

Ordinary Differential Equations

These notes are my own summary of [Brendle's](#) notes on ODEs, and Stanford Math 63. Note that I use some pretty idiosyncratic definitions, for my own clarity.

We consider first order ordinary differential equations equations of the following form, fixing x_0 :

$$\frac{d\phi(x_0)(t)}{dt} = F(\phi(x_0)(t), t) \quad (1)$$

where F is some particular function that we fix, and the goal is to find a function $\phi(x_0)$ which satisfies the equation and also satisfies $\phi(x_0)(0) = x_0$. All n th order differential equations (featuring higher derivatives) can be reduced to first order equations of multiple variables, so that's why we focus on equations of the first order. In general

$$\begin{aligned} \phi &\in (\mathbb{R}^n \rightarrow (\mathbb{R} \rightarrow \mathbb{R}^n)) \\ t &\in \mathbb{R} \\ F &\in ((\mathbb{R}^n, \mathbb{R}) \rightarrow \mathbb{R}^n) \end{aligned} \quad (2)$$

Often we consider the subset of *autonomous* cases:

$$\frac{d\phi(x_0)(t)}{dt} = F(\phi(x_0)(t)) \quad (3)$$

Often we write (3) as:

$$\phi(x_0)'(t) = F(\phi(x_0)(t)) \quad (4)$$

or even:

$$\phi(x_0)' = F(\phi(x_0)) \quad (5)$$

Note that we are assuming that the initial time t is 0 without loss of generality, because given an ODE where $\psi(x_0)(t_0) = x_0$:

$$\psi(x_0)'(t) = F(\psi(x_0)(t), t) \quad (6)$$

We can obtain a new ODE of the same form for $\phi(x_0)(t) = \psi(x_0)(t + t_0)$.

I'll write ϕ_F to refer to the solution to the ODE defined in terms of some F , i.e. the unique maximal solution (when it exists). We can also uncurry ϕ , and write an ODE as:

$$D_2\phi_F(x_0, t) = F(\phi_F(x_0, t), t) \quad (7)$$

Also note that when $F(x(t), t) = F(x(t), t')$ for any t , we say that the corresponding ODE is autonomous and abuse notation by writing $F(x(t))$, omitting the second argument.

Existence and uniqueness theorems:

Peano existence theorem: If F is continuous, a solution exists. We can prove this by constructing it from a series of linear interpolations, and using the Arzela-Ascoli theorem to show that this series of functions converges, and moreover converges to a solution. This is boring and fiddly to prove, so I'll omit it.

Picard existence theorem: If F is continuous in both arguments and locally Lipschitz in t , $\phi_F(x, t)$ exists in an interval around 0 and is continuously differentiable in t . We prove this by considering the

$$\phi(x_0)'(t) = F(\phi(x_0)(t), t) \Rightarrow \phi(x_0)(t) = x_0 + \int_0^t F(\phi(x_0)(t), t) dt \quad (8)$$

Let $A : C_1 \rightarrow C_1$ be an operator mapping defined as $A[y] = \int_0^t F(y(t), t) dt$

If A has a fix point, then that fix point is a solution to the ODE defined by F . If A is a contraction map on the metric space of C_1 then it has a fix point. So we need only show it is a contraction map. This can be done with some careful analytic argumentation which is omitted here.

Note that this also gives a method (called Picard iteration) for constructing the solutions to particular differential equations, by analysing the sequence produced by repeated applications of A .

Now consider a solution like:

$$\phi_F(x)(t) \quad (9)$$

Think of this as a path through the space, with the first input representing the initial condition and the second representing the time. Note that ϕ_F is a semigroup in t (and if F is uniformly Lipschitz, then it's even a group):

$$\phi_F(t + s, x) = \phi_F(t, x) \circ \phi_F(s, x); \quad \phi_F(0, x) = x \quad (10)$$

Note: this path cannot intersect any other path representing a solution with a different initial condition.

