Chapter 16
The Derivatives

16.1 Introduction

Differential calculus is a theory which has its origin in the solution of two old problems - one of drawing a tangent line to a curve and the other of calculating the velocity of non-uniform motion of a particle. In both the problems, the curves involved are continuous curves and the process used is the limiting process. So the objects of study in the differential calculus are continuous functions. These problems were solved in a certain sense by Isaac Newton (English, 1642-1727) and Gottfried Willhelm Leibnitz (German, 1646-1716) and in the process differential calculus is discovered.

Increment

A small change in the value of a variable is known as the increment. A change in the value of the variable may increase or decrease. But in either case, we use the term increment to denote the change. As for example, if the value of the variable changes from 2.0 to 2.0004 or from 2.0004 to 2.0 the change known as the increment is 0.0004.

Let $y = f(x)$ be the function of x. If the value of x changes, the value of y will also change accordingly. Generally we denote by Δx (read as 'delta x') to change in x and Δy the corresponding change in y. If x changes to $x + \Delta x$, y will change to $y + \Delta y$ and $f(x)$ changes to $f(x + \Delta x)$ that is, if $f(x) = 2x + 3$, then $f(x + \Delta x) = 2(x + \Delta x) + 3$

For example: If $y = 3x + 7$ and x changes from 3 to 3.01 then the change in y is given below.

When $x = 3$, $\quad y = 3 \times 3 + 7 = 16$

When $x = 3.01$, $\quad y = 3 \times 3.01 + 7 = 16.03$

Now change in $y = \Delta y = 16.03 - 16 = 0.03$

Thus if $y = f(x)$ and x change to $x + \Delta x$, y changes to $y + \Delta y$ and $f(x)$ to $f(x + \Delta x)$ then

$$y + \Delta y = f(x + \Delta x)$$

and $$\Delta y = f(x + \Delta x) - y = f(x + \Delta x) - f(x)$$

16.2 Derivative of a Function

Let $y = f(x)$ be a continuous function of x. Let Δx and Δy be the small increments in x and y respectively. Then

$$y = f(x) \Rightarrow \quad y + \Delta y = f(x + \Delta x)$$

so that $$\Delta y = f(x + \Delta x) - f(x)$$

$$\Rightarrow \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\Rightarrow \lim_{x\to 0} \frac{\Delta y}{\Delta x} = \lim_{x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

If the limiting value of the R.H.S. exists, it is called the derivative or differential coefficient of y with respect to x and is denoted by $\frac{dy}{dx}$.

$$\therefore \frac{dy}{dx} = \lim_{x\to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad \text{provided that the limit exists.}$$

The process of obtaining the derivative of a function is called the **differentiation.** The above process of finding the derivative of a function is called the 'derivative of the function by **first principle** or **by definition.**'

Thus we have the following definition of the derivative of a function.

Definition. Let the function f be defined in the interval (a, b). Then the *derivative* or the *differential coefficient of the function* f at a point x of the interval is defined to be the limiting value of

$$\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Again if a is a fixed point then, the derivative of $f(x)$ at $x = a$ denoted by $f'(a)$ is defined by

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

provided that the limit exists.

The symbols used to denote the derivative of f with respect to x are

$$f'(x), \quad \frac{d\,f(x)}{dx}, \quad y', \quad \frac{dy}{dx}.$$

16.3 Derivative of Algebraic Functions

Derivative of x^n from First Principles (Using Limit Theorem)

Let $\quad y = x^n \quad$ ...... (i)

Let Δx be a small increment in x and Δy, a corresponding increment in y. Then,

$$y + \Delta y = (x + \Delta x)^n \quad \text{...... (ii)}$$

Subtracting (i) from (ii), we get

$$\Delta y = (x+\Delta x)^n - x^n$$

or, $$\frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

Now, $\dfrac{dy}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{(x + \Delta x)^n - x^n}{(x + \Delta x) - x}$

$= \lim\limits_{(x + \Delta x) \to x} \dfrac{(x + \Delta x)^n - x^n}{(x + \Delta x) - x} = nx^{n-1}$ $\left(\because \lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = n\,a^{n-1}\right)$

Now we shall show that the derivative of a constant is zero.

Let $y = f(x) = c$, a constant

Then $y + \Delta y = c$

$\Delta y = c - y = c - c = 0$

$\dfrac{\Delta y}{\Delta x} = 0$

$\therefore \dfrac{dy}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = 0$

Example 1

Find from first principles the derivative of $(ax + b)^n$.

Solution :

Let $y = (ax + b)^n$

Let Δx and Δy be the small increments in x and y respectively. Then,

$y + \Delta y = \{a(x + \Delta x) + b\}^n$

$\Delta y = (ax + a\Delta x + b)^n - (ax + b)^n$

or $\dfrac{\Delta y}{\Delta x} = \dfrac{(ax + a\Delta x + b)^n - (ax + b)^n}{\Delta x}$

$\dfrac{dy}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{(ax + a\Delta x + b)^n - (ax + b)^n}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{(ax + a\Delta x + b)^n - (ax + b)^n}{(ax + a\Delta x + b) - (ax + b)} \cdot a$

$= \lim\limits_{(ax + a\Delta x + b) \to (ax + b)} \dfrac{(ax + a\Delta x + b)^n - (ax + b)^n}{(ax + a\Delta x + b) - (ax + b)} \cdot a$

$= n.(ax + b)^{n-1} . a$ $\left(\because \lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}\right)$

$= n.a\,(ax + b)^{n-1}$

Example 2

Find from first principles, the derivative of $\dfrac{1}{\sqrt{x+2}}$

Solution :

Let $y = \dfrac{1}{\sqrt{x+2}}$.

Let Δx be a small increment in x and Δy be the corresponding small increment in y. Then

$$y + \Delta y = \frac{1}{\sqrt{(x+\Delta x+2)}}$$

$$\Delta y = \frac{1}{\sqrt{(x+\Delta x+2)}} - \frac{1}{\sqrt{(x+2)}}$$

$$= \frac{\sqrt{(x+2)} - \sqrt{(x+\Delta x+2)}}{\sqrt{(x+\Delta x+2)}\,\sqrt{(x+2)}}$$

$$= \frac{[\sqrt{x+2} - \sqrt{x+\Delta x+2}]\,[\sqrt{x+2} + \sqrt{x+\Delta x+2}]}{\sqrt{x+\Delta x+2}\,\sqrt{x+2}\,[\sqrt{x+2} + \sqrt{x+\Delta x+2}]}$$

$$= \frac{x+2-x-\Delta x-2}{\sqrt{x+\Delta x+2}\,\sqrt{x+2}\,[\sqrt{x+2} + \sqrt{x+\Delta x+2}]}$$

$$= \frac{-\Delta x}{\sqrt{x+\Delta x+2}\,\sqrt{x+2}\,[\sqrt{x+2} + \sqrt{x+\Delta x+2}]}$$

$$\frac{\Delta y}{\Delta x} = \frac{-1}{\sqrt{x+\Delta x+2}\,\sqrt{x+2}\,[\sqrt{x+2} + \sqrt{x+\Delta x+2}]}$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-1}{\sqrt{x+\Delta x+2}\,\sqrt{x+2}\,[\sqrt{x+2} + \sqrt{x+\Delta x+2}]}$$

$$= \frac{-1}{\sqrt{x+2}\,\sqrt{x+2}\,[\sqrt{x+2} + \sqrt{x+2}]}$$

$$= \frac{-1}{2(x+2)^{3/2}}$$

16.4 Rules of Differentiation

Here we shall deduce some fundamental formulae of differentiation.

I. The Sum Rule

Let $f(x)$ and $g(x)$ be any two differentiable functions of x.

Let $\quad h(x) = f(x) \pm g(x)$

Then $\quad h'(x) = f'(x) \pm g'(x)$

or, $\quad \dfrac{d}{dx}h(x) = \dfrac{d}{dx}f(x) \pm \dfrac{d}{dx}g(x)$.

Proof.

Let a be a fixed point. Then, by definition,

$$h'(a) = \lim_{x \to a} \frac{h(x)-h(a)}{x-a}$$
$$= \lim_{x \to a} \frac{[f(x) \pm g(x)] - [f(a) \pm g(a)]}{x-a}$$
$$= \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \pm \lim_{x \to a} \frac{g(x)-g(a)}{x-a}$$
$$= f'(a) \pm g'(a).$$

But a is an arbitrary fixed number so

$$h'(x) = f'(x) \pm g'(x)$$

or, $\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x).$

Put $f(x) = u$ and $g(x) = v$. Then we have

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

From this sum Rule, we can deduce a formula for the differentiation of 'a constant times a function'.

(i) $\frac{d(2u)}{dx} = \frac{d(u+u)}{dx} = \frac{du}{dx} + \frac{du}{dx} = 2 \cdot \frac{du}{dx}$

(ii) $\frac{d(3u)}{dx} = \frac{d(2u+u)}{dx} = \frac{d(2u)}{dx} + \frac{du}{dx} = 2 \cdot \frac{du}{dx} + \frac{du}{dx} = 3\frac{du}{dx}$.

(iii) $\frac{d(4u)}{dx} = \frac{d(3u+u)}{dx} = \frac{d(3u)}{dx} + \frac{du}{dx} = 3 \cdot \frac{du}{dx} + \frac{du}{dx} = 4\frac{du}{dx}$

So in general, we have $\frac{d(nu)}{dx} = n\frac{du}{dx}$ for any integer n.

(This rule holds not only for an integer, but also for an rational number).

Example 3

Find the derivative of $5x^3 + 4x^2 - 2x + 7$

Solution:

Let $y = 5x^3 + 4x^2 - 2x + 7$

Differentiating both sides w.r.t. 'x'.

