3. Eulerian and Hamiltonian Graphs

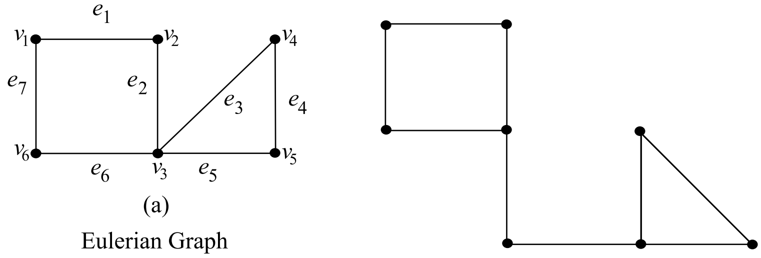
There are many games and puzzles which can be analysed by graph theoretic concepts. In fact, the two early discoveries which led to the existence of graphs arose from puzzles, namely, the Konigsberg Bridge Problem and Hamiltonian Game, and these puzzles also resulted in the special types of graphs, now called Eulerian graphs and Hamiltonian graphs. Due to the rich structure of these graphs, they find wide use both in research and application.

3.1 Euler Graphs

A closed walk in a graph G containing all the edges of G is called an *Euler line* in G. A graph containing an Euler line is called an *Euler graph*.

We know that a walk is always connected. Since the Euler line (which is a walk) contains all the edges of the graph, an Euler graph is connected except for any isolated vertices the graph may contain. As isolated vertices do not contribute anything to the understanding of an Euler graph, it is assumed now onwards that Euler graphs do not have any isolated vertices and are thus connected.

Example Consider the graph shown in Figure 3.1. Clearly, $v_1\ e_1\ v_2\ e_2\ v_3\ e_3\ v_4\ e_4\ v_5\ e_5\ v_3\ v_6\ e_7\ v_1$ in (a) is an Euler line, whereas the graph shown in (b) is non-Eulerian.



(a)
Eulerian Graph

(b)
Non-Eulerian Graph

Fig. 3.1

The following theorem due to Euler [74] characterises Eulerian graphs. Euler proved the necessity part and the sufficiency part was proved by Hierholzer [115].

Theorem 3.1 (Euler) A connected graph G is an Euler graph if and only if all vertices of G are of even degree.

Proof

Necessity Let $G(V, E)$ be an Euler graph. Thus G contains an Euler line Z, which is a closed walk. Let this walk start and end at the vertex $u \in V$. Since each visit of Z to an intermediate vertex v of Z contributes two to the degree of v and since Z traverses each edge exactly once, $d(v)$ is even for every such vertex. Each intermediate visit to u contributes two to the degree of u, and also the initial and final edges of Z contribute one each to the degree of u. So the degree $d(u)$ of u is also even.

Sufficiency Let G be a connected graph and let degree of each vertex of G be even. Assume G is not Eulerian and let G contain least number of edges. Since $\delta \geq 2$, G has a cycle. Let Z be a closed walk in G of maximum length. Clearly, $G - E(Z)$ is an even degree graph. Let C_1 be one of the components of $G - E(Z)$. As C_1 has less number of edges than G, it is Eulerian and has a vertex v in common with Z. Let Z' be an Euler line in C_1. Then $Z' \cup Z$ is closed in G, starting and ending at v. Since it is longer than Z, the choice of Z is contradicted. Hence G is Eulerian.

Second proof for sufficiency Assume that all vertices of G are of even degree. We construct a walk starting at an arbitrary vertex v and going through the edges of G such that no edge of G is traced more than once. The tracing is continued as far as possible. Since every vertex is of even degree, we exit from the vertex we enter and the tracing clearly cannot stop at any vertex but v. As v is also of even degree, we reach v when the tracing comes to an end. If this closed walk Z we just traced includes all the edges of G, then G is an Euler graph. If not, we remove from G all the edges in Z and obtain a subgraph Z' of G formed by the remaining edges. Since both G and Z have all their vertices of even degree, the degrees of the vertices of Z' are also even. Also, Z' touches Z at least at one vertex say u, because G is connected. Starting from u, we again construct a new walk in Z'. As all the vertices of Z' are of even degree, therefore this walk in Z' terminates at vertex u. This walk in Z'combined with Z forms a new walk, which starts and ends at the vertex v and has more edges than Z. This process is repeated till we obtain a closed walk that traces all the edges of G. Hence G is an Euler graph (Fig. 3.2) ❑

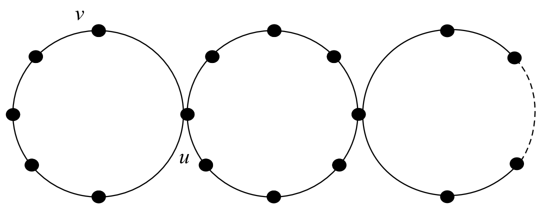
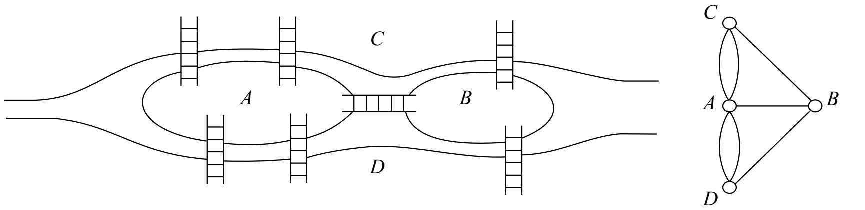


Fig. 3.2

3.2 Konigsberg Bridge Problem

Two islands A and B formed by the Pregal river (now Pregolya) in Konigsberg (then the capital of east Prussia, but now renamed Kaliningrad and in west Soviet Russia) were connected to each other and to the banks C and D with seven bridges. The problem is to start at any of the four land areas, A, B, C, or D, walk over each of the seven bridges exactly once and return to the starting point.

Euler modeled the problem representing the four land areas by four vertices, and the seven bridges by seven edges joining these vertices. This is illustrated in Figure 3.3.

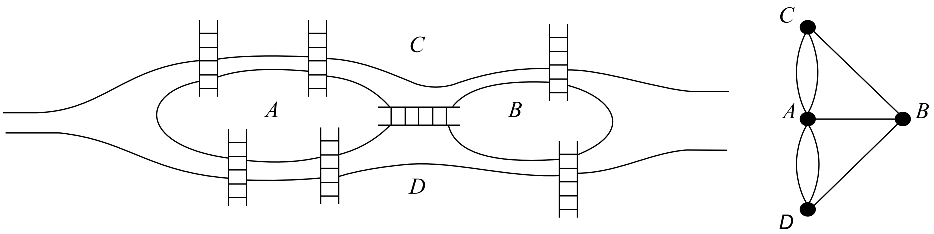


Konigsberg Bridges Underlying Graph

Fig. 3.3

We see from the graph G of the Konigsberg bridges that not all its vertices are of even degree. Thus G is not an Euler graph, and implies that there is no closed walk in G containing all the edges of G. Hence it is not possible to walk over each of the seven bridges exactly once and return to the starting point.

