Chapter 15
Limits and Continuity

15.1 Introduction

We have discussed a function and its graph in chapter I. As a review, we give the definition of a function, its domain, range and its graph once again before defining the limit of a function.

We deal with limits and continuity which are quite fundamental for the development of calculus. These two concepts are closely linked together with the involvement of the concept of limit in the definition of continuity. So in the sequence, limit comes first and it is proper to begin with some discussion about it. The discussion is initiated with some examples so as to give some intuitive idea about it. Then follows the precise definition of the limit. The same line of approach is being followed in the case of continuity as well. We shall also mention some limit theorems and properties of continuous functions without proof.

Function

Let X and Y be two non-empty sets. Then a function f from X to Y is a rule which assigns a unique element of Y to each element of X. The unique element of Y which f assigns corresponding to an element $x \in X$ is denoted by $f(x)$. So, we also write $y = f(x)$. The symbol $f: X \to Y$ usually means 'f is a function from X to Y'. The element $f(x)$ of Y is called the image of x under the function f.

Value of the Function

If f is a function from X to Y and $x = a$ is an element in the domain of f, then the image $f(a)$ corresponding to $x = a$ is said to be the value of the function at $x = a$.

If the value of the function $f(x)$ at $x = a$ denoted by $f(a)$ is a finite number, then $f(x)$ exists or is defined at $x = a$ otherwise, $f(x)$ does not exist or is not defined at $x = a$.

For example :

i) $y = f(x) = 3x + 5$ exists or is defined at $x = 2$ as

$f(2) = 3 \times 2 + 5 = 11$ is a finite number.

ii) $y = f(x) = \dfrac{1}{x-1}$ is not defined at $x = 1$ as

$f(1) = \dfrac{1}{0}$ is not a finite number.

Hence $f(x)$ does not exist or is undefined at $x = 1$.

15.2 Indeterminate Forms

Consider the function $y = f(x) = \frac{x^2 - 1}{x - 1}$ and $x = 1$.

When $x = 1$, $y = f(0) = \frac{0}{0}$.

This does not give any number. It simply indicates that the number in the numerator and denominator are each zero. So, there are some functions that take the form $\frac{0}{0}$ for some value of x. Such form is said to be an **indeterminate form.** Other indeterminate forms which a function may take for some values of x are $\frac{\infty}{\infty}$, $\infty - \infty$, 1^∞ and 0^∞.

Meaning of $x \to a$

Before giving the meaning of $x \to a$, we consider an example to illustrate the meaning of $x \to 2$. Let x be a variable and let us make the variable x to take the values 1.9, 1.99, 1.999, 1.9999, ... As the number of 9's increases, the value of x will be nearer and nearer to 2 but will never be 2. In such a situation, the numerical difference between x and 2 will be very small. Again we let the variable x take the values 2.1, 2.01, 2.001, 2.0001, ... As number of zeros increases, but due to the presence of 1 at the end, the value of x will be nearer and nearer to 2 but will never be equal to 2. In such a situation also, the numerical difference between x and 2 will be sufficiently small. Thus if x takes the value greater than 2 or less than 2 but the numerical difference between x and 2 is sufficiently small, then we say that x approaches 2 or x tends to 2 and is written as $x \to 2$.

Let x be a variable and 'a' a constant number. If x takes a value such that the numerical difference between x and a is sufficiently small, then we say that x tends to a and is written as $x \to a$.

15.3 Limit of a Function

We use the concept of the limit of a sequence to understand the meaning of the limit of a function.

First, we consider the function $y = f(x) = 2x + 3$.

Considering the sequence of values of x to be 0.5, 0.75, 0.9, 0.99, 0.999, 0.9999,... whose limit is 1, we see the corresponding values of $f(x)$ are 4, 4.5, 4.8, 4.98, 4.998, 4.9998, ... which go nearer and nearer to 5 when x is very near to 1. So, when x is sufficiently close to 1, $f(x)$ is very close to 5.

Again if we consider the sequence of values of x to be 2, 1.5, 1.25, 1.1, 1.01, 1.001, 1.0001, ... whose limit is 1, we shall find the corresponding values of $f(x)$ to be 7, 6, 5.5, 5.2, 5.02, 5.002, 5.0002, ... which go nearer and nearer to 5 when x is very close to 1. So, when x is sufficiently close to 1, $f(x)$ is very close to 5. That is, when $x \to 1$, $f(x) \to 5$. In symbol, we write

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} (2x + 3) = 5$$

Hence, we have the following definition of limit of a function.

The number 'l' to which the value of a function $f(x)$ approaches when x approaches a certain number 'a' is said to be the limiting value of $f(x)$.

We can define limit of a function in the following way also.

A function $f(x)$ is said to tend to a limit 'l' when $x \to a$ if the numerical difference between $f(x)$ and l can be made as small as we please by making x sufficiently close to a and we write

$$\lim_{x \to a} f(x) = l$$

Meaning of Infinity (∞)

Let us consider the function $y = f(x) = \frac{1}{x}$

If we consider the sequence of values of x to be 1, 0.5, 0.1, 0.01, 0.001, 0.0001, ... whose limit is 0, we see that the corresponding values of $f(x)$ are 1, 2, 10, 100, 1000, 10000, ... which go on increasing. If we take x small enough, the corresponding value of $f(x)$ will be large enough. Taking the value of x to be sufficiently close to 0, the value of $f(x)$ will be greater than any positive number, however large. In such a case, we say that as x tends to zero, $f(x)$ tends to infinity and is indicated by the symbol, $f(x) \to \infty$ as $x \to 0$

or, $$\lim_{x \to 0} \frac{1}{x} = \infty$$

Infinity as a Limit of a Function

Let $f(x)$ be a function of x. Making x sufficiently close to a, if the value of $f(x)$ obtained **is greater** than any **pre-assigned number**, however large, we say that the limit of $f(x)$ is infinity as x tends to a. Symbolically, we write

$$\lim_{x \to a} f(x) = \infty$$

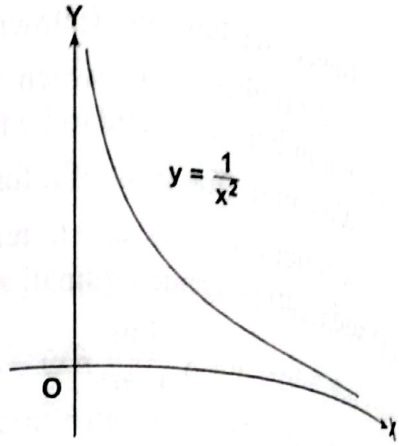
Limit at Infinity

Let us consider the function $f(x) = \frac{1}{x^2}$ and we see its nature when the value of x goes on increasing. The following table shows the values of x and the corresponding values of y.

x	1	10	100	1000	...
$f(x)$	1	0.01	0.0001	0.000001	...

