Mathematics-III

Module-I (18 hours)

Partial differential equation of first order, Linear partial differential equation, Non-linear partial differential equation, Homogenous and non-homogeneous partial differential equation with constant co-efficient, Cauchy type, Monge's method, Second order partial differential equation The vibrating string, the wave equation and its solution, the heat equation and its solution, Two dimensional wave equation and its solution, Laplace equation in polar, cylindrical and spherical coordinates, potential.

Module-II (12 hours)

Complex Analysis:
Analytic function, Cauchy-Riemann equations, Laplace equation, conformal mapping, Complex integration: Line integral in the complex plane, Cauchy's integral theorem, Cauchy's integral formula, Derivatives of analytic functions

Module –III (10 hours)

Power Series, Taylor's series, Laurent's series, Singularities and zeros, Residue integration method, evaluation of real integrals

Contents

Formation of Partial Differential Equations

In practice, there are <u>two methods</u> to form a partial differential equation.

(i) By the elimination of arbitrary constants.

(ii) By the elimination of arbitrary functions.

(i) <u>Formation of Partial Differential Equations by the elimination of arbitrary constants method:</u>

* Let f(x,y,z,a,b)=0 be an equation containing 2 arbitrary constants "a" and "b".

* To eliminate 2 constants we require at least 3 equations hence we partially differentiate the above equation with respect to (w.r.t) 'x' and w.r.t 'y' to obtain 2 more equations.

* From the three equations we can eliminate the constants "a" and "b".

NOTE 1: *If the number of arbitrary constants to be eliminated is equal to the number independent variables, elimination of constants gives a first order partial differential equation. But if the number of arbitrary constants to be eliminated is greater than the number of independent variables, then elimination of constants gives a second or higher order partial differential equation.*

NOTE 2: *In this chapter we use the following notations*:

$p = \partial z/\partial x$, $q = \partial z/\partial y$, $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/(\partial x \partial y)$, $t = \partial^2 z/\partial y^2$

METHOD TO SOLVE PROBLEMS:

Step 1: Differentiate the given question first w.r.t 'x' and then w.r.t 'y'.

Step 2: We know $p = \partial z/\partial x$ and $q = \partial z/\partial y$.

Step 3: Now find out a and b values in terms of p and q.

Step 4: Substitute these values in the given equation.

Step 5: Hence the final equation is in terms of p and q and free of arbitrary constants "a" and "b" which is the required partial differential equation.

(ii) Formation of Partial Differential Equations by the elimination of arbitrary functions method:

* Here it is the arbitrary function that gets eliminated instead of the arbitrary constants "a" and "b".

NOTE: *The elimination of 1 arbitrary function from a given partial differential equation gives a first order partial differential equation while the elimination of the 2 arbitrary functions from a given relation gives second or higher order partial differential equations*.

METHOD TO SOLVE PROBLEMS:

Step 1: Differentiate the given question first w.r.t 'x' and then w.r.t 'y'.

Step 2: We know p = ∂z/∂x and q = ∂z/∂y.

Step 3: Now find out the value of the differentiated function (f'') from both the equations separately. [(f'') =?]

Step 4: Equate the other side of the differentiated function (f'') which is in terms of p in one equation and q in other.

Step 5: Hence the final equation is in terms of p and q and free of the arbitrary function which is the required p.d.e.

** Incase there are 2 arbitrary functions involved, then do single differentiation i.e. p = ∂z/∂x, q = ∂z/∂y, then also do double differentiation i.e.* $r = \partial^2 z/\partial x^2,\ t = \partial^2 z/\partial y^2$ *and then eliminate (f'') and (f'''') from these equations.*

Worked out Examples

Elimination of arbitrary constants:

Ex 1:

$$or\ 2z = xz_x + yz_y\ which\ is\ the\ required\ p.d.e.$$

Elimination of arbitrary functions:

Ex 2: $\boldsymbol{Form\ the\ p.d.e\ from\ the\ following\ equation\ z = x^n f\left(\frac{y}{x}\right)}$

$$solution: By\ differentiation, z_x = nx^{n-1}.f + x^n\left(\frac{-y}{x^2}\right).f'$$

$$z_y = \frac{1}{x}x^n.f' \quad or \quad f' = \frac{z_y}{x^{n-1}}$$

$$Eliminating\ f', \quad z_x = nx^{n-1}.f - x^{n-2}.y.\frac{z_y}{x^{n-1}}$$

$$or\ z_x = nx^{n-1}.\frac{z}{x^n} - x^{n-2}.y.\frac{z_y}{x^{n-1}}$$

$$\mathrm{or}\ z_x = \frac{nz}{x} - \frac{yz_y}{x}$$

$$or\quad xp = nz - yq\ \ which\ is\ the\ required\ p.d.e.$$

Partial Differential Equations of First Order

The general form of first order p.d.e is F(x, y,z,p,q)=0 .... (1) where x,y are the independent variables and z is dependent variable .

Types of Solution of first order partial differential equations

Complete Solution

Any function f(x,y,z,a,b)=0 ....(2) involving arbitrary constants a and b satisfying p.d.e (1) is known as complete solution or complete integral of (1).

General Solution

Any function F (u,v)=0.......(3) satisfying p.d.e (1) is known as general solution or of (1).

Linear partial differential equations of First Order

Larange's Linear Equation

The equation of the form Pp+Qq=R......(1) where P,Q,R are functions of x,y,z is called Lagrange's partial differential equation. Any function F(u,v)=0 .....(2) where u=u(x,y,z) and v=v(x,y,z) satisfying (1) is the general solution .

Methods of obtaining General Solution

$$\boldsymbol{Step\ 1: Rewrite\ the\ equation\ in\ standard\ form\ Pp + Qq = R}$$

$$\boldsymbol{Step\ 2: Form\ the\ Lagrange's\ auxillary\ equation(A.E)\ \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \dots\dots(3)}$$

$$\boldsymbol{Step\ 3:\ u(x.y,z) = c_1\ and\ v(x.y,z) = c_2\quad where\ \frac{u}{v} \neq constant\ are\ the\ complete}$$

$$\boldsymbol{solutions\ of\ equations\ (3)}$$

$$\boldsymbol{To\ find\ u\ and\ v\ we\ follow\ the\ cases\ given\ below}$$

$$\boldsymbol{Case\ 1: One\ of\ the\ variables\ is\ either\ absent\ or\ cancells\ out\ from\ the\ set\ of\ auxillary\ equations.}$$

$$\boldsymbol{Case\ 2: If\ u = c_1\ is\ known\ but\ v = c_2\ is\ not\ possible\ by\ case 1,\ then\ use\ u = c_1\ to\ get\ v = c_2.}$$

$$\boldsymbol{Case\ 3: Introducing\ Lagrange's\ multipliers\ P_1, Q_1, R_1\ which\ are\ functions\ of\ x, y, z\ or\ constants,\ each\ fraction\ in\ (3) is\ equal\ to}$$

$\frac{P_1 dx +}{\quad}$ (4).

(4) (3)

(1) $F(u\ v)$

Worked Out Examples

$'$

— — —

— — —

— —

—

$F\left(- \ —\right)$

$\ln(x \quad y)$

$— \quad — \quad \frac{}{\ln(x \quad y)}$

$x ln(x \quad y)$

$(x \quad x ln(x \quad y) \quad z)$

$(\mathbf{z} \quad \mathbf{y})\mathbf{p} \quad (\mathbf{x} \quad \mathbf{z})\mathbf{q}$

$F(x \qquad)$

NON LINEAR P.D.Es OF FIRST ORDER

$f(p \; q)$

$f(a \; q)$

$\emptyset(a)$, $\emptyset(a)$

$\emptyset(a)y$

$Solution$: $The\ equation\ is\ of\ the\ form\ f(a,q) = 0$

$Putting\ p = a\ we\ have\ a^3 - q^3 = 0\ or\ q = a\ .Thus\ the\ solution\ is$

$$z = ax + ay + c$$

$Form\ 2$: $f(z,p,q) = 0$

$Assume\ that\ q = ap, substituting\ \ equation\ it\ in\ the\ equation$

$we\ have\ f(z,p,ap) = 0\ .Then\ p = \ \emptyset(z).Now\ dz = p\ dx + q\ dy$

$or\ \ dz = p\ dx + ap\ dy = p\ (dx + a\ dy\ )$

$or\ \frac{dz}{\emptyset(z)} = (dx + a\ dy\ )\ ,Integrating\ we\ get\ x + ay = \int \frac{dz}{\emptyset(z)} + c$

Ex : $Solve\ p^2z^2 + q^2 = \ p^2q$

$Solution$: $The\ equation\ is\ of\ the\ form\ f(z,p,q) = 0$

$Substituting\ q = ap\ in\ the\ equation\ we\ get\ p^2z^2 + a^2p^2 = \ p^2ap$

$$or\ p = \frac{(\ z^2 + a^2)}{a}\ .Hence\ the\ solution\ is\ \ x + ay = \int \frac{a\ dz}{(\ z^2 + a^2)} + c$$

$$= \tan^{-1}\frac{z}{a} + c$$

$Form\ 3$: $f(x,p) = g(y,q)$

$Assuming\ f(x,p) = g(y,q) = a\ \ we\ have\ \ p = \emptyset(x,a)\ and$

$q = \varphi(y,a).\ \ Now\ dz = p\ dx + q\ dy = \emptyset(x,a)\ dx + \varphi(y,a)\ dy$

$$Integrating\ we\ get\ \ z = \int \emptyset(x,a)\ dx + \int \varphi(y,a)\ dy + c$$

Ex : $Solve\ yp + xp + pq = 0$

$Solution$: $The\ equation\ is\ of\ the\ form\ \ (x + p)q = \ -yp$

$$or \frac{x + p}{p} = \frac{-y}{q} = a.Then\ p = \frac{x}{a - 1}\ and\ q = \frac{-y}{a}$$

$$Now\ dz = p\ dx + q\ dy = \frac{x}{a - 1}\ dx + \left(\frac{-y}{a}\right) dy$$

$$\int \frac{}{} \qquad \int \left(\frac{}{}\right) dy$$

$f(p\ q)$

$f(a\ b)$

$f(x \qquad q)$

$.(1)$

$$or\ \frac{dp}{-p(32p^2z+18q^2z+8z)}=\frac{dq}{q(32p^2z+18q^2z+8z)}=\frac{dx}{-p(32pz^2)-q(18qz^2)}$$
$$=\frac{dy}{-32pz^2}=\frac{dz}{-18qz^2}$$

$Taking\ the\ first\ and\ second\ fractions, \left(\frac{1}{p}\right)dp=\left(\frac{1}{q}\right)dq\ so\ that\quad p=aq\ \ldots..(2)$

$Solving\ (1)and(2)for\ p\ and\ q\ ,we\ have\ \ q=\dfrac{2(1-z^2)^{\frac{1}{2}}}{z(16a^2+9)^{\frac{1}{2}}}\ \ and$

$p=\dfrac{2a(1-z^2)^{\frac{1}{2}}}{z(16a^2+9)^{\frac{1}{2}}}\ .Hence\ dz=p\ dx+q\ dy=\dfrac{2(1-z^2)^{\frac{1}{2}}}{z(16a^2+9)^{\frac{1}{2}}}(a\ dx+dy)\,.$

$or\ \ \dfrac{z(16a^2+9)^{\frac{1}{2}}}{2(1-z^2)^{\frac{1}{2}}}\ dz=a\ dx+dy\ .Integrating\ we\ get$

$(16a^2+9)^{\frac{1}{2}}(1-z^2)^{\frac{1}{2}}+2(ax+y)=b\ which\ is\ the\ complete\ integral\ of(1).$

Homogenous partial differential Equations with constant coefficients

$General\ Form: A_0\dfrac{\partial^n z}{\partial x^n}+A_1\dfrac{\partial^{n-1}z}{\partial x^{n-1}\partial y}+\cdots+A_n\dfrac{\partial^n z}{\partial y^n}=f(x,y)\ \ldots(1)where$

$A_0, A_1\ \ldots\ldots A_n are\ constants\ .This\ equation\ can\ be\ rewritten\ as$

$(A_0D^n+A_1D^{n-1}D'+\cdots+\ A_nDD'^n)z=f(x,y)\ \ldots(2)$

$i,e\ F(D,D')z=f(x,y)\ \ \ldots(3)where\ D=\dfrac{\partial}{\partial x}\ ,D'=\dfrac{\partial}{\partial y}\ \ \ and$

$F(D,D')=(A_0D^n+A_1D^{n-1}D'+\cdots+\ A_nDD'^n)$

Solution of Homogenous partial differential Equations with constant coefficients

The complete solution of $eq^n(1)$is $z=z_c+z_p$ where $z_c$is called the

complementary function(C.F) which is the solution of $F(D, D')z = 0 \ldots (4)$

and z_p is the particular integral(P.I)which is the solution of

$F(D, D')z = f(x, y)$.

Working Rules to find the complementary function

Consider the equation $F(D, D')z = 0$. *Its auxillay* eq^n *is* $F(D, D') = 0$.

Putting $D = m$ *and* $D' = 1$ *we have the auxillay* eq^n *is* $F(m) = 0 \ldots\ldots (5)$

Solving (5) *we have* $= m_1, m_2, \ldots\ldots\ldots m_n$ *which are the roots of* (5).

Now the following cases arise.

Case 1: If all the roots are distinct then

$z_c = \emptyset_1(y + m_1x) + \emptyset_2(y + m_2x) + \cdots + \emptyset_n(y + m_nx)$ where

$\emptyset_1, \emptyset_2, \ldots \emptyset_n$ are arbitrary functions .

Case 2: If all the roots are equal say then

$$z_c = \emptyset_1(y + mx) + x\emptyset_2(y + mx) + \cdots + x^n\emptyset_n(y + mx)$$

Case 3: If k roots are equal say $m = m'$ *and others are distinct then*

$$z_c = \emptyset_1(y + m'x) + x\emptyset_2(y + m'x) + \cdots + x^k\emptyset_k(y + m'x) + \emptyset_{k+1}(y + m_{k+1}x) + \cdots + \emptyset_n(y + m_nx) .$$

Ex 1: Solve $(D^2 - 2DD' - 15D'^2)z = 0$

Solution: Putting $D = m$ *and* $D' = 1$ *in the given equation we have its*

auxillary equation is $m^2 - 2m - 15 = 0$. *Its roots are* $m = 5, -3$

Thus the solution of the equation is $z_c = \emptyset_1(y+5x) + \emptyset_2(y-3x)$.

Ex 2: *Solve* $\left(D^3 + 3D^2D' - 4D'^3\right)z = 0$

Solution: *Putting* $D = m$ *and* $D' = 1$ *in the given equation we have its auxillary equation is* $(m^3 + 3m^2 - 4) = 0$. *Solving* $m = 1, -2, -2$

Thus the solution of the equation is $z_c = \emptyset_1(y+x) + \emptyset_2(y-2x) + x\emptyset_3(y-2x)$.

Working Rules to find the Particular Integral.

Consider the equation $F(D, D')z = f(x,y)$. *Then* $z_p = \dfrac{1}{F(D,D')} f(x,y)$.

Case 1: *If* $f(x,y) = e^{ax+by}$ *then* $z_p = \dfrac{1}{F(D,D')} e^{ax+by}$ *or* $z_p = \dfrac{1}{F(a,b)} e^{ax+by}$

provided $F(a,b) \neq 0$.

If $F(a,b) = 0$, *then* $F(D,D') = \left(D - \frac{a}{b}D'\right)^r g(D,D')$ *so that* $g(a,b) \neq 0$.

Then $z_p = \dfrac{1}{g(a,b)} \dfrac{x^r}{r!} e^{ax+by}$

Case 2: *If* $f(x,y) = \sin(ax+by)$ *or* $\cos(ax+by)$

then $\dfrac{1}{F(D^2, DD', D'^2)} \sin(ax+by) = \dfrac{1}{F(-a^2, -ab, -b^2)} \sin(ax+by)$

provided $F(-a^2, -ab, -b^2) \neq 0$

Case 3: *If* $f(x,y) = x^m y^n$ *where m and n are positive constants.*

then $z_p = \dfrac{1}{F(D,D')} x^m y^n = F(D,D')^{-1} x^m y^n$

(*a*) *If* $n < m$, $\dfrac{1}{F(D,D')}$ *is expanded in powers of* $\dfrac{D'}{D}$.

$$(a) If\ m < n, \frac{1}{F(D,D')} is\ expanded\ in\ powers\ of\ \frac{D}{D'}.$$

$Case\ 4: If\ f(x,y) = e^{ax+by}V(x,y)$

$$then\ z_p = \frac{1}{F(D,D')}e^{ax+by}V(x,y) = e^{ax+by}\frac{1}{F(D+a,D'+b)}V(x,y)$$

Worked Out Examples

$Ex\ 1: Solve\ (D^2 + 5DD' + 6D'^2)z = e^{x-y}.$

$Solution: A.E\ is\ (m^2 + 5m + 6) = 0. Its\ roots\ are\ m = -3, -2$

$$then\ z_c = \emptyset_1(y-3x) + \emptyset_2(y-2x).$$

$$z_p = \frac{1}{(D^2 + 5DD' + 6D'^2)}e^{x-y} = \frac{1}{(1^2 + 5.1.(-1) + 6.(-1)^2)}e^{x-y}$$

$$= \frac{1}{2}e^{x-y}$$

$$Then\ the\ general\ solution\ is\ z = z = z_c + z_p$$

$$= \emptyset_1(y-3x) + \emptyset_2(y-2x) + \frac{1}{2}e^{x-y}.$$

$Ex\ 2: Solve\ (4D^2 + 12DD' + 9D'^2)z = e^{3x-2y}.$

$Solution: A.E\ is\ (4m^2 + 12m + 9) = 0. Its\ roots\ are\ m = -\frac{3}{2}, -\frac{3}{2}$

$$then\ z_c = \emptyset_1\left(y - \frac{3}{2}x\right) + x\emptyset_2\left(y - \frac{3}{2}x\right).$$

$$z_p = \frac{1}{(4D^2 + 12DD' + 9D'^2)}e^{3x-2y} = \frac{1}{4\left[D - \left(-\frac{3}{2}\right)D'\right]^2}e^{3x-2y}$$

$$= \frac{1}{4}\frac{x^2}{2!}e^{3x-2y}$$

$$Then\ the\ general\ solution\ is\ z = z = z_c + z_p$$

$$= \emptyset_1\left(y - \frac{3}{2}x\right) + x\emptyset_2\left(y - \frac{3}{2}x\right) + \frac{1}{4}\frac{x^2}{2!}e^{3x-2y}.$$

Ex 3: **Solve** $(D^2 - 2DD' + D'^2)z = \sin(x + 2y)$

Solution: *A.E is* $(m^2 - 2m + 1) = 0$. *Its roots are* $m = 1, 1$

then $z_c = \emptyset_1(y + x) + x\emptyset_2(y + x)$.

$$z_p = \frac{1}{(D^2 - 2DD' + D'^2)}\sin(x + 2y) = \frac{1}{(-1^2 - 2.(-1).2 - 2^2)}\sin(x + 2y)$$

$$= -\sin(x + 2y)$$

$$Then\ the\ general\ solution\ is\ z = z = z_c + z_p$$

$$= \quad \emptyset_1(y + x) + x\emptyset_2(y + x) - \sin(x + 2y).$$

Ex 4: **Solve** $(2D^2 - 5DD' + 2D'^2)z = 5\sin(2x + y)$

Solution: *A.E is* $(2m^2 - 5m + 2) = 0$. *Its roots are* $m = 2, \frac{1}{2}$

then $z_c = \emptyset_1(y + 2x) + x\emptyset_2\left(y + \frac{1}{2}x\right)$.

$$z_p = \frac{1}{(2D^2 - 5DD' + 2D'^2)}\ 5\sin(2x + y)$$

$$= \frac{5}{2}\frac{1}{\left(D - \frac{1}{2}D'\right)(D - 2D')}\ 5\sin(2x + y) \quad \text{as } F(-a^2, -ab, -b^2) \neq 0$$

$$\text{for } a = 2, b = \frac{1}{2}$$

$$\text{Now } z_p = \frac{5}{(2D - D')(D - 2D')}\ \sin(2x + y).$$

$$\text{Now let } u = \frac{5}{(2D - D')}\ \sin(2x + y) = 5\int \sin\left(2x + \frac{c - x}{2}\right)dx$$

$$= 5\int sin\left(\frac{3x+c}{2}\right)dx = 5.\frac{2}{3}\left(-cos\left(\frac{3x+c}{2}\right)\right) = -\frac{10}{3}cos(2x+y)$$

$$Hence\ z_p = \frac{-10}{3(D-2D')}cos(2x+y) = \frac{-10}{3}\int cos\ c\ dx$$

$$= \frac{-10x}{3}cos\ c = \frac{-10x}{3}cos\ (2x+y)$$

$$Then\ the\ general\ solution\ is\ z = z = z_c + z_p$$

$$= \emptyset_1(y+2x) + x\emptyset_2\left(y+\frac{1}{2}x\right) + \frac{-10x}{3}cos\ (2x+y).$$

$N:B-For\ the\ above\ example\ we\ apply\ the\ formula\ \dfrac{1}{D-mD'}f(x,y) = \int f(x,c-mx)dx\ ,then\ in\ the\ result\ c\ is\ substituted\ back\ by\ y+mx.$

$$Ex\ 5: Solve\ \left(D^2 - D'^2\right)z = x^2y^2$$

$$Solution: A.E\ is\ (m^2-1) = 0\ .\ \text{Its roots are } m = \pm 1.\ \text{So}$$

$$z_c = \emptyset_1(y+x) + x\emptyset_2(y-x).$$

$$Now\ z_p = \frac{1}{(D^2 - D'^2)}\ x^2y^2 = \frac{1}{D^2\left(1-\frac{D'^2}{D^2}\right)}x^2y^2$$

$$= \frac{1}{D^2}\left(1-\frac{D'^2}{D^2}\right)^{-1}\ x^2y^2$$

$$= \frac{1}{D^2}\left(1-\frac{D'^2}{D^2}+\frac{D'^4}{D^2}+\cdots\right)x^2y^2 = \frac{1}{D^2}\left(x^2y^2 - \frac{1}{6}x^4\right) = \frac{x^4}{12}y^2 - \frac{1}{180}x^6$$

$$Then\ the\ general\ solution\ is\ z = z = z_c + z_p$$

$$= \emptyset_1(y+x) + x\emptyset_2(y-x) + \frac{x^4}{12}y^2 - \frac{1}{180}x^6.$$

$Ex\ 6: Solve\ \left(D^2 - DD' - 2D'^2\right)z = (y-1)e^x$

$Solution: A.E\ is\ (m^2 - m - 2) = 0\ \text{or}\ m = -1, 2$

$Then\ z_c = \emptyset_1(y-x) + \emptyset_2(y+2x)$

$$Now\ z_p = \frac{1}{\left(D^2 - DD' - 2D'^2\right)}(y-1)e^x$$

$$= \frac{e^x}{(D+1)^2 - (D+1)D' - 2D'^2}(y-1)$$

$$= e^x \frac{1}{1 + \left(D^2 + 2D - DD' - D' - 2D'^2\right)}(y-1)$$

$$= e^x\left[1 - \left(\left(D^2 + 2D - DD' - D' - 2D'^2\right)\right)\right](y-1)$$

$$= e^x(y - 1 + 1) = ye^x.$$

$Then\ the\ general\ solution\ is\ z = z = z_c + z_p$

$$= \emptyset_1(y-x) + \emptyset_2(y+2x) + ye^x.$$

<u>Cauchy type Differential Equation</u>

The Partial differential Equation of the form

$F(xD, yD')z = f(x, y)$ *is Cauchy type differential equation. This equation can be transformed to the equations with constant coefficients by putting* $u = \ln x$ *and* $v = \ln y$ *so that* $xD = D_u$ *and* $x^2D^2 = D_u(D_u - 1)$ *and so on.*

Simillarly $yD' = D_v, y^2D'^2 = D_v(D_v - 1)$ *and so on. Here* $D_u = \frac{\partial}{\partial u}, D_v = \frac{\partial}{\partial v}$. *The transformed equation in u and v can be solved by previous methods*

Then in the solution we substitute back $u = \ln x$ *and* $v = \ln y$ *to get original solution.*

MONGE'S METHOD

In this section we shall solve the equation of the form $Rr + Ss + Tt = V \ldots (1)$ *by Monge's Method. Let us assume that an intermediate integral of the equation* (1) *exists and it is of the Form* $u = f(v) \ldots (2)$. *For any function z of x and y we have*

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy = r\,dx + s\,dy \ldots (3)$$

$$dq = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy = s\,dx + t\,dy \ldots (4)$$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = p\,dx + q\,dy \ldots (5)$$

$$\text{giving } r = \frac{dp - s\,dy}{dx} \quad \text{and } t = \frac{dq - s\,dx}{dy}$$

Substituting values of r and t in equation (1), *we have*

$$R\,\frac{dp - s\,dy}{dx} + Ss + T\,\frac{dq - s\,dx}{dy} = V$$

$$\text{or } R\,dp\,dy - Rs\,dy^2 + Ss\,dx\,dy + T\,dq\,dx - Ts\,dx^2 = V\,dx\,dy \quad \text{or}$$

$$(R\,dp\,dy + T\,dq\,dx - V\,dx\,dy) - s(Rdy^2 - S\,dx\,dy + T\,dx^2) = 0 \ldots (6).$$

Any relation between x, y and z that satisfies (6) *must necessarily*

$$satisfy\ the\ equations\ (R\,dp\,dy + T\,dq\,dx - V\,dx\,dy) = 0 \ldots (7)$$

$$and\ (Rdy^2 - S\,dx\,dy + T\,dx^2) = 0 \ldots (8)$$

The equations (7) *and* (8) *are known as Monge's subsidiary equations of equation* (1). *Therefore the complete solution of equation* (1) *also satisfies* (7) *and* (8) *and vice − versa. Now let us proceed to obtain solutions of equations* (7) *and* (8). *In general equation* (8) *can be resolved into two equations* $dy - m_1\,dx = 0$ *and* $dy - m_2 dx = 0$. *There now arises two cases.*

Case 1: *If* m_1 *and* m_2 *are distinct, then* $dy - m_1\,dx = 0$ *and equation* (7), *if necessary by use of* (5) *leads to two integrals of the type* $u_1 = a$ *and* $v_1 = b$. *These give an intermediate integral of the type* $u_1 = f(v_1) \ldots (9)$. *Simillarly* $dy - m_2 dx = 0$ *and equation* (7) *lead to another intermediate integral of the type* $u_2 = f(v_2) \ldots (10)$.

