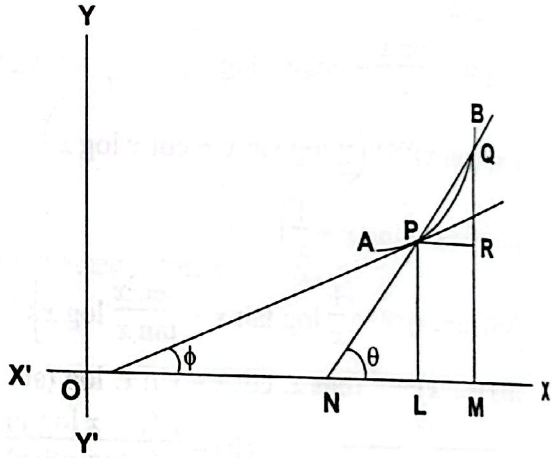
Chapter 17
Application of Derivatives

17.1 Introduction

The application of derivatives plays a very important role in the field of Science, engineering, economics, commerce and so on. The problems of the type "the increase in the length of an iron rod due to the increase in temperature"; "the increasing and decreasing tendency of the cost functions"; "the maximum profit to be made with the help of minimum investment" and so on can betterly be solved with the help of derivatives.

17.2 Geometrical Interpretation of Derivative of a Function

Let AB be a continuous curve given by $y = f(x)$ and P, Q be any two neighbouring points in it. Let the coordinates of P and Q be (x, y) and $(x + \Delta x, y + \Delta y)$. When a point moves along the curve from the point P to the point Q, it moves horizontaly through the distance PR and vertically through the distance RQ.



$$PR = LM = OM - OL = x + \Delta x - x$$
$$RQ = QM - RM = y + \Delta y - PL$$
$$= y + \Delta y - y$$

Δx and Δy are called the increments in x and y respectively.

Also $\Delta y = f(x + \Delta x) - f(x)$

If we join the points P and Q, we get secant PQ which makes an angle θ with the x-axis, i.e. $\angle QNM = \theta$. So $\angle QPR = \angle QNM = \theta$ and

$$\tan\theta = \frac{QR}{PR} = \frac{\Delta y}{\Delta x},$$

which is the slope of the secant PQ. As Q moves along the curve and approaches P, the secant rotates about P. The limiting position of the secant, when Q ultimately coincides with P, is the tangent at P, making the angle ø with the x-axis. In that situation, Δx, Δy tend to zero. So

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \tan\theta = \tan ø$$

Also, $$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

and $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

$$\therefore \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \tan \phi$$

$$= \text{slope of the tangent at P}$$

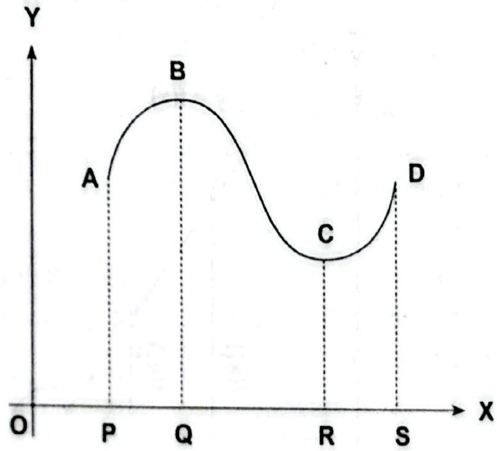
i.e. the derivative of y with respect to x, denoted by $\frac{dy}{dx}$ gives the slope of a tangent to the curve $y = f(x)$ at a point.

17.3 Monotonicity of Functions

A function can be presented in various forms. One of the most important forms of the presentation is the graph. To know the nature of the graph of the function, we must have the idea of increasing and decreasing tendency of the function on an interval.

Later on, the increasing and decreasing nature of the function are also used in finding the maximum and the minimum values of the function.

Here, in this section as an application, we use derivatives to examine the increasing and decreasing tendency of the function in an interval.



The figure given aside is the continuous curve represented by the given function $y = f(x)$. Consider the points A, B, C and D on the curve where OP = x_1, OQ = x_2, OR = x_3, OS = x_4 and PA = $f(x_1)$, QB = $f(x_2)$, RC = $f(x_3)$ and SD = $f(x_4)$

From the figure, we see that the curve (i.e. the function) increases from A to B then decreases from B to C and again increases from C to D.

That is, in the part AB, $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$

in the part BC, $x_3 > x_2 \Rightarrow f(x_3) < f(x_2)$

and in the part CD, $x_4 > x_3 \Rightarrow f(x_4) > f(x_3)$

The increasing and decreasing tendency of the function is known as the **monotonic functions** or **monotonicity of the function.**

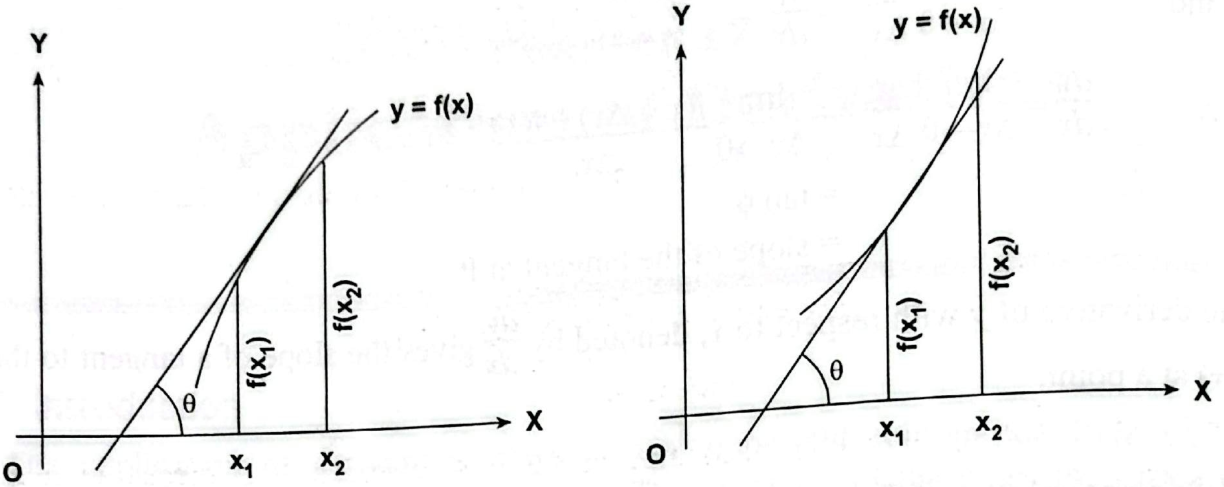
Now, we have the following definitions of increasing and decreasing functions.

