Ans 1. Given X is the number of heads on the first two coins and therefore it can take values $\{0, 1, 2\}$.

And Y denotes the number of heads on the last two coins and therefore it can take values $\{0, 1, 2\}$

(a) Let $P_{X,Y}(x,y)$ be the probability whe $X = x$ and $Y = y$

i.e $P_{X,Y}(x,y) = \mathbb{P}(X=x, Y=y)$

$P_{X,Y}(0,0) = \mathbb{P}(X=0, Y=0) = \frac{1}{8}$

$P_{X,Y}(0,1) = \mathbb{P}(X=0, Y=1) = \frac{1}{8}$; $P_{X,Y}(1,0) = \mathbb{P}(X=1, Y=0) = \frac{1}{8}$

Similarly, we can find other probabilities and they are given below:
Hence the joint pmf is given in the table below.

X \ Y	0	1	2
0	$\frac{1}{8}$	$\frac{1}{8}$	0
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$
2	0	$\frac{1}{8}$	$\frac{1}{8}$

(b) Since we have the joint pmf of X and Y.

Consider $P_X(x)$ to be the marginal pmf of X and $P_Y(y)$ to be the marginal pmf of Y.

$$P_x(0) = \mathbb{P}(X=0) = \sum_{y=0}^{2} \mathbb{P}(X=0, Y=y)$$

$$= \frac{1}{8} + \frac{1}{8} + 0 = \frac{1}{4}$$

$$P_x(1) = \mathbb{P}(X=1) = \sum_{y=0}^{2} \mathbb{P}(X=1, Y=y) \quad ; \quad P_x(2) = \mathbb{P}(X=2) = \sum_{y=0}^{2} \mathbb{P}(X=2, Y=y)$$

$$= \frac{1}{8} + \frac{2}{8} + \frac{1}{8} = \frac{1}{2} \qquad\qquad = 0 + \frac{1}{8} + \frac{1}{8}$$

$$= \frac{1}{4}$$

Similar values can be calculated for P_Y. And they are listed in the below table along with the joint pmf of X and Y.

X \ Y	0	1	2	$P_X(x)$
0	$\frac{1}{8}$	$\frac{1}{8}$	0	$\frac{1}{4}$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{1}{2}$
2	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$
$P_Y(y)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	1

$$P_Y(0) = \mathbb{P}(Y=0) = \sum_{x=0}^{2} \mathbb{P}(X=x, Y=0)$$
$$= \frac{1}{8} + \frac{1}{8} + 0 = \frac{1}{4}$$
$$P_Y(1) = \mathbb{P}(Y=1) = \sum_{x=0}^{2} \mathbb{P}(X=x, Y=1)$$
$$= \frac{1}{8} + \frac{2}{8} + \frac{1}{8} = \frac{1}{2}$$
$$P_Y(2) = \mathbb{P}(Y=2) = \sum_{x=0}^{2} \mathbb{P}(X=x, Y=2)$$
$$= 0 + \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

We can check that our derived marginal pmf's are correct because

$$\sum_{x=0}^{2} P_X(x) = \sum_{y=0}^{2} P_Y(y) = 1$$

(c) We know that;

$$P_{X|Y=y}(x) = \frac{P_{X,Y}(x,y)}{P_Y(y)} = \frac{\mathbb{P}(X=x, Y=y)}{\sum_{x=0}^{2} \mathbb{P}(X=x, Y=y)}$$

Case $Y=0$

$$\mathbb{P}(X=0|Y=0) = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{2}$$
$$\mathbb{P}(X=1|Y=0) = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{2}$$
$$\mathbb{P}(X=2|Y=0) = \frac{0}{\frac{1}{4}} = 0$$

In general distribution would be

$$P_{X|Y=y}(x) = \frac{\mathbb{P}(X=x, Y=y)}{\sum_{x=0}^{2} \mathbb{P}(X=x, Y=y)}$$

$$\therefore P_{X|Y=0}(x) = \begin{cases} \frac{1}{2}, & x=0 \\ \frac{1}{2}, & x=1 \\ 0, & x=2 \end{cases}$$ or $$P_{X|Y=0}(x) = 4 \cdot P_{X,Y}(x,0)$$

