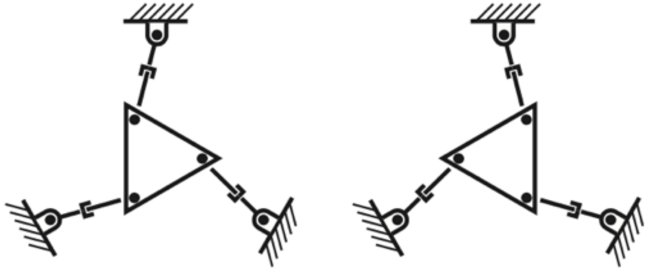
Important concepts, symbols, and equations

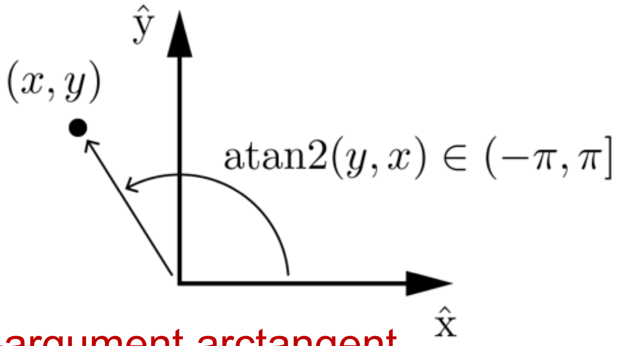
- **inverse kinematics (IK)**: given $T_{sd} \in SE(3)$, find θ such that $T(\theta) = T_{sd}$.

- unlike FK, IK for serial chains could have zero, one, or multiple solutions

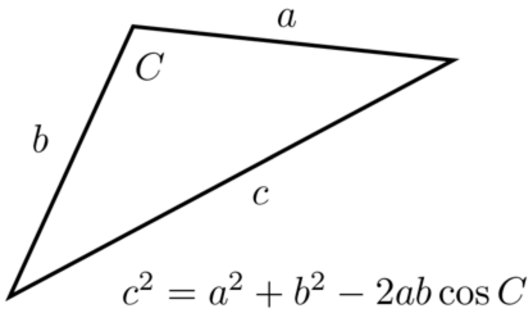
situation is reversed
for closed chains:



