CS325 - Project 1

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Proof of Claim 1

> Claim 1: y_i is not visible iff $\exists j, k$ such that $j < i < k$ and $y* > m_i x * + b_i$ where $(x*, y*)$ is the intersection of y_j and j_k.

$A \equiv y_i$ is not visible
$B \equiv \exists j, k$ such that $j < i < k$ and $y* < m_i x * + b_i$ where $(x*, y*)$ is the intersection of y_j and y_k.
$A \Leftrightarrow B$

First Prove $A \Rightarrow B$

Direct Proof:

Let y_i be a line that is not visible.
Then $l < i < n$ because y_i and y_n are always visible.
Let k be the smallest index greater than i such that y_k is visible.
e.g. $y_1, y_2, ..., y_k, y_{k+1}, ..., y_{n-1}, y_n$
Let $(x*, y*)$ be the left most point on y_k that is visible.
Let j be the greatest index such that y_i intersects y_k at $(x*, y*)$ is visible.
Because y_i through y_{k-1} are not visible (by definition of k_j) $j < i < k$.
Since $x*, y*$ is visible and y_i is not visible, $m_i x + b_i < y*$.

Prove $B \Rightarrow A$

Direct Proof:

Since $m_i < m_k$, the intersection point of y_i and y_k is left of $x*$.
Since $m_i < m_k$, $m_i x + b_i < m_k x + b_k$ $\forall x > \bar{x}$.
Likewise since $m_i > m_j$, y_i and y_j intersect at $(\bar{\bar{x}}, \bar{\bar{y}})$ right of $x*$ $(\bar{\bar{x}} > x*)$.
$\therefore m_i x + b_i < m_j x + b_j$; $\forall x < \bar{\bar{x}}$.
$\therefore y_i$ is not visible.
$y_k + y_j$ intersect at $m_k x + b_k = m_j x + b_j$
$x = \dfrac{(b_j - b_k)}{(m_k - m_j)}$
Is $m_j \left(\dfrac{b_j - b_k}{m_k - m_j} \right) + b_j > m_i \left(\dfrac{b_j - b_k}{m_k - m_j} \right) + b_i$
If $m_k > m_j$ then instead compare $m_j(b_j - b_k) + b_j(m_k - m_j) > m_i(b_j + b_k)$

Proof of Claim 2

> Claim 2: If $\{y_1, y_2, ..., y_{j_t}\}$ is the visible subset of $\{y_1, y_2, ..., y_{i-1}\}(t \leq i-1)$ then $\{y_1, y_2, ..., y_{j_k}, y_i\}$ is the visible subset of $\{y_1, y_2, ..., y_i\}$ where y_{j_k} is the last line such that $y_{j_k}(x*) > y_i(x*)$ where $(x*, y_{j_k}(x*))$ is the point of intersection of lines y_{j_k} and $y_{j_{k-1}}$.

Proof by Inductions:

Known:
The lines with the greatest and least slope magnitudes are always visible.
The array is sorted from least to greatest slope magnitude.
Base Case:
If $i \leq 2$ then all lines are visible.
Inductive Hypothesis:
Let $\{y_1, y_2, ..., y_{j_m}\}$ be the visible subset of $\{y_1, y_2, ..., y_{o-1}\}(t \leq o-1)$ then $\{y_1, y_2, ..., y_{j_n}, y_i\}$ is the visible subset of $\{y_1, y_2, ..., y_i\}$ where y_{j_n} is the last line such that $y_{j_n}(x*) > y_o(x*)$ where $(x*, y_{j_n}(x*))$ is the point of intersection of lines y_{j_n} and $y_{j_{n-1}}$.

If $y_o(x*) > y_o(x*)$ then $y_{j_n}(x*) < y_o(x*)$ then
Assume that any array of length n ¡ i is correctly flagged visible. Apply the Axiom of Induction: A is an array of length o.

If $y_o(x*) > y_o(x*)$ then
If $y_{j_n}(x*) < y_{o+1}(x*)$ then remove $y_{j_n}(x*)$ from the visibility array and recurse.
If $y_{j_n}(x*) \geq y_{o+1}(x*)$

If $i = 3$ then let A$_v$\{y$_1, y_2$\} be the visible subset of set A $\{y_1, y_2\}$.
Let the new set B be $\{y_1, y_2, y_3\}$.
Let the new initial visible set of B, B_{v_0}, be $A_v + y_3$, that is $\{y_1, y_2, y_3\}$.
It's initial visible subset will be A$_v$.
Let(x*, y$_2(x*)$) be the point of intersection of y_2 and $y_{2-1} = y_1$.
Then let $y_{2+1}(x*) = y_3(x*)$.
Then let y_3 be y coordinate at point $(x*, y_3(x*))$, above the intersection of y_2 and $y_{2-1} = y_1$.
If $y_3(x*) > y_2(x*)$ then remove y_2 from B_{v_0} and
Otherwise add y_3 to B_{v_0}

Group Proof of Claim 2

Claim 2: If $\{y_1, y_2, ..., y_{j_t}\}$ is the visible subset of $\{y_1, y_2, ..., y_{i-1}\}(t \leq i-1)$ then $\{y_1, y_2, ..., y_{j_k}, y_i\}$ is the visible subset of $\{y_1, y_2, ..., y_i\}$ where y_{j_k} is the last line such that $y_{j_k}(x*) > y_i(x*)$ where $(x*, y_{j_k}(x*))$ is the point of intersection of lines y_{j_k} and $y_{j_{k-1}}$.

Proof by Induction:

Let A be $\{y_1, y_2, ..., y_{i-1}\}$ and A^+ be $\{y_1, y_2, ..., y_i\}$.
Let V be $\{y_{j_1}, y_{j_2}, ..., y_{j_t}\}$ and V^+ be $\{y_{j_1}, y_{j_2}, ..., y_{j_k}, y_i\}$.

Base Case:

If $size(A) < 2$, V^+ trivially contains V and y_i because none of the line's slopes are the same and one or two lines with different slopes cannot cover one another.

Inductive Hypothesis:

The claim is true when A (and V) have at least two lines.

Apply the Axiom of Induction:

Pick off smallest with $size(A) = size(V) \geq 2$.
There are two possibilities:

1. y_i does not cover a line in V
   So, $V = \{y_1, ..., y_{z-1}, y_z\}$ and $y_z(x^*) \geq y_i$ where x^* is the intersection of y_{z-1} and y_z
   So, y_z is not covered. $V^+ = V \cup y_i$.

2. y_i covers a line in V
   So $V = \{y_1, ..., y_{z-1}, y_z\}$ and $y_z(x*) < y_i(x*)$
   So y_z is covered.
   To determine visibility, recurse with same y_i and A but remove y_z from V.