Similarly for other cases answer would be,

$$P_{X|Y=1}(x) = \begin{cases} \frac{1}{4}, & x=0 \\ \frac{1}{2}, & x=1 \\ \frac{1}{4}, & x=2 \end{cases}$$; $$P_{X|Y=2}(x) = \begin{cases} 0, & x=0 \\ \frac{1}{2}, & x=1 \\ \frac{1}{2}, & x=2 \end{cases}$$

or $$P_{X|Y=1}(x) = 2 \cdot P_{X,Y}(x,1)$$ or $$P_{X|Y=2} = 4 \cdot P_{X,Y}(x,2)$$

$P_{Y|X}(x,y)$ The distribution would be this :-

$$P_{Y|X=x}(y) = \frac{\mathbb{P}(X=x, Y=y)}{\sum_{y=0}^{2} \mathbb{P}(X=x, Y=y)}$$

Case $X=0$

$$P_{Y|X=0}(0) = \frac{1/8}{1/4} = \frac{1}{2}$$

$$P_{Y|X=0}(1) = \frac{1/8}{1/4} = \frac{1}{2}$$

$$P_{Y|X=0}(2) = \frac{0}{1/4} = 0$$

$$\therefore P_{Y|X=0}(y) = \begin{cases} \frac{1}{2}, & y=0 \\ \frac{1}{2}, & y=1 \\ 0, & y=2 \end{cases}$$

or $P_{Y|X=0}(y) = 4 \cdot P_{X,Y}(0,y)$

Similarly for other cases

$$P_{Y|X=1}(y) = \begin{cases} \frac{1}{4}, & y=0 \\ \frac{1}{2}, & y=1 \\ \frac{1}{4}, & y=2 \end{cases} \quad ; \quad P_{Y|X=2}(y) = \begin{cases} 0, & y=0 \\ \frac{1}{2}, & y=1 \\ \frac{1}{2}, & y=2 \end{cases}$$

or $P_{Y|X=1}(y) = 2 \cdot P_{X,Y}(1,y)$ or $P_{Y|X=2} = 4 \cdot P_{X,Y}(2,y)$

(d) $$E[Y|X=1] = \sum_{y=0}^{2} y\, P_{Y|X=1}(y)$$

$$= 2 \sum_{y=0}^{2} y\, P_{X,Y}(1,y)$$

$$= 2\left[0 \cdot P_{X,Y}(1,0) + 1 \cdot P_{X,Y}(1,1) + 2 \cdot P_{X,Y}(1,2)\right]$$

$$= 2\left[0 + \frac{2}{8} + 2 \cdot \left(\frac{1}{8}\right)\right] = 1$$

$$E[X|Y=1] = \sum_{x=0}^{2} x \cdot P_{X|Y=1}(x)$$

$$= 2 \sum_{x=0}^{2} x\, P_{X,Y}(x,1)$$

$$= 2\left[0 \cdot P_{X,Y}(0,1) + 1 \cdot P_{X,Y}(1,1) + 2 \cdot P_{X,Y}(2,1)\right]$$

$$= 2\left[0 + \frac{2}{8} + 2\left(\frac{1}{8}\right)\right] = 1$$

(e) Correlation Coefficient between X and Y, i.e $\rho_{X,Y} = \dfrac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y}$

where $Cov(X,Y)$ = Covariance of X & Y = $E(XY) - E(X)\,E(Y)$

$\sigma_X^2 = Var(X)$, $\sigma_Y^2 = Var(Y)$

$$E[X] = \sum_{x=0}^{2} x\,P_X = 0\cdot\frac{1}{4} + 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} = 1$$

$$E[Y] = \sum_{y=0}^{2} y\,P_Y = 0\cdot\frac{1}{4} + 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} = 1$$

