Chapter 2
Functions

2.1 Review of Ordered Pairs and Cartesian Products

Ordered Pair

A *pair* consists of two elements. Some examples of pair are

(Nepal, Kathmandu), {Sita, Ram}, (3, 4), $\{a, b\}$, $\{\{a\}, \{a, b\}\}$.

A pair having one element as the first and the other as the second is called an **ordered pair**. An ordered pair having a as the first element and b as the second element is denoted by (a, b).

An ordered pair (a, b) is generally not the same as the ordered pair (b, a). But, this will happen so when the two elements are **identical**. Thus, (3, 4) is different from (4, 3); but (3, 3) is the same as (3, 3).

Two ordered pairs (a, b) and (c, d) are said to be **equal** *if and only if* $a = c$ and $b = d$.

Cartesian Product

Given two sets A and B, the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the **Cartesian product** of A and B, and is denoted by $A \times B$. It is read "A cross B".

In the set-builder notation, we have

$$A \times B = \{(a, b): a \in A, b \in B\}.$$

Relations

Any subset of a Cartesian product $A \times B$ of two sets A and B is called a **relation**. A relation from a set A to a set B is denoted by $x\mathfrak{R}y$, if $x \in A$ and $y \in B$, or simply by $\mathfrak{R}$ if $(x, y) \in \mathfrak{R}$.

In particular, a relation from a set A to itself is called a relation **on** A.

Domain and Range

The **domain** of a relation $\mathfrak{R}$ is the set of all first members of the pairs (x, y) of $\mathfrak{R}$. It is denoted by Dom($\mathfrak{R}$).

Symbolically, $\text{Dom}(\mathfrak{R}) = \{x: (x, y) \in \mathfrak{R} \text{ for some } y \in B\}$

The **range** of a relation $\mathfrak{R}$ is the set of all second members of the pairs (x, y) of $\mathfrak{R}$. It is denoted by Ran($\mathfrak{R}$).

Symbolically, $\text{Ran}(\mathfrak{R}) = \{y : (x, y) \in \mathfrak{R} \text{ for some } x \in A\}$

Inverse Relation

Since a relation is a subset of a Cartesian product, we can think of a set formed by interchanging the first and second members of a relation. This implies that every relation from a set A to a set B has a relation from B to A. Given a relation

$$\mathfrak{R} = \{(x, y): x \in A, y \in B\} \subset A \times B,$$

we can define a relation of the form

$$\{(y, x): y \in B, x \in A\} \subset B \times A.$$

Such a relation is denoted by $\mathfrak{R}^{-1}$, read "script R inverse", and is called **inverse relation** from B to A of $\mathfrak{R}$.

2.2 Function

Consider two sets A and B. Any non-empty subset $\mathcal{R}$ of the Cartesian product A × B is known as a relation from A to B. A special but very important type of relation is that which associates *each element of the set* A *with a unique element of* B. This may therefore be visualized as a *refinement* of the concept of a relation. Let us first see a concrete case of such a refinement before we give a formal definition of what is known as a **function.**

Example 1

Let A = {1, 2, 3} and B = {2, 4, 6}. From the Cartesian product

A × B = {**(1,2)**, (1,4), (1,6), (2,2), **(2,4)**, (2,6), (3,2), (3,4), **(3,6)**},

Let us make the following *selection* (or *refinement*)

{(1, 2), (2, 4), (3, 6)}.

This is a *relation* from A to B. Here each first element is uniquely associated with a second element. The *rule* or *relation* corresponding to the present assignment is

"*To each first element there corresponds a unique second element that is two times the first element.*"

A **function** from a set A to a set B is a *relation* or *rule* that associates each element of A with a *unique* element of B.

In other words, a **function** from a set A to a set B is a relation f such that for each $a \in A$, ∃ a unique $b \in B$ s.t. $(a, b) \in f$.

Here the symbol, " ∃ " is read " *there exists* " and " s.t. " stands for "*such that* ".

Also, a function from a set A to set B is a relation in which no two ordered pairs have the same first coordinate.

Symbolically, a **function** f from a set A to a set B is denoted by

$$f: A \to B \quad \text{or} \quad A \xrightarrow{f} B.$$

If $y \in B$ is associated with an element x of A, we write it as

$$y = f(x),$$

which is read " y equals f of x ". Here $f(x)$ is known as the **image** of f at x or **value** of f at x.

If b is the unique element of B corresponding to an element a of A under the function f, a is called the **pre-image** of b under f and b the **image of** a under f. We also write

$$b = f(a).$$

It is to be noted that if f is a function from A to B, then **no** element of A is related to **more than one** element of B; although **more than one** element of A can be related to the **same** element of B. In each case, A is called the **domain** of f and B its **co-domain.**

The subset of B that contains only those elements of B that have pre-images in A is often called the **range** of f. Obviously,

$$\text{range of } f \subseteq B.$$

Thus in the function $f: A \to B$, the set of values of $x \in A$ for which the function is defined, is said to be the domain of the function and is denoted by $D(f)$.

$\therefore D(f) = \{x: x \in A \text{ for which } f(x) \text{ is defined}\}$

For example: If $y = f(x) = \sqrt{x-1}$ then $f(x)$ is defined for $x - 1 \geq 0$ i.e. $x \geq 1$.

$\therefore D(f) = \{x: x \geq 1\} = [1, \infty)$

The set of values of $y \in B$ corresponding to each $x \in A$ which runs over the domain of the function is said to be the range of the function and is denoted by $R(f)$.

$\therefore R(f) = \{f(x): x \in D(f)\}$

For example: If $y = f(x) = \sqrt{x-1}$
then the range of $f = R(f) = \{y: y \geq 0\} = [0, \infty)$.

In the function $f: A \to B$, A itself is the domain of f and the set of values of $y \in B$ which are the images of the elements of A, is the range of f. So, range is, of course a subset of B.

A function is also called a **mapping** or in special contexts, a **transformation** or an **operator**.

The fact that y is the image of an element x under a function f is also indicated by

$$x \mapsto y \quad \text{or} \quad x \mapsto f(x).$$

Now, we have the following definitions

Function: A relation is said to be a function from set A to set B if every element of set A associates with a unique element of set B. A function from set A to set B is denoted by $f: A \to B$.

2.3 Domain and Range of a Function

Domain of the function: The set A is known as the domain of the function. The domain of the function is denoted by dom (f). Thus,

$$\text{dom } (f) = \{x: x \in A\}$$

Co-domain of the function: The set B is known as the co-domain of the function.

Range of the function: The set of values of $y = f(x) \in B$ for every $x \in A$ is known as the range of the function f. It is denoted by range (f). Thus,

$$\text{range } (f) = \{y: y \in B, y = f(x) \text{ for all } x \in A\}$$

Image: The element $y \in B$ with which the element $x \in A$ associates, is known as the image of x under f. It is also known as the value of f at x.

Pre-image: The element $x \in A$ which associates with $y \in B$, is known as the pre-image of y under f.

Equal function: Two functions f and g are said to be equal i.e. $f = g$ if domain of f = domain of g and $f(x) = g(x)$ for all x belonging to the domain of f (or domain of g).

Types of Functions

There are three types of functions that are of special importance.

a) One to one or Injective Function:

A function f from a set A to another set B i.e. $f: A \to B$ is said to be **one to one (1-1)** or **injective** if **distinct elements** (or **pre-images**) in A have **distinct images** in B.

In symbols, for any $x, y \in A$,

$$x \neq y \quad \Rightarrow \quad f(x) \neq f(y);$$

or, equivalently, $$f(x) = f(y) \quad \Rightarrow \quad x = y.$$

In other words, a function f is said to be **one-one** or **injective** if

$$(x, f(x)), (y, f(y)) \in f \quad \Rightarrow \quad x = y.$$

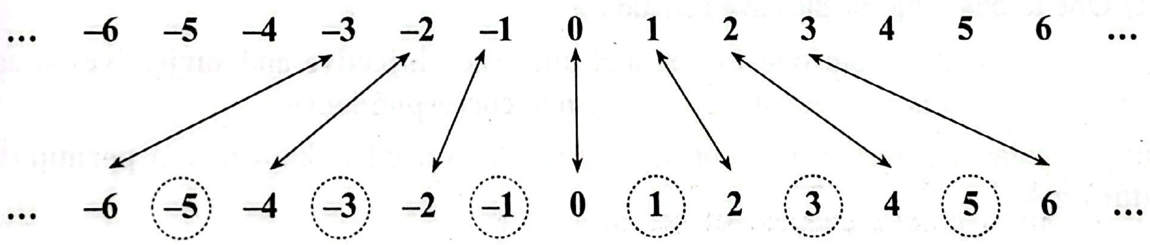
Thus, under one to one function all elements of A are related to different elements of B.

Examples:

1. The function

$$f: Z \to Z, \qquad \text{where } Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

defined by $f(x) = 2x$ and diagrammatically represented by



is one to one, since

$$f(x) = f(y) \quad \Rightarrow \quad 2x = 2y \quad \Rightarrow \quad x = y.$$

b) Onto or Surjective Function

A function f from a set A to another set B i.e., $f: A \to B$ is said to be **onto** or **surjective, if every element** of B is an image of **at least one element** of A , i.e., every element of B has a pre-image or, if $f(A) = B$.

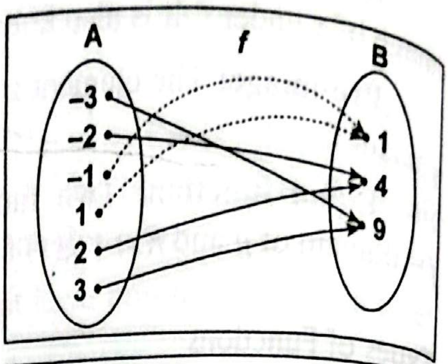
Sometimes such a function becomes a **many-one onto function.**

Examples:

1. If A = {–3, –2, –1,1, 2, 3} and B = {1, 4, 9}, then the function

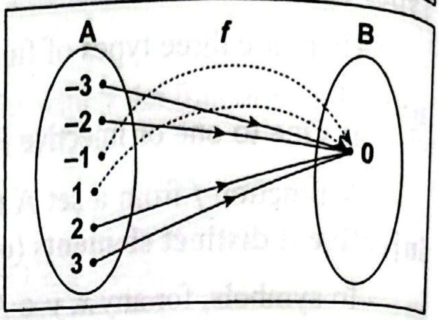
$$f: A \to B,$$

defined by $f(x) = x^2$, is **onto** (moreover, it is **many-one onto**). Here, $f(A) = B$.