Values of p and q in general can be determined in terms of x, y and z which when substituted in equation (5) *and integrated gives the complete integral of equation* (1).

Case 2: *If* $m_1 = m_2$, *then equation* (8) *is a perfect square, we get only one intermediate integral containing p, q and x, y, z of the form* $Pp + Qq = R$. *The solution can now be obtained by forming Lagrange's subsidiary equations.*

$(\boldsymbol{i})\boldsymbol{an}$

$(\boldsymbol{ii})$

$(\boldsymbol{ii})\boldsymbol{gi} \quad (\boldsymbol{dy} \quad \boldsymbol{dx})(\boldsymbol{dy} \quad \boldsymbol{dx})$

$(\boldsymbol{i})\boldsymbol{gi}$

$(\boldsymbol{i})$

$(\boldsymbol{x} \quad \boldsymbol{y})\boldsymbol{an}$

$(\boldsymbol{x} \quad \boldsymbol{y}).$

$$- \quad (\boldsymbol{x} \quad \boldsymbol{y}) \quad - \quad (\boldsymbol{x} \quad \boldsymbol{y})$$

$$- \quad (\boldsymbol{x} \quad \boldsymbol{y}) \quad - \quad (\boldsymbol{x} \quad \boldsymbol{y}).$$

$$- \quad (\boldsymbol{x} \quad \boldsymbol{y})\boldsymbol{d}(\boldsymbol{x} \quad \boldsymbol{y}) \quad - \quad (\boldsymbol{x} \quad \boldsymbol{y})\boldsymbol{d}(\boldsymbol{x} \quad \boldsymbol{y})$$

$$(\boldsymbol{x} \quad \boldsymbol{y}) \qquad (\boldsymbol{x} \quad \boldsymbol{y})$$

$$are \quad q dp\, dy = 0 \ldots (i) and$$

$$q dy^2 + p\, dx\, dy = 0 \ldots (ii)$$

$$Eq^n (ii) gives\ p\, dx + q\, dy = 0$$

$$But\ dz = p\, dx + q\, dy = 0\ or\ z = const. Again\ (i) gives\ dp = 0$$

$$or\ p = const. \quad \therefore p = f_1\,(z) is\ one\ intermediate\ integral.$$

$But\ p = f_1\,(z)\ is\ linear\ of\ first\ order. By\ Lagrange's method, we\ have$

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dz}{f_1\,(z)}$$

$$\therefore dy = 0\ or\ y = c_1$$

$$and\ dx = \frac{dz}{f_1\,(z)}\ . Integrating\ f_2(z) + c_2 = x.$$

$$Hence\ the\ general\ solution\ is\ x - f_2(z) = -g(y)\ or\ z = f\{x + g(y)\}.$$

Partial Differential Equations

In this chapter we are going to take a very brief look at one of the more common methods for solving simple partial differential equations.

The Heat Equation

The first partial differential equation that we'll be looking at once we get started with solving will be the heat equation, which governs the temperature distribution in an object. We are going to give several forms of the heat equation for reference purposes, but we will only be really solving one of them.

We will start out by considering the temperature in a 1-D bar of length *L*. What this means is that we are going to assume that the bar starts off at $x = 0$ and ends when we reach $x = L$. We are also going to so assume that at any location, *x* the temperature will be constant an every point in the cross section at that *x*. In other words, temperature will only vary in *x* and we can hence consider the bar to be a 1-D bar. Note that with this assumption the actual shape of the cross section (*i.e.* circular, rectangular, *etc.*) doesn't matter.

Note that the 1-D assumption is actually not all that bad of an assumption as it might seem at first glance. If we assume that the lateral surface of the bar is perfectly insulated (*i.e.* no heat can flow through the lateral surface) then the only way heat can enter or leave the bar as at either end. This means that heat can only flow from left to right or right to left and thus creating a 1-D temperature distribution.

The assumption of the lateral surfaces being perfectly insulated is of course impossible, but it is possible to put enough insulation on the lateral surfaces that there will be very little heat flow through them and so, at least for a time, we can consider the lateral surfaces to be perfectly insulated.

Let's now get some definitions out of the way before we write down the first form of the heat equation.

$$\begin{aligned}
u(x,t) &= \text{Temperature at any point } x \text{ and any time } t \\
c(x) &= \text{Specific Heat} \\
\rho(x) &= \text{Mass Density} \\
\varphi(x,t) &= \text{Heat Flux} \\
Q(x,t) &= \text{Heat energy generated per unit volume per unit time}
\end{aligned}$$

The mass density $\rho(x)$, is the mass per unit volume of the material. As with the specific heat we're going to initially assume that the mass density may not be uniform throughout the bar.

The heat flux, $\varphi(x,t)$, is the amount of thermal energy that flows to the right per unit surface area per unit time. The "flows to the right" bit simply tells us that if $\varphi(x,t) > 0$ for some *x* and *t* then the heat is flowing to the right at that point and time. Likewise if $\varphi(x,t) < 0$ then the heat will be flowing to the left at that point and time.

The final quantity we defined above is $Q(x,t)$ and this is used to represent any external sources or sinks (*i.e.* heat energy taken out of the system) of heat energy. If $Q(x,t) > 0$ then heat energy is being added to the system at that location and time and if $Q(x,t) < 0$ then heat energy is being removed from the system at that location and time.

With these quantities the heat equation is,

$$c(x)\rho(x)\frac{\partial u}{\partial t} = -\frac{\partial \varphi}{\partial x} + Q(x,t) \tag{1}$$

While this is a nice form of the heat equation it is not actually something we can solve. In this form there are two unknown functions, *u* and φ, and so we need to get rid of one of them. With **Fourier's law** we can easily remove the heat flux from this equation.

Fourier's law states that,

$$\varphi(x,t) = -K_0(x)\frac{\partial u}{\partial x}$$

where $K_0(x) > 0$ is the **thermal conductivity** of the material and measures the ability of a given material to conduct heat. The better a material can conduct heat the larger $K_0(x)$ will be. As noted the thermal conductivity can vary with the location in the bar. Also, much like the specific heat the thermal conductivity can vary with temperature, but we will assume that the total temperature change is not so great that this will be an issue and so we will assume for the purposes here that the thermal conductivity will not vary with temperature.

Fourier's law does a very good job of modeling what we know to be true about heat flow. First, we know that if the temperature in a region is constant, *i.e.* $\frac{\partial u}{\partial x} = 0$, then there is no heat flow.

Next, we know that if there is a temperature difference in a region we know the heat will flow from the hot portion to the cold portion of the region. For example, if it is hotter to the right then we know that the heat should flow to the left. When it is hotter to the right then we also know that $\frac{\partial u}{\partial x} > 0$ (*i.e.* the temperature increases as we move to the right) and so we'll have $\varphi < 0$ and so the heat will flow to the left as it should. Likewise, if $\frac{\partial u}{\partial x} < 0$ (*i.e.* it is hotter to the left) then we'll have $\varphi > 0$ and heat will flow to the right as it should.

Finally, the greater the temperature difference in a region (*i.e.* the larger $\frac{\partial u}{\partial x}$ is) then the greater the heat flow.

So, if we plug Fourier's law into (1), we get the following form of the heat equation,

$$c(x)\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right) + Q(x,t) \tag{2}$$

Note that we factored the minus sign out of the derivative to cancel against the minus sign that was already there. We cannot however, factor the thermal conductivity out of the derivative since it is a function of *x* and the derivative is with respect to *x*.

Solving (2) is quite difficult due to the non uniform nature of the thermal properties and the mass density. So, let's now assume that these properties are all constant, *i.e.*,

$$c(x) = c \qquad \rho(x) = \rho \qquad K_0(x) = K_0$$

where *c*, ρ and K_0 are now all fixed quantities. In this case we generally say that the material in the bar is **uniform**. Under these assumptions the heat equation becomes,

$$c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad (3)$$

For a final simplification to the heat equation let's divide both sides by $c\rho$ $c\rho$ and define the **thermal diffusivity** to be,

$$k = \frac{K_0}{c\rho}$$

The heat equation is then,

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + \frac{Q(x,t)}{c\rho} \qquad (4)$$

To most people this is what they mean when they talk about the heat equation and in fact it will be the equation that we'll be solving. Well, actually we'll be solving (4) with no external sources, *i.e.* $Q(x,t) = 0$, but we'll be considering this form when we start discussing separation of variables in a couple of sections. We'll only drop the sources term when we actually start solving the heat equation.

Now that we've got the 1-D heat equation taken care of we need to move into the initial and boundary conditions we'll also need in order to solve the problem. .

The initial condition that we'll use here is,

$$u(x,0) = f(x)$$

and we don't really need to say much about it here other than to note that this just tells us what the initial temperature distribution in the bar is.

The boundary conditions will tell us something about what the temperature and/or heat flow is doing at the boundaries of the bar. There are four of them that are fairly common boundary conditions.

The first type of boundary conditions that we can have would be the **prescribed temperature** boundary conditions, also called **Dirichlet conditions**. The prescribed temperature boundary conditions are,

$$u(0,t) = g_1(t) \qquad\qquad u(L,t) = g_2(t)$$

The next type of boundary conditions are **prescribed heat flux**, also called **Neumann conditions**. Using Fourier's law these can be written as,

$$-K_0(0)\frac{\partial u}{\partial x}(0,t) = \varphi_1(t) \qquad\qquad -K_0(L)\frac{\partial u}{\partial x}(L,t) = \varphi_2(t)$$

If either of the boundaries are **perfectly insulated,** *i.e.* there is no heat flow out of them then these boundary conditions reduce to,

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad\qquad \frac{\partial u}{\partial x}(L,t) = 0$$

and note that we will often just call these particular boundary conditions **insulated** boundaries and drop the "perfectly" part.

The third type of boundary conditions use **Newton's law of cooling** and are sometimes called **Robins conditions**. These are usually used when the bar is in a moving fluid and note we can consider air to be a fluid for this purpose.

Here are the equations for this kind of boundary condition.

$$-K_0(0)\frac{\partial u}{\partial x}(0,t) = -H\left[u(0,t) - g_1(t)\right] \qquad -K_0(L)\frac{\partial u}{\partial x}(L,t) = H\left[u(L,t) - g_2(t)\right]$$

where *H* is a positive quantity that is experimentally determined and $g_1(t)$ and $g_2(t)$ give the temperature of the surrounding fluid at the respective boundaries.

Note that the two conditions do vary slightly depending on which boundary we are at. At $x = 0$ $x = 0$ we have a minus sign on the right side while we don't at $x = L$. To see why this is let's first assume that at $x = 0$ we have $u(0,t) > g_1(t)$. In other words the bar is hotter than the surrounding fluid and so at $x = 0$ the heat flow (as given by the left side of the equation) must be to the left, or negative since the heat will flow from the hotter bar into the cooler surrounding liquid. If the heat flow is negative then we need to have a minus sign on the right side of the equation to make sure that it has the proper sign.

If the bar is cooler than the surrounding fluid at $x = 0$, *i.e.* $u(0,t) < g_1(t)$ we can make a similar argument to justify the minus sign. We'll leave it to you to verify this.

If we now look at the other end, $x = L$, and again assume that the bar is hotter than the surrounding fluid or, $u(L,t) > g_2(t)$. In this case the heat flow must be to the right, or be positive, and so in this case we can't have a minus sign. Finally, we'll again leave it to you to verify that we can't have the minus sign at $x = L$ is the bar is cooler than the surrounding fluid as well.

The final type of boundary conditions that we'll need here are **periodic** boundary conditions. Periodic boundary conditions are,

$$u(-L,t) = u(L,t) \qquad\qquad \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$$

Note that for these kinds of boundary conditions the left boundary tends to be $x = -L$ instead of $x = 0$ as we were using in the previous types of boundary conditions. The periodic boundary conditions will arise very naturally from a couple of particular geometries that we'll be looking at down the road.

We will now close out this section with a quick look at the 2-D and 3-D version of the heat equation. However, before we jump into that we need to introduce a little bit of notation first.

The **del** operator is defined to be,

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} \qquad\qquad \nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

depending on whether we are in 2 or 3 dimensions. Think of the del operator as a function that takes functions as arguments (instead of numbers as we're used to). Whatever function we "plug" into the operator gets put into the partial derivatives.

So, for example in 3-D we would have,

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial k}\vec{k}$$

This of course is also the gradient of the function $f(x,y,z)$.

The del operator also allows us to quickly write down the divergence of a function. So, again using 3-D as an example the divergence of $f(x,y,z)$ can be written as the dot product of the del operator and the function. Or,

$$\nabla \cdot f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial k}$$

Finally, we will also see the following show up in the our work,

$$\nabla \cdot (\nabla f) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial k}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

This is usually denoted as,

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

and is called the **Laplacian**. The 2-D version of course simply doesn't have the third term.

Okay, we can now look into the 2-D and 3-D version of the heat equation and where ever the del operator and or Laplacian appears assume that it is the appropriate dimensional version.

The higher dimensional version of (1) is,

$$c\rho\frac{\partial u}{\partial t} = -\nabla \cdot \varphi + Q \tag{5}$$

and note that the specific heat, *c*, and mass density, ρ, are may not be uniform and so may be functions of the spatial variables. Likewise, the external sources term, *Q*, may also be a function of both the spatial variables and time.

Next, the higher dimensional version of Fourier's law is,

$$\varphi = -K_0 \nabla u$$

where the thermal conductivity, K_0, is again assumed to be a function of the spatial variables.

If we plug this into (5) we get the heat equation for a non uniform bar (*i.e.* the thermal properties may be functions of the spatial variables) with external sources/sinks,

$$c\rho\frac{\partial u}{\partial t} = \nabla \cdot \left(K_0 \nabla u\right) + Q \tag{6}$$

If we now assume that the specific heat, mass density and thermal conductivity are constant (*i.e.* the bar is uniform) the heat equation becomes,

$$\frac{\partial u}{\partial t} = k\nabla^2 u + \frac{Q}{cp} \tag{7}$$

where we divided both sides by $c\rho$ to get the thermal diffusivity, *k* in front of the Laplacian.

The initial condition for the 2-D or 3-D heat equation is,

$$u\left(x,y,t\right) = f\left(x,y\right) \qquad \text{or} \qquad u\left(x,y,z,t\right) = f\left(x,y,z\right)$$

depending upon the dimension we're in.

The prescribed temperature boundary condition becomes,

$$u\left(x,y,t\right) = T\left(x,y,t\right) \qquad \text{or} \qquad u\left(x,y,z,t\right) = T\left(x,y,z,t\right)$$

where $\left(x,y\right)$ or $\left(x,y,z\right)$, depending upon the dimension we're in, will range over the portion of the boundary in which we are prescribing the temperature.

The prescribed heat flux condition becomes,

$$-K_0 \nabla u \cdot \vec{n} = \varphi\left(t\right)$$

where the left side is only being evaluated at points along the boundary and $\vec{n}$ $\vec{n}$ is the outward unit normal on the surface.

Newton's law of cooling will become,

$$-K_0 \nabla u \cdot \vec{n} = H\left(u - u_B\right)$$

where H is a positive quantity that is experimentally determine, u_B u_B is the temperature of the fluid at the boundary and again it is assumed that this is only being evaluated at points along the boundary.

We don't have periodic boundary conditions here as they will only arise from specific 1-D geometries.

We should probably also acknowledge at this point that we'll not actually be solving (7) at any point, but we will be solving a special case of it in the Laplace's Equation section.

The Wave Equation

In this section we want to consider a vertical string of length L that has been tightly stretched between two points at $x = 0$ and $x = L$.

Because the string has been tightly stretched we can assume that the slope of the displaced string at any point is small. So just what does this do for us? Let's consider a point x on the string in its equilibrium position, *i.e.* the location of the point at $t = 0$. As the string vibrates this point will be displaced both vertically and horizontally, however, if we assume that at any point the slope of the string is small then the horizontal displacement will be very small in relation to the vertical displacement. This means that we can now assume that at any point x on the string the displacement will be purely vertical. So, let's call this displacement $u(x,t)$.

We are going to assume, at least initially, that the string is not uniform and so the mass density of the string, $\rho(x)$ may be a function of x.

Next we are going to assume that the string is perfectly flexible. This means that the string will have no resistance to bending. This in turn tells us that the force exerted by the string at any point x on the endpoints will be tangential to the string itself. This force is called the **tension** in the string and its magnitude will be given by $T(x,t)$.

Finally, we will let $Q(x,t)$ represent the vertical component per unit mass of any force acting on the string. Provided we again assume that the slope of the string is small the vertical displacement of the string at any point is then given by,

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x,t)\frac{\partial u}{\partial x}\right) + \rho(x)Q(x,t) \tag{1}$$

This is a very difficult partial differential equation to solve so we need to make some further simplifications.

First, we're now going to assume that the string is perfectly elastic. This means that the magnitude of the tension, $T(x,t)$, will only depend upon how much the string stretches near x.

Again, recalling that we're assuming that the slope of the string at any point is small this means that the tension in the string will then very nearly be the same as the tension in the string in its equilibrium position. We can then assume that the tension is a constant value, $T(x,t) = T_0$.

Further, in most cases the only external force that will act upon the string is gravity and if the string light enough the effects of gravity on the vertical displacement will be small and so will also assume that $Q(x,t) = 0$. This leads to

$$\rho \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2}$$

If we know divide by the mass density and define,

$$c^2 = \frac{T_0}{\rho}$$

we arrive at the 1-D **wave equation**,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{2}$$

In the previous section when we looked at the heat equation he had a number of boundary conditions however in this case we are only going to consider one type of boundary conditions. For the wave equation the only boundary condition we are going to consider will be that of prescribed location of the boundaries or,

$$u(0,t) = h_1(t) \qquad\qquad u(L,t) = h_2(t)$$

The initial conditions (and yes we meant more than one…) will also be a little different here from what we saw with the heat equation. Here we have a 2nd order time derivative and so we'll also need two initial conditions. At any point we will specify both the initial displacement of the string as well as the initial slope of the string. The initial conditions are then,

$$u(x,0) = f(x) \qquad\qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

For the sake of completeness we'll close out this section with the 2-D and 3-D version of the wave equation. We'll not actually be solving this at any point, but since we gave the higher dimensional version of the heat equation (in which we will solve a special case) we'll give this as well.

The 2-D and 3-D version of the wave equation is,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

where ∇^2 is the Laplacian.

Separation of Variables

Okay, it is finally time to at least start discussing one of the more common methods for solving basic partial differential equations. The method of **Separation of Variables** cannot always be used and even when it can be used it will not always be possible to get much past the first step in the method. However, it can be used to easily solve the 1-D heat equation with no sources, the 1-D wave equation, and the 2-D version of Laplace's Equation, $\nabla^2 u = 0$.

In order to use the method of separation of variables we must be working with a linear homogenous partial differential equations with linear homogeneous boundary conditions. At this point we're not going to worry about the initial condition(s) because the solution that we initially get will rarely satisfy the initial condition(s). As we'll see however there are ways to generate a solution that will satisfy initial condition(s) provided they meet some fairly simple requirements.

The method of separation of variables relies upon the assumption that a function of the form,

$$u(x,t) = \varphi(x)G(t) \tag{1}$$

will be a solution to a linear homogeneous partial differential equation in *x* and *t*. This is called a **product solution** and provided the boundary conditions are also linear and homogeneous this will also satisfy the boundary conditions. However, as noted above this will only rarely satisfy the initial condition, but that is something for us to worry about in the next section.

So, let's do a couple of examples to see how this method will reduce a partial differential equation down to two ordinary differential equations.

Example 1 Use Separation of Variables on the following partial differential equation.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad u(0,t) = 0 \qquad u(L,t) = 0$$

Solution

So, we have the heat equation with no sources, fixed temperature boundary conditions (that are also homogeneous) and an initial condition. The initial condition is only here because it belongs here, but we will be ignoring it until we get to the next section.

The method of separation of variables tells us to assume that the solution will take the form of the product,

$$u(x,t) = \varphi(x)G(t)$$

so all we really need to do here is plug this into the differential equation and see what we get.

$$\frac{\partial}{\partial t}\left(\varphi(x)G(t)\right) = k\frac{\partial^2}{\partial x^2}\left(\varphi(x)G(t)\right)$$
$$\varphi(x)\frac{dG}{dt} = kG(t)\frac{d^2\varphi}{dx^2}$$

As shown above we can factor the $\varphi(x)$ out of the time derivative and we can factor the $G(t)$ out of the spatial derivative. Also notice that after we've factored these out we no longer have a partial derivative left in the problem. In the time derivative we are now differentiating only $G(t)$ with respect to *t* and this is now an ordinary derivative. Likewise, in the spatial derivative we are now only differentiating $\varphi(x)$ with respect to *x* and so we again have an ordinary derivative.

At this point it probably doesn't seem like we've done much to simplify the problem. However, just the fact that we've gotten the partial derivatives down to ordinary derivatives is liable to be good thing even if it still looks like we've got a mess to deal with.

Speaking of that apparent (and yes I said apparent) mess, is it really the mess that it looks like? The idea is to eventually get all the *t*'s on one side of the equation and all the *x*'s on the other side. In other words we want to "separate the variables" and hence the name of the method. In this case let's notice that if we divide both sides by $\varphi(x)G(t)$ we get want we want and we should point out that it won't always be as easy as just dividing by the product solution. So, dividing out gives us,

$$\frac{1}{G}\frac{dG}{dt} = k\frac{1}{\varphi}\frac{d^2\varphi}{dx^2} \qquad \Rightarrow \qquad \frac{1}{kG}\frac{dG}{dt} = \frac{1}{\varphi}\frac{d^2\varphi}{dx^2}$$

Notice that we also divided both sides by *k*. This was done only for convenience down the road. It doesn't have to be done and nicely enough if it turns out to be a bad idea we can always come back to this step and put it back on the right side. Likewise, if we don't do it and it turns out to maybe not be such a bad thing we can always come back and divide it out. For the time being however, please accept our word that this was a good thing to do for this problem. We will discuss the reasoning for this after we're done with this example.

Now, while we said that this is what we wanted it still seems like we've got a mess. Notice however that the left side is a function of only *t* and the right side is a function only of *x* as we wanted. Also notice these two functions must be equal.

Let's think about this for a minute. How is it possible that a function of only t's can be equal to a function of only x's regardless of the choice of t and/or x that we have? This may seem like an impossibility until you realize that there is one way that this can be true. If both functions (*i.e.* both sides of the equation) were in fact constant and not only a constant, but the same constant then they can in fact be equal.

So, we must have,

$$\frac{1}{kG}\frac{dG}{dt} = \frac{1}{\varphi}\frac{d^2\varphi}{dx^2} = -\lambda$$

where the $-\lambda$ is called the **separation constant** and is arbitrary.

The next question that we should now address is why the minus sign? Again, much like the dividing out the k above, the answer is because it will be convenient down the road to have chosen this. The minus sign doesn't have to be there and in fact there are times when we don't want it there.

So how do we know it should be there or not? The answer to that is to proceed to the next step in the process (which we'll see in the next section) and at that point we'll know if would be convenient to have it or not and we can come back to this step and add it in or take it our depending what we chose to do here.

Okay, let's proceed with the process. The next step is to acknowledge that we can take the equation above and split it into the following two ordinary differential equations.

$$\frac{dG}{dt} = -k\lambda G \qquad\qquad \frac{d^2\varphi}{dx^2} = -\lambda\varphi$$

Both of these are very simple differential equations, however because we don't know what λ λ is we actually can't solve the spatial one yet. The time equation however could be solved at this point if we wanted to, although that won't always be the case. At this point we don't want to actually think about solving either of these yet however.

The last step in the process that we'll be doing in this section is to also make sure that our product solution, $u(x,t) = \varphi(x)G(t)$, satisfies the boundary conditions so let's plug it into both of those.

$$u(0,t) = \varphi(0)G(t) = 0 \qquad\qquad u(L,t) = \varphi(L)G(t) = 0$$

Let's consider the first one for a second. We have two options here. Either $\varphi(0) = 0$ or $G(t) = 0$ for every t. However, if we have $G(t) = 0$ for every t then we'll also have $u(x,t) = 0$, *i.e.* the trivial solution, and as we discussed in the previous section this is definitely a solution to any linear homogeneous equation we would really like a non-trivial solution.

Therefore we will assume that in fact we must have $\varphi(0)=0$. Likewise, from the second boundary condition we will get $\varphi(L)=0$ to avoid the trivial solution. Note as well that we were only able to reduce the boundary conditions down like this because they were homogeneous. Had they not been homogeneous we could not have done this.

So, after applying separation of variables to the given partial differential equation we arrive at a 1st order differential equation that we'll need to solve for $G(t)$ and a 2nd order boundary value problem that we'll need to solve for $\varphi(x)$. The point of this section however is just to get to this point and we'll hold off solving these until the next section.

Let's summarize everything up that we've determined here.

$$\frac{dG}{dt} = -k\lambda G \qquad \qquad \frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$$
$$\varphi(0) = 0 \qquad \varphi(L) = 0$$

and note that we don't have a condition for the time differential equation and is not a problem. Also note that we rewrote the second one a little.

So just what have we learned here? By using separation of variables we were able to reduce our linear homogeneous partial differential equation with linear homogeneous boundary conditions down to an ordinary differential equation for one of the functions in our product solution (1), $G(t)$ in this case, and a boundary value problem that we can solve for the other function, $\varphi(x)$ in this case.