Increasing Function

A function $y = f(x)$ is said to be increasing in the interval (a, b) if for every $x_1, x_2 \in (a, b)$

$$x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$$

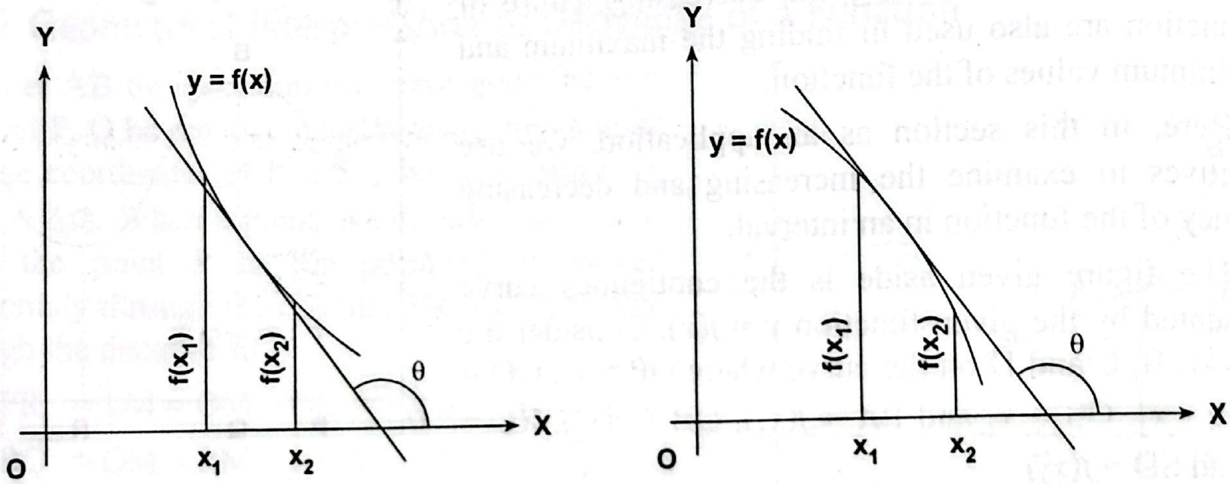
This result shows that as x increases, y i.e. $f(x)$ also increases. So, the slope of the tangent at any point of such a curve is positive. Thus, $y = f(x)$ is an increasing function if $\frac{dy}{dx} = f'(x) > 0$.



Decreasing Function

A function $y = f(x)$ is said to be decreasing in the interval (a, b) if for every $x_1, x_2 \in (a, b)$

$$x_2 > x_1 \quad \Rightarrow \quad f(x_2) < f(x_1)$$



This result shows that as x increases, y i.e. $f(x)$ decreases. So, the slope of the tangent at any point of such a curve is negative. Thus $y = f(x)$ is a decreasing function if $\frac{dy}{dx} = f'(x) < 0$.

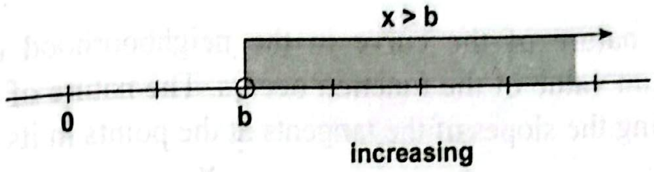
A function is increasing (or decreasing) at the point $x = a$ means the function is increasing (or decreasing) at $x = a$ which is an interior point of an interval.

The following table shows the sign of the derivative of the given function $y = f(x)$ defined in an interval (a, b) and the nature of the curve represented by the given function.

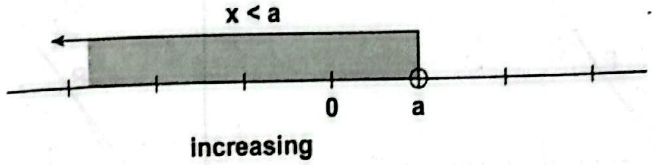
Sign of the derivative	Nature of $f(x)$
$f'(x) > 0$	Function is increasing.
$f'(x) < 0$	Function is decreasing.

17.4 Intervals of Monotonicity

If the function $y = f(x)$ is increasing for $x > b$ $(b > 0)$, then the function is increasing on the interval (b, ∞)

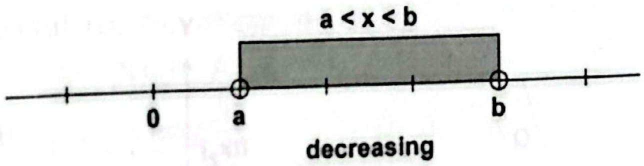


Again if the function $y = f(x)$ is increasing for $x < a$ $(a > 0)$, then the function is increasing on the interval $(-\infty, a)$.

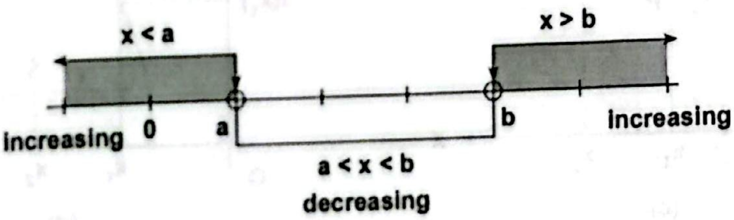


Thus the function $y = f(x)$ is increasing for $x < a$ and $x > b$ means increasing for $x \in (-\infty, a) \approx (b, \infty)$

But if the function $y = f(x)$ is decreasing for $a < x < b$, then the function is dencreasing on (a, b).



Thus the intervals in which the function $y = f(x)$ are increasing and decreasing are shown below:



Example

Show that the function $f(x) = \frac{1}{2}x^2 - 3x$ increases in the interval $(3, \infty)$ and decreases in the interval $(-\infty, 3)$.

Solution :

We have $\quad f(x) = \frac{1}{2}x^2 - 3x \quad \Rightarrow \quad f'(x) = x - 3$

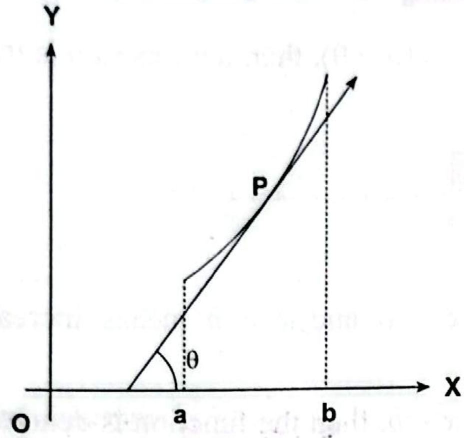
For $x > 3$, $\quad f'(x) > 0 \quad$ and $\quad$ for $x < 3$, $\quad f'(x) < 0$

$\therefore f(x)$ is increasing function for $x > 3$ i.e. on the interval $(3, \infty)$ and $f(x)$ decreases for $x < 3$ i.e. on the interval $(-\infty, 3)$.

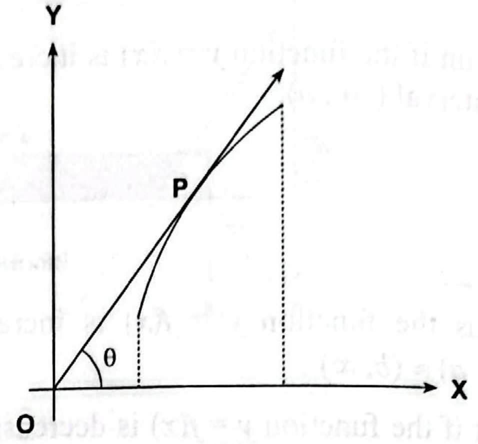
Note: At $x = 4$, $f'(x) = 4 - 3 = 1 > 0$, so $f(x)$ is increasing at $x = 4$ which is a point on $(3, \infty)$.

