DEPARTMENT OF MATHEMATICAL SCIENCES, IIT (BHU)
ODD SEMESTER 2023-24
MA101 – ENGINEERING MATHEMATICS I
SOLUTIONS TO END-TERM EXAM

Saturday, 18 November 2023, 08:30 – 11:30

Marks:50

Instructions:

- Start each question on a new page.
- Answer all questions. Give proper justification. Don't copy from others.

(1) (a) Compute

$$\lim_{n\to\infty}(2^n + 3^n + 5^n + 7^n)^{1/n}.$$

(3 Marks)

Solution. Observe that $7 \leq (2^n+3^n+5^n+7^n)^{1/n} \leq 7\cdot 4^{1/n}$. Since $\lim_{n\to\infty} 4^{1/n} = 1$, it follows from the Sandwich Theorem that $\lim_{n\to\infty}(2^n+3^n+5^n+7^n)^{1/n} = 7$. ◆

(b) Find all $x \in \mathbb{R}$ for which the following series converges: (3 Marks)

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)(2n+1)} x^n.$$

Solution. Let $a_n = \frac{n}{(n+1)(2n+1)} x^n$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|.$$

By the Ratio Test the series converges if $|x| < 1$ and diverges if $|x| > 1$. If $x = 1$, the series diverges by limit comparision with the harmonic series. If $x = -1$, the series converges by the alternating series test. ◆

(2) Use the ε-δ definition to prove that

(a) $\lim_{x\to 8} \sqrt[3]{x} = 2$ (3 Marks)

Solution. Let $\varepsilon > 0$ be given. Take $\delta = \min\{1, 7\varepsilon\}$. Then $0 < |x - 8| < \delta$ implies

$$\begin{aligned}\left|\sqrt[3]{x} - 2\right| &= \frac{|x-8|}{x^{2/3} + 2x^{1/3} + 4}\\ &\leq \frac{|x-8|}{7^{2/3} + 2 \cdot 7^{1/3} + 4}\\ &\leq \frac{|x-8|}{7}\\ &< \varepsilon.\end{aligned}$$

◆

(b) $\lim\limits_{(x,y)\to(2,3)} \left(x^2 + y^2\right) = 13$ **(3 Marks)**

Solution. Let $\varepsilon > 0$ be given. Take $\delta = \min\{1, \varepsilon/12\}$. Then

$$\begin{aligned}\left|(x^2+y^2) - 13\right| &\leq \left|x^2 - 4\right| + \left|y^2 - 9\right|\\ &= |x+2|\,|x-2| + |y+3|\,|y-3|\\ &\leq 5\,|x-2| + 7\,|y-3|\\ &\leq 12\sqrt{(x-2)^2 + (y-3)^2}\\ &< \varepsilon.\end{aligned}$$

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(3) Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function, and suppose $f''(x) \geq 0$ for all $x \in \mathbb{R}$. Prove that if $t \in [0,1]$ then **(3 Marks)**

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \qquad x, y \in \mathbb{R}.$$

Solution. If $t \in \{0,1\}$ or if $x = y$, the result is a tautology. We may assume that $x < y$. Otherwise the roles of x and y may be reversed in the foregoing argument.

The hypothesis and the Mean Value Theorem imply that f' is increasing. Let $t \in (0,1)$. Then $x < tx + (1-t)y < y$. By the Mean Value Theorem, there exist $c \in (x, tx + (1-t)y)$ and $d \in (tx + (1-t)y, y)$ such that

$$\frac{f(tx+(1-t)y) - f(x)}{tx + (1-t)y - x} = f'(c), \quad \text{and}$$
$$\frac{f(y) - f(tx+(1-t)y)}{y - tx - (1-t)y} = f'(d).$$

Since $c < d$ and f' is increasing, it follows that

$$\frac{f(tx+(1-t)y) - f(x)}{tx + (1-t)y - x} \leq \frac{f(y) - f(tx+(1-t)y)}{y - tx - (1-t)y}.$$

Since $t, 1-t > 0$, it follows that

$$t(f(tx+(1-t)y) - f(x)) \leq (1-t)(f(y) - f(tx+(1-t)y)),$$

and so

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y).$$

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(4) Do not use trigonometric functions (such as sin and cos) in solving this problem.

Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function, and suppose $f''(x) + f(x) = 0$ for all x.

(a) Prove that f is infinitely differentiable. `(1 Mark)`

Solution. We proceed by induction. By assumption $f^{(1)}$ exists. Assume now that $n \ge 2$ and that $f^{(1)}, \dots, f^{(n-1)}$ exist. Then

$$f^{(n)} = (f'')^{(n-2)} = -f^{(n-2)},$$

so $f^{(n)}$ exists. This completes the induction step. Therefore $f^{(n)}$ exists for all $n \in \mathbb{N}$. □◆

(b) Find the Maclaurin series of f (it is not uniquely determined, but will have two parameters). `(1 Mark)`

Solution. Let $a = f(0)$ and $b = f'(0)$. Then

$$f^{(n)}(0) = \begin{cases} a & \text{if } n \equiv 0 \pmod 4 \\ b & \text{if } n \equiv 1 \pmod 4 \\ -a & \text{if } n \equiv 2 \pmod 4 \\ -b & \text{if } n \equiv 3 \pmod 4). \end{cases}$$

Therefore, the Maclaurin coefficients are given by

$$c_{2k} = \frac{(-1)^k a}{(2k)!}, \qquad \text{and}$$

$$c_{2k+1} = \frac{(-1)^k b}{(2k+1)!}.$$

where $k = 0, 1, 2, \dots$, and the Maclaurin series is given by

$$a \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + b \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}.$$

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(c) Prove that $f(x)^2 + f'(x)^2$ is constant. Deduce that there exists a constant C such that $\left|f^{(n)}(x)\right| \le C$ for all $x \in \mathbb{R}$ and $n = 0, 1, 2, \dots$. `(2 Marks)`

Solution. Since

$$\frac{d}{dx}\left(f(x)^2 + f'(x)^2\right) = 2f(x)f'(x) + 2f'(x)f''(x) = 0,$$

it follows from the Mean Value Theorem that $f(x)^2 + f'(x)^2 = C^2$. Since $f(x)^2, f'(x)^2 \ge 0$, it follows that $f'(x)^2, f(x)^2 \le C^2$, whence $|f'(x)|, |f(x)| \le C$.

If $n \in \mathbb{N}$ is even, then $\left|f^{(n)}(x)\right| = |f(x)| \leq C$, and if $n \in \mathbb{N}$ is odd, then $\left|f^{(n)}(x)\right| = |f'(x)| \leq C$. ◆

(d) Prove that the Maclaurin series of f converges to f everywhere. `(2 Marks)`

Solution. Let $R_n(x)$ denote the remainder term in Taylor formula for f. By Taylor's theorem there exists c between 0 and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

It follows from the previous part that

$$|R_n(x)| \leq \frac{C\,|x|^{n+1}}{(n+1)!}.$$

Therefore $R_n(x) \to 0$ for all $x \in \mathbb{R}$ and hence the Maclaurin series of f converges to f everywhere. ◆

(5) Let $f(x, y) = x^2y + xy^2$.

(a) Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. `(1 Mark)`

Solution. $\frac{\partial f}{\partial x} = 2xy + y^2$ and $\frac{\partial f}{\partial y} = x^2 + 2xy$ ◆

(b) Let $\vec{u} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Find the directional derivative of f at $(2, 1)$ in the direction $\vec{u}$, i.e., compute $D_{\vec{u}}f(2, 1)$. `(1 Mark)`

Solution. **Using the fact that f is differentiable:** First note that $\vec{\nabla} f(2, 1) = 5\widehat{\mathbf{i}} + 8\widehat{\mathbf{j}}$. Since $\|\vec{u}\| = 1$, we have

$$D_{\vec{u}}f(2, 1) = \vec{\nabla} f(2, 1) \cdot \vec{u} = (5\widehat{\mathbf{i}} + 8\widehat{\mathbf{j}}) \cdot \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \frac{5}{2} + 4\sqrt{3}.$$