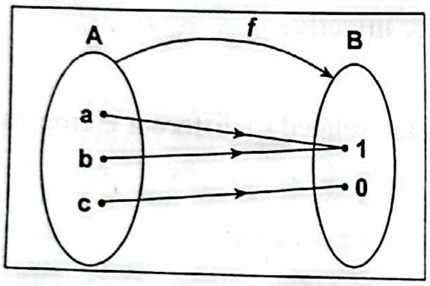


2. A special case of many-one onto or simply onto function is the function $f: A \to B$ such that $f(x) = c \in B$ for every $x \in A$. Such a function is called a **constant function.** The following figure shows a constant function

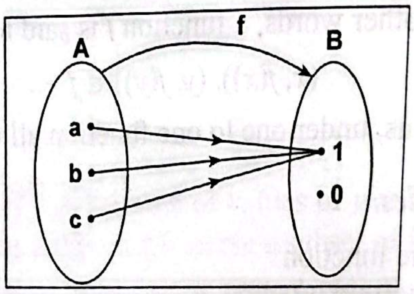
$$f: A = \{-3, -2, -1, 1, 2, 3\} \to B = \{0\}$$



3. The following arrow diagrams show functions defined on the same set A = {*a*, *b*, *c*}. One of them is onto and the other is into



Onto $f(A) = B$



Into $f(A) \subset B$

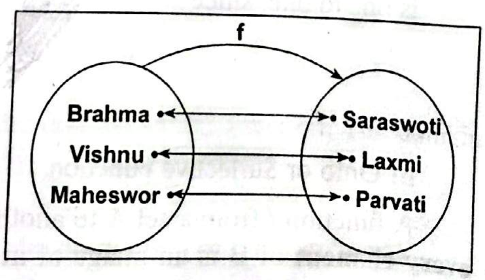
c) One to one Onto or Bijective Function

A function that is both **one to one** and **onto** (i.e., **injective** and **surjective**) is called a **bijective** function. It is also known as a **one-to-one correspondence.**

In particular, a bijective function from a set A to itself is known as a **permutation** or **operator on** A.

Examples:

1. Consider two sets A = {Bramha, Vishnu, Maheswor} and B = {Saraswoti, Laxmi, Parvati}. The following arrow diagram shows a one-to-one correspondence or a bijective function from A to B.



2. Consider a function $f: A \to A$ defined by

$$f(x) = x \qquad \text{for all } x \in A$$

Obviously, it is well-defined and is *both one-one and onto*. That is, it is a bijective function. This function is known as the **identity function.**

Examples:

1. If A = {−3, −2, −1, 1, 2, 3} and B = {1, 4, 9}, then the function

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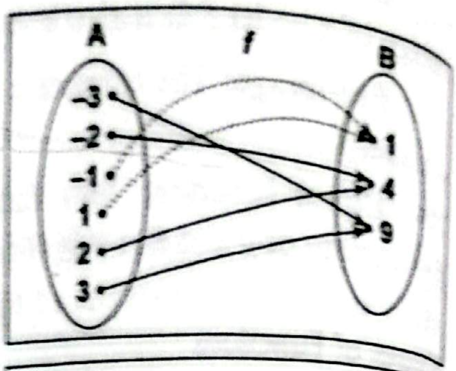

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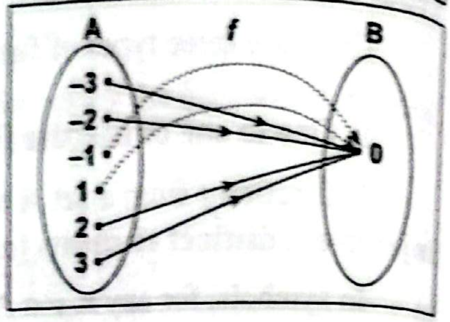

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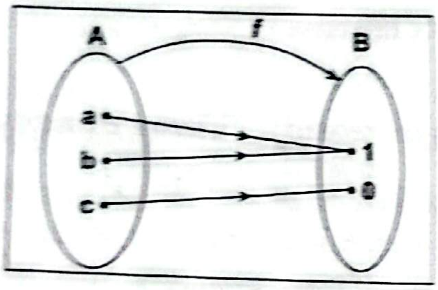

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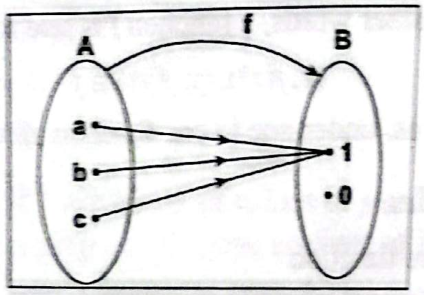

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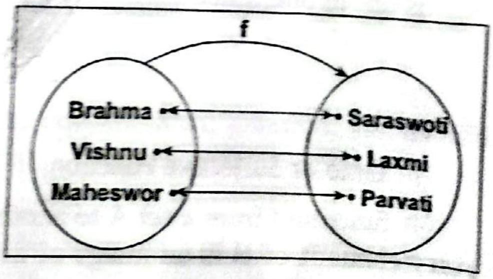

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$$f(x) = x \qquad \text{for all } x \in A$$

Obviously, it is well-defined and is *both one-one and onto.* That is, it is a bijective function. This function is known as the **identity function.**

3. The function $f: N \to \{3\}$ defined by

$$f(x) = 3 \qquad \text{for all } x \in N,$$

the set of counting numbers, is obviously well defined. It is not one to one but onto. So, it is a not a bijective function.

Worked Out Examples

Example 1

Let A = {a, b, c} and B = {x, y, z, p, q}. Which of the following relations are functions from A to B. Give reason

a) $R_1 = \{(a, p), (b, q), (c, x)\}$ b) $R_2 = \{(a, p), (a, q), (c, y)\}$

c) $R_3 = \{(a, p), (b, p)\}$

Also, find the domain and the range of the function.

Solution:

a) $a \in A \to p \in B$, $b \in A \to q \in B$ and $c \in A \to x \in B$

Since every element of A associates with a unique element of B, so R_1 is a function from A to B.

Here, domain is A and range is {p, q, x}.

b) In R_2, two ordered pairs have the same first coordinate, so R_2 is not a function

c) Since $c \in A$ associates with no element of B, so R_3 is not a function.

Example 2

Let $f(x) = x + 1$ be a function defined in the closed interval $-1 \le x \le 1$. Find

a) $f(-1)$ b) $f(0)$ c) $f(1)$ d) $f(2)$.

Solutions:

a) $f(-1) = (-1) + 1 = 0$

b) $f(0) = 0 + 1 = 1$

c) $f(1) = 1 + 1 = 2$

d) $f(2)$ is not defined since 2 does not belong to the domain of definition of f (i.e, $\mathcal{D}(f) = [-1, 1]$).

Example 3

Let the function $f: R \to R$ be defined by $f(x) = \begin{cases} 3 + 2x & \text{for } -3/2 \le x < 0 \\ 3 - 2x & \text{for } 0 \le x < 3/2 \\ -3 - 2x & \text{for } x \ge 3/2 \end{cases}$

Find a) $f(-3/2)$ b) $f(0)$ c) $f(3/2)$

d) $\dfrac{f(h) - f(0)}{h}$ for $0 \le h < 3/2$.

Solutions:

a) Since $-3/2 \in [-3/2, 0)$ or $-3/2 \le x < 0$, we use the first formula

$$f(x) = 3 + 2x.$$

Hence, $f(-3/2) = 3 + 2(-3/2) = 3 - 3 = 0.$

b) Since $0 \in [0, 3/2)$ or $0 \le x < 3/2$, we use the second formula

$$f(x) = 3 - 2x.$$

Hence, $f(0) = 3 - 2.0 = 3.$

c) Since $3/2 \in [3/2, \infty)$ or $3/2 \le x$, we use the third formula

$$f(x) = -3 - 2x.$$

Hence, $f(3/2) = -3 - 2.(3/2) = -6.$

d) For $0 \le h < 3/2$, we have to use the second formula.

Hence, $\dfrac{f(h) - f(0)}{h} = \dfrac{3 - 2h - 3}{h} = \dfrac{-2h}{h} = -2.$

Example 4

Let A = {0, 1, 2, 3, 4, 5, 6} and a function f: A → Q is defined by $f(x) = x/2$. Find the range of f.

Solution:

By given, $f(x) = \dfrac{x}{2}$ and A = {0, 1, 2, 3, 4, 5, 6}

When x = 0, 1, 2, 3, 4, 5, 6 the values of $f(x)$

i.e. $f(0), f(1), f(2), f(3), f(4), f(5)$ and $f(6)$ are $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$ and 3 respectively.

∴ range of $f = \{f(0), f(1), f(2), f(3), f(4), f(5), f(6)\}$

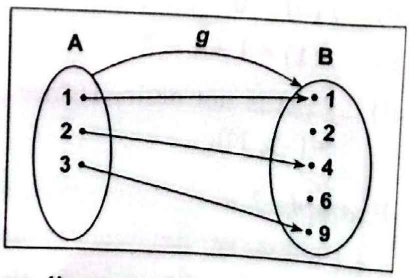
∴ $f(A) = \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\right\}$

Example 5

Determine whether the functions f and g defined below are equal or not:

a) $f(x) = x^2$ where $\{x : x = 1, 2, 3\}$

b) Venn-diagram



Solution:

The functions f and g are defined by

$f(x) = x^2$ where $\{x: x = 1, 2, 3\}$ and the Venn-diagram.

Since f and g have the same domain {1, 2, 3} and assign the same image to each element in the domain, so $f = g$.

Example 6

Let a function $f: A \to A$ be defined by $f(x) = x^3$, where $A = \{-1, 0, 1\}$. Find the range of the function. Is the function one to one, onto or both ?

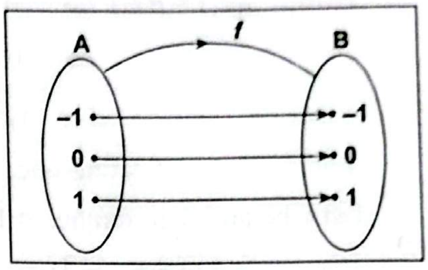
Solution:

When $x = -1$, $f(-1) = (-1)^3 = -1$

$x = 0$, $f(0) = 0^3 = 0$

$x = 1$, $f(1) = 1^3 = 1$

$\therefore$ the range of $f = R(f) = \{-1, 0, 1\}$



Since different elements of A (domain) have different images in A (co-domain) so f is one to one function.

Also since, $f(A) = A$, so f is onto. Hence f is one-one and onto both.

Example 7

Let $f: R \to R$ be defined by $f(x) = |x - 1|$. Show that f is neither one-one nor onto function.

Solution:

$$f(x) = |x - 1|$$

Consider two element $2, 0 \in R$.

