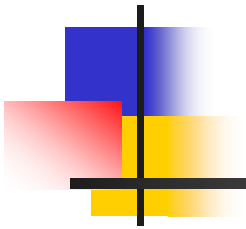
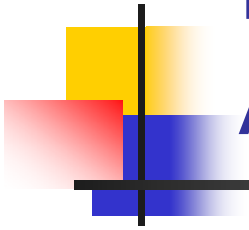


Regular Expressions

Reading: Chapter 3

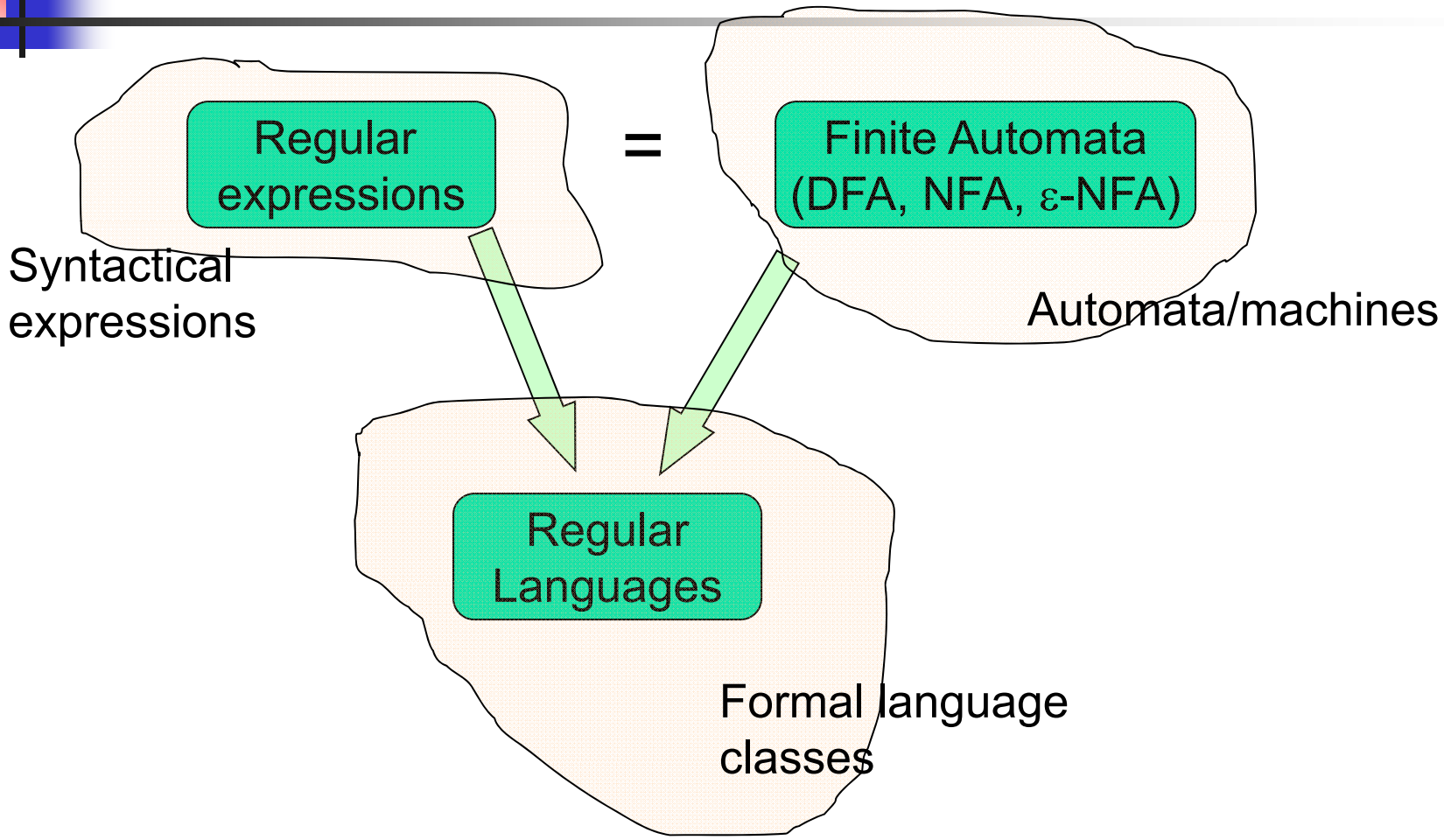
Regular Expressions vs. Finite Automata

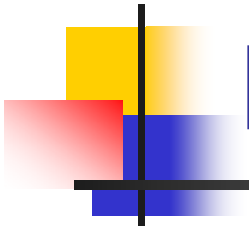


- Offers a declarative way to express the pattern of any string we want to accept
  - E.g., 01*+ 10*
- Automata => more machine-like