Using the definition of directional derivative:

$$\begin{aligned}
D_{\vec{u}}f(2, 1) &= \lim_{h\to 0} \frac{f((2, 1) + h\vec{u}) - f(2, 1)}{h} \\
&= \lim_{h\to 0} \frac{f\left(2 + \frac{h}{2}, 1 + \frac{h\sqrt{3}}{2}\right) - f(2, 1)}{h} \\
&= \lim_{h\to 0} \frac{\left(2 + \frac{h}{2}\right)^2\left(1 + \frac{h\sqrt{3}}{2}\right) + \left(2 + \frac{h}{2}\right)\left(1 + \frac{h\sqrt{3}}{2}\right)^2}{h} \\
&= \lim_{h\to 0} \left[\left(\frac{5}{2} + 4\sqrt{3}\right) + h\left(\frac{7}{4} + \frac{3\sqrt{3}}{2}\right) + h^2\left(\frac{\sqrt{3}+3}{8}\right)\right] \\
&= \frac{5}{2} + 4\sqrt{3}.
\end{aligned}$$

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(c) Prove that f is differentiable at $(2,1)$ using the definition. `(3 Marks)`

Solution.

$$\begin{aligned} & f(2+\Delta x, 1+\Delta y) - f(2,1) \\ =\ & (2+\Delta x)^2(1+\Delta y) + (2+\Delta x)(1+\Delta y)^2 - 6 \\ =\ & 5\Delta x + 8\Delta y + (\Delta x + 6\Delta y + \Delta x\Delta y + \Delta y^2)\Delta x + (\Delta y)\Delta y \\ =\ & \frac{\partial f}{\partial x}(2,1)\Delta x + \frac{\partial f}{\partial y}(2,1)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \end{aligned}$$

where $\varepsilon_1 = \Delta x + 6\Delta y + \Delta x\Delta y + \Delta y^2 \to 0$ as $\Delta x, \Delta y \to 0$ and $\varepsilon_2 = \Delta y \to 0$ as $\Delta x, \Delta y \to 0$. ◆

(6) Let

$$f(x,y) = \begin{cases} \frac{x^2y+xy^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. `(2 Marks)`

Solution. If $(x,y) \neq (0,0)$, then by the power, product and quotient rules,

$$\frac{\partial f}{\partial x}(x,y) = \frac{y^4 + 2xy^3 - x^2y^2}{(x^2+y^2)^2},$$

and

$$\frac{\partial f}{\partial y}(x,y) = \frac{x^4 - x^2y^2 + 2yx^3}{(x^2+y^2)^2}.$$

Also

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0,$$

and

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0}\frac{f(0,0+k) - f(0,0)}{k} = \lim_{k\to 0}\frac{0-0}{k} = 0.$$

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(b) Let $\vec{u} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Find the directional derivative of f at $(0,0)$ in the direction $\vec{u}$, i.e., compute $D_{\vec{u}}f(0,0)$. `(1 Mark)`

Solution.

$$\lim_{h\to 0}\frac{f(0+hu, 0+hv) - f(0,0)}{h} = \frac{u^2v + uv^2}{u^2+v^2} = \frac{\sqrt{3}+3}{8}.$$

◆

(c) Is f differentiable at $(0,0)$? `(3 Marks)`

Solution.

$$\begin{aligned}
&f(0+\Delta x, 0+\Delta y) - f(0,0) \\
&= \frac{(\Delta x)^2(\Delta y) + (\Delta x)(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} \\
&= \left(\frac{(\Delta x)(\Delta y)}{(\Delta x)^2 + (\Delta y)^2}\right)\Delta x + \left(\frac{(\Delta x)(\Delta y)}{(\Delta x)^2 + (\Delta y)^2}\right)\Delta y \\
&= \frac{\partial f}{\partial x}(0,0)\Delta x + \frac{\partial f}{\partial y}(0,0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y
\end{aligned}$$

where

$$\varepsilon_1 = \varepsilon_2 = \left(\frac{(\Delta x)(\Delta y)}{(\Delta x)^2 + (\Delta y)^2}\right).$$

However, by the two path test, it is **not true** that $\varepsilon_1 \to 0$ as $\Delta x, \Delta y \to 0$. Therefore f is **not differentiable** at $(0,0)$. ◆

(7) Let $D = \{(x,y) \mid x^2 + y^2 + z^2 \leq 16\}$ and suppose $f : D \to \mathbb{R}$ is given by

$$f(x,y) = x^2 + y^2 + z^2 - 4(x+y).$$

Find the absolute maximum points and the maximum value of f on D. Find the absolute minimum points and the minimum value of f on D. `(3 Marks)`

Solution. The question is wrong. Everyone gets 3 marks.