Example 2 Use Separation of Variables on the following partial differential equation.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x) \qquad \frac{\partial u}{\partial x}(0,t) = 0 \qquad \frac{\partial u}{\partial x}(L,t) = 0$$

Solution

In this case we're looking at the heat equation with no sources and perfectly insulated boundaries.So, we'll start off by again assuming that our product solution will have the form,

$$u(x,t) = \varphi(x)G(t)$$

and because the differential equation itself hasn't changed here we will get the same result from plugging this in as we did in the previous example so the two ordinary differential equations that we'll need to solve are,

$$\frac{dG}{dt} = -k\lambda G \qquad\qquad \frac{d^2\varphi}{dx^2} = -\lambda\varphi$$

Now, the point of this example was really to deal with the boundary conditions so let's plug the product solution into them to get,

$$\frac{\partial\left(G(t)\,\varphi(x)\right)}{\partial x}(0,t) = 0 \qquad\qquad \frac{\partial\left(G(t)\,\varphi(x)\right)}{\partial x}(L,t) = 0$$

$$G(t)\frac{d\varphi}{dx}(0) = 0 \qquad\qquad G(t)\frac{d\varphi}{dx}(L) = 0$$

Now, just as with the first example if we want to avoid the trivial solution and so we can't have $G(t) = 0$ for every *t* and so we must have,

$$\frac{d\varphi}{dx}(0) = 0 \qquad\qquad \frac{d\varphi}{dx}(L) = 0$$

Here is a summary of what we get by applying separation of variables to this problem.

$$\frac{dG}{dt} = -k\lambda G \qquad\qquad \frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$$

$$\frac{d\varphi}{dx}(0) = 0 \qquad \frac{d\varphi}{dx}(L) = 0$$

Next, let's see what we get if use periodic boundary conditions with the heat equation.

Example 3 Use Separation of Variables on the following partial differential equation.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad u(-L,t) = u(L,t) \qquad \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$$

Solution

First note that these boundary conditions really are homogeneous boundary conditions. If we rewrite them as,

$$u(-L,t) - u(L,t) = 0 \qquad\qquad \frac{\partial u}{\partial x}(-L,t) - \frac{\partial u}{\partial x}(L,t) = 0$$

it's a little easier to see.

Now, again we've done this partial differential equation so we'll start off with,

$$u(x,t) = \varphi(x)G(t)$$

and the two ordinary differential equations that we'll need to solve are,

$$\frac{dG}{dt} = -k\lambda G \qquad \frac{d^2\varphi}{dx^2} = -\lambda\varphi$$

Plugging the product solution into the rewritten boundary conditions gives,

$$G(t)\,\varphi(-L) - G(t)\,\varphi(L) = G(t)\left[\varphi(-L) - \varphi(L)\right] = 0$$
$$G(t)\frac{d\varphi}{dx}(-L) - G(t)\frac{d\varphi}{dx}(L) = G(t)\left[\frac{d\varphi}{dx}(-L) - \frac{d\varphi}{dx}(L)\right] = 0$$

and we can see that we'll only get non-trivial solution if,

$$\varphi(-L) - \varphi(L) = 0 \qquad \frac{d\varphi}{dx}(-L) - \frac{d\varphi}{dx}(L) = 0$$
$$\varphi(-L) = \varphi(L) \qquad \frac{d\varphi}{dx}(-L) = \frac{d\varphi}{dx}(L)$$

So, here is what we get by applying separation of variables to this problem.

$$\frac{dG}{dt} = -k\lambda G \qquad \frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$$
$$\varphi(-L) = \varphi(L) \qquad \frac{d\varphi}{dx}(-L) = \frac{d\varphi}{dx}(L)$$

Let's now take a look at what we get by applying separation of variables to the wave equation with fixed boundaries.

Example 4 Use Separation of Variables on the following partial differential equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x) \qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$
$$u(0,t) = 0 \qquad u(L,t) = 0$$

Solution

Now, as with the heat equation the two initial conditions are here only because they need to be here for the problem. We will not actually be doing anything with them here and as mentioned previously the product solution will rarely satisfy them. We will be dealing with those in a later section when we actually go past this first step. Again, the point of this example is only to get down to the two ordinary differential equations that separation of variables gives.

So, let's get going on that and plug the product solution, $u(x,t) = \varphi(x)h(t)$ (we switched the *G* to an *h* here to avoid confusion with the *g* in the second initial condition) into the wave equation to get,

$$\frac{\partial^2}{\partial t^2}\left(\varphi(x)h(t)\right) = c^2\frac{\partial^2}{\partial x^2}\left(\varphi(x)h(t)\right)$$
$$\varphi(x)\frac{d^2h}{dt^2} = c^2 h(t)\frac{d^2\varphi}{dx^2}$$
$$\frac{1}{c^2 h}\frac{d^2h}{dt^2} = \frac{1}{\varphi}\frac{d^2\varphi}{dx^2}$$

Note that we moved the c^2 to the right side for the same reason we moved the *k* in the heat equation. It will make solving the boundary value problem a little easier.

Now that we've gotten the equation separated into a function of only *t* on the left and a function of only *x* on the right we can introduce a separation constant and again we'll use $-\lambda$ so we can arrive at a boundary value problem that we are familiar with. So, after introducing the separation constant we get,

$$\frac{1}{c^2 h}\frac{d^2h}{dt^2} = \frac{1}{\varphi}\frac{d^2\varphi}{dx^2} = -\lambda$$

The two ordinary differential equations we get are then,

$$\frac{d^2h}{dt^2} = -\lambda c^2 h \qquad\qquad \frac{d^2\varphi}{dx^2} = -\lambda\varphi$$

The boundary conditions in this example are identical to those from the first example and so plugging the product solution into the boundary conditions gives,

$$\varphi(0) = 0 \qquad\qquad \varphi(L) = 0$$

Applying separation of variables to this problem gives,

$$\frac{d^2h}{dt^2} = -\lambda c^2 h \qquad\qquad \frac{d^2\varphi}{dx^2} = -\lambda\varphi$$
$$\varphi(0) = 0 \qquad \varphi(L) = 0$$

Next, let's take a look at the 2-D Laplace's Equation.

Example 5 Use Separation of Variables on the following partial differential equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 \le x \le L \qquad 0 \le y \le H$$
$$u(0,y) = g(y) \qquad u(L,y) = 0$$
$$u(x,0) = 0 \qquad u(x,H) = 0$$

Solution

This problem is a little (well actually quite a bit in some ways) different from the heat and wave equations. First, we no longer really have a time variable in the equation but instead we usually consider both variables to be spatial variables and we'll be assuming that the two variables are in the ranges shown above in the problems statement. Note that this also means that we no longer have initial conditions, but instead we now have two sets of boundary conditions, one for x and one for y.

Also, we should point out that we have three of the boundary conditions homogeneous and one nonhomogeneous for a reason. When we get around to actually solving this Laplace's Equation we'll see that this is in fact required in order for us to find a solution.

For this problem we'll use the product solution,

$$u(x,y) = h(x)\varphi(y)$$

It will often be convenient to have the boundary conditions in hand that this product solution gives before we take care of the differential equation. In this case we have three homogeneous boundary conditions and so we'll need to convert all of them. Because we've already converted these kind of boundary conditions we'll leave it to you to verify that these will become,

$$h(L) = 0 \qquad \varphi(0) = 0 \qquad \varphi(H) = 0$$

Plugging this into the differential equation and separating gives,

$$\frac{\partial^2}{\partial x^2}\left(h(x)\,\varphi(y)\right)+\frac{\partial^2}{\partial y^2}\left(h(x)\,\varphi(y)\right)=0$$

$$\varphi(y)\frac{d^2h}{dx^2}+h(x)\frac{d^2\varphi}{dy^2}=0$$

$$\frac{1}{h}\frac{d^2h}{dx^2}=-\frac{1}{\varphi}\frac{d^2\varphi}{dy^2}$$

Okay, now we need to decide upon a separation constant. Note that every time we've chosen the separation constant we did so to make sure that the differential equation

$$\frac{d^2\varphi}{dy^2}+\lambda\varphi=0$$

would show up. Of course, the letters might need to be different depending on how we defined our product solution (as they'll need to be here). We know how to solve this eigenvalue/eigenfunction problem as we pointed out in the discussion after the first example. However, in order to solve it we need two boundary conditions.

So, for our problem here we can see that we've got two boundary conditions for $\varphi(y)$ but only one for $h(x)$ and so we can see that the boundary value problem that we'll have to solve will involve $\varphi(y)$ and so we need to pick a separation constant that will give use the boundary value problem we've already solved. In this case that means that we need to choose λ for the separation constant. If you're not sure you believe that yet hold on for a second and you'll soon see that it was in fact the correct choice here.

Putting the separation constant gives,

$$\frac{1}{h}\frac{d^2h}{dx^2}=-\frac{1}{\varphi}\frac{d^2\varphi}{dy^2}=\lambda$$

The two ordinary differential equations we get from Laplace's Equation are then,

$$\frac{d^2h}{dx^2}=\lambda h \qquad\qquad -\frac{d^2\varphi}{dy^2}=\lambda\varphi$$

and notice that if we rewrite these a little we get,

$$\frac{d^2h}{dx^2}-\lambda h=0 \qquad\qquad \frac{d^2\varphi}{dy^2}+\lambda\varphi=0$$

We can now see that the second one does now look like one we've already solved (with a small change in letters of course, but that really doesn't change things).

So, let's summarize up here.

$$\frac{d^2h}{dx^2} - \lambda h = 0 \qquad\qquad \frac{d^2\varphi}{dy^2} + \lambda\varphi = 0$$
$$h(L) = 0 \qquad\qquad \varphi(0) = 0 \qquad \varphi(H) = 0$$

So, we've finally seen an example where the constant of separation didn't have a minus sign and again note that we chose it so that the boundary value problem we need to solve will match one we've already seen how to solve so there won't be much work to there.

All the examples worked in this section to this point are all problems that we'll continue in later sections to get full solutions for. Let's work one more however to illustrate a couple of other ideas. We will not however be doing any work with this in later sections however, it is only here to illustrate a couple of points.

Example 6 Use Separation of Variables on the following partial differential equation.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - u$$
$$u(x,0) = f(x) \qquad u(0,t) = 0 \qquad -\frac{\partial u}{\partial x}(L,t) = u(L,t)$$

Solution

Note that this is a heat equation with the source term of $Q(x,t) = -c\rho u$ and is both linear and homogenous. Also note that for the first time we've mixed boundary condition types. At $x = 0$ we've got a prescribed temperature and at $x = L$ we've got a Newton's law of cooling type boundary condition. We should not come away from the first few examples with the idea that the boundary conditions at both boundaries always the same type. Having them the same type just makes the boundary value problem a little easier to solve in many cases.

So we'll start off with,

$$u(x,t) = \varphi(x)G(t)$$

and plugging this into the partial differential equation gives,

$$\frac{\partial}{\partial t}\left(\varphi(x)G(t)\right)=k\frac{\partial^2}{\partial x^2}\left(\varphi(x)G(t)\right)-\varphi(x)G(t)$$

$$\varphi(x)\frac{dG}{dt}=kG(t)\frac{d^2\varphi}{dx^2}-\varphi(x)G(t)$$

Now, the next step is to divide by $\varphi(x)G(t)$ and notice that upon doing that the second term on the right will become a one and so can go on either side. Theoretically there is no reason that the one can't be on either side, however from a practical standpoint we again want to keep things a simple as possible so we'll move it to the *t* side as this will guarantee that we'll get a differential equation for the boundary value problem that we've seen before. So, separating and introducing a separation constant gives,

$$\frac{1}{k}\left(\frac{1}{G}\frac{dG}{dt}+1\right)=\frac{1}{\varphi}\frac{d^2\varphi}{dx^2}=-\lambda$$

The two ordinary differential equations that we get are then (with some rewriting),

$$\frac{dG}{dt}=-(\lambda k+1)G \qquad \frac{d^2\varphi}{dx^2}=-\lambda\varphi$$

Now let's deal with the boundary conditions.

$$G(t)\,\varphi(0)=0$$

$$G(t)\frac{d\varphi}{dx}(L)+G(t)\varphi(L)=G(t)\left[\frac{d\varphi}{dx}(L)+\varphi(L)\right]=0$$

and we can see that we'll only get non-trivial solution if,

$$\varphi(0)=0 \qquad \frac{d\varphi}{dx}(L)+\varphi(L)=0$$

So, here is what we get by applying separation of variables to this problem.

$$\frac{dG}{dt}=-(\lambda k+1)G \qquad \frac{d^2\varphi}{dx^2}+\lambda\varphi=0$$

$$\varphi(0)=0 \qquad \frac{d\varphi}{dx}(L)+\varphi(L)=0$$

Solving the Heat Equation

In this section we will now solve those ordinary differential equations and use the results to get a solution to the partial differential equation. We will be concentrating on the heat equation in this section and will do the wave equation and Laplace's equation in later sections.

The first problem that we're going to look at will be the temperature distribution in a bar with zero temperature boundaries. We are going to do the work in a couple of steps so we can take our time and see how everything works.

The first thing that we need to do is find a solution that will satisfy the partial differential equation and the boundary conditions. At this point we will not worry about the initial condition. The solution we'll get first will not satisfy the vast majority of initial conditions but as we'll see it can be used to find a solution that will satisfy a sufficiently nice initial condition.

Example 1 Find a solution to the following partial differential equation that will also satisfy the boundary conditions.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad u(0,t) = 0 \qquad u(L,t) = 0$$

Solution

The first thing we technically need to do here is apply separation of variables. Even though we did that in the previous section let's recap here what we did.

First, we assume that the solution will take the form,

$$u(x,t) = \varphi(x)G(t)$$

and we plug this into the partial differential equation and boundary conditions. We separate the equation to get a function of only t on one side and a function of only x on the other side and then introduce a separation constant. This leaves us with two ordinary differential equations.

We did all of this in Example 1 of the previous section and the two ordinary differential equations are,

$$\frac{dG}{dt} = -k\lambda G \qquad\qquad \frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$$

$$\varphi(0) = 0 \qquad \varphi(L) = 0$$

The time dependent equation can really be solved at any time, but since we don't know what λ

λ is yet let's hold off on that one. Also note that in many problems only the boundary value problem can be solved at this point so don't always expect to be able to solve either one at this point.

The spatial equation is a boundary value problem and we know from our work in the previous chapter that it will only have non-trivial solutions (which we want) for certain values of λ, which we'll recall are called eigenvalues. Once we have those we can determine the non-trivial solutions for each λ, *i.e.* eigenfunctions.

Now, we actually solved the spatial problem,

$$\frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$$
$$\varphi(0) = 0 \qquad \varphi(L) = 0$$

in Example 1 of the Eigenvalues and Eigenfunctions section of the previous chapter for $L = 2\pi$. So, because we've solved this once for a specific *L* and the work is not all that much different for a general *L* we're not going to be putting in a lot of explanation here and if you need a reminder on how something works or why we did something go back to Example 1 from the Eigenvalues and Eigenfunctions section for a reminder.

We've got three cases to deal with so let's get going.

$\underline{\lambda > 0}$

In this case we know the solution to the differential equation is,

$$\varphi(x) = c_1 \cos\left(\sqrt{\lambda}\, x\right) + c_2 \sin\left(\sqrt{\lambda}\, x\right)$$

Applying the first boundary condition gives,

$$0 = \varphi(0) = c_1$$

Now applying the second boundary condition, and using the above result of course, gives,

$$0 = \varphi(L) = c_2 \sin\left(L\sqrt{\lambda}\right)$$

Now, we are after non-trivial solutions and so this means we must have,

$$\sin\left(L\sqrt{\lambda}\right) = 0 \qquad \Rightarrow \qquad L\sqrt{\lambda} = n\pi \quad n = 1, 2, 3, \ldots$$

The positive eigenvalues and their corresponding eigenfunctions of this boundary value problem are then,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

Note that we don't need the c_2 in the eigenfunction as it will just get absorbed into another constant that we'll be picking up later on.

$\underline{\lambda = 0}$

The solution to the differential equation in this case is,

$$\varphi(x) = c_1 + c_2 x$$

Applying the boundary conditions gives,

$$0 = \varphi(0) = c_1 \qquad 0 = \varphi(L) = c_2 L \qquad \Rightarrow \qquad c_2 = 0$$

So, in this case the only solution is the trivial solution and so $\lambda = 0$ is not an eigenvalue for this boundary value problem.

$\underline{\lambda < 0}$

Here the solution to the differential equation is,

$$\varphi(x) = c_1 \cosh\left(\sqrt{-\lambda}\, x\right) + c_2 \sinh\left(\sqrt{-\lambda}\, x\right)$$

Applying the first boundary condition gives,

$$0 = \varphi(0) = c_1$$

and applying the second gives,

$$0 = \varphi(L) = c_2 \sinh\left(L\sqrt{-\lambda}\right)$$

So, we are assuming $\lambda < 0$ and so $L\sqrt{-\lambda} \neq 0$ and this means $\sinh\left(L\sqrt{-\lambda}\right) \neq 0$. We therefore we must have $c_2 = 0$ and so we can only get the trivial solution in this case.

Therefore, there will be no negative eigenvalues for this boundary value problem.

The complete list of eigenvalues and eigenfunctions for this problem are then,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

Now let's solve the time differential equation,

$$\frac{dG}{dt} = -k\lambda_n G$$

and note that even though we now know λ we're not going to plug it in quite yet to keep the mess to a minimum. We will however now use λ_n to remind us that we actually have an infinite number of possible values here.

This is a simple linear (and separable for that matter) 1st order differential equation and so we'll let you verify that the solution is,

$$G(t) = c\mathbf{e}^{-k\lambda_n t} = c\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Now that we've gotten both of the ordinary differential equations solved we can finally write down a solution. Note however that we have in fact found infinitely many solutions since there are infinitely many solutions (*i.e.* eigenfunctions) to the spatial problem.

Our product solution are then,

$$u_n(x,t) = B_n \sin\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t} \qquad n = 1, 2, 3, \ldots$$

denoted the product solution u_n to acknowledge that each value of *n* will yield a different solution. Also note that we've changed the *c* in the solution to the time problem to B_n to denote the fact that it will probably be different for each value of *n* as well and because had we kept the c_2 with the eigenfunction we'd have absorbed it into the *c* to get a single constant in our solution.

So, there we have it. The function above will satisfy the heat equation and the boundary condition of zero temperature on the ends of the bar.

The problem with this solution is that it simply will not satisfy almost every possible initial condition we could possibly want to use. That does not mean however, that there aren't at least a few that it will satisfy as the next example illustrates.

Example 2 Solve the following heat problem for the given initial conditions.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x) \qquad u(0,t) = 0 \qquad u(L,t) = 0$$

(a) $f(x) = 6\sin\left(\frac{\pi x}{L}\right)$

(b) $f(x) = 12\sin\left(\frac{9\pi x}{L}\right) - 7\sin\left(\frac{4\pi x}{L}\right)$

Solution

a) This is actually easier than it looks like. All we need to do is choose $n = 1$ and $B_1 = 6$ in the product solution above to get,

$$u(x,t) = 6\sin\left(\frac{\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{\pi}{L}\right)^2 t}$$

and we've got the solution we need. This is a product solution for the first example and so satisfies the partial differential equation and boundary conditions and will satisfy the initial condition since plugging in $t = 0$ will drop out the exponential.

(b) This is almost as simple as the first part. Recall from the Principle of Superposition that if we have two solutions to a linear homogeneous differential equation (which we've got here) then their sum is also a solution. So, all we need to do is choose *n* and B_n as we did in the first part to get a solution that satisfies each part of the initial condition and then add them up. Doing this gives,

$$u(x,t) = 12\sin\left(\frac{9\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{9\pi}{L}\right)^2 t} - 7\sin\left(\frac{4\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{4\pi}{L}\right)^2 t}$$

We'll leave it to you to verify that this does in fact satisfy the initial condition and the boundary conditions.

So, we've seen that our solution from the first example will satisfy at least a small number of highly specific initial conditions.

Now, let's extend the idea out that we used in the second part of the previous example a little to see how we can get a solution that will satisfy any sufficiently nice initial condition. The

Principle of Superposition is, of course, not restricted to only two solutions. For instance the following is also a solution to the partial differential equation.

$$u(x,t) = \sum_{n=1}^{M} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

and notice that this solution will not only satisfy the boundary conditions but it will also satisfy the initial condition,

$$u(x,0) = \sum_{n=1}^{M} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Let's extend this out even further and take the limit as $M \to \infty$. Doing this our solution now becomes,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

This solution will satisfy any initial condition that can be written in the form,

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

This may still seem to be very restrictive, but the series on the right should look awful familiar to you after the previous chapter. The series on the left is exactly the Fourier sine series we looked at in that chapter. Also recall that when we can write down the Fourier sine series for any piecewise smooth function on $0 \le x \le L$.

So, provided our initial condition is piecewise smooth after applying the initial condition to our solution we can determine the B_n as if we were finding the Fourier sine series of initial condition. So we can either proceed as we did in that section and use the orthogonality of the sines to derive them or we can acknowledge that we've already done that work and know that coefficients are given by,

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 1, 2, 3, \ldots$$

So, we finally can completely solve a partial differential equation.

Example 3 Solve the following BVP.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = 20 \qquad u(0,t) = 0 \qquad u(L,t) = 0$$

Solution

There isn't really all that much to do here as we've done most of it in the examples and discussion above.

First, the solution is,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

The coefficients are given by,

$$B_n = \frac{2}{L}\int_0^L 20\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left(\frac{20L\left(1-\cos(n\pi)\right)}{n\pi}\right) = \frac{40\left(1-(-1)^n\right)}{n\pi}$$

If we plug these in we get the solution,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{40\left(1-(-1)^n\right)}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

While the example itself was very simple, it was only simple because of all the work that we had to put into developing the ideas that even allowed us to do this. Because of how "simple" it will often be to actually get these solutions we're not actually going to do anymore with specific initial conditions. We will instead concentrate on simply developing the formulas that we'd be required to evaluate in order to get an actual solution.

So, having said that let's move onto the next example. In this case we're going to again look at the temperature distribution in a bar with perfectly insulated boundaries. We are also no longer going to go in steps. We will do the full solution as a single example and end up with a solution that will satisfy any piecewise smooth initial condition.

Example 4 Find a solution to the following partial differential equation.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad \frac{\partial u}{\partial x}(0,t) = 0 \qquad \frac{\partial u}{\partial x}(L,t) = 0$$

Solution

We applied separation of variables to this problem in Example 2 of the previous section. So, after assuming that our solution is in the form,

$$u(x,t) = \varphi(x)G(t)$$

and applying separation of variables we get the following two ordinary differential equations that we need to solve.

$$\frac{dG}{dt} = -k\lambda G \qquad \begin{array}{l} \dfrac{d^2\varphi}{dx^2} + \lambda\varphi = 0 \\ \dfrac{d\varphi}{dx}(0) = 0 \qquad \dfrac{d\varphi}{dx}(L) = 0 \end{array}$$

We solved the boundary value problem in Example 2 of the Eigenvalues and Eigenfunctions section of the previous chapter for $L = 2\pi$ so as with the first example in this section we're not going to put a lot of explanation into the work here. If you need a reminder on how this works go back to the previous chapter and review the example we worked there. Let's get going on the three cases we've got to work for this problem.

$\underline{\lambda > 0}$

The solution to the differential equation is,

$$\varphi(x) = c_1\cos\left(\sqrt{\lambda}\,x\right) + c_2\sin\left(\sqrt{\lambda}\,x\right)$$

Applying the first boundary condition gives,

$$0 = \frac{d\varphi}{dx}(0) = \sqrt{\lambda}\,c_2 \qquad \Rightarrow \qquad c_2 = 0$$

The second boundary condition gives,

$$0 = \frac{d\varphi}{dx}(L) = -\sqrt{\lambda}\, c_1 \sin\left(L\sqrt{\lambda}\right)$$

Recall that $\lambda > 0$ and so we will only get non-trivial solutions if we require that,

$$\sin\left(L\sqrt{\lambda}\right) = 0 \qquad \Rightarrow \qquad L\sqrt{\lambda} = n\pi \qquad n = 1, 2, 3, \ldots$$

The positive eigenvalues and their corresponding eigenfunctions of this boundary value problem are then,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

$\underline{\lambda = 0}$

The general solution is

$$\varphi(x) = c_1 + c_2 x$$

Applying the first boundary condition gives,

$$0 = \frac{d\varphi}{dx}(0) = c_2$$

Using this the general solution is then,

$$\varphi(x) = c_1$$

and note that this will trivially satisfy the second boundary condition. Therefore $\lambda = 0$ is an eigenvalue for this BVP and the eigenfunctions corresponding to this eigenvalue is,

$$\varphi(x) = 1$$

$\underline{\lambda < 0}$

The general solution here is,

$$\varphi(x) = c_1 \cosh\left(\sqrt{-\lambda}\, x\right) + c_2 \sinh\left(\sqrt{-\lambda}\, x\right)$$

Applying the first boundary condition gives,

$$0 = \frac{d\varphi}{dx}(0) = \sqrt{-\lambda}\,c_2 \quad \Rightarrow \quad c_2 = 0$$

The second boundary condition gives,

$$0 = \frac{d\varphi}{dx}(L) = \sqrt{-\lambda}\,c_1 \sinh\left(L\sqrt{-\lambda}\right)$$

We know that $L\sqrt{-\lambda} \neq 0$ and so $\sinh\left(L\sqrt{-\lambda}\right) \neq 0$. Therefore we must have $c_1 = 0$ and so, this boundary value problem will have no negative eigenvalues.

So, the complete list of eigenvalues and eigenfunctions for this problem is then,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$
$$\lambda_0 = 0 \qquad \varphi_0(x) = 1$$

and notice that we get the $\lambda_0 = 0$ eigenvalue and its eigenfunction if we allow $n = 0$ in the first set and so we'll use the following as our set of eigenvalues and eigenfunctions.