Then, $f(2) = 1$ and $f(0) = 1$

Since $f(2) = f(0) \not\Rightarrow 2 = 0$

so, f is not one-one function.

For all $x \in R$, the set of values of $f(x)$ are the non-negative real numbers.

So, range of $f = [0, \infty) \subset R$

$\therefore f$ is not onto.

Hence f is neither one-one nor onto function.

Example 8

Let $f: N \to N$ be defined by $f(x) = 3x$ for all $x \in N$ where N is the set of natural numbers. Show that f is one-one but not onto function.

Solution:

Let $x_1, x_2 \in N$

then $f(x_1) = f(x_2) \quad \Rightarrow 3x_1 = 3x_2 \quad \Rightarrow x_1 = x_2$

$\therefore f$ is one-one function.

Again, $f(x) = 3x$

$\Rightarrow \quad y = 3x$, where y is any element $\in N$

$\Rightarrow \quad x = \frac{1}{3}y$

But, for some $y \in N$, $x \notin N$, so f is not onto.

Example 9

Prove that a function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x - 1$ is one-one and onto (i.e. bijective).

Solution:

Let $x_1, x_2 \in \mathbb{R}$.

Then $f(x_1) = 3x_1 - 1$ and $f(x_2) = 3x_2 - 1$

Now, $f(x_1) = f(x_2) \Rightarrow 3x_1 - 1 = 3x_2 - 1$

$\Rightarrow 3x_1 = 3x_2$

$\Rightarrow x_1 = x_2$

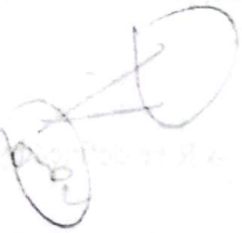
$\therefore$ f is one-one.

Let y be any real number $\in \mathbb{R}$. Then,

$f(x) = y = 3x - 1$

or, $3x = y + 1$

$x = \dfrac{y+1}{3} \in \mathbb{R}$ for all $y \in \mathbb{R}$.

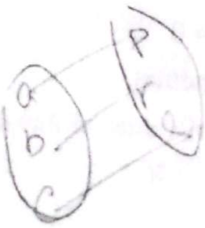


and $f\left(\dfrac{y+1}{3}\right) = 3.\left(\dfrac{y+1}{3}\right) - 1 = y$

$\therefore$ y is the image of $\dfrac{y+1}{3}$.

$\therefore$ f is onto.

Hence f is one-one and onto (i.e. bijective).



EXERCISE 2.1

1. Let X = $\{a, b, c\}$ and Y = $\{p, r, s\}$. Determine which of the following relations from X to Y are functions. Give reasons for your answers:

   a) $R_1 = \{(a, p)\ (a, r), (b, r), (c, s)\}$
   b) $R_2 = \{(a, p), (b, r)\}$
   c) $R_3 = \{(a, s), (b, s), (c, s)\}$
   d) $R_4 = \{(a, p), (b, p), (c, s)\}$
   e) $R_5 = \{(a, p), (b, r), (c, s)\}$

   Find the domain and range of these relations which are functions.

2. If $f: A \to B$ where A and B $\subset \mathbb{R}$, is defined by $f(x) = 1 - x$, find the images of $1, \frac{3}{2}, -1, 2 \in A$.

3. Let i) $f(x) = x + 2$ ii) $f(x) = 2|x| + 3x$ in the interval $-1 \le x \le 2$. Find

   a) $f(-1)$ b) $f(0)$ c) $f(1)$
   d) $f(2)$ e) $f(-2)$ f) $f(3)$.

4. i) Let the function f: R → R be defined by

$$f(x) = \begin{cases} 3 + 2x & \text{for } -1/2 \le x < 0 \\ 3 - 2x & \text{for } 0 \le x < 1/2 \\ -3 - 2x & \text{for } x \ge 1/2 \end{cases}$$

Find a) $f(-1/2)$ b) $f(0)$ c) $f(1/2)$

d) $\dfrac{f(h) - f(0)}{h}$ for $0 \le h < 1/2$.

ii) Let the function f: R → R be defined by $f(x) = \begin{cases} 4x - 2 & \text{for } x \ge 1 \\ 2x & \text{for } x < 1 \end{cases}$

Find a) $f(2)$ b) $f(1)$ c) $f(0)$ d) $f(-1)$

e) $\dfrac{f(h) - f(1)}{h}$ for $1 < h$.

5. Let A = {–1, 0, 2, 4, 6} and a function f: A → R is defined by

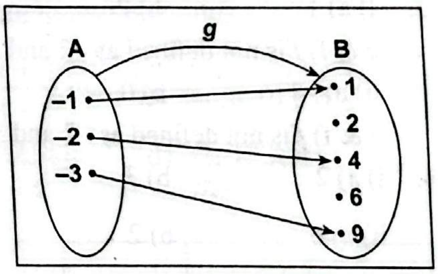
i) $y = f(x) = \dfrac{x}{x+2}$ ii) $y = f(x) = \dfrac{x(x+1)}{x+2}$

Find the range of f.

6. Determine whether the functions f and g defined below are equal or not

a) $f(x) = x^2$ where A = { x: x = –1, –2, –3}

b) Venn-diagram



7. Determine whether or not each of the following functions is one to one.

a) Let A = {– 2, – 1, 0, 1, 2} and B = {4, 0, 1}. A function f: A → B is defined by

$$\left.\begin{matrix} f(-2) \\ f(2) \end{matrix}\right\} = 4, \qquad f(0) = 0, \qquad \left.\begin{matrix} f(-1) \\ f(1) \end{matrix}\right\} = 1$$

b) Let A be the set of positive integers and B the set of squares of positive integers. A function f: A → B is defined by $f(x) = x^2$;

c) f: [–2, 2] → R is defined by $f(x) = x^2$

d) f: [0, 3] → R is defined by $f(x) = x^2$

8. Examine whether the following functions are one to one, onto, both or neither.

i) f: A → B defined by $f(x) = x^2$ where A = {1, –3, 3} and B = {1, 9}

ii) f: N → N defined by $f(x) = 2x$

iii) f: (–2, 2) → R defined by $f(x) = x^2$

iv) f: Q → Q defined by $f(x) = 6x + 5$

v) f: R → R defined by $f(x) = x^3$

9. a) Let a function f: A → B be defined by $f(x) = \frac{x^2}{6}$ with A = {–2, –1, 0, 1, 2} and B = $\{0, \frac{1}{6}, \frac{2}{3}\}$. Find the range of f. Is the function f one to one and onto both ?

b) Let a function f: A → B be defined by $f(x) = \frac{x+1}{2x-1}$ with A = {–1, 0, 1, 2, 3, 4} and B = $\{-1, 0, \frac{4}{5}, \frac{5}{7}, 1, 2, 3\}$. Find the range of f. Is the function f one to one and onto both? If not, how can the function be made one-one and onto both?

Answers

1. a) No, first two ordered pairs have the same first coordinate.
   b) No, c ∈ X corresponds with no element of Y.
   c) Yes, every element of X associates with a unique element of Y; D(f) = $\{a, b, c\}$, R(f) = $\{s\}$
   d) Yes, Same as c. D(f) = $\{a, b, c\}$, R(f) = $\{p, s\}$
   e) Yes, same as (c); D(f) = $\{a, b, c\}$, R(f) = $\{p, r, s\}$
2. $0, -\frac{1}{2}, 2, -1$
3. i) a) 1 b) 2 c) 3 d) 4
   e & f) f is not defined as –2 and 3 do not belong to thedomain of definition
   ii) a) –1 b) 0 c) 5 d) 10
   e & f) f is not defined as –2 and 3 do not belong to the domain of definition.
4. i) a) 2 b) 3 c) –4 d) – 2
   ii) a) 6 b) 2 c) 0 d) –2 e) $\frac{4(h-1)}{h}$
5. a) $\{-1, 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\}$ b) $\{0, \frac{3}{2}, \frac{10}{3}, \frac{21}{4}\}$
6. Since $f(x) = \{(-1, 1), (-2, 4), (-3, 9)\} = g(x)$, the two functions are equal
7. a) No, since $f(-2) = 4 = f(2)$ does not imply $-2 = 2$
   b) Yes, $x \neq y$ implies $x^2 \neq y^2$. So f is one to one.
   c) No, since $f(-2) = 4 = f(2)$ does not imply $-2 = 2$. That is f is not one to one.
   d) Yes, $x \neq y$ implies $x^2 \neq y^2$. So f is one to one.
8. i) onto function ii) one to one function iii) Neither
   iv) one to one and onto function v) one to one and onto function
9. a) R(f) = $\{0, \frac{1}{6}, \frac{2}{3}\}$, onto only b) R($f$) = $\{-1, 0, \frac{4}{5}, \frac{5}{7}, 1, 2\}$; No, one to one only

Multiple Choice Question

1. Let A = $\{a, b, c\}$ and B = $\{x, y, z\}$. From the relations given below, the fun
B is
a) $\{(a, x), (b, y), (c, z)\}$ b) $\{(a, x), (a, y), (b, z)\}$
c) $\{(a, x), (b, y), (b, z)\}$ d) $\{(a, x, (a, y), (a, z)\}$

range ...

a) $\{-1, 0, 1, 2, 3\}, \{1, -2\}$

c) $\{-1, 2\}, \{1, -1\}$

d) $\{2, 3\}, \{10, -1\}$

6. The function $f: R \to R$ defined by $f(x) = x^2$ is

a) one to one

b) onto

c) one to one and onto

d) neither one to one nor onto

7. Let $f: A \to R$ where $A = \{0, 2, 3, 4\}$ is defined by $f(x) = \dfrac{x+2}{x-1}$ then the range of f is

a) $\{-2, 2, 3, 4\}$

b) $\{-2, 1, 2, 4\}$

c) $\{-2, 2, 5/2, 4\}$

d) $-2, 1, 3, 4\}$

Answers

1. a	2. c	3. b	4. d	5. a	6. b	7. c			

2.4 Inverse Function

The notions of inverse image of an element and inverse relation can be easily extended to the case of a function also.

a) Inverse Image of An Element

Given a function $f: A \to B$

the **inverse image** of an element $y \in B$ with respect to f is defined as the set of elements in A which have y as their image. It is usually denoted by $f^{-1}(y)$ and is read "*f inverse of y*".

