$\epsilon - \delta$ definition of continuity and topological definition of continuity for the reals

In this page, I will be talking about the $\epsilon - \delta$ definition and how it relates to the topological definition of continuity specifically for the real numbers.

> **Definition 1 ($\epsilon - \delta$ Continuity).** A function $f : A \to \mathrm{R}$ with $A \subseteq \mathrm{R}$ is continuous if, for all $x_0 \in A$ and $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta)$ it follows that $f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$

> **Definition 2 (Topological Definition of Continuity).** Let (X, τ) and (Y, B) be topological spaces, then $f : X \to Y$ is continuous if for any $B \subseteq Y$ that is an open set under the topology B it follows that the preimage of B in X denoted $A = f^{-1}(B)$ is also open under the topology τ.

Now I want to tell yo u a story about this. One day, I wanted to prove that the first definition implies the second definition. When I was trying to do that, I got the reference for the second definition from the automatic top results in a Google search. The issue with that is that the definition Google gave me was not very rigorous which made me forget about the fact that I need to care about the topologies when talking about the pure topological definition. Here was the definition Google gave

> $f : X \to Y$ is continuous if the pre-image of every open set in Y is open in X

When trying to prove this, I forgot to actually think about the topologies. I was thinking that if I wanted to prove that the $\epsilon - \delta$ definition implies the topological definition, I would simply be proving that the pre-image of every open intervals in R (which comes from the fact that every open set is the union of open intervals) under f such that every element in the open interval has an element in the domain of the function which corresponds to that element is strictly an open set which is NOT TRUE. If that's the case, then any function from the rationals to the reals is discontinuous which is not necessarily true (I still thought it was true, though) and that functions with only one possible input cannot be continuous. Heck, by that definition, constant functions will not be continuous since there is no open interval in R which contains only one element which is required for the part which says that "every element in the open interval has an element in the domain of the function which corresponds to that element in the open interval."

A part (since most of it simply happened from sheer stupidity) of the previous train of thought makes a tiny bit more sense when looking at my actual misunderstanding. When you have a function $f : A \to \mathrm{R}$ where $A \subseteq \mathrm{R}$, then if you're trying to describe it's continuity using the topological definition of continuity, you would have to describe the topology to use for A. This would mean that earlier, I was thinking that you would always use the same topology that you would use for R. The topology in which open sets are the way we know them to be but that is not the case. If it were the case, then the topological statement would imply that the pre-image of every open interval in R is a strictly open set in A.

The right topology to use is something called the "subset topology" or "relative topology" for A which is the topology in which open sets in A denoted S is defined as any set where there exists an open set in $\mathbb{R}$ that you can intersect with A to be strictly equal to S. More formally, if $(\mathbb{R}, \tau)$ is the standard topological space for $\mathbb{R}$ and $A \subseteq \mathbb{R}$, then the relative topology of A is denoted $\mathbb{O}$ is denoted $\mathbb{O} = \{U \cap A | U \in \tau\}$. For example, if A is the set of all rationals, then intuitively, you would want open sets in $\mathbb{Q}$ to be any set S such that for all $q_0 \in S$ it follows that there exists an $\epsilon > 0$ such that for any *rational* $q \in (q_0 - \epsilon, q_0 + \epsilon)$ it follows that $q \in S$.

This makes a lot of sense when going back to the $\epsilon - \delta$ definition of continuity. Suppose f is continuous by the $\epsilon - \delta$ definition of limits. Then for any $a \in A, \epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in (a - \delta, a + \delta) \cap A$ implies $f(x) \in (f(a) - \epsilon, f(a) + \epsilon)$. This is basically saying that the pre-image of the open interval $(f(a) - \epsilon, f(a) + \epsilon)$ strictly contains the set $(a - \delta, a + \delta) \cap X$ which is an open set under the space (A, τ_A) and so it lines up with the topological definition since it follows the pattern that is the pre-image of an open set is related to open sets. You can just go a bit further from there to prove that the $\epsilon - \delta$ definition implies the topological definition.

I suppose I simply wasn't thinking much about how much it makes sense since my mind was just thinking of it as facts. That it HAS to be true and I'm just being stupid for not being able to prove it. One thing which also caused me to get even more confused is the fact that I actually did not have a very good understanding of the funtion declaration notation. I thought that $f : X \longrightarrow Y$ doesn't necessarily mean that f is defined for all of X and on some occasions, I even thought that it is deined for all X but then Y is xtrictly equal to $f(X)$.

The lesson to be taken here for me after going through all of that is

- Understand the "function declaration notation" better. $f : X \longrightarrow Y$ is saying that f is defined for all elements of X.
- Be more mindul about the specific topologies you're dealing with when dealing with topological spaces in topology.
- Be more mindful when looking at definitions/problems especially when it comes from unfamiliar sources. If it comes from unfamiliar sources, then the statement of the problem from that source may depend on previous assumptions, concepts, notations and so on defined on the source which may cause some confusion just like what happened in here.