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(5x^3 + 4x^2 - 2x + 7)$$
$$= \frac{d(5x^3)}{dx} + \frac{d(4x^2)}{dx} - \frac{d(2x)}{dx} + \frac{d(7)}{dx}$$
$$= 5 \cdot \frac{dx^3}{dx} + 4 \cdot \frac{d(x^2)}{dx} - 2 \cdot \frac{dx}{dx}.$$

or, $\frac{dy}{dx} = 15x^2 + 8x - 2.$

II. The Product Rule

Let $f(x)$ and $g(x)$ be any two differentiable functions of x. Let $h(x) = f(x) \cdot g(x)$. Then

$$h'(x) = f(x).g'(x) + g(x).f'(x)$$

or, $$\frac{d}{dx}h(x) = f(x) \cdot \frac{d}{dx}g(x) + g(x) \cdot \frac{d}{dx}f(x).$$

Proof: Let a be a fixed point. Then, by definition

$$h'(a) = \lim_{x\to a} \frac{h(x) - h(a)}{x - a}$$

$$= \lim_{x\to a} \frac{f(x)\, g(x) - f(a)\, g(a)}{x - a}$$

$$= \lim_{x\to a} \frac{f(x)\, g(x) - f(x)\, g(a) + f(x)\, g(a) - f(a)\, g(a)}{x - a}$$

$$= \lim_{x\to a} f(x) \lim_{x\to a} \frac{g(x) - g(a)}{x - a} + g(a) \cdot \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

$$= f(a)\, g'(a) + g(a)\, f'(a).$$

But a is an arbitrary fixed number, so

$$h'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x).$$

$$\frac{d}{dx}\{f(x) \cdot g(x)\} = f(x) \cdot \frac{d}{dx}g(x) + g(x) \cdot \frac{d}{dx}f(x).$$

Put $f(x) = u$ and $g(x) = v$. Then we get

$$\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Example 4

Find the derivative of $(3x^2 - 5x)(2x + 3)$

Solution:

Let $u = 3x^2 - 5x$ and $v = 2x + 3$. Then we have

$$\frac{d}{dx}(u.v) = u.\frac{dv}{dx} + v\frac{du}{dx}$$

$$= (3x^2 - 5x)\frac{d}{dx}(2x + 3) + (2x + 3)\frac{d}{dx}(3x^2 - 5x)$$

$$= (3x^2 - 5x)\,2 + (2x + 3)(6x - 5)$$

$$= 18x^2 - 2x - 15$$

III. The Power Rule

If u is the function of x, then $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$.

Proof: By the Product Rule, we have

$$\frac{d}{dx}(u^2) = \frac{d}{dx}(u.u) = u\frac{du}{dx} + u\frac{du}{dx} = 2u\frac{du}{dx}.$$

Also, $\frac{d}{dx}(u^3) = \frac{d(u.u^2)}{dx} = u\frac{du^2}{dx} + u^2\frac{du}{dx}$

$= u.2u\frac{du}{dx} + u^2\frac{du}{dx}$

$= 3u^2\frac{du}{dx}$.

Similarly, for any natural number n, we have $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$

(But in fact, it is true for any rational number n).

Example 5

Find the derivative of $(4x^3 + 5)^{3/2}$.

Solution:

Let $u = 4x^3 + 5$

$\because \frac{du}{dx} = \frac{d}{dx}(4x^3 + 5) = 12x^2$.

$\therefore \frac{d}{dx}(4x^3 + 5)^{3/2} = \frac{d(u^{3/2})}{dx}$

$= \frac{3}{2}u^{1/2}\frac{du}{dx} = \frac{3}{2}(4x^3 + 5)^{1/2} . 12x^2$

$= 18x^2\sqrt{4x^3 + 5}$

IV. The Quotient Rule

Let $f(x)$ and $g(x)$ be any two differentiable functions of x. Let $h(x) = \frac{f(x)}{g(x)}$. Then

$$h'(x) = \frac{g(x)\,f'(x) - f(x)\,g'(x)}{\{g(x)\}^2}$$

Proof: We have, by definition,

$$h'(a) = \lim_{x\to a}\frac{h(x) - h(a)}{x - a}$$

$$= \lim_{x\to a}\frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x - a} \qquad g(a) \neq 0$$

$$= \lim_{x\to a}\frac{g(a)\,f(x) - f(a)\,g(x)}{(x - a)\,g(x)\,g(a)}$$

$$= \lim_{x\to a}\frac{g(a)\,f(x) - g(a)\,f(a) + g(a)\,f(a) - f(a)\,g(x)}{(x - a)\,g(x)\,g(a)}$$

$$= \lim_{x\to a}\left[\frac{1}{g(a)\,g(x)}\left\{g(a)\frac{f(x) - f(a)}{x - a} - f(a)\frac{g(x) - g(a)}{x - a}\right\}\right]$$

$$= \frac{1}{g(a)\, g(a)} \{g(a)\, f'(a) - f(a)\, g'(a)\}$$

$$= \frac{g(a)\, f'(a) - f(a)\, g'(a)}{\{g(a)\}^2}$$

$\therefore$ In general, we have

$$h'(x) = \frac{g(x)\, f'(x) - f(x)\, g'(x)}{\{g(x)\}^2}$$

or, $$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{\{g(x)\}^2}$$

Put $u = f(x)$ and $v = g(x)$. Then we have $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Example 6

Find the derivative of $\frac{4x^2 + 3}{3x^2 - 2}$.

Solution:

Let $u = 4x^2 + 3$ and $v = 3x^2 - 2$. Then $\frac{du}{dx} = 8x$ and $\frac{dv}{dx} = 6x$

$$\therefore \quad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(3x^2 - 2)\,.\,8x - (4x^2 + 3)\,.\,6x}{(3x^2 - 2)^2}$$

$$= \frac{-34x}{(3x^2 - 2)^2}$$

V. The Chain Rule

If $y = f(u)$ and $u = g(x)$, where f and g are differentiable functions, then $\frac{dy}{dx}$ exists and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u)\frac{du}{dx}$$

Proof: We have $u = g(x)$. Let Δx be a small increment in x and Δu be the correspondin[g] small increment in u. Then

$$u + \Delta u = g(x + \Delta x)$$

or, $$\Delta u = g(x + \Delta x) - u = g(x + \Delta x) - g(x)$$

since $g(x)$ is differentiable, it is continuous. So,

$$\lim_{\Delta x \to 0} \Delta u = \lim_{\Delta x \to 0} [g(x + \Delta x) - g(x)]$$

$$= g(x) - g(x) = 0$$

Thus $\Delta u \to 0$, as $\Delta x \to 0$. Then

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left[\frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}\right]$$

$$= \left(\lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u}\right)\left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\right)$$

$$= \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 7

Find $\frac{dy}{dx}$, if $y = 4u^2 - 3u + 5$ and $u = 2x^2 - 3$.

Solution:

We have $\frac{dy}{du} = 8u - 3$ and $\frac{du}{dx} = 4x$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (8u - 3)\, 4x$$

$$= \{8(2x^2 - 3) - 3\}\, 4x$$

$$= 64x^3 - 108x.$$

16.5 Derivative of Parametric function

The function in which the variables (say) x and y are expressed in terms of a third variable, called a parameter, is known as the parametric functions.

Let $x = f(t)$ and $y = g(t)$ be the parametric functions, t being the parameter. If $\frac{dx}{dt}$ and $\frac{dy}{dt}$ exist and $\frac{dx}{dt} \neq 0$, then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

Example 8

Find $\frac{dy}{dx}$ when $x = 2at^3$ and $y = 3at^2$.

Solution:

$$x = 2at^3 \quad \Rightarrow \quad \frac{dx}{dt} = \frac{d}{dt}(2at^3) = 6at^2$$

$$y = 3at^2 \quad \Rightarrow \quad \frac{dy}{dt} = \frac{d}{dx}(3at^2) = 6at$$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6at}{6at^2} = \frac{1}{t}$

16.6 Derivatives of Implicit Function

Let $f(x, y)$ be an arbitrary function of two variables x and y and let

$$f(x, y) = 0 \qquad \text{...... (i)}$$

This equation may or may not be solvable for y. But we can differentiate (i) term by term with respect to x and solve for $\frac{dy}{dx}$. This process of finding the value of $\frac{dy}{dx}$ without solving the equation for y is called *implicit differentiation.*

Example 9

Use implicit differentiation to find $\frac{dy}{dx}$ in $2x^2 - 3y^2 = 16$.

Solution:

The given equation is $\quad 2x^2 - 3y^2 = 16$

Differentiating both sides with respect to x, we get

$$\frac{2dx^2}{dx} - \frac{3dy^2}{dy} \cdot \frac{dy}{dx} = \frac{d(16)}{dx}$$

or, $\quad 4x - 6y \cdot \frac{dy}{dx} = 0$

$\because \quad \frac{dy}{dx} = \frac{4x}{6y} = \frac{2x}{3y}$.

16.7 Higher Order Derivatives

Let $y = f(x)$ be a differentiable function. Then $\frac{dy}{dx} = f'(x)$ is called the *first derivative* or the first differential coefficient of $f(x)$ with respect to x. If we differentiate $\frac{dy}{dx}$ again, we get $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ written as $\frac{d^2y}{dx^2}$ or $f''(x)$, which is called the *second derivative* or the second differential coefficient of $f(x)$ with respect to x and so on.

Example 10

Find the second and higher derivatives of $\quad y = 5x^4 - 3x^2 + 11$

Solution:

We have $\quad y = 5x^4 - 3x^2 + 11$

$\therefore \quad \frac{dy}{dx} = 20x^3 - 6x, \quad$ the first derivative

$$\frac{d^2y}{dx^2} = 60x^2 - 6, \quad \text{the second derivative}$$

$$\frac{d^3y}{dx^3} = 120x, \quad \text{the third derivative}$$

$$\frac{d^4y}{dx^4} = 120, \quad \text{the fourth derivative}$$

$$\frac{d^5y}{dx^5} = 0, \quad \text{the fifth derivative}$$

and all other higher derivatives are also zero.

Worked Out Examples

Example 1

Find from first principles, the derivative of $2x^2 + 3x - 6$.