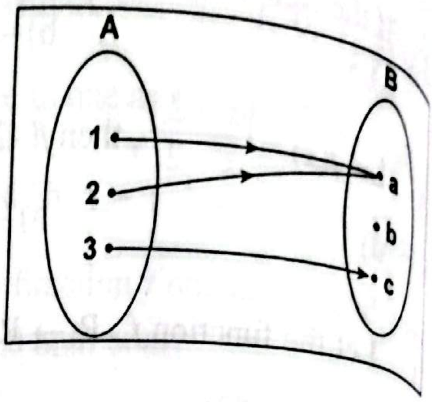
(Note that $f^{-1} \neq 1/f$)

In symbols, if a function is defined by $f: A \to B$, then

$$f^{-1}(y) = \{x \in A : y = f(x)\}.$$

Examples:

1. Let $f: A \to B$ be defined by the arrow diagram



   Then, the inverse of a under f, i.e., $f^{-1}(a) = \{1, 2\}$; $f^{-1}(b)$ is the null set ϕ and $f^{-1}(c) = \{3\}$.

2. Let a function $f: R \to R$ be defined by $f(x) = x^2$.

   Then, $f^{-1}(9) = \{-3, 3\}$, since 9 is the image of both -3 and 3. We further note that $f^{-1}(-1) = \phi$, since there is no real number whose square is -1.

b) Inverse Function

Let $f: A \to B$ be a one to one and onto (i.e., injective and surjective) function. Then, since f is onto, corresponding to each element $b \in B$, there is *at least* one element $a \in A$. But, f is one to one, and so a is the only (or unique) element of A corresponding to the element $b \in B$. We thus have a rule which associates each element b of B with a **unique element** a of A, i.e., a function from B to A. We often denote such a function by f^{-1} (read "*ef*" inverse). In other words we have a function of the type:

$$f^{-1}: B \to A.$$

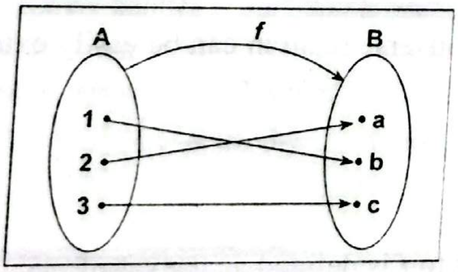
This function is known as the **inverse function** of f.

Thus if $f: A \to B$ be one to one and onto, then a function can be defined from B to A such that every element of B associates with a unique element of A, then the function defined from B to A is known as the inverse function of f and is denoted by f^{-1}.

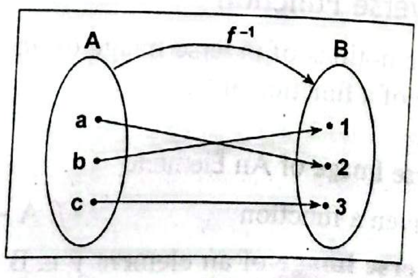
In short, when a function $f: A \to B$ is bijective, there exists a function $f^{-1}: B \to A$ called the inverse function of f.

Examples:

1. Let $f: A \to B$ be a one to one and onto (i.e., injective and surjective) function defined by the following arrow diagram on the left side:



(a)

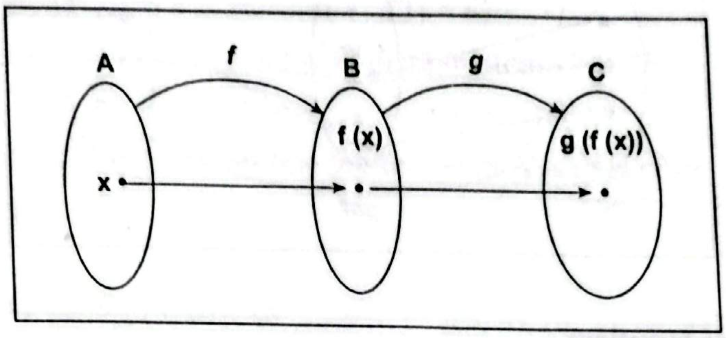


(b)

Obviously, the inverse function of f is well-defined and the arrow diagram on the right represents it.

2.5 Composition of Functions

Let f: A → B and g: B → C be any two functions. We can represent the two functions diagrammatically as shown below:



g(f(x))

Suppose $x \in$ A, then its image (function value) $f(x) \in$ B. B being the domain of g, we can find the image of $f(x) \in$ B under g, that is we can find $g(f(x))$ belonging to C. In other words, we can associate an element $x \in$ A with a unique element $g(f(x)) \in$ C. Consequently, we have a function from A to C. This new function is known as the **composite function** of f and g (**not** g **and** f); and it is denoted by

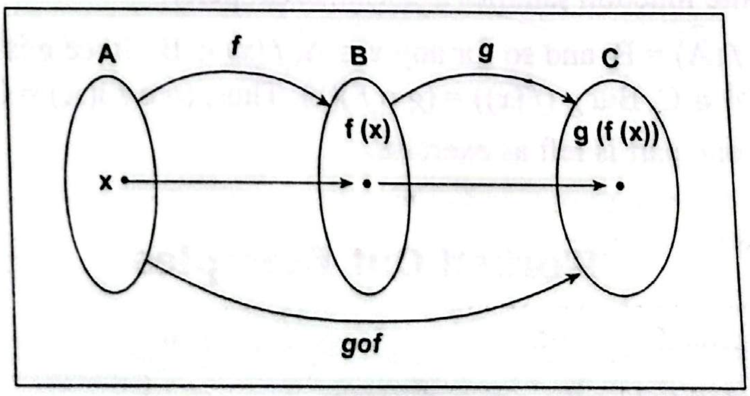
$$(gof), \text{ (read } g \textit{ oh } f) \text{ or } (g(f)).$$

In short, if f: A → B and g: B → C be any two functions, then the **composite function** of f and g (also known as the **product function** or **function of a function**) is the function,

$$g \circ f: A \to C \quad (\text{read " } g \textit{ oh } f \text{ from A to C "})$$

defined by the equation $(g \circ f)(x) = g(f(x))$.

Schematically, the situation described above may be illustrated by the following diagram:



Examples

1. Let A, B and C denote the sets of real numbers. Suppose

$$f: A \to B \text{ and } \qquad g: B \to C$$

are defined by $f(x) = x - 1$ and $g(x) = x^2$. Then,

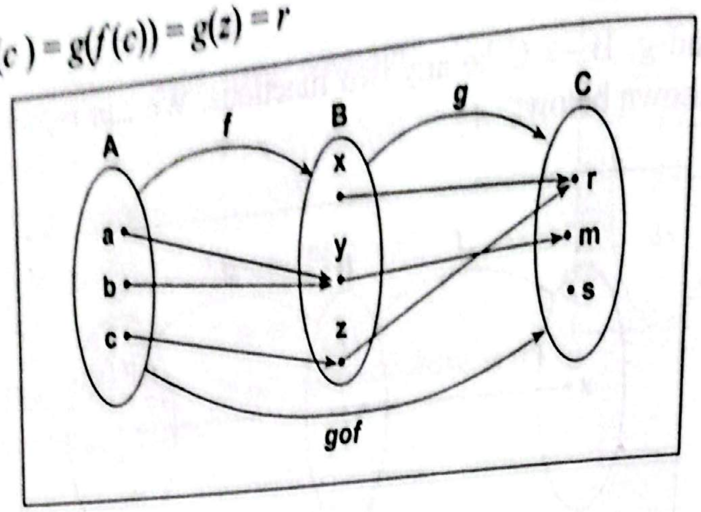
$$(g \circ f)(x) = g(f(x)) = g(x - 1) = (x - 1)^2.$$

2. Suppose f: A → B and g: B → C are defined by the following diagram:

Here, $(gof)(a) = g(f(a)) = g(v) = m$

$(gof)(b) = g(f(b)) = g(v) = m$

$(gof)(c) = g(f(c)) = g(z) = r$



Properties of Composite Functions

In what follows, we assume that the product or composite functions do exist.

i) If f: A → B, g: B → C and h: C → D are given functions, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

ii) If f: A → B and g ; B → C and are given functions, then g o f is onto or one-one according as each of f and g is onto or one-one.

A brief sketch of the proof of each property mentioned above is as follows:

i) By assumption each side of $h \circ (g \circ f) = (h \circ g) \circ f$ is well defined. The equality of the two sides can easily be seen if we note that

$$(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$$

and $$((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x))) \quad \text{for any } x \in A.$$

Hence, composite function satisfies associative property.

ii) Since f is onto, $f(A) = B$, and so for any $x \in A$, $f(x) \in B$. Since g is onto, $g(B) = C$ for any $f(x) \in B$, $g(f(x)) \in C$. But $g(f(x)) = (g \circ f)(x)$. Thus, $(g \circ f)(A) = C$.

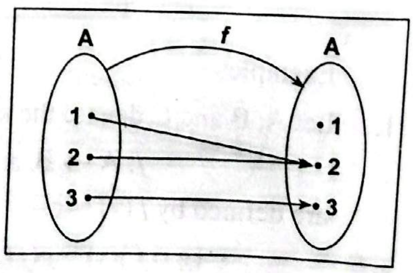
Proof of the second part is left as exercise.

Worked Out Examples

Example 1

Let f: A → A be defined by the arrow diagram

Find a) $f^{-1}(2)$ b) $f^{-1}(3)$ c) $f^{-1}(1)$ d) $f^{-1}(2,3)$.



Solutions:

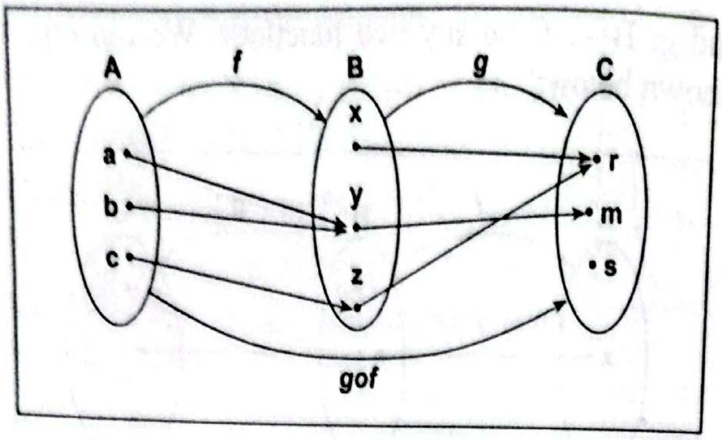
a) $f^{-1}(2) = \{1, 2\}$ b) $f^{-1}(3) = \{3\}$

c) $f^{-1}(1) = \phi$ d) $f^{-1}(2,3) = \{1, 2, 3\}$.