Note Two additional bridges have been built since Euler's day. The first has been built between land areas C and D and the second between the land areas A and B. Now in the graph of Konigsberg bridge problem with nine bridges, every vertex is of even degree and the graph is thus Eulerian. Hence it is now possible to walk over each of the nine bridges exactly once and return to the starting point (Fig 3.4).



Konigsberg Bridges Underlying Graph

Fig. 3.4

The following characterisation of Eulerian graphs is due to Veblen [254].

Theorem 3.2 A connected graph G is Eulerian if and only if its edge set can be decomposed into cycles.

Proof Let $G(V, E)$ be a connected graph and let G be decomposed into cycles. If k of these cycles are incident at a particular vertex v, then $d(v) = 2k$. Therefore the degree of every vertex of G is even and hence G is Eulerian.

Conversely, let G be Eulerian. We show G can be decomposed into cycles. To prove this, we use induction on the number of edges.

Since $d(v) \geq 2$ for each $v \in V$, G has a cycle C. Then $G - E(C)$ is possibly a disconnected graph, each of whose components $C_1, C_2, \ldots, C_k$ is an even degree graph and hence Eulerian. By the induction hypothesis, each C_i is a disjoint union of cycles. These together with C provide a partition of $E(G)$ into cycles. ❑

The following result is due to Toida [244].

Theorem 3.3 If W is a walk from vertex u to vertex v, then W contains an odd number of $u - v$ paths.

Proof Let W be a walk which we consider as a graph in itself, and not as a subgraph of some other graph. Let u and v be initial and final vertices of the walk W. Clearly, $d(u|W)$ and $d(v|W)$ are odd, and $d(w|W)$ is even, for every $w \in V(W) - \{u, v\}$. We count the number of distinct $u - v$ walks in W. These walks are the subgraphs of W.

When we take a $u - v$ walk by successively selecting the edges $e_1, e_2, \ldots, e_s$, initial vertex of e_1 being u and terminal vertex of e_s being v, for each edge there are an odd number of choices. The total number of such edges is the product of these odd numbers and is therefore odd. Now from these walks, we find the $u - v$ paths. If a $u - v$ walk W_1 is not a path, then it contains one or more cycles. The traversal of these cycles in the two possible alternative directions (clockwise and anticlockwise) produces in all an even number of walks, all with the same edge set as W_1. Omitting these even number of walks which are not paths from the total odd collection of $u - v$ walks, gives an odd number of $u - v$ paths. ❑

Toida [244] proved the necessity part and McKee [157] the sufficiency part of the next characterisation. The second proof of this result can be found in Fleischner [79], [80].

Theorem 3.4 A connected graph is Eulerian if and only if each of its edges lies on an odd number of cycles.

Proof

Necessity Let G be a connected Eulerian graph and let $e = uv$ be any edge of G. Then $G - e$ is a $u - v$ walk W, and so $G - e = W$ contains an odd number of $u - v$ paths. Thus each of the odd number of $u - v$ paths in W together with e gives a cycle in G containing e and these are the only such cycles. Therefore there are an odd number of cycles in G containing e.

Sufficiency Let G be a connected graph so that each of its edges lies on an odd number of cycles. Let v be any vertex of G and $E_v = \{e_1, \ldots, e_d\}$ be the set of edges of G incident on v, then $|E_v| = d(v) = d$. For each i, $1 \le i \le d$, let k_i be the number of cycles of G containing e_i. By hypothesis, each k_i is odd. Let $c(v)$ be the number of cycles of G containing v. Then clearly $c(v) = \frac{1}{2} \sum_{i=1}^{d} k_i$ implying that $2c(v) = \sum_{i=1}^{d} k_i$. Since $2c(v)$ is even and each k_i is odd, d is even. Hence G is Eulerian. ❑

Corollary 3.1 The number of edge–disjoint paths between any two vertices of an Euler graph is even.

A consequence of Theorem 3.4 is the result of Bondy and Halberstam [37], which gives yet another characterisation of Eulerian graphs.

Corollary 3.2 A graph is Eulerian if and only if it has an odd number of cycle decompositions.

Proof In one direction, the proof is trivial. If G has an odd number of cycle decompositions, then it has at least one, and hence G is Eulerian.

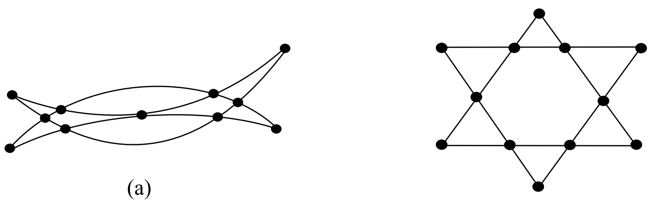
Conversely, assume that G is Eulerian. Let $e \in E(G)$ and let $C_1, \ldots, C_r$ be the cycles containing e. By Theorem 3.4, r is odd. We proceed by induction on $m = |E(G)|$, with G being Eulerian.

If G is just a cycle, then the result is true. Now assume that G is not a cycle. This means that for each i, $1 \le i \le r$, by the induction assumption, $G_i = G - E(C_i)$ has an odd number, say s_i, of cycle decompositions. (If G_i is disconnected, apply the induction assumption to each of the nontrivial components of G_i). The union of each of these cycle decompositions of G_i and C_i yields a cycle decomposition of G. Hence the number of cycle decompositions of G containing C_i is s_i, $1 \le i \le r$. Let $s(G)$ denote the number of cycle decompositions of G. Then

$$s(G) \equiv \sum_{i=1}^{r} s_i \equiv r (\bmod 2) \qquad (\text{since } s_i \equiv 1 (\bmod 2))$$

$$\equiv 1 (\bmod 2).$$ ❑

Two examples of Euler graphs are shown in Figure 3.5.



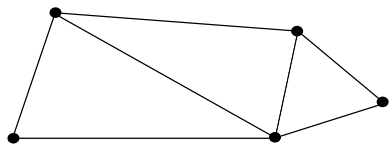
(a) (b)

Two Euler graphs

Fig. 3.5

3.3 Unicursal Graphs

An open walk that includes (or traces) all edges of a graph without retracing any edge is called a unicursal line or open Euler line. A connected graph that has a unicursal line is called a unicursal graph. Figure 3.6 shows a unicursal graph.



Unicursal graph

Fig. 3.6

Clearly by adding an edge between the initial and final vertices of a unicursal line, we get an Euler line.

The following characterisation of unicursal graphs can be easily derived from Theorem 3.1.

Theorem 3.5 A connected graph is unicursal if and only if it has exactly two vertices of odd degree.

Proof Let G be a connected graph and let G be unicursal. Then G has a unicursal line, say from u to v, where u and v are vertices of G. Join u and v to a new vertex w of G to get a graph H. Then H has an Euler line and therefore each vertex of H is of even degree. Now, by deleting the vertex w, the degree of vertices u and v each get reduced by one, so that u and v are of odd degree.

Conversely, let u and v be the only vertices of G with odd degree. Join u and v to a new vertex w to get the graph H. So every vertex of H is of even degree and thus H is Eulerian. Therefore, $G = H - w$ has a $u - v$ unicursal line so that G is unicursal. ❑

The following result is the generalisation of Theorem 3.5.