17.5 Extreme Values of the Function

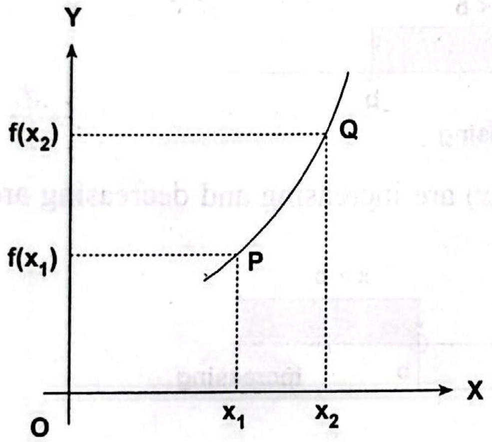
The extreme value of a function is the maximum or minimum value of a function at a point. The basic problem that we are going to solve in this section is to find a maximum or minimum value of a function f in an interval (a, b). To understand the problem properly, it is necessary to know the nature of the curve in the neighbourhood of the point where the maximum or the minimum value of the function occurs. The nature of the curve at a particular point is known by studying the slopes of the tangents at the points in its neighbourhood.



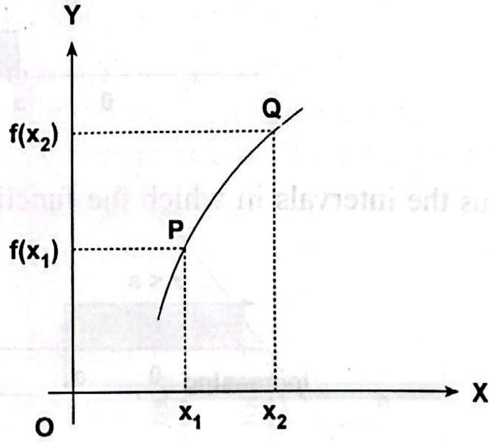
(a)



(b)

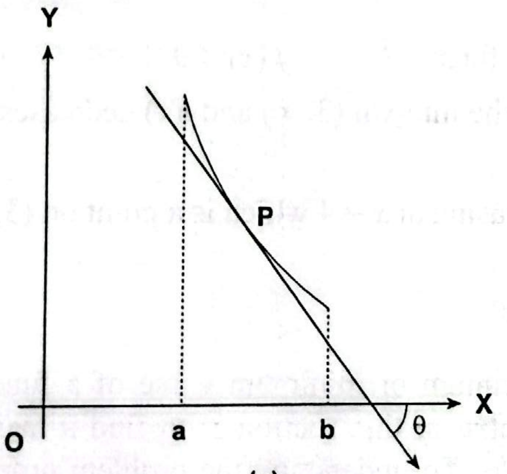


(c)

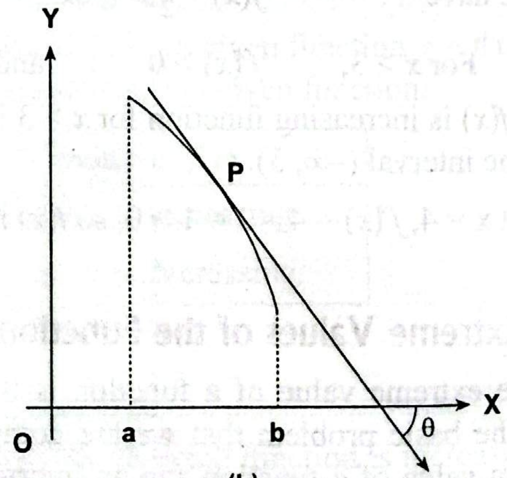


(d)

The slope or the gradient of a tangent line at a point of the graph of a function f may be positive, negative or zero. If the gradient of $f(x)$ is positive the inclination θ of the tangent line is positive and the tangent line slopes upwards (or goes up-hill) from left to right as shown in the figures. In the figures, within the interval (a, b), where the slope of the tangent line at any point is positive, $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$, i.e., the function is increasing. So we can say that if $f'(x)$ is positive at a certain point, the function f is increasing in its neighbourhood.



(a)



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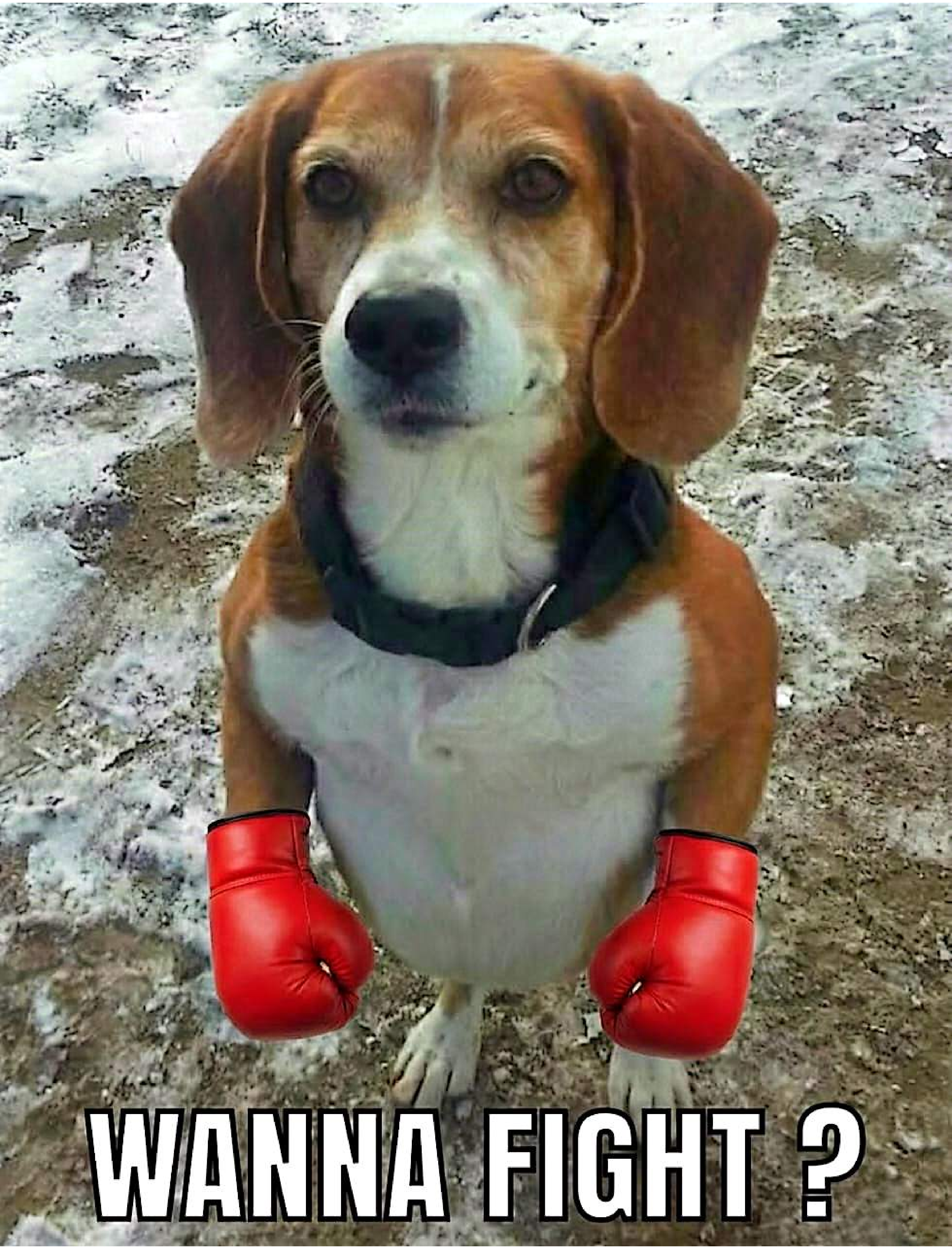
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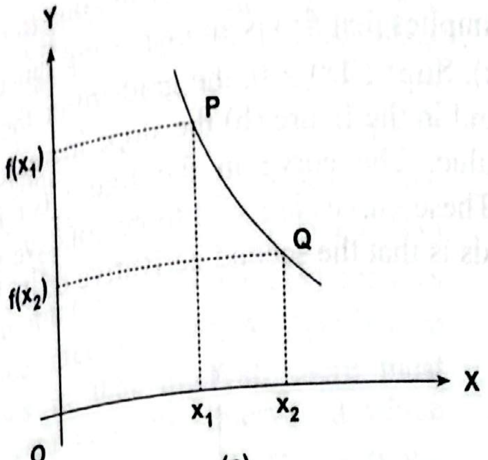