  < input: string , output: [accept/reject] >
- Regular expressions => more program syntax-like
- Unix environments heavily use regular expressions
  - E.g., bash shell, grep, vi & other editors, sed
- Perl scripting – good for string processing
- Lexical analyzers such as Lex or Flex

Regular Expressions





Language Operators

- Union of two languages:
  - **L U M** = all strings that are either in L or M
  - Note: A union of two languages produces a third language

- Concatenation of two languages:
  - **L . M** = all strings that are of the form *xy* s.t., x $\in$ L and y $\in$ M
  - The *dot* operator is usually omitted
    - i.e., **LM** is same as L.M

"i" here refers to how many strings to concatenate from the parent language L to produce strings in the language L^i

Kleene Closure (the * operator)

- Kleene Closure of a given language L:
  - $L^0 = \{\varepsilon\}$
  - $L^1 = \{w \mid \text{for some } w \in L\}$
  - $L^2 = \{ w_1w_2 \mid w_1 \in L, w_2 \in L \text{ (duplicates allowed)}\}$
  - $L^i = \{ w_1w_2\ldots w_i \mid \text{all w's chosen are} \in L \text{ (duplicates allowed)}\}$
  - (Note: the choice of each w_i is independent)
  - **$L^* = \bigcup_{i\geq 0} L^i$** (arbitrary number of concatenations)

Example:

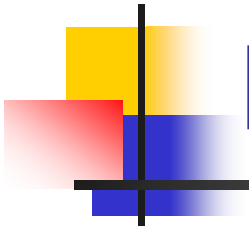
- Let L = { 1, 00}
  - $L^0 = \{\varepsilon\}$
  - $L^1 = \{1,00\}$
  - $L^2 = \{11,100,001,0000\}$
  - $L^3 = \{111,1100,1001,10000,000000,00001,00100,0011\}$
  - **$L^* = L^0 \cup L^1 \cup L^2 \cup \ldots$**

Kleene Closure (special notes)

- L^* is an infinite set iff $|L| \geq 1$ and $L \neq \{\varepsilon\}$ **Why?**
- If $L=\{\varepsilon\}$, then $L^* = \{\varepsilon\}$ **Why?**
- If $L = \Phi$, then $L^* = \{\varepsilon\}$ **Why?**

Σ^* denotes the set of all words over an alphabet Σ

- Therefore, an abbreviated way of saying there is an arbitrary language L over an alphabet Σ is:
  - $L \subseteq \Sigma^*$

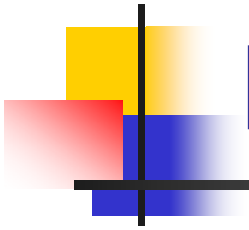


Building Regular Expressions

- Let E be a regular expression and the language represented by E is L(E)
- Then:
  - (E) = E
  - L(E + F) = L(E) U L(F)
  - L(E F) = L(E) L(F)
  - L(E*) = (L(E))*

Example: how to use these regular expression properties and language operators?

- ***L = { w | w is a binary string which does not contain two consecutive 0s or two consecutive 1s anywhere)***
  - E.g., w = 01010101 is in L, while w = 10010 is not in L
- Goal: Build a regular expression for L
- Four cases for w:
  - Case A: w starts with 0 and |w| is even
  - Case B: w starts with 1 and |w| is even
  - Case C: w starts with 0 and |w| is odd
  - Case D: w starts with 1 and |w| is odd
- Regular expression for the four cases:
  - Case A: (01)*
  - Case B: (10)*
  - Case C: 0(10)*
  - Case D: 1(01)*
- Since L is the union of all 4 cases:
  - Reg Exp for L = (01)* + (10)* + 0(10)* + 1(01)*
- If we introduce ε then the regular expression can be simplified to:
  - Reg Exp for L = (ε +1)(01)*(ε +0)