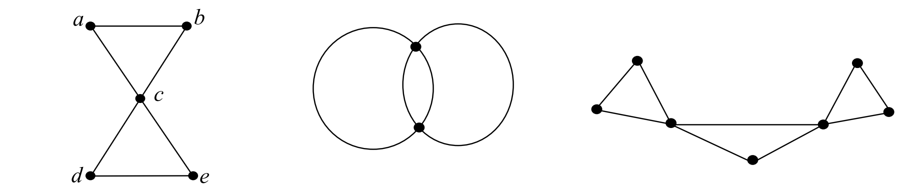
Theorem 3.6 In a connected graph G with exactly $2k$ odd vertices, there exists k edge disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph.

Proof Let G be a connected graph with exactly $2k$ odd vertices. Let these odd vertices be named $v_1, v_2, \ldots, v_k$; $w_1, w_2, \ldots, w_k$ in any arbitrary order. Add k edges to G between the vertex pairs (v_1, w_1), (v_2, w_2), $\ldots$, (v_k, w_k) to form a new graph H, so that every vertex of H is of even degree. Therefore H contains an Euler line Z.

Now, if we remove from Z the k edges we just added (no two of these edges are incident on the same vertex), then Z is divided into k walks, each of which is a unicursal line. The first removal gives a single unicursal line, the second removal divides that into two unicursal lines, and each successive removal divides a unicursal line into two unicursal lines, until there are k of them. Hence the result. ❑

3.4 Arbitrarily Traceable Graphs

An Eulerian graph G is said to be arbitrarily traceable (or randomly Eulerian) from a vertex v if every walk with initial vertex v can be extended to an Euler line of G. A graph is said to be arbitrarily traceable if it is arbitrarily traceable from every vertex (Fig. 3.7).



(a) Arbitrarily traceable graph from c

(b) Arbitrarily traceable graph from all vertices

(c) Euler graph, not arbitrarily traceable

Fig. 3.7

The following characterisation of arbitrarily traceable graphs is due to Ore [174]. Such graphs were also characterised by Chartrand and White [56] .

Theorem 3.7 An Eulerian graph G is arbitrarily traceable from a vertex v if and only if every cycle of G passes through v.

Proof

Necessity Let the Eulerian graph G be arbitrarily traceable from a vertex v. Assume there is a cycle C not passing through v. Let $H = G - E(C)$. Then every vertex of H has an even degree and the component of H containing v is Eulerian. This component of H can be traversed as an Euler line Z, starting and ending with v and contains all those edges of G which are incident at v. Clearly, this $v-v$ walk cannot be extended to contain the edges of C also, contradicting that G contains v. Thus every cycle in G contains v.

Sufficiency Let every cycle of the Eulerian graph G pass through the vertex v of G. We show that G is arbitrarily traceable from v. Assume, on the contrary, that G is not arbitrarily traceable from v. Then there is a $v-v$ closed walk W of G containing all the edges of G incident with v and yet not containing all the edges of G. Let one such edge be incident at a vertex u on W. So every vertex of $H = G - E(W)$ is of even degree and v is an isolated vertex of H and u is not. The component of H containing u is therefore Eulerian subgraph of G not passing through v, contradicting the assumption. Hence the result follows. ❑

Corollary 3.3 Cycles are the only arbitrarily traceable graphs.

3.5 Sub-Eulerian Graphs

A graph G is said to be *sub-Eulerian* if it is a spanning subgraph of some Eulerian graph.

The following characterisation of sub-Eulerian graphs is due to Boesch, Suffel and Tindell [28].

Theorem 3.8 A connected graph G is sub-Eulerian if and only if G is not spanned by a complete bipartite graph.

Proof

Necessity We prove that no spanning supergraph H of an odd complete bipartite graph G is Eulerian. Let $V_1 \cup V_2$ be the bipartition of the vertex set of G. Since degree of each vertex of G is odd, and G is complete bipartite, therefore $|V_1|$ and $|V_2|$ are odd. If H_1 is the induced subgraph of H on V_1, then at least one vertex, say v, of V_1 has even degree in H_1, since $|V_1|$ is odd. But then $d(v|H) = d(v|H) + |V_2|$, which is odd. Therefore H is not Eulerian.

Sufficiency Refer Boesch et. al., [28]. ❑

Super-Eulerian graphs

A non-Eulerian graph G is said to be *super-Eulerian* if it has a spanning Eulerian subgraph.

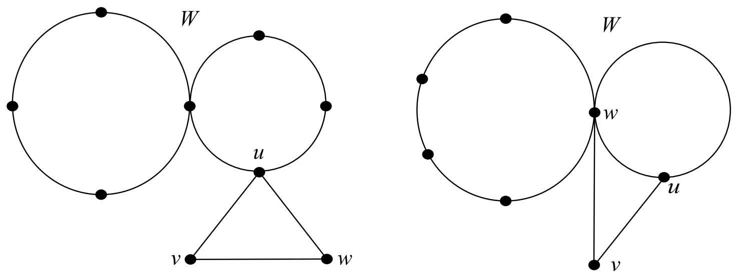
The following sufficient conditions for super-Eulerian graphs are due to Lesniak-Foster and Williams [148].

Theorem 3.9 If a graph G is such that $n \geq 6$, $\delta \geq 2$ and $d(u) + d(v) \geq n - 1$, for every pair of non-adjacent vertices u and v, then G is super-Eulerian.

The following result is due to Balakrishnan and Paulraja [12].

Theorem 3.10 If G is any connected graph and if each edge of G belongs to a triangle in G, then G has a spanning Eulerian subgraph.

Proof Since G has a triangle, G has a closed walk. Let W be the longest closed walk in G. Then W must be a spanning Eulerian subgraph of G. If not, there exists a vertex $v \notin W$ and v is adjacent to a vertex u of W. By hypothesis, uv belongs to a triangle, say uvw. If none of the edges of this triangle is in W, then $W \cup \{uv, vw, wu\}$ yields a closed walk longer than W (Fig. 3.8). If $uw \in W$, then $(W - uw) \cup \{uv, vw\}$ would be a closed walk longer than W. This contradiction proves that W is a spanning closed walk in G. ❑

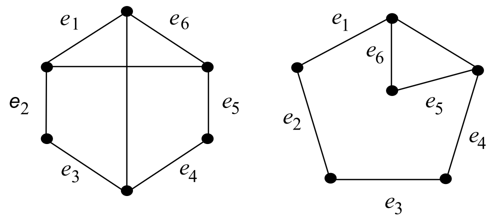


(a) WU {*uv*, *vw*, *wu*} (b) (W – *uw*) U {*uv*, *vw*}

Fig. 3.8

3.6 Hamiltonian Graphs

A cycle passing through all the vertices of a graph is called a *Hamiltonian cycle*. A graph containing a Hamiltonian cycle is called a *Hamiltonian graph*. A path passing through all the vertices of a graph is called a *Hamiltonian path* and a graph containing a Hamiltonian path is said to be *traceable*. Examples of Hamiltonian graphs are given in Figure 3.9.