- closed-form (analytical) IK

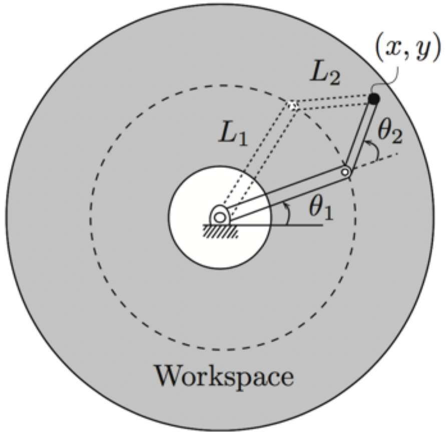


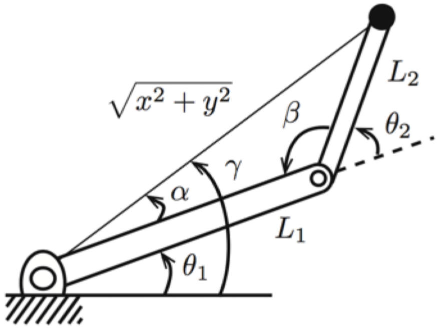
two-argument arctangent



law of cosines

IK for a 2R robot
γ from atan2
α, β from law of cosines



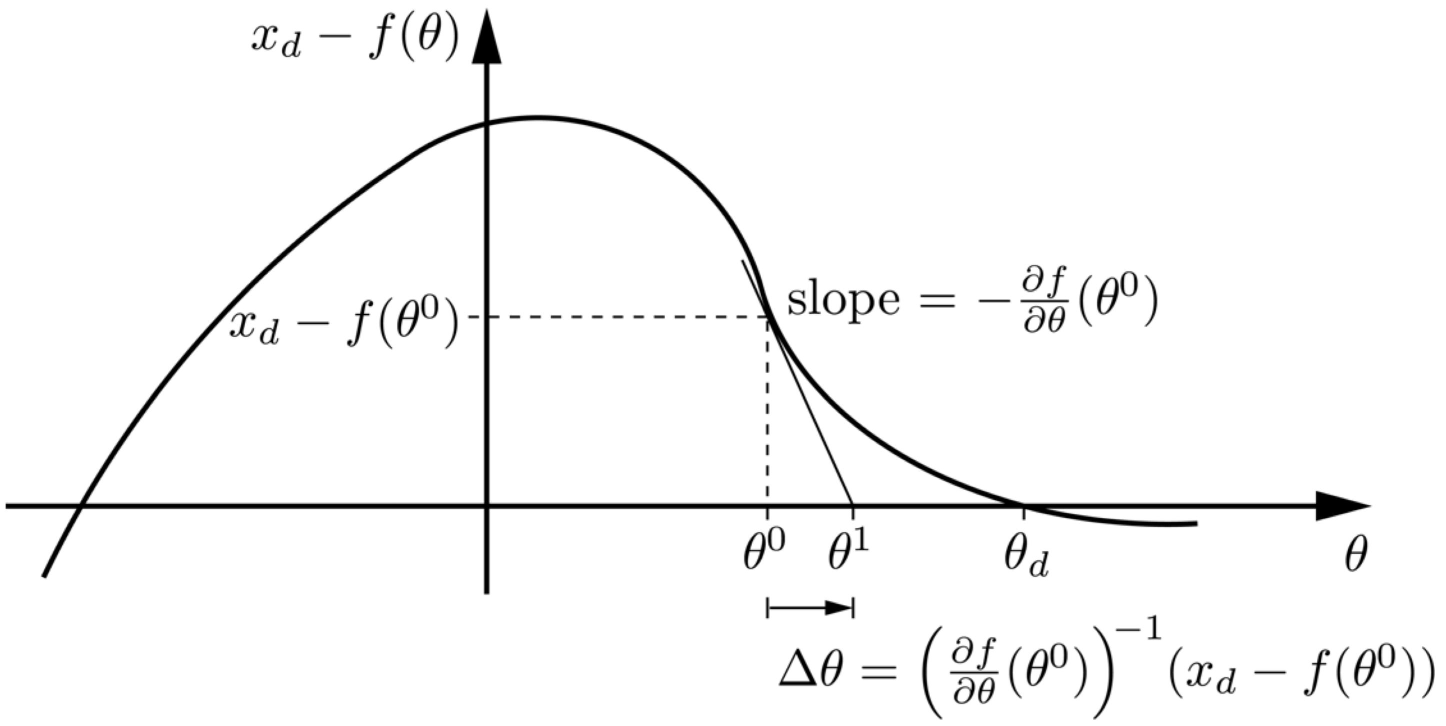


Important concepts, symbols, and equations (cont.)

- **numerical IK**: iteratively refine an initial guess θ^0 to find θ^k such that $T(\theta^k) \approx T_{sd}$.

- **Newton-Raphson root finding**

x_d : desired function value
$f(\theta)$: actual function value at θ



Important concepts, symbols, and equations (cont.)

- vector Taylor expansion: $$x_d = f(\theta_d) = f(\theta^0) + \underbrace{\left.\frac{\partial f}{\partial \theta}\right|_{\theta^0}}_{J(\theta^0)} \underbrace{(\theta_d - \theta^0)}_{\Delta\theta} + \text{h.o.t.}$$

linear approximation: $$J(\theta^0)\,\Delta\theta = x_d - f(\theta^0) \qquad (1)$$

linear correction to the guess θ^0: $$\Delta\theta = J^{-1}(\theta^0)\,(x_d - f(\theta^0))$$

- if J is not invertible, use the **pseudoinverse**: $$\Delta\theta^* = J^{\dagger}(\theta^0)\,(x_d - f(\theta^0))$$

  If there exists a $\Delta\theta$ exactly satisfying (1), then $\Delta\theta^*$ has the smallest 2-norm among all solutions.