Example 2

If A = {1, 2, 3}, B = {4, 5, 6} and f: A → B is a function such that $f(1) = 4$, $f(2) = 5$ and $f(3) = 6$. Write down f^{-1}: B → A as a set of ordered pairs.

$$(gof)(c) = g(f(c)) = g(z) = r$$



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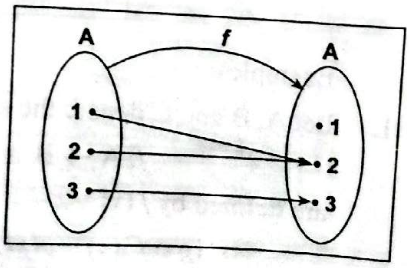

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Solution:

$1 \in A$ corresponds with $4 \in B$ $(\because f(1) = 4)$

$2 \in A$ corresponds with $5 \in B$ $(\because f(2) = 5)$

$3 \in A$ corresponds with $6 \in B$ $(\because f(3) = 6)$

The distinct elements of A correspond with distinct elements of B.

$\therefore$ f is one-one.

Since each element of B has at least one pre-image in A, so f is onto.

$\therefore$ f is one-one and onto.

So, f^{-1} exists.

Since, $f = \{(1, 4), (2, 5), (3, 6)\}$, so $f^{-1} = \{(4, 1), (5, 2), (6, 3)\}$

Example 3

Let a function $f: R \to R$ be defined by $y = f(x) = 2x - 3, x \in R$. Find a formula that defines the inverse function f^{-1}.

Solution:

Let $x_1, x_2 \in R$ (domain).

Then $f(x_1) = 2x_1 - 3$ and $f(x_2) = 2x_2 - 3$

Now, $f(x_1) = f(x_2)$

$\Rightarrow 2x_1 - 3 = 2x_2 - 3$

$\Rightarrow 2x_1 = 2x_2$

$\Rightarrow x_1 = x_2$

$\therefore f$ is one-one function.

Again let $k \in R$

Then, $k = 2x - 3$

$\Rightarrow 2x = k + 3$

$\Rightarrow x = \dfrac{k+3}{2} \in R$

$\therefore f$ is onto function.

Hence f is one-one and onto function. Since f is one-one and onto function, so f^{-1} exists and $f^{-1}: R \to R$ so that x is the image of y under f^{-1} i.e. $x = f^{-1}(y)$

Solving for x in terms of y, we have

$$x = \frac{y+3}{2} \quad \text{or,} \quad f^{-1}(y) = \frac{y+3}{2}$$

which is a formula defining the inverse function. But since y is a dummy variable and can be replaced by x. So, in terms of x, the inverse function is defined by

$$f^{-1}(x) = \frac{x+3}{2}$$

Example 4

a) Let f: R → R and g: R → R be defined by $f(x) = x + 1$ and $g(x) = x^3$. Find $(g \circ f)(x)$ and $(f \circ g)(x)$.

b) Let f: R → R and g: R → R be defined by $f = \{(5, 2), (6, 3)\}$ and $g = \{(2, 5), (3, 6)\}$, find fog and gof.

Solutions:

a) $(g \circ f)(x) = g(f(x)) = g(x + 1) = (x + 1)^3$.

$(f \circ g)(x) = f(g(x)) = g(x^3) = x^3 + 1$.

Clearly, $(g \circ f)(x) \neq (f \circ g)(x)$.

b) **For fog**

$f = \{(5, 2), (6, 3)\}$ and $g = \{(2, 5), (3, 6)\}$ 2~3

$g(2) = 5,\quad g(3) = 6$

and $f(5) = 2,\quad f(6) = 3$

Now, $(fog)(2) = f(g(2)) = f(5) = 2$

$(fog)(3) = f(g(3)) = f(6) = 3$

$\therefore$ fog = $\{(2, 2), (3, 3)\}$

For gof

$f(5) = 2,\quad f(6) = 3$

and $g(2) = 5,\quad g(3) = 6$

Now, $(gof)(5) = g(f(5)) = g(2) = 5$

$(gof)(6) = g(f(6)) = g(3) = 6$

$\therefore$ gof = $\{(5, 5), (6, 6)\}$

Example 5

Find the domain and the range of the following functions:

a) $y = f(x) = x^2 - 6x + 6$

b) $y = f(x) = \frac{1}{x-1}$

c) $y = f(x) = \frac{x^2 - 4}{x - 2}$

d) $y = f(x) = \sqrt{6 - x - x^2}$

Solution:

a) $y = f(x) = x^2 - 6x + 6$

(c) ∞)

The given function is a polynomial of degree two in x. y is defined for all $x \in$ R, so domain of f = dom (f) = R = $(-\infty, \infty)$

Again, $y = x^2 - 6x + 6$

$y + 3 = (x - 3)^2$

$y = -3 + (x - 3)^2$

Since $(x - 3)^2 \geq 0$ so for all $x \in$ R, $y \geq -3$

$\therefore$ range of f = R(f) = $[-3, \infty)$

$[-3, \infty)$

$= \frac{4ac - b^2}{4a}$

$= \frac{4 \times 1 \times 6 - (6)^2}{4 \times 1}, \infty)$

$= \frac{24 - 36}{4}$

$= -\frac{12}{4}$ 3

b) $y = \dfrac{1}{x-1}$ $D(f) = R - \{1\}$

The given function is defined for all values of x except at $x = 1$ For $x \neq 1$,

$\therefore$ domain of the function = D(f) = R – {1}

Again, $y = \dfrac{1}{x-1}$

$\Rightarrow \quad x = \dfrac{1}{y} + 1$

$\Rightarrow \quad x = \dfrac{1+y}{y}$

$y \neq 0$ for all $x \in D(f)$

$\therefore$ range of f = R(f) = R – {0}

c) The function will not be defined when $x - 2 = 0$ i.e. $x = 2$.

So, for all values of x except at $x = 2$, $y = f(x)$ exists.

$\therefore$ domain of the function = D(f) = R – {2}

If $x \neq 2$, then $y = \dfrac{x^2-4}{x-2} = x + 2$

Since $x = 2$ is not in the domain of the function, so $y = 4$ will not be in the range of the function.

$\therefore$ range of f = R – {4}

d) $f(x) = \sqrt{6 - x - x^2}$

$= \sqrt{\dfrac{25}{4} - \left(\dfrac{1}{4} + x + x^2\right)}$

$= \sqrt{\left(\dfrac{5}{2}\right)^2 - \left(x + \dfrac{1}{2}\right)^2}$

For $\left(\dfrac{5}{2}\right)^2 - \left(x + \dfrac{1}{2}\right)^2 < 0$, y will be imaginary. So, y will be defined only for

$\left(\dfrac{5}{2}\right)^2 - \left(x + \dfrac{1}{2}\right)^2 \geq 0$

$\Rightarrow \quad \left(x + \dfrac{1}{2}\right)^2 \leq \left(\dfrac{5}{2}\right)^2$

$\Rightarrow \quad -\dfrac{5}{2} \leq x + \dfrac{1}{2} \leq \dfrac{5}{2}$

$\Rightarrow \quad -3 \leq x \leq 2$

$\therefore$ the domain of the function = [–3, 2]

Again, $y^2 = \left(\dfrac{5}{2}\right)^2 - \left(x + \dfrac{1}{2}\right)^2$

$\Rightarrow \quad \left(x + \dfrac{1}{2}\right)^2 = \left(\dfrac{5}{2}\right)^2 - y^2$

Since $\left(x+\frac{1}{2}\right)^2 \geq 0$ for all $x \in R$

so, $\left(\frac{5}{2}\right)^2 - y^2 \geq 0 \quad \Rightarrow \quad y^2 \leq \left(\frac{5}{2}\right)^2$

Since y is a positive square root, so, $0 \leq y \leq \frac{5}{2}$

$\therefore$ the range of $f = [0, 5/2]$

Alternative method:

If $6 - x - x^2 < 0$, y will be imaginary. So, y will be defined only when

$$6 - x - x^2 \geq 0$$

$\Rightarrow \quad (x + 3)(2 - x) \geq 0$

The corresponding equation is

$$(x + 3)(2 - x) = 0$$

$\therefore \quad x = -3, 2$

Thus, we may have the following three intervals $(-\infty, -3]$, $[-3, 2]$ and $[2, \infty)$

Interval	Value of		
	$x + 3$	$2 - x$	$(x + 3)(2 - x)$
$(-\infty, -3]$	≤ 0	> 0	≤ 0
$[-3, 2]$	≥ 0	≥ 0	≥ 0
$[2, \infty)$	> 0	≤ 0	≤ 0

From the above table, the required interval is $[-3, 2]$

$\therefore$ domain of the function $= [-3, 2]$

The range can be obtained as in the above method.

Example 6

Find the domain and the range of the function $y = \frac{1}{x^2 - 1}$.

Solution: $(-\infty, -1) \cup (1, \infty)$

$$y = \frac{1}{x^2 - 1}$$

y is defined for all values of x except when $x^2 - 1 = 0$

or, $x^2 = 1$ i.e. at $x = -1, 1$.

$\therefore$ domain of the function $= D(f) = \Re - \{-1, 1\}$

Again, $\quad y = \frac{1}{x^2 - 1}$

$\Rightarrow \quad yx^2 - (y + 1) = 0$

For real values of x,

$0 - 4.y(-(y + 1)) \geq 0 \quad \Rightarrow \quad 4y^2 + 4y \geq 0$

$\Rightarrow \quad (2y+1)^2 \geq 1 \quad \Rightarrow \quad |2y+1| \geq 1$

If $2y + 1 > 0$, $\quad 2y + 1 \geq 1 \quad \Rightarrow \quad y \geq 0$

But $y = 0$ is not possible. So $y > 0 \Rightarrow y \in (0, \infty)$

Again, if $2y + 1 < 0$,

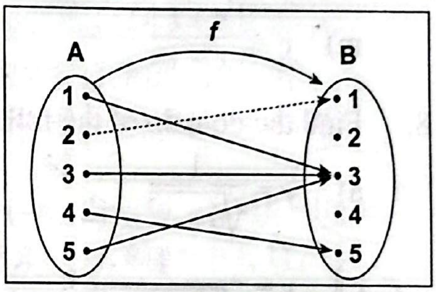
$2y + 1 \leq -1 \quad \Rightarrow \quad 2y \leq -2$

$\Rightarrow \quad y \leq -1 \quad \Rightarrow \quad y \in (-\infty, -1]$

$\therefore$ the range of the function $= R(f) = (-\infty, -1] \cup (0, \infty)$

EXERCISE 2.2

1. Let $f: A \to A$ be defined by the arrow diagram



Find a) $f^{-1}(1)$ b) $f^{-1}(3)$
c) $f^{-1}(5)$ d) $f^{-1}(2)$
e) $f^{-1}(1, 3, 5)$ f) $f^{-1}(2, 4)$.