Hamiltonian Graphs

Fig. 3.9

If the last edge of a Hamiltonian cycle is dropped, we get a Hamiltonian path. However, a non-Hamiltonian graph can have a Hamiltonian path, that is, Hamiltonian paths cannot always be used to form Hamiltonian cycles. For example, in Figure 3.10, G_1 has no Hamiltonian path, and so no Hamiltonian cycle; G_2 has the Hamiltonian path $v_1v_2v_3v_4$, but has no Hamiltonian cycle, while G_3 has the Hamiltonian cycle $v_1v_2v_3v_4v_1$.

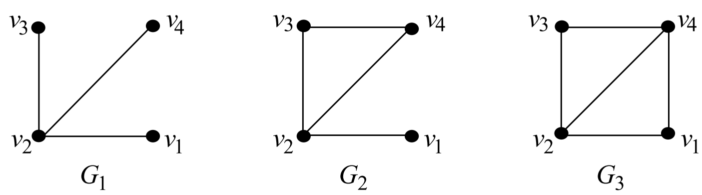
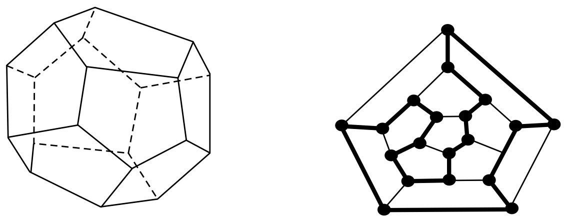


Fig. 3.10

Hamiltonian graphs are named after Sir William Hamilton, an Irish Mathematician (1805−1865), who invented a puzzle, called the Icosian game, which he sold for 25 guineas to a game manufacturer in Dublin. The puzzle involved a dodecahedron on which each of the 20 vertices was labelled by the name of some capital city in the world. The aim of the game was to construct, using the edges of the dodecahedron a closed walk of all the cities which traversed each city exactly once, beginning and ending at the same city. In other words, one had essentially to form a Hamiltonian cycle in the graph corresponding to the dodecahedron. Figure 3.11 shows such a cycle.



(a) (b)

Dedecahedron and its graph shown with the Hamiltonian cycle

Fig. 3.11

Clearly, the n-cycle C_n with n distinct vertices (and n edges) is Hamiltonian. Now, given any Hamiltonian graph G, the supergraph G' (obtained by adding in new edges between non-adjacent vertices of G) is also Hamiltonian. This is because any Hamiltonian cycle in G is also a Hamiltonian cycle of G'. For instance, K_n is a supergraph of an n-cycle and so K_n is Hamiltonian.

A multigraph or general graph is Hamiltonian if and only if its underlying graph is Hamiltonian, because if G is Hamiltonian, then any Hamiltonian cycle in G remains a Hamiltonian cycle in the underlying graph of G. Conversely, if the underlying graph of a graph G is Hamiltonian, then G is also Hamiltonian.

Let G be a graph with n vertices. Clearly, G is a subgraph of the complete graph K_n. From G, we construct step by step supergraphs of G to get K_n, by adding an edge at each step between two vertices that are not already adjacent (Fig. 3.12).

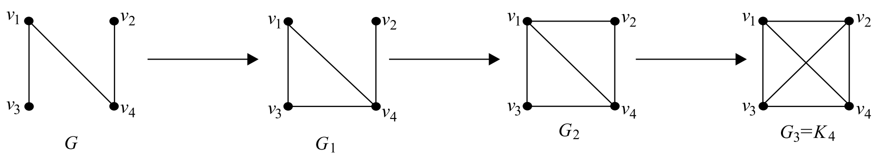


Fig. 3.12

Now, let us start with a graph G which is not Hamiltonian. Since the final outcome of the procedure is the Hamiltonian graph K_n, we change from a non-Hamiltonian graph to a Hamiltonian graph at some stage of the procedure. For example, the non-Hamiltonian graph G_1 above is followed by the Hamiltonian graph G_2. Since supergraphs of Hamiltonian graphs are Hamiltonian, once a Hamiltonian graph is reached in the procedure, all the subsequent supergraphs are Hamiltonian.

Definition: A simple graph G is called *maximal non-Hamiltonian* if it is not Hamiltonian and the addition of an edge between any two non-adjacent vertices of it forms a Hamiltonian graph. For example, G_1 above is maximal non-Hamiltonian. Figure 3.13 shows a maximal non-Hamiltonian graph.

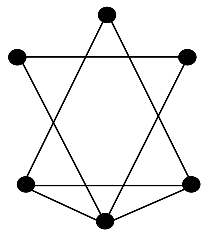


Fig. 3.13

It follows from the above procedure that any non-Hamiltonian graph with n-vertices is a subgraph of a maximal non-Hamiltonian graph with n vertices.

The above procedure is used to prove the following sufficient conditions due to Dirac [68].

Theorem 3.11 (Dirac) If G is a graph with n vertices, where $n \geq 3$ and $d(v) \geq n/2$, for every vertex v of G, then G is Hamiltonian.

Proof Assume that the result is not true. Then for some value $n \geq 3$, there is a non-Hamiltonian graph H in which $d(v) \geq n/2$, for every vertex of H. In any spanning super graph K (i.e., with the same vertex set) of H, $d(v) \geq n/2$ for every vertex of K, since any proper supergraph of this form is obtained by adding more edges. Thus there is a maximal non-Hamiltonian graph G with n vertices and $d(v) \geq n/2$ for every v in G. Using this G, we obtain a contradiction.

Clearly, $G \neq K_n$, as K_n is Hamiltonian. Therefore there are non-adjacent vertices u and v in G. Let $G+uv$ be the supergraph of G by adding an edge between u and v. Since G is maximal non-Hamiltonian, $G+uv$ is Hamiltonian. Also, if C is a Hamiltonian cycle of $G+uv$, then C contains the edge uv, since otherwise C is a Hamiltonian cycle of G, which is not possible. Let this Hamiltonian cycle C be $u = v_1, v_2, \ldots, v_n = v, u$.

Now, let $S = \{v_i \in C :$ there is an edge from u to v_{i+1} in $G\}$ and $T = \{v_j \in C :$ there is an edge from v to v_j in $G\}$.

Then $v_n \notin T$, since otherwise there is an edge from v to $v_n = v$, that is a loop, which is impossible.

Also $v_n \notin S$, (taking v_{n+1} as v_1), since otherwise we again get a loop from u to $v_1 = u$. Therefore, $v_n \in S \cup T$ (Fig. 3.14).

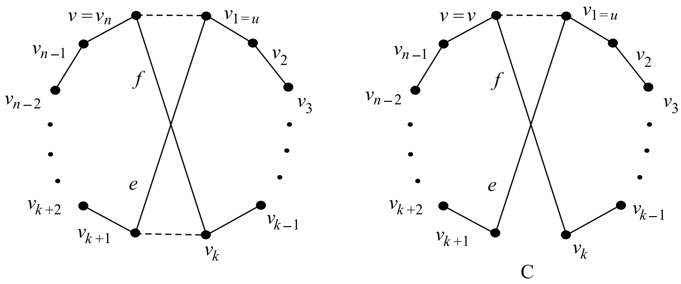


Fig. 3.14

Let $|S|$, $|T|$ and $|S \cup T|$ be the number of elements in S, T and $S \cup T$ respectively. So $|S \cup T| < n$. Also, for every edge incident with u, there corresponds one vertex v_i in S. Therefore, $|S| = d(u)$. Similarly, $|T| = d(v)$.