Solution :

Let $y = 2x^2 + 3x - 6$

Let Δx and Δy be the small increments in x and y respectively. Then,

$$y + \Delta y = 2(x + \Delta x)^2 + 3(x + \Delta x) - 6$$

or, $$\Delta y = 2(x + \Delta x)^2 + 3(x + \Delta x) - 6 - 2x^2 - 3x + 6$$

$$= 4x\Delta x + 2(\Delta x)^2 + 3\Delta x$$

or, $$\frac{\Delta y}{\Delta x} = 4x + 2\Delta x + 3$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (4x + 2\Delta x + 3) = 4x + 3$$

$\therefore$ $$\frac{dy}{dx} = 4x + 3$$

Example 2

Find from first principles, the derivative of $(2 - 3x)^{1/2}$.

Solution :

Let $y = \sqrt{2 - 3x}$

Let Δx and Δy be the small increments in x and y respectively. Then,

$$y + \Delta y = \sqrt{2 - 3(x + \Delta x)}$$

$$\Delta y = \sqrt{2 - 3(x + \Delta x)} - \sqrt{2 - 3x}$$

or, $$\frac{\Delta y}{\Delta x} = \frac{\sqrt{2 - 3(x + \Delta x)} - \sqrt{2 - 3x}}{\Delta x}$$

$$= \frac{2-3(x+\Delta x)-2+3x}{\Delta x\left(\sqrt{2-3(x+\Delta x)}+\sqrt{2-3x}\right)}$$

$$= \frac{-3}{\left(\sqrt{2-3(x+\Delta x)}+\sqrt{2-3x}\right)}$$

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{-3}{\sqrt{2-3(x+\Delta x)}+\sqrt{2-3x}}$$

$$= -\frac{3}{2\sqrt{2-3x}}$$

Alternative Method

Let $y = (2-3x)^{1/2}$

$y + \Delta y = \{2-3(x+\Delta x)\}^{1/2}$

where Δx and Δy are the small increments in x and y respectively.

$$\Delta y = \{2-3(x+\Delta x)\}^{1/2} - (2-3x)^{1/2}$$

or, $$\frac{\Delta y}{\Delta x} = \frac{(2-3x-3\Delta x)^{1/2}-(2-3x)^{1/2}}{\Delta x}$$

$$= \frac{(2-3x-3\Delta x)^{1/2}-(2-3x)^{1/2}}{(2-3x-3\Delta x)-(2-3x)} \times -3$$

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

$$= \lim_{(2-3x-3\Delta x)\to(2-3x)}\frac{(2-3x-3\Delta x)^{1/2}-(2-3x)^{1/2}}{(2-3x-3\Delta x)-(2-3x)} \times -3$$

$$= \frac{1}{2}\cdot(2-3x)^{-1/2}\cdot(-3)$$

$$= -\frac{3}{2\sqrt{2-3x}} \qquad \left(\because \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right)$$

Example 3

Find the derivative of $\frac{\sqrt{x}+a}{\sqrt{x}+b}$.

Solution :

Let $y = \frac{\sqrt{x}+a}{\sqrt{x}+b}$

Then, $$\frac{dy}{dx} = \frac{(\sqrt{x}+b)\frac{d}{dx}(\sqrt{x}+a)-(\sqrt{x}+a)\frac{d}{dx}(\sqrt{x}+b)}{(\sqrt{x}+b)^2}$$

$$= \frac{(\sqrt{x}+b)\cdot\frac{1}{2\sqrt{x}}-(\sqrt{x}+a)\frac{1}{2\sqrt{x}}}{(\sqrt{x}+b)^2} = \frac{b-a}{2\sqrt{x}\,(\sqrt{x}+b)^2}$$

Example 4

Find the derivative of $\dfrac{1}{x+\sqrt{x^2-a^2}}$.

Solution :

Let $y = \dfrac{1}{x+\sqrt{x^2-a^2}}$

Then, $y = \dfrac{1}{x+\sqrt{x^2-a^2}} \times \dfrac{x-\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}$

$= \dfrac{1}{a^2}(x-\sqrt{x^2-a^2})$

$$\frac{dy}{dx} = \frac{1}{a^2}\frac{d}{dx}\{x-\sqrt{x^2-a^2}\}$$

$$= \frac{1}{a^2}\left\{\frac{d}{dx}(x) - \frac{d(x^2-a^2)^{1/2}}{d(x^2-a^2)} \cdot \frac{d(x^2-a^2)}{dx}\right\}$$

$$= \frac{1}{a^2}\left\{1 - \frac{1}{2\sqrt{x^2-a^2}} \times 2x\right\}$$

$$= \frac{\sqrt{x^2-a^2}-x}{a^2\sqrt{x^2-a^2}}$$

Example 5

Find the derivative of $\sqrt{\dfrac{1-x}{1+x}}$.

Solution :

Let $y = \sqrt{\dfrac{1-x}{1+x}} = \left(\dfrac{1-x}{1+x}\right)^{1/2}$

Then, $\dfrac{dy}{dx} = \dfrac{d\left(\frac{1-x}{1+x}\right)^{1/2}}{d\left(\frac{1-x}{1+x}\right)} \cdot \dfrac{d\left(\frac{1-x}{1+x}\right)}{dx}$

$$= \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2} \frac{(1+x)\frac{d}{dx}(1-x)-(1-x)\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= \frac{1}{2}\cdot\frac{(1-x)^{-1/2}}{(1+x)^{-1/2}}\cdot\frac{-2}{(1+x)^2}$$

$$= -\frac{1}{\sqrt{1-x}\,.\,(1+x)^{3/2}}$$

Example 6

Find $\dfrac{dy}{dx}$ if $x = t + \dfrac{1}{t}$ and $y = t - \dfrac{1}{t}$.

Solution :

$$x = t + \frac{1}{t}$$

$$\frac{dx}{dt} = \frac{d}{dt}\left(t + \frac{1}{t}\right) = 1 - \frac{1}{t^2}$$

Again, $y = t - \frac{1}{t}$

$$\frac{dy}{dt} = \frac{d}{dt}\left(t - \frac{1}{t}\right) = 1 + \frac{1}{t^2}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = \frac{t^2 + 1}{t^2 - 1}$$

Example 7

Find $\frac{dy}{dx}$ for $x^3 + y^3 - 3axy = 0$

Solution :

$$x^3 + y^3 - 3axy = 0$$

$$\frac{d}{dx}(x^3 + y^3 - 3axy) = 0$$

or, $\frac{dx^3}{dx} + \frac{dy^3}{dy}\frac{dy}{dx} - 3a\left\{x\frac{dy}{dx} + y\frac{dx}{dx}\right\} = 0$

or, $3x^2 + 3y^2\frac{dy}{dx} - 3ax\frac{dy}{dx} - 3ay = 0$

or, $3(y^2 - ax)\frac{dy}{dx} = 3(ay - x^2)$

$\therefore \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$

Example 8

Find $\frac{dy}{dx}$ for $xy^2 = (x + 2y)^3$

Solution :

We have, $xy^2 = (x + 2y)^3$

Then, $\frac{d}{dx}(xy^2) = \frac{d}{dx}(x + 2y)^3$

$x\frac{dy^2}{dy} \cdot \frac{dy}{dx} + y^2\frac{d}{dx}(x) = \frac{d(x + 2y)^3}{d(x + 2y)} \cdot \frac{d(x + 2y)}{dx}$

$x.2y\frac{dy}{dx} + y^2.1 = 3(x + 2y)^2.\left(1 + 2\frac{dy}{dx}\right)$

or, $2xy\dfrac{dy}{dx} + y^2 = 3(x + 2y)^2 + 6(x + 2y)^2\dfrac{dy}{dx}$

or, $\{2xy - 6(x + 2y)^2\}\dfrac{dy}{dx} = 3(x + 2y)^2 - y^2$

$$\frac{dy}{dx} = \frac{3(x + 2y)^2 - y^2}{2xy - 6(x + 2y)^2}$$

$$= \frac{\dfrac{3(x + 2y)^3}{x + 2y} - y^2}{2xy - \dfrac{6(x + 2y)^3}{x + 2y}}$$

$$= \frac{3xy^2 - y^2(x + 2y)}{2xy(x + 2y) - 6xy^2} \qquad (\because xy^2 = (x + 2y)^3)$$

$$= \frac{2y^2(x - y)}{2xy(x - y)} = \frac{y}{x}$$

Example 9

Differentiate $(3x - 1)^2$ w.r.t. $2x + 1$.

Solution :

Let $y = (3x - 1)^2$

Then, $\dfrac{dy}{dx} = \dfrac{d}{dx}(3x - 1)^2$

$$= \frac{d(3x - 1)^2}{d(3x - 1)} \cdot \frac{d(3x - 1)}{dx}$$

$$= 2.(3x - 1).3.1 = 6(3x - 1)$$

Again let $z = 2x + 1$

Then, $\dfrac{dz}{dx} = \dfrac{d}{dx}(2x + 1) = 2$

Now, $\dfrac{d(3x - 1)^2}{d(2x + 1)} = \dfrac{dy}{dz} = \dfrac{dy/dx}{dz/dx} = \dfrac{6(3x - 1)}{2}$

$$= 3(3x - 1)$$

2 marks (sure?)