2. a) If A = {0, 1, 2, 3,}, B = {10, 13, 16, 19} and f: A → B is a function such that $f(0) = 10, f(1) = 13, f(2) = 16, f(3) = 19$, write down f^{-1}: B → A as a set of ordered pairs.

b) If X = {–1, 1, 2, 4}, Y = $\left\{\frac{1}{5}, \frac{2}{5}, \frac{1}{2}, 1\right\}$ and f: X → Y is a function such that $f(-1) = \frac{1}{5}, f(1) = 1, f(2) = \frac{1}{2}, f(4) = \frac{2}{5}$, write down f^{-1}: Y → X as a set of ordered pairs.

3. Let a function f: R → R be defined by

a) $f(x) = x + 1, x \in R$ b) $f(x) = 2x + 3, x \in R$
c) $f(x) = 2x + 5, x \in R$ d) $f(x) = x^3 + 5, x \in R$
e) $f(x) = 3x - 2, x \in R$

Find a formula that defines the inverse function f^{-1}.

4. Let f: R → R and g: R → R be defined by

a) $f(x) = 2x + 1$ and $g(x) = 3x - 1$ b) $f(x) = 3x^2 - 4$ and $g(x) = 2x - 5$
c) $f(x) = x^3 - 1$ and $g(x) = x^2$ d) $f(x) = x^2 + 1$ and $g(x) = x^5$

Find $(gof)(x)$ and $(fog)(x)$.

5. Let f: R → R and g: R → R be defined by

a) f = {(1, 5), (2, 6), (3, 7), (4, 6)} and g = {(5, 1), (6, 2), (7, 3)}, find gof and fog.
b) f = {(1, 2), (3, 5), (4, 1)} and g = {(2, 3), (5, 1), (1, 3)}, find gof and fog.

6. a) If $f: R \to R$ be defined by $f(x) = \frac{1}{1-x}$, $(x \neq 1)$, show that $(fof)\left(\frac{1}{2}\right) = -1$

b) If $f: R \to R$ be defined by $f(x) = \frac{x-1}{x+1}$, $(x \neq -1)$ show that $(fof)(4) = -\frac{1}{4}$.

7. Find the domain and the range of the following functions defined in the set of real numbers (i.e. $f: R \to R$)

a) $y = 3x + 1$ b) $y = x^2 - 1$ c) $y = x^3$

d) $y = -x^2 + 4x - 3$ e) $y = \sqrt{x-2}$ f) $y = \frac{1}{x+1}$

g) $y = (3 - x)^2 - 8$ h) $y = 3 - (x + 1)^2$ i) $y = \frac{1}{\sqrt{3-x}}$

j) $y = \frac{x^2 - 16}{x - 4}$ k) $y = \sqrt{x^2 - 2x - 8}$ l) $y = \sqrt{21 - 4x - x^2}$

m) $y = \frac{|x-1|}{x-1}$

8. Find the domain of the following functions

a) $y = \frac{1}{\sqrt{1-x^2}}$ b) $y = \sqrt{x^2 - 25}$

c) $y = \frac{1}{\sqrt{x^2 + 6x + 8}}$ $\frac{1}{0}$ d) $y = \frac{x-1}{x^2-1}$

9. Show that $f: R \to R$ defined by $f(x) = cx + d$, where $c\ (\neq 0)$ and d are real numbers, is one to one and onto. Find f^{-1}. Also show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

10. i) Let $f: R - \{2\} \to R - \{3\}$ be defined by $f(x) = \frac{3x}{x-2}$. Show that f is bijective. Also, find f^{-1}.

ii) Let $f: R - \{0\} \to R - \{2\}$ be defined by $f(x) = \frac{2x+3}{x}$. Show that f is bijective. Also, find f^{-1}.

Practical Work/Activities

A. Let A = {1, 2, 6}, B = {2, 5, 8} and f = {(1, 2), (3, 5), (6, 8)}. Answer the following questions:

a) Is f a function?

b) If function, whether it is one to one, onto or both?

c) Find the inverse of f if possible.

d) Find the composite functions fof^{-1} and $f^{-1}of$.

e) How are fof^{-1} and $f^{-1}of$ related?

B. Let $f: R \to R$, $g: R \to R$ and f = {(0, 1), (2, 5), (3, 8)} and gof = {(0, 5), (2, 8), (3, 0)} Find g. Present gof in arrow diagram. Can fog be defined? Give reason.

a<0
a>0

C. Let $f : R \to R$, $g : R \to R$ and $g = \{(2, 9), (7, 6), (5, 2)\}$ and $fog = \{(2, 2), (5, 5), (7, 7)\}$. Find f and gof. Present fog and gof in arrow diagram. How are fog and gof related?

Answers

1. a) $\{2\}$ b) $\{1, 3, 5\}$ c) $\{4\}$ d) ϕ e) $\{1, 2, 3, 4, 5\}$ f) ϕ
2. a) $\{(10, 0), (13, 1), (16, 2), (19, 3)\}$ b) $\left\{\left(\frac{1}{5}, -1\right), (1, 1), \left(\frac{1}{2}, 2\right), \left(\frac{2}{5}, 4\right)\right\}$
3. a) $f^{-1}(x) = x - 1$ b) $f^{-1}(x) = \frac{(x-3)}{2}$ c) $f^{-1}(x) = \frac{(x-5)}{2}$ d) $f^{-1}(x) = (x-5)^{1/3}$ e) $f^{-1}(x) = \frac{(x+2)}{3}$
4. a) $6x + 2, 6x - 1$ b) $6x^2 - 13, 12x^2 - 60x + 71$ c) $(x^3 - 1)^2, x^6 - 1$ d) $(x^2 + 1)^5, x^{10} + 1$
5. a) $gof = \{(1, 1), (2, 2), (3, 3), (4, 2)\}$; $fog = \{(5, 5), (6, 6), (7, 7)\}$
   b) $gof = \{(1, 3), (3, 1), (4, 3)\}$, $fog = \{(2, 5), (5, 2), (1, 5)\}$

$x = \frac{-b}{2a} = 0$.

7. a) $D(f) = R = (-\infty, \infty)$, $R(f) = (-\infty, \infty) = R$ b) $D(f) = R = (-\infty, \infty)$, $R(f) = [-1, \infty)$
   c) $D(f) = R = (-\infty, \infty)$, $R(f) = R = (-\infty, \infty)$ d) $D(f) = R = (-\infty, \infty)$, $R(f) = (-\infty, 1]$
   e) $D(f) = [2, \infty)$, $R(f) = [0, \infty)$ f) $D(f) = R - \{-1\}$, $R(f) = R - \{0\}$
   g) $D(f) = R = (-\infty, \infty)$, $R(f) = [-2, \infty)$ h) $D(f) = R = (-\infty, \infty)$, $R(f) = (-\infty, 3]$
   i) $D(f) = (-\infty, 3)$, $R(f) = (0, \infty)$
   j) $D(f) = R - \{4\}$, $R(f) = R - \{8\}$ k) $D(f) = (-\infty, -2] \cup [4, \infty)$, $R(f) = [0, \infty)$
   l) $D(f) = [-7, 3]$, $R(f) = [0, 5]$ m) $D(f) = R - \{1\}$, $R(f) = \{-1, 1\}$
8. a) $D(f) = (-1, 1)$ b) $D(f) = (-\infty, -5] \cup [5, \infty)$
   c) $D(f) = (-\infty, -4) \cup (-2, \infty)$ d) $D(f) = R - \{-1, 1\}$
9. $\frac{x-d}{c}$ 10. i) $\frac{2x}{x-3}$ ii) $\frac{3}{x-2}$

Multiple Choice Question

1. Let $X = \{2, 5, 8\}$ and $Y = \{10, 15, 20\}$ and $f : X \to Y$ is a function such that $f(2) = 15$, $f(5) = 10$ and $f(8) = 20$ then $f^{-1} : Y \to X$ is
   a) $\{(15, 2), (5, 10), (8, 20)\}$ b) $\{(2, 15), (5, 10), (8, 20)\}$
   c) $\{(15, 2), (10, 5), (20, 8)\}$ d) $\{(15, 2), (10, 5), (8, 20)\}$
2. Let $f : R \to R$ be defined by $f(x) = 5x - 2$ then f^{-1} is defined because f is
   a) one to one b) onto
   c) one to one and into d) one to one and onto
3. Let $f : R \to R$ be defined by $f(x) = \frac{1-3x}{2}$. Then f^{-1} is equal to
   a) $\frac{1+3x}{2}$ b) $\frac{2x+1}{3}$ c) $\frac{1-2x}{3}$ d) $\frac{1-3x}{2}$

2.6 Introduction of types of Algebraic Functions

Functions, that can be formed from a real variable x with the help of some algebraic operations such as addition, subtraction, multiplication, division and extraction of roots, are called the *algebraic functions*. Some special types of algebraic functions are defined below.

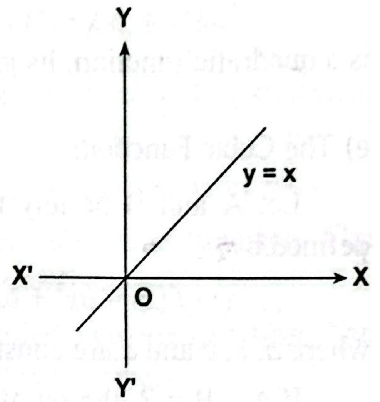
a) The Identity Function:

Let A be any set. The function f: A → A defined by

$$y = f(x) = x \quad \text{for} \quad x \in A$$

is called the **identity function**. It is usually denoted by I_A.

If A = R, the set of real numbers, the graph of I_R is as given aside.



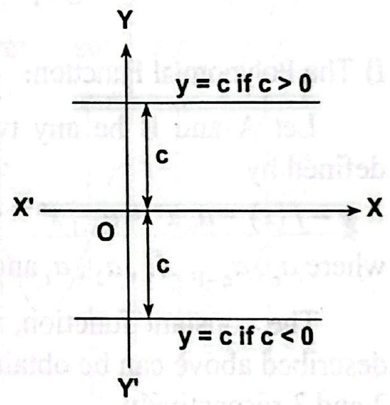
b) The Constant Function:

Let A be any set and B = $\{c\}$. Then, the function f: A → B defined by

$$y = f(x) = c \quad \text{for} \quad x \in A$$

is called the **constant function**. In other words, a function is said to be a constant function if all its functional values are the same (i.e., if the range of the function is a singleton set).

If A = R, the set of real numbers and c is a real number, the graph of $y = f(x) = c$ is a straight line parallel to the x-axis at a distance of c units from the x–axis:



c) The Linear Function:

Let A and B be any two sets. Then, a function f: A → B defined by

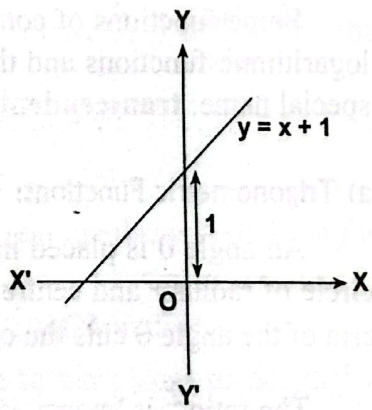
$$y = f(x) = mx + c \qquad \text{for} \quad x \in A,$$

where m and c are constants, is called a **linear function**.