Now, if v_k is a vertex belonging to both S and T, there is an edge e joining u to v_{k+1} and an edge f joining v to v_k. This implies that $C' = v_1,\ v_{k+1},\ v_{k+2},\ \ldots,\ v_n,\ v_k,\ v_{k-1},\ \ldots,\ v_2,\ v_1$ is a Hamiltonian cycle in G, which is a contradiction as G is non-Hamiltonian. This shows that there is no vertex v_k in $S \cap T$, so that $S \cap T = \Phi$.

Thus $|S \cup T| = |S| + |T| - |S \cap T|$ gives $|S| + |T| = |S \cup T|$, so that $d(u) + d(v) < n$. This is a contradiction, because $d(u) \geq n/2$ for all u in G, and so $d(u) + d(v) \geq n/2 + n/2$ giving $d(u) + d(v) \geq n$. Hence the theorem follows. ❑

The following result is due to Ore [176].

Theorem 3.12 (Ore) Let G be a graph with n vertices and let u and v be non-adjacent vertices in G such that $d(u) + d(v) \geq n$. Let $G + uv$ denote the super graph of G obtained by joining u and v by an edge. Then G is Hamiltonian if and only if $G + uv$ is Hamiltonian.

Proof Let G be a graph with n vertices and suppose u and v are non-adjacent vertices in G such that $d(u) + d(v) \geq n$. Let $G + uv$ be the super graph of G obtained by adding the edge uv. Let G be Hamiltonian. Then obviously $G + uv$ is Hamiltonian. Conversely, let $G + uv$ be Hamiltonian. We have to show that G is Hamiltonian. Then, as in Theorem 3.11, we get $d(u) + d(v) < n$, which contradicts the hypothesis that $d(u) + d(v) \geq n$. Hence G is Hamiltonian. ❑

The following is the proof of Bondy [35] of Theorem 3.12, and this proof bears a close resemblance to the proof of Dirac's theorem given by Newman [170], but is more direct.

Proof (Bondy [35]) Consider the complete graph K on the vertex set of G in which the edges of G are coloured blue and the remaining edges of K are coloured red. Let C be a

Hamiltonian cycle of K with as many blue edges as possible. We show that every edge of C is blue, in other words, that C is Hamiltonian cycle of G.

Suppose to the contrary, C has a red edge uu^- (where u^- is the successor of u on C). Consider the set S of vertices joined to u by blue edges (that is, the set of neighbours of u in G). The successor u^- of u on C must be joined by a blue edge to some vertex v^- of S^-, because if u^- is adjacent in C only to vertices $V-(S^-U\{u^-\})$, $d_G(u)+d_G(u^-)=|N_G(u)|+|N_G(u^-)| \leq |S|+(|V|-|S^-|-1)=|V(G)|-1$, contradicting the hypothesis that $d_G(u)+d_G(u^-) \geq |V(G)|$, u and u^- being non-adjacent in G. But now the cycle C obtained from C by exchanging the edges uu^- and vv^- has more blue edges than C, which is a contradiction. ❑

Definition: Let G be a graph with n vertices. If there are two non-adjacent vertices u_1 and v_1 in G such that $d(u_1)+d(v_1) \geq n$, join u_1 and v_1 by an edge to form the super graph G_1. Now, if there are two non-adjacent vertices u_2 and v_2 in G_1 such that $d(u_2)+d(v_2) \geq n$, join u_2 and v_2 by an edge to form supergraph G_2. Continue in this way, recursively joining pairs of non-adjacent vertices whose degree sum is at least n until no such pair remains. The final supergraph thus obtained is called the *closure* of G and is denoted by $c(G)$.

The example in Figure 3.15 illustrates the closure operation.

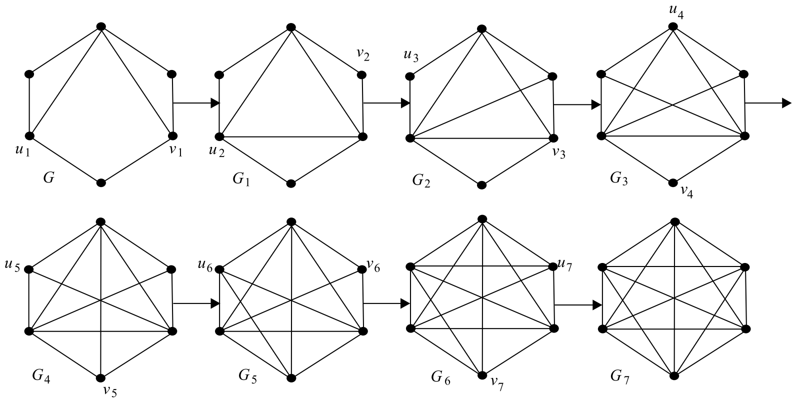


Fig. 3.15

We observe in this example that there are different choices of pairs of non-adjacent vertices u and v with $d(u)+d(v) \geq n$. Therefore the closure procedure can be carried out in several different ways and each different way gives the same result.

In the graph shown in Figure 3.16, $n=7$ and $d(u)+d(v)<7$, for any pair u, v of adjacent vertices. Therefore, $c(G)=G$.

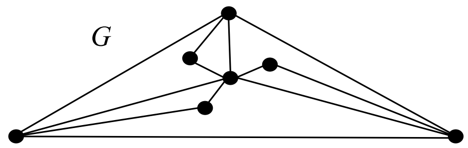


Fig. 3.16

The importance of $c(G)$ is given in the following result due to Bondy and Chvatal [36].

Theorem 3.13 A graph G is Hamiltonian if and only if its closure $c(G)$ is Hamiltonian.

Proof Let $c(G)$ be the closure of the graph G. Since $c(G)$ is a supergraph of G, therefore, if G is Hamiltonian, then $c(G)$ is also Hamiltonian.

Conversely, let $c(G)$ be Hamiltonian. Let $G, G_1, G_2, \ldots, G_{k-1}, G_k = c(G)$ be the sequence of graphs obtained by performing the closure procedure on G. Since $c(G) = G_k$ is obtained from G_{k-1} by setting $G_k = G_{k-1} + uv$, where u, v is a pair of non adjacent vertices in G_{k-1} with $d(u) + d(v) \geq n$, therefore it follows that G_{k-1} is Hamiltonian. Similarly G_{k-2}, so G_{k-3}, ..., G_1 and thus G is Hamiltonian. ❑

Corollary 3.4 Let G be a graph with n vertices with $n \geq 3$. If $c(G)$ is complete, then G is Hamiltonian.