The proof of this follows from a more specialized case: if $x_0 = y_0$, then for $F : U \rightarrow \mathbb{R}^n$ then

$\forall t : \phi_F(x_0)(t) = \phi_F(y_0)(t)$ at least where the domains of definition agree. The proof illustrates some useful tools, so I'll include it:

First suppose that $\phi_F(x_0) : I \rightarrow \mathbb{R}^n$ and $\phi_F(y_0)(t) : J \rightarrow \mathbb{R}^n$ for open intervals I and J both containing 0. Then suppose that $\phi_F(x_0) : I \cap J$ and $\phi_F(y_0)(t) : I \cap J$ are not equal. Then for $\tau = \inf\{t \in I \cap J : \phi_F(x_0)(t) \neq \phi_F(y_0)(t)\}$, $\phi_F(x_0)(\tau) = \phi_F(y_0)(\tau)$. Let $\bar{x} = \phi_F(x_0)(\tau)$. Then for a ball $B_r(\bar{x}) \subset U$, we can find $\delta > 0$ such that for t , $\phi_F(x_0)(t) \in B_r(\bar{x})$ and $\phi_F(y_0)(t) \in B_r(\bar{x})$. We then set L as $\sup_{x \in B_r(\bar{x})} \|DF(x)\|_{op}$, which gives us that $\forall t \in [\tau, \tau + \delta] : \|F(\phi_F(x_0)(t)) - F(\phi_F(y_0)(t))\| \leq L\|\phi_F(x_0)(t) - \phi_F(y_0)(t)\|$

We now note that if $\|F(z(t))\| \leq L\|z(t)\|$ then:

$$\begin{aligned} \frac{d}{dt} \|z(t)\|^2 &= \frac{d}{dt} z(t)^T z(t) = 2z(t)^T \frac{d}{dt} z(t) \\ &= \langle 2z(t), F(z(t)) \rangle \leq 2\|z(t)\| \cdot \|F(z(t))\| \leq 2\|z(t)\| \cdot L\|z(t)\| = 2L\|z(t)\|^2 \end{aligned} \quad (11)$$

We can now be rather clever, and note that $\frac{d}{dt} (e^{-2Lt} \|z(t)\|^2) \leq 2Le^{-2Lt} \|z(t)\|^2 - 2Le^{-2Lt} \|z(t)\|^2 = 0$, so $e^{-2Lt} \|z(t)\|^2$ is monotone decreasing.

With $z(t) = \phi_F(x_0)(t) - \phi_F(y_0)(t)$, we have that $e^{-2Lt} \|z(t)\|^2$ is monotone decreasing, which means that, since $e^{-2L\tau} \|\phi_F(x_0)(\tau) - \phi_F(y_0)(\tau)\|^2 = 0$, $\phi_F(x_0)(t) = \phi_F(y_0)(t)$ for $t \in [\tau, \tau + \delta]$, since otherwise the expression would be positive (norms are positive).

The fact that paths can never intersect is useful because it means we can often constrain the movement of a path. For example, see problem 1 of pset 3, Math63.md.

We can now introduce some powerful results about the domain of solutions of ODEs. For continuously differentiable (hence locally Lipschitz) $F : U \rightarrow \mathbb{R}^n$, solutions $\phi_F(x, t)$ where F is autonomous which are defined for $t \in (\alpha, \beta)$ for β finite, $\|\phi_F(x, t)\|$ either goes to infinity in finite time t , or $d(U, \phi_F(x, t))$ converges to 0, i.e. the path converges to the boundary of U . Note that we assume U is open, and that $d(U, y)$ for a set U and a point y is $\inf_{z \in U} d(y, z)$.

That's a mouthful. The idea is that if the path doesn't continue for infinite time, the solution must either converge onto the boundary of the (finite) open domain, or go to infinity in finite time.

The contrapositive is equally useful: if $\phi_F(x, t)$ stays within a closed set, or equivalently $\|\phi_F(x, t)\|$ stays within a closed interval, then the solution exists for all $t > 0$.