$$E[XY] = \sum_{x=0}^{2}\sum_{y=0}^{2} xy\,P_{X,Y}(x,y)$$

$$= 0\cdot0\cdot\frac{1}{8} + 0\cdot1\cdot\frac{1}{8} + 0\cdot2\cdot0 + 1\cdot0\cdot\frac{1}{8} + 1\cdot1\cdot\frac{2}{8} + 1\cdot2\cdot\frac{1}{8}$$
$$+ 2\cdot0\cdot0 + 2\cdot1\cdot\frac{1}{8} + 2\cdot2\cdot\frac{1}{8}$$

$$= \frac{5}{4}$$

$\therefore\ Cov(X,Y) = E[XY] - E[X]\cdot E[Y] = \frac{5}{4} - 1\cdot1 = \frac{1}{4}$

for $Var(X)$; $E[X^2] = \sum_{x=0}^{2} x^2 P_X = 0\cdot\frac{1}{4} + 1\cdot\frac{1}{2} + 4\cdot\frac{1}{4} = \frac{3}{2}$

$Var(X) = E[X^2] - (E[X])^2 = \frac{3}{2} - 1^2 = \frac{1}{2} \Rightarrow \sigma_X = \frac{1}{\sqrt{2}}$

For $Var(Y)$; $E[Y^2] = \sum_{y=0}^{2} y^2 P_Y = 0\cdot\frac{1}{4} + 1\cdot\frac{1}{2} + 4\cdot\frac{1}{4} = \frac{3}{2}$

$Var(Y) = \frac{3}{2} - 1^2 = \frac{1}{2} \Rightarrow \sigma_Y = \frac{1}{\sqrt{2}}$

$$\therefore\ \rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{Var(X)\cdot Var(Y)}} = \frac{\cancel{5}\,\frac{1}{4}}{\sqrt{1/4}} = \frac{1}{2}$$

Ans 2. $X \sim U(0,1) \Rightarrow$ p.d.f of X i.e $f_X(x) = 1$ for $0 \le x \le 1$ and 0 otherwise

Conditional p.d.f

$$g_{Y|X}(y|x) = \begin{cases} \frac{1}{x} & y \in [0,x] \\ 0 & \text{otherwise} \end{cases}$$

$$E[Y^k | X] = \int_0^x y^k \, g_{Y|X}(y|x) \, dy$$

Substituting the value of p.d.f.

$$E[Y^k|X] = \int_0^x \frac{y^k}{x} dy = \frac{1}{x} \int_0^x y^k dy = \frac{1}{x}\left[\frac{y^{k+1}}{k+1}\right]_0^x$$

$$= \frac{x^k}{k+1}$$

$$\therefore E[Y^k|X] = \frac{x^k}{k+1} \rightarrow (i)$$

For the next part

$E[Y^k]$;

Claim: $E[A] = E[E[A|B]]$ (Law of total expectation)

Proof: For discrete random variables A, B

$$E[E(A|B)] = E\left[\sum_{a \in A} a \, \mathbb{P}(A=a|B)\right]$$

$$= \sum_{b \in B}\left[\sum_{a \in A} a . \mathbb{P}(A=a|B=b)\right] . \mathbb{P}(B=b)$$

$$= \sum_{a \in A}\sum_{b \in B} a . \mathbb{P}(A=a|B=b) . \mathbb{P}(B=b)$$

$$= \sum_{a \in A} a . \sum_{b \in B} \frac{\mathbb{P}(A=a, B=b)}{\mathbb{P}(B=b)} . \mathbb{P}(B=b)$$

$$= \sum_{a \in A} a . \sum_{b \in B} \mathbb{P}(A=a, B=b)$$

$$= \sum_{a \in A} a . \mathbb{P}(A=a)$$

$$= E(A)$$

For continuous random variables the proof only requires the summations to be changed to integrals.