From the above table, we see that when the value of x increases, the corresponding value of $f(x)$ decreases. When the value of x becomes large enough, the corresponding value of $f(x)$ becomes small enough. That is, taking the value of x to be sufficiently large i.e. the value greater than any positive number, however large, the value of $f(x)$ can be made sufficiently close to 0. In such a situation, we say that as x tends to infinity, $f(x)$ tends to zero and is indicated by the symbol, $f(x) \to 0$ when $x \to \infty$.

or, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x^2} = 0$



A function $f(x)$ is said to tend to 'l' when $x \to \infty$ if $f(x)$ can be made close to 'l' when x is greater than any pre-assigned number, however large. Symbolically, we write,

$$\lim_{x\to\infty} f(x) = l$$

15.4 Algebraic Properties of a Limit

Let $f(x)$ and $g(x)$ be two functions of x such that $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$, then we have the following theorems on limits :

i) The limit of the sum (or difference) of the functions $f(x)$ and $g(x)$ is the sum (or difference) of the limits of the functions. i.e.

$$\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = l \pm m$$

ii) The limit of the product of the functions $f(x)$ and $g(x)$ is the product of the limits of the functions. i.e.

$$\lim_{x\to a} [f(x) \cdot g(x)] = \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right) = l.m$$

iii) The limit of the quotient of the function $f(x)$ and $g(x)$ is the quotient of the limits of the functions, provided that the limit of the denominator is not zero i.e.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{l}{m} \quad \text{provided that} \quad \lim_{x\to a} g(x) = m \neq 0.$$

iv) The limit of the nth root of a function $f(x)$ is the nth root of the limit of the function. i.e.

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{l}$$

15.5 Basic Theorems on Limits (A. Algebraic function)

1. For all rational values of n, $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

The proof of this theorem consists of the following three cases.

Case I :

When n is a positive integer :

By actual division, $\frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}.a + x^{n-3}.a^2 + \ldots + a^{n-1}$

Now, $\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{x\to a}[x^{n-1}+x^{n-2}.a+x^{n-3}.a^2+\dots a^{n-1}]$

$=a^{n-1}+a^{n-1}+a^{n-1}+\dots+a^{n-1}$

$=n\,a^{n-1}$

Case II :

When n is a negative integer :

Let $n=-m$ where m is a positive integer

Then, $\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{x\to a}\frac{x^{-m}-a^{-m}}{x-a}$

$$=\lim_{x\to a}\frac{\frac{1}{x^m}-\frac{1}{a^m}}{x-a}$$

$$=\lim_{x\to a}\frac{a^m-x^m}{x^m a^m(x-a)}$$

$$=\lim_{x\to a}\left[-\frac{x^m-a^m}{x-a}\times\frac{1}{x^m a^m}\right]$$

$$=-\left(\lim_{x\to a}\frac{x^m-a^m}{x-a}\right)\left(\lim_{x\to a}\frac{1}{x^m a^m}\right)$$

$$=-m.\,a^{m-1}\frac{1}{a^m.a^m} \qquad \text{(using case I)}$$

$$=(-m)\,a^{(-m)-1}$$

$$=n\,a^{n-1}$$

Case III :

When n is a rational fraction:

Let $n=\frac{p}{q}$ where p and q are integers and $q\neq 0$.

Then, $\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{x\to a}\frac{x^{p/q}-a^{p/q}}{x-a}$

$$=\lim_{x\to a}\frac{(x^{1/q})^p-(a^{1/q})^p}{x-a}$$

Put $x^{1/q}=y$ and $a^{1/q}=b$ so that $x=y^q$ and $a=b^q$

when $x\to a,\ y\to b$

Now, $\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{y\to b}\frac{y^p-b^p}{y^q-b^q}=\lim_{y\to b}\frac{\frac{y^p-b^p}{y-b}}{\frac{y^q-b^q}{y-b}}$

$$= \frac{\lim\limits_{y \to b} \frac{y^p - b^p}{y - b}}{\lim\limits_{y \to b} \frac{y^q - b^q}{y - b}} = \frac{p\, b^{p-1}}{q\, b^{q-1}} = \frac{p}{q} \cdot b^{p-q}$$

$$= \frac{p}{q}\, b^{q(p/q-1)} = \frac{p}{q} \cdot (b^q)^{p/q - 1} = n\, a^{n-1}.$$

$\therefore$ for all rational values of n,

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = n\, a^{n-1}$$

Example 1.

Find the limiting value of $f(x) = 3x - 2$, when x approaches 3 or evaluate $\lim\limits_{x \to 3} f(x)$.

Solution:

When x approaches 3, $3x$ approaches $3 \times 3 = 9$. So $3x - 2$ approaches $9 - 2 = 7$.

Therefore, $\lim\limits_{x \to 3} f(x) = \lim\limits_{x \to 3} (3x - 2) = 7$

Example 2.

Find the limiting value of $f(x) = 3x^2 - 5x + 6$, when x approaches 2.

Solution:

When x approaches 2, $3x^2$ approaches $3 \times 2^2 = 12$, and $5x$ approaches $5 \times 2 = 10$

Therefore, $\lim\limits_{x \to 2} (3x^2 - 5x + 6) = 12 - 10 + 6 = 8$

Example 3.

Evaluate $\lim\limits_{x \to 3} \frac{4x - 5}{2x + 3}$

Solution:

When x approaches 3, $4x - 5$ approaches $4 \times 3 - 5 = 7$ and $2x + 3$ approaches $2 \times 3 + 3 = 9$.

Therefore, $\lim\limits_{x \to 3} \frac{4x - 5}{2x + 3} = \frac{7}{9}$

Example 4.

Evaluate $\lim\limits_{x \to 0} \frac{5x^2 + 3x}{x}$

Solution:

When $x = 0$, the function $\frac{5x^2 + 3x}{x}$ takes the form $\frac{0}{0}$, which is indeterminate. Therefore,

$\lim\limits_{x \to 0} \frac{5x^2 + 3x}{x} = \lim\limits_{x \to 0} \frac{x(5x + 3)}{x}$

$$= \lim_{x \to 0} (5x + 3)$$

$$= 0 + 3 = 3$$

Example 5.

Evaluate $\lim_{x \to a} \frac{x^5 - a^5}{x^4 - a^4}$

Solution:

Following the argument used in the solution of Ex. 4.

We have, $\lim_{x \to a} \frac{x^5 - a^5}{x^4 - a^4}$

$$= \lim_{x \to a} \frac{(x-a)(x^4 + x^3a + x^2a^2 + xa^3 + a^4)}{(x-a)(x^3 + x^2a + xa^2 + a^3)}$$

$$= \lim_{x \to a} \frac{x^4 + x^3a + x^2a^2 + xa^3 + a^4}{x^3 + x^2a + xa^2 + a^3}$$

$$= \frac{5a^4}{4a^3} = \frac{5a}{4}$$

Example 6.

Evaluate $\lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{x^{1/2} - a^{1/2}}$

Solution:

The given function takes the indeterminate form $\frac{0}{0}$, when $x = a$. But

$$\lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{x^{1/2} - a^{1/2}} = \lim_{x \to a} \frac{(x^{1/6})^2 - (a^{1/6})^2}{(x^{1/6})^3 - (a^{1/6})^3}$$

$$= \lim_{x \to a} \frac{(x^{1/6} - a^{1/6})(x^{1/6} + a^{1/6})}{(x^{1/6} - a^{1/6})(x^{2/6} + x^{1/6}a^{1/6} + a^{2/6})}$$

$$= \lim_{x \to a} \frac{x^{1/6} + a^{1/6}}{x^{1/3} + x^{1/6}a^{1/6} + a^{1/6}}$$

$$= \frac{2a^{1/6}}{3a^{1/3}} = \frac{2}{3a^{1/6}}$$

Example 7

Evaluate $\lim_{x \to a} \frac{\sqrt{x+a} - \sqrt{3x-a}}{x-a}$

Solution:

The given function takes the indeterminate form $\frac{0}{0}$, when $x = a$. But

$$\lim_{x \to a} \frac{\sqrt{x+a} - \sqrt{3x-a}}{x-a}$$

$$= \lim_{x\to a} \frac{(\sqrt{x+a}-\sqrt{3x-a})(\sqrt{x+a}+\sqrt{3x-a})}{(x-a)(\sqrt{x+a}+\sqrt{3x-a})}$$

$$= \lim_{x\to a} \frac{x+a-3x+a}{(x-a)(\sqrt{x+a}+\sqrt{3x-a})}$$

$$= \lim_{x\to a} \frac{-2(x-a)}{(x-a)(\sqrt{x+a}+\sqrt{3x-a})}$$

$$= \lim_{x\to a} \frac{-2}{\sqrt{x+a}+\sqrt{3x-a}}$$

$$= \frac{-2}{\sqrt{2a}+\sqrt{2a}} = \frac{-1}{\sqrt{2a}}.$$

Example 8

Evaluate $\lim\limits_{x\to\infty} \dfrac{3x^2+2x+1}{4x^2+x+5}$

Solution:

The given function takes the indeterminate form $\frac{\infty}{\infty}$, when $x = \infty$.