Some Simple ODEs and their solutions

For some simple forms of F , we can derive a general solution. For example, suppose:

$$F(x, t) = ax, \quad a \in \mathbb{R} \quad (12)$$

Then, to find a general solution, first note

$$\frac{d}{dt}(e^{-at}\phi(x_0, t)) = -ae^{-at}\phi(x_0, t) + e^{-at}\frac{d}{dt}\phi(x_0, t) = e^{-at}(\frac{d}{dt}\phi(x_0, t) - a\phi(x_0, t)) = 0 \quad (13)$$

This means that $e^{-at}\phi(x_0, t) = c$ for some constant c , so $\phi(x_0, t) = ce^{at}$. Then $x_0 = \phi(x_0, 0) = c$.

Now consider a more general case:

$$F(x, t) = a(t)x(t) + f(t) \quad (14)$$

for a and f continuous. We can apply a similar approach to before:

$$\begin{aligned} \frac{d}{dt}(e^{-\int_0^t a(s)ds}\phi(x_0, t)) &= -a(t)e^{-\int_0^t a(s)ds}\phi(x_0, t) + e^{-\int_0^t a(s)ds} \cdot (a(t)\phi(x, t) + f(t)) \\ &= e^{-\int_0^t a(s)ds}(a(t)\phi(x, t) + f(t) - a(t)\phi(x_0, t)) = e^{-\int_0^t a(s)ds}f(t) \\ &\Leftrightarrow e^{-\int_0^t a(s)ds}\phi(x_0, t) = \int_0^t e^{-\int_0^s a(s)ds}f(t) + c \\ &\Leftrightarrow \phi(x_0, t) = e^{\int_0^t a(s)ds}(\int_0^t e^{-\int_0^s a(s)ds}f(t) + c) \end{aligned} \quad (15)$$

Further:

$$x_0 = \phi(x_0, 0) = c \quad (16)$$

Now for one final class of cases:

$$F(x(t), t) = f(x(t)) \cdot g(t) \quad (17)$$

where both g and f are continuous, and f has an open domain without 0. Then suppose there exists a τ such that $\tau'(k) = \frac{1}{f(k)}$. What τ is may depend on the F , of course - that is, I don't think there's a general expression for it. Then:

$$\begin{aligned} \frac{d}{dt}\tau(\phi(x_0, t)) &= \frac{1}{f(\phi(x_0, t))}D_2\phi(x_0, t) = g(t) \\ &\Rightarrow \tau(\phi(x_0, t)) = \int g(t)dt + c \\ &\Rightarrow \phi(x_0, t) = \tau^{-1}(\int g(t)dt + c) \end{aligned} \quad (18)$$

This approach is known as separation of variables. As an example, suppose you see

$\frac{\partial}{\partial t}\phi(x_0, t) = t(1 + \phi(x_0, t)^2)$. Then you can identify $f(x) = 1 + x^2, g(t) = t$. Further, you can remember from your trigonometry that $\int \tan^{-1}(x)dx = \frac{1}{1+x^2}$. Using the above,

$$\frac{d}{dt}\tan^{-1}x(t) = \frac{1}{1+x(t)^2}x'(t) = t \Rightarrow x(t) = \tan(\frac{t^2}{2} + c)$$

Duhamel's Principle

For a linear operator ψ , consider $F_1(x) = \psi(x)$ and $F_2(x, t) = F_1(x(t)) + g(t)$. Duhamel's principle (also known as the Variation of Parameters principle) provides a connection between ϕ_{F_1} and ϕ_{F_2} . In particular:

$$\phi_{F_2}(x_0, t) = \phi_{F_1}(x_0, t) + \int_{t_1}^t \phi_{F_1}(g(s), t_2 - s) ds \quad (19)$$

We can verify this by differentiating, taking care to use Leibniz' rule to differentiate under the integral:

$$\begin{aligned} D_2 \phi_{F_2}(x_0, t) &= D_2 \phi_{F_1}(x_0, t) + \int_{t_1}^t D_1 \phi_{F_1}(g(s), t - s) ds + \phi_{F_1}(g(t), 0) \\ &= \psi(\phi_{F_1}(x_0, t)) + \int_{t_1}^t \psi(\phi_{F_1}(g(s), t - s)) ds + g(t) \\ &= \psi(\phi_{F_1}(x_0, t)) + \int_{t_1}^t \phi_{F_1}(g(s), t - s) ds + g(t) \\ &= \psi(\phi_{F_1}(x_0, t) + \int_{t_1}^t \phi_{F_1}(g(s), t - s) ds) + g(t) \\ &= \psi(\phi_{F_2}(x_0, t)) + g(t) \end{aligned} \quad (20)$$