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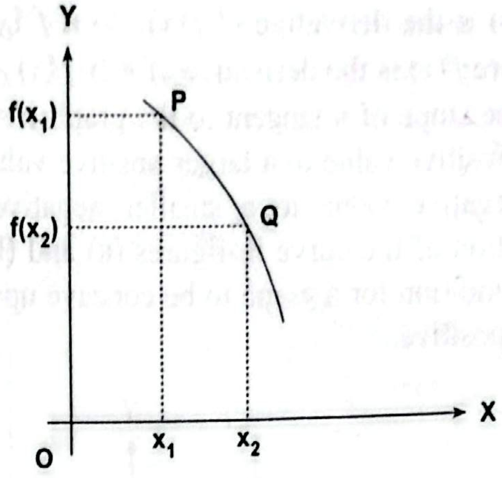
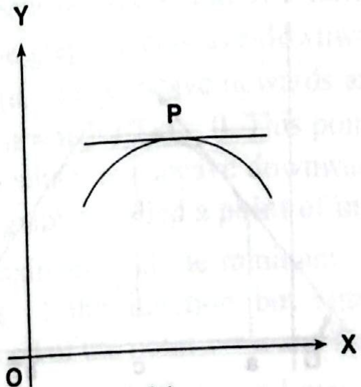


(d)

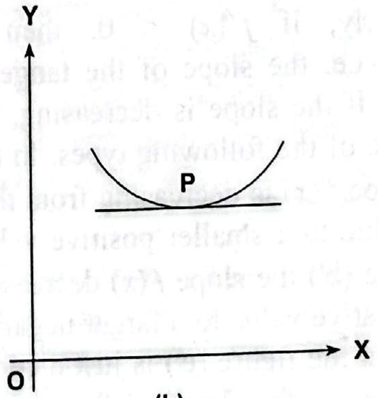
Similarly, if $f'(x)$ is negative, the inclination θ of the tangent line is negative and the tangent line slopes downwards (or goes down-hill) from left to right as shown in the figures. In the figures, within the interval (a, b), where the slope of the tangent line at any point is negative,

$$x_2 > x_1 \quad \Rightarrow \quad f(x_2) < f(x_1),$$

i.e., the function f is decreasing in its neighbourhood. So we can say that if $f'(x)$ is negative at a certain point, the function f is decreasing in its neighbourhood.

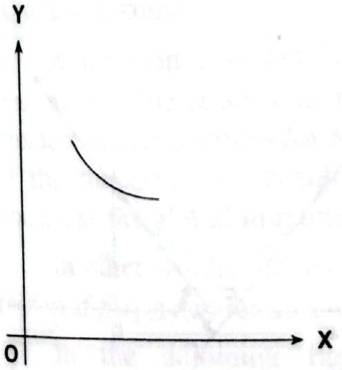


(a)

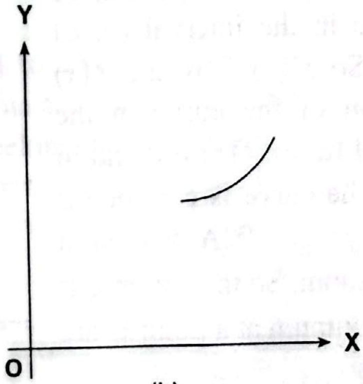


(b)

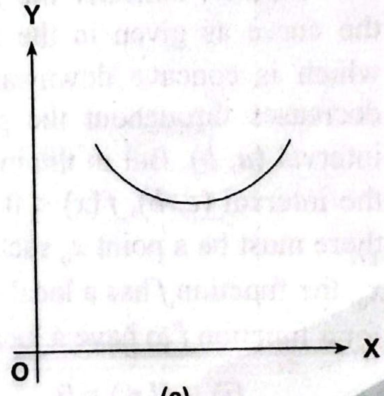
When $f'(x) = 0$, the tangent line neither goes 'uphill' nor goes 'downhill'. It remains horizontal or stationary. The points at which $f'(x) = 0$ or $f'(x)$ is not defined are called critical (or stationary) points. We shall come back to these points while dealing with maxima and minima of a function.



(a)

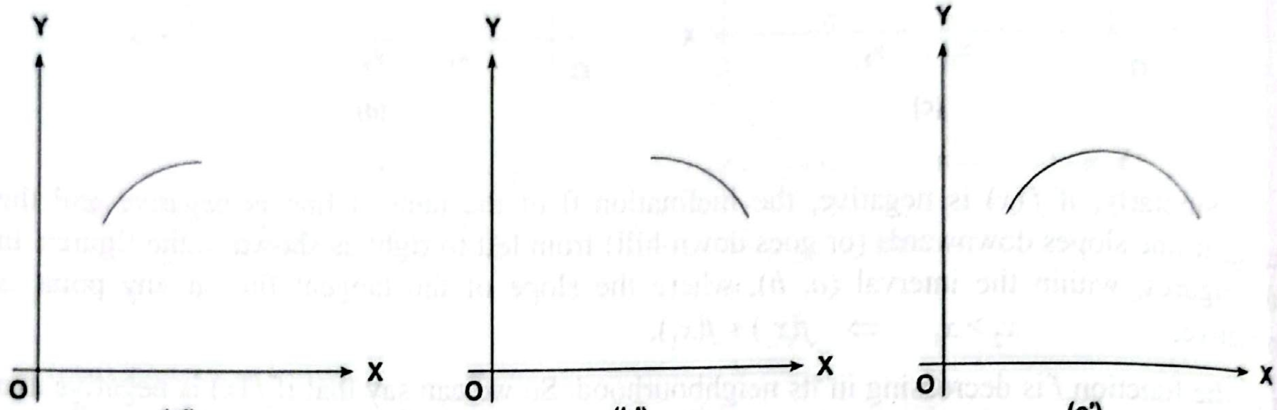


(b)

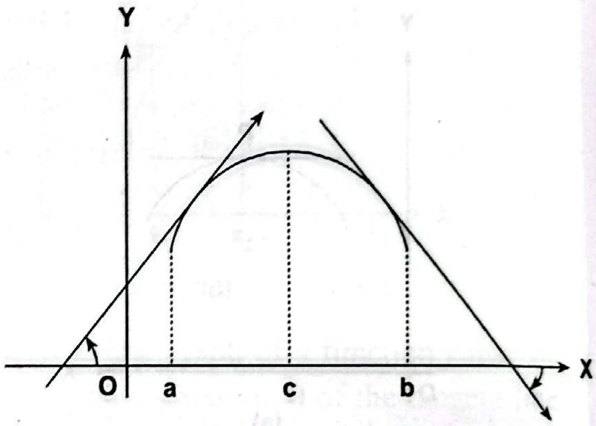


(c)

$f''(x)$ is the derivative of $f'(x)$. So if $f''(x) > 0$, $f'(x)$ is increasing (analogous to the above case where $f'(x)$ is the derivative of $f(x)$. $f'(x) > 0$ implies that $f(x)$ is an increasing function). But $f'(x)$ is the slope of a tangent to the graph $y = f(x)$. So if $f''(x) > 0$, the gradient or the slope of smaller positive value to a larger positive value and in the figure (b) the slope increases from a larger negative value to a smaller negative value. The curve in the figure (c) is just a combination of the curve in figures (a) and (b). These curves are said to be concave upwards. So the condition for a graph to be concave upwards is that the second derivative of the function must be positive.