We get $\lim\limits_{x\to\infty} \dfrac{3+\frac{2}{x}+\frac{1}{x^2}}{4+\frac{1}{x}+\frac{5}{x^2}} = \dfrac{3+0+0}{4+0+0} = \dfrac{3}{4}$

Example 9

Evaluate $\lim\limits_{x\to\infty} (\sqrt{x+a}-\sqrt{x})$

Solution:

The given function takes the indeterminate form $\infty - \infty$, when $x = \infty$. So by multiplying the numerator and the denominator by $\sqrt{x+a}+\sqrt{x}$, we have

$$\lim_{x\to\infty} \frac{(\sqrt{x+a}-\sqrt{x})(\sqrt{x+a}+\sqrt{x})}{(\sqrt{x+a}+\sqrt{x})}$$

$$= \lim_{x\to\infty} \frac{x+a-x}{\sqrt{x+a}+\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{a}{\sqrt{x+a}+\sqrt{x}}$$

$$= \frac{a}{\infty+\infty} = \frac{a}{\infty}$$

$$= 0$$

EXERCISE 15.1

1. Find the following limits:

(a) $\lim\limits_{x\to 2}(2x^2+3x-14)$ 0

(b) $\lim\limits_{x\to 5}(x^2+2x-9)$ 26

(c) $\lim\limits_{x\to 1}\dfrac{3x^2+2x-4}{x^2+5x-4}$

(d) $\lim\limits_{x\to 3}\dfrac{6x^2+3x-12}{2x^2+x+1}$ 51/22

2. Compute the following limits :

(a) $\lim\limits_{x\to 0}\dfrac{4x^3-x^2+2x}{3x^2+4x}$

(b) $\lim\limits_{x\to 4}\dfrac{x^3-64}{x^2-16}$

(c) $\lim\limits_{x\to a}\dfrac{x^{2/3}-a^{2/3}}{x-a}$

(d) $\lim\limits_{x\to 1}\dfrac{x^2+3x-4}{x-1}$

(e) $\lim\limits_{x\to 2}\dfrac{x^2-5x+6}{x^2-x-2}$

(f) $\lim\limits_{x\to 2}\dfrac{x^2-4x+4}{x^2-7x+10}$

(g) $\lim\limits_{x\to a}\dfrac{\sqrt{3x}-\sqrt{2x+a}}{2(x-a)}$

(h) $\lim\limits_{x\to a}\dfrac{\sqrt{2x}-\sqrt{3x-a}}{\sqrt{x}-\sqrt{a}}$

(i) $\lim\limits_{x\to 1}\dfrac{\sqrt{2x}-\sqrt{3-x^2}}{x-1}$

(j) $\lim\limits_{x\to 2}\dfrac{\sqrt{x}-\sqrt{6-x^2}}{x-2}$

(k) $\lim\limits_{x\to 64}\dfrac{\sqrt[6]{x}-2}{\sqrt[3]{x}-4}$

(l) $\lim\limits_{x\to a}\dfrac{\sqrt{3a-x}-\sqrt{x+a}}{4(x-a)}$

3. Calculate the following limits:

(a) $\lim\limits_{x\to\infty}\dfrac{2x^2}{3x^2+2}$

(b) $\lim\limits_{x\to\infty}\dfrac{3x^2-4}{4x^2}$

(c) $\lim\limits_{x\to\infty}\dfrac{4x^2+3x+2}{5x^2+4x-3}$

(d) $\lim\limits_{x\to\infty}\dfrac{5x^2+2x-7}{3x^2+5x+2}$

4. Calculate the following limits:

(a) $\lim\limits_{x\to\infty}(\sqrt{x}-\sqrt{x-3})$

(b) $\lim\limits_{x\to\infty}(\sqrt{x-a}-\sqrt{x-b})$

(c) $\lim\limits_{x\to\infty}(\sqrt{3x}-\sqrt{x-5})$

(d) $\lim\limits_{x\to\infty}\sqrt{x}\,(\sqrt{x}-\sqrt{x-a})$

(e) $\lim\limits_{x\to\infty}(\sqrt{x-a}-\sqrt{bx})$

5. (a) $\lim\limits_{x\to 2}\dfrac{x-\sqrt{8-x^2}}{\sqrt{x^2+12}-4}$

(b) $\lim\limits_{x\to 1}\dfrac{x-\sqrt{2-x^2}}{2x-\sqrt{2+2x^2}}$.

Answers

1. (a) 0. (b) 26. (c) $\frac{1}{2}$ (d) $\frac{51}{22}$
2. (a) $\frac{1}{2}$ (b) 6 (c) $\frac{2}{3}a^{-1/3}$ (d) 5 (e) $-\frac{1}{3}$ (f) 0
   (g) $\frac{1}{4\sqrt{3a}}$ (h) $-\frac{1}{\sqrt{2}}$ (i) $\sqrt{2}$ (j) $\frac{5}{2\sqrt{2}}$ (k) $\frac{1}{4}$ (l) $-\frac{1}{4\sqrt{2a}}$
3. (a) $\frac{2}{3}$ (b) $\frac{3}{4}$ (c) $\frac{4}{5}$ (d) $\frac{5}{3}$
4. (a) 0 (b) 0 (c) ∞ (d) a/2 (e) ∞ if $b \neq 1$ and 0 if $b = 1$
   (If the limiting value of a function is ∞, we say that the limit of the function does not exist)
5. (a) 4 (b) 2

Multiple Choice Questions

1. The one which is not an indeterminate form is

   a) $\frac{0}{0}$ b) $\infty + \infty$ c) $\infty - \infty$ d) $0 \times \infty$

2. $\lim\limits_{x \to a} f(x) = l \Rightarrow$

   a) $\lim\limits_{x \to a^-} f(x) = l$ b) $\lim\limits_{x \to a^+} f(x) = l$

   c) $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x) = l$ d) $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x)$

3. $\lim\limits_{x \to a} f(x)$ does not exist if

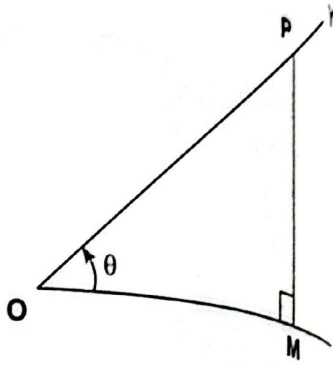
   a) $\lim\limits_{x \to a^-} f(x) = -\infty$ b) $\lim\limits_{x \to a^+} f(x) = \infty$

   lim lim

B. Limits of Trigonometric Functions

Standard Results

(i) $\lim\limits_{\theta\to 0} \sin\theta = 0$ (ii) $\lim\limits_{\theta\to 0} \cos\theta = 1$



Let OX be the initial line and $\angle XOY = \theta$. Take any point P on the line OY. From P draw PM perpendicular to OX. Then,

$$\sin\theta = \frac{MP}{OP} \quad \text{and} \quad \cos\theta = \frac{OM}{OP}$$

When θ is small, MP will be small and P will be near to M. When θ is small enough, MP will be small enough and P will be very close to M. This implies that as $\theta\to 0$, $MP\to 0$ and $OP\to OM$.