If A = B = R, the set of real numbers, the function defined by

$$y = f(x) = x + 1$$

is a linear function. Its graph is a straight as shown in figure given aside.



d) The Quadratic Function:

Let A and B be any two sets. Then, a function f: A → B defined by

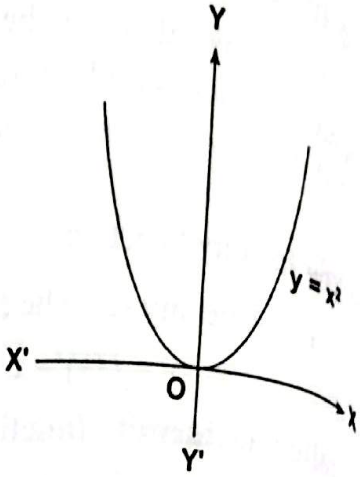
$$y = f(x) = ax^2 + bx + c \quad \text{for } x \in A,$$

where a, b and c are constants, is called a **quadratic function.**

If A = B = R, the set of real numbers, the function defined by

$$y = f(x) = x^2$$

is a quadratic function. Its graph is shown as aside.



e) The Cubic Function:

Let A and B be any two sets. Then, a function f: A → B defined by

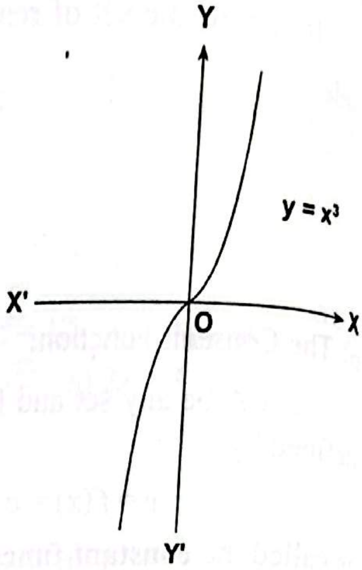
$$y = f(x) = ax^3 + bx^2 + cx + d \quad \text{for } x \in A,$$

where a, b, c and d are constants, is called a **cubic function.**

If A = B = R, the set of real numbers, the function defined by

$$y = f(x) = x^3$$

is a cubic function. Its graph is shown aside.



f) The Polynomial Function:

Let A and B be any two sets. Then, a function f: A → B defined by

$$y = f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x^1 + a_0 \text{ for } x \in A,$$

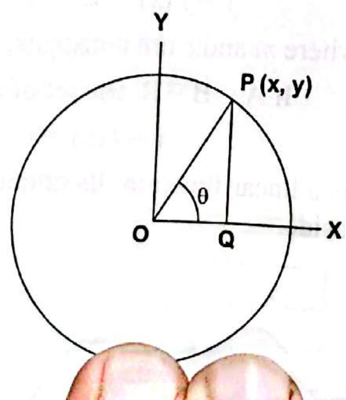
where $a_n, a_{n-1}, \ldots, a_2, a_1$ and a_0 are constants, is called a **polynomial function.**

The constant function, the linear function, the quadratic function and the cubic function described above can be obtained as special cases of the polynomial function by putting $n = 0, 1, 2$ and 3 respectively.

2.7 Transcendental Functions

Some functions of common interest are the trigonometric functions, exponential function, logarithmic functions and the hyperbolic functions. They are not algebraic. They are given the special name: **transcendental functions.**

a) Trigonometric Functions:



An angle θ is placed in the standard position together with a circle of radius r and centre at the origin O. Suppose the terminal arm of the angle θ cuts the circle at the point P(x, y).

The ratio $\frac{y}{r}$ is known as the sine function of an angle θ and is written as sin θ. Thus, $\sin \theta = \frac{y}{r}$.

In the same way, $\frac{x}{r}$ is known as the cosine function of θ and is written as cos θ. Thus, cos θ $= \frac{x}{r}$.

The other functions tangent, cotangent, cosecant and secant function of θ written as tan θ, cot θ, cosec θ and sec θ respectively are defined as

$$\tan\theta = \frac{y}{x}, \quad \cot\theta = \frac{x}{y}, \quad \operatorname{cosec}\theta = \frac{r}{y} \quad \text{and} \quad \sec\theta = \frac{r}{x} \quad (x \neq 0, y \neq 0).$$

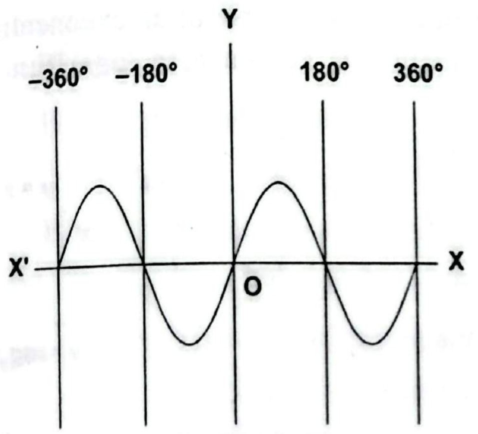
The six functions defined above are known as trigonometric function or trigonometric ratios.

Besides the above definitions we define sine and cosine functions in the following way.

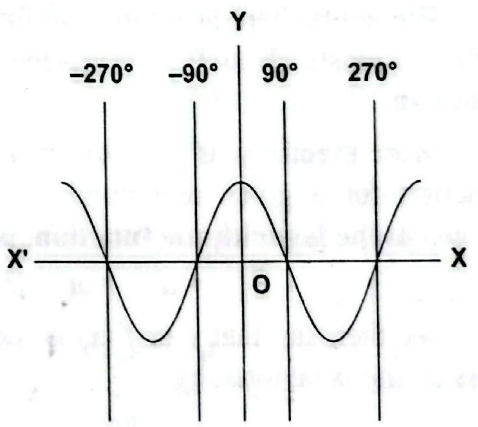
A function $f: R \to R$ defined by $f(x) = \sin x$ for all $x \in R$ is called **sine function**. The domain and the range of the sine function are $(-\infty, \infty)$ and $[-1, 1]$ respectively.

A function $f: R \to R$ defined by $f(x) = \cos x$ for all $x \in R$ is called a **cosine function**. The domain and the range of the cosine functions are $(-\infty, \infty)$ and $[-1, 1]$ respectively.

The graphs of some of the trigonometric functions are shown in the following figures:



The sine graph



The cosine graph

b) Exponential Function:

For every real number $a > 0$, the **exponential function** f with base a, is defined by the formula

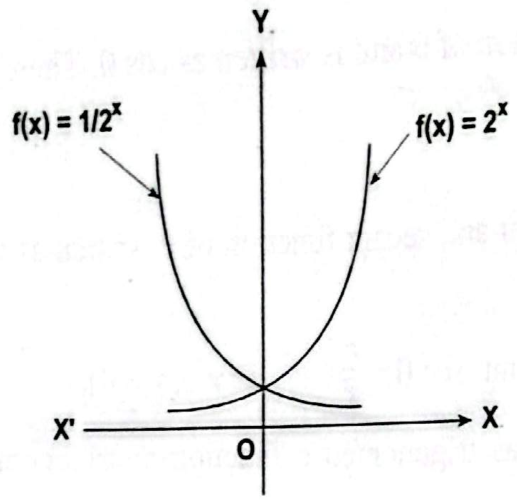
$$y = f(x) = a^x, \qquad x \in R.$$

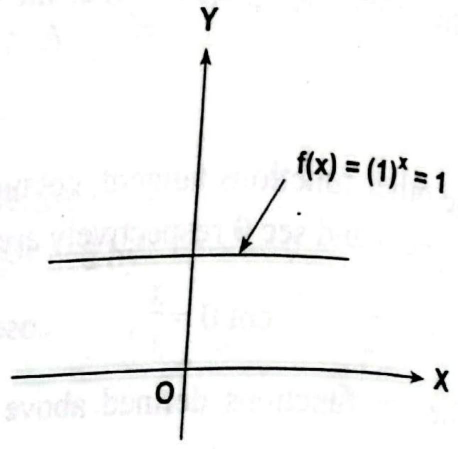
We may rewrite this definition in the following standard form:

Let $f: R \to R$ be defined by $y = f(x) = a^x, x \in R$ for a given real number $a > 0$. Then f is called an **exponential function** with base a.

Three typical examples of exponential functions are 2^x, $1/2^x$ and e^x, where e is an irrational number lying between 2 and 3. Its value to ten places of decimal is $e = 2.7182818284....$

The graphs of $y = f(x) = 2^x$ and $y = f(x) = 1/2^x$ are given below:





For the typical case $a = 1$, the graph of $y = f(x) = (1)^x = 1$ is a horizontal straight line at a distance of 1 unit from the x-axis.

The graphs of $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$ can be obtained by plotting the points with the values of x as the x-coordinates and the corresponding values of y as the y-coordinates.

c) The Logarithmic Function:

We define here as an inverse function. It can be proved that the inverse of an exponential function exists (How?). The inverse of an exponential function is known as a **logarithmic function.**

More precisely, if $y = f(x) = a^x$ defines an exponential function for a given real number $a > 0$, then its inverse, known as the **logarithmic function**, is defined by

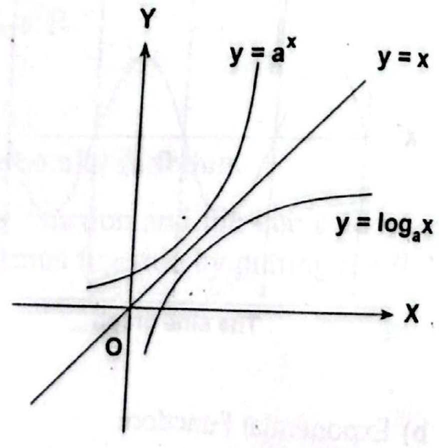
$$x = a^y \quad \text{or} \quad a^{f(x)}.$$

We then say that y or $f(x)$ *is* the **logarithm** of x to the **base** a, and is denoted by

$$\log_a x.$$

Thus, the exponential equation $x = a^y$ or $a^{f(x)}$ carries the same meaning as the logarithmic equation

$$y = \log_a x.$$



Logarithms are used for speeding up computation when the uses of electronic calculators or computers were not so common as today. Logarithm to the base 10 is known as **common logarithm**; and that to the base e is known as **natural logarithm**. If the base of a logarithm is e, it is generally omitted. We often write

$$\boxed{\log_e x = \log x = \ln x}$$

The graphs of $y = a^x$ and $y = \log_a x$ are given aside.