The partial derivatives of ϕ :

Now consider:

$$D_2 \phi_F(x, t) = \frac{\partial \phi_F(x, t)}{\partial t} \quad (21)$$

$$D_2 \phi_F(x, t) = F(\phi(x, t)) \quad (22)$$

The derivative with respect to initial condition can be proven to exist as well, and moreover to be continuous, (I don't prove either here).

$$D_1 \phi_F(x, t)_{ij} = \frac{\partial \phi_F(x_i, t)}{\partial x_j} \quad (23)$$

Intuitively, this is asking: how does the whole path change if I vary the initial condition by just a tiny bit. This would seem reminiscent of functional analysis.

What can we say about this derivative? Well:

$$D_2 D_1 \phi_F(x, t)_{ij} = \frac{\partial}{\partial t} \frac{\partial \phi_F(x_i, t)}{\partial x_j} = \frac{\partial}{\partial x_j} \frac{\partial \phi_F(x_i, t)}{\partial t} = \frac{\partial}{\partial x_j} F_i(\phi(x, t)) = DF(\phi(x, t))_{ik} \cdot D_1 \phi_F(x, t)_{kj} \quad (24)$$

This itself is a differential equation. To make this clear, write $H(t) = D_1 \phi_F(x, t)_{ij}$ and $A(t) = DF(\phi(x, t))$. Then:

$$H'(t) = A(t)H(t) \quad (25)$$

To solve this in closed form, we need to use some linear algebra; cue the next section.

Linear Differential Equations

Linear differential equations have an F of the following form, where $\psi(t)$ is a linear operator, and f is continuous:

$$F(x(t), t) = \psi(t)(x(t)) + f(t); \quad x(0) = x_0 \quad (26)$$

The approach to solving these is to use matrix exponentials (see LinearAlgebra.md).

For a simple example, suppose $F(x(t), t) = \psi(x(t))$. Then, with some more analysis, we can show that $\phi_F(x, t) = xe^{t\psi}$ is the unique solution as we'd expect.

For a harder example, consider an ODE deriving from $F(x, t) = \psi(x) + f(t)$. We can derive a solution by:

$$\begin{aligned} \frac{d}{dt}(e^{-t\psi} \phi(x_0, t)) &= e^{-t\psi} (D_2 \phi(x_0, t) - \psi(\phi(x_0, t))) = e^{-t\psi} f(t) \\ \Rightarrow \phi(x_0, t) &= e^{t\psi} \left(\int_0^t e^{-s\psi} f(s) ds + x_0 \right) \end{aligned} \quad (27)$$

Now returning to $H'(t) = A(t)H(t)$, the solution, analogous to the scalar case, is $e^{\int_0^t A(s) ds}$, which means that $D_1 \phi_F(x, t) = e^{\int_0^t DF(\phi(x, s)) ds}$

Differential Flow

We can view ϕ as defining a flow in \mathbb{R}^n , and ask questions like: does the image of a set of some volume in \mathbb{R}^n under ϕ have the same volume? In fluid dynamics, this is the question of compressability. So we are asking: what are the conditions for an incompressible flow? More formally, given a set X and abusing notation so that $\phi(X, t)$ is the image of X under $\lambda x \rightarrow \phi(x, t)$:

$$Vol(\phi(x, t)) = \int_{\phi(X, t)} 1 dx = \int_X |\det D_1 \phi(x, t)| dx \quad (28)$$

So what we want to know is the rate of change of the Jacobian $J(x, t) = \det D_1 \phi(x, t)$ that is, $\frac{d}{dt} \det D_1 \phi(x, t)$. If this is 0, then the volume never changes.