By definition $\dfrac{dy}{dx} = \lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$

ratio of Δy & Δx must be found first

EXERCISE 16.1

1. Find, from definition, the derivatives of the following:

or first principle

(i) $3x^2$	(ii) $x^2 - 2$	(iii) $x^2 + 5x - 3$
(iv) $3x^2 - 2x + 1$	(v) $\dfrac{1}{x}$	(vi) $\dfrac{3}{2x^2}$
(vii) $\dfrac{1}{x - 1}$	(viii) $\dfrac{1}{5 - x}$	(ix) $\dfrac{1}{2x + 3}$

(x) $\frac{ax+b}{x}$ (xi) $x^{1/2}$ (xii) $x+\sqrt{x}$

(xiii) $(1+x)^{1/2}$ (xiv) $(2x+3)^{1/2}$ (xv) $(1+x^2)^{1/2}$

(xvi) $\frac{1}{x^{1/2}}$ (xvii) $\frac{1}{(1-x)^{1/2}}$ (xviii) $\frac{1}{(3x+4)^{1/2}}$

(xix) $\frac{ax+b}{\sqrt{x}}$

2. Find the derivatives of the following:

(i) x^5 (ii) $5x^7$

(iii) $3x^2-5x+7$ (iv) $\frac{3x^3+2x-1}{2x^2}$

(v) $2x^{3/4}-3x^{1/2}-5x^{1/4}$ (vi) $\frac{2x+3x^{3/4}+x^{1/2}+1}{x^{1/4}}$

3. Use the product rule to calculate the derivatives of :

(i) $3x^2(2x-1)$ (ii) $(2x^2+1)(3x^2-2)$

(iii) $(3x^4+5)(4x^5-3)$ (iv) $(3x^2+5x-1)(x^2+3)$

(v) $(a+\sqrt{x})(a-\sqrt{x})$

4. Use the quotient rule to find the derivatives of :

(i) $\frac{x}{1+x}$ (ii) $\frac{x^2}{1-x^2}$ (iii) $\frac{x^2-a^2}{x^2+a^2}$

(iv) $\frac{3}{x^2}$ (v) $\frac{x^2-2x}{x+1}$

5. Use the general power rule to calculate the derivatives of :

(i) $(2x+3)^2$ (ii) $(3-2x)^3$ (iii) $(3x^2+2x-1)^4$

(iv) $(2x^2+3x-3)^{-6}$ (v) $\sqrt{(8-5x)}$ (vi) $(2x^2-3x+1)^{3/4}$

(vii) $\frac{1}{\sqrt{(ax^2+bx+c)}}$ (viii) $\frac{1}{\sqrt[3]{3x^2-4x-1}}$ (ix) $\frac{1}{\sqrt{a^n-x^n}}$

(x) $\frac{1}{\sqrt{(x+a)}-\sqrt{x}}$ (xi) $\frac{1}{x-\sqrt{a^2+x^2}}$ (xii) $\frac{1}{\sqrt{x+a}+\sqrt{x-a}}$

6. Use the chain rule to calculate $\frac{dy}{dx}$, if

(i) $y=2u^2-3u+1$ and $u=2x^2$. (ii) $y=2u^2+3$ and $u=3x^2-1$

(iii) $y=5t^2+6t-7$ and $t=x^3-2$ (iv) $y=\frac{t-1}{2t}$ and $t=\sqrt{x+1}$

(v) $y=(2u^2+3)^{1/3}$ and $u=\sqrt{(2x+1)}$

(vi) $y=\frac{t}{t^2-1}$ and $t=3x^2+1$.

implicit function is the function having both x y i.e. f(x,y)

7. Use implicit differentiation to obtain $\frac{dy}{dx}$ in the following.

(i) $x^2 + y^2 = 16$ (ii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(iii) $y^2 = 4ax$ (iv) $2x^2 + 3xy + 2y^2 = 0$

(v) $x^2 + 2x^2y = y^3$ (vi) $x^3y^6 = (x + y)^9$

(vii) $x^2y + xy^2 = a^3$ (viii) $x^3 + y^3 = 3xy^2$

(ix) $x^2y^2 = x^2 + y^2$

8. Find $\frac{dy}{dx}$ when

a) $x = 2a(t^2 + 1), y = 2at^3$ b) $x = 2t^2 - 1, y = t^4 - 3$

c) $x = u^3 + 3u + 1, y = 3u^2 + 1$ d) $x = \frac{2a}{t}, y = \frac{1}{t^2}$

e) $x = u^2 - 1, y = \frac{u-1}{2u}$ f) $x = \frac{2au}{1+u}, y = \frac{2au}{1-u}$

9. i) Differentiate x^6 w.r.t. x^2. ii) Differentiate $(3x + 1)^4$ w.r.t. $(3x + 1)$

iii) Differentiate $2x^6 - 3x^4 + x^2$ w.r.t. x^3 iv) Differentiate $(5x - 1)^6$ w.r.t. $(x + 1)^2$

Answers

1. (i) $6x$ (ii) $2x$ (iii) $2x + 5$ (iv) $6x - 2$ (v) $-\frac{1}{x^2}$

(vi) $-\frac{3}{x^3}$ (vii) $-\frac{1}{(x-1)^2}$ (viii) $\frac{1}{(5-x)^2}$ (ix) $-\frac{2}{(2x+3)^2}$ (x) $-\frac{b}{x^2}$

(xi) $\frac{1}{2}x^{-1/2}$ (xii) $1 + \frac{1}{2}x^{-1/2}$ (xiii) $\frac{1}{2}(1 + x)^{-1/2}$ (xiv) $(2x + 3)^{-1/2}$ (xv) $\frac{x}{\sqrt{1+x^2}}$

(xvi) $-\frac{1}{2x^{3/2}}$ (xvii) $\frac{1}{2(1-x)^{3/2}}$ (xviii) $\frac{-3}{2(3x+4)^{3/2}}$ (xix) $\frac{ax-b}{2x^{3/2}}$

2. (i) $5x^4$ (ii) $35x^6$ (iii) $6x - 5$ (iv) $\frac{3x^3 - 2x + 2}{2x^3}$

(v) $\frac{6x^{1/2} - 6x^{1/4} - 5}{4x^{3/4}}$ (vi) $\frac{3}{2x^{1/4}} + \frac{3}{2x^{1/2}} + \frac{1}{4x^{3/4}} - \frac{1}{4x^{5/4}}$

3. (i) $6x(3x - 1)$ (ii) $2x(12x^2 - 1)$ (iii) $4x^3(27x^5 + 25x - 9)$ (iv) $12x^3 + 15x^2 + 16x + 15$

(v) -1

4. (i) $\frac{1}{(1+x)^2}$ (ii) $\frac{2x}{(1-x^2)^2}$ (iii) $\frac{4a^2x}{(x^2+a^2)^2}$ (iv) $-\frac{6}{x^3}$ (v) $\frac{x^2+2x-2}{(x+1)^2}$

5. (i) $4(2x + 3)$ (ii) $-6(3 - 2x)^2$ (iii) $8(3x + 1)(3x^2 + 2x - 1)^3$

(iv) $-6(4x + 3)(2x^2 + 3x - 3)^{-7}$ (v) $\frac{-5}{2\sqrt{8-5x}}$ (vi) $\frac{3}{4}(4x - 3)(2x^2 - 3x + 1)^{-1/4}$

(vii) $-\frac{1}{2}(2ax + b)(ax^2 + bx + c)^{-3/2}$ (viii) $-\frac{2}{3}(3x - 2)(3x^2 - 4x - 1)^{-4/3}$

(ix) $\frac{1}{2}nx^{n-1}(a^n - x^n)^{-3/2}$ (x) $\frac{1}{2a}\left(\frac{1}{\sqrt{x+a}} + \frac{1}{\sqrt{x}}\right)$ (xi) $-\frac{1}{a^2}\left(1 + \frac{x}{\sqrt{a^2+x^2}}\right)$

(xii) $\frac{1}{4a}\left(\frac{1}{\sqrt{x+a}} - \frac{1}{\sqrt{x-a}}\right)$

6. (i) $4x(8x^2-3)$ (ii) $24x(3x^2-1)$ (iii) $6x^2(5x^3-7)$

(iv) $\frac{1}{4(x+1)^{3/2}}$ (v) $\frac{4}{3}(4x+5)^{-2/3}$ (vi) $-\frac{2(9x^4+6x^2+2)}{3x^3(3x^2+2)^2}$

7. (i) $-\frac{x}{y}$ (ii) $\frac{b^2x}{a^2y}$ (iii) $\frac{2a}{y}$ (iv) $-\frac{(4x+3y)}{(3x+4y)}$ (v) $\frac{2x(1+2y)}{3y^2-2x^2}$

(vi) $\frac{y}{x}$ (vii) $-\frac{y(2x+y)}{x(x+2y)}$ (viii) $\frac{y^2-x^2}{y(y-2x)}$ (ix) $\frac{x(1-y^2)}{y(x^2-1)}$

8. a) $\frac{3t}{2}$ b) t^2 c) $\frac{2u}{u^2+1}$ d) $\frac{-1}{at}$ e) $\frac{1}{4u^3}$ f) $\frac{(1+u)^2}{(1-u)^2}$

9. (i) $3x^4$ (ii) $4(3x+1)^3$ (iii) $4x^3-4x+\frac{2}{3x}$ (iv) $\frac{15(5x-1)^5}{x+1}$

Multiple Choice Questions

1. The derivative of $y = f(x)$ w.r.t. x is denoted by
   a) $\frac{dy}{dx}$ b) y' c) $\frac{df}{dx}$ d) all of the above

2. By the definition, $\frac{dy}{dx} =$
   a) $\frac{\Delta y}{\Delta x}$ b) $\frac{f(x+\Delta x)-f(x)}{\Delta x}$
   c) $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$, provided that the limit exists
   d) none of the above

3. If $y = f(x) = 3x - 1$ and the value of x changes from 2 to 2.01 then $\Delta y =$
   a) 0.01 b) 0.02 c) 0.03 d) 0.04

0.01 then $\Delta y =$

16.8 Derivatives of the Trigonometrical Functions

(i) Derivative of sin x

Let $y = \sin x$ .... (i)

Let Δx be a small increment in x and Δy be the corresponding increment in y. Then

$y + \Delta y = \sin(x + \Delta x)$ .... (ii)

Now subtracting (i) from (ii) we get

$$\Delta y = \sin(x + \Delta x) - \sin x = 2\sin\frac{\Delta x}{2}\cdot\cos\frac{2x + \Delta x}{2}$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\cdot\cos\frac{2x + \Delta x}{2}$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\left(\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\cdot\cos\frac{2x + \Delta x}{2}\right) = \cos x$$

Hence $\dfrac{d(\sin x)}{dx} = \cos x$ sin

(ii) Derivative of cos x

With the same process that we followed to derive the derivative of sin x, we get the derivative of cos x, as $\dfrac{d(\cos x)}{dx} = -\sin x$

(iii) Derivative of tan x

Let $y = \tan x$ .... (i)

Let Δx be a small increment in x and Δy be the corresponding increment in y. Then

$y + \Delta y = \tan(x + \Delta x)$ ...... (ii)

Now subtracting (i) from (ii), we get

$$\Delta y = \tan(x + \Delta x) - \tan x$$

$$= \frac{\sin(x + \Delta x)}{\cos(x + \Delta x)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin(x + \Delta x)\cos x - \cos(x + \Delta x)\sin x}{\cos(x + \Delta x)\cos x}$$

$$= \frac{\sin(x + \Delta x - x)}{\cos(x + \Delta x)\cos x}$$

$$= \frac{\sin \Delta x}{\cos(x + \Delta x)\cos x}$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{\sin \Delta x}{\Delta x} \cdot \frac{1}{\cos(x + \Delta x)\cos x}$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left(\frac{\sin \Delta x}{\Delta x} \cdot \frac{1}{\cos(x + \Delta x)\cos x} \right)$$

$$= \frac{1}{\cos x \,.\, \cos x} = \sec^2 x$$

Hence, $\dfrac{d(\tan x)}{dx} = \sec^2 x$

(iv) Derivative of cot x

With the same process that we followed to derive the derivative of $\tan x$, we can get the derivative of $\cot x$, as $\dfrac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$

(v) Derivative of sec x

Let $\quad y = \sec x = \dfrac{1}{\cos x}$ ...... (i)

Let Δx be a small increment in x and Δy be the corresponding increment in y.