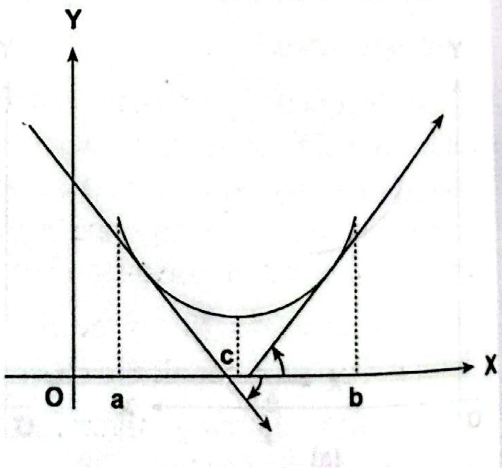


(a') (b') (c')



Similarly, if $f''(x) < 0$, then $f'(x)$ is decreasing, i.e. the slope of the tangent line is decreasing. If the slope is decreasing, the curve must be one of the following types. In the figure (a'), the slope $f'(x)$ is decreasing from the greater positive value to a smaller positive value, while in the figure (b') the slope $f'(x)$ decreases from a smaller negative value to a larger negative value. The curve in the figure (c') is just a combination of the curves in the figures (a') and (b'). These curves are said to be concave downwards. So the condition for a curve to be concave downwards is that the second derivative $f''(x)$ of the function f must be negative.

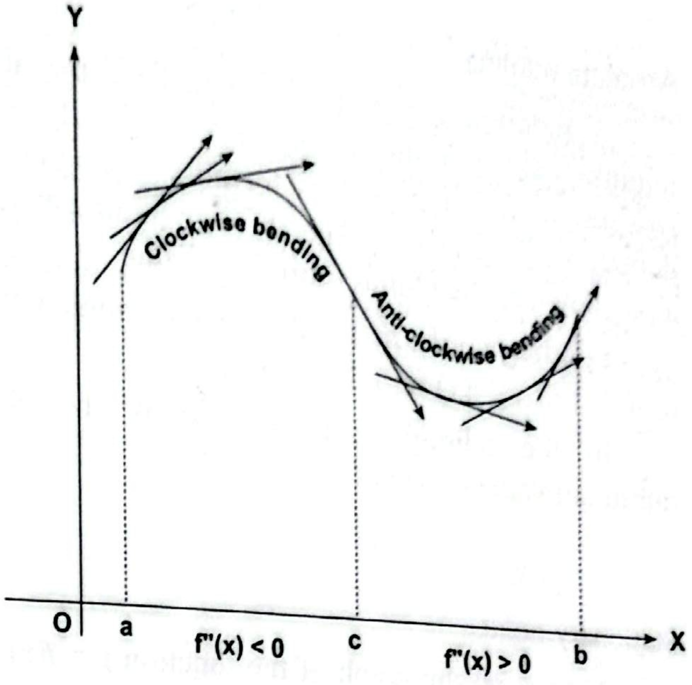
Thus we see that if $f''(x) < 0$ in an interval (a, b) the graph of f is concave downwards in the interval and if $f''(x) > 0$ in the interval, the graph of f is concave upwards in the interval.



Further, consider the function f with a portion of the curve as given in the figure in the interval (a, b) which is concave downwards. So $f''(x) < 0$ and $f'(x)$ decreases throughout the portion of the curve in the interval (a, b). But in the interval (a, c), $f'(x) > 0$ and in the interval (c, b), $f'(x) < 0$. As the curve is continuous there must be a point x_0 such that $f'(x_0) = 0$. At that point x_0, the function f has a local maximum. So the conditions for a function f to have a local maximum at a point x_0 are

(i) $f'(x_0) = 0$ (ii) $f''(x_0) < 0$

Similarly, let us consider a function f with a portion of the curve concave upwards in the interval (a, b). As the curve is concave upwards, $f''(x) > 0$ and so $f'(x)$ is increasing in the interval (a, b). But $f'(x) < 0$ in the interval (a, c) and $f'(x) > 0$ in the interval (c, b), i.e. $f'(x)$ changes sign in the interval (a, b). As $f'(x)$ is also continuous in that interval, there must exist a point x_0 in the interval at which $f'(x)$ takes a value zero, i.e., $f'(x_0) = 0$. So, $x_0 \in (a, b)$ is the point at which the function f has a local minimum. Hence the conditions for a function f to have a local minimum at x_0 are



(i) $f'(x_0) = 0$

(ii) $f''(x_0) > 0$.

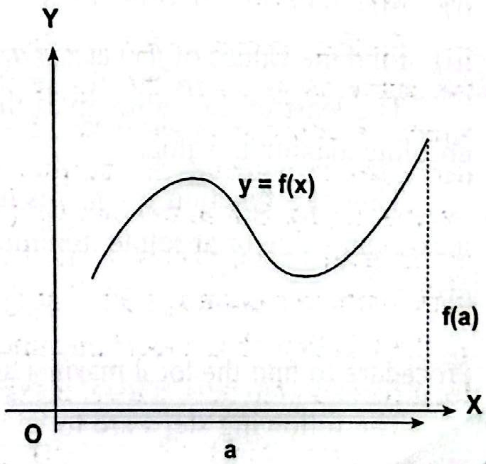
Again consider the graph of a function f in the following figure. In the interval (a, c), the portion of the graph is concave downwards and so $f''(x) < 0$, while the portion of the graph in the interval (c, b) is concave upwards and so $f''(x) > 0$. So naturally there must exist a point in the interval at which $f''(x) = 0$. This point is of considerable importance. It separates the portion of the graph which is concave downwards from the portion which is concave upwards. Such a point of the graph is called a **point of inflection.**

The maximum and the minimum values of the function need not be the greatest and the least values of the function but simply the maximum and the minimum values on the neighbourhood of the point $x = a$ and hence termed as the local maxima and local minima.

A function may have several maximum and minimum values but they occur alternatively. But besides these maximum and the minimum values, there may be the values greater than the maximum value and less than the minimum value i.e. these may be the greatest and the least values of the function which are termed as absolute maximum value and the absolute minimum value of the function.

Absolute maxima

A function $y = f(x)$ is said to have the absolute maximum value or absolute maxima at $x = a$ if $f(a)$ is the greatest of all its values for all x belonging to the domain of the function. The absolute maximum value is also known as the **global maximum value**.



In other words, $f(a)$ is the absolute maximum value of $f(x)$ if $f(a) \geq f(x)$ for all $x \in D(f)$.