  If there is no $\Delta\theta$ exactly satisfying (1), then $\Delta\theta^*$ comes closest in the 2-norm sense.

Important concepts, symbols, and equations (cont.)

Special cases of pseudoinverse for $J \in \mathbb{R}^{m \times n}$ (m e-e velocity directions, n joints):

- If J is full rank and square: $J^\dagger = J^{-1}$
- If J is full rank and tall ($m > n$): $J^\dagger = (J^{\mathrm{T}} J)^{-1} J^{\mathrm{T}} \in \mathbb{R}^{n \times m}$ (the "left inverse")
- If J is full rank and wide ($n > m$): $J^\dagger = J^{\mathrm{T}} (J J^{\mathrm{T}})^{-1} \in \mathbb{R}^{n \times m}$ (the "right inverse")

Important concepts, symbols, and equations (cont.)

- Numerical inverse kinematics, coordinate version:

  (a) **Initialization**: Given $x_d \in \mathbb{R}^m$ and an initial guess $\theta^0 \in \mathbb{R}^n$, set $i = 0$.

  (b) Set $e = x_d - f(\theta^i)$. While $\|e\| > \epsilon$ for some small ϵ:

  - Set $\theta^{i+1} = \theta^i + J^\dagger(\theta^i)e$.
  - Increment i.

- Numerical inverse kinematics, geometric version:

  (a) **Initialization**: Given T_{sd} and an initial guess $\theta^0 \in \mathbb{R}^n$, set $i = 0$.

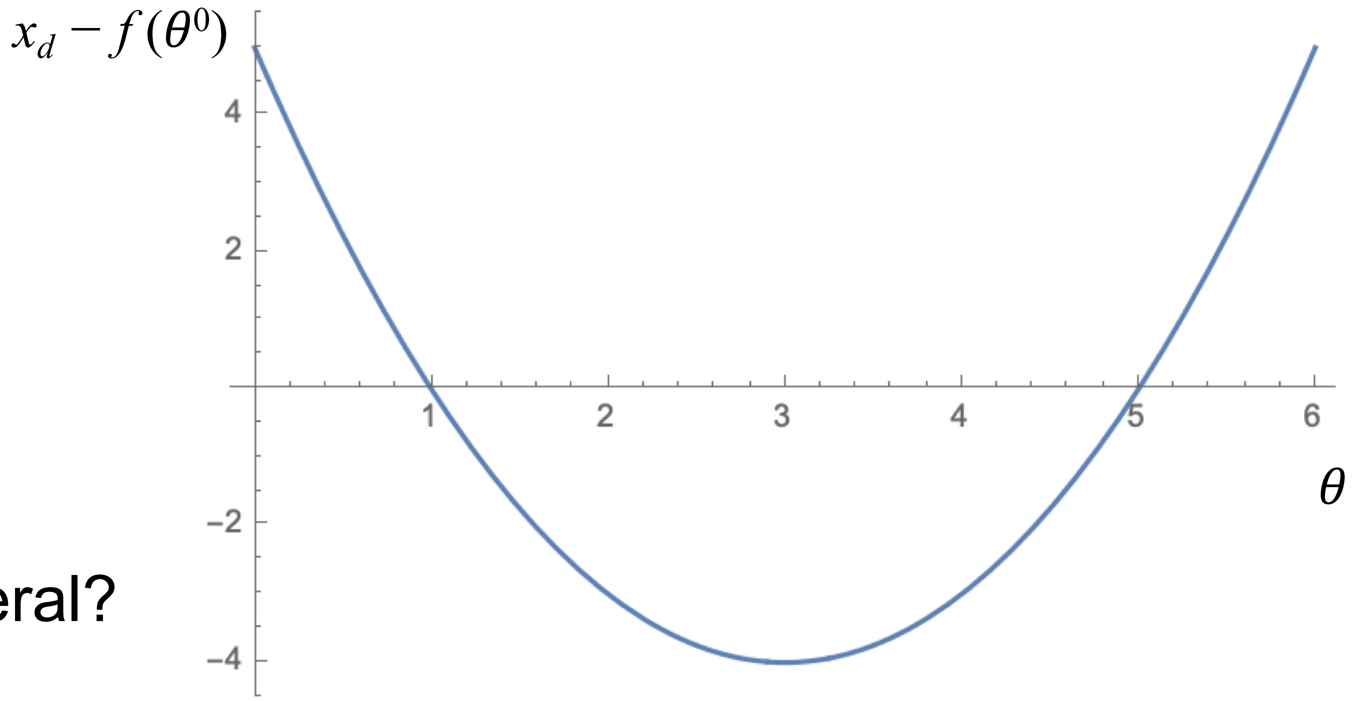
  (b) Set $[\mathcal{V}_b] = \log\left(T_{sb}^{-1}(\theta^i)T_{sd}\right)$. While $\|\omega_b\| > \epsilon_\omega$ or $\|v_b\| > \epsilon_v$ for small $\epsilon_\omega, \epsilon_v$:

  - Set $\theta^{i+1} = \theta^i + J_b^\dagger(\theta^i)\mathcal{V}_b$.
  - Increment i.

Illustrate Newton-Raphson root finding for initial guesses $\theta^0 = 3.6$ and $\theta^0 = 0$.

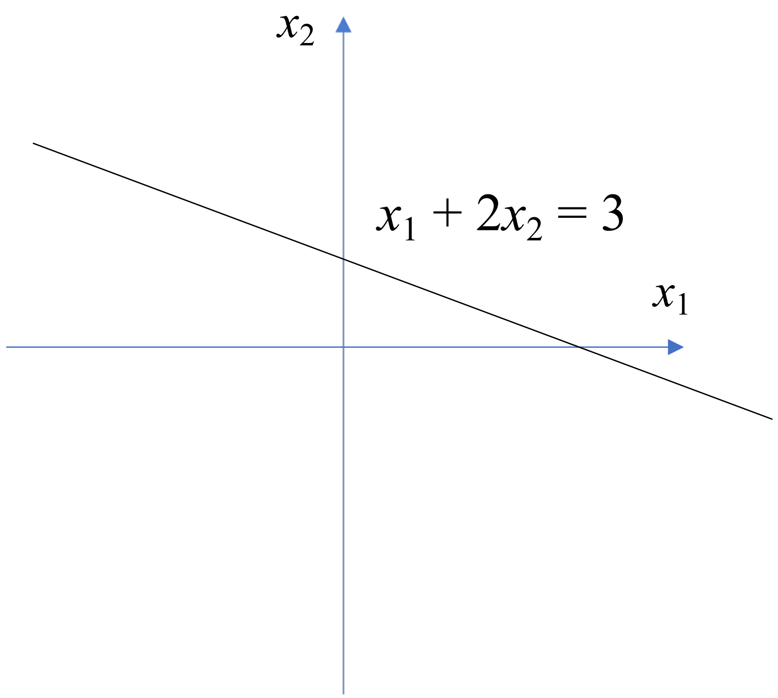
What if $\theta^0 = 3.1$ and this were a joint angle?

For a robot controller, what's a good choice for the initial guess θ^0 in general?