$\therefore$ $E[y^k] = E[E[y^k|x]]$ (By Law of total expectation proved above)

From earlier we know by equation (i) $E[y^k|x] = \frac{x^k}{k+1}$

$\therefore$ $E[y^k] = E\left[\frac{x^k}{k+1}\right] = \frac{1}{k+1} E[x^k]$

Now we know; $E[x^k] = \int_0^1 x^k . 1 dx = \left[\frac{x^{k+1}}{k+1}\right]_0^1 = \frac{1}{k+1}$

$\therefore$ $E[y^k] = \frac{1}{(k+1)^2} \rightarrow$ (ii)

$E[y^k] = \frac{1}{(k+1)^2}$ and $E[y^k|x] = \frac{x^k}{k+1}$

Ans 3. X and Y are iid random variables with pdf;

$$f(x) = e^{-x}, \quad x > 0$$

(a) Let $M = \min(X,Y)$, $N = \max(X,Y)$

$$\mathbb{P}(M \le m, N \le n) = \begin{cases} \mathbb{P}(\min(X,Y) \le m, \max(X,Y) \le n) & 0 < m \le n \\ 0 & \text{otherwise} \end{cases}$$

$\therefore$ $\mathbb{P}(M \le m, N \le n) = \mathbb{P}(X \le m, Y \le n) + \mathbb{P}(X \le n, Y \le m) - \mathbb{P}(X \le m, Y \le m)$ [By inclusion exclusion principle]

$= F_X(m) \cdot F_Y(n) + F_X(n) \cdot F_Y(m) - F_X(m) \cdot F_Y(m)$

[F_X, F_Y are c.d.f for X, Y respectively]

$$F_X(a) = \int_0^a e^{-x}\,dx = 1 - e^{-a} \quad \forall a \ge 0$$

$\therefore$ $\mathbb{P}(M \le m, N \le n) = 2(1 - e^{-m})(1 - e^{-n}) - (1 - e^{-m})^2$

$\therefore$ Joint CDF of (M,N) is

$$F_{M,N}(m,n) = \begin{cases} 2(1-e^{-m})(1-e^{-n}) - (1-e^{-m})^2 & 0 < m \le n \\ 0 & \text{otherwise} \end{cases}$$

Joint PDF of (M,N) is

$$f_{M,N}(m,n) = \frac{\partial^2}{\partial m \partial n} F_{M,N}(m,n)$$

$$= \frac{\partial^2}{\partial m \partial n}\left[2(1-e^{-m})(1-e^{-n}) - (1-e^{-m})^2\right]$$

$\therefore$

$$f_{M,N}(m,n) = \begin{cases} 2e^{-(m+n)} & 0 < m \le n \\ 0 & \text{otherwise} \end{cases}$$

(b) $f_M(m) = \int_m^{\infty} f_{M,N}(m,n)\,dn = \int_m^{\infty} 2e^{-(m+n)}\,dn$

$= 2e^{-2m}, \quad m \geq 0$

$\therefore$ Marginal PDF of $\min(X,Y) = 2e^{-2m}$ where $m = \min(X,Y)$

$f_N(n) = \int_0^n 2e^{-(m+n)}\,dm = 2e^{-n}(1-e^{-n})$

$\therefore$ Marginal PDF of $\max(X,Y) = 2e^{-n}(1-e^{-n})$ where $n = \max(X,Y)$

(c) Let $T := \dfrac{\min(X,Y)}{\max(X,Y)}$ Now T must belong to $[0,1]$

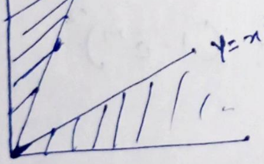
$\because \min(X,Y) \leq \max(X,Y)$

For any $0 \leq t \leq 1$ For CDF of T

$F_T(t) = \mathbb{P}(T \leq t) = \mathbb{P}(\min(X,Y) \leq t\max(X,Y))$

$= \mathbb{P}((X \leq tY) \cap (X \leq Y)) + \mathbb{P}((Y \leq tX) \cap (Y < X))$

$= 1 - \int_0^{\infty}\int_{xt}^{x/t} f_{X,Y}(x,y)\,dy\,dx$



Shaded region is to be considered.