Once again, let's define $H_x(t) = D_1 \phi(x, t)$. Then:

$$H'_x(t)_{ij} = DF(\phi(x, t))_{ik} \cdot H_x(t)_{kj} \quad (29)$$

First, $H_x(0)_{ij} = D_1 \phi(x, 0)_{ij} = \frac{dx_i}{dx_j} = \delta_{i=j}$. This means that the initial value of H is the identity matrix I . So now we know that if $\frac{\partial}{\partial t} J(x, t)$ is 0, then the phase flow is incompressible and thus a volume of the space never changes under the flow.

OK, so now we would like to analyze $\frac{\partial}{\partial t} J(x, t)$ further.

Jacobi's formula (see LinearAnalysis.md) is: $\frac{d}{dt} \det \phi = \text{tr}(\det(\phi) \phi^{-1} \frac{d\phi}{dt})$

This means that $\frac{d}{dt} \det H_x(t) = \text{tr}(\det(H_x(t)) H_x(t)^{-1} DF(x, t) \cdot H_x(t)) = \text{tr}(DF(x, t)) \cdot \det(H_x(t))$.
Or:

$$\det H_x(t) = \det H_x(0) e^{\int_0^t \text{tr}(DF(x(s))) ds} \quad (30)$$

In other words, if the differential of F is traceless, or in physics terminology, the divergence (trace of differential of F) is 0, then the phase flow is incompressible.

Note for example that in a Hamiltonian system, divergence is 0. (See e.g. Math63.md, Pset 4.7)

Stability Theory

An equilibrium point p satisfies $F(p) = 0$.

We can then ask questions like: do points near a given equilibrium point stay near it over time (stability)? Do they converge to it when nearby (asymptotic stability)?

Formally, an equilibrium point p is stable if $\forall \epsilon > 0, \exists \delta > 0 : \forall x \in B_\delta(p) : (\forall t : \phi_F(x, t) \in B_\epsilon(p))$. Here, I use $B_\alpha(x)$ to denote the open ball in \mathbb{R}^n of radius α centered at x .

An equilibrium point p is asymptotic if $\exists \delta : \forall x \in B_\delta(p) : \lim_{t \rightarrow \infty} \phi(x, t) = 0$. In other words, if there's an open ball around p such that all points in that ball get sucked to p .

There is no entailment in either direction between being asymptotic and being stable. For example, a flow which is composed of circular paths orbiting p results in p being stable, but not asymptotic. Conversely, there may be a flow in which some paths in any ball always go far away before converging to p .

Note that a equilibrium point at 0 also a solution, namely $\phi_F(0, t) = 0$. This is a nice perspective, because later we'll see convergence of ODE solutions not just to points, but to other solutions, so we can take convergence to equilibrium points as a simple case which we generalize.

We can first establish a number of useful and intuitive results about the behaviour of $e^{t\psi}$ when the eigenvalues of the linear operator ψ have certain properties. For instance, suppose they all have negative real part. Then, making use of the ΛN decomposition:

$$\lim_{t \rightarrow \infty} \|e^{t\psi} x\| = \lim_{t \rightarrow \infty} \|e^{t\lambda_i}\| \cdot \left\| \sum_{k=0}^{n-1} \frac{t^k (\psi - \lambda_i I)^k}{k!} \right\| = \lim_{t \rightarrow \infty} e^{t \operatorname{Re}(\lambda_i)} \cdot \left\| \sum_{k=0}^{n-1} \frac{t^k (\psi - \lambda_i I)^k}{k!} \right\| = 0 \quad (31)$$

The exponential term dominates, and this shrinks to 0 with t . So $\lim_{t \rightarrow \infty} e^{t\psi} = 0$ for this ϕ . Similarly, if all the eigenvalues have positive real part, the size grows exponentially.

This means that for $F(x, t) = \psi(x(t))$, $\phi_F(x, t)$ is asymptotically stable.

Now consider an arbitrary linear operator $\psi : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that every eigenvalue has non-zero real part. Recall the generalized eigendecomposition, namely that \mathbb{C}^n is a direct sum of eigenspaces. Consider the sum of only the subspaces which have negative eigenvalues, and call this subspace Φ_- . Similarly for Φ^+ . We call the projection maps into these two subspace P_- and P_+ respectively. Further, $\ker P_- \oplus \ker P_+ = \mathbb{C}^n$. And as we've already established: $\lim_{t \rightarrow \infty} e^{t\psi} P_- = 0$ and $\lim_{t \rightarrow -\infty} e^{t\psi} P_+ = 0$.