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right) \qquad n = 0, 1, 2, 3, \ldots$$

The time problem here is identical to the first problem we looked at so,

$$G(t) = c\,\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Our product solutions will then be,

$$u_n(x,t) = A_n \cos\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t} \qquad n = 0, 1, 2, 3, \ldots$$

and the solution to this partial differential equation is,

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

If we apply the initial condition to this we get,

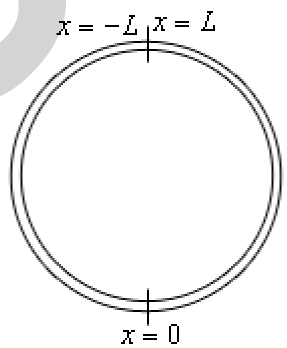
$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

and we can see that this is nothing more than the Fourier cosine series for $f(x)$ on $0 \le x \le L$ and so again we could use the orthogonality of the cosines to derive the coefficients or we could recall that we've already done that in the previous chapter and know that the coefficients are given by,

$$A_n = \begin{cases} \frac{1}{L}\int_0^L f(x)\,dx & n = 0 \\ \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n \ne 0 \end{cases}$$

The last example that we're going to work in this section is a little different from the first two. We are going to consider the temperature distribution in a thin circular ring. We will consider the lateral surfaces to be perfectly insulated and we are also going to assume that the ring is thin enough so that the temperature does not vary with distance from the center of the ring.

So, what does that leave us with? Let's set $x = 0$ as shown below and then let *x* be the arc length of the ring as measured from this point.



We will measure *x* as positive if we move to the right and negative if we move to the left of $x = 0$. This means that at the top of the ring we'll meet where $x = L$ (if we move to the right) and $x = -L$ (if we move to the left). By doing this we can consider this ring to be a bar of length 2*L* and the heat equation that we developed earlier in this chapter will still hold.

At the point of the ring we consider the two "ends" to be in **perfect thermal contact**. This means that at the two ends both the temperature and the heat flux must be equal. In other words we must have,

$$u(-L,t) = u(L,t) \qquad\qquad \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$$

If you recall from the section in which we derived the heat equation we called these periodic boundary conditions. So, the problem we need to solve to get the temperature distribution in this case is,

Example 5 Find a solution to the following partial differential equation.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad u(-L,t) = u(L,t) \qquad \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$$

Solution

We applied separation of variables to this problem in Example 3 of the previous section. So, if we assume the solution is in the form,

$$u(x,t) = \varphi(x)G(t)$$

we get the following two ordinary differential equations that we need to solve.

$$\frac{dG}{dt} = -k\lambda G \qquad\qquad \frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$$

$$\varphi(-L) = \varphi(L) \qquad \frac{d\varphi}{dx}(-L) = \frac{d\varphi}{dx}(L)$$

As we've seen with the previous two problems we've already solved a boundary value problem like this one back in the Eigenvalues and Eigenfunctions section of the previous chapter, Example 3 to be exact with $L = \pi$. So, if you need a little more explanation of what's going on here go back to this example and you can see a little more explanation.

We again have three cases to deal with here.

$\underline{\lambda > 0}$

The general solution to the differential equation is,

$$\varphi(x) = c_1 \cos\left(\sqrt{\lambda}\, x\right) + c_2 \sin\left(\sqrt{\lambda}\, x\right)$$

Applying the first boundary condition and recalling that cosine is an even function and sine is an odd function gives us,

$$\begin{aligned} c_1 \cos\left(-L\sqrt{\lambda}\right) + c_2 \sin\left(-L\sqrt{\lambda}\right) &= c_1 \cos\left(L\sqrt{\lambda}\right) + c_2 \sin\left(L\sqrt{\lambda}\right) \\ -c_2 \sin\left(L\sqrt{\lambda}\right) &= c_2 \sin\left(L\sqrt{\lambda}\right) \\ 0 &= 2c_2 \sin\left(L\sqrt{\lambda}\right) \end{aligned}$$

At this stage we can't really say anything as either c_2 c_2 or sine could be zero. So, let's apply the second boundary condition and see what we get.

$$\begin{aligned} -\sqrt{\lambda}\, c_1 \sin\left(-L\sqrt{\lambda}\right) + \sqrt{\lambda}\, c_2 \cos\left(-L\sqrt{\lambda}\right) &= -\sqrt{\lambda}\, c_1 \sin\left(L\sqrt{\lambda}\right) + \sqrt{\lambda}\, c_2 \cos\left(L\sqrt{\lambda}\right) \\ \sqrt{\lambda}\, c_1 \sin\left(L\sqrt{\lambda}\right) &= -\sqrt{\lambda}\, c_1 \sin\left(L\sqrt{\lambda}\right) \\ 2\sqrt{\lambda}\, c_1 \sin\left(L\sqrt{\lambda}\right) &= 0 \end{aligned}$$

We get something similar. However notice that if $\sin\left(L\sqrt{\lambda}\right) \neq 0$ then we would be forced to have $c_1 = c_2 = 0$ and this would give us the trivial solution which we don't want.

This means therefore that we must have $\sin\left(L\sqrt{\lambda}\right) = 0$ which in turn means (from work in our previous examples) that the positive eigenvalues for this problem are,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, 3, \ldots$$

Now, there is no reason to believe that $c_1 = 0$ or $c_2 = 0$. All we know is that they both can't be zero and so that means that we in fact have two sets of eigenfunctions for this problem corresponding to positive eigenvalues. They are,

$$\varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right) \qquad \varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

$\underline{\lambda = 0}$

The general solution in this case is,

$$\varphi(x) = c_1 + c_2 x$$

Applying the first boundary condition gives,

$$\begin{aligned} c_1 + c_2(-L) &= c_1 + c_2(L) \\ 2Lc_2 = 0 \qquad &\Rightarrow \qquad c_2 = 0 \end{aligned}$$

The general solution is then,

$$\varphi(x) = c_1$$

and this will trivially satisfy the second boundary condition. Therefore $\lambda = 0$ is an eigenvalue for this BVP and the eigenfunctions corresponding to this eigenvalue is,

$$\varphi(x) = 1$$

$\underline{\lambda < 0}$

For this final case the general solution here is,

$$\varphi(x) = c_1 \cosh\left(\sqrt{-\lambda}\, x\right) + c_2 \sinh\left(\sqrt{-\lambda}\, x\right)$$

Applying the first boundary condition and using the fact that hyperbolic cosine is even and hyperbolic sine is odd gives,

$$\begin{aligned} c_1 \cosh\left(-L\sqrt{-\lambda}\right) + c_2 \sinh\left(-L\sqrt{-\lambda}\right) &= c_1 \cosh\left(L\sqrt{-\lambda}\right) + c_2 \sinh\left(L\sqrt{-\lambda}\right) \\ -c_2 \sinh\left(-L\sqrt{-\lambda}\right) &= c_2 \sinh\left(L\sqrt{-\lambda}\right) \\ 2c_2 \sinh\left(L\sqrt{-\lambda}\right) &= 0 \end{aligned}$$

Now, in this case we are assuming that $\lambda < 0$ and so $L\sqrt{-\lambda} \neq 0$. This turn tells us that $\sinh\left(L\sqrt{-\lambda}\right) \neq 0$. We therefore must have $c_2 = 0$.

Let's now apply the second boundary condition to get,

$$\sqrt{-\lambda}\, c_1 \sinh\left(-L\sqrt{-\lambda}\right) = \sqrt{-\lambda}\, c_1 \sinh\left(L\sqrt{-\lambda}\right)$$

$$2\sqrt{-\lambda}\, c_1 \sinh\left(L\sqrt{-\lambda}\right) = 0 \qquad \Rightarrow \qquad c_1 = 0$$

By our assumption on λ we again have no choice here but to have $c_1 = 0$ and so for this boundary value problem there are no negative eigenvalues.

Summarizing up then we have the following sets of eigenvalues and eigenfunctions and note that we've merged the $\lambda = 0$ case into the cosine case since it can be here to simplify things up a little.

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right) \qquad n = 0, 1, 2, 3, \ldots$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

The time problem is again identical to the two we've already worked here and so we have,

$$G(t) = c\,\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Now, this example is a little different from the previous two heat problems that we've looked at. In this case we actually have two different possible product solutions that will satisfy the partial differential equation and the boundary conditions. They are,

$$u_n(x,t) = A_n \cos\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t} \qquad n = 0, 1, 2, 3, \ldots$$

$$u_n(x,t) = B_n \sin\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t} \qquad n = 1, 2, 3, \ldots$$

The Principle of Superposition is still valid however and so a sum of any of these will also be a solution and so the solution to this partial differential equation is,

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t} + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

If we apply the initial condition to this we get,

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

and just as we saw in the previous two examples we get a Fourier series. The difference this time is that we get the full Fourier series for a piecewise smooth initial condition on $-L \le x \le L$. As noted for the previous two examples we could either rederive formulas for the coefficients using the orthogonality of the sines and cosines or we can recall the work we've already done. There's really no reason at this point to redo work already done so the coefficients are given by,

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\tfrac{n\pi x}{L}\right)dx \qquad n = 1,2,3,\ldots$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\tfrac{n\pi x}{L}\right)dx \qquad n = 1,2,3,\ldots$$

Note that this is the reason for setting up *x* as we did at the start of this problem. A full Fourier series needs an interval of $-L \le x \le L$ whereas the Fourier sine and cosines series we saw in the first two problems need $0 \le x \le L$.

Heat Equation with Non-Zero Temperature Boundaries

In this section we want to expand one of the cases from the previous section a little bit. In the previous section we look at the following heat problem.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x) \qquad u(0,t) = 0 \qquad u(L,t) = 0$$

Now, there is nothing inherently wrong with this problem, but the fact that we're fixing the temperature on both ends at zero is a little unrealistic. The other two problems we looked at, insulated boundaries and the thin ring, are a little more realistic problems, but this one just isn't all that realistic so we'd like to extend it a little.

What we'd like to do in this section is instead look at the following problem.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x) \qquad u(0,t) = T_1 \qquad u(L,t) = T_2 \tag{1}$$

In this case we'll allow the boundaries to be any fixed temperature, T_1 or T_2. The problem here is that separation of variables will no longer work on this problem because the boundary conditions are no longer homogeneous. Recall that separation of variables will only work if both the partial differential equation and the boundary conditions are linear and homogeneous. So, we're going to need to deal with the boundary conditions in some way before we actually try and solve this.

Luckily for us there is an easy way to deal with them. Let's consider this problem a little bit. There are no sources to add/subtract heat energy anywhere in the bar. Also our boundary conditions are fixed temperatures and so can't change with time and we aren't prescribing a heat flux on the boundaries to continually add/subtract heat energy. So, what this all means is that there will not ever be any forcing of heat energy into or out of the bar and so while some heat energy may well naturally flow into our out of the bar at the end points as the temperature changes eventually the temperature distribution in the bar should stabilize out and no longer depend on time.

Or, in other words it makes some sense that we should expect that as $t \to \infty$ our temperature distribution, $u(x,t)$ should behave as,

$$\lim_{t\to\infty} u(x,t) = u_E(x)$$

where $u_E(x)$ is called the **equilibrium temperature**. Note as well that is should still satisfy the heat equation and boundary conditions. It won't satisfy the initial condition however because it is the temperature distribution as $t \to \infty$ whereas the initial condition is at $t = 0$. So, the equilibrium temperature distribution should satisfy,

$$0 = \frac{d^2 u_E}{dx^2} \qquad u_E(0) = T_1 \qquad u_E(L) = T_2 \tag{2}$$

This is a really easy 2nd order ordinary differential equation to solve. If we integrate twice we get,

$$u_E(x) = c_1 x + c_2$$

and applying the boundary conditions (we'll leave this to you to verify) gives us,

$$u_E(x) = T_1 + \frac{T_2 - T_1}{L} x$$

What does this have to do with solving the problem given by (1) above? We'll let's define the function,

$$v(x,t) = u(x,t) - u_E(x) \tag{3}$$

where $u(x,t)$ is the solution to (1) and $u_E(x)$ is the equilibrium temperature for (1).

Now let's rewrite this as,

$$u(x,t) = v(x,t) + u_E(x)$$

and let's take some derivatives.

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial u_E}{\partial t} = \frac{\partial v}{\partial t} \qquad\qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u_E}{\partial x^2} = \frac{\partial^2 v}{\partial x^2}$$

In both of these derivatives we used the fact that $u_E(x)$ is the equilibrium temperature and so is independent of time *t* and must satisfy the differential equation in (2).

What this tells us is that both $u(x,t)$ and $v(x,t)$ must satisfy the same partial differential equation. Let's see what the initial conditions and boundary conditions would need to be for $v(x,t)$.

$$\begin{aligned} v(x,0) &= u(x,0) - u_E(x) = f(x) - u_E(x) \\ v(0,t) &= u(0,t) - u_E(0) = T_1 - T_1 = 0 \\ v(L,t) &= u(L,t) - u_E(L) = T_2 - T_2 = 0 \end{aligned}$$

So, the initial condition just gets potentially messier, but the boundary conditions are now homogeneous! The partial differential equation that $v(x,t)$ must satisfy is,

$$\begin{aligned} &\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} \\ &v(x,0) = f(x) - u_E(x) \qquad\qquad v(0,t) = 0 \qquad\qquad v(L,t) = 0 \end{aligned}$$

We saw how to solve this in the previous section and so we the solution is,

$$v(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

where the coefficients are given by,

$$B_n = \frac{2}{L}\int_0^L \left(f(x) - u_E(x)\right)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 1,2,3,\ldots$$

The solution to (1) is then,

$$\begin{aligned} u(x,t) &= u_E(x) + v(x,t) \\ &= T_1 + \frac{T_2 - T_1}{L}x + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t} \end{aligned}$$

and the coefficients are given above.

Laplace's Equation

The next partial differential equation that we're going to solve is the 2-D Laplace's equation,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

A natural question to ask before we start learning how to solve this is does this equation come up naturally anywhere? The answer is a very resounding yes! If we consider the 2-D heat equation,

$$\frac{\partial u}{\partial t} = k\nabla^2 u + \frac{Q}{cp}$$

We can see that Laplace's equation would correspond to finding the equilibrium solution (*i.e.* time independent solution) if there were not sources. So, this is an equation that can arise from physical situations.

How we solve Laplace's equation will depend upon the geometry of the 2-D object we're solving it on. Let's start out by solving it on the rectangle given by $0 \le x \le L$, $0 \le y \le H$. For this geometry Laplace's equation along with the four boundary conditions will be,

$$\begin{aligned} &\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ &u(0,y) = g_1(y) \qquad u(L,y) = g_2(y) \\ &u(x,0) = f_1(x) \qquad u(x,H) = f_2(x) \end{aligned} \tag{1}$$

Example 1 Find a solution to the following partial differential equation.

$$\nabla^2 u_4 = \frac{\partial^2 u_4}{\partial x^2} + \frac{\partial^2 u_4}{\partial y^2} = 0$$

$$u_4(0,y) = g_1(y) \qquad u_4(L,y) = 0$$

$$u_4(x,0) = 0 \qquad u_4(x,H) = 0$$

Solution

We'll start by assuming that our solution will be in the form,

$$u_4(x,y) = h(x)\varphi(y)$$

and then recall that we performed separation of variables on this problem (with a small change in notation) back in Example 5 of the Separation of Variables section. So from that problem we know that separation of variables yields the following two ordinary differential equations that we'll need to solve.

$$\frac{d^2h}{dx^2} - \lambda h = 0 \qquad \frac{d^2\varphi}{dy^2} + \lambda\varphi = 0$$

$$h(L) = 0 \qquad \varphi(0) = 0 \qquad \varphi(H) = 0$$

Note that in this case, unlike the heat equation we must solve the boundary value problem first. Without knowing what λ is there is no way that we can solve the first differential equation here with only one boundary condition since the sign of λ will affect the solution.

Let's also notice that we solved the boundary value problem in Example 1 of Solving the Heat Equation and so there is no reason to resolve it here. Taking a change of letters into account the eigenvalues and eigenfunctions for the boundary value problem here are,

$$\lambda_n = \left(\frac{n\pi}{H}\right)^2 \qquad \varphi_n(y) = \sin\left(\frac{n\pi y}{H}\right) \qquad n = 1,2,3,\ldots$$

Now that we know what the eigenvalues are let's write down the first differential equation with λ λ plugged in.

$$\frac{d^2h}{dx^2} - \left(\frac{n\pi}{H}\right)^2 h = 0$$
$$h(L) = 0$$

Because the coefficient of $h(x)$ in the differential equation above is positive we know that a solution to this is,

$$h(x) = c_1 \cosh\left(\frac{n\pi x}{H}\right) + c_2 \sinh\left(\frac{n\pi x}{H}\right)$$

However, this is not really suited for dealing with the $h(L) = 0$ boundary condition. So, let's also notice that the following is also a solution.

$$h(x) = c_1 \cosh\left(\frac{n\pi(x-L)}{H}\right) + c_2 \sinh\left(\frac{n\pi(x-L)}{H}\right)$$

You should verify this by plugging this into the differential equation and checking that it is in fact a solution. Applying the lone boundary condition to this "shifted" solution gives,

$$0 = h(L) = c_1$$

The solution to the first differential equation is now,

$$h(x) = c_2 \sinh\left(\frac{n\pi(x-L)}{H}\right)$$

and this is all the farther we can go with this because we only had a single boundary condition. That is not really a problem however because we now have enough information to form the product solution for this partial differential equation.

A product solution for this partial differential equation is,

$$u_n(x,y) = B_n \sinh\left(\frac{n\pi(x-L)}{H}\right)\sin\left(\frac{n\pi y}{H}\right) \qquad n = 1,2,3,\ldots$$

The Principle of Superposition then tells us that a solution to the partial differential equation is,

$$u_4(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi(x-L)}{H}\right)\sin\left(\frac{n\pi y}{H}\right)$$

and this solution will satisfy the three homogeneous boundary conditions.

To determine the constants all we need to do is apply the final boundary condition.

$$u_4(0,y) = g_1(y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi(-L)}{H}\right)\sin\left(\frac{n\pi y}{H}\right)$$

Now, in the previous problems we've done this has clearly been a Fourier series of some kind and in fact it still is. The difference here is that the coefficients of the Fourier sine series are now,

$$B_n \sinh\left(\frac{n\pi(-L)}{H}\right)$$

instead of just B_n. We might be a little more tempted to use the orthogonality of the sines to derive formulas for the B_n, however we can still reuse the work that we've done previously to get formulas for the coefficients here.

Remember that a Fourier sine series is just a series of coefficients (depending on *n*) times a sine. We still have that here, except the "coefficients" are a little messier this time that what we saw when we first dealt with Fourier series. So, the coefficients can be found using exactly the same formula from the Fourier sine series section of a function on $0 \le y \le H$ we just need to be careful with the coefficients.

$$B_n \sinh\left(\frac{n\pi(-L)}{H}\right) = \frac{2}{H}\int_0^H g_1(y)\sin\left(\frac{n\pi y}{H}\right)dy \qquad n = 1,2,3,\ldots$$

$$B_n = \frac{2}{H\sinh\left(\frac{n\pi(-L)}{H}\right)}\int_0^H g_1(y)\sin\left(\frac{n\pi y}{H}\right)dy \qquad n = 1,2,3,\ldots$$

The formulas for the B_n are a little messy this time in comparison to the other problems we've done but they aren't really all that messy.

Let's do one of the other problems here so we can make a couple of points.

Example 2 Find a solution to the following partial differential equation.

$$\nabla^2 u_3 = \frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial y^2} = 0$$

$$u_3(0,y) = 0 \qquad\qquad u_3(L,y) = 0$$

$$u_3(x,0) = 0 \qquad\qquad u_3(x,H) = f_2(x)$$

Solution

Okay, for the first time we've hit a problem where we haven't previous done the separation of variables so let's go through that. We'll assume the solution is in the form,

$$u_3(x,y) = h(x)\,\varphi(y)$$

We'll apply this to the homogeneous boundary conditions first since we'll need those once we get reach the point of choosing the separation constant. We'll let you verify that the boundary conditions become,

$$h(0) = 0 \qquad\qquad h(L) = 0 \qquad\qquad \varphi(0) = 0$$

Next, we'll plug the product solution into the differential equation.

$$\frac{\partial^2}{\partial x^2}\left(h(x)\,\varphi(y)\right) + \frac{\partial^2}{\partial y^2}\left(h(x)\,\varphi(y)\right) = 0$$

$$\varphi(y)\frac{d^2 h}{dx^2} + h(x)\frac{d^2\varphi}{dy^2} = 0$$

$$\frac{1}{h}\frac{d^2 h}{dx^2} = -\frac{1}{\varphi}\frac{d^2\varphi}{dy^2}$$

Now, at this point we need to choose a separation constant. We've got two homogeneous boundary conditions on *h* so let's choose the constant so that the differential equation for *h* yields a familiar boundary value problem so we don't need to redo any of that work. In this case, unlike the u_4 case, we'll need $-\lambda$.

This is a good problem in that is clearly illustrates that sometimes you need λ as a separation constant and at other times you need $-\lambda$. Not only that but sometimes all it takes is a small change in the boundary conditions it force the change.

So, after adding in the separation constant we get,

$$\frac{1}{h}\frac{d^2 h}{dx^2} = -\frac{1}{\varphi}\frac{d^2\varphi}{dy^2} = -\lambda$$

and two ordinary differential equations that we get from this case (along with their boundary conditions) are,

$$\frac{d^2h}{dx^2} + \lambda h = 0 \qquad \frac{d^2\varphi}{dy^2} - \lambda\varphi = 0$$
$$h(0) = 0 \quad h(L) = 0 \qquad \varphi(0) = 0$$

Now, as we noted above when we were deciding which separation constant to work with we've already solved the first boundary value problem. So, the eigenvalues and eigenfunctions for the first boundary value problem are,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad h_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1,2,3,\ldots$$

The second differential equation is then,

$$\frac{d^2\varphi}{dx^2} - \left(\frac{n\pi}{L}\right)^2 \varphi = 0$$
$$\varphi(0) = 0$$

Because the coefficient of the φ is positive we know that a solution to this is,

$$\varphi(y) = c_1 \cosh\left(\frac{n\pi y}{L}\right) + c_2 \sinh\left(\frac{n\pi y}{L}\right)$$

In this case, unlike the previous example, we won't need to use a shifted version of the solution because this will work just fine with the boundary condition we've got for this. So, applying the boundary condition to this gives,

$$0 = \varphi(0) = c_1$$

and this solution becomes,

$$\varphi(y) = c_2 \sinh\left(\frac{n\pi y}{L}\right)$$

The product solution for this case is then,

$$u_n(x,y) = B_n \sinh\left(\frac{n\pi y}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1,2,3,\ldots$$

The solution to this partial differential equation is then,

$$u_3(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi y}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

Finally, let's apply the nonhomogeneous boundary condition to get the coefficients for this solution.

$$u_3(x,H) = f_2(x) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi H}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

As we've come to expect this is again a Fourier sine (although it won't always be a sine) series and so using previously done work instead of using the orthogonality of the sines to we see that,

$$B_n \sinh\left(\frac{n\pi H}{L}\right) = \frac{2}{L}\int_0^L f_2(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 1,2,3,\ldots$$

$$B_n = \frac{2}{L\sinh\left(\frac{n\pi H}{L}\right)}\int_0^L f_2(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 1,2,3,\ldots$$

We've worked two of the four cases that would need to be solved in order to completely solve (1). As we've seen each case was very similar and yet also had some differences. We saw the use of both separation constants and that sometimes we need to use a "shifted" solution in order to deal with one of the boundary conditions.

Before moving on let's note that we used prescribed temperature boundary conditions here, but we could just have easily used prescribed flux boundary conditions or a mix of the two. No matter what kind of boundary conditions we have they will work the same.

As a final example in this section let's take a look at solving Laplace's equation on a disk of radius *a* and a prescribed temperature on the boundary. Because we are now on a disk it makes sense that we should probably do this problem in polar coordinates and so the first thing we need to so do is write down Laplace's equation in terms of polar coordinates.

Laplace's equation in terms of polar coordinates is,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

This is a lot more complicated than the Cartesian form of Laplace's equation and it will add in a few complexities to the solution process, but it isn't as bad as it looks. The main problem that we've got here really is that fact that we've got a single boundary condition. Namely,

$$u(a,\theta) = f(\theta)$$

This specifies the temperature on the boundary of the disk. We are clearly going to need three more conditions however since we've got a 2nd derivative in both *r* and θ.

When we solved Laplace's equation on a rectangle we used conditions at the end points of the range of each variable and so it makes some sense here that we should probably need the same kind of conditions here as well. The range on our variables here are,

$$0 \le r \le a \qquad\qquad -\pi \le \theta \le \pi$$

Note that the limits on θ are somewhat arbitrary here and are chosen for convenience here. Any set of limits that covers the complete disk will work, however as we'll see with these limits we will get another familiar boundary value problem arising. The best choice here is often not known until the separation of variables is done. At that point you can go back and make your choices.

We now need conditions for $r = 0$ and $\theta = \pm\pi$. First, note that Laplace's equation in terms of polar coordinates is singular at $r = 0$ (*i.e.* we get division by zero). However, we know from physical considerations that the temperature must remain finite everywhere in the disk and so let's impose the condition that,

$$\left|u(0,\theta)\right| < \infty$$

This may seem like an odd condition and it definitely doesn't conform to the other boundary conditions that we've seen to this point, but it will work out for us as we'll see.

Now, for boundary conditions for θ we'll do something similar to what we did for the 1-D head equation on a thin ring. The two limits on θ are really just different sides of a line in the disk and so let's use the periodic conditions there. In other words,

$$u(-\pi,t) = u(\pi,t) \qquad\qquad \frac{\partial u}{\partial r}(-\pi,t) = \frac{\partial u}{\partial r}(\pi,t)$$

With all of this out of the way let's solve Laplace's equation on a disk of radius *a*.