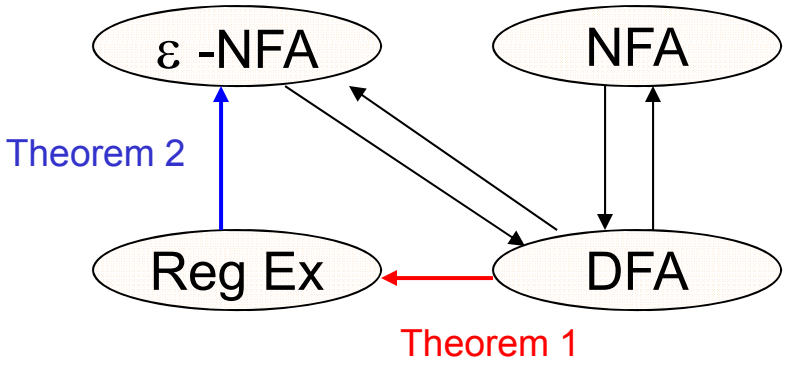
Precedence of Operators

- Highest to lowest
  - * operator (star)
  - . (concatenation)
  - + operator
- Example:
  - 01* + 1 = (0 . ((1)*)) + 1

Finite Automata (FA) & Regular Expressions (Reg Ex)

- To show that they are interchangeable, consider the following theorems:
  - *Theorem 1: For every DFA A there exists a regular expression R such that L(R)=L(A)*
  - *Theorem 2: For every regular expression R there exists an ε -NFA E such that L(E)=L(R)*

Proofs in the book

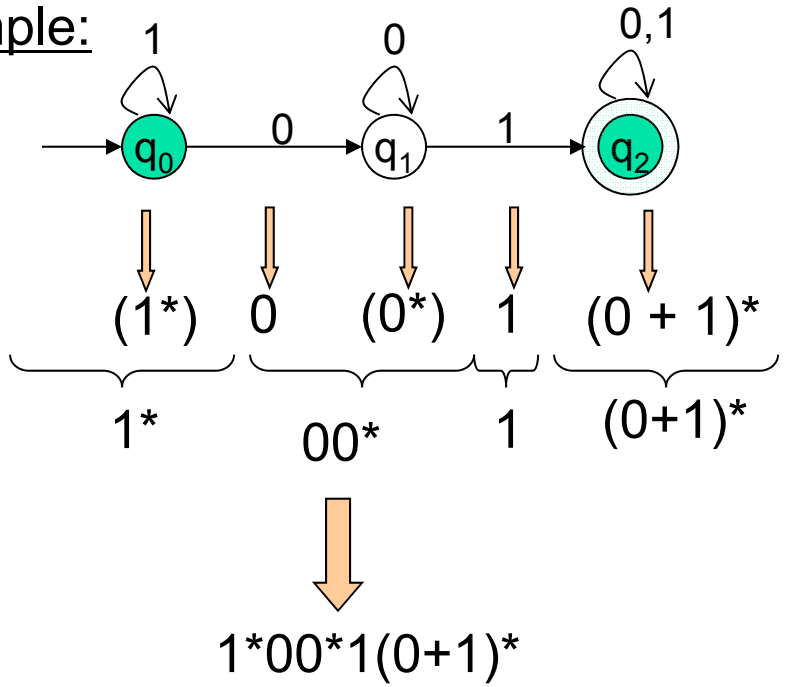


Kleene Theorem

DFA to RE construction

Informally, trace all distinct paths (traversing cycles only once)
from the start state to *each of the* final states
and enumerate all the expressions along the way

Example:

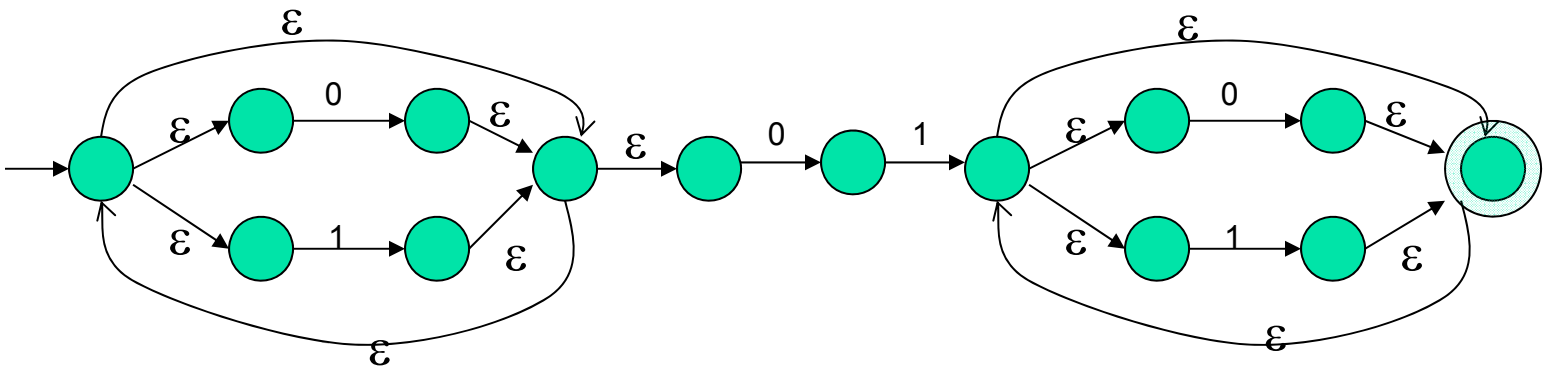


Q) What is the language?