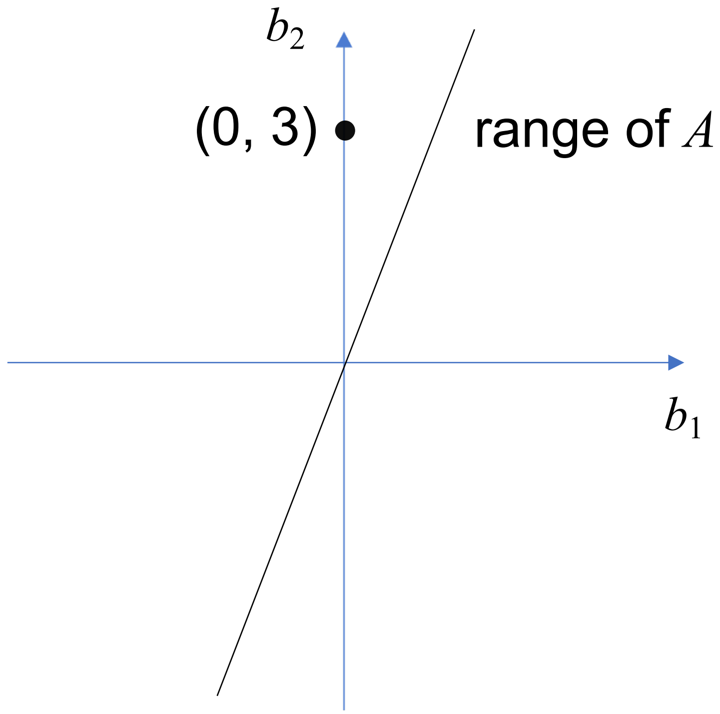


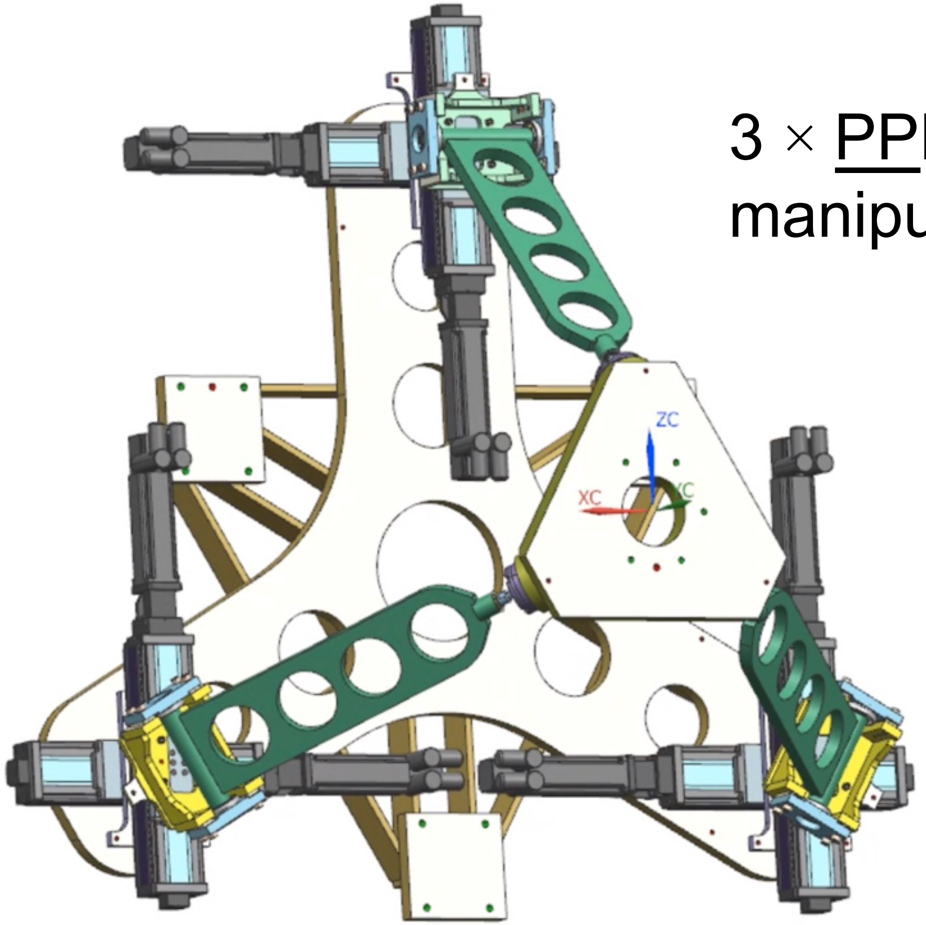
Graphically find a "good" solution to $Ax = b$, e.g., $x = A^{\dagger}b$.

1) $A = [1 \;\; 2]$, $b = 3$



2) $A = [1 \;\; 2]^{\mathrm{T}}$, $b = [0 \;\; 3]^{\mathrm{T}}$





3 × PPRS parallel manipulator

IK? FK?

KUKA Systems North America LLC
(patent pending)