Example 3 Find a solution to the following partial differential equation.

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\left|u(0,\theta)\right| < \infty \qquad\qquad u(a,\theta) = f(\theta)$$

$$u(-\pi,t) = u(\pi,t) \qquad\qquad \frac{\partial u}{\partial r}(-\pi,t) = \frac{\partial u}{\partial r}(\pi,t)$$

Solution

In this case we'll assume that the solution will be in the form,

$$u(\theta,r) = \varphi(\theta)G(r)$$

Plugging this into the periodic boundary conditions gives,

$$\varphi(-\pi) = \varphi(\pi) \qquad \frac{d\varphi}{d\theta}(-\pi) = \frac{d\varphi}{d\theta}(\pi)$$
$$|G(0)| < \infty$$

Now let's plug the product solution into the partial differential equation.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\big(\varphi(\theta)G(r)\big)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\big(\varphi(\theta)G(r)\big) = 0$$
$$\varphi(\theta)\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right) + G(r)\frac{1}{r^2}\frac{d^2\varphi}{d\theta^2} = 0$$

This is definitely more of a mess that we've seen to this point when it comes to separating variables. In this case simply dividing by the product solution, while still necessary, will not be sufficient to separate the variables. We are also going to have to multiply by r^2 to completely separate variables. So, doing all that, moving each term to one side of the equal sign and introduction a separation constant gives,

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\varphi}\frac{d^2\varphi}{d\theta^2} = \lambda$$

We used λ as the separation constant this time to get the differential equation for φ to match up with one we've already done.

The ordinary differential equations we get are then,

$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right) - \lambda G = 0 \qquad \frac{d^2\varphi}{d\theta^2} + \lambda\varphi = 0$$
$$|G(0)| < \infty \qquad \varphi(-\pi) = \varphi(\pi) \qquad \frac{d\varphi}{d\theta}(-\pi) = \frac{d\varphi}{d\theta}(\pi)$$

Now, we solved the boundary value problem above in Example 3 of the Eigenvalues and Eigenfunctions section of the previous chapter and so there is no reason to redo it here. The eigenvalues and eigenfunctions for this problem are,

$$\lambda_n = n^2 \qquad \varphi_n(\theta) = \sin(n\theta) \qquad n = 1, 2, 3, \ldots$$
$$\lambda_n = n^2 \qquad \varphi_n(\theta) = \cos(n\theta) \qquad n = 0, 1, 2, 3, \ldots$$

Plugging this into the first ordinary differential equation and using the product rule on the derivative we get,

$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right) - n^2 G = 0$$
$$r\left(r\frac{d^2G}{dr^2} + \frac{dG}{dr}\right) - n^2 G = 0$$
$$r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - n^2 G = 0$$

This is an Euler differential equation and so we know that solutions will be in the form $G(r) = r^p$ provided p is a root of,

$$p(p-1) + p - n^2 = 0$$
$$p^2 - n^2 = 0 \qquad \Rightarrow \qquad p = \pm n \qquad n = 0, 1, 2, 3, \ldots$$

So, because the $n = 0$ case will yield a double root, versus two real distinct roots if $n \neq 0$ we have two cases here. They are,

$$G(r) = c_1 + c_2 \ln r \qquad n = 0$$
$$G(r) = \overline{c}_1 r^n + \overline{c}_2 r^{-n} \qquad n = 1, 2, 3, \ldots$$

Now we need to recall the condition that $|G(0)| < \infty$. Each of the solutions above will have $G(r) \to \infty$ as $r \to 0$ Therefore in order to meet this boundary condition we must have $c_2 = \overline{c}_2 = 0$.

Therefore, the solution reduces to,

$$G(r) = c_1 r^n \qquad n = 0, 1, 2, 3, \ldots$$

and notice that with the second term gone we can combine the two solutions into a single solution.

So, we have two product solutions for this problem. They are,

$$u_n(\theta, r) = A_n r^n \cos(n\theta) \qquad n = 0, 1, 2, 3, \ldots$$
$$u_n(\theta, r) = B_n r^n \sin(n\theta) \qquad n = 1, 2, 3, \ldots$$

Our solution is then the sum of all these solutions or,

$$u(\theta, r) = \sum_{n=0}^{\infty} A_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n r^n \sin(n\theta)$$

Applying our final boundary condition to this gives,

$$u(a, \theta) = f(\theta) = \sum_{n=0}^{\infty} A_n a^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n a^n \sin(n\theta)$$

This is a full Fourier series for $f(\theta)$ on the interval $-\pi \le \theta \le \pi$, *i.e.* $L = \pi$. Also note that once again the "coefficients" of the Fourier series are a little messier than normal, but not quite as messy as when we were working on a rectangle above. We could once again use the orthogonality of the sines and cosines to derive formulas for the A_n and B_n or we could just use the formulas from the Fourier series section with $L = \pi$ to get,

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta$$
$$A_n a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta)\, d\theta \qquad n = 1, 2, 3, \ldots$$
$$B_n a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta)\, d\theta \qquad n = 1, 2, 3, \ldots$$

Upon solving for the coefficients we get,

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta$$
$$A_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta)\, d\theta \qquad n = 1, 2, 3, \ldots$$
$$B_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta)\, d\theta \qquad n = 1, 2, 3, \ldots$$

Prior to this example most of the separation of variable problems tended to look very similar and it is easy to fall in to the trap of expecting everything to look like what we'd seen earlier. With this example we can see that the problems can definitely be different on occasion so don't get too locked into expecting them to always work in exactly the same way.

Before we leave this section let's briefly talk about what you'd need to do on a partial disk. The periodic boundary conditions above were only there because we had a whole disk. What if we only had a disk between say $\alpha \le \theta \le \beta$.

When we've got a partial disk we now have two new boundaries that we not present in the whole disk and the periodic boundary conditions will no longer make sense. The periodic boundary conditions are only used when we have the two "boundaries" in contact with each other and that clearly won't be the case with a partial disk.

So, if we stick with prescribed temperature boundary conditions we would then have the following conditions

$$\begin{aligned} &\left|u(0,\theta)\right| < \infty & & \\ &u(a,\theta) = f(\theta) & &\alpha \le \theta \le \beta \\ &u(r,\alpha) = g_1(r) & &0 \le r \le a \\ &u(r,\beta) = g_2(r) & &0 \le r \le a \end{aligned}$$

Also note that in order to use separation of variables on these conditions we'd need to have $g_1(r) = g_2(r) = 0$ to make sure they are homogeneous.

As a final note we could just have easily used flux boundary conditions for the last two if we'd wanted to. The boundary value problem would be different, but outside of that the problem would work in the same manner.

We could also use a flux condition on the $r = a$ boundary but we haven't really talked yet about how to apply that kind of condition to our solution. Recall that this is the condition that we apply to our solution to determine the coefficients. It's not difficult to use we just haven't talked about this kind of condition yet. We'll be doing that in the next section.

Vibrating String

This will be the final partial differential equation that we'll be solving in this chapter. In this section we'll be solving the 1-D wave equation to determine the displacement of a vibrating string. There really isn't much in the way of introduction to do here so let's just jump straight into the example.

Example 1 Find a solution to the following partial differential equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

$$u(0,t) = 0 \qquad u(L,t) = 0$$

Solution

One of the main differences here that we're going to have to deal with is the fact that we've now got two initial conditions. That is not something we've seen to this point, but will not be all that difficult to deal with when the time rolls around.

We've already done the separation of variables for this problem, but let's go ahead and redo it here so we can say we've got another problem almost completely worked out.

So, let's start off with the product solution.

$$u(x,t) = \varphi(x)h(t)$$

Plugging this into the two boundary conditions gives,

$$\varphi(0) = 0 \qquad \varphi(L) = 0$$

Plugging the product solution into the differential equation, separating and introducing a separation constant gives,

$$\frac{\partial^2}{\partial t^2}\left(\varphi(x)h(t)\right) = c^2 \frac{\partial^2}{\partial x^2}\left(\varphi(x)h(t)\right)$$

$$\varphi(x)\frac{d^2 h}{dt^2} = c^2 h(t)\frac{d^2\varphi}{dx^2}$$

$$\frac{1}{c^2 h}\frac{d^2 h}{dt^2} = \frac{1}{\varphi}\frac{d^2\varphi}{dx^2} = -\lambda$$

We moved the c^2 to the left side for convenience and chose $-\lambda$ for the separation constant so the differential equation for φ would match a known (and solved) case.

The two ordinary differential equations we get from separation of variables are then,

$$\frac{d^2h}{dt^2}+c^2\lambda h=0 \qquad\qquad \begin{aligned}&\frac{d^2\varphi}{dx^2}+\lambda\varphi\\&\varphi(0)=0 \qquad \varphi(L)=0\end{aligned}$$

We solved the boundary value problem above in Example 1 of the Solving the Heat Equation section of this chapter and so the eigenvalues and eigenfunctions for this problem are,

$$\lambda_n=\left(\frac{n\pi}{L}\right)^2 \qquad \varphi_n(x)=\sin\left(\frac{n\pi x}{L}\right) \qquad n=1,2,3,\ldots$$

The first ordinary differential equation is now,

$$\frac{d^2h}{dt^2}+\left(\frac{n\pi c}{L}\right)^2h=0$$

and because the coefficient of the *h* is clearly positive the solution to this is,

$$h(t)=c_1\cos\left(\frac{n\pi c\,t}{L}\right)+c_2\sin\left(\frac{n\pi c\,t}{L}\right)$$

Because there is no reason to think that either of the coefficients above are zero we then get two product solutions,

$$\begin{aligned}u_n(x,t)&=A_n\cos\left(\frac{n\pi c\,t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\\u_n(x,t)&=B_n\sin\left(\frac{n\pi c\,t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\end{aligned} \qquad n=1,2,3,\ldots$$

The solution is then,

$$u(x,t)=\sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi c\,t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)+B_n\sin\left(\frac{n\pi c\,t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\right]$$

Now, in order to apply the second initial condition we'll need to differentiate this with respect to *t* so,

$$\frac{\partial u}{\partial t}=\sum_{n=1}^{\infty}\left[-\frac{n\pi c}{L}A_n\sin\left(\frac{n\pi c\,t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)+\frac{n\pi c}{L}B_n\cos\left(\frac{n\pi c\,t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\right]$$

If we now apply the initial conditions we get,

$$u(x,0) = f(x) = \sum_{n=1}^{\infty}\left[A_n \cos(0)\sin\left(\frac{n\pi x}{L}\right) + B_n \sin(0)\sin\left(\frac{n\pi x}{L}\right)\right] = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$
$$\frac{\partial u}{\partial t}(x,0) = g(x) = \sum_{n=1}^{\infty}\frac{n\pi c}{L} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Both of these are Fourier sine series. The first is for $f(x)$ on $0 \le x \le L$ while the second is for $g(x)$ on $0 \le x \le L$ with a slightly messy coefficient. As in the last few sections we're faced with the choice of either using the orthogonality of the sines to derive formulas for A_n and B_n or we could reuse formula from previous work.

It's easier to reuse formulas so using the formulas form the Fourier sine series section we get,

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 1,2,3,\ldots$$
$$\frac{n\pi c}{L} B_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 1,2,3,\ldots$$

Upon solving the second one we get,

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 1,2,3,\ldots$$
$$B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 1,2,3,\ldots$$

So, there is the solution to the 1-D wave equation and with that we've solved the final partial differential equation in this chapter.

Two dimensional wave Equation:

Examples of two dimensional waves:

Motion of a stretched elastic membrane such as a drumhead.

Two dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \text{where } c^2 = \frac{T}{\rho}$$

Solution of two dimensional wave equation (Using Double Fourier Series):

We know the two dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \text{where } c^2 = \frac{T}{\rho} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots[1$$

And let its solution be $u(x, y, t)$

And the boundary conditions are

$u = 0$ on the boundary ofd the membrane for all $t \geq 0$ ....[2

The two initial conditions

$u(x, y, 0) = f(x, y)$ (where $f(x, y)$ will be the given initial displacement) ....[3

$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x, y)$ where $g(x, y)$ will be the given initial velocity ....[4

The solution $u(x, y, t)$means the displacement of the point (x,y) of the membrane from rest $(u = 0)$at time t.

Step-1:

Let the solution is $u(x, y, t) = F(x, y)G(t)$ ....[5

Substituting the equation (5) in wave equation(1) we have

$F(x, y)\frac{\partial^2 G}{\partial t^2} = c^2(F_{xx}G + F_{yy}G)$ where subscripts denotes the partial derivatives.

Dividing c^2FG to both side we get

$\frac{\frac{\partial^2 G}{\partial t^2}}{c^2 G} = \frac{1}{F}(F_{xx} + F_{yy})$ as it is a proportionality we can consider as a constant

$\frac{\frac{\partial^2 G}{\partial t^2}}{c^2 G} = \frac{1}{F}\left(F_{xx} + F_{yy}\right) = \gamma^2$ say

This gives two equations one is for time function $G(t)$

i.e. $\frac{\partial^2 G}{\partial t^2} + \lambda^2 G = 0$ where $\lambda = c\gamma$ ....[6

and other is the amplitude function $F(x, y)$

i.e. $F_{xx} + F_{yy} + \gamma^2 F = 0$ (It is called as **Helmholtz Equation**) ....[7

Now let us solve the equation (7) by variable separation method.

Let $F(x, y) = H(x)Q(y)$ ....[8

Substituting equation (8) in equation (7) we get

$$\frac{d^2H}{dx^2}Q = -\left(H\frac{d^2Q}{dy^2} + \gamma^2 HQ\right)$$

Now dividing HQ in the above equation we get

$$\frac{1}{H}\ \frac{d^2H}{dx^2}Q = -\frac{1}{Q}\left(H\frac{d^2Q}{dy^2} + \gamma^2 Q\right)$$

As by this the separation is possible we can consider this as proportional constant as

$$\frac{1}{H}\ \frac{d^2H}{dx^2} = -\frac{1}{Q}\left(\frac{d^2Q}{dy^2} + \gamma^2 Q\right) = -k^2$$

From this we can yield two ordinary differential equation for H and Q as

$\frac{d^2H}{dx^2} + k^2 H = 0$ ....[9

$\frac{d^2Q}{dy^2} + p^2Q = 0$ where $p^2 = \gamma^2 - k^2$ ....[10

Step-2

The general solution of equation (9) is $H(x) = A\, coskx + B\, Sinkx$

The general solution of equation (10) is $Q(y) = C\, cos\, py + Dsin\, py$

Where A B C D are constants. which are to be determined due to boundary conditions.

In two dimensional wave for all time $t \geq 0$ the deflection $u(x,y) = 0$ .That means the membrane is not vibrating at the boundary. As solution of $F(x,y) = H(x)Q(y)$ says solution about x,y we use the required boundary condition for x and y i.e.

$$H(0) = 0, \qquad H(a) = 0, \qquad Q(y) = 0, \qquad Q(b) = 0$$

Using this boundary condition in the general solution H(x) we get

$$H(0) = A\, Cos0 + B\, sin0 = 0 \Rightarrow A = 0$$

$$H(a) = A\, coska + B\, sinka = 0 \Rightarrow H(a) = 0 + B\, sinka = 0 \Rightarrow B\, sinka = 0$$

If we take $B \neq 0$ then $sinka = 0 \Rightarrow k = \frac{m\pi}{a}$ where m is an integer. Hence $H_m(x) = sin\frac{m\pi x}{a}$

In similar fashion we can get solution for , $Q(y)$ by using the corresponding boundary condition $Q(y) = 0$, $Q(b) = 0$ we get $Q_n(y) = sin\frac{n\pi y}{b}$

Hence the equation (8) reduces to the form

$F_{mn}(x,y) = H_m(x)Q_n(y) = sin\frac{m\pi x}{a}\, sin\frac{n\pi y}{b}$ for m and n are +ve integer ....[11

As in equation (60 and (100 we have considered $\lambda = c\gamma$ and $p^2 = \gamma^2 - k^2$ then this two gives

$$\lambda = c\sqrt{k^2 + p^2} \quad \Rightarrow \lambda = \lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \quad ....[12$$

Hence using equation (12) in the solution of (6) we get

$$G_{mn}(t) = B_{mn}cos\lambda_{mn}t + B_{mn}{}^{*}sin\lambda_{mn}t$$

Hence as per the equation (5) the solution is

$$u_{mn}(x,y,t) = (B_{mn}cos\lambda_{mn}t + B_{mn}{}^{*}sin\lambda_{mn}t)\, sin\frac{m\pi x}{a}\, sin\frac{n\pi y}{b} \quad ....[13$$

Step-3

Now using the Fourier series concept and initial velocity and deflection we get

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(B_{mn}cos\lambda_{mn}t + B_{mn}{}^{*}sin\lambda_{mn}t)\, sin\frac{m\pi x}{a}\, sin\frac{n\pi y}{b}$$

....[17

Then for $t = 0$we have $u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\, sin\frac{m\pi x}{a}\, sin\frac{n\pi y}{b} = f(x,y)$....................[18

Using generalized Euler formula for coefficient B_{mn} we get

$$B_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)sin\frac{m\pi x}{a}\, sin\frac{n\pi y}{b}\, dxdy \quad ....[19$$

Again using the condition $\left.\frac{\partial u}{\partial t}\right|_{t=0}$ we get

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}{}^{*}\, \lambda_{mn}\, sin\frac{m\pi x}{a}\, sin\frac{n\pi y}{b} = g(x,y) \quad ....[20$$

Using generalized Euler formula for coefficient $B_{mn}{}^{*}$ we get

$B^*_{mn} = \frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a g(x,y)sin\frac{m\pi x}{a}\ sin\frac{n\pi y}{b}dxdy$ ....[21

Ex: Find the vibration of a rectangular membrane of sides a=4ft and b=2ft. if the tension is 12.5lb/ft, the density is 2.5slugs/ft^2, the initial velocity is 0, and he initial displacement is

$f(x,y) = 0.1(4x - x^2)(2y - y^2)ft$

Ans:

We know the two dimensional wave equation is

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}u_{mn}(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(B_{mn}cos\lambda_{mn}t + B_{mn}{}^*sin\lambda_{mn}t)\ sin\frac{m\pi x}{a}\ sin\frac{n\pi y}{b}$$

For t=0 initial deflection is given as $f(x,y) = 0.1(4x - x^2)(2y - y^2)ft$ and initial velocity is zero.. So $B_{mn}{}^* = 0$ Hence we have to find only B_{mn}

$$B_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)sin\frac{m\pi x}{a}\ sin\frac{n\pi y}{b}dxdy$$

$$= \frac{4}{4.2}\int_0^2\int_0^4 0.1(4x - x^2)(2y - y^2)sin\frac{m\pi x}{4}\ sin\frac{n\pi y}{2}dxdy$$

$$= \frac{1}{20}\int_0^4(4x - x^2)\ sin\frac{m\pi x}{4}dx\int_0^2(2y - y^2)\ sin\frac{n\pi y}{2}dy$$

Integrating both the integrals by parts we get

The 1st integral is $\frac{128}{m^3\pi^3}[1 - (-1)^m = \frac{256}{m^3\pi^3}$ where m is odd

The 2nd integral is $\frac{16}{n^3\pi^3}[1 - (-1)^n = \frac{32}{n^3\pi^3}$ where n is odd

Hence $B_{mn} = \frac{1}{20}.\frac{256}{m^3\pi^3}.\frac{32}{n^3\pi^3}$ and

$$\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} = c\pi\sqrt{\frac{m^2}{4^2} + \frac{n^2}{2^2}}$$

Again we can get the value of c from the formula $c^2 = \frac{T}{\rho} = \frac{12.5}{2.5} = 5$

Hence the solution is

$$u(x,y,t) = \frac{256\ .32}{20}\sum_{m\ as\ odd}\sum_{n\ as\ odd}\frac{1}{m^3\pi^3}\ \cos\left(5\pi\sqrt{\frac{m^2}{4^2} + \frac{n^2}{2^2}}\right)t\ \sin\ sin\frac{m\pi x}{4}\ sin\frac{n\pi y}{2}$$

Ex:

Laplacian in polar coordinates:

In the discussion of two dimensional wave equation we have considered the rectangular membrane with the boundary condition. Sometimes we come across the vibrating circular membrane like drum heads. To discuss about circular membrane it is better to consider polar coordinate form of the equation.

Conversion of laplacian to polar form:

The laplacian is $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

Let $x = r\,cos\theta \qquad y = r\,sin\theta$ ……………………………………………………………[*

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$\Rightarrow u_x = u_r r_x + u_\theta \theta_x$ as x is a function of two variables r and θ

Similarly again differentiating u_x with respect to x we get

$$u_{xx} = (u_r r_x)_x + (u_\theta \theta_x)_x$$
$$=(u_r)_x r_x + u_r (r_x)_x \;+\; (u_\theta)_x \theta_x + u_\theta(\theta_x)_x$$
$$=\left(\frac{\partial}{\partial r}u_r\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}u_r\frac{\partial \theta}{\partial x}\right)r_x + u_r r_{xx} + \left(\frac{\partial}{\partial r}u_\theta\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}u_\theta\frac{\partial \theta}{\partial x}\right)\theta_x + u_\theta\theta_{xx}$$
$$=(u_{rr}r_x + u_{r\theta}\theta_x)r_x + u_r r_{xx} + (u_{r\theta}r_x + u_{\theta\theta}\theta_x)\theta_x + u_\theta\theta_{xx} \;\text{………..[1}$$

The above equation contains r_x, r_{xx}, θ_x, θ_{xx}, which are to be determined.

From the parametric form given in (*) we get $r = \sqrt{x^2+y^2}$ and $\theta = tan^{-1}\frac{y}{x}$

Hence $\frac{\partial r}{\partial x} = r_x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$ and $\theta_x = \frac{1}{1+\left(y/x\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{r^2}$

Again differentiating the above two equation we get

$$r_{xx} = \frac{r - x r_x}{r^2} = \frac{1}{r} - \frac{x^2}{r^3} = \frac{y^2}{r^3} \qquad \text{and} \qquad \theta_{xx} = -y\left(-\frac{2}{r^3}\right)r_x = \frac{2xy}{r^4}$$

Now substituting all these values in equation (1) and assuming continuity of the 1st and 2nd partial derivatives we have $u_{r\theta} = u_{\theta r}$ we get

$$u_{xx} = \frac{x^2}{r^2}u_{rr} - 2\frac{xy}{r^3}u_{r\theta} + \frac{y^2}{r^4}u_{\theta\theta} + \frac{y^2}{r^3}u_r + 2\frac{xy}{r^4}u_\theta \;\text{…………………………………….[2}$$

In similar manner again we can find

$$u_{yy} = \frac{y^2}{r^2}u_{rr} + 2\frac{xy}{r^3}u_{r\theta} + \frac{x^2}{r^4}u_{\theta\theta} + \frac{x^2}{r^3}u_r - 2\frac{xy}{r^4}u_\theta \;\text{…………………………..[3}$$

Then by adding equation (2) and (3) we get the laplacian of u in polar coordinates as

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \;\text{……………………………………………………………[4}$$

Q: Show that the Laplacian of u in cylindrical coordinate is

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$
$$=u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}$$

Q: Show that an alternative form of laplacian in polar coordinate is

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Circular Membrane (Use of Fourier-Bessel Series):

Two dimensional wave equation in polar form is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$$

It's solution means we have to find the deflection $u(r,t)$ that are radially symmetric ,that do not depend on θ. So the equation reduces to

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \text{...} [1$$

The boundary condition is

$u(R,t) = 0$ for all $t \geq 0$ ....[2

$u(R,0) = f(r)$ initial deflection f(r) ....[3

$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r)$ where initial velocity g(r)(r) ....[4

It's solution is $u(r,t) = W_m(r)G_m(t) = (a_m \cos\lambda_m t + b_m \sin\lambda_m t)\, J_0(k_m r)$ ....[5

Where $a_m = \frac{2}{R^2 J_1^2(\alpha_m)} \int_0^R r f(r) J_0\left(\frac{\alpha_m}{R} r\right) dr$

Ex . Find the vibration of a circular drumhead of radius 1ft and density 2 slugs/ft^2 if the tension is 8lb/ft, the initial velocity is 0, and initial displacement is $f(r) = 1 - r^2$.

Ans $c^2 = \frac{T}{\rho} = \frac{8}{2} = 4$ Also $b_m = 0$. Since the initial velocity is o and R=1

$$\text{Now } a_m = \frac{2}{R^2 J_1^2(\alpha_m)} \int_0^R r f(r) J_0\left(\frac{\alpha_m}{R} r\right) dr$$

$$= \frac{2}{J_1^2(\alpha_m)} \int_0^1 r(1 - r^2) J_0(a_m r) dr$$

$$= \frac{4 J_2(\alpha_m)}{\alpha_m^2 J_1^2(\alpha_m)} = \frac{8}{\alpha_m^3 J_1(\alpha_m)}$$

$$J_2(\alpha_m) = \frac{2}{\alpha_m} J_1(\alpha_m) - J_0(\alpha_m) = \frac{2}{\alpha_m} J_1(\alpha_m)$$

Laplace's Equation in cylindrical and Spherical coordinates:

Laplace equation in in three dimensional coordinate system is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \text{ ....} [1$$

$$\Rightarrow \nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$$

This equation has great application in physics and engineering. The theory of solution of this equation is called potential theory. The solution of equation (1) that have continuous 2nd order partial derivatives are called harmonic function. Generally this equation come across in gravitation, electrostatics, steady state heat flow, and fluid flow.