We can interpret these as saying: if you go forward in time in Φ_- or backward in time in Φ_+ , you'll hit the 0 equilibrium.

In fact, we can make the stronger statements that for $\alpha > 0$ such that $\forall i : \alpha < |\operatorname{Re}(\lambda_i(\psi))|$, it is the case that $\lim_{t \rightarrow \infty} e^{\alpha t} e^{t\psi} P_- = 0$ and $\lim_{t \rightarrow -\infty} e^{-\alpha t} e^{t\psi} P_+ = 0$. These are statements about the rate of the convergence. They're intuitive: the convergence happens at an exponential rate greater dominated by the largest (in absolute value) eigenvalue.

Now consider ODEs defined by $F(x(t), t) = \psi(x(t)) + f(t)$ where f is continuous and τ -periodic. Again, eigenvalues of ψ have negative real parts). We know from our exploration of linear ODEs that the general solution is:

$$\phi(x_0, t) = e^{t\psi} \left(\int_0^t e^{-s\psi} f(s) ds + x_0 \right) \quad (32)$$

Now consider shifting the input by τ :

$$\begin{aligned}
\phi(x_0, t + \tau) &= e^{(t+\tau)\psi} \left(\int_0^{t+\tau} e^{-s\psi} f(s) ds + x_0 \right) = e^{(t+\tau)\psi} \left(\int_{\tau}^{t+\tau} e^{-s\psi} f(s) ds + \int_0^{\tau} e^{-s\psi} f(s) ds + x_0 \right) \quad (33) \\
&= e^{(t+\tau)\psi} \left(\int_0^t e^{-(s+\tau)\psi} f(s + \tau) ds + \int_0^{\tau} e^{-s\psi} f(s) ds + x_0 \right) \\
&= e^{(t+\tau)\psi} \left(e^{-\tau\psi} \int_0^t e^{-s\psi} f(s) ds + \int_0^{\tau} e^{-s\psi} f(s) ds + x_0 \right)
\end{aligned}$$

If we look at this last equation, we can see that if:

$$\int_0^{\tau} e^{-s\psi} f(s) ds + x_0 = e^{-\tau\psi} x_0 \quad (34)$$

then

$$\phi(x_0, t + \tau) = e^{t\psi} \left(\int_{t_0}^t e^{-s\psi} f(s) ds + x_0 \right) = \phi(x_0, t) \quad (35)$$

So let's set x_0 such that it satisfies this property:

$$\int_0^{\tau} e^{-s\psi} f(s) ds + x_0 = e^{-\tau\psi} x_0 \Leftrightarrow \int_0^{\tau} e^{-s\psi} f(s) ds = (e^{-\tau\psi} - I)x_0 \Leftrightarrow x_0 = (e^{-\tau\psi} - I)^{-1} \int_0^{\tau} e^{-s\psi} f(s) ds \quad (36)$$

Finally, we know that $\xi = (e^{-\tau\psi} - I)$ exists because if the kernel of ξ were not empty, then 1 would be an eigenvalue of $e^{-\tau k\psi}$ for every $k \in \mathbb{Z}$. But since we know that $\forall i : \text{Re}(\lambda_i(\psi)) < 0$, $\lim_{k \rightarrow \infty} e^{-\tau k\psi} = 0$. So the kernel of ξ is empty, meaning that it is invertible.

Further, we can show that $\phi(x_0, t)$, for this value of x_0 , is an *asymptotic* solution, in the sense that all other solutions converge to it in large positive time. There's a lovely way to show this. Just consider the behavior of $\chi(t) = \phi(x_0, t) - \phi(x_1, t)$, where x_1 is the initial condition corresponding to some other solution:

$$\chi'(t) = \psi(\phi(x_0, t) - \phi(x_1, t)) \quad (37)$$

We can *solve* this (note the tool we're employing: consider the derivative of a function we want to analyze, obtain a differential equation, solve it using existing techniques) to obtain $\chi'(t) = (x_0 - x_1)e^{t\psi}$ which we know from above converges to 0.