Then $\quad y + \Delta y = \dfrac{1}{\cos(x + \Delta x)}$ ...... (ii)

Now subtracting (i) from (ii), we get

$$\Delta y = \frac{1}{\cos(x + \Delta x)} - \frac{1}{\cos x}$$

$$= \frac{\cos x - \cos(x + \Delta x)}{\cos(x + \Delta x)\cos x}$$

$$= \frac{2 \sin \frac{2x + \Delta x}{2} \sin \frac{\Delta x}{2}}{\cos x . \cos(x + \Delta x)}$$

$$\text{or, } \frac{\Delta y}{\Delta x} = \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \frac{\sin \frac{2x + \Delta x}{2}}{\cos x . \cos(x + \Delta x)}$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \frac{\sin \frac{2x + \Delta x}{2}}{\cos x . \cos(x + \Delta x)} \right)$$

$$= \frac{\sin x}{\cos x \,.\, \cos x} = \tan x \sec x.$$

Hence, $\dfrac{d(\sec x)}{dx} = \sec x . \tan x$

f) Derivative of cosec x

With the same process that we followed to derive the derivative of sec x, we can get the derivative of cosec x, as $\frac{d(\text{cosec } x)}{dx} = -\text{cosec } x.\cot x$

16.9 Derivatives of Inverse Trigonometric Functions

Let $y = f(x)$ be the given function. Again let Δx and Δy be the small increments in x and y respectively.

a) Derivative of $\sin^{-1}x$

$$\text{Let} \quad y = \sin^{-1}x, \quad \text{then}$$

$$y + \Delta y = \sin^{-1}(x + \Delta x)$$

$$\Delta y = \sin^{-1}(x + \Delta x) - y$$

$$\Rightarrow \quad \Delta y = \sin^{-1}(x + \Delta x) - \sin^{-1}x$$

$$\Rightarrow \quad \frac{\Delta y}{\Delta x} = \frac{\sin^{-1}(x + \Delta x) + \sin^{-1}x}{\Delta x}$$

$$\Rightarrow \quad \lim_{x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{\sin^{-1}(x + \Delta x) - \sin^{-1}x}{\Delta x} \quad \ldots\ldots\text{(i)}$$

Put $\sin^{-1}x = u$ and $\sin^{-1}(x + \Delta x) = u + h$

so that $x = \sin u$, $x + \Delta x = \sin(u + h)$

and $\Delta x = \sin(u + h) - \sin u$

When $\Delta x \to 0$, $h \to 0$

Now, from (i)

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{h\to 0}\frac{u + h - u}{\sin(u + h) - \sin u}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{h\to 0}\frac{h}{2\cos\left(u + \frac{h}{2}\right)\sin\frac{h}{2}}$$

$$= \lim_{h\to 0}\frac{1}{\cos\left(u + \frac{h}{2}\right)\frac{\sin h/2}{h/2}}$$

$$= \frac{1}{\cos u\,.\,1} = \frac{1}{\sqrt{1 - \sin^2 u}} = \frac{1}{\sqrt{1 - x^2}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}} \quad (|x| < 1)$$

b) Derivative of $\cos^{-1}x$

With the process used as in (a), we have $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}}$

c) Derivative of $\tan^{-1}x$

Let $y = \tan^{-1}x$, then

$$y + \Delta y = \tan^{-1}(x + \Delta x)$$

$$\Rightarrow \Delta y = \tan^{-1}(x + \Delta x) - \tan^{-1}x$$

$$\Rightarrow \frac{\Delta y}{\Delta x} = \frac{\tan^{-1}(x + \Delta x) - \tan^{-1}x}{\Delta x}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\tan^{-1}(x + \Delta x) - \tan^{-1}x}{\Delta x} \quad \ldots\ldots(i)$$

Put $\tan^{-1}x = u$ and $\tan^{-1}(x + \Delta x) = u + h$

so that $x = \tan u$, $x + \Delta x = \tan(u + h)$

and $\Delta x = \tan(u + h) - \tan u$

When $\Delta x \to 0, h \to 0$

Now, from (i)

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{u + h - u}{\tan(u + h) - \tan u}$$

$$\Rightarrow \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{h}{\frac{\sin(u+h)}{\cos(u+h)} - \frac{\sin u}{\cos u}}$$

$$= \lim_{h \to 0} \frac{\cos(u+h)\cos u\, h}{\sin(u+h)\cos u - \cos(u+h)\sin u}$$

$$= \lim_{h \to 0} \frac{\cos(u+h)\cos u\, h}{\sin(u + h - u)}$$

$$= \lim_{h \to 0} \frac{\cos(u+h)\cos u}{\sin h/h} = \cos^2 u$$

$$= \frac{1}{\sec^2 u} = \frac{1}{1 + \tan^2 u} = \frac{1}{1 + x^2}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2}$$

d) Derivative of $\cot^{-1}x$

With the process used as in (c), we have $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1 + x^2}$

e) Derivative of $\mathrm{cosec}^{-1}x$

Let $y = \mathrm{cosec}^{-1}x$. Then,

$$y + \Delta y = \mathrm{cosec}^{-1}(x + \Delta x)$$

$$\Rightarrow \Delta y = \mathrm{cosec}^{-1}(x + \Delta x) - \mathrm{cosec}^{-1}x$$

$$\Rightarrow \frac{\Delta y}{\Delta x} = \frac{\mathrm{cosec}^{-1}(x + \Delta x) - \mathrm{cosec}^{-1}x}{\Delta x}$$

$$\Rightarrow \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{\text{cosec}^{-1}(x+\Delta x)-\text{cosec}^{-1}x}{\Delta x} \quad ......(i)$$

Put $\text{cosec}^{-1}x = u$ and $\text{cosec}^{-1}(x+\Delta x) = u+h$

so that $x = \text{cosec}\, u$, $x+\Delta x = \text{cosec}\,(u+h)$

and $\Delta x = \text{cosec}\,(u+h) - \text{cosec}\, u$

When $\Delta x \to 0, h \to 0$

Now from (i),

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{h\to 0}\frac{u+h-u}{\text{cosec}\,(u+h)-\text{cosec}\,u}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{h\to 0}\frac{\sin(u+h)\,.\,\sin u\,.\,h}{\sin u - \sin(u+h)}$$

$$= \lim_{h\to 0}\frac{\sin(u+h)\,.\,\sin u\,.\,h}{2\cos\left(u+\frac{h}{2}\right)\sin\left(-\frac{h}{2}\right)}$$

$$= -\lim_{h\to 0}\frac{\sin(u+h)\,.\,\sin u}{\cos\left(u+\frac{h}{2}\right)\frac{\sin h/2}{h/2}}$$

$$= -\frac{\sin^2 u}{\cos u\,.\,1} = -\frac{1}{\cot u\,\text{cosec}\,u}$$

$$= -\frac{1}{\text{cosec}\,u\sqrt{\text{cosec}^2u-1}} = -\frac{1}{x\sqrt{x^2-1}}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\text{cosec}^{-1}x) = -\frac{1}{x\sqrt{x^2-1}} \quad (|x|>1)$$

f) Derivative of $\sec^{-1}x$

With the process used as in (e), we have $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$

Worked Out Examples

Example 1

Find, from first principles, the differential coefficient of $\sin(ax+b)$.

Solution:

Let $y = \sin(ax+b)$

If Δx is a small increment in x and Δy the corresponding increment in y, then

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{\sin[a(x+\Delta x)+b]-\sin(ax+b)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2\sin\left(\frac{ax + a\Delta x + b - ax - b}{2}\right)\cos\left(\frac{ax + a\Delta x + b + ax + b}{2}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{\sin\frac{a\Delta x}{2}}{\frac{a\,\Delta x}{2}}\right) . \, a \, . \cos\left(ax + b + \frac{a\,\Delta x}{2}\right)$$

$= 1 \, . \, a \cos(ax + b)$

$= a \cos(ax + b).$

Example 2

Find the derivative of $\sin(ax^2 - b)$

Solution:

Let $y = \sin(ax^2 - b)$. Put $u = ax^2 - b$.