Laplacian in cylindrical coordinates:

In cylindrical coordinate system we consider $x = r\, cos\theta \quad y = r\, sin\theta \quad and \quad z = z$

Already we have derived the laplacian in polar coordinate in two dimensional as

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Then just by adding the component for z coordinate we get laplacian in polar coordinate in three dimension as

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \text{ ....} [2$$

Laplacian in Spherical Coordinates:

Spherical symmetry (a ball as region T bounded by a sphere S) requires spherical coordinates $r,\ \theta,\ \emptyset$ related to x,y,z by

$$x = r\cos\theta\sin\emptyset, \quad y = r\sin\theta\sin\emptyset, \quad z = r\cos\emptyset$$

Then using the chan rule and proceeding as two dimensional case we get the laplacian in spherical coordinates is

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \emptyset^2} + \frac{\cot\emptyset}{r^2}\frac{\partial u}{\partial \emptyset} + \frac{1}{r^2\sin^2\emptyset}\frac{\partial^2 u}{\partial \theta^2}$$

This equation can also be written as

$$\nabla^2 u = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\emptyset}\frac{\partial}{\partial \emptyset}\left(\sin\emptyset\frac{\partial u}{\partial \emptyset}\right) + \frac{1}{\sin^2\emptyset}\frac{\partial^2 u}{\partial \theta^2}\right]$$

Solution of partial Differential equation by laplace Transform.

Laplace transform can also be used to solve partial differential equation. As Laplace transform is defined only for $t \geq 0$ it can be applied to partial differential equation when one of the independent variables ranges over the +ve axis

Working procedure to solve a PDE by using LT:

Step-1 take laplace transform with respect to one of the two variables, usually t. This gives an ordinary differential equation for the transform of the unknown function.

Step-2 Solving that ordinary differential equation, obtain the transform of the unknown function.

Step-3 Taking the inverse transform , obtain the solution of the given equation.

Ex. Solve the PDE by using LT

$$\frac{\partial u}{\partial x} + 2x\frac{\partial y}{\partial t} = 2x \qquad u(x,0) = 1, \quad u(0,t) = 1$$

Ans:

Step-1 Taking the LT of both side we get

$$L\left(\frac{\partial u}{\partial x}\right) + L\left(2x\frac{\partial u}{\partial t}\right) = L(2x\ ) \quad \text{..........[1}$$

Let us denote $L\{u(x,t)\} = U(x,s)$ and $L\left(\frac{\partial u}{\partial x}\right) = \frac{\partial U}{\partial x}$ also we use the LT of derivatives as $L\left\{\frac{\partial u}{\partial t}\right\} = sU - u(x,0)$ now using all this condition to equation (1) we get

$$\frac{\partial U}{\partial x} + 2x(sU - 1) = \frac{2x}{s} \quad \text{..........[2}$$

$$\Rightarrow \frac{\partial U}{\partial x} + 2xsU - 2x = \frac{2x}{s}$$

$$\Rightarrow \frac{\partial U}{\partial x} + 2xsU = 2x(1 + \frac{1}{s}) \quad \text{.......... [3}$$

This involves partial differential equation w.r.t only one variable. and hence it is a linear differential equation.

So the integrating factor IF $= e^{\int 2xsdx} = e^{sx^2}$

So the solution of equation (3) is

$U(IF) = \int (IF)2x\left(1 + \frac{1}{s}\right)dx + c(s)$ where c(s) is a constant of integration

$$\Rightarrow Ue^{sx^2}\int e^{sx^2}.2x\left(1+\frac{1}{s}\right)dx+c(s)$$

$$=\left(1+\frac{1}{s}\right)\int e^{sx^2}.2xdx+c(s)=\left(s+\frac{1}{s}\right)\left(e^{sx^2}/s\right)+c(s)$$

$$\Rightarrow U(x,s)=\left(s+\frac{1}{s^2}\right)+e^{-sx^2}\mathrm{c(s)} \text{ ....[4}$$

Given that $u(0,t)=1 \Rightarrow L\{u(0,t)\}=L\{1\}\Rightarrow U(0,s)=\frac{1}{s}$

So equation (4) gives $\frac{1}{s}=s+\frac{1}{s^2}+c(s)$

$$\Rightarrow c(s)=\frac{1}{s}-s-\frac{1}{s^2}\Rightarrow c(s)=-\frac{1}{s^2}$$

Using equation (4) gives $U=\frac{s+1}{s^2}+e^{-sx^2}\left(-\frac{1}{s^2}\right)$

$$\Rightarrow U(x,s)=\frac{s+1}{s^2}-\frac{e^{-sx^2}}{\frac{1}{s^2}}=\frac{1}{s}+\frac{1}{s^2}-\frac{e^{-sx^2}}{s^2}$$

Taking the inverse laplace transform we get

$$U(x,t)=1+t+yu(t-x^2)$$

COMPLEX NUMBERS AND FUNCTIONS, CONFORMAL MAPPING

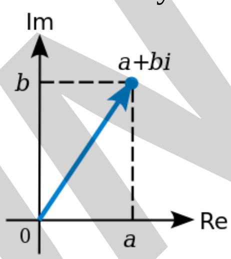
12.1 Complex number, Complex plane

A complex number is a number that can be expressed in the form a + bi, where a and b are real numbers and i is the imaginary unit, that satisfies the equation $x^2=-1$, that is, $i^2=-1$. In this expression, a is the real part and b is the imaginary part of the complex number.

The real number a is called the *real part* of the complex number $a+bi$; the real number b is called the *imaginary part* of $a+bi$. The real part of a complex number z is denoted by $\mathrm{Re}(z)$; the imaginary part of a complex number z is denoted by $\mathrm{Im}(z)$. For example,

The set of all complex numbers is denoted by $\mathbf{C}$ or $\mathbb{C}$.

A complex number can be viewed as a point or position vector in a two-dimensional Cartesian coordinate system called the complex plane or Argand diagram



Two complex numbers are equal if and only if both their real and imaginary parts are equal.
The complex conjugate of the complex number $z=x+yi$ is defined to be $x-yi$. It is denoted or z^*.
Formally, for any complex number z:
(a+bi) + (c+di) = (a+c) + (b+d)i

Example: (3 + 2i) + (1 + 7i) = (4 + 9i)
(a+bi) - (c+di) = (a-c) + (b-d)i
Example: (3 + 2i) - (1 + 7i) = (2 -5i)
(a+bi)(c+di) = (ac-bd) + (ad+bc)i
Example: (3 + 2i)(1 + 7i) = (3×1 - 2×7) + (3×7 + 2×1)i = -11 + 23i

$$\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)}$$

Example

$$\frac{2 + 3\mathbf{i}}{4 - 5\mathbf{i}}$$

Multiply top and bottom by the conjugate of 4 - 5i :

$$\frac{2+i3}{4-i5} = \frac{(2+i3)(4+i5)}{(4-i5)(4+i5)}$$

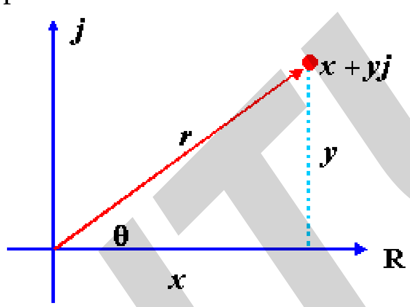
And then back into a + bi form:

$\frac{-7}{41} + i\frac{22}{41}$

Polar form of a complex numbers, powers and roots

Polar Form of a Complex Number

complex numbers in terms of a distance from the origin and a direction (or angle) from the positive horizontal axis.



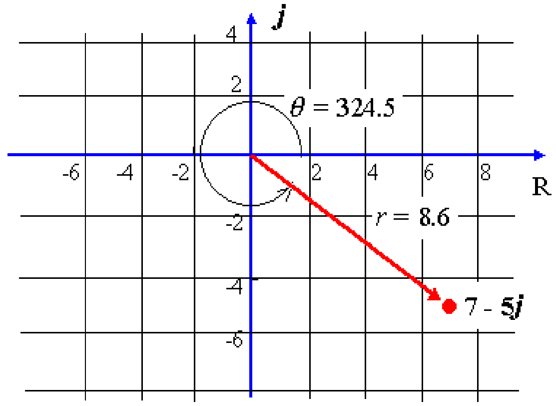
We find the real (horizontal) and imaginary (vertical) components in terms of r (the length of the vector) and θ(the angle made with the real axis):
From Pythagoras, we have:$r^2 = x^2 + y^2$ and basic trigonometry gives us:
tan θ=$\frac{y}{x}$
r is the **absolute value** (or **modulus**) of the complex number
θ is the **argument** of the complex number.

Example 1

Find the polar form and represent graphically the complex number 7−5j.
Ans)

We need to find r and θ.

$r = |z| = \sqrt{7^2 + (-5)^2} = \sqrt{74}$ $\qquad$ $\vartheta = Arg(z) = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left(\frac{-5}{7}\right)$



De Moivre's Theorem: $\left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)$

Example 1: $$(1+i)^7 = \left[\sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\right]^7$$
$$= \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^7$$
$$= \left(\sqrt{2}\right)^7\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$$
$$= 8\sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$
$$= 8 - 8i$$

Example 2: $$i^{1/3} = \left[1(0 + 1i)\right]^{1/3}$$
$$= \left[1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\right]^{1/3} \text{ or } \left[1\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right)\right]^{1/3} \text{ or } \left[1\left(\cos\frac{9\pi}{2} + i\sin\frac{9\pi}{2}\right)\right]^{1/3}$$
$$= 1^{1/3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \text{ or } 1^{1/3}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) \text{ or } 1^{1/3}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$
$$= \frac{\sqrt{3}}{2} + \frac{1}{2}i \quad \text{or} \quad -\frac{\sqrt{3}}{2} + \frac{1}{2}i \quad \text{or} \quad -i$$

***Fundamental Theorem of Algebra*.**

Let z = r (cos θ + i sin θ) be an arbitrary complex number. Then the n nth roots w of z are each of the form

$w = \sqrt[n]{r}\,\{\cos[(\theta + 2\pi m)/n] + i\sin[(\theta + 2\pi m)/n]\}$

where m is an integer ranging from 0 to n-1.

Example 1: Compute the two square roots of i.

Solution: It is easy to see that i has the polar form cos π/2 + i sin π/2. Thus, by Lemma 6.6.4, its square roots are cos π/4 + i sin π/4 = √2/2 + √2/2 i and cos 3π/4 + i sin 3π/4 = -√2/2 - √2/2 i.

Example 2 Compute the three cube roots of -8.

Solution: Since -8 has the polar form 8 (cos π + i sin π), its three cube roots have the form $\sqrt[3]{8}$ {cos[(π + 2πm)/3] + i sin[(π + 2πm)/3]} for m=0, 1, and 2. Thus the roots are 2 (cos π/3 + i sin π/3) = 1 + √3 i, 2 (cos π + i sin π) = -2, and 2 (cos 5π/3 + i sin 5π/3) = 1 - √3 i.

Derivative analytic function

Differention

Using our imagination, we take our lead from elementary calculus and define the derivative of $f(z)$ at z_0 , written $f'(z_0)$, by

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

provided that the limit exists. If it does, we say that the function $f(z)$ is differentiable at z_0. If we write $\Delta z = z - z_0$,

then we can express

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

If we let $w = f(z)$ and $\Delta w = f(z) - f(z_0)$, then we can use the Leibniz's notation $\frac{dw}{dz}$ for the derivative:

$$f'(z_0) = \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}.$$

Explore the Derivative.

Example: Use the limit definition to find the derivative of $f(z) = z^3$.

$f(z)=z^3 \quad at \quad z = z_0$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{z^3 - z_0^3}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{(z^2 + z z_0 + z_0^2)(z - z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} (z^2 + z z_0 + z_0^2)$$

$$= (z_0^2 + z_0 z_0 + z_0^2)$$

$$= 3 z_0^2$$

Example 3.2. Show that the function $f(z) = \bar{z}$ is nowhere differentiable.
Refer [page no- 667 in kreyszig]

ANALYTIC FUNCTION

A function $f(z)$ is said to be analytic in a domain D if $f(z)$ is defined and differentiable at all points of D. The function $f(z)$ is said to be analytic at a point $z = z_0$ in D if $f(z)$ is analytic in a neighborhood of z_0 .

Def (Entire Function). If $f(z)$ is analytic on the whole complex plane then $f(z)$ is said to be an entire function.

Points of non-analyticity for a function are called singular points.

Most special functions are analytic (at least in some range of the complex plane). Typical examples of analytic functions are:

1) Any polynomial (real or complex) is an analytic function. This is because if a polynomial has degree *n*, any terms of degree larger than *n* in its Taylor series expansion must immediately vanish to 0, and so this series will be trivially convergent. Furthermore, every polynomial is its own Maclaurin series.
2) The exponential function is analytic. Any Taylor series for this function converges not only for *x* close enough to x_0 (as in the definition) but for all values of *x* (real or complex).
3) The trigonometric functions, logarithm, and the power functions are analytic on any open set of their domain.

Typical examples of functions that are not analytic are:

1) The absolute value function when defined on the set of real numbers or complex numbers is not everywhere analytic because it is not differentiable at 0.Piecewise defined functions (functions given by different formulas in different regions) are typically not analytic where the pieces meet.
2) The complex conjugate function z → z* is not complex analytic, although its restriction to the real line is the identity function and therefore real analytic, and it is real analytic as a function from **R**² to **R**².

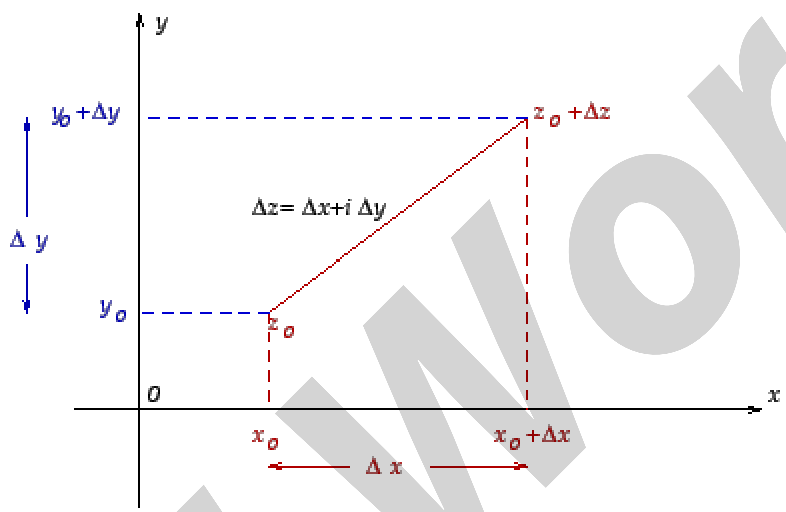
Cauchy Riemann equation, Laplace equation

Cauchy-Riemann Equations.

Recall the definition of the derivative of a function $f(z)$ at a point :

(z_0) $\quad f(z_0+\Delta z) - f(z_0)$

Denote: $f(z)$ $u(x\ y)$ $iv(x\ y)$
where , Δx , , $u(x\ y)$ and $v(x\ y)$ are real.



Suppose that ; thus we have:

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{[u(x_0+\Delta x, y_0) + iv((x_0+\Delta x, y_0)] - [u(x_0, y_0) + iv((x_0, y_0)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v((x_0+\Delta x, y_0) - v((x_0, y_0)}{\Delta x}$$

$$= u_x(x_0, y_0) + iv_x(x_0, y_0)$$

Suppose that $\Delta x = 0$; thus we have:

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{[u(x_0, y_0+\Delta y) + iv((x_0, y_0+\Delta y)] - [u(x_0, y_0) + iv((x_0, y_0)]}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v((x_0, y_0+\Delta y) - v((x_0, y_0)}{i\Delta y}$$

$$= -iv_y(x_0, y_0) + v_y(x_0, y_0)$$

Hence we have the so-called Cauchy-Riemann Equations:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases}$$

which can be wriiten in the following form, with a notation frequently used in Calculus:

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

If $f(z) = u(x,y) + iv(x,y)$ is derivable at $z_0 = x_0 + iy_{0,}$, then u and v verify the Cauchy-Riemann Equations at (x_0, y_0).

Example Let $(z) = z^2$. As a polynomial function, $f(z)$ is derivable over the whole of $\mathbb{C}$.

Let us check the Cauchy-Riemann equations. Denote $= x + iy \quad x, y \in R$. Then we have:

$$f(z) = (x+iy)^2 = \underbrace{x^2 - y^2}_{=u(x,y)} + i \underbrace{2xy}_{=v(x,y)}.$$

It follows that:

$$\begin{cases} u_x = 2x = v_y \\ u_y = -2y = -2v_x \end{cases}$$

at every point in the plane, i.e. Cauchy-Riemann equations hold everywhere.

Example Let $f(z) = |z|^2$. If $z = x + iy \quad x, y \in R$ then:

$$f(z) = x^2 + y^2 = \underbrace{x^2 + y^2}_{u(x,y)} + i \cdot \underbrace{0}_{v(x,y)}$$

Let us check at which points the Cauchy-Riemann equations are verified. We have: $u_x = 2x$, $u_y = 2y$, $v_x = 0$ and $v_y = 0$. Cauchy-Riemann equations are verified if, and only if, $\begin{cases} 2x = 0 \\ 2y = 0 \end{cases}$, i.e. $x = y = 0$. The only point where f can be differentiable is the origin.

There is a kind of inverse theorem:

Theorem *If $f(z) = u(x,y) + iv(x,y)$ verifies the Cauchy-Riemann Formulas at z_0 and if the partial derivatives of u and v are continuous at (x_0, y_0), the f is derivable at $z_0$and $f'(z_0) = u_x(x_0, y_0) + v_x(x_0, y_0)$.*

Example Let $f(z) = z^2$; the function f is derivable at any point and $f'(z) = 2z$.

If $z = x + iy$, then $f(z) = (x+iy)^2 = \underbrace{x^2 - y^2}_{u(x,y)} + i\,\underbrace{2xy}_{v(x,y)}$. Then: $u_x = 2x$, $u_y = -2y$, $v_x = 2y$ and v_y $2y$. These partial derivatives verify the C-R equations.

By that way, we have a new proof of the differentiability of f at every point.

Example Let $f(z) = z|z|^2$, for any $z \in \mathbb{C}$. We work as in the previous examples:

$$f(z) = (x+iy)(x^2+y^2) = \underbrace{(x^3 + xy^2)}_{=u(x,y)} + i\,\underbrace{(x^2y + y^3)}_{=v(x,y)}$$

We compute the first partial derivatives:

$$\begin{cases} u_x = 3x^2 + y^2 \\ u_y = 2xy \end{cases} \quad \text{and} \quad \begin{cases} v_x = 2xy \\ v_y = x^2 + 3y^2 \end{cases}$$

We solve Cauchy-Riemann equations:

$$\begin{cases} 3x^2 + y^2 = x^2 + 3y^2 \\ 2xy = -2xy \end{cases} \iff [\; x = 0 \quad \text{or} \quad y = o \;]$$

The subset of the plane where f can be differentiable is the union of the two coordinate axes. As the first partial derivatives of u and v are continuous at every point in the plane, f is differentiable at every point on one of the coordinate axes.

Cauchy-Riemann equations in polar form:

Instead of

$$f(z) = u(x,y) + iv(x,y)$$

$$f(z) = u(r,\theta) + iv(r,\theta) \qquad z = x + iy = r(\cos\theta + i\sin\theta)$$

.

$$\begin{cases} \frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta} \\ \frac{\partial v}{\partial r} = -\frac{1}{r} \cdot \frac{\partial u}{\partial \theta} \end{cases}$$

Geometry of analytic function conformal mapping

A conformal mapping, also called a conformal map, conformal transformation, angle-preserving transformation, or biholomorphic map, is a transformation $w = f(z)$ that preserves local angles. An analytic function is conformal at any point where it has a nonzero derivative. Conversely, any conformal mapping of a complex variable which has continuous partial derivatives is analytic.

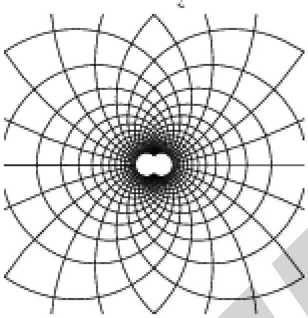
For example, let

$$w(z) = A z^n = A r^n e^{i n \theta},$$

the real and imaginary parts then give

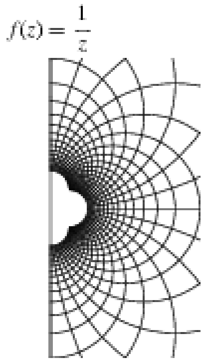
$$\phi = A r^n \cos(n\theta)$$
$$\psi = A r^n \sin(n\theta).$$

$f(z) = \frac{1}{z^2}$



For $n = -2$,

$$\phi = \frac{A}{r^2} \cos(2\theta)$$
$$\psi = -\frac{A}{r^2} \sin(2\theta),$$



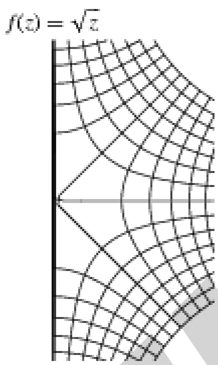
For $n = -1$,

=

=

$$\emptyset = \frac{A}{r} cos\theta \qquad and\ \ \psi = -\frac{A}{r} sin\theta$$

This solution consists of two systems of circles, and ϕ is the potential function for two parallel opposite charged line charges .



For $n = 1/2$,

$\phi =$

$$A\, r^{1/2} \cos\left(\frac{\theta}{2}\right) = A \sqrt{\frac{\sqrt{x^2+y^2} + x}{2}}$$

$\psi =$

$$A\, r^{1/2} \sin\left(\frac{\theta}{2}\right) = A \sqrt{\frac{\sqrt{x^2+y^2} - x}{2}}\ .$$

Exponential function , Trigonometric function, Hyperbolic function, logarithimic function

The Complex Exponential Function

Recall that the real exponential function can be represented by the power series $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$. Thus it is only natural to define the complex exponential e^z, also written as $\exp(z)$, in the following way:

Def(ExponentialFunction).

The definition of exp(z)

$$\exp(z) = e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n.$$

Clearly, this definition agrees with that of the real exponential function when z is a real number.

properties: The function $\exp(z) = e^z$ is an entire function satisfying the following conditions:

(i). $\frac{d}{dz} \exp(z) = \exp(z) = e^z$, using Leibniz notation $\frac{d}{dz} e^z = e^z$.

(ii). $\exp(z_1 + z_2) = \exp(z_1) \exp(z_2)$, i.e. $e^{z_1+z_2} = e^{z_1} e^{z_2}$.

(iii). If θ is a real number, then $e^{i\theta} = \cos\theta + i \sin\theta$.

The exponential function is a solution to the differential equation $f'(z) = f(z)$ with the initial condition $f(0) = 1$.

We now explore some additional properties of $\exp(z) = e^z$.

(i) $e^{z+i2n\pi} = e^z$, for all z, provided n is an integer,

(ii) $e^z = 1$, if and only if $z = i 2 n \pi$, where n is an integer, and

(iii) $e^{z_1} = e^{z_2}$, if and only if $z_2 = z_1 + i 2 n \pi$, for some integer n.

Example 1. For any integer n, the points $z_n = \frac{5}{4} + i\left(\frac{11\pi}{6} + 2n\pi\right)$ are mapped onto a single point

$$w_0 = e^{z_n} = \exp(z_n)$$

$$= \exp\left[\frac{5}{4} + i\left(\frac{11\pi}{6} + 2n\pi\right)\right]$$

$$= e^{5/4}\left[\cos\left(\frac{11\pi}{6} + 2n\pi\right) + i\sin\left(\frac{11\pi}{6} + 2n\pi\right)\right]$$

$$= e^{5/4}\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$

$$= \frac{\sqrt{3}}{2}e^{5/4} - i\frac{1}{2}e^{5/4}$$

$$= 3.02272566908 - 1.74517147873\, i$$

in the w plane, as indicated in Figure 1.

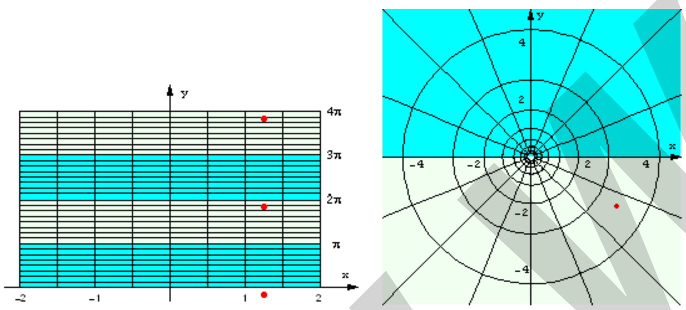


Figure 1 The points $\{z_n\}$ in the z plane (i.e., the xy plane) and their image $w_0 = \exp(z_n)$ in the w plane (i.e., the uv plane).

Let's look at the range of the exponential function. If $z = x + i y$, we see from identity that e^z can never equal zero, as e^x is never zero, and the cosine and sine functions are never zero at the same point. Suppose, then, that $w = e^z \neq 0$. If we write w in its exponential form as $w = \rho e^{i\phi}$, identity gives

$$\rho e^{i\phi} = e^x e^{iy}.$$

Using identity, and property $\rho = e^x$ and $\phi = y + 2n\pi$, where n is an integer. Therefore, $\rho = |e^z| = e^x$,

and $\phi \in \arg(e^z) = \{\text{Arg}(e^z) + 2n\pi : \text{ n is an integer}\}$.

Solving these equations for x and y,

yields

$$x = \ln\rho \text{ and } y = \phi + 2n\pi,$$

where n is an integer. Thus, for any complex number $w \neq 0$, there are infinitely many complex numbers $z = x + iy$ such that $w = e^z$. From the previous equations, we see that the numbers z are

$$z = x + iy = \ln \rho + i(\phi + 2n\pi)$$

or

$$z = \ln |w| + i(\operatorname{Arg} w + 2n\pi),$$

where n is an integer. Hence

$$\exp[\ln |w| + i(\operatorname{Arg} w + 2n\pi)] = w$$

$$e^{\ln|w| + i(\operatorname{Arg} w + 2n\pi)} = w$$

In summary, the transformation $w = e^z$

maps the complex plane (infinitely often) onto the set of non zero complex numbers. If we restrict the solutions in equation .so that only the principal value of the argument, $-\pi < \operatorname{Arg} w \leq \pi$, is used, the transformation $w = e^z = e^{x+iy}$ maps the horizontal strip $\{(x, y) : -\pi < y \leq \pi\}$, one-to-one and onto the range set $S = \{w : w \neq 0\}$. This strip is called the fundamental period strip and is shown in Figure 2.