There can be more than one Hamiltonian cycle in a given graph, but the interest lies in the edge-disjoint Hamiltonian cycles. The following result gives the number of edge-disjoint Hamiltonian cycles in a complete graph with odd number of vertices.

Theorem 3.14 In a complete graph with n vertices there are $(n-1)/2$ edge-disjoint Hamiltonian cycles, if n is an odd number, $n \geq 3$.

Proof A complete graph G of n vertices has $n(n-1)/2$ edges and a Hamiltonian cycle in G contains n edges. Therefore the number of edge-disjoint Hamiltonian cycles in G cannot exceed $(n-1)/2$. When n is odd, we show there are $(n-1)/2$ edge-disjoint Hamiltonian cycles.

The subgraph of a complete graph with n vertices shown in Figure 3.17 is a Hamiltonian cycle. Keeping the vertices fixed on a circle, rotate the polygonal pattern clockwise by $\frac{360}{n-1}, 2.\frac{360}{n-1}, \ldots, \frac{n-3}{2}.\frac{360}{n-1}$ degrees. We see that each rotation produces a Hamiltonian cycle that has no edge in common with any of the previous ones. Therefore, there are $(n-3)/2$ new Hamiltonian cycles, all disjoint from the one in Figure 3.17, and also edge-disjoint among themselves. Thus there are $(n-1)/2$ edge disjoint Hamiltonian cycles. ❑

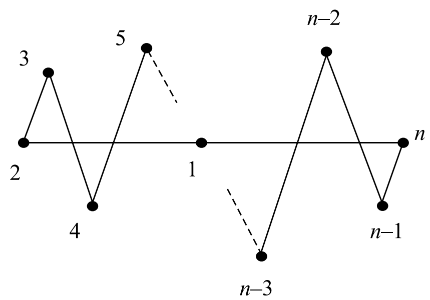


Fig. 3.17

The next result involving degrees give the sufficient conditions for a graph to be Hamiltonian.

Theorem 3.15 Let $D = [d_i]_1^n$ be a degree sequence of a graph $G = (V, E)$, $d_1 \le d_2 \le \ldots \le d_n$. Each of the following gives the sufficient conditions for G to be Hamiltonian.

A. $1 \le k \le n \Rightarrow d_k \ge \dfrac{n}{2}$ (Dirac [68])

B. $uv \notin E \Rightarrow d(u) + d(v) \ge n$ (Ore [176])

C. $1 \le k \le \frac{n}{2} \Rightarrow d_k > k$ (Posa [210]).

D. $j < k$, $d_j \le j$ and $d_k \le k - 1 \Rightarrow d_j + d_k \ge n$ (Bondy [33])

E. $d_k \le k < \frac{n}{2} \Rightarrow d_{n-k} \ge n - k$ (Chvatal [59])

F. For every i and j with $1 \le i \le n$, $1 \le j \le n$, $i + j \ge n$, $v_i v_j \notin E$, $d(v_i) \le i$ and $d(v_j) \le j - 1 \Rightarrow d(v_i) + d(v_j) \ge n$ (Las Vergnas [256].

G. $c(G)$ is complete (Bondy and Chvatal [36]).

Proof We first prove that

$$A \overset{(i)}{\Rightarrow} B \overset{(ii)}{\Rightarrow} C \overset{(iii)}{\Rightarrow} D \overset{(iv)}{\Rightarrow} E \overset{(v)}{\Rightarrow} F \overset{(vi)}{\Rightarrow} G.$$

i. This can be easily established.

ii. Assume that (C) is not true, so that there exists a k with $1 \le k < \frac{n}{2}$ and $d_k \le k$. Then the induced subgraph on the vertices $v_1, v_2, \ldots, v_k$ is a complete graph. For, if there are vertices i and j with $1 \le i < j \le k$ and $v_i v_j \notin E$, then $d_i + d_j \le 2d_k < n$, contradicting (B). Since $d_k \le k$, each v_i, $1 \le i \le k$, is adjacent to at most one v_j, $k + 1 \le j \le n$. Also, $n - k > k$, because $k < \frac{n}{2}$. Therefore there is a vertex v_j, $k + 1 \le j \le n$ not adjacent to any of the vertices $v_1, v_2, \ldots, v_k$. For this v_j, we have $d_j \le n - k - 1$. But then $d_j + d_k \le (n - k - 1) + k = n - 1$. Thus there is a $v_j v_k \notin E$ with $d_j + d_k \le n - 1$, contradicting (B). Hence proving (ii).

iii. Assume that (D) is not true, so that there exist j and k with $j < k$, $d_j \le j$, $d_k \le k - 1$ and $d_j + d_k < n$. This gives $i = d_j < \frac{n}{2}$. But then $d_j \le j$ gives $d_{d_j} \le d_j$, since the sequence is non-decreasing. Therefore, $d_i \le d_j = i$. Thus there is an i, $1 \le i \le \frac{n}{2}$ with $d_i \le i$, contradicting (C). This proves (iii).

iv. If (E) is not true, there is a k with $d_k \le k < \frac{n}{2}$ and $d_{n-k} \le n - k - 1$. Then $d_k + d_{n-k} \le n - 1$. Setting $n - k = j$, we have $k < j$, $d_k \le k$, $d_j \le j - 1$ and $d_j + d_k \le n - 1$. This contradicts (D) and so (iv) is proved.

v. Assume that (F) is not true, so that there is a pair of vertices v_i and v_j, $i < j$ with $v_i v_j \notin E$ and violating (F). Choose i to be the least such possible integer. Then by minimality of i, $d_{i-1} > i - 1$. Thus $d_i \ge d_{i-1} \ge i$ and since $d_i \le i$, we obtain $d_i = i$. If

$i \geq \frac{n}{2}$, we get $d_i + d_j \geq 2d_i \geq n$, contradicting the violation of (F). Therefore, $i < \frac{n}{2}$. Thus there is an i, $1 \leq i < \frac{n}{2}$ with $d_i = i$. Now, if (E) is satisfied, we have $d_{n-i} \geq n - i$ and since $j \geq n - i$, we obtain $d_j \geq d_{n-i} \geq n - i$. By minimality of i, $d_i = i$ and we have $d_j + d_i \geq (n-i) + i = n$, again contradicting the violation of (F). Thus negation of (F) implies negation of (E) and (V) is established.

vi. Assume that $c(G) = H$ is not complete. Let v_i and v_j be non-adjacent vertices in H such that (a) j is as large as possible and (b) i is as large as possible subject to (a). Then $i < j$, and since H is the closure of G, therefore

$$d(v_i|H) + d(v_j|H) \leq n - 1, \tag{3.15.1}$$

$$d(v_i|G) + d(v_j|G) \leq n - 1.$$

By the choice of j, v_i is adjacent in H to all v_k with $k > j$, so that

$$d(v_i|H) \geq n - j. \tag{3.15.2}$$

Again, by the choice of i, v_j is adjacent in H to all v_k with $k > i$, $k \neq j$, so that

$$d(v_j|H) \geq n - i - 1. \tag{3.15.3}$$

From (3.15.1) and (3.15.2), we have

$$d(v_j|G) \leq d(v_j|H) \leq (n-1) - (n-j) = j - 1.$$

From (3.15.1) and (3.15.3), we have

$$d(v_j|G) \leq d(v_i|H) \leq (n-1) - (n-i-1) = i.$$

From (3.15.2) and (3.15.3), we have

$$i + j \geq (2n-1) - d(v_i|H) - d(v_j|H) \geq n. \qquad \text{(using (1))}$$

Therefore i and j contradict the given conditions. Thus $H = c(G)$ is complete. This proves (vi).