Note that one is the reflection of the other on the line $y = x$. An important particular case is

$$\log_a a = 1, \quad \text{since } a^1 = a.$$

Logarithms possess certain properties that are fundamental in many branches of mathematics. Some of them are stated and proved aside.

Theorem 1.

For any positive numbers x, y and a,

$$\log_a(xy) = \log_a x + \log_a y.$$

$\log_{20} 400 = 2$

Proof. For any positive numbers x, y and a, put

$$\log_a x = b \quad \text{and} \quad \log_a y = c.$$

Then by definition $x = a^b$ and $y = a^c$.

So $\quad xy = a^b.a^c = a^{b+c}$.

Hence, $\log_a (xy) = b + c = \log_a x + \log_a y$.

Theorem 2.

For any positive number a and x,

$$\log_a x^p = p \log_a x, \qquad \text{where } p \text{ is any real number.}$$

Proof. For any positive numbers a and x, put $\log_a x = b$.

Then, $x = a^b$ and so $x^p = (a^b)^p = a^{bp}$.

Hence, $\log_a x^p = bp = pb = p \log_a x$.

Theorem 3.

For any positive numbers x, y and a,

$$\log_a(x/y) = \log_a x - \log_a y.$$

Proof. For any positive numbers x, y and a, put

$$\log_a x = b \quad \text{and} \quad \log_a y = c.$$

Then by definition $x = a^b$ and $y = a^c$.

So $\quad x/y = a^b/a^c = a^{b-c}$.

Hence, $\log_a (x/y) = b - c = \log_a x - \log_a y$.

Theorem 4.

For any positive numbers a, b and x,

$$\log_a x = \log_a b.\ \log_b x.$$

$\log_{10} 10 = 1$

$x = 10$

Proof. For any positive numbers a, b and x, put

$$\log_b x = m.$$

Then by definition $x = b^m$,

Hence, $\log_a x = \log_a b^m = m \log_a b = \log_a b.\ \log_b x$.

d) Hyperbolic Functions:

With the help of an exponential function, a new function is defined which is known as the hyperbolic function. They are so named as the functions are closely related to the conic section i.e. hyperbola.

Certain combinations of exponential functions e^x and e^{-x} play very important roles in pure and applied mathematics. We consider the following six combinations:

a) $\frac{e^x - e^{-x}}{2}$ b) $\frac{e^x + e^{-x}}{2}$ c) $\frac{e^x - e^{-x}}{e^x + e^{-x}}$ d) $\frac{e^x + e^{-x}}{e^x - e^{-x}}$.

e) $\frac{2}{e^x - e^{-x}}$ f) $\frac{2}{e^x + e^{-x}}$.

These functions have properties somewhat similar to those of trigonometric functions. Trigonometric functions are related to circle and hence they are also known as **circular functions**. In the same way, the above six combinations are related to a hyperbola; and hence the name hyperbolic functions. They are defined as follows:

a) Hyperbolic sine of x : $\sinh x = \frac{e^x - e^{-x}}{2}$

b) Hyperbolic cosine of x : $\cosh x = \frac{e^x + e^{-x}}{2}$.

c) Hyperbolic tangent of x : $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

d) Hyperbolic cotangent of x : $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

e) Hyperbolic cosecant of x : $\text{cosech}\, x = \frac{2}{e^x - e^{-x}}$

f) Hyperbolic secant of x : $\text{sech}\, x = \frac{2}{e^x + e^{-x}}$.

We list below some of the basic identities and formulae related to these functions:

a) $\cosh^2 x - \sinh^2 x = 1$
b) $\tanh^2 x + \text{sech}^2 x = 1$
c) $\coth^2 x - \text{cosech}^2 x = 1$
d) $\sinh(-x) = -\sinh x$
e) $\cosh(-x) = \cosh x$
f) $\sinh 2x = 2 \sinh x \cosh x$
g) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$
h) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
i) $\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$

As an illustration of the technique of proving the above results, we consider the first one. From definition, we have

$$
\begin{aligned}
\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \left(\frac{e^x + e^{-x} + e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x} - e^x + e^{-x}}{2}\right) \\
&= \left(\frac{2e^x}{2}\right)\left(\frac{2e^{-x}}{2}\right) \\
&= e^x . e^{-x} = e^0 \\
&= 1
\end{aligned}
$$

Worked Out Examples

Example 1

a) Prove that: $\cosh 2x = 2\cosh^2 x - 1$

b) Prove that: $\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$.

Solution:

a) $\cosh 2x = \dfrac{e^{2x}+e^{-2x}}{2}$

$$= \frac{e^{2x}+2+e^{-2x}-2}{2} = \frac{(e^x+e^{-x})^2-2}{2}$$

$$= 2\left(\frac{e^x+e^{-x}}{2}\right)^2 - 1 = 2\cosh^2 x - 1$$

b) Here $\tanh 2x = \dfrac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}} = \dfrac{(e^x+e^{-x})(e^x-e^{-x})}{\frac{(e^x-e^{-x})^2+(e^x+e^{-x})^2}{2}}$

$$= \frac{\dfrac{2(e^x-e^{-x})}{(e^x+e^{-x})}}{1+\dfrac{(e^x-e^{-x})^2}{(e^x+e^{-x})^2}} = \frac{2\tanh x}{1+\tanh^2 x}.$$

Example 2

Prove that $\log_{a}\sqrt{a3\sqrt{a2}} = 2$

Solution:

$\log_{a}\sqrt{a3\sqrt{a2}} = \log_{a}\sqrt{a3.a} = \log_{a}\sqrt{a4}$

$= \log_a a^2 = 2\log_a a$

$= 2 \times 1 = 2$

e

Example 3

Prove that

a) $\log_a (x^3y^2z) = 3\log_a x + 2\log_a y + \log_a z$

b) $\log a^2/bc + \log b^2/ca + \log c^2/ab = 0$

c) $\log \dfrac{a+b}{3} = \dfrac{1}{2}(\log a + \log b)$, if $a^2 + b^2 = 7ab$.

Solutions:

a) $\log_a (x^3y^2z) = \log_a x^3y^2 + \log_a z$

$= \log_a x^3 + \log_a y^2 + \log_a z$

$= 3\log_a x + 2\log_a y + \log_a z$

b) Here, $\log a^2/bc + \log b^2/ca + \log c^2/ab$

$$= \log (a^2/bc)(b^2/ca)(c^2/ab)$$

$$= \log \frac{a^2b^2c^2}{bc.ca.ab}$$

$$= \log 1 = 0 \ (a^o = 1)$$

c) Here, $a^2 + b^2 = 7ab$ or $a^2 + b^2 + 2ab = 9ab$.

So, $(a + b)^2/3^2 = ab$

Taking log of both sides, we get

$$\log[(a + b)^2/3^2] = \log ab.$$

or, $2 \log(a + b)/3 = \log a + \log b$

or, $\log\frac{a+b}{3} = \frac{1}{2}(\log a + \log b)$

Example 4

If $f(x) = \log\frac{1+x}{1-x}\ (-1 < x < 1)$, show that $f(a) + f(b) = f\left(\frac{a+b}{1+ab}\right)\ (|a| < 1, |b| < 1)$

Solution:

Since, $f(x) = \log\frac{1+x}{1-x}$

so $f(a) = \log\frac{1+a}{1-a}$ and $f(b) = \log\frac{1+b}{1-b}$

Now, $f(a) + f(b) = \log\frac{1+a}{1-a} + \log\frac{1+b}{1-b}$

$$= \log\left(\frac{1+a}{1-a}\cdot\frac{1+b}{1-b}\right) \quad = \log\frac{1+a+b+ab}{1-a-b+ab}$$

$$= \log\frac{(1+ab)+(a+b)}{(1+ab)-(a+b)} \quad = \log\left(\frac{1+\frac{a+b}{1+ab}}{1-\frac{a+b}{1+ab}}\right)$$

$$= f\left(\frac{a+b}{1+ab}\right)$$

Example 5

If $x = \log_a bc, y = \log_b ca, z = \log_c ab$, prove that $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1$

Solution:

$x + 1 = \log_a bc + \log_a a = \log_a abc$

$$= \frac{1}{\log_{abc} a} \qquad (\because \log_b a \,.\, \log_a b = 1)$$

Similarly, $y + 1 = \frac{1}{\log_{abc} b}$ and $z + 1 = \frac{1}{\log_{abc} c}$

Now, $\dfrac{1}{x+1}+\dfrac{1}{y+1}+\dfrac{1}{z+1} = \log_{abc}a + \log_{abc}b + \log_{abc}c$

$= \log_{abc}abc$

$= 1$

EXERCISE 2.3

1. Prove that
   a) $\log_a (xy^3/z^2) = \log_a x + 3\log_a y - 2\log_a z$
   b) $\log_a 2x + 3(\log_a x - \log_a y) = \log_a (2x^4/y^3)$
   c) $\log_a x^2 - 2\log_a \sqrt{x} = \log_a x$
   d) $a^{\log_a x} = x$
   e) $\log_a a^x = x$
   f) $(\log a)^2 - (\log b)^2 = \log(ab).\log(a/b)$
   g) $\log(1+2+3) = \log 1 + \log 2 + \log 3$
   h) $x^{\log y - \log z}.y^{\log z - \log x}.z^{\log x - \log y} = 1$
   i) $(yz)^{\log y - \log z}.(zx)^{\log z - \log x}.(xy)^{\log x - \log y} = 1$
   j) $\log_a\sqrt{a\sqrt{a\sqrt{a^2}}} = 1$

$\log_a x = y$

$a^y = x$

2. If $a^2 + b^2 = 2ab$, show that $\log\dfrac{a+b}{2} = \dfrac{\log a + \log b}{2}$

3. If $f(x) = \log\dfrac{1-x}{1+x}$ $(-1 < x < 1)$, show that $f\left(\dfrac{2ab}{1+a^2b^2}\right) = 2f(ab)$ where $|ab| < 1$.

4. If $x = \log_{2a} a$, $y = \log_{3a} 2a$ and $z = \log_{4a} 3a$, prove that $xyz + 1 = 2yz$

5. If $\dfrac{\log x}{y-z} = \dfrac{\log y}{z-x} = \dfrac{\log z}{x-y}$, prove that $x^x y^y z^z = 1$.

$\log_8 32 = x$

$32 = 8^x$

Multiple Choice Question

1. The value of $\log_8 32 =$

a) 4 b) $\frac{3}{2}$ c) $\frac{5}{3}$ d) $\frac{5}{2}$

2. If $\log_{x-3} 81 = 4$, then $x =$

a) 3 b) 4 c) 5 d) 6