Therefore, (i) $\lim\limits_{\theta\to 0} \sin\theta = \lim\limits_{\theta\to 0} \frac{MP}{OP} = 0$

and (ii) $\lim\limits_{\theta\to 0} \cos\theta = \lim\limits_{\theta\to 0} \frac{OM}{OP} = 1$

(iii) $\lim\limits_{\theta\to\alpha} \sin\theta = \sin\alpha$

Put $\theta = \alpha + h$ so that when $\theta\to\alpha$, $h\to 0$.

Now, $\lim\limits_{\theta\to\alpha} \sin\theta = \lim\limits_{h\to 0} \sin(\alpha + h)$

$= \lim\limits_{h\to 0} \{\sin\alpha \cos h + \cos\alpha \sin h\}$

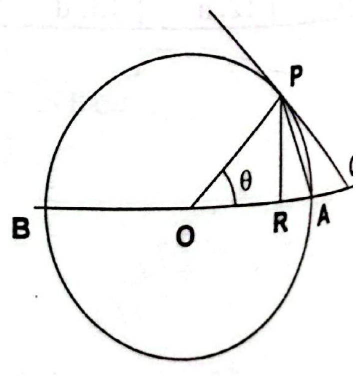
$= \sin\alpha \lim\limits_{h\to 0} \cos h + \cos\alpha \lim\limits_{h\to 0} \sin h$

$= \sin\alpha \,.\, 1 + \cos\alpha \,.\, 0 = \sin\alpha$

$\therefore \lim\limits_{\theta\to\alpha} \sin\theta = \sin\alpha$

Theorem.

$\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ where θ is measured in radian.



Let ABC be a circle of radius r and AP be an arc which subtends an angle θ at the centre O. Let PQ be the tangent at the point P of the circle which meets BA produced at Q. Join PA and draw PR perpendicular to BA. Then

Area of $\Delta OPA \leq$ Area of sector $OAP \leq$ Area of ΔOPQ

Now, Area of $\Delta OPA = \frac{1}{2} OA.\, PR$

$= \frac{1}{2} r^2 \sin\theta$ ($\because OA = r$, $PR = r\sin\theta$)

Area of sector OAP $= \frac{1}{2} r^2 \theta$

Area of ΔOPQ $= \frac{1}{2}$OP.PQ $= \frac{1}{2} r^2 \tan\theta$

$$\therefore \quad \frac{1}{2} r^2 \sin\theta \le \frac{1}{2} r^2 \theta \le \frac{1}{2} r^2 \tan\theta$$

$$1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$$

or,

$$1 \ge \frac{\sin\theta}{\theta} \ge \cos\theta$$

or,

$$\lim_{\theta\to 0} 1 \ge \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \ge \lim_{\theta\to 0} \cos\theta$$

or,

$$1 \ge \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \ge 1$$

$$\therefore \quad \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

$\frac{\tan\theta}{\theta} = 1$

C. Limits of Logarithmic and Exponential Functions

For the limits of logarithmic and exponential functions, we recall the following definition of e.

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

If we put $n = \frac{1}{h}$ so that when $n\to\infty$, $h\to 0$

then
$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{h\to 0} (1+h)^{1/h} = e$$

Some Standard Results

a) $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = \lim_{x\to 0} \frac{1}{x} \log(1+x)$$

$$= \lim_{x\to 0} \log(1+x)^{1/x}$$

$$= \log\left\{\lim_{x\to 0} (1+x)^{1/x}\right\}$$

$$= \log e = 1$$

b) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

Put $e^x - 1 = y$ then $e^x = 1 + y$ and $x = \log(1+y)$ so that when $x\to 0$, $y\to 0$.

Now, $\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{y\to 0} \frac{y}{\log(1+y)}$

$= \lim_{y\to 0} \frac{1}{\frac{1}{y}\log(1+y)}$

$= \frac{1}{1} = 1$

c) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log a$

Put $a^x - 1 = y$ then $a^x = 1 + y$ which implies $x \log a = \log(1+y)$

and $x = \frac{\log(1+y)}{\log a}$ so that when $x \to 0, y \to 0$.

Now, $\lim_{x\to 0} \frac{a^x - 1}{x} = \lim_{y\to 0} \frac{y}{\frac{\log(1+y)}{\log a}}$

$= \log a \lim_{y\to 0} \frac{1}{\frac{1}{y}\log(1+y)}$

$= \log a \,.\, 1$

$= \log a$

CO-Ord Functions

Worked Out Examples

Example 1.

Show that $\lim_{x\to 0} \frac{\tan x}{x} = 1$

Solution:

$\frac{\tan x}{x}$ takes the indeterminate form $\frac{0}{0}$ at $x = 0$. So we write,

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0}\left(\frac{\sin x}{x} \cdot \frac{1}{\cos x}\right)$$

$$= 1 \,.\, \frac{1}{\cos 0} = 1$$

Example 2.

Show that $\lim_{x\to 0} \frac{1 - \cos 3x}{3x^2} = \frac{3}{2}$

Solution:

$\frac{1 - \cos 3x}{3x^2}$ takes the indeterminate form $\frac{0}{0}$ at $x = 0$. So we write

$$\lim_{x\to0}\frac{1-\cos 3x}{3x^2}=\lim_{x\to0}\frac{2\sin^2\frac{3x}{2}}{3x^2}$$

$$=\frac{2}{3}\lim_{x\to0}\left(\frac{\sin\frac{3x}{2}}{x}\right)^2=\frac{2}{3}\lim_{x\to0}\left(\frac{\sin\frac{3x}{2}}{\frac{3x}{2}}\cdot\frac{3}{2}\right)^2$$

$$=\frac{2}{3}\cdot\left(1.\frac{3}{2}\right)^2=\frac{3}{2}.$$

Example 3.

Evaluate $\lim_{x\to0}\frac{\operatorname{cosec} x-\cot x}{x}$

Solution:

$\frac{\operatorname{cosec} x-\cot x}{x}$ takes the indeterminate form $\frac{\infty-\infty}{0}$ when $x=0$.

So we write,

$$\lim_{x\to0}\frac{\operatorname{cosec} x-\cot x}{x}=\lim_{x\to0}\frac{1}{x}\left(\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right)$$

$$=\lim_{x\to0}\frac{1-\cos x}{x\sin x}$$

$$=\lim_{x\to0}\frac{2\sin^2\frac{x}{2}}{x.2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\lim_{x\to0}\frac{\tan\frac{x}{2}}{2\cdot\frac{x}{2}}=\frac{1}{2}.$$

Example 4

Evaluate : $\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{\tan(x-a)}$

Solution :

The given function takes the form $\frac{0}{0}$ when $x=a$.