$f_T(t) = \dfrac{d}{dt}F_T(t)$

$= -\int_0^{\infty} \dfrac{d}{dt}\left[\int_{xt}^{x/t} f_{X,Y}(x,y)\,dy\right] dx$

$f_T(t) = \int_0^{\infty}\left(\dfrac{x}{t^2} f_{XY}\left(x, \dfrac{x}{t}\right) + x f_{XY}(x, xt)\right) dx$ [Applying Leibnitz Rule]

$f_T(t) = \int_0^{\infty}\left[\dfrac{x}{t^2} e^{-\left(\frac{1+t}{t}\right)x} + x e^{-(1+t)x}\right] dx$

$$f_T(t) = \int_0^\infty x \left(\frac{e^{-\left(\frac{1+t}{t}\right)x}}{t^2} + e^{-(1+t)x} \right) dx$$

On applying integration by parts and plugging in the limits we get.

$$f_T(t) = \left[\frac{-((t^2+t)x+t)\, e^{-x-xt} - ((t+1)x+t)\, e^{-x-\frac{x}{t}}}{t(t+1)^2} \right]_0^\infty$$

$$f_T(t) = \frac{2}{(t+1)^2}$$

$\therefore$ Pdf of $T := \frac{\min(X,Y)}{\max(X,Y)}$ $= \begin{cases} \frac{2}{(t+1)^2} & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$ where $t = \frac{\min(X,Y)}{\max(X,Y)}$

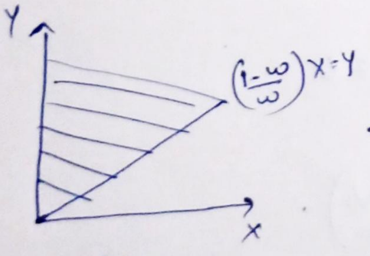
Let $W := \frac{X}{X+Y}$ $\quad$ W must belong to $[0,1)$

$\because X < X+Y$ iff $Y > 0 \ \forall x$

CDF of W will be

$$F_W(\omega) = \mathbb{P}(W \le \omega) = \mathbb{P}\left(\frac{X}{X+Y} \le \omega\right)$$

$$= \mathbb{P}\left(X \le \left(\frac{\omega}{1-\omega}\right) Y\right)$$



$$= \int_0^\infty \int_0^{\left(\frac{\omega}{1-\omega}\right)y} f_{XY}(x,y)\, dx\, dy$$

Shaded region is to be considered

For PDF of W; i.e $f_W(\omega)$

$$f_W(\omega) = \frac{d}{dw} F_W(\omega) = \int_0^\infty \frac{d}{dw} \left[\int_0^{\left(\frac{\omega}{1-\omega}\right)y} f_{XY}(x,y)\, dx \right] dy$$

$$f_W(\omega) = \int_0^\infty \left[\frac{+y}{(1-\omega)^2} f_{XY}\left(\frac{\omega}{1-\omega} y, y\right) \right] dy$$

$$= \int_0^\infty \frac{+y}{(1-\omega)^2} e^{-\left(\frac{\omega}{1-\omega}+1\right)y} dy$$

On applying integration by parts and solving the integral we get

$$f_W(\omega) = \left[-\frac{(y\bar{+}\omega+1)}{(1-\omega)} e^{-\frac{y}{1-w}} \right]_0^\infty$$

$$f_W(\omega) = 1$$

PDF for $W = \frac{X}{X+Y}$ $= f_W(\omega) = \begin{cases} 1 ; & 0 \le w < 1 \text{ where } w = \frac{X}{X+Y} \\ 0 & \text{otherwise} \end{cases}$

(d) Define:

$U = X+Y \quad , V = X-Y$

$\therefore X = \frac{U+V}{2} \quad , Y = \frac{U-V}{2}$; $\frac{U+V}{2} \geq 0 \Rightarrow V \geq -U$
also $\frac{U-V}{2} \geq 0 \Rightarrow V \leq U$

$\frac{\partial X}{\partial U} = \frac{1}{2}, \quad \frac{\partial X}{\partial V} = \frac{1}{2}, \quad \frac{\partial Y}{\partial U} = \frac{1}{2}, \quad \frac{\partial Y}{\partial V} = -\frac{1}{2}$

Jacobian, $\frac{\partial(X,Y)}{\partial(U,V)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2} \neq 0$

Joint PDF of $U,V = \left|-\frac{1}{2}\right| f_{X,Y}\left(\frac{U+V}{2}, \frac{U-V}{2}\right)$.