Next, we can prove results about equilibrium points based on information about DF . The idea is to use knowledge about the linearization of the system at the 0 equilibrium (obtained by taking the first order Taylor expansion, i.e. the derivative) to gain knowledge about the system as a whole. This turns out to be a very profitable approach. To see this, suppose $\phi(x, t) = 0$ is a solution (i.e. 0 is an equilibrium point). Further suppose that $\forall i : \text{Re}(\lambda_i(DF(0))) < 0$. That is, all real parts of the Jacobian's eigenvalues are negative. We can then prove asymptotic stability as follows:

Since $DF(0)$ has all real parts of its eigenvalues negative, $e^{tDF(0)}$ tends to 0 with large t , and so for some τ , $\|D_1\phi_F(x, \tau)\|_{op} < \frac{1}{2}$

Further, because ϕ is continuously differentiable in both argument, we can find an η such that $\forall y \in B_\eta : \sup_{t \in [0, \tau]} \|\phi(y, t)\| < \epsilon$. In other words, $\phi(y, t)$ stays in B_ϵ until time τ .

OK now we invoke the fact that any continuously differentiable f with an open domain is locally Lipschitz in a sufficiently small ball around any point, and note that $\frac{1}{2}$ is the Lipschitz constant for points in $B_\eta(0)$. This means that $\|\phi(x, \tau)\| \leq \frac{1}{2}\|x\|$ for $x \in B_\delta(0)$ where δ is a positive number smaller than η .

TODO: clarify the above

OK, now we've shown that if you start inside $B_\delta(0)$, the flow takes you closer. It's now just an inductive argument (omitted) which shows that 0 is asymptotically stable.

The next result is called the stable manifold theorem. The idea is to note that for linear ODEs, there exists a stable subspace relative to the equilibrium point 0, such that all points on this subspace converge to 0 under the differential flow. For non-linear ODEs, things are not so simple, but delightfully, there is a stable *submanifold* of the space with the same property that the flow converges to the equilibrium (and exponentially fast), which is tangential to the stable subspace at the origin.

We now investigate what this stable submanifold W_- looks like. The approach is to define $g(t) = F(\phi_F(x_0, t)) - \psi(\phi_F(x_0, t))$, so that our non-linear equation can be written as a perturbation from a linear approximation: $D_2\phi(x_0, t) = \psi(\phi(x_0, t)) + g(t)$.

This allows us to use Duhamel's principle, to obtain: $\phi_F(x_0, t) = e^{t\psi}x_0 + \int_{t_0}^t e^{(t-s)\psi}g(s)ds$. Recall that P_- and P_+ refer to the projections into the stable and unstable subspaces respectively, which together "span" the space. What we now show is that we can derive an integral equation for which the solution is also a solution to $\phi_F(x, t)$, and moreover which being a solution for is both a necessary and sufficient condition of being on the stable manifold.

Assume x_0 lies on the stable manifold. Then project $\phi_F(x_0, t) = e^{tA}x_0 + \int_{t_0}^t e^{(t-s)A}g(s)ds$ onto the stable subspace, setting $t_0 = 0$:

$$P_- \phi_F(x_0, t) = e^{tP_- \psi} P_- x_0 + \int_0^t e^{(t-s)P_- \psi} P_- g(s) ds \quad (38)$$

Similarly, we can project onto the unstable subspace, here taking $t_0 \rightarrow \infty$ and noting that the first term on the right of the equation will vanish, given our assumption that x_0 lies on the stable manifold (which consists of points destined for the origin):

$$P_+ \phi_F(x_0, t) = - \int_t^\infty e^{(t-s)P_+ \psi} P_+ g(s) ds \quad (39)$$

Summing these two equations gives our desired integral equation, namely:

$$\phi_F(x_0, t) = e^{tP_- \psi} P_- x_0 + \int_0^t e^{(t-s)P_- \psi} P_- g(s) ds - \int_t^\infty e^{(t-s)P_+ \psi} P_+ g(s) ds \quad (40)$$