Now, $\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{\tan(x-a)}$

$$=\lim_{x\to a}\left\{\frac{\sqrt{x}-\sqrt{a}}{\tan(x-a)}\cdot\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}\right\}$$

$$=\lim_{x\to a}\left\{\frac{(x-a)\cos(x-a)}{\sin(x-a)}\cdot\frac{1}{\sqrt{x}+\sqrt{a}}\right\}$$

$$= \lim_{(x-a)\to 0} \frac{1}{\frac{\sin(x-a)}{x-a}} \lim_{(x-a)\to 0} \cos(x-a) \times \lim_{x\to a} \frac{1}{\sqrt{x}+\sqrt{a}}$$

$$= \frac{1}{1} \cdot 1 \cdot \frac{1}{\sqrt{a}+\sqrt{a}}$$

$$= \frac{1}{2\sqrt{a}}$$

Example 5

Evaluate $\lim_{x\to\theta} \frac{x\sin\theta - \theta\sin x}{x-\theta}$

Solution :

$\frac{x\sin\theta - \theta\sin x}{x-\theta}$ takes the indeterminate form $\frac{0}{0}$ at $x = \theta$.

Put $x - \theta = h$ then $x = \theta + h$ so that when $x \to \theta$, $h \to 0$.

Now, $\lim_{x\to\theta} \frac{x\sin\theta - \theta\sin x}{x-\theta}$

$$= \lim_{h\to 0} \frac{(\theta+h)\sin\theta - \theta\sin(\theta+h)}{h}$$

$$= \lim_{h\to 0} \frac{h\sin\theta + \theta\{\sin\theta - \sin(\theta+h)\}}{h}$$

$$= \lim_{h\to 0} \left[\sin\theta + \frac{\theta}{h} 2\cos\left(\frac{2\theta+h}{2}\right)\sin\left(\frac{-h}{2}\right)\right]$$

$$= \lim_{h\to 0} \sin\theta - \lim_{h\to 0} \theta\cos\left(\theta+\frac{h}{2}\right)\frac{\sin h/2}{h/2}$$

$$= \sin\theta - \theta\cos\theta$$

Example 6

Evaluate : $\lim_{x\to 0} \frac{e^{3x}-1}{x.5^x}$

Solution:

$\lim_{x\to 0} \frac{e^{3x}-1}{x.5^x}$ [form $\frac{0}{0}$]

$$= \lim_{x\to 0} \frac{e^{3x}-1}{3x} \cdot 3 \cdot \frac{1}{5^x}$$

$$= \left(\lim_{x\to 0} \frac{e^{3x}-1}{3x} \cdot 3\right)\left(\lim_{x\to 0} \frac{1}{5^x}\right)$$

$$= 1.3.1$$

$$= 3$$

Example 7

Evaluate : $\lim_{x\to 0} \frac{a^x - b^x}{x}$

Solution:

$$\lim_{x\to 0} \frac{a^x - b^x}{x} \qquad [\text{form } \tfrac{0}{0}]$$

$$= \lim_{x\to 0} \frac{(a^x - 1) - (b^x - 1)}{x}$$

$$= \lim_{x\to 0} \frac{a^x - 1}{x} - \lim_{x\to 0} \frac{b^x - 1}{x}$$

$$= \log a - \log b$$

$$= \log\left(\frac{a}{b}\right)$$

Example 8

Evaluate : $\lim_{x\to 1} \frac{\log x}{x-1}$

Solution:

$$\lim_{x\to 1} \frac{\log x}{x-1} \qquad [\text{form } \tfrac{0}{0}]$$

Put $x - 1 = y$ then $x = 1 + y$ so that when $x\to 1, y\to 0$.

Now, $\lim_{x\to 1} \frac{\log x}{x-1} = \lim_{y\to 0} \frac{\log(1+y)}{y}$

$= 1$

$\frac{1+x}{x} = 1$

EXERCISE 15.2

Evaluate the following:

1. $\lim_{x\to 0} \frac{\sin ax}{x}$
2. $\lim_{x\to 0} \frac{\tan bx}{x}$
3. $\lim_{x\to 0} \frac{\sin mx}{\sin nx}$
4. $\lim_{x\to 0} \frac{\tan ax}{\tan bx}$
5. $\lim_{x\to 0} \frac{\sin px}{\tan qx}$
6. $\lim_{x\to a} \frac{\sin(x-a)}{x^2 - a^2}$
7. $\lim_{x\to p} \frac{x^2 - p^2}{\tan(x-p)}$
8. $\lim_{x\to 0} \frac{\sin ax.\cos bx}{\sin cx}$
9. $\lim_{x\to 0} \frac{1 - \cos x}{x^2}$
10. $\lim_{x\to 0} \frac{1 - \cos 6x}{x^2}$

11. $\lim\limits_{x\to 0} \dfrac{1-\cos 9x}{x^2}$

12. $\lim\limits_{x\to 0} \dfrac{\cos ax - \cos bx}{x^2}$

13. $\lim\limits_{x\to 0} \dfrac{\sin ax - \sin bx}{x}$

14. $\lim\limits_{x\to 0} \dfrac{1-\cos px}{1-\cos qx}$

15. $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{x^3}$

16. $\lim\limits_{x\to 0} \dfrac{\tan 2x - \sin 2x}{x^3}$

17. $\lim\limits_{x\to \frac{\pi}{2}} (\sec x - \tan x)$

18. $\lim\limits_{x\to \frac{\pi}{4}} \dfrac{\sec^2 x - 2}{\tan x - 1}$

19. $\lim\limits_{x\to \frac{\pi}{4}} \dfrac{2-\operatorname{cosec}^2 x}{1-\cot x}$

20. $\lim\limits_{x\to y} \dfrac{\tan x - \tan y}{x-y}$

21. $\lim\limits_{x\to y} \dfrac{\sin x - \sin y}{x-y}$

22. $\lim\limits_{x\to y} \dfrac{\cos x - \cos y}{x-y}$

23. $\lim\limits_{x\to \theta} \dfrac{x\cot\theta - \theta\cot x}{x-\theta}$

24. $\lim\limits_{x\to \theta} \dfrac{x\cos\theta - \theta\cos x}{x-\theta}$

25. $\lim\limits_{x\to 1} \dfrac{1+\cos \pi x}{\tan^2 \pi x}$

26. $\lim\limits_{x\to \theta} \dfrac{x\tan\theta - \theta\tan x}{x-\theta}$

27. $\lim\limits_{\theta\to \frac{\pi}{4}} \dfrac{\cos\theta - \sin\theta}{\theta - \frac{\pi}{4}}$

28. $\lim\limits_{x\to c} \dfrac{\sqrt{x}-\sqrt{c}}{\sin x - \sin c}$

29. Find the limits of

a) $\lim\limits_{x\to 0} \dfrac{e^{6x}-1}{x}$

b) $\lim\limits_{x\to 0} \dfrac{e^{2x}-1}{x.2^{x+1}}$

c) $\lim\limits_{x\to 0} \dfrac{e^{ax}-e^{bx}}{x}$

d) $\lim\limits_{x\to 0} \dfrac{a^x+b^x-2}{x}$

30. Evaluate the limits of

a) $\lim\limits_{x\to 2} \dfrac{x-2}{\log(x-1)}$

b) $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\cos x}{\log\left(x-\frac{\pi}{2}+1\right)}$

Practical Work/Activities

A. A problem on the purification of the impurities present in the pond is given by

$$C(p) = \frac{8100p}{100-p}$$

where p is the percentage impurities present in the polluted water and C(p), the Rs. needed to remove p% of the impurities.

a) Find the cost of removing 90% of th…

Answers

1. a 2. b 3. $\frac{m}{n}$ 4. $\frac{a}{b}$ 5. $\frac{p}{q}$ 6. $\frac{1}{2a}$

7. $2p$ 8. $\frac{a}{c}$ 9. $\frac{1}{2}$ 10. 18 11. $\frac{81}{2}$ 12. $\frac{1}{2}(b^2 - a^2)$