$\therefore \quad \frac{du}{dx} = 2ax$

Also $\quad y = \sin u \qquad \therefore \frac{dy}{du} = \cos u$

Now, $\quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u . \, 2ax = 2ax \cos(ax^2 - b)$

This can be worked out directly also

Let $\quad y = \sin(ax^2 - b)$. Then

$$\frac{dy}{dx} = \frac{d \sin(ax^2 - b)}{dx} = \frac{d \sin(ax^2 - b)}{d(ax^2 - b)} \cdot \frac{d(ax^2 - b)}{dx}$$

$= \cos(ax^2 - b) \, . \, 2ax$

$= 2ax \cos(ax^2 - b)$

Example 3

Find the derivative of $\sqrt{\tan x^2}$

Solution:

Let $y = \sqrt{\tan x^2}$. Put $u = \tan x^2$ and $v = x^2$

Then $\quad y = \sqrt{u} = u^{1/2}$. So

$\frac{dy}{du} = \frac{1}{2} u^{-1/2} = \frac{1}{2} (\tan x^2)^{-1/2}$

$\frac{du}{dv} = \frac{d \tan v}{dv} = \sec^2 v = \sec^2 x^2.$

$\frac{dv}{dx} = \frac{dx^2}{dx} = 2x$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$

$$= \frac{1}{2}(\tan x^2)^{-1/2} \sec^2 x^2 \,.\, 2x$$

$$= \frac{x \sec^2 x^2}{\sqrt{\tan x^2}}$$

Alt. Method. $\frac{dy}{dx} = \frac{d(\tan x^2)^{1/2}}{dx}$

$$= \frac{d(\tan x^2)^{1/2}}{d(\tan x^2)} \cdot \frac{d(\tan x^2)}{dx^2} \cdot \frac{dx^2}{dx}$$

$$= \frac{1}{2}(\tan x^2)^{-1/2} \sec^2 x^2 \,.\, 2x$$

$$= \frac{x \sec^2 x^2}{\sqrt{\tan x^2}}$$

Example 4

Find the derivative of $x^2 \sec(ax - b)$

Solution:

Let $y = x^2 \sec(ax - b)$. Then

$$\frac{dy}{dx} = x^2 \frac{d \sec(ax-b)}{dx} + \sec(ax-b) \,.\, \frac{dx^2}{dx}$$

$$= x^2 \,.\, \frac{d \sec(ax-b)}{d(ax-b)} \cdot \frac{d(ax-b)}{dx} + \sec(ax-b) \,.\, 2x$$

$$= x^2 \sec(ax-b) \tan(ax-b) \,.\, a + 2x \sec(ax-b)$$

$$= x \sec(ax-b) \{ax \tan(ax-b) + 2\}$$

Example 5

Find the derivative of $\frac{1 + \sin x}{1 - \sin x}$

Solution:

Let $y = \frac{1 + \sin x}{1 - \sin x}$. Then

$$\frac{dy}{dx} = \frac{(1 - \sin x) \,.\, \frac{d(1 + \sin x)}{dx} - (1 + \sin x) \frac{d(1 - \sin x)}{dx}}{(1 - \sin x)^2}$$

$$= \frac{(1 - \sin x)\cos x + (1 + \sin x)\cos x}{(1 - \sin x)^2}$$

$$= \frac{2\cos x}{(1 - \sin x)^2}$$

Example 6

Find the derivative of $\frac{1 - \tan x}{\sec x}$.

Solution:

Let $y = \dfrac{1 - \tan x}{\sec x} = \cos x\,(1 - \tan x)$

$= \cos x - \sin x$

$\therefore \dfrac{dy}{dx} = \dfrac{d\cos x}{dx} - \dfrac{d\sin x}{dx}$

$= -\sin x - \cos x.$

Example 7

Find $\dfrac{dy}{dx}$, when $x - y = \sin xy$.

Solution:

We have $x - y = \sin xy$.

Differentiating both sides with respect to x, we get

$$\frac{d(x-y)}{dx} = \frac{d(\sin xy)}{dx}$$

$$\text{or, } 1 - \frac{dy}{dx} = \frac{d(\sin xy)}{d(xy)} \cdot \frac{d(xy)}{dx}$$

$$= \cos xy\left(y + x\frac{dy}{dx}\right)$$

$$= y\cos xy + x\cos xy \cdot \frac{dy}{dx}$$

$$\text{or, } (1 + x\cos xy)\frac{dy}{dx} = 1 - y\cos xy$$

$$\therefore \frac{dy}{dx} = \frac{1 - y\cos xy}{1 + x\cos xy}$$

Example 8

Find $\dfrac{dy}{dx}$, when $y = 2\theta - \tan\theta$ and $x = \tan\theta$

Solution:

We have $y = 2\theta - \tan\theta$ and $x = \tan\theta$

$\dfrac{dy}{d\theta} = 2 - \sec^2\theta$ and $\dfrac{dx}{d\theta} = \sec^2\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= \frac{2 - \sec^2\theta}{\sec^2\theta} = 2\cos^2\theta - 1$$

$$= \cos 2\theta$$

Example 9

Find $\frac{dy}{dx}$ when

i) $y = \tan^{-1}\frac{2x}{1-x^2}$

ii) $y = x^2 \sin^{-1}x$

Solution:

i) $y = \tan^{-1}\frac{2x}{1-x^2}$

Put $x = \tan\theta$. Then,

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta)$$

$$= 2\theta = 2\tan^{-1}x$$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1}x) = 2\frac{d}{dx}(\tan^{-1}x) = 2 \cdot \frac{1}{1+x^2}$$

$$= \frac{2}{1+x^2}$$

ii) $y = x^2 \sin^{-1}x$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 \sin^{-1}x)$$

$$= x^2\frac{d}{dx}(\sin^{-1}x) + \sin^{-1}x\frac{d}{dx}(x^2)$$

$$= x^2 \cdot \frac{1}{\sqrt{1-x^2}} + \sin^{-1}x \cdot 2x$$

$$= x\left(\frac{x}{\sqrt{1-x^2}} + 2\sin^{-1}x\right)$$

EXERCISE 16.2

1. Find, from the first principles, the differential coefficients of

(i) $\sin 4x$ (ii) $\cos(ax - b)$ (iii) $\tan(3x - 4)$

(iv) $\sin\frac{3x}{2}$ (v) $\tan\frac{5x}{3}$ (vi) $\cos^2 x$

(vii) $\sin^2 3x$ (viii) $\sqrt{\sin 2x}$ (ix) $\sqrt{\sec x}$

2. Find the derivatives of

(i) $\sin(4x-5)$ (ii) $\cos(ax+b)$ (iii) $\tan(5x^2+6)$

(iv) $\cot\sqrt{x}$ (v) $\sec\frac{1}{x}$ (vi) $\operatorname{cosec}\frac{ax^2}{b}$

(vii) $\sin^5(ax^2-c)$ (viii) $\cos^3(2ax-3b)$ (ix) $a\sqrt{\tan(5x-7)}$

(x) $\sec^5\left(\frac{ax+b}{c}\right)$ (xi) $\operatorname{cosec}^n\left(\frac{px^2-q}{r}\right)$

3. Find the differential coefficients of

(i) $\tan(\cos 5x)$ (ii) $\cos(\sin(3x^2+2))$ (iii) $\sin(\tan(ax+b))$

(iv) $\sec^2(\tan\sqrt{x})$ (v) $\tan^5(\sin(px-q))$

(vi) $\operatorname{cosec}^3(\cot 4x)$

(vii) $\cot(\sqrt{\tan 3x})$ (viii) $\sin^2(\cos 6x)$

4. Find the derivatives of

(i) $(x^2+3x)\sin 5x$ (ii) $x^3\tan(2x^3+3x)$ (iii) $a\sqrt{x}\cos(ax^2-b)$

(iv) $(x+\sin 2x)\sec 3x^2$ (v) $ax^3\operatorname{cosec}(p-qx)$ (vi) $\frac{1}{\sqrt{x}}\cdot\sin\sqrt{x}$

(vii) $\frac{x^2-1}{\cos 4x}$ viii) $\frac{\sec nx}{ax-b}$

5. Find the differential coefficients of

(i) $\frac{1-2\sin^2\frac{x}{2}}{\cos^2 x}$ (ii) $\frac{\sin 2nx}{\cos^2 nx}$ (iii) $\frac{\sin ax-\sin bx}{\cos ax+\cos bx}$

(iv) $\frac{1-\cos x}{1+\cos x}$ (v) $\sqrt{\frac{1-\sin x}{1+\sin x}}$ (vi) $\frac{\cos 2x+1}{\sin 2x}$

(vii) $\frac{\cos 2x}{1-\sin 2x}$ (viii) $\frac{1}{\sec x-\tan x}$ (ix) $\frac{\sec x+\tan x}{\sec x-\tan x}$

(x) $\frac{1+\tan x}{1-\tan x}$

6. Find the derivatives of

(i) $\sin 6x\cos 4x$ (ii) $\sin 2mx\sin 2nx$

(iii) $\cos 7x\,.\,\cos 5x$ (iv) $\sin 3x\,.\,\cos 5x$

7. Find $\frac{dy}{dx}$, when

(i) $x+y=\sin y$ (ii) $x+y=\cos(x-y)$

(iii) $x-y=\tan xy$ (iv) $x^2y=\sec xy^2$

(v) $x^2y^2=\tan(ax+by)$ (vi) $x^2+y^2=\sin xy$

(vii) $x^2y^2=\tan xy$ (viii) $xy=\sec(x-y)$

(ix) $xy=\tan(x^2+y^2)$

8. Find $\frac{dy}{dx}$, when

(i) $x = a\cos^2\theta$, $y = b\sin^2\theta$

(ii) $x = 2a\sin t\cos t$, $y = b\cos 2t$

(iii) $x = 2a\tan\theta$, $y = a\sec^2\theta$

(iv) $x = \tan t$, $y = \sin t\cos t$

(v) $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$

(vi) $x = a(\tan t - t\sec^2 t)$, $y = a\sec^2 t$

(vii) $x = a(t + \sin t)$, $y = a(1 - \cos t)$

9. Differentiate

(i) $\sin x$ with respect to $\cos x$.