Eyeballing this informally suggests that with $t \rightarrow \infty$, $\phi_F(x_0, t) \rightarrow 0$, if x_0 is sufficiently close in norm to 0, which is true (after some careful argumentation) and shows that solutions to the above integral equation lie on the stable manifold. Note that this is a *new* equation. That is, it is not simply a rejigging of $\phi_F(x_0, t) = \psi(\phi(x_0, t)) + g(t)$, since to get it, we **assumed** that the solution lay on the stable manifold. In other words, it is not the case that all solutions of $\phi_F(x_0, t)$ are solutions of the above integral equation. However, it is true that a solution of the above integral equation is a solution ϕ_F which preserves the stable manifold. Fortunately, we can construct solutions to this integral equation much in the same way as Picard iteration, i.e. by finding a fix point of the appropriate functional.

Moreover, for every point in the stable subspace $P_- x_0$, there's a corresponding version of the above integral equation and also a corresponding solution ϕ_F with $P_- x_0 + P_+ \phi(x_0, 0)$ lying on the stable manifold. So this determines the stable manifold: for every point in the stable subspace, solve the corresponding integral equation, and that gives you a point x which is on the stable manifold.

This is all very informal, because the maths starts getting to be a bit of a trudge at this point, and honestly, you get the general idea.

A final note is that the tangent space to the stable manifold at 0 is the stable subspace.

Lyapunov Functions and Gradient Systems:

Suppose that there is a smooth function L that is convex and attains a local minimum among points in some $B_r(p)$, for $r > 0$ and p an equilibrium point.

We proceed by first choosing a positive $\epsilon \leq r$ and noting that $L(p) < \inf_{x \in \partial B_\epsilon(p)} L(x)$, where ∂A is the boundary of a set A . We then use the continuity of L to establish that there must exist a positive $\delta < \epsilon$ such that $\sup_{x \in B_\delta(p)} L(x) < \inf_{x \in \partial B_\epsilon(p)} L(x)$.

By way of contradiction, suppose that for some point $x_0 \in B_\delta(p)$, $\exists t : \phi(x, t) \notin B_\epsilon(p)$. Then for $\tau = \inf\{t \geq 0 : \phi(x_0, t) \notin B_\epsilon(p)\}$, $\phi(x_0, \tau) \in \partial B_\epsilon(p)$ and for positive $t \leq \tau$, $\phi(x_0, t) \in B_\epsilon(p)$.

That means that $\inf_{x \in \partial B_\epsilon(p)} L(x) \leq L(\phi(x_0, \tau))$. But by the local convexity of L , we also know that $L(\phi(x_0, 0)) = x_0 \leq \sup_{x \in B_\delta(p)} L(x)$.

If $L(\phi(x_0, \tau))$ were less than or equal to $L(\phi(x_0, 0))$, then chaining together these inequalities would yield a contradiction, since we assumed that $\sup_{x \in B_\delta(p)} L(x) < \inf_{x \in \partial B_\epsilon(p)} L(x)$, and it would follow that p must be stable (since the path cannot leave B_ϵ).

A sufficient condition for $L(\phi(x_0, \tau)) \leq L(\phi(x_0, 0))$ is that $\langle \nabla Lx, F(x) \rangle \leq 0$ for $x \in B_r(p)$, since $\frac{d}{dt} L(\phi(x_0, t)) = \langle \nabla L\phi(x_0, t), F(\phi(x_0, t)) \rangle$. This condition (along with the local convexity), makes L a Lyapunov function.

So if L is a Lyapunov function for p , then we can then prove that p is stable. If, additionally, $\langle \nabla Lx, F(x) \rangle \neq 0$, p is asymptotically stable. To see this further point

An extremely useful corollary is that if $F = -\nabla V$, for some real-valued V , and V attains a minimum at p , then V is a Lyapunov function, since $\langle \nabla Vx, F(x) \rangle = \langle \nabla Vx, -\nabla V(x) \rangle = -\|V(x)\|^2 \leq 0$.