$= \frac{1}{2} e^{-u}, \quad u > 0, -u < v < u$

Conditional Density $f_{V|U}(v|u) = \frac{f_{U,V}(u,v)}{f_U(u)}$

Marginal PDF of U $f_U(u) = \int_{-u}^{u} \frac{1}{2} e^{-u} \, dv = \frac{1}{2} e^{-u}(2u) = ue^{-u}$

$\therefore \boxed{f_{V|U}(v|u) = \frac{\frac{1}{2}e^{-u}}{ue^{-u}} = \frac{1}{2u}}$; $u > 0$ & $v \in (-u, u)$

(e) From the earlier parts ;

$$f_U(u) = u\bar{e}^{u}, \; u>0 \xrightarrow{} (i) \quad \& \quad f_Z(z) = 1 \xrightarrow{} (ii) \; ; \; z \in [0,1)$$

are the respective PDFs for $U = X+Y$ & $Z = \frac{X}{X+Y}$

Joint PDF of U, Z

$$X = UZ \quad , Y = U - UZ$$

$$\frac{\partial X}{\partial U} = z \; , \frac{\partial X}{\partial z} = U, \; \frac{\partial Y}{\partial U} = 1-z \; , \frac{\partial Y}{\partial z} = -U$$

$$\therefore \text{Jacobian} \quad \frac{\partial(x,Y)}{\partial(U,Z)} = \begin{bmatrix} z & U \\ 1-z & -U \end{bmatrix} \Rightarrow \det(J) = -U \neq 0$$

$$\therefore \text{Joint PDF of } U, Z = f_{U,Z}(U,z) = |-U| \, f_{X,Y}(Uz, U(1-z))$$

$$= u \, e^{-(Uz + U - Uz)}$$

$$f_{U,Z}(U,z) = Ue^{-U} \xrightarrow{} (iii)$$

For two random variables to be independent

$$f_{U,Z}(u,z) = f_U(u) \cdot f_Z(z)$$

From results (i), (ii), (iii) we can see that the independente criterion is being satisfied

Hence $U = X+Y$ and $Z = \frac{X}{X+Y}$ are two independent random variables

Ans 4

$X \sim P(\lambda)$

$$\mathbb{P}(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$M_x(t) = e^{\lambda(e^t-1)}$$

$$E(X) = \frac{d}{dt} M_x(t)\Big|_{t=0} = \lambda \to \text{i)}$$

$$E(X^2) = \frac{d^2}{dt^2} M_x(t)\Big|_{t=0} = \lambda^2+\lambda \to \text{ii)}$$

$$E(X^3) = \frac{d^3}{dt^3} M_x(t)\Big|_{t=0} = \lambda^3+3\lambda^2+\lambda \to \text{iii)}$$

From equations i) ii) iii)

$$\lambda^3 = E(X^3) - 3E(X^2) + 2E(X) = E(X^3-3X^2+2X)$$

$\Rightarrow$ ~~X^3-3X~~ Define

$$Y_N = \frac{1}{N}\sum_{i=1}^{N}\left[X_i^3-3X_i^2+2X_i\right]$$

$$E(Y_N) = \frac{1}{N}E\left[\left(\sum_{i=1}^{N} X_i^3-3X_i^2+2X_i\right)\right] = \frac{1}{N}\cdot N\lambda^3 = \lambda^3$$

$\therefore$ Y_N is an unbiased estimator for λ^3.

* Unbiased estimator for $e^{-\lambda}$.