(ii) $\tan x$ with respect to $\sec x$

(iii) $\sec^2 x$ with respect to $\tan x$

(iv) $\text{cosec}\, x$ with respect to $\cot x$

10. Find the derivatives of

a) $\cos x^\circ$

b) $x\sin x^\circ$

11. Find the derivatives of

a) $\sin^{-1}\frac{2x}{1+x^2}$

b) $\cos^{-1}\frac{1-x^2}{1+x^2}$

c) $\tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right)$

d) $\sec^{-1}\frac{1}{\sqrt{1-x^2}}$

e) $\sin^{-1}(1-2x^2)$

f) $\cos^{-1}(4x^3-3x)$

g) $\sin^{-1}(3x-4)$

h) $\sec^{-1}(\tan x)$

Answers

1. (i) $4\cos 4x$ (ii) $-a\sin(ax-b)$ (iii) $3\sec^2(3x-4)$ (iv) $\frac{3}{2}\cos\frac{3x}{2}$

(v) $\frac{5}{3}\sec^2\frac{5x}{3}$ (vi) $-\sin 2x$ (vii) $3\sin 6x$ (viii) $\frac{\cos 2x}{\sqrt{\sin 2x}}$

(ix) $\frac{1}{2}\sqrt{\sec x}\,.\tan x$

2. (i) $4\cos(4x-5)$ (ii) $-a\sin(ax+b)$ (iii) $10x\sec^2(5x^2+6)$

(iv) $-\frac{1}{2\sqrt{x}}\text{cosec}^2\sqrt{x}$ (v) $-\frac{1}{x^2}\sec\frac{1}{x}\tan\frac{1}{x}$ (vi) $-\frac{2ax}{b}\text{cosec}\frac{ax^2}{b}\cot\frac{ax^2}{b}$

(vii) $10ax\sin^4(ax^2-c)\cos(ax^2-c)$ (viii) $-6a\cos^2(2ax-3b)\sin(2ax-3b)$

(ix) $\frac{5a\sec^2(5x-7)}{2\sqrt{\tan(5x-7)}}$ (x) $\frac{5a}{c}\sec^5\frac{ax+b}{c}\tan\frac{ax+b}{c}$

(xi) $\frac{-2npx}{r}\text{cosec}^n\left(\frac{px^2-q}{r}\right)\cot\left(\frac{px^2-q}{r}\right)$

3. (i) $-5\sec^2(\cos 5x)\sin 5x$ (ii) $-6x\sin(\sin(3x^2+2))\cos(3x^2+2)$

(iii) $a\cos(\tan(ax+b))\sec^2(ax+b)$ (iv) $\frac{1}{\sqrt{x}}\sec^2(\tan\sqrt{x})\tan(\tan\sqrt{x})\sec^2\sqrt{x}$

(v) $5p\tan^4(\sin(px-q))\sec^2(\sin(px-q))\cos(px-q)$

$\pi = 180^\circ$ ...

$1^\circ = \frac{\pi}{180}$

(vi) $12\ \text{cosec}^3(\cot 4x) \cot(\cot 4x)\ \text{cosec}^2 4x$ (vii) $\dfrac{-3}{2\sqrt{\tan 3x}} \text{cosec}^2\left(\sqrt{\tan 3x}\right) \sec^2 3x$

(viii) $-6 \sin(2\cos 6x) \sin 6x$

4. (i) $(2x + 3) \sin 5x + 5(x^2 + 3x) \cos 5x$ (ii) $3x^2 \tan(2x^3 + 3x) + 3x^3(2x^2 + 1) \sec^2(2x^3 + 3x)$

(iii) $\dfrac{a}{2\sqrt{x}} \cos(ax^2 - b) - 2a^2 x^{3/2} \sin(ax^2 - b)$

(iv) $(1 + 2\cos 2x) \sec 3x^2 + 6x(x + \sin 2x) \sec 3x^2 . \tan 3x^2$

(v) $3ax^2\ \text{cosec}(p - qx) + aqx^3\ \text{cosec}(p - qx) \cot(p - qx)$

(vi) $\dfrac{1}{2x}\left(\cos\sqrt{x} - \dfrac{1}{\sqrt{x}} \sin\sqrt{x}\right)$ (vii) $\dfrac{2x\cos 4x + 4(x^2 - 1)\sin 4x}{\cos^2 4x}$

(viii) $\dfrac{n(ax - b) \sec nx \tan nx - a \sec nx}{(ax - b)^2}$

5. (i) $\sec x \tan x$ (ii) $2n \sec^2 nx$ (iii) $\dfrac{1}{2}(a - b) \sec^2 \dfrac{1}{2}(a - b)x$

(iv) $\tan\dfrac{1}{2}x \sec^2\dfrac{1}{2}x$ (v) $-\dfrac{1}{2}\sec^2\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right)$ (vi) $-\text{cosec}^2 x$

(vii) $\sec^2\left(\dfrac{\pi}{4} + x\right)$ (viii) $\sec x(\tan x + \sec x)$

(ix) $2\sec x(\sec x + \tan x)^2$ (x) $\sec^2\left(\dfrac{\pi}{4} + x\right)$

6. (i) $5\cos 10x + \cos 2x$ (ii) $(m + n)\sin 2(m + n)x - (m - n)\sin 2(m - n)x$

(iii) $-6\sin 12x - \sin 2x$ (iv) $4\cos 8x - \cos 2x$

7. (i) $\dfrac{1}{\cos y - 1}$ (ii) $\dfrac{1 + \sin(x - y)}{\sin(x - y) - 1}$ (iii) $\dfrac{1 - y\sec^2 xy}{1 + x\sec^2 xy}$

(iv) $\dfrac{2xy - y^2\sec xy^2 \tan xy^2}{2xy \sec xy^2 \tan xy^2 - x^2}$ (v) $\dfrac{2xy^2 - a\sec^2(ax + by)}{b\sec^2(ax + by) - 2x^2 y}$ (vi) $\dfrac{2x - y\cos xy}{x\cos xy - 2y}$

(vii) $-\dfrac{y}{x}$ (viii) $\dfrac{\sec(x - y)\tan(x - y) - y}{x + \sec(x - y)\tan(x - y)}$ (ix) $\dfrac{y - 2x\sec^2(x^2 + y^2)}{2y\sec^2(x^2 + y^2) - x}$

8. (i) $\dfrac{-b}{a}$ (ii) $\dfrac{-b}{a}\tan 2t$ (iii) $\tan\theta$ (iv) $\cos^2 t(\cos^2 t - \sin^2 t)$

(v) $\tan t$ (vi) $-\dfrac{1}{t}$ (vii) $\tan\dfrac{1}{2}t$

9. (i) $-\cot x$ (ii) $\text{cosec}\, x$ (iii) $2\tan x$ (iv) $\cos x$

10. a) $-\dfrac{\pi}{180}\sin x^\circ$ b) $\dfrac{\pi x}{180}\cos x^\circ + \sin x^\circ$

11. a) $\dfrac{2}{1 + x^2}$ b) $\dfrac{2}{1 + x^2}$ c) 1 d) $\dfrac{1}{\sqrt{1 - x^2}}$ e) $\dfrac{-2}{\sqrt{1 - x^2}}$ f) $-\dfrac{3}{\sqrt{1 - x^2}}$

g) $\dfrac{3}{\sqrt{1 - (3x - 4)^2}}$ h) $\dfrac{\sec^2 x}{\tan x\sqrt{\tan^2 x - 1}}$

Multiple Choice Questions

1. If $y = \tan\dfrac{2x}{3}$, then $\dfrac{dy}{dx} =$

$3 \quad 2x \quad 2 \quad 2x \quad 2 \quad 2x \quad 2 \quad 2x \tan\dfrac{2x}{3}$

16.10 Derivatives of Exponential and Logarithmic Functions

Any function of the form

$$f(x) = a^x$$

is called an exponential function, in which the base a is a constant and the index x is a variable. The inverse of an exponential function is called a logarithmic function which is denoted by $\log_a x$.

So, if $y = a^x$, we have $\log_a y = x$

There is a special type of exponential function e^x, where e is the limiting value $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. The value of e lies between 2 and 3 and is approximately 2.718. The corresponding logarithmic function is called the natural logarithmic function and is denoted by $\log x$, base e being understood.

Thus $\quad e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

Further put $n = \frac{1}{h}$ so that $h = \frac{1}{n}$

When $n\to\infty$, $h\to 0$.

So, $\quad e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{h\to 0}(1+h)^{1/h}$

The following properties of the logarithmic functions can be easily deduced.

1. $\log_a(x.y) = \log_a x + \log_a y$
2. $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. $\log_a x^n = n \log_a x$
4. $\log_a a = 1$
5. $\log_a m = \log_a b . \log_b m$
6. $\log_a 1 = 0$

Now, we find the derivatives of the exponential and the logarithmic functions by first principles using the following limit theorems

1. $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$
2. $\lim_{x\to 0}\frac{e^x-1}{x} = 1$
3. $\lim_{x\to 0}\frac{a^x-1}{x} = \log a$

I. Derivative of e^x (from First Principles)

Let $\quad y = e^x \quad$ .... (1)

Let Δx be a small increment in x and Δy, a corresponding increment in y. Then,

$y + \Delta y = e^{x+\Delta x} \quad$ .... (2)

Subtracting (1) from (2), we get

$$\Delta y = e^{x+\Delta x} - e^x = e^x.e^{\Delta x} - e^x = e^x(e^{\Delta x} - 1)$$

or, $\quad \frac{\Delta y}{\Delta x} = \frac{e^x(e^{\Delta x}-1)}{\Delta x}$

Now, $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{e^x(e^{\Delta x} - 1)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{e^x(e^{\Delta x} - 1)}{\Delta x}$$

$$= e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x}$$

$$= e^x.1 = e^x \qquad \left(\because \lim_{x \to 0} \frac{e^x - 1}{x} = 1\right)$$

II. Derivative of Natural Logarithmic Function

Let $y = \log x$

Let Δx and Δy be the small increments in x and y respectively.