13. $a-b$ 14. $\frac{p^2}{q^2}$ 15. $\frac{1}{2}$ 16. 4 17. 0 18. 2

19. 2 20. $\sec^2 y$ 21. $\cos y$ 22. $-\sin y$ 23. $\cot\theta + \frac{\theta}{\sin^2\theta}$

24. $\cos\theta + \theta\sin\theta$ 25. $\frac{1}{2}$ 26. $\tan\theta - \theta\sec^2\theta$ 27. $-\sqrt{2}$

28. $\frac{\sec c}{2\sqrt{c}}$ 29. a) 6 b) 1 c) $a-b$ d) $\log(ab)$ 30. a) 1 b) -1

Multiple Choice Questions

1. The one which is not true is

a) $\lim_{x\to 0} \cos x = 1$ b) $\lim_{x\to 0} \sin x = 0$ c) $\lim_{x\to 0} \frac{\sin x}{x} = 1$ d) $\lim_{x\to 0} \frac{\cos x}{x} =$

2. In the relation, $\lim_{x\to 0} \frac{\sin x}{x} = 1$, x is measured in

b) radian c) grade d) all of the ab

a) ∞ b) log 2

11. $\lim_{x \to \infty} x(e^{1/x} - 1) =$

a) 1 b) 0 c) ∞ d) 2

Answers

1. d	2. b	3. c	4. d	5. b	6. d	7. a	8. c	9. d	10. c
11. a									

Right hand limit and Left hand limit

A function $f(x)$ is said to have the right hand limit l_1 at $x = a$ as x approaches a through value greater than a (i.e. x approaches a from the right) and symbolically it is written as $\lim_{x \to a^+} f(x) = l_1$. The right hand limit of $f(x)$ at $x = a$ is also written as $\lim_{x \to a+0} f(x)$ or $f(a + 0)$.

A function $f(x)$ is said to have the left hand limit l_2 at $x = a$ as x approaches a through value less than a (i.e. x approaches a from left) and symbolically it is written as $\lim_{x \to a^-} f(x) = l_2$. The left hand limit of $f(x)$ at $x = a$ is also written as $\lim_{x \to a-0} f(x)$ or $f(a - 0)$.

The necessary and sufficient condition for $f(x)$ to have a limit at $x = a$ is $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ should exist and coincide. That is, $\lim_{x \to a} f(x)$ exists if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$

Example 1

Calculate the limit of the function f at the point specified below:

$$f(x) \begin{cases} = 3x^2 - 1, & \text{when } x \le 2 \\ = 4x + 3, & \text{when } x > 2 \end{cases} \text{ at } x = 2$$

Solution:

Left-hand limit at $x = 2$ is

$$\lim_{x\to 2-0} f(x) = \lim_{x\to 2-0} (3x^2 - 1) = 12 - 1 = 11$$

Right-hand limit at $x = 2$ is

$$\lim_{x\to 2+0} f(x) = \lim_{x\to 2+0} (4x + 3) = 8 + 3 = 11.$$

As the left-hand limit is equal to the right-hand limit, the limit of the function $f(x)$ at the point $x = 2$ exists and is equal to 11; i.e., $\lim_{x\to 2} f(x) = 11$.

Example 2

Find the limit if it exists $\lim_{x\to 2} \frac{x-2}{|x-2|}$

Solution :

Let $f(x) = \frac{x-2}{|x-2|}$

By the definition,

$$|x-2| = \begin{cases} x-2 & \text{if } x > 2 \\ -(x-2) & \text{if } x < 2 \end{cases}$$

Then, $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x-2}{x-2}$

$= 1$

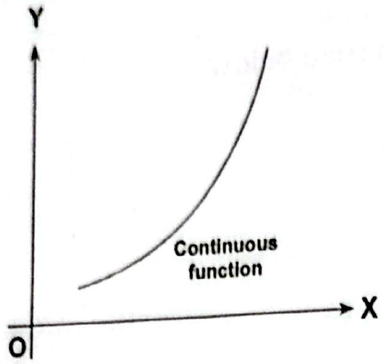
$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x-2}{-(x-2)}$

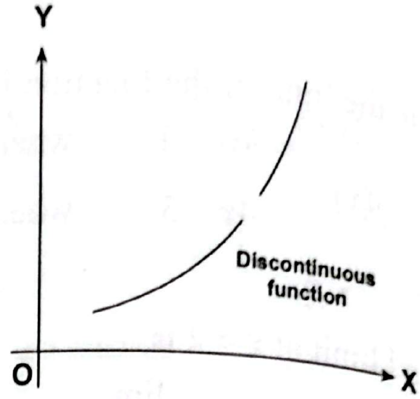
$= -1$

Since, $\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$, so $\lim_{x\to 2} f(x)$ does not exist.

15.6 Continuity of a Function

The intuitive idea of a continuous functions f in the interval $[a, b]$ gives the impression that the graph of the function f in this interval is a smooth curve without any break in it. Actually this curve is such that it can be drawn by the continuous motion of pencil without lifting it in a sheet of paper. Similarly, a discontinuous function gives the picture consisting of disconnected curves.





Definition. The function $f(x)$ is said to be continuous at the point $x = x_0$, if and only if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

This definition of continuity of the function $f(x)$ at $x = x_0$ implies that

a) $\lim_{x \to x_0} f(x)$ exists i.e. $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ are finite and equal.

b) $f(x_0)$ exists.

c) $\lim_{x \to x_0} f(x) = f(x_0)$

Hence $f(x)$ will be continuous at $x = x_0$ if $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0)$

If any one of the above conditions is not satisfied then the function $f(x)$ is said to be discontinuous at that point.

15.7 Types of discontinuities

A discontinuous function may be of the following types :

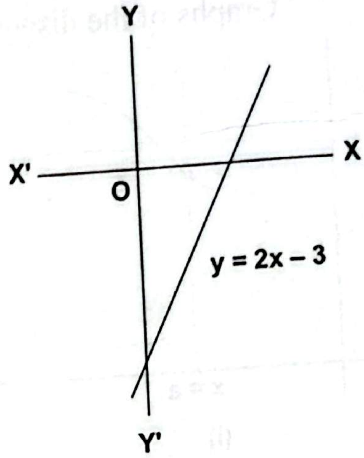
i) If $\lim_{x \to x_0} f(x)$ does not exist i.e. $\lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x)$ then $f(x)$ is said to be an **ordinary discontinuity** or a **jump.**

ii) If $\lim_{x \to x_0} f(x) \neq f(x_0)$ then the function $f(x)$ is said to have a **removable discontinuity** at $x = x_0$. This type of discontinuity can be removed by redefining the function.

iii) If $\lim_{x \to x_0} f(x) \to \infty$ or $-\infty$, then $f(x)$ is said to have **infinite discontinuity** at $x = x_0$.

Let us see the following examples to have the idea about the continuity and the different types of discontinuities :

i) The graph of $y = 2x - 3$ is given aside. It is continuous at every point.

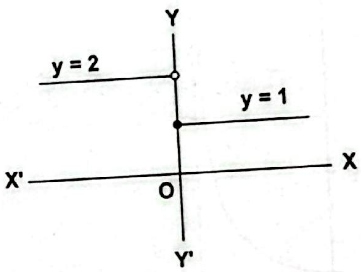


ii) A function $f(x)$ is defined below

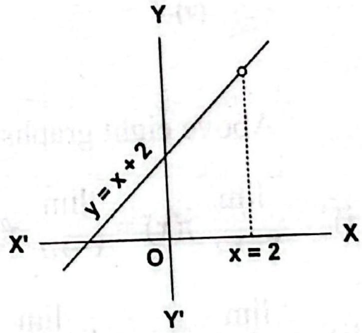
$$f(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 2 & \text{for } x < 0 \end{cases}$$

Its graph is given aside.