~~Let~~ Define $Y_n = I\{X_1=0\}$ (Indicator ~~random~~ variable)

$$Y_n = \begin{cases} 1 & \text{if } X_1=0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[Y_n] = 1\cdot\mathbb{P}(X_1=0) + 0\cdot\mathbb{P}(X_1\neq 0) \Rightarrow \mathbb{P}(X_1=0) = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda}$$

$\therefore$ Y_n is an unbiased estimator of $e^{-\lambda}$

There does not exist an unbiased estimator for $\frac{1}{\lambda}$

Proof: Assume there exists an unbiased estimator $\hat{\theta}(X)$ for $\frac{1}{\lambda}$

$\Rightarrow E[\theta(X)] = \frac{1}{\lambda} \quad \forall \lambda > 0$

$E[\theta(X)] = \sum_{K=0}^{\infty} \theta(K) \, \mathbb{P}(X=K) = \sum_{K=0}^{\infty} \theta(K) \frac{e^{-\lambda} \lambda^{K}}{K!} = \frac{1}{\lambda}$

But as $\lambda \to 0$ $\quad \frac{1}{\lambda} \to \infty$.

E

But $E[\theta(X)]$ cannot tend to infinity

$\therefore$ There does not exist an unbiased estimator for $\frac{1}{\lambda}$

Ans 5

$$P(X_n = \pm 2k^{\alpha}) = \frac{1}{2}$$

$$E[X_n] = 2k^{\alpha} \cdot \frac{1}{2} + (-2k^{\alpha}) \cdot \frac{1}{2} = 0$$

$$E[X_n^2] = 4k^{2\alpha} \cdot \frac{1}{2} + 4k^{2\alpha} \cdot \frac{1}{2} = 4k^{2\alpha}$$

$$\therefore Var(X_n) = 4k^{2\alpha}$$

$$B_n = Var(X_1 + X_2 + \dots + X_n) = 4nk^{2\alpha} \quad [\because X_i \text{ are independent}]$$

$$\lim_{n\to\infty} \frac{B_n}{n^2} = \cancel{\lim_{n\to\infty} \frac{4k^{2\alpha}}{n} = 0}$$

$\therefore$ ~~we can say that~~

$$\frac{1}{n^2} \sum_{i=1}^{n} 4k^{2\alpha} = \frac{4k^{2\alpha}}{n}$$

$$\lim_{n\to\infty} \frac{B_n}{n^2} = 0 \quad \text{as } k \text{ is constant}$$

$\therefore$ Any α holds true ~~as long~~?

Ans 6

To Show: The estimator $\theta(n) = \frac{2}{n(n+1)} \sum_{i=1}^{n} iX_i$ based on sample of size n is consisted for μ

$$E[\theta(n)] = E\left[\frac{2}{n(n+1)} \sum_{i=1}^{n} iX_i\right]$$

$$= \frac{2}{n(n+1)} \sum_{i=1}^{n} i\,E(X_i) \quad [\because \text{Using Linearity of Expectation}]$$

$$= \frac{2\mu}{n(n+1)} \sum_{i=1}^{n} i \quad (\because E(X_i) = \mu)$$

$$= \frac{2\mu}{\cancel{n(n+1)}} \cdot \frac{\cancel{n(n+1)}}{2} = \mu$$

$$\therefore E[\theta(n)] = \mu$$

Now

$$Var(\theta(n)) = Var\left(\frac{2}{n(n+1)} \sum_{i=1}^{n} iX_i\right)$$

$$= \left(\frac{2}{n(n+1)}\right)^2 \sum_{i=1}^{n} i^2\, Var(X_i)$$

$$\leq \left(\frac{4}{n^2(n+1)^2}\right)\left(\frac{n(2n+1)(n+1)}{6}\right) Var(X_i) \quad \left[\text{Assuming } Var(X_i) \text{ is maximum among all } i\right]$$

$$Var(\theta(n)) \leq \frac{4\sigma^2(2n+1)}{6n(n+1)} \quad [\text{let } \sigma^2 = \max(Var(X_i))]$$

$$\Rightarrow \lim_{n\to\infty} Var(\theta(n)) \to 0 \quad \& \quad E[\theta(n)] = \mu$$

$\therefore$ We can say that for any population having mean μ the estimator $\frac{2}{n(n+1)} \sum_{i=1}^{n} iX_i$ is consistent for parameter μ.