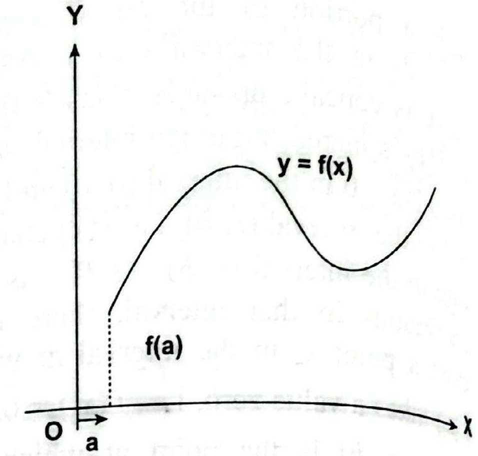
In the adjoining figure, $f(a)$ is the absolute maximum value of $f(x)$ at $x = a$.

Absolute minima

A function $y = f(x)$ is said to have the absolute minimum value or absolute minima at $x = a$ if $f(a)$ is the smallest of all of its values for all x belonging to the domain of the function. The absolute minimum value is also known as the **global minimum value**.

In other words, $f(a)$ is the absolute minimum value of $f(x)$ if $f(a) \le f(x)$ for all $x \in D(f)$.

In the adjoining figure, $f(a)$ is the absolute minimum value of $f(x)$ at $x = a$.



Stationary point

A point on the graph of the function $y = f(x)$ where the tangent is parallel to the x-axis is known as the stationary point or critical point. At the stationary point, $\frac{dy}{dx} = f'(x) = 0$

Local maxima

A function $y = f(x)$ is said to have the local maximum value or local maxima at $x = a$ if $f(a) > f(a \pm h)$ for sufficiently small positive value of h. The local maxima is also known as the relative maxima of the function.

Local minima

A function $y = f(x)$ is said to have the local minimum value or local minima at $x = a$ if $f(a) < f(a \pm h)$ for sufficiently small positive value of h. The local minima is also known as the relative minima of the function.

Procedure to find the absolute maxima and minima

Let $y = f(x)$ be the given function defined in an interval $[a, b]$.

i) Find the first derivative of $f(x)$ i.e. find $f'(x)$.

ii) Making $f'(x) = 0$, solve for x to get the stationary points. Let the values of x be c and d.

iii) Find the values of $f(x)$ at $x = a, b, c$ and d.

The least of the value gives the absolute minimum value and the greatest value gives the absolute maximum value.

Thus if a function $y = f(x)$ is defined in an interval $[a, b]$, then $f(x)$ may have the absolute maximum value or absolute minimum value at $x = a$, $x = b$ or an interior point $c \in [a, b]$ where $f'(c) = 0$

Procedure to find the local maxima and local minima

The following steps are to be used in finding the maxima and minima of the function $f(x)$ at a point.

i) Find $f'(x)$ and $f''(x)$ of the given function $y = f(x)$.

ii) Making $f'(x) = 0$, solve for x to get the stationary points. Let one of the stationary point be a i.e. $x = a$.

iii) Find $f''(a)$. If $f''(a) < 0$, then $f(x)$ has maximum value at $x = a$ and the maximum value = $f(a)$

If $f''(a) > 0$, then $f(x)$ has minimum value at $x = a$ and the minimum value = $f(a)$.

If $f''(a) = 0$ and $f'''(a) \neq 0$, then $f(x)$ has no maximum and no minimum value at $x = a$.

The condition for the function $y = f(x)$ to have the maximum, minimum or no maxima no minima at the point $x = a$ are given below.

Conditions	For the function $y = f(x)$			
	Maxima	Minima	No max. or no min.	
First order derivative	$\frac{dy}{dx} = f'(x) = 0$	$\frac{dy}{dx} = f'(x) = 0$	$\frac{dy}{dx} = f'(x) = 0$	$\frac{dy}{dx} = f'(x) \neq 0$
Second order derivative	$\frac{d^2y}{dx^2} = f''(x) < 0$	$\frac{d^2y}{dx^2} = f''(x) > 0$	$\frac{d^2y}{dx^2} = f''(x) = 0$	
Third order derivative			$\frac{d^3y}{dx^3} = f'''(x) \neq 0$	

Alternative method to find the local maxima and local minima with first derivative

For the maximum or minimum value of the function $y = f(x)$, we use the following steps:

i) Find $f'(x)$ or $\frac{dy}{dx}$.

ii) Making $f'(x) = 0$, solve for x. Let $x = a$ be one of the stationary points.

iii) Note the sign of $\frac{dy}{dx}$ when x changes its value from $a - h$ to $a + h$.

iv) If $\frac{dy}{dx}$ changes its sign from +ve to –ve, y or $f(x)$ has the maximum value at $x = a$. If $\frac{dy}{dx}$ changes its sign from –ve to +ve, $f(x)$ has minimum value at $x = a$.

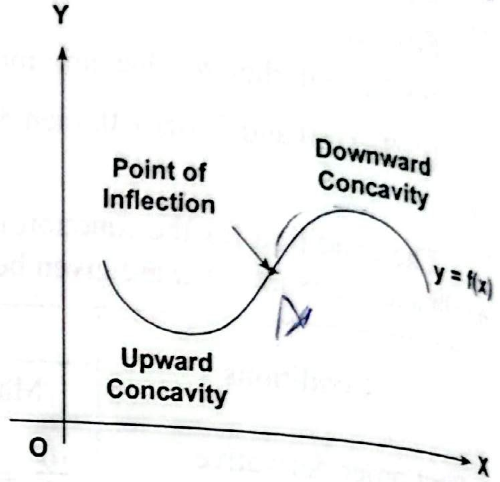
The maximum or minimum value of $f(x)$ at $x = a$ is $f(a)$.

If $\frac{dy}{dx}$ does not change its sign, then $f(x)$ has no maximum or minimum value.

17.6 Concavity

The graph of the function $y = f(x)$ defined in an interval (a, b) is **concave upward** (or convex downward) if each point (except the point of contact) on the graph (i.e. curve) lies above any tangent to it in that interval.

But if each point on the graph (i.e. curves) lies below any tangent to the curve in the interval, then the graph of the function is said to be **concave downward** (or convex upward).



The graph of the function $y = f(x)$ will be concave upward or concave downward according as

$$\frac{d^2y}{dx^2} = f''(x) > 0 \qquad \text{or} \qquad \frac{d^2y}{dx^2} = f''(x) < 0.$$

Procedure to find the concave upward and concave downward of the graph of the function

Let $y = f(x)$ be the given function. For the concave upward or concave downward of the graph of the function in an interval we use the following steps:

i) Find the second derivative of $f(x)$ i.e. find $f''(x)$.

ii) Find the interval within which $f''(x) > 0$. Then we conclude that the graph of $y = f(x)$ will be concave upward in the interval.

iii) Again find the interval within which $f''(x) < 0$. Then we conclude that the graph of $y = f(x)$ will be concave downward in that interval.

17.7 Point of inflection

The point which divides the graph of the function from the shape of upward concavity (or downward concavity) to the downward concavity (or upward concavity) is known as the point of inflection. If $y = f(x)$ be the given function which represent a continuous curve, then the point where $\frac{d^2y}{dx^2} = f''(x) = 0$ and $\frac{d^3y}{dx^3} = f'''(x) \neq 0$ is said to be the point of inflection.

Worked Out Examples

Example 1

Show that the function $f(x) = x^2 - 3x + 4$ is increasing at the point $x = 2$ and is decreasing at the point $x = 1$.