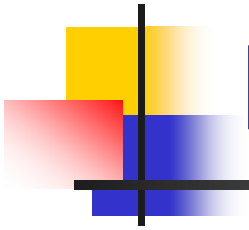
RE to ε-NFA construction

Example: (0+1)*01(0+1)*

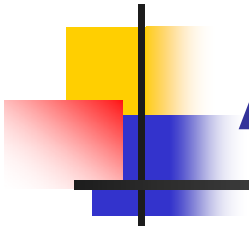
(0+1)* 01 (0+1)*



Algebraic Laws of Regular Expressions

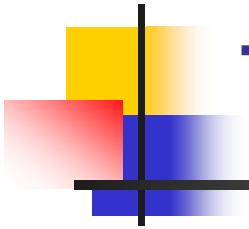


- Commutative:
  - $E+F = F+E$
- Associative:
  - $(E+F)+G = E+(F+G)$
  - $(EF)G = E(FG)$
- Identity:
  - $E+\Phi = E$
  - $\varepsilon E = E \varepsilon = E$
- Annihilator:
  - $\Phi E = E\Phi = \Phi$



Algebraic Laws…

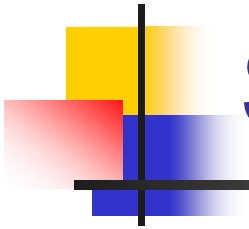
- Distributive:
  - E(F+G) = EF + EG
  - (F+G)E = FE+GE
- Idempotent: E + E = E
- Involving Kleene closures:
  - $(E^*)^*$ = E^*
  - Φ^* = ε
  - ε^* = ε
  - E^+ =EE^*
  - E? = ε +E



True or False?

Let R and S be two regular expressions. Then:

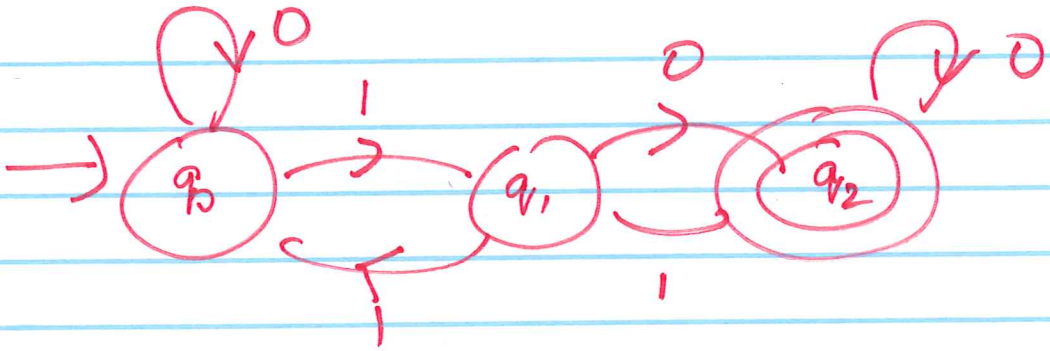
1. $((R^*)^*)^* = R^*$?
2. $(R+S)^* = R^* + S^*$?
3. $(RS + R)^* RS = (RR^*S)^*$?



Summary

- Regular expressions
- Equivalence to finite automata
- DFA to regular expression conversion
- Regular expression to ε-NFA conversion
- Algebraic laws of regular expressions
- Unix regular expressions and Lexical Analyzer

DFA:



1) Language Description:

All binary strings satisfying two conditions:

i) 10 is present as a substring; and

ii) the last occurrence of substring 11 (if exists) must be followed by at least one occurrence of 10.

2) Regular Expression:

Case A) 11 does not occur anywhere:
Case B) 11 occurs at least once

A) $0^* 10 (0+10)^*$

B) $(0+1)^* 11 \; 0^* 10 \; (0+10)^*$

Complete Reg. Exp(L) = $0^* 10 (0+10)^* + (0+1)^* 11 \; 0^* 10 (0+10)^*$

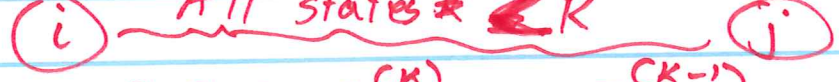
$= \boxed{(\varepsilon + (0+1)^* 11) \; 0^* 10 (0+10)^*}$

Q) How to convert a DFA $\Longrightarrow$ Reg. Exp.?