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(8) Let $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$.

(a) Find the critical points of f. `(2 Marks)`

Solution. We compute

$$\begin{aligned}
f_x &= 6x(y-1) \\
f_y &= 3x^2 + 3y^2 - 6y
\end{aligned}$$

Now $f_x = 0$ implies $x = 0$ or $y = 1$. If $x = 0$, then $f_y = 0$ implies $y = 0$ or $y = 2$. If $y = 1$ then $f_y = 0$ implies $x = 1$ or $x = -1$.
Therefore the critical points are $(0,0), (0,2), (1,1), (-1,1)$ ◆

(b) For each critical point, determine whether it is a local maximum, local minimum or a saddle point. `(4 Marks)`

Solution. We will use the second derivative test to classify the critical points.

$$\begin{aligned}
f_{xx} &= 6(y-1) \\
f_{xy} &= 6x \\
f_{yy} &= 6(y-1).
\end{aligned}$$

Therefore the discriminant is $D = f_{xx}f_{yy} - f_{xy}^2 = 36\left[(y-1)^2 - x^2\right]$.
At $(0,0)$, $f_{xx} = -6 < 0$ and $D = 36 > 0$, so $(0,0)$ is a local maximum.
At $(0,2)$, $f_{xx} = 6 > 0$ and $D = 36 > 0$, so $(0,2)$ is a local minimum.

At $(1,1)$, $f_{xx}=0$ and $D=-36<0$, so $(1,1)$ is a saddle point.
At $(-1,1)$, $f_{xx}=0$ and $D=-36<0$, so $(-1,1)$ is a saddle point. ◆

(9) Determine whether the following improper integrals converge.

(a) $\displaystyle\int_1^\infty \frac{1}{\sqrt{e^x-2^x}}\,dx$ `(3 Marks)`

Solution. Let $f(x)=\frac{1}{\sqrt{e^x-2^x}}$ and $g(x)=e^{-x/2}$. Then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{1}{\sqrt{1-(2/e)^x}}=1<\infty,$$

since $2<e$. Since $\int_1^\infty g(x)$ converges, it follows from the *Limit Comparison Test* that $\int_1^\infty f(x)$ converges ◆

(b) $\displaystyle\int_\pi^{4\pi}\frac{\sin x}{\sqrt[3]{\pi-x}}\,dx$ `(3 Marks)`

Solution. Since $\int_{2\pi}^{4\pi}\frac{\sin x}{\sqrt[3]{x-\pi}}\,dx$ is not improper, it suffices to consider the convergence of $\int_\pi^{2\pi}\frac{\sin x}{\sqrt[3]{x-\pi}}\,dx$. Since $0\le\sin x\le x$ on the interval $[0,\pi]$, we have

$$0\le\frac{\sin(\pi-x)}{\pi-x}(\pi-x)^{2/3}\le(\pi-x)^{2/3}.$$

Let $g(x)=(\pi-x)^{2/3}$. Since $\int_\pi^{2\pi}g(x)\,dx$ converges, it follows from the direct comparison test that $\int_\pi^{2\pi}f(x)\,dx$ converges. ◆

(10) Evaluate $\displaystyle\int_0^a\int_x^a\frac{x}{\sqrt{x^2+y^2}}\,dy\,dx,\quad a>0.$ `(3 Marks)`

Solution.

$$\begin{aligned}\int_0^a\int_x^a\frac{x\,dy\,dx}{\sqrt{x^2+y^2}}&=\int_0^a\int_0^y\frac{x}{\sqrt{x^2+y^2}}\,dx\,dy\\&=\int_0^a\int_{y^2}^{2y^2}\frac{du}{2\sqrt{u}}\,dy\\&=\int_0^a\left(\sqrt{2y^2}-\sqrt{y^2}\right)dy\\&=\int_0^a(\sqrt{2}-1)y\,dy\\&=\frac{\sqrt{2}-1}{2}a^2.\end{aligned}$$

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