Solution :

$$f(x) = x^2 - 3x + 4$$

Then, $\quad f'(x) = 2x - 3$

At $x = 2$, $\quad f'(x) = 2 \times 2 - 3 = 1 > 0$

$\therefore f(x)$ is increasing at $x = 2$.

Again at $x = 1, f'(x) = 2 \times 1 - 3 = -1 < 0$

$\therefore f(x)$ is decreasing at $x = 1$.

Example 2

Show that the function $f(x) = x - \frac{1}{x}$ is increasing for all $x \in R$ $(x \neq 0)$.

Solution :

$$f(x) = x - \frac{1}{x}$$

Then, $\quad f'(x) = 1 + \frac{1}{x^2}$ $\quad$ which is positive for $x \in R$ except $x = 0$

$\therefore f(x)$ is increasing for all $x \in R$ $(x \neq 0)$, that is for

$$x \in (-\infty, 0) \approx (0, \infty)$$

Example 3

Find the interval in which the function $f(x) = 2x^3 - 15x^2 + 36x + 1$ is increasing or decreasing.

Solution :

$$f(x) = 2x^3 - 15x^2 + 36x + 1$$

Then, $f'(x) = 6x^2 - 30x + 36$

$= 6(x^2 - 5x + 6)$

$= 6(x - 2)(x - 3)$

$f'(x) = 0$ gives $\quad x = 2$ and 3 which are the stationary points.

For $x > 3$, $\quad f'(x) > 0$

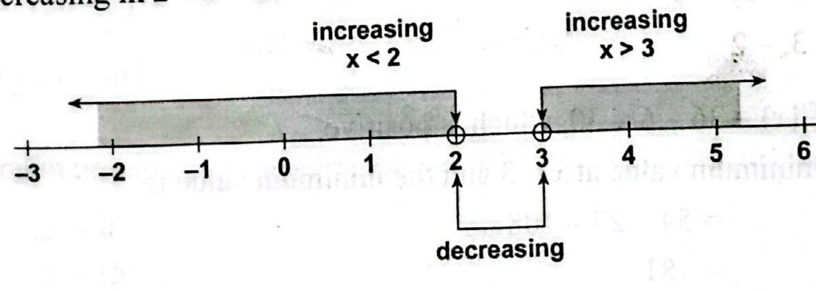
$\therefore f(x)$ is increasing for $x > 3$ i.e. on the interval $(3, \infty)$

Again for $x < 2$, $\quad f'(x) > 0$

$\therefore f(x)$ is increasing for $x < 2$ i.e. on the interval $(-\infty, 2)$

For $2 < x < 3$, $\quad f'(x) < 0$

$\therefore f(x)$ is decreasing in $2 < x < 3$.



$\therefore f(x)$ is increasing on $(-\infty, 2) \approx (3, \infty)$ and decreasing on $x \in (2, 3)$.

Example 4

Find the absolute maximum (greatest value) and the absolute minimum value (least value) of the function $f(x) = 2x^3 - 9x^2 + 12x + 20$ defined on an interval $[-1, 5]$

Solution :

$$f(x) = 2x^3 - 9x^2 + 12x + 20$$

Then, $f'(x) = 6x^2 - 18x + 12$

$f'(x) = 0$ gives

$6x^2 - 18x + 12 = 0$

$\Rightarrow\ x^2 - 3x + 2 = 0$ p''

$\Rightarrow\ (x - 1)(x - 2) = 0$

$\therefore\ x = 1, 2$

When $x = -1$, $f(x) = 2(-1)^3 - 9(-1)^2 + 12(-1) + 20 = -3$

$x = 5$, $f(x) = 2(5)^3 - 9(5)^2 + 12(5) + 20 = 105$

$x = 1$, $f(x) = 2(1)^3 - 9(1)^2 + 12(1) + 20 = 24$

$x = 2$, $f(x) = 2(2)^3 - 9(2)^2 + 12(2) + 20 = 72$

$\therefore$ absolute max. value = 105 and absolute min. value = -3

Example 5

Calculate the maximum and minimum values of $f(x) = 2x^3 - 3x^2 - 36x$

Solution:

We have $f(x) = 2x^3 - 3x^2 - 36x$

$\therefore\ f'(x) = 6x^2 - 6x - 36$

$f''(x) = 12x - 6$

For the maximum or minimum value of $f(x)$,

$f'(x) = 0$

$\therefore\ 6x^2 - 6x - 36 = 0$

$x^2 - x - 6 = 0$

or, $(x - 3)(x + 2) = 0$

$\therefore\ x = 3, -2$

(i) When $x = 3$, $f''(x) = 36 - 6 = 30$, which is positive.

So $f(x)$ has a minimum value at $x = 3$ and the minimum value is

$= 54 - 27 - 108$

$= -81$

(ii) When $x = -2$, $f''(x) = -24 - 6 = -30$, which is negative.

So $f(x)$ has a maximum value at $x = -2$ and the maximum value is

$= -16 - 12 + 72 = 44$

Example 6

Show that $f(x) = x^3 - 3x^2 + 6x + 4$ has neither a maximum nor a minimum value.

Solution :

Here, $f(x) = x^3 - 3x^2 + 6x + 4$

$$f'(x) = 3x^2 - 6x + 6 = 3(x^2 - 2x + 2)$$

$$= 3\{(x-1)^2 + 1\}$$

which is always positive for all real values of x and can never be zero.

$\therefore f(x)$ has neither a maximum nor a minimum value.

Example 7

Let $f(x) = 2x^3 - 6x^2 + 5$. Find where the graph is concave downward and where it is concave upward.

Solution:

We have $f(x) = 2x^3 - 6x^2 + 5$

$$f'(x) = 6x^2 - 12x$$

$$f''(x) = 12x - 12 = 12(x-1)$$

$f''(x) = 0$ gives $x = 1$ which is the point of inflection.

For $x > 1, f''(x) > 0$ and for $x < 1, f''(x) < 0$.

Hence the graph is concave downwards if $x < 1$ and is concave upwards if $x > 1$.

Example 8

Find the maximum area of a rectangular plot of land which can be enclosed by a rope of length 60 metres.

Solution:

Let the sides of the rectangular plot of land be x and y. So

$$2x + 2y = 60$$

or $x + y = 30$

The area of the land is

$$A = xy = x(30 - x) = 30x - x^2$$

$\therefore \dfrac{dA}{dx} = 30 - 2x$ and $\dfrac{d^2A}{dx^2} = -2$

For the maximum or minimum value of A, $\dfrac{dA}{dx} = 0$

$$30 - 2x = 0$$

or $x = 15$

and $y = 30 - x = 30 - 15 = 15$

Since $\dfrac{d^2A}{dx^2} = -2 < 0$, so A has maximum value when $x = 15$.

$\therefore$ the maximum area $= 15 \times 15 = 225$ sq.m.

EXERCISE 17.1

1. i) Examine whether the function $f(x) = 15x^2 - 14x + 1$ is increasing or decreasing at $x = \frac{2}{5}$ and $x = \frac{5}{2}$.

   ii) Show that the function $f(x) = 2x^3 - 24x + 15$ is increasing at $x = 3$ and decreasing at $x = \frac{3}{2}$.