Anyway, the following will be my proof that these two definitions are equivalent.

> **Definition 3 (Standard Topology of $\mathbb{R}$).** τ is the standard topology if R if τ is the set of all open sets in $\mathbb{R}$

> **Definition 4 (Relative Topology).** If (X, τ) is a topological space and $A \subseteq X$, then the relative topology of A under τ is denoted $\tau_A = \{U \cap A | U \in \tau\}$

Notice that this implies any $B \in A$ is an open set in (A, τ_A) if and only if, for all $b \in B$, there exists an ϵ such that $(b - \epsilon, b + \epsilon) \cap A \subseteq B$

> **Lemma 1 (Open Sets are Intersections of Open Intervals).** Suppose A is an open set in $\mathbb{R}$, then there exists a set of open intervals S such that $\bigcup_{I \in S} I = A$ and for all $a \in A$, there exists an $I \in S$ where $a \in I$.

Proof. By definition, A satisfy for all $a \in A$ there exists an $\epsilon > 0$ such that for all $x \in (a - \epsilon, a + \epsilon)$, $x \in A$. Now define $S = \{(a - \epsilon, a + \epsilon) | a \in A, \epsilon > 0 \text{ satisfies } (a - \epsilon, a + \epsilon) \subseteq A\}$. Since such an ϵ

always exists for any $a \in A$, then there is always an open interval I in S which contains a for any $a \in A$ (which is already one part of the lemma proved) meaning $a \in \bigcup_{I \in S} I$ and so A is a subset of the union. By definition, everything in S is an interval in the form of $(a - \epsilon, a + \epsilon) \subseteq A$ so since we're literally just unionizing a set of subsets of A then the result, $\bigcup_{I \in S} I$ must also be a subset of A.

> **Theorem 1 ($\epsilon - \delta$ Continuity is Equivalent to Topological Continuity).** Suppose (R, τ) is the standard topological space of the reals and (X, τ_X) is a topological space with $X \subseteq \mathrm{R}$ and τ_X is the relative topology of X under R and there's the function $f : X \longrightarrow \mathrm{R}$. Then f satisfy for all $x_0 \in \mathrm{X}, \epsilon > 0$ there exists a $\delta > 0$ where for all $x \in (x_0 - \delta, x_0 + \delta) \cap X$ it follows that $f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$ if and only if the pre-image of every open set in (R, τ) is open in (X, τ_X)

Proof. ($\Longrightarrow$) Suppose f satisfy for all $x_0 \in X, \epsilon > 0$ there exists a $\delta > 0$ where for all $x \in (x_0 - \delta, x_0 + \delta) \cap X$ it follows that $f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$. We want to prove that

- Take any open set B in R and denote it's preimage under f to be $A = f^{-1}(B)$ which is known to be a subset of X.
- If A is empty, then we already know that it's an open set.
- If A is non-empty, we want to prove that A is open under the topology τ_X.
- We know that for all $a \in A$, since $a \in X$ as well, then for all $\epsilon > 0$ there exists a $\delta > 0$ such that $f((a - \delta, a + \delta) \cap X) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$
- By the openness of B, we know that for all $b \in B$ there exists an $\epsilon > 0$ where $(b - \epsilon, b + \epsilon) \in B$
- Since $f(a) \in B$, we can pick the same epsilon as before and the corresponding delta and know that $B \supseteq (f(a) - \epsilon, f(a) + \epsilon) \supseteq f((a - \delta, a + \delta) \cap X)$ which implies $(a - \delta, a + \delta) \cap X \subseteq A$
- It can be concluded that A is closed under the relative topology τ_X since the previous bullet point applies for all $a \in A$.

($\Longleftarrow$) Suppose the pre-image of every open set in (R, τ) is open in (X, τ_X).

- We want to prove that for any $x_0 \in X, \epsilon > 0$ there exists a $\delta > 0$ such that $x \in (x_0 - \delta, x_0 + \delta) \cap X$ implies $f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$
- Take any $x_0 \in X, \epsilon > 0$. Since $(f(x_0) - \epsilon, f(x_0) + \epsilon)$ is an open set since it's an open interval, then it's pre-image $S = f^{-1}((f(x_0) - \epsilon, f(x_0) + \epsilon))$ is known to also be an open set under the space (X, τ_X) and it also known that $x_0 \in S$ because S is the pre-image of a set which contains $f(x_0)$.
- Since S is an open set under the space (X, τ_X) and $x_0 \in S$ then by definition there exists a $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta) \cap X$ it follows that $x \in S$ therefore $f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$
- Since such a delta can be found for any $x_0 \in A, \epsilon > 0$, then we have proven that the $\epsilon - \delta$ definition is also true.

And yeah we're done with everything now. Note that the lemma didn't actually end up being used. I thought it would end up being used since I wrote it before writing fully thinking out the actual proof. I'll keep it in here because why not.