$\therefore$ the function $f(x)$ is discontinuous at $x = 0$. There is a jump at $x = 0$.



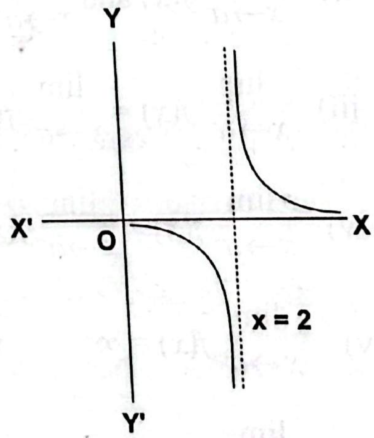
iii) The graph of $y = f(x) = \dfrac{x^2 - 4}{x - 2}$ is given aside. But $f(2) = \dfrac{0}{0}$ which is an indeterminate form. So, $f(x)$ is not defined at $x = 2$ and hence $f(x)$ is discontinuous at $x = 2$.



iv) The graph of $y = f(x) = \dfrac{1}{x - 2}$ is given aside. Here

$$\lim_{x \to 2^-} \frac{1}{x-2} \to -\infty$$

and $$\lim_{x \to 2^+} \frac{1}{x-2} \to \infty.$$



15.8 Graphs of Discontinuous Functions

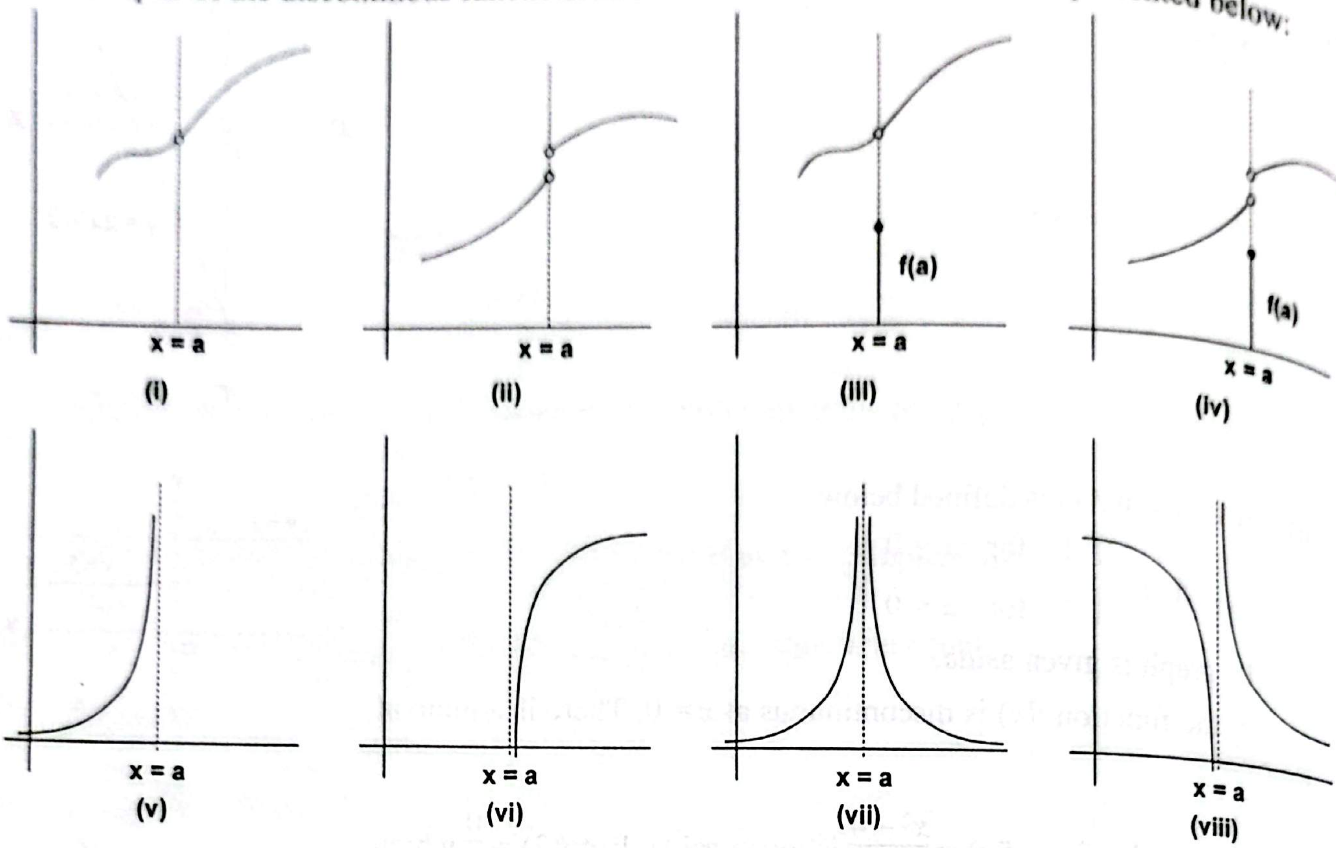
Function $y = f(x)$ may be discontinuous at a point $x = a$ under the following situations

a) $f(a)$ does not exist

b) $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$ i.e. $\lim\limits_{x \to a} f(x)$ does not exist.

c) $\lim\limits_{x \to a} f(x) \neq f(a)$ i.e. $\lim\limits_{x \to a^+} f(x) \neq f(a)$

Graphs of the discontinous functions due to the different conditions are presented below:



Above eight graphs have the following discontinuities

i) $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x)$ but $f(a)$ is not defined.

ii) $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exists but $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x)$ i.e. $\lim\limits_{x\to a} f(x)$ does not exist.

iii) $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) \neq f(a)$

iv) $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x) \neq f(a)$

v) $\lim\limits_{x\to a^-} f(x) = \infty$ i.e. $\lim\limits_{x\to a^-} f(x)$ does not exist.

vi) $\lim\limits_{x\to a^+} f(x) = -\infty$ i.e. $\lim\limits_{x\to a^+} f(x)$ does not exist.

vii) $\lim\limits_{x\to a^-} f(x) = \infty$ and $\lim\limits_{x\to a^+} f(x) = \infty$, i.e. both $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ do not exist.

viii) $\lim\limits_{x\to a^-} f(x) = -\infty$ and $\lim\limits_{x\to a^+} f(x) = \infty$. i.e. both $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ do not exist.

Properties of Continuous functions

If $f(x)$ and $g(x)$ are continuous functions at $x = a$, then

a) $f \pm g$ is continuous at $x = a$.

b) cf is continuous at $x = a$, $c \in R$.

c) fg is continuous at $x = a$.

d) f/g is continuous at $x = a$ provided that $g(a) \neq 0$.

e) Every polynomial function is continuous.

Worked Out Examples

Example 1.