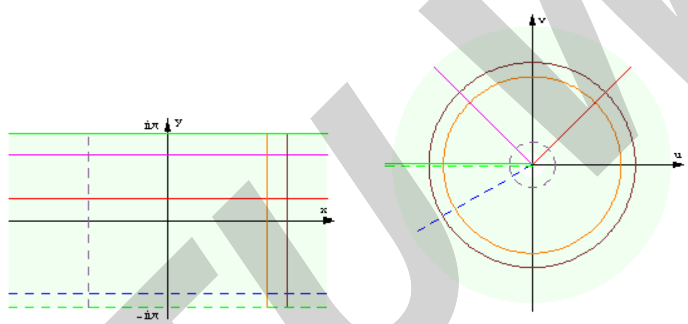


Figure 2 The fundamental period strip for the mapping $w = e^z = \exp(z)$.

The horizontal line $z = t + ib$, for $-\infty < t < \infty$ in the z plane, is mapped onto the ray $w = e^t e^{ib} = e^t(\cos b + i \sin b)$ that is inclined at an angle $\phi = b$ in the w plane. The vertical segment $z = a + i\theta$, for $-\pi < \theta \leq \pi$ in the z plane, is mapped onto the circle centered at the origin with radius e^a in the w plane. That is, $w = e^a e^{i\theta} = e^a(\cos\theta + i\sin\theta)$.

Example 2. Consider a rectangle $R = \{(x, y) : a \leq x \leq b \text{ and } c \leq y \leq d\}$, where $-\pi < c < d \leq \pi$. Show that the transformation $w = e^z = e^{x+iy}$ maps the rectangle R onto a portion of an annular region bounded by two rays.

Solution. The image points in the w plane satisfy the following relationships involving the modulus and argument of w:

$$e^a = \left|e^{a+iy}\right| \le \left|e^{x+iy}\right| \le \left|e^{b+iy}\right| = e^b,$$ and

$$c = \operatorname{Arg}\left(e^{x+ic}\right) \le \operatorname{Arg}\left(e^{x+iy}\right) \le \operatorname{Arg}\left(e^{x+id}\right) \le d,$$

which is a portion of the annulus $\{\rho e^{i\phi} : e^a \le \rho \le e^b\}$ in the w plane subtended by the rays $\phi = c$ and $\phi = d$. In Figure 3, we show the image of the rectangle

$$R^* = \left\{(x, y) : -1 \le x \le 1 \text{ and } \frac{-\pi}{4} \le y \le \frac{\pi}{3}\right\}.$$

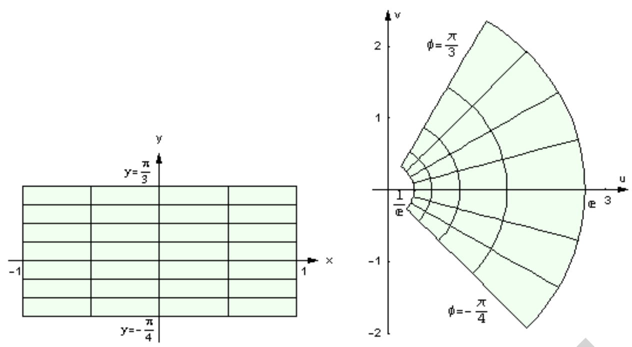


Figure 3 The image of R^* under the transformation $w = e^z = \exp(z)$.

Trigonometric and Hyperbolic Functions

Based on the success we had in using power series to define the complex exponential , we have reason to believe this approach will be fruitful for other elementary functions as well. The power series expansions for the real-valued sine and cosine functions are

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},$$ and

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Thus, it is natural to make the following definitions.

Definition. The series for Sine and Cosine are

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!},$$ and

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

Clearly, these definitions agree with their real counterparts when z is real. Additionally, it is easy to show that $\cos(z)$ and $\sin(z)$ are entire functions.

With these definitions in place, it is now easy to create the other complex trigonometric functions, provided the denominators in the following expressions do not equal zero.

Definition . $\tan z = \dfrac{\sin z}{\cos z}$, $\cot z = \dfrac{\cos z}{\sin z}$, $\sec z = \dfrac{1}{\cos z}$, and $\csc z = \dfrac{1}{\sin z}$.

$\sin(z)$ and $\cos(z)$ are entire functions, with $\dfrac{d}{dz}\sin(z) = \cos(z)$ and $\dfrac{d}{dz}\cos(z) = -\sin(z)$.

We now list several additional properties, For all complex numbers z,

$$\sin(-z) = -\sin(z)$$

$$\cos(-z) = \cos(z)$$

$$(\cos z)^2 + (\sin z)^2 = 1$$

For all complex numbers z for which the expressions are defined,

$$\frac{d}{dz}\tan z = (\sec z)^2,$$

$$\frac{d}{dz}\cot z = -(\csc z)^2,$$

$$\frac{d}{dz}\sec z = \sec z\ \tan z,$$

$$\frac{d}{dz}\csc z = -\cot z\ \csc z.$$

To establish additional properties, it will be useful to express $\cos z$ and $\sin z$ in the Cartesian form $u + i v$.

$$e^{iz} = \cos z + i \sin z,$$

for all z,

Whether z is real or complex. Hence, $e^{-iz} = \cos(-z) + i\sin(-z) = \cos z - i\sin z$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2},$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

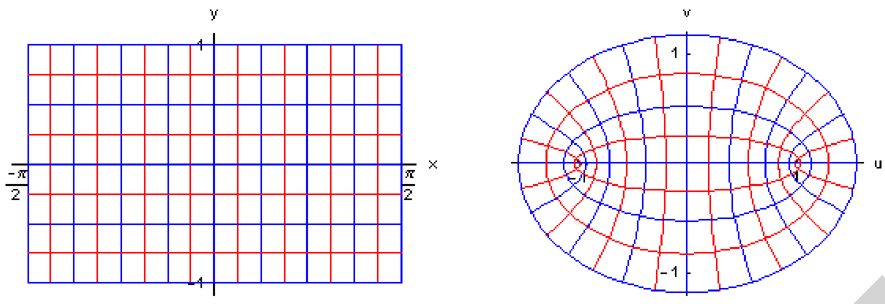


Figure The mapping $w = u + iv = \sin z$.

These equations in turn are used to obtain the following important identities

$$\sin z = \sin(x + iy) = \sin x \cosh y + i\cos x \sinh y,$$

$$\cos z = \cos(x + iy) = \cos x \cosh y - i\sin x \sinh y..$$

If z, z_1 and z_2 are any complex numbers, then

$$\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos z_1 \sin z_2,$$
$$\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin z_1 \sin z_2,$$

$$\sin(2z) = 2\sin z \cos z,$$

$$\cos(2z) = (\cos z)^2 - (\sin z)^2,$$

$$\sin\left(\frac{\pi}{2} + z\right) = \sin\left(\frac{\pi}{2} - z\right) = \cos z.$$

A solution to the equation $f(z) = 0$ is called a zero of the given function f. As we now show, the zeros of the sine and cosine function are exactly where you might expect them to be. We have $\sin z = 0$ iff $z = n\pi$, where n is any integer, and $\cos z = 0$ iff $z = \left(n + \frac{1}{2}\right)\pi$, where n is any integer.

We show the result for $\cos z$ and leave the result for $\sin z$ as an exercise. When we use Identity $\cos z = 0$ iff

$$0 = \cos x \cosh y - i\sin x \sinh y.$$

Equating the real and imaginary parts of this equation gives

$0 = \cos x \cosh y$ and $0 = \sin x \sinh y$.

The real-valued function cosh y is never zero, so the equation $0 = \cos x \cosh y$ implies that $0 = \cos x$, from which we obtain $x = \left(n + \frac{1}{2}\right)\pi$ for any integer n .

Using the values $z = x + i y = \left(n + \frac{1}{2}\right)\pi + i y$ in the equation $0 = \sin x \sinh y$ yields

$$0 = \sin\left(\left(n + \frac{1}{2}\right)\pi\right)\sinh y = (-1)^n \sinh y.$$

which implies that $y = 0$, so the only zeros for $\cos z$ are the values $z = \left(n + \frac{1}{2}\right)\pi$ for n an integer.

What does the mapping $w = \sin z$ look like? We can get a graph of the mapping $w = \sin z = \sin(x + i y) = \sin x \cosh y + i \cos x \sinh y$ by using parametric methods. Let's consider the vertical line segments in the z plane obtained by successfully setting $x = -\frac{\pi}{2} + \frac{k\pi}{12}$ for $k = 0, 1, \ldots, 12$, and for each x value and letting y vary continuously, $3 \le y \le 3$. In the exercises we ask you to show that the images of these vertical segments are hyperbolas in the uv plane, we give a more detailed analysis of the mapping $w = \sin z$.

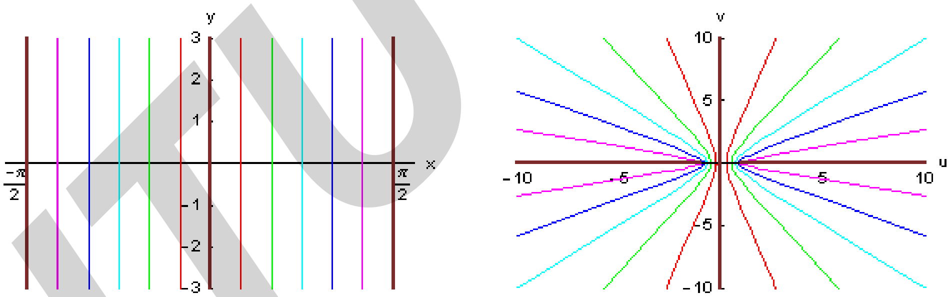


Figure Vertical segments mapped onto hyperbolas by $w = \sin z$.

$$\tan z = \tan(x + i y) = \frac{\sin(x + i y)}{\cos(x + i y)} = \frac{\sin x \cosh y + i \cos x \sinh y}{\cos x \cosh y - i \sin x \sinh y}.$$

If we multiply each term on the right by the conjugate of the denominator, the simplified result is

$$\tan z = \frac{\cos x \sin x + i \cosh y \sinh y}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}$$

We leave it as an exercise to show that the identities $\cosh^2 y - \sinh^2 y = 1$ and $\sinh 2y = 2\cosh y \sinh y$ can be used in simplifying Equation to get

$$\tan z = \frac{\sin(2x)}{\cos(2x) + \cosh(2y)} + i\frac{\sinh(2y)}{\cos(2x) + \cosh(2y)}$$

As with $\sin z$, we obtain a graph of the mapping $w = \tan z$ parametrically. Consider the vertical line segments in the z plane obtained by successively setting $x = -\frac{\pi}{4} + \frac{k\pi}{16}$ for $k = 0, 1, \ldots, 8$, and for each z value letting y vary continuously, $3 \le y \le 3$. In the exercises we ask you to show that the images of these vertical segments are circular arcs in the uv plane, as Figure investigation of the mapping $w = \tan z$.

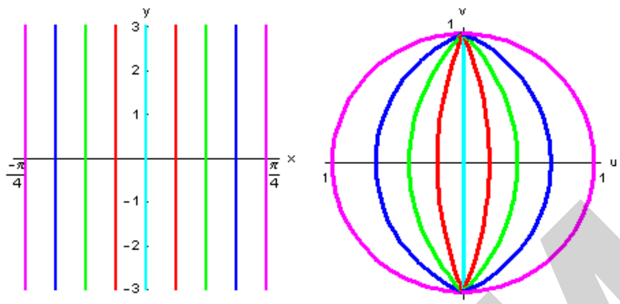


Figure Vertical segments mapped onto circular arcs by $w = \tan z$.

Definition complexhyperbolicfunctions

$$\cosh z = \frac{1}{2}(e^z + e^{-z})$$

$$\sinh z = \frac{1}{2}(e^z - e^{-z})$$

With these definitions in place, we can now easily create the other complex hyperbolic trigonometric functions, provided the denominators in the following expressions are not zero.

Definition Identities for the hyperbolic trigonometric functions are

$$\tanh z = \frac{\sinh z}{\cosh z},$$
$$\coth z = \frac{\cosh z}{\sinh z},$$

$$\operatorname{sech} z = \frac{1}{\cosh z},$$

$$\operatorname{csch} z = \frac{1}{\sinh z}.$$

The derivatives of the hyperbolic functions follow the same rules as in calculus:

$$\frac{d}{dz}\cosh z = \sinh z, \qquad \text{and} \qquad \frac{d}{dz}\operatorname{Sinh} z = \cosh z,$$

$$\frac{d}{dz}\operatorname{Tanh} z = \operatorname{sech}^2 z, \qquad \text{and} \qquad \frac{d}{dz}\coth z = -\operatorname{csch}^2 z,$$

$$\frac{d}{dz}\operatorname{sech} z = -\operatorname{sech} z \operatorname{Tanh} z, \qquad \text{and} \qquad \frac{d}{dz}\operatorname{csch} z = -\operatorname{csch} z \coth z.$$

The hyperbolic cosine and hyperbolic sine can be expressed as

$$\cosh z = \cosh(x + i y) = \cos y \cosh x + i \sinh x \sin y$$

$$\sinh z = \sinh(x + i y) = \sinh x \cos y + i \cosh x \sin y$$

Some of the important identities involving the hyperbolic functions are

Complex Logarithm function

In Section 1, we showed that, if w is a nonzero complex number, then the equation $w = e^z$ has infinitely many solutions. Because the function $f(z) = e^z$ is a many-to-one function, its inverse (the logarithm) is multivalued.

Definition . (Multivalued Logarithm) For $z \neq 0$, we define the function $\log(z)$ as the inverse of the exponential function; that is,

$\log(z) = w$ if and only if $z = e^w$.

$$w = \log(z) = \ln|z| + i\theta, \quad \text{for} \quad z \neq 0,$$

where $\theta \in \arg(z)$ and $\ln|z|$ denotes the natural logarithm of the positive number |z|. Because $\arg(z)$ is the set $\arg(z) = \{\mathrm{Arg}(z) + 2n\pi : \text{where n is an integer}\}$, we can express the set of values comprising $\log(z)$ as

$$\log(z) = \{\ln|z| + i(\mathrm{Arg}(z) + 2n\pi) : \text{where n is an integer}\},$$

or

$$\log(z) = \ln|z| + i\arg(z) \quad \text{for} \quad z \neq 0,$$

Recall that Arg is defined so that for $z \neq 0$, we have $-\pi < \mathrm{Arg}(z) \leq \pi$. We call any one of the values given in Identities of a logarithm of z. Notice that the different values of $\log(z)$ all have the same real part and that their imaginary parts differ by the amount $2\pi n$, where n is an integer. When $n = 0$, we have a special situation.

Definition . (Principal Value of the Logarithm) For $z \neq 0$, we define the principal value of the logarithm as follows:

$$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z) \quad \text{where} \quad |z| > 0 \text{ and } -\pi < \mathrm{Arg}(z) \leq \pi.$$

The domain for the function $w = f(z) = \mathrm{Log}(z)$ is the set of all nonzero complex numbers in the z-plane, and its range is the horizontal strip $\{w : -\pi < \mathrm{Im}(w) \leq \pi\}$ in the w-plane, and is shown in Figure 5.A. We stress again that $\mathrm{Log}(z)$ is a single-valued function and corresponds to setting $n = 0$ in equation . As we demonstrated in Section 2, the function $\mathrm{Arg}(z)$ is discontinuous at each point along the negative x-axis, hence so is the function $\mathrm{Log}(z)$. In fact, because any branch of the multi-valued function $\mathrm{Arg}(z)$ is discontinuous along some ray, a corresponding branch of the logarithm will have a discontinuity along that same ray.

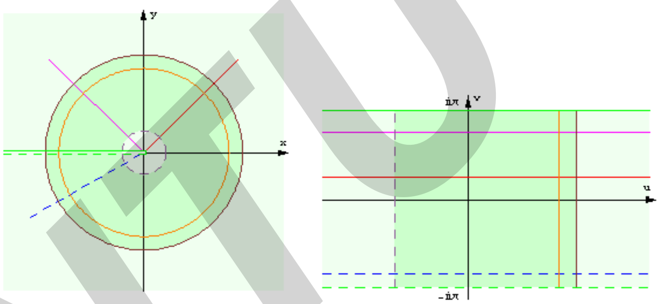


Figure The principal branch of the logarithm $w = \mathrm{Log}(z)$.

Caution. A phenomenon inherent in constructing an logarithm function: It must have a discontinuity! This is the case because as we saw in Section 2, any branch we choose for $\arg(z)$ is necessarily a discontinuous function. The principal branch, $\mathrm{Log}(z)$, is discontinuous at each point along the negative x-axis.

Example. Find the values of

$\text{Log}(1+i)$ and $\log(1+i)$.

Solution. By standard computations, we have

$$\log(1+i) = \{\ln|1+i| + i\,(\text{Arg}(1+i) + 2n\pi) : \text{where } n \text{ is an integer}\}$$

$$= \left\{\ln\sqrt{2} + i\left(\frac{\pi}{4} + 2n\pi\right) : \text{where } n \text{ is an integer}\right\}$$

and

$$\log(i) = \{\ln|i| + i\,(\text{Arg}(i) + 2n\pi) : \text{where } n \text{ is an integer}\}$$

$$= \left\{ i\left(\frac{\pi}{2} + 2n\pi\right) : \text{where } n \text{ is an integer}\right\}$$

The principal values are

$$\text{Log}(1+i) = \ln\sqrt{2} + i\,\frac{\pi}{4} = \frac{\ln 2}{2} + i\,\frac{\pi}{4}$$

and

$$\text{Log}(i) = i\,\frac{\pi}{2}$$

Extra Example 2. The transformation w = Log(z) maps the z-plane punctured at the origin onto the horizontal strip in the w-plane.

We now investigate some of the properties of log(z) and Log(z).

$\exp(\text{Log}\,z) = z$ for all z~=0

and $\text{Log}(\exp z) = z$, provided $-\pi < \text{Arg}(z) \le \pi$,

and that the mapping w = Log(z) is one-to-one from domain $D = \{z : |z| > 0\}$ in the z plane onto the horizontal strip $\{w : -\pi < \text{Im}(w) \le \pi\}$ in the w plane.

The following example illustrates that, even though Log(z) is not continuous along the negative real axis, it is still defined there.

Example. Identity reveals that

(a) $\text{Log}(-e) = \ln|-e| + i\,\text{Arg}(-e) = 1 + i$, and

(b) $\text{Log}(-1) = \ln|-1| + i\,\text{Arg}(-1) = i$.

When $z = x + i\,0$, where x is a positive real number, the principal value of the complex logarithm of z is

$$\text{Log}(x + i\,0) = \ln x + i\,\text{Arg}(x) = \ln x + i\,0 = \ln x,$$

where $x > 0$. Hence Log is an extension of the real function ln(x) to the complex case. Are there other similarities? Let's use complex function theory to find the derivative of Log(z). When we use polar coordinates for $z = r e^{i\theta} \neq 0$,

$$\begin{aligned} \mathrm{Log}(z) &= \mathrm{Log}\left(r e^{i\theta}\right) \\ &= \ln\left|r e^{i\theta}\right| + i\,\mathrm{Arg}\left(r e^{i\theta}\right) \\ &= \ln r + i\,\theta, \quad \text{for } r > 0 \text{ and } -\pi < \theta \le \pi \\ &= U(r, \theta) + i\,V(r, \theta) \end{aligned}$$

where $U(r, \theta) = \ln r$ and $V(r, \theta) = \theta$. Because $\mathrm{Arg}(z)$ is discontinuous only at points in its domain that lie on the negative real axis, U and V have continuous partials for any point (r, θ) in their domain, provided $r e^{i\theta}$ is not on the negative real axis, that is, provided $-\pi < \theta < \pi$ (Note the strict inequality for here.). In addition, the polar form of the Cauchy-Riemann equations holds in this region, since $U_r(r, \theta) = \frac{1}{r}$, $U_\theta(r, \theta) = 0$, $V_r(r, \theta) = 0$, $V_\theta(r, \theta) = 1$ it follows that

$$U_r(r, \theta) = \frac{1}{r} = \frac{1}{r} V_\theta(r, \theta)$$

and

$$U_\theta(r, \theta) = 0 = \frac{-1}{r} 0 = -\frac{-1}{r} V_r(r, \theta)$$

$$\begin{aligned} \frac{d}{dz}\mathrm{Log}(z) &= e^{-i\theta}\left(U_r(r, \theta) + i\,U_\theta(r, \theta)\right) \\ &= e^{-i\theta}\left(\frac{1}{r} + 0\,i\right) = \frac{1}{r e^{i\theta}} \\ &= \frac{1}{z} \end{aligned}$$

provided $r > 0$ and $-\pi < \theta < \pi$. Thus the principal branch of the complex logarithm has the derivative we would expect. Other properties of the logarithm carry over, but only in specified regions of the complex plane.

Example . Show that the identity $\mathrm{Log}(z_1 z_2) = \mathrm{Log}(z_1) + \mathrm{Log}(z_2)$ is not always valid.

Solution. Let $z_1 = -\sqrt{3} + i$ and $z_2 = -1 + i\sqrt{3}$. Then

$$\begin{aligned} \mathrm{Log}(z_1 z_2) &= \mathrm{Log}\left[\left(-\sqrt{3} + i\right)\left(-1 + i\sqrt{3}\right)\right] \\ &= \mathrm{Log}(-4i) = \ln 4 + i\frac{-\pi}{2} \\ &= \ln 4 - i\frac{\pi}{2} \end{aligned}$$

but

Since $\ln 4 - i\frac{\pi}{2} \neq \ln 4 + i\frac{3\pi}{2}$,

this is a counter example for which

$\mathrm{Log}(z_1 z_2) \neq \mathrm{Log}(z_1) + \mathrm{Log}(z_2)$.

(i) The identity $\mathrm{Log}(z_1 z_2) = \mathrm{Log}(z_1) + \mathrm{Log}(z_2)$ holds true if and only if $-\pi < \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) \leq \pi$

(ii) Let z_1 and z_2 be nonzero complex numbers. The multivalued function $\log(z)$ obeys the familiar properties of logarithms:

$$\log(z_1 z_2) = \log(z_1) + \log(z_2),$$

$$\log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2),$$

and

$$\begin{aligned}\mathrm{Log}(z_1) + \mathrm{Log}(z_2) &= \left(\mathrm{Log}[2] + i\frac{5\pi}{6}\right) + \left(\mathrm{Log}[2] + i\frac{2\pi}{3}\right) \\ &= 2\,\mathrm{Log}[2] + i\left(\frac{5\pi}{6} + \frac{2\pi}{3}\right) \\ &= \ln 4 + i\frac{3\pi}{2}\end{aligned}$$

$$\log\left(\frac{1}{z}\right) = -\log(z).$$

We can construct many different branches of the multivalued logarithm function that are continuous and differentiable except at points along any preassigned ray $\{r e^{i\alpha} : r > 0\}$. If we let α denote a real fixed number and choose the value of $\theta \in \arg(z)$, that lies in the range $\alpha < \theta \leq \alpha + 2\pi$, then the function $\log_\alpha(z)$ defined by

$$\log_\alpha(z) = \ln r + i\theta$$

where $z = r e^{i\theta} \neq 0$, and $\alpha < \theta \leq \alpha + 2\pi$, is a single-valued branch of the logarithm function. The branch cut for $\log_\alpha(z)$ is the ray $\{r e^{i\alpha} : r > 0\}$, and each point along this ray is a point of discontinuity of $\log_\alpha(z)$. Because $\exp(\log_\alpha(z)) = z$, we conclude that the mapping $w = \log_\alpha(z)$ is a one-to-one mapping of the domain $|z| > 0$ onto the horizontal strip $\{w : \alpha < \mathrm{Im}(w) \leq \alpha + 2\pi\}$. If $\alpha < c < d < \alpha + 2\pi$, then the function $w = \log_\alpha(z)$ maps the set $D = \{r e^{i\alpha} : 0 < a < r < b, \text{ and } c < \theta < d\}$ one-to-one and onto the rectangle $R = \{u + iv : \ln a < u < \ln b, \text{ and } c < v < d\}$. Figure 5.4 shows the mapping $w = \log_\alpha(z)$, its branch cut $\{r e^{i\alpha} : r > 0\}$, the set D, and its image R.

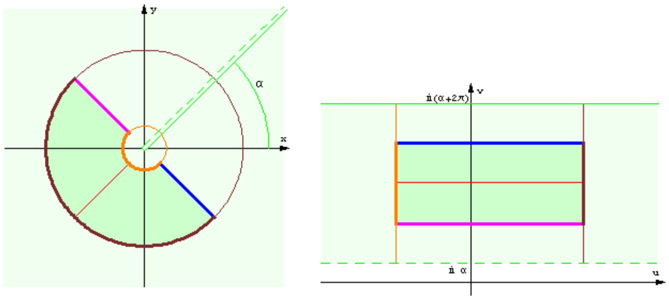


Figure 5.4 The branch $w = \log_\alpha(z)$ of the logarithm.