Approach:

Let $R_{ij}^{(k)}$ ← Reg. Exp. for all strings that go from state i to state j, visiting only states numbered no more than k.

case (i):



$$R.E: \quad R_{ij}^{(k)} = R_{ij}^{(k-1)}$$

case (ii):

$$R.E: \quad R_{ij}^{(k)} = R_{ik}^{(k-1)} \left(R_{kk}^{(k-1)}\right)^{*} R_{kj}^{(k-1)}$$

$$\therefore \quad R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)} \left(R_{kk}^{(k-1)}\right)^{*} R_{kj}^{(k-1)}$$

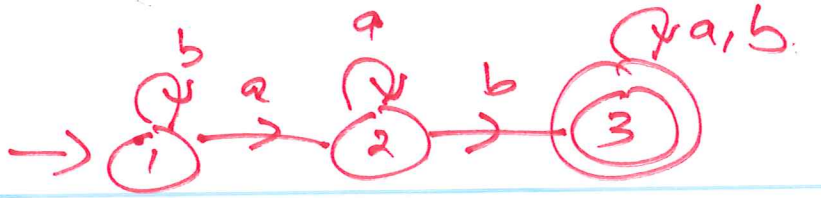
Assuming all states are numbered from 1 to n, s.t 1 is the start state, and f is any final state

$$\Rightarrow \boxed{\text{Reg. Exp}(DFA) = \bigcup_{\forall f \in F} R_{1f}^{(n)}}$$

Base case:

$$R_{ij}^{(0)} = \begin{cases} a_1 + a_2 + \dots a_k : & i \xrightarrow{a_1, a_2, \dots a_k} j \\ \varepsilon : & i = j \\ \emptyset & \end{cases}$$

Example:



Reg. Exp: $= R_{13}^{(3)}$

$$R_{13}^{(3)} = R_{13}^{(2)} + R_{13}^{(2)} (R_{33}^{(2)})^* R_{33}^{(2)}$$

$$R_{13}^{(2)} = R_{13}^{(1)} + R_{12}^{(1)} (R_{22}^{(1)})^* R_{23}^{(1)}$$

$$R_{13}^{(1)} = R_{13}^{(0)} + R_{11}^{(0)} (R_{11}^{(0)})^* R_{13}^{(0)}$$

$$= \phi + (\varepsilon + b)(\varepsilon + b)^* \phi = \phi \rightarrow (1)$$

$$R_{12}^{(1)} = R_{12}^{(0)} + R_{11}^{(0)} (R_{11}^{(0)})^* (R_{12}^{(0)})$$

$$= a + (\varepsilon + b)(\varepsilon + b)^* a = a + (\varepsilon + b) b^* a \rightarrow (2)$$

$$R_{22}^{(1)} = R_{22}^{(0)} + R_{21}^{(0)} (R_{11}^{(0)})^* R_{12}^{(0)}$$

$$= (\varepsilon + a) + \phi (R_{11}^{(0)})^* R_{12}^{(0)} = \varepsilon + a \rightarrow (3).$$

$$R_{23}^{(1)} = R_{23}^{(0)} + R_{21}^{(0)} (R_{11}^{(0)})^* R_{13}^{(0)}$$

$$= b + \phi - - = b \rightarrow (4)$$

$$\Rightarrow R_{13}^{(2)} = \phi + (a + (\varepsilon + b) b^* a)(\varepsilon + a)^* b$$

$$= (a + (\varepsilon + b) b^* a) a^* b$$

$$= (a + b^* a) a^* b = a^+ b + b^* a^+ b \rightarrow (5).$$

$$R_{33}^{(2)} = R_{33}^{(1)} + R_{32}^{(1)} (R_{22}^{(1)})^* R_{23}^{(1)} = R_{33}^{(1)}$$

$$R_{33}^{(1)} = R_{33}^{(0)} + R_{31}^{(0)} (R_{11}^{(0)})^* R_{13}^{(0)}$$

$$= (\varepsilon + a + b) + \phi \cdots = \varepsilon + a + b \rightarrow (6).$$

$$\Rightarrow R_{13}^{(3)} = (a^+ b + b^* a^+ b)(\varepsilon + (\varepsilon + a + b)^* (\varepsilon + a + b))$$

$$= (a^+ b + b^* a^+ b)(\varepsilon + (a + b)^*) \rightarrow \text{final}.$$