Test the continuity or discontinuity of the following functions by calculating the left hand limit, the right hand limit and the value of the function at the points mentioned :

i) $f(x) = 2x^2 - 3x + 10$ at $x = 1$ ii) $f(x) = \frac{1}{x-2}$ at $x = 2$

iii) $f(x) = |x - 2|$ at $x = 2$

Solution :

i) Left hand limit at $x = 1$ is

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (2x^2 - 3x + 10)$$
$$= 2 - 3 + 10 = 9$$

Right hand limit at $x = 1$ is

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x^2 - 3x + 10)$$
$$= 2 - 3 + 10 = 9$$

$\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$ are finite and equal so $\lim_{x\to 1} f(x)$ exists.

$$\therefore \lim_{x\to 1} f(x) = 9$$

Also, $f(1) = 2 \times 1 - 3 \times 1 + 10 = 9$

$$\lim_{x\to 1} f(x) = f(1)$$

Hence $f(x)$ is continuous at $x = 1$.

ii) Left hand limit at $x = 2$ is

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{1}{x-2}$$
$$= -\infty \text{ which does not exist.}$$

Hence $f(x)$ is discontinuous at $x = 2$.

iii) $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} |x-2| = \lim\limits_{x\to 2^-} (2-x) = 0$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} |x-2| = \lim\limits_{x\to 2^+} (x-2) = 0$

and $f(2) = 2 - 2 = 0$

$\therefore \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = f(2)$

Hence $f(x)$ is continuous at $x = 2$.

Example 2.

A function $f(x)$ is defined as follows :

$$f(x) = \begin{cases} 2x+3 & \text{for } x<1 \\ 4 & \text{for } x=1 \\ 6x-1 & \text{for } x>1 \end{cases}$$

Is the function continuous at $x = 1$? If not, how can you make it continuous ?

Solution :

Left hand limit at $x = 1$ is

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (2x+3) = 2\times 1 + 3 = 5$

Right hand limit at $x = 1$ is

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (6x-1) = 6\times 1 - 1 = 5$

$\lim\limits_{x\to 1^-} f(x)$ and $\lim\limits_{x\to 1^+} f(x)$ are finite and equal.

So, $\lim\limits_{x\to 1} f(x)$ exist and $\lim\limits_{x\to 1} f(x) = 5$

But $f(1) = 4$

$\therefore \lim\limits_{x\to 1} f(x) \neq f(1)$

Hence $f(x)$ is not continuous at $x = 1$.

This is a case of **removable discontinuity.**

The given function will be continuous if $f(1)$ is also equal to 5. Thus the given functio can be made continuous by defining the function in the following way :

$$f(x) = \begin{cases} 2x+3 & \text{for } x<1 \\ 5 & \text{for } x=1 \\ 6x-1 & \text{for } x>1 \end{cases}$$

Example 3

A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} x^2 - 1 & \text{for } x < 3 \\ 2kx & \text{for } x \geq 3 \end{cases}$$

Find the value of k so that $f(x)$ is continuous at $x = 3$.

Solution :

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2 - 1) = 9 - 1 = 8$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 2kx = 6k$$

and $\quad f(3) = 6k$

Since $f(x)$ is continuous at $x = 3$,

so $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) \quad \Rightarrow \quad 6k = 8$

$\therefore \quad k = \frac{4}{3}$

EXERCISE 15.4

1. Test the continuity or discontinuity of the following functions by calculating the left-hand limits, the right-hand limits and the values of the functions at points specified:

(i) $f(x) = x^2$ at $x = 4$

(ii) $f(x) = 2 - 3x^2$ at $x = 0$

(iii) $f(x) = 3x^2 - 2x + 4$ at $x = 1$

(iv) $f(x) = \frac{1}{2x}$ at $x = 0$

(v) $f(x) = \frac{1}{x-2}$ at $x \neq 2$

(vi) $f(x) = \frac{1}{3x}$ at $x \neq 0$

(vii) $f(x) = \frac{1}{1-x}$ at $x = 1$

(viii) $f(x) = \frac{1}{x-3}$ at $x = 3$

(ix) $f(x) = \frac{x^2-9}{x-3}$ at $x = 3$

(x) $f(x) = \frac{x^2-16}{x-4}$ at $x = 4$

(xi) $f(x) = \frac{|x-2|}{x-2}$ at $x = 2$

(xii) $f(x) = \frac{x}{|x|}$ at $x = 0$

2. Discuss the continuity of functions at the points specified:

(i) $f(x) = \left.\begin{matrix} 2 - x^2 & \text{for } x \leq 2 \\ x - 4 & \text{for } x > 2 \end{matrix}\right\}$ at $x = 2$

(ii) $f(x) = \left.\begin{array}{ll} 2x^2 + 1 & \text{for } x \le 2 \\ 4x + 1 & \text{for } x > 2 \end{array}\right\}$ at $x = 2$

(iii) $f(x) = \left.\begin{array}{ll} 2x & \text{for } x \le 3 \\ 3x - 3 & \text{for } x > 3 \end{array}\right\}$ at $x = 3$

(iv) $f(x) = \left.\begin{array}{ll} 2x + 1 & \text{for } x < 1 \\ 2 & \text{for } x = 1 \\ 3x & \text{for } x > 1 \end{array}\right\}$ at $x = 1$.

3. i) A function f(x) is defined as follows: $f(x) = \begin{cases} x^2 + 2 & \text{for } x < 5 \\ 20 & \text{for } x = 5 \\ 3x + 12 & \text{for } x > 5 \end{cases}$

Show that $f(x)$ has removable discontinuity at $x = 5$.

ii) A function $f(x)$ is defined as follows : $f(x) = \begin{cases} 2x - 3 & \text{for } x < 2 \\ 2 & \text{for } x = 2 \\ 3x - 5 & \text{for } x > 2 \end{cases}$

Is the function $f(x)$ continuous at $x = 2$? If not, how can the function $f(x)$ be made continuous at $x = 2$?

4. (i) A function $f(x)$ is defined as follows: $f(x) = \begin{cases} kx + 3 & \text{for } x \ge 2 \\ 3x - 1 & \text{for } x < 2 \end{cases}$

Find the value of k so that $f(x)$ is continuous at $x = 2$.

(ii) A function $f(x)$ is defined as follows: $f(x) = \begin{cases} \dfrac{2x^2 - 18}{x - 3} & \text{for } x \neq 3 \\ k & \text{for } x = 3 \end{cases}$

Find the value of k so that $f(x)$ is continuous at $x = 3$.

Practical Work/Activities

A. Observe the following mathematical symbolic representation

$\lim\limits_{x \to 0^+} f(x)$, $\lim\limits_{x \to 0^-} f(x)$ and $f(0)$ where $f(x) = \dfrac{1}{x}$

Answer the following questions

a) Express each of the above mathematical notations in the sentence form.

b) Evaluate each of them if possible.

c) Is the function $f(x) = \dfrac{1}{x}$ continuous at $x = 0$?

d) If not continuous, what type of discontinuity it is?

e) Can they be shown in the graph?

Answers

1 (i) continuous (ii) continuous (iii) continuous (iv) discontinuous
(v) continuous (vi) continuous (vii) discontinuous (viii) discontinuous
(ix) discontinuous (x) discontinuous (xi) discontinuous (xii) discontinuous

2. (i) continuous (ii) continuous (iii) continuous (iv) discontinuous

3. ii) No, $f(x) = \begin{cases} 2x-3 & \text{for } x<2 \\ 1 & \text{for } x=2 \\ 3x-5 & \text{for } x>2 \end{cases}$

4. (i) 1 (ii) 12

Multiple Choice Questions

1. A function $f(x)$ is continuous at $x = a$ if

   a) $f(a)$ is defined

   b) $\lim\limits_{x\to a} f(x)$ exists

   c) $\lim\limits_{x\to a} f(x) = f(a)$

   d) all of them

2. The condition under which a function $y = f(x)$ will be discontinuous at the point is

   a) $f(x) = \pm\infty$

   b) $\lim\limits_{x\to a} f(x)$ does not exist

   c) $\lim\limits_{x\to a} f(x) \neq f(a)$

   d) all of them

$f(x)$ is discontinous at $x = a$ due to $\lim\limits_{x\to a} f(x) \neq f(a)$, then the discont