2. i) Test whether the function $f(x) = 2x^2 - 4x + 3$ is increasing or decreasing on the interval (1, 4].

   ii) Examine whether the function $f(x) = 16x - \frac{4}{3}x^3$ is increasing or decreasing on the interval $(-\infty, -2)$.

   iii) Show that the function $f(x) = -x^3 + 6x^2 - 13x + 20$ is decreasing for all $x \in R$.

   iv) Show that the function $f(x) = 4x - \frac{9}{x} + 6$ is increasing for all $x \in R$ except at $x = 0$.

3. Find the intervals in which the following functions are increasing or decreasing.

   i) $f(x) = 3x^2 - 6x + 5$ ii) $f(x) = x^4 - \frac{1}{3}x^3$

   iii) $f(x) = 5x^3 - 135x + 22$ iv) $f(x) = 6 + 12x + 3x^2 - 2x^3$

   v) $f(x) = x^3 - 12x$ defined on [–3, 5]

4. Find the absolute maximum and the absolute minimum values of the following function on the given intervals:

   i) $f(x) = 3x^2 - 6x + 4$ on $[-1, 2]$

   ii) $f(x) = 2x^3 - 9x^2$ on $[-2, 4]$

   iii) $f(x) = x^3 - 6x^2 + 9x$ on $[0, 5]$

   iv) $f(x) = 2x^3 - 15x^2 + 36x + 10$ on $[1, 4]$

5. Find the local maxima and minima and points of inflection:

   (i) $f(x) = 3x^2 - 6x + 3$ (ii) $f(x) = x^3 - 12x + 8$

   (iii) $f(x) = x^3 - 6x^2 + 3$ (iv) $f(x) = 2x^3 - 15x^2 + 36x + 5$

   (v) $f(x) = 2x^3 - 9x^2 - 24x + 3$ (vi) $f(x) = 4x^3 - 15x^2 + 12x + 7$

   (vii) $f(x) = 4x^3 - 6x^2 - 9x + 1$ on the inverval $(-1, 2)$

   (viii) $f(x) = x + \frac{100}{x} - 5$

6. Show that the following functions have neither maximum nor minimum value.

   i) $f(x) = x^3 - 6x^2 + 24x + 4$ ii) $f(x) = x^3 - 6x^2 + 12x - 3$

7. Determine where the graph is concave upwards and where it is concave downwards of the following functions:

(i) $f(x) = x^4 - 2x^3 + 5$ (ii) $f(x) = x^4 - 8x^3 + 18x^2 - 24$

(iii) $y = 3x^5 + 10x^3 + 15x$ (iv) $f(x) = x^3 - 9x^2$ defined on $[-2, 5]$

8. A man who has 144 metres of fencing material wishes to enclose a rectangular garden. Find the maximum area he can enclose.

9. Show that the rectangle of largest possible area, for a given perimeter, is a square.

10. A window is in the form of a rectangle surmounted by a semi-circle. If the total perimeter is 9 metres, find the radius of the semi-circle for the greatest window area.

11. A closed cylindrical can is to be made so that its volume is 52 cm³. Find its dimensions if the surface is to be a minimum.

12. A gardener having 120 m. of fencing wishes to enclose a rectangular plot of land and also to erect a fence across the land parallel to two of the sides. Find the maximum area he can enclose.

13. Find two numbers whose sum is 10 and the sum of whose squares is minimum.

$(10-x)^2+x^2$

$=100-20x+x^2+x^2 = 100-20x+2x^2$

Practical Work/Activities

A. Given the perimeter of the rectangular plot of land, find the dimension of the land where the area is maximum. Perform above result experimentally i.e. with various values of length and breadth. What will be the shape of the land when the area is maximum?

B. Given the sum of the height and the radius, find the height and radius of the cylinder when its volume is maximum.

Answers

1. (i) Decreasing at $x = \frac{2}{5}$ and increasing at $x = \frac{5}{2}$

2. (i) Increasing (ii) Decreasing

3. (i) Increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$

(ii) Increasing on $\left(\frac{1}{4}, \infty\right)$ and decreasing on $\left(-\infty, \frac{1}{4}\right)$

(iii) Increasing on $(-\infty, -3) \cup (3, \infty)$ and decreasing on $(-3, 3)$

(iv) Decreasing on $(-\infty, -1) \cup (2, \infty)$ and increasing on $(-1, 2)$

(v) Increasing on $[-3, -2) \cup (2, 5]$ and decreasing on $(-2, 2)$

4. (i) Absolute max = 13, Absolute min. = 1 (ii) Absolute max = 0, Absolute min. = −52

(iii) Absolute max = 20, Absolute min. = 0 (iv) Absolute max = 42, Absolute min. = 33

5. (i) min. value = 0 at x = 1. No point of inflection.

(ii) max. value = 24 at x = −2; min. value = −8 at x = 2. The point of inflection is at x = 0

(iii) max. value = 3 at x = 0; min. value = −29 at x = 4. The point of inflection is at x = 2

(iv) max. value = 33 at x = 2; min. value = 32 at x = 3. The point of inflection is at $x = \frac{5}{2}$.

(v) max. value = 16 at x = –1; min. value = –109 at x = 4. The point of inflection is at $x = \frac{3}{2}$.

(vi) max. value = $\frac{39}{4}$ at $x = \frac{1}{2}$; min. value = 3 at x = 2. The point of inflection is at $x = \frac{5}{4}$.

(vii) Max. value = $\frac{7}{2}$ at $x = -\frac{1}{2}$, Min. value = $-\frac{25}{2}$ at $x = \frac{3}{2}$. The point of inflection is at $x = \frac{1}{2}$

(viii) min. value = 15 at x = 10; max. value = –25 at x = –10. No point of inflection.

7. (i) Concave upwards for x > 1, Concave upwards for x < 0 and Concave downwards for 0 < x < 1.
   (ii) Concave upwards for x < 1; Concave downwards for 1 < x < 3; Concave upwards for x > 3.
   (iii) Concave upwards for x > 0; Concave downwards for x < 0.
   (iv) Concave upward on (3, 5] and concave down on [–2, 3).

8. 1296 sq. m

10. $\frac{9}{4+\pi}$ metres

11. height = 2 radius where radius = $\left(\frac{26}{\pi}\right)^{1/3}$

12. 600 m^2

13. 5, 5

Multiple Choice Questions

1. $f(x)$ is increasing in (a, b) if
   a) $f'(x) > 0$ for $x \in (a, b)$
   b) $f'(x) < 0$ for $x \in (a, b)$
   c) $f'(x) = 0$ for $x \in (a, b)$
   d) all of the above
2. $f(x)$ is decreasing in (a, b) if for $x_1, x_2 \in (a, b)$
   a) $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$
   b) $x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$
   c) $x_2 > x_1 \Rightarrow f(x_1) = f(x_2)$
   d) $x_1 = x_2 \Rightarrow f(x_1) > f(x_2)$
3. For the function $y = f(x) = x^2 + 3x - 6, f'(2) = 4$, so at the point $x = 2, f(x)$ is
   a) stationary
   b) increasing
   c) decreasing
   d) cannot be concluded
4. For the function $y = f(x) = x^2 - 5x + 4, f(0) = 4$ and $f(2) = 2$ so for $0, 2 \in I, f(x)$ is
   a) decreasing
   b) increasing
   c) stationary
   d) none of the above
5. $x = a$ on the curve represented by $y = f(x)$ is called the stationary point if
   a) $f'(a) > 0$
   b) $f'(a) < 0$
   c) $f'(a) = 0$
   d) $f'(a) = \infty$
6. The turning points of the curve $y = \frac{1}{\ldots}$
   a) 2, 6

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