Then,

$$y + \Delta y = \log(x + \Delta x)$$

or, $$\Delta y = \log(x + \Delta x) - \log x$$

or, $$\frac{\Delta y}{\Delta x} = \frac{\log(x + \Delta x) - \log x}{\Delta x}$$

$$= \frac{\log\left(\frac{x + \Delta x}{x}\right)}{\Delta x} = \frac{1}{\Delta x} \log\left(1 + \frac{\Delta x}{x}\right)$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \log\left(1 + \frac{\Delta x}{x}\right) = \lim_{\Delta x \to 0} \frac{\log\left(1 + \frac{\Delta x}{x}\right)}{\frac{\Delta x}{x} . x}$$

$$= \frac{1}{x} \lim_{\Delta x \to 0} \frac{\log\left(1 + \frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}} = \frac{1}{x} . 1 = \frac{1}{x}$$

III. Derivative of Logarithmic Functions $\log_a x$

l ogaP

Let $y = \log_a x$

$= \log_a e . \log x$, by changing the base from a to e.

$$\therefore \frac{dy}{dx} = \log_a e \frac{d(\log x)}{dx}$$

$$= (\log_a e) . \frac{1}{x}$$

IV. Derivative of a^x (from First Principle)

Let $y = a^x$

Let Δx and Δy be the small increments in x and y respectively. Then,

$$y + \Delta y = a^{x+\Delta x}$$

or, $\Delta y = a^{x+\Delta x} - a^x$

$$= a^x (a^{\Delta x} - 1)$$

$$\frac{\Delta y}{\Delta x} = \frac{a^x (a^{\Delta x} - 1)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} a^x \frac{(a^{\Delta x} - 1)}{\Delta x}$$

$$= a^x . \log a \qquad \left(\because \lim_{\Delta x \to 0} \frac{a^x - 1}{x} = \log a\right)$$

Worked Out Examples

Example 1.

Find the derivative of $\log(5x^2 + 6)$

Solution:

Let $y = \log(5x^2 + 6)$. Put $u = 5x^2 + 6$

Then $y = \log u$ and $u = 5x^2 + 6$.

$$\because \frac{dy}{du} = \frac{d \log u}{du} = \frac{1}{u}, \quad \frac{du}{dx} = 10x.$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot 10x = \frac{10x}{5x^2 + 6}$$

Alt. Method

Let $y = \log(5x^2 + 6)$

$$\frac{dy}{dx} = \frac{d \log(5x^2 + 6)}{dx}$$

$$= \frac{d \log(5x^2 + 6)}{d(5x^2 + 6)} \cdot \frac{d(5x^2 + 6)}{dx}$$

$$= \frac{1}{5x^2 + 6} \cdot 10x = \frac{10x}{5x^2 + 6}$$

Example 2.

Find the derivative of $\sin 3x . \log(ax + b)$

Solution:

Let $y = \sin 3x . \log(ax + b)$

$$\because \frac{dy}{dx} = \frac{d(\sin 3x)}{d(3x)} \cdot \frac{d(3x)}{dx} \cdot \log(ax + b) + \sin 3x . \frac{d \log(ax + b)}{d(ax + b)} \cdot \frac{d(ax + b)}{dx}$$

$$= \cos 3x \,.\, 3 \,.\, \log(ax+b) + \sin 3x \,.\, \frac{1}{ax+b} \,.\, a$$

$$= 3 \cos 3x.\, \log(ax+b) + \frac{a \sin 3x}{ax+b}$$

Example 3.

Find the derivative of $\frac{\sin x}{\log(3x+4)}$

Solution:

Let $y = \frac{\sin x}{\log(3x+4)}$

$$\because \quad \frac{dy}{dx} = \frac{\log(3x+4)\,.\,\frac{d \sin x}{dx} - \sin x.\,\frac{d \log(3x+4)}{dx}}{\{\log(3x+4)\}^2}$$

$$= \frac{\log(3x+4)\,.\,\cos x - \sin x\,\frac{3}{3x+4}}{\{\log(3x+4)\}^2}$$

$$= \frac{(3x+4)\cos x \log(3x+4) - 3\sin x}{(3x+4)\{\log(3x+4)\}^2}$$

Example 4

Find the derivative of e^{ax+b}

Solution:

Let $y = e^{ax+b}$. Put $u = ax + b$

$\because \quad y = e^u$ and $u = ax + b$

$\frac{dy}{du} = \frac{de^u}{du}$ and $\frac{du}{dx} = \frac{d(ax+b)}{dx}$

$\frac{dy}{du} = e^u$ and $\frac{du}{dx} = a.$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \,.\, a = ae^{ax+b}.$$

Example 5.

Find the derivative of $e^{5x} \,.\, \sin 6x$

Solution:

Let $y = e^{5x} \sin 6x$. Then

$$\frac{dy}{dx} = \frac{d\,e^{5x}}{dx} \,.\, \sin 6x + e^{5x} \,.\, \frac{d \sin 6x}{dx}$$

$$= 5.e^{5x} \sin 6x + 6e^{5x} \cos 6x$$

$$= e^{5x} (5 \sin 6x + 6 \cos 6x)$$

EXERCISE 16.3

1. Find, from the first principle, the derivatives of:

   (i) $\log(ax+b)$ (ii) $\log_5 x$ (iii) $\log\frac{x}{10}$

   (iv) e^{ax+b} (v) $e^{x/3}$

2. Find the derivatives of

   (i) $\log(\sin x)$ (ii) $\log(x+\tan x)$ (iii) $\log(1+e^{5x})$

   (iv) $\log(\log x)$ (v) $\log(\sec x)$ (vi) $\log(1+\sin^2 x)$

   (vii) $\ln(e^{ax}+e^{-ax})$ (viii) $\log\left(\sqrt{a^2+x^2}+b\right)$

   (ix) $\log\left(\sqrt{a+x}+\sqrt{a-x}\right)$ x) $\ln|x-4|$

3. Find the differential coefficient of

   (i) $e^{\sin x}$ (ii) $e^{\sqrt{\cos x}}$ (iii) $e^{(1+\log x)}$

   (iv) $e^{\sin(\log x)}$ (v) $\tan(\log x)$ (vi) $\sin(1+e^{ax})$

   (vii) $\cos(\log\sec x)$ (viii) $\sec(\log\tan x)$ (ix) $\sin\log\sin e^{x^2}$

4. Differentiate the following with respect to x:

   (i) $x^2\log(1+x)$ (ii) x^5e^{ax} (iii) $\sin ax.\log x$

   (iv) $e^{ax}\cos bx$ (v) $(\tan x+x^2)\log x$ (vi) $(\sin x+\cos x)e^{ax}$

5. Calculate the derivatives of

   (i) $\frac{\log x}{\sin x}$ (ii) $\frac{\log(ax+b)}{e^{px}}$ (iii) $\frac{e^{ax}}{\cos bx}$

   (iv) $\frac{\sin ax}{1+\log x}$ (v) $\frac{\log x}{a^2+x^2}$ (vi) $\frac{x^n}{e^{ax+b}}$

6. Find $\frac{dy}{dx}$, when

   (i) $xy=\log(x^2+y^2)$ (ii) $x^2+y^2=\log(x+y)$

   (iii) $e^{xy}=xy$ (iv) $x=e^{\cos 2t}, y=e^{\sin 2t}$

   (v) $x=\cos(\log t), y=\log(\cos t)$

   (vi) $x=\log t+\sin t, y=e^t+\cos t$.

Practical Work/Activities

A. A function $y=f(x)$ is given to the students in the class. (Consider different type of function by different students.) Answer the following questions:

a) Draw the graph of ...

c) Find the point ... tangent is parallel to x-axis if possible.

d) Find the point on the curve where the tangent is perpendicular to x-axis if possible.

e)

Answers

1. (i) $\frac{a}{ax+b}$ (ii) $\frac{\log_5 e}{x}$ (iii) $\frac{1}{x}$ (iv) ae^{ax+b} (v) $\frac{1}{3}e^{x/3}$

2. (i) $\cot x$ (ii) $\frac{1+\sec^2 x}{x+\tan x}$ (iii) $\frac{5e^{5x}}{1+e^{5x}}$ (iv) $\frac{1}{x\log x}$ (v) $\tan x$

(vi) $\frac{2\sin x\cos x}{1+\sin^2 x}$ (vii) $\frac{a(e^{ax}-e^{-ax})}{e^{ax}+e^{-ax}}$ (viii) $\frac{x}{\sqrt{a^2+x^2}\,(\sqrt{a^2+x^2}+b)}$

(ix) $\frac{1}{2(\sqrt{a+x}+\sqrt{a-x})}\left(\frac{1}{\sqrt{a+x}}-\frac{1}{\sqrt{a-x}}\right)$ (x) $\frac{1}{x-4}$

3. (i) $e^{\sin x}\cos x$ (ii) $\frac{-e^{\sqrt{\cos x}}\sin x}{2\sqrt{\cos x}}$ (iii) $\frac{e^{(1+\log x)}}{x}$ (iv) $\frac{1}{x}e^{\sin(\log x)}.\cos(\log x)$

(v) $\frac{\sec^2(\log x)}{x}$ (vi) $ae^{ax}\cos(1+e^{ax})$ (vii) $-\tan x.\sin(\log\sec x)$

(viii) $\frac{\sec(\log\tan x)\tan(\log\tan x)\sec^2 x}{\tan x}$ (ix) $2x\,e^{x^2}\cot(e^{x^2})\cos\log\sin(e^{x^2})$

4. (i) $2x\log(1+x)+\frac{x^2}{1+x}$ (ii) $5x^4e^{ax}+ax^5e^{ax}$

(iii) $a\cos ax.\log x+\frac{\sin ax}{x}$ (iv) $(a\cos bx-b\sin bx)e^{ax}$

(v) $(\sec^2 x+2x)\log x+\frac{\tan x+x^2}{x}$ (vi) $(\cos x-\sin x)e^{ax}+a(\sin x+\cos x)e^{ax}$

5. (i) $\frac{\sin x-x\cos x.\log x}{x\sin^2 x}$ (ii) $\frac{a-p(ax+b)\log(ax+b)}{(ax+b)e^{px}}$

(iii) $\frac{(a\cos bx+b\sin bx)e^{ax}}{\cos^2 bx}$ (iv) $\frac{ax(1+\log x)\cos ax-\sin ax}{x(1+\log x)^2}$

(v) $\frac{a^2+x^2-2x^2\log x}{x(a^2+x^2)^2}$ (vi) $\frac{(n-ax)x^{n-1}}{e^{ax+b}}$

6. (i) $\frac{2x-x^2y-y^3}{x^3+xy^2-2y}$ (ii) $\frac{1-2x^2-2xy}{2xy+2y^2-1}$

(iii) $-\frac{y}{x}$ (iv) $-e^{\sin 2t-\cos 2t}.\cot 2t$

(v) $\frac{t\tan t}{\sin(\log t)}$ (vi) $\frac{t(e^t-\sin t)}{1+t\cos t}$