By Theorem 3.13 it follows that if (G) holds, then G is Hamiltonian. ❑

The next result is due to Nash-Williams [168].

Theorem 3.16 (Nash-Williams) Every k-regular graph on $2k+1$ vertices is Hamiltonian.

Proof Let G be a k-regular graph on $2k+1$ vertices. Add a new vertex w and join it by an edge to each vertex of G. The resulting graph H on $2k+2$ vertices has $\delta = k+1$. Thus by Theorem 3.15 (A), H is Hamiltonian. Removing w from H, we get a Hamiltonian path, say $v_0 v_1 \ldots v_{2k}$.

Assume that G is not Hamiltonian, so that (a) if $v_0v_i \in E$, then $v_{i-1}v_{2k} \notin E$, (b) if $v_0v_i \notin E$, then $v_{i-1}v_{2k} \in E$, since $d(v_0) = d(v_{2k}) = k$.

The following cases arise.

Case (i) v_0 is adjacent to $v_1, v_2, \ldots, v_k$, and v_{2k} is adjacent to $v_k, v_{k+1}, \ldots, v_{2k-1}$. Then there is an i with $1 \le i \le k$ such that v_i is not adjacent to some v_j for $0 \le j \le k (j \ne i)$. But $d(v_i) = k$. So v_i is adjacent to v_j for some j with $k+1 \le j \le 2k-1$. Then the cycle C given by $v_iv_{i-1} \ldots v_0v_{i+1} \ldots v_{j-1}v_{2k}v_{2k+1} \ldots v_j$ is a Hamiltonian cycle of G (Fig 3.18).

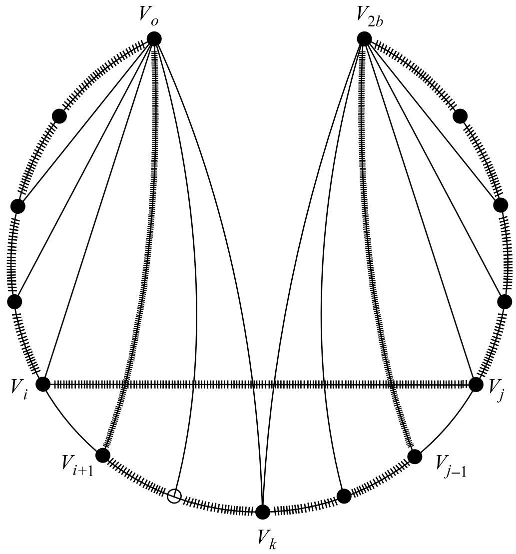


Fig. 3.18

Case (ii) There is an i with $1 \le i \le 2k-1$ such that $v_{i+1}v_0 \in E$, but $v_iv_0 \notin E$. Then by (b), $v_{i-1}v_{2k} \in E$. Thus G contains the $2k$-cycle $v_{i-1}v_{i-2} \ldots v_0v_{i+1}$. Renaming the $2k$-cycle C as $u_1u_2 \ldots u_{2k}$ and let u_0 be the vertex of G not on C. Then u_0 cannot be adjacent to two consecutive vertices on C and hence u_0 is adjacent to every second vertex on C, say $u_1, u_3, \ldots, u_{2k-1}$. Replacing u_{2i} by u_0, we obtain another maximum cycle C' of G and hence u_{2i} must be adjacent to $u_1, u_3, \ldots, u_{2k-1}$. But then u_1 is adjacent to $u_0, u_2, \ldots, u_{2k}$, implying $d(u_1) \ge k+1$. This is a contradiction and hence G is Hamiltonian. ❑

3.7 Pancyclic Graphs

Definition: A graph G of order $n(\ge 3)$ is *pancyclic* if G contains all cycles of lengths from 3 to n. G is called *vertex-pancyclic* if each vertex v of G belongs to a cycle of every length ℓ, $3 \le \ell \le n$.

Example Clearly, a vertex-pancyclic graph is pancyclic. However, the converse is not true. Figure 3.19 displays a pancyclic graph that is not vertex-pancyclic.

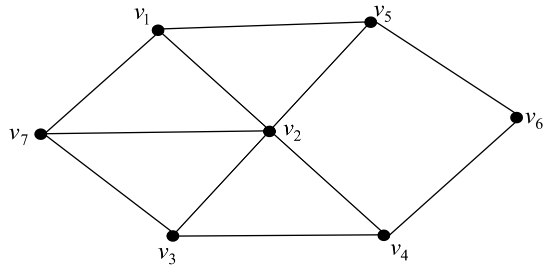


Fig. 3.19

The result of pancyclic graphs was initiated by Bondy [34], who showed that Ore's sufficient condition for a graph G to be Hamiltonian (Theorem 6.2.5) actually implies much more. Note that if $\delta \geq \frac{n}{2}$, then $m \geq \frac{n^2}{2}$. The proof of the following result due to Thomassen can be found in Bollobas [29].

Theorem 3.17 Let G be a simple Hamiltonian graph on n vertices with at least $\left[\frac{n^2}{2}\right]$ edges. Then G is either pancyclic or else is the complete bipartite graph $K_{\frac{n}{2},\,\frac{n}{2}}$. In particular, if G is Hamiltonian and $m > \frac{n^2}{4}$, then G is pancyclic.

Proof The result can easily be verified for $n = 3$. We may therefore assume that $n \geq 4$. We apply induction on n. Suppose the result is true for all graphs of order at most $n-1 (n \geq 4)$, and let G be a graph of order n.

First, assume that G has a cycle $C = v_0 v_1 \ldots v_{n-2} v_0$ of length $n-1$. Let v be the (unique) vertex of G not belonging to C. If $d(v) \geq \frac{n}{2}$, v is adjacent to two consecutive vertices on C and hence G has a cycle of length 3. Suppose for some r, $2 \leq r \leq \frac{n-1}{2}$, C has no pair of vertices u and w on C adjacent to v in G with $d_C(u, w) = r$. Then if $v_{i_1}, v_{i_2}, \ldots v_{i_{d(v)}}$ are the vertices of C that are adjacent to v in G (recall that C contains all the vertices of G except v), then $v_{i_1+r}, v_{i_2+r}, \ldots, v_{i_{d(v)}+r}$ are nonadjacent to v in G, where the suffixes are taken modulo $(n-1)$. Thus, $2d(v) \leq n-1$, a contradiction. Hence, for each r, $2 \leq r \leq \frac{n-1}{2}$, C has a pair of vertices u and w on C adjacent to v in G with $d_C(u, w) = r$. Thus for each r, $2 \leq r \leq \frac{n-1}{2}$, G has a cycle of length $r+2$ as well as a cycle of length $n-1-r+2 = n-r+1$ (Fig. 3.20). Thus G is pancyclic.

