. What is the smallest gain K_p that ensures a steady-state position error of no more than 0.1 radians? Give units.

$$0.1\text{ rad} = \frac{c}{K_p} = \frac{3\text{ rad/s}}{K_p} \rightarrow \boxed{K_p = 30\text{ s}^{-1}}$$

To eliminate steady-state error, you decide to use a PI controller. What gains K_p and K_i should you choose to achieve critical damping and a settling time of 0.1 s? Give units.

$$0.1\text{ s} = \frac{4}{\zeta\omega_n} = \frac{4}{\omega_n} \rightarrow \omega_n = 40\text{ s}^{-1} \quad \boxed{K_i = \omega_n^2 = 1600\text{ s}^{-2}}$$

crit. damp: $\zeta = 1$ $\quad \zeta = 1 = K_p/2\sqrt{K_i} = K_p/80\text{ s}^{-1} \rightarrow \boxed{K_p = 80\text{ s}^{-1}}$

Explain how to estimate the error integral if the controller's frequency is $1/T$.

$$e_{int}(k+1) = e_{int}(k) + e(k)T$$

Why not choose arbitrarily large K_p and K_i to achieve arbitrarily fast settling?

instability due to large gains and delays; chattering/saturation; nonlinear behavior

How well would a PI controller track a quadratic joint trajectory? (Not a ramp.)

It would have a constant steady-state error, just like a P controller for a ramp desired trajectory