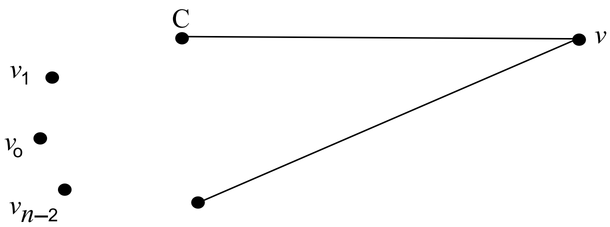


Fig. 3.20

If $d(v) \leq \frac{n-1}{2}$, then $G[V(C)]$, the subgraph of G induced by $V(C)$ has at least $\frac{n^2}{4} - \frac{n-1}{2} > \frac{(n-1)^2}{4}$ edges. So, by the induction assumption, $G[V(C)]$ is pancyclic and hence G is pancyclic. (By hypothesis, G is Hamiltonian).

Next, assume that G has no cycle of length $n-1$. Then G is not pancyclic. In this case, we show that G is $K_{\frac{n}{2},\frac{n}{2}}$.

Let $C = v_0 v_1 v_2 \ldots v_{n-1} v_0$ be a Hamilton cycle of G. We claim that of the two pairs $v_i v_k$ and $v_{i+1} v_{k+2}$ (where suffixes are taken modulo n), at most only one of them can be an edge of G. Otherwise, $v_k v_{k-1} v_{k-2} \ldots v_{i+1} v_{k+2} v_{k+3} v_{k+4} \ldots v_i v_k$ is an $(n-1)$-cycle in G, a contradiction. Hence, if $d(v_i) = r$, then there are r vertices adjacent to v_i in G and hence at least r vertices (including v_{i+1} since $v_i v_{i-1} \in E(G)$) that are nonadjacent to v_{i+1}. Thus, $d(v_{i+1}) \leq n - r$ and $d(v_i) + d(v_{i+1}) \leq n$.

Summing the last inequality over i from 0 to $n-1$, we get $4m \leq n^2$. But by hypothesis, $4m \geq n^2$. Hence, $m = \frac{n^2}{4}$ and so n must be even.

This gives $d(v_i) + d(v_{i+1}) = n$ for each i, and thus for each i and k, exactly one of $v_i v_k$ and $v_{i+1} v_{k+2}$ is an edge of G. (3.17.1)

Thus, if $G \neq K_{\frac{n}{2},\frac{n}{2}}$, then certainly there exist i and j such that $v_i v_j \in E$ and $i \equiv j \pmod 2$. Hence for some j, there exists an even positive integer s such that $v_{j+1} v_{j+1+s} \in E$. Choose s to be the least even positive integer with the above property. Then $v_j v_{j+1+s} \notin E$. Hence, $s \geq 4$ (as $s = 2$ would mean that $v_j v_{j+1} \notin E$). Again, by (3.17.1), $v_{j-1} v_{j+s-3} = v_{j-1} v_{j-1+(s-2)} \in E(G)$ contradicting the choice of s. Thus, $G = K_{\frac{n}{2},\frac{n}{2}}$. The last part follows from the fact that $|E(K_{\frac{n}{2},\frac{n}{2}},)| = \frac{n^2}{4}$. ❑

Theorem 3.18 Let $G \neq K_{\frac{n}{2},\frac{n}{2}}$, be a simple graph with $n \geq 3$ vertices and let $d(u) + d(v) \geq n$ for every pair of non-adjacent vertices of G. Then G is pancyclic.

Proof By Ore's Theorem (Theorem 3.12), G is Hamiltonian. We show that G is pancyclic by first proving that $m \geq \frac{n^2}{4}$ and then invoking Theorem 3.17. This is true if $\delta \geq \frac{n}{2}$ (as $2m = \sum_{i=1}^{n} d_i \geq \delta n \geq n^2/2$). So assume that $\delta < \frac{n}{2}$.

Let S be the set of vertices of degree δ in G. For every pair (u, v) of vertices of degree δ, $d(u) + d(v) < \frac{n}{2} + \frac{n}{2} = n$. Hence by hypothesis, S induces a clique of G and $|S| \leq \delta + 1$. If $|S| = \delta + 1$, then G is disconnected with $G[S]$ as a component, which is impossible (as G is Hamiltonian). Thus, $|S| \leq \delta$. Further, if $v \in S$, v is nonadjacent to $n - 1 - \delta$ vertices of G. If u is such a vertex, $d(v) + d(u) \geq n$ implies that $d(u) \geq n - \delta$. Further, v is adjacent to at least one vertex $w \notin S$ and $d(w) \geq \delta + 1$, by the choice of S. These facts give that

$$2m = \sum_{i=1}^{n} d_i \geq (n - \delta - 1)(n - \delta) + \delta^2 + (\delta + 1),$$

where the last $(\delta + 1)$ comes out of the degree of w. Thus,

$$2m \geq n^2 - n(2\delta + 1) + 2\delta^2 + 2\delta + 1,$$

which implies that

$$4m \geq 2n^2 - 2n(2\delta + 1) + 4\delta^2 + 4\delta + 2$$
$$= (n - (2\delta + 1)^2 + n^2 + 1$$
$$\geq n^2 + 1, \text{ since } n > 2\delta.$$

Consequently, $m > \frac{n^2}{4}$, and by Theorem 3.17, G is pancyclic. ❑

3.8 Exercises

1. Prove that the wheel W_n is Hamiltonian for every $n \geq 2$, and n-cube Q_n is Hamiltonian for each $n \geq 2$.
2. If G is a k-regular graph with $2k-1$ vertices, then prove that G is Hamiltonian.
3. Show that if a cubic graph G has a spanning closed walk, then G is Hamiltonian.
4. If $G = G(X, Y)$ is a bipartite Hamiltonian graph, then show that $|X| = |Y|$.
5. Prove that for each $n \geq 1$, the complete tripartite graph $K_{n,2n,3n}$ is Hamiltonian, but $K_{n,2n,3n+1}$ is not Hamiltonian.
6. How many spanning cycles are there in the complete bipartite graphs $K_{3,3}$ and $K_{4,3}$?
7. Prove that a graph G with $n \geq 3$ vertices is arbitrarily traceable if and only if it is one of the graphs C_n, K_n or $K_{n,n}$ with $n = 2p$.
8. Prove that a graph G with $n \geq 3$ vertices is randomly traceable if and only if it is randomly Hamiltonian.
9. Find the closure of the graph given in Figure 3.2. Is it Hamiltonian?
10. Does there exist an Eulerian graph with

    i. an even number of vertices and an odd number of edges,
    ii. and odd number of vertices and an even number of edges.

    Draw such a graph if it exists.
11. Characterise graphs which are both Eulerian and Hamiltonian.
12. Characterise graphs which possess Hamiltonian paths but not Hamiltonian cycles.
13. Characterise graphs which are unicursal but not Eulerian.
14. Give an example of a graph which is neither pancyclic nor bipartite, but whose n-closure is complete.