We can easily compute the derivative of any branch of the multivalued logarithm. For a particular branch $w = \log_\alpha(z)$ for $z = re^{i\theta} \neq 0$, and $\alpha < \theta \le \alpha + 2\pi$ (note the strict inequality for θ), we start with $z = \exp(w)$ in Equations and differentiate both sides to get

$$\begin{aligned} 1 &= \frac{d}{dz} z \\ &= \frac{d}{dz} \exp(\log_\alpha(z)) \\ &= \exp(\log_\alpha(z)) \frac{d}{dz} \log_\alpha(z) \\ &= z \frac{d}{dz} \log_\alpha(z) \end{aligned}$$

Solving for $\frac{d}{dz} \log_\alpha(z)$ gives

$$\frac{d}{dz} \log_\alpha(z) = \frac{1}{z}, \quad \text{for} \quad \text{for } z = e^{i\theta} \neq 0, \text{ and } \alpha < \theta \le \alpha + 2\pi.$$

The Riemann surface for the multivalued function $w = \log(z)$ is similar to the one we presented for the square root function. However, it requires infinitely many copies of the z plane cut along the negative x axis, which we label S_k for $k = \ldots, -n, \ldots, -1, 0, 1, \ldots, n, \ldots$. Now, we stack these cut planes directly on each other so that the corresponding points have the same position. We join the sheet S_k to S_{k+1} as follows. For each integer k, the edge of the sheet S_k in the upper half-plane is joined to the edge of the sheet S_{k+1} in the lower half-plane. The Riemann surface for the domain of $\log(z)$ looks like a spiral staircase that extends upward on the sheets $S_1, S_2, \ldots$ and downward on the sheets $S_{-1}, S_{-2}, \ldots$, as shown in Figure . We use polar coordinates for z on each sheet. For S_k, we use

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta), \quad \text{where}$$

$$r = |z| \text{ and } 2\pi k - \pi < \theta \le \pi + 2\pi k.$$

Again, for S_k, the correct branch of $\log(z)$ on each sheet is

$\log(z) = \ln r + i\,\theta$, where

$r = |z|$ and $2\pi k - \pi < \theta \le \pi + 2\pi k$.

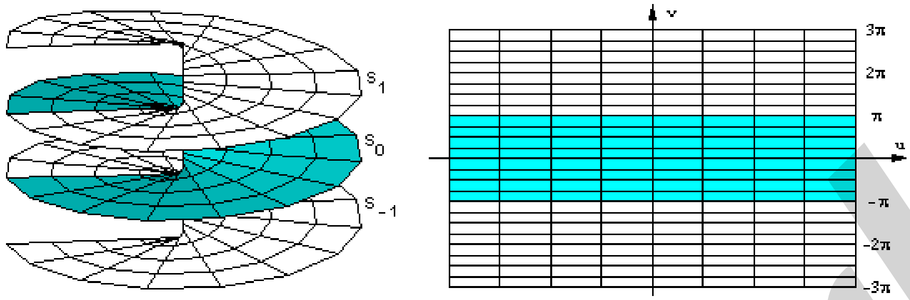


Figure The Riemann surface for the mapping $w = \log(z)$.

12.7 Linear fractional transformation

Bilinear Transformations - Mobius Transformations

Another important class of elementary mappings was studied by August Ferdinand Möbius (1790-1868). These mappings are conveniently expressed as the quotient of two linear expressions and are commonly known as linear fractional or bilinear transformations. They arise naturally in mapping problems involving the function arctan(z). In this section, we show how they are used to map a disk one-to-one and onto a half-plane. An important property is that these transformations are conformal in the entire complex plane except at one point.

Let $a, b, c,$ and d denote four complex constants with the restriction that $ad \ne bc$. Then the function

$$w = S(z) = \frac{az + b}{cz + d}$$

is called a bilinear transformation, a Möbius transformation, or a linear fractional transformation.

If the expression for S(z) is multiplied through by the quantity $cz + d$, then the resulting expression has the bilinear form

$cwz - az + dw - b = 0$.

We collect terms involving z and write $z(cw - a) = -dw + b$. Then, for values of

$$w \ne \frac{a}{c}$$

the inverse transformation is given by

$z = S^{-1}(w) = \dfrac{-dw + b}{cw - a}$.

We can extend $S(z)$ and $S^{-1}(w)$ to mappings in the extended complex plane. The value $S(\infty)$ should be chosen to equal the limit of $S(z)$ as $z \to \infty$. Therefore we define

$$S(\infty) = \lim_{z\to\infty} S(z) = \lim_{z\to\infty} \frac{a + \frac{b}{z}}{c + \frac{d}{z}} = \frac{a}{c},$$

and the inverse is $S^{-1}\left(\frac{a}{c}\right) = \infty$. Similarly, the value $S^{-1}(\infty)$ is obtained by

$$S^{-1}(\infty) = \lim_{w\to\infty} S^{-1}(w) = \lim_{w\to\infty} \frac{-d + \frac{b}{w}}{c - \frac{a}{w}} = \frac{-d}{c},$$

and the inverse is $S\left(\frac{-d}{c}\right) = \infty$. With these extensions we conclude that the transformation $w = S(z)$ is a one-to-one mapping of the extended complex z-plane onto the extended complex w-plane.

We now show that a bilinear transformation carries the class of circles and lines onto itself.

If S(z) is an arbitrary bilinear transformation . $c = 0$, thenS(z) reduces to a linear transformation, which carries lines onto lines and circles onto circles. If $c \neq 0$, then we can write S(z) in the form

$$S(z) = \frac{az + b}{cz + d}$$

$$= \frac{c(az + b)}{c(cz + d)}$$

$$= \frac{acz + bc}{c(cz + d)}$$

$$= \frac{acz + ad - ad + bc}{c(cz + d)}$$

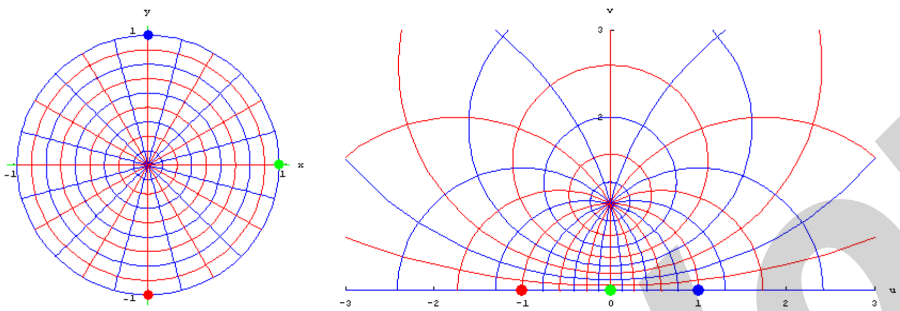
$$= \frac{a(cz + d) - ad + bc}{c(cz + d)}$$

$$= \frac{a}{c} + \frac{bc - ad}{c}\,\frac{1}{cz + d}$$

The condition $ad \neq bc$ precludes the possibility that S(z) reduces to a constant. Now S(z) can be considered as a composition of functions.

It is a linear mapping $\xi = cz + d$, followed by the reciprocal transformation $Z = \dfrac{1}{\xi}$, followed by $w = \frac{a}{c} + \frac{bc - ad}{c} Z$. In above we showed that each function in this composition maps the class of circles and lines onto itself; it follows that the bilinear transformation S(z) has this property. A half-plane can be considered to be a family of parallel lines and a disk as a family of circles. Therefore we conclude that a bilinear transformation maps the class of half-planes and disks onto itself. Example 10.3 illustrates this idea.

Example Show that $w = S(z) = \frac{i(1-z)}{1+z}$ maps the unit disk $D: |z| < 1$ one-to-one and onto the upper half-plane $\mathrm{Im}(w) > 0$.



Solution. We first consider the unit circle $C: |z| = 1$, which forms the boundary of the disk and find its image in the w plane.

If we write $S(z) = \frac{-iz + i}{z + 1}$, then we see that $a = -i$, $b = i$, $c = 1$, and $d = 1$. we find that the inverse is given by

$$z = S^{-1}(w) = \frac{-dw + b}{cw - a} = \frac{-(1)w + (i)}{(1)w - (-i)} = \frac{-w + i}{w + i}.$$

If $|z| = 1$, then the above Equation satisfy $\left|\frac{-w + i}{w + i}\right| = 1$ which yields the equation

$$|w + i| = |-w + i|.$$

Squaring both sides of above Equation , we obtain

$$|u + iv + i| = |-u - iv + i|$$

$$|u + i(1 + v)|^2 = |-u + i(1 - v)|^2$$

$$u^2 + (1+v)^2 = (-u)^2 + (1-v)^2$$

$$u^2 + (1+v)^2 = u^2 + (1-v)^2$$

$$(1+v)^2 = (1-v)^2$$

$$1 + 2v + v^2 = 1 - 2v + v^2$$

$$4v = 0$$

$$v = 0$$

which is the equation of the u axis in the w plane.

The circle C divides the z plane into two portions, and its image is the u axis, which divides the w plane into two portions. The image of the point $z = 0$ is $w = S(0) = i$, so we expect that the interior of the circle C is mapped onto the portion of the w plane that lies above the u axis. To show that this outcome is true, we let $|z| < 1$. Hence the image values must satisfy the inequality $|-w + i| < |w + i|$, which we write as

$$d_1 = |w - i| < |w - (-i)| = d_2 .$$

If we interpret d_1 as the distance from w to i and d_2 as the distance from w to $-i$, then a geometric argument shows that the image point w must lie in the upper half-plane $\mathrm{Im}(w) > 0$, as shown in Figure 10.5. As S(z) is one-to-one and onto in the extended complex plane, it follows that S(z) maps the disk onto the half-plane.

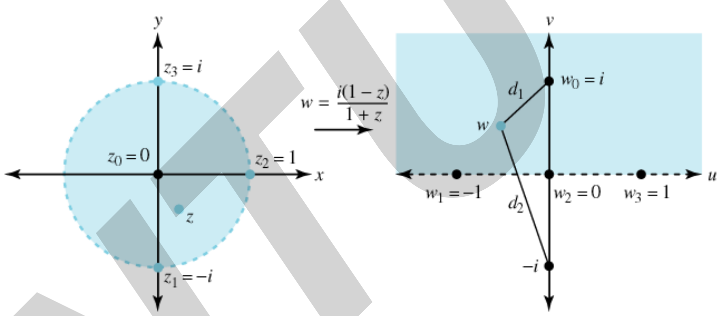


image $|z| < 1$ under $w = \dfrac{i(1-z)}{1+z}$, the points $z_1 = -i,\ z_2 = 1,\ z_3 = i$ are mapped onto the points $w_1 = -1,\ w_2 = 0,\ w_3 = 1$, respectively.

The general formula for a bilinear transformation (Equation appears to involve four independent coefficients: $a, b, c,$ and d. But as S(z) is not identically constant, either $a \neq 0$ or $c \neq 0$, we can express the transformation with three unknown coefficients and write either

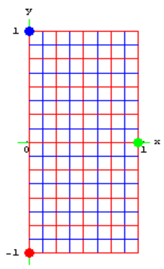
$$S(z) = \frac{z + \frac{b}{a}}{\frac{c}{a} z + \frac{d}{a}} \quad \text{or} \quad S(z) = \frac{\frac{a}{c} z + \frac{b}{c}}{z + \frac{d}{c}},$$

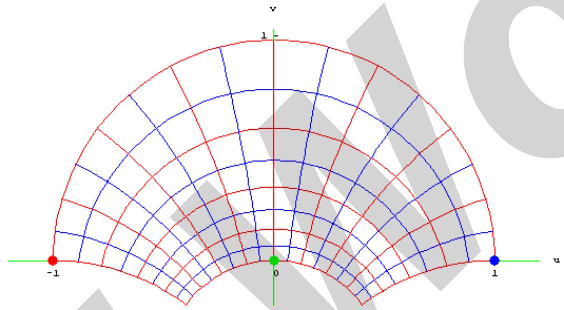
respectively. Doing so permits us to determine a unique a bilinear transformation if three distinct image values $S(z_1) = w_1$, $S(z_2) = w_2$, and $S(z_3) = w_3$ are specified. To determine such a mapping, we can conveniently use an implicit formula involving z and w.

Theorem (The Implicit Formula). There exists a unique bilinear transformation that maps three distinct points $z_1, z_2,$ and z_3 onto three distinct points $w_1, w_2,$ and w_3, respectively. An implicit formula for the mapping is given by the equation

$$\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}.$$

Example Construct the bilinear transformation w = S(z) that maps the points $z_1 = -i,\ z_2 = 1,\ z_3 = i$ onto the points $w_1 = -1,\ w_2 = 0,\ w_3 = 1$, respectively.





Solution. We use the implicit formula

$$\frac{(z - (-i))(1 - i)}{(z - i)(1 - (-i))} = \frac{(w - (-1))(0 - 1)}{(w - 1)(0 - (-1))}$$

$$\frac{(z + i)(1 - i)}{(z - i)(1 + i)} = \frac{(w + 1)(0 - 1)}{(w - 1)(0 + 1)}$$

$$\frac{(z + i)(1 - i)}{(z - i)(1 + i)} = \frac{w + 1}{-w + 1}.$$

Expanding this equation, collecting terms involving w and zw on the left and then simplify.

$$(z - i)(1 + i)(w + 1) = (z + i)(1 - i)(-w + 1)$$

$$(1 + i) z w + (1 - i) w + (1 + i) z + (1 - i)$$
$$=$$
$$(-1 + i) z w + (-1 - i) w + (1 - i) z + (1 + i)$$

$$z\,w + i\,z\,w + w - i\,w + z + i\,z + 1 - i$$
$$=$$
$$-z\,w + i\,z\,w - w - i\,w + z - i\,z + 1 + i$$

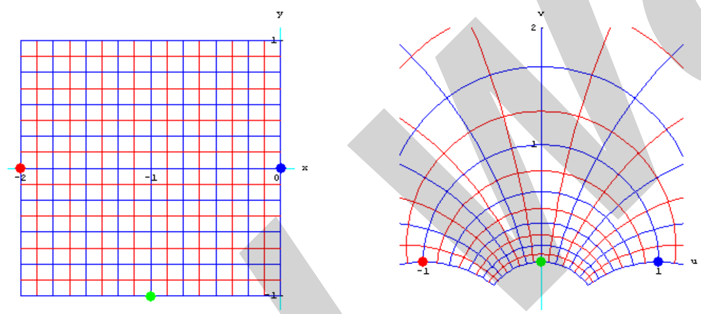
$$2\,z\,w + 2\,w = -2\,i\,z + 2\,i$$

$$z\,w + w = -i\,z + i$$

$$w\,(1 + z) = i\,(1 - z)$$

Therefore the desired bilinear transformation is $w = S(z) = \dfrac{i\,(1 - z)}{1 + z}$.

Example Find the bilinear transformation w = S(z) that maps the points $z_1 = -2,\ z_2 = -1 - i,\ z_3 = 0$ onto the points $w_1 = -1,\ w_2 = 0,\ w_3 = 1$, respectively.



Solution. Again, we use the implicit formula, Equation , and write

$$\frac{(z - (-2))\,((-1 - i) - 0)}{(z - 0)\,((-1 - i) - (-2))} = \frac{(w - (-1))\,(0 - 1)}{(w - 1)\,(0 - (-1))}$$

$$\frac{(z + 2)\,(-1 - i)}{(z)\,(-1 - i + 2)} = \frac{(w + 1)\,(-1)}{(w - 1)\,(1)}$$

$$\frac{z + 2}{z}\;\frac{-1 - i}{1 - i} = \frac{1 + w}{1 - w}$$

Using the fact that

$$\frac{-1 - i}{1 - i} = \frac{1}{i},$$

we rewrite this equation as $\dfrac{z + 2}{i\,z} = \dfrac{1 + w}{1 - w}$.

We now expand the equation and obtain

$$(z+2)(1-w) = i z(1+w)$$

$$z + 2 - z w - 2w = i z + i z w$$

$$z - i z + 2 = z w + i z w + 2w$$

$$(1-i)z + 2 = w(z + i z + 2)$$

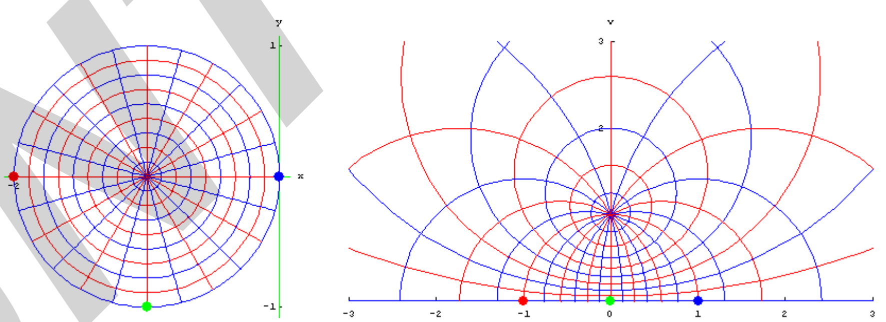
$$(1-i)z + 2 = w((1+i)z + 2)$$

which can be solved for w in terms of z, giving the desired solution $w = S(z) = \dfrac{(1-i)z+2}{(1+i)z+2}$.

We let D be a region in the z plane that is bounded by either a circle or a straight line C. We further let $z_1, z_2,$ and z_3 be three distinct points that lie on C and have the property that an observer moving along C from z_1 to z_3 through z_2 finds the region D to be on the left. If C is a circle and D is the interior of C, then we say that C is positively oriented. Conversely, the ordered triple (z_1, z_2, z_3) uniquely determines a region that lies to the left of C.

We let G be a region in the w plane that is bounded by either a circle of a straight line K. We further let $w_1, w_2,$ and w_3 be three distinct points that lie on K such that an observer moving along K from w_1 to w_3 through w_2 finds the region G to be on the left. Because a bilinear transformation is a conformal mapping that maps the class of circles and straight lines onto itself, we can use the implicit formula to construct a bilinear transformation $w = S(z)$ that is a one-to-one mapping of D onto G.

Example . Show that the mapping $w = S(z) = \dfrac{(1-i)z+2}{(1+i)z+2}$ maps the disk $D: |z+1| < 1$ one-to-one and onto the upper half plane $\operatorname{Im}(w) > 0$.



Solution. For convenience, we choose the ordered triple $z_1 = -2,\ z_2 = -1 - i,\ z_3 = 0$, which gives the circle $C: |z+1| = 1$ a positive orientation and the disk D a left orientation. From Example 10.5, the corresponding image points are

$$w_1 = S(z_1) = S(-2) = -1,$$

$$w_2 = S(z_2) = S(-1-i) = 0,$$

$$w_3 = S(z_3) = S(0) = 1.$$

Because the ordered triple of points $w_1 = -1,\ w_2 = 0,\ w_3 = 1$, lie on the u axis, it follows that the image of circle C is the u axis. The points $w_1 = -1,\ w_2 = 0,\ w_3 = 1$ give the upper half-plane $G: \mathrm{Im}(w) > 0$ a left orientation. Therefore $w = S(z) = \dfrac{(1-i)z + 2}{(1+i)z + 2}$ maps the disk D onto the upper half-plane G. To check our work, we choose a point z_0 that lies in D and find the half-plane in which its image, w_0 lies. The choice $z_0 = -1$ yields $w_0 = S(z_0) = i$. Hence the upper half-plane is the correct image. This situation is illustrated in Figure 10.6.

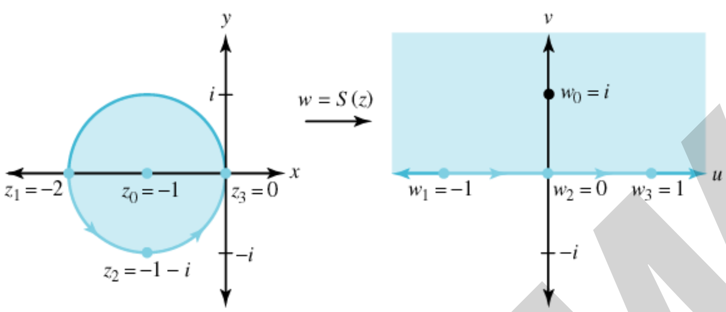


Figure The bilinear mapping $w = S(z) = \dfrac{(1-i)z + 2}{(1+i)z + 2}$

Corollary (The Implicit Formula with a point at Infinity). In equation the point at infinity can be introduced as one of the prescribed points in either the z plane or the w plane.

Proof. Case 1. If $z_3 = \infty$, then we can write $\dfrac{(z_2 - z_3)}{(z - z_3)} = \dfrac{(z_2 - \infty)}{(z - \infty)} = 1$ and substitute this expression to obtain $\dfrac{(z - z_1)(z_2 - \infty)}{(z - \infty)(z_2 - z_1)} = \dfrac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}$ which can be rewritten as $\dfrac{(z - z_1)(z_2 - \infty)}{(z_2 - z_1)(z - \infty)} = \dfrac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}$ and simplifies to obtain

$$\frac{z - z_1}{z_2 - z_1} = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}.$$

Case 2. If $w_3 = \infty$, then we can write $\dfrac{(w_2 - w_3)}{(w - w_3)} = \dfrac{(w_2 - \infty)}{(w - \infty)} = 1$ and substitute this expression to obtain

$$\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \frac{(w - w_1)(w_2 - \infty)}{(w - \infty)(w_2 - w_1)}$$

which can be rewritten as

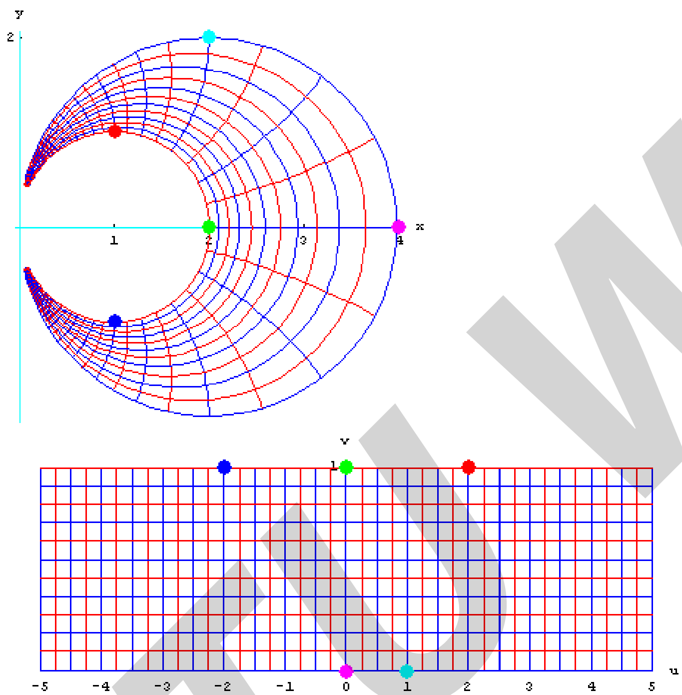
$$\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \frac{(w - w_1)(w_2 - \infty)}{(w_2 - w_1)(w - \infty)}$$

and simplifies to obtain

$$\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \frac{w - w_1}{w_2 - w_1}.$$

Above equation is sometimes used to map the crescent-shaped region that lies between the tangent circles onto an infinite strip.

Example Find the bilinear transformation $w = S(z)$ that maps the crescent-shaped region that lies inside the disk $D: |z-2| < 2$ and outside the circle $|z-1| = 1$ onto a horizontal strip.



Solution. For convenience we choose $z_1 = 4,\ z_2 = 2+2i,\ z_3 = 0$ and the image values $w_1 = 0,\ w_2 = 1,\ w_3 = \infty$, respectively. The ordered triple $z_1 = -4,\ z_2 = 2+2i,\ z_3 = 0$ gives the circle $C: |z-2| = 2$ a positive orientation and the disk $D: |z-2| < 2$ has a left orientation. The image points $w_1 = 0,\ w_2 = 1,\ w_3 = \infty$ all lie on the extended u axis, and they determine a left orientation for the upper half-plane $\mathrm{Im}(w) > 0$. Therefore we can use the second implicit formula to write

$$\frac{(z - 4)(2 + 2i - 0)}{(z - 0)(2 + 2i - 4)} = \frac{w - 0}{1 - 0},$$

which determines a mapping of the disk $D: |z-2| < 2$ onto the upper half-plane $\mathrm{Im}(w) > 0$. Use the fact that

$$\frac{2+2i}{-2+2i} = -i$$

to simplify the preceding equation and get

$$\frac{z-4}{z}\,\frac{2+2i}{-2+2i} = \frac{z-4}{z}(-i) = \frac{w}{1}$$

which can be written in the form $w = S(z) = \dfrac{-iz + i4}{z}$.

A straightforward calculation shows that the points $z_4 = 1 - i,\ z_5 = 2,\ z_6 = 1 + i$ are mapped onto the points

$$w_4 = S(z_4) = S(1-i) = -2+i,$$

$$w_5 = S(z_5) = S(2) = i,$$

$$w_6 = S(z_6) = S(1+i) = 2+i,$$

respectively. The points $w_4 = -2+i,\ w_5 = i,\ w_6 = 2+i$ lie on the horizontal line $\mathrm{Im}(w) > 1$ in the upper half-plane. Therefore the crescent-shaped region is mapped onto the horizontal strip $0 < \mathrm{Im}(w) < 1$, as shown in Figure 10.7.

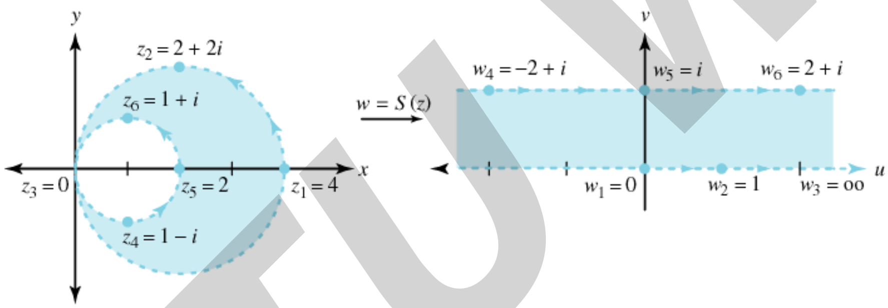


Figure The mapping $w = S(z) = \dfrac{-iz + i4}{z}$.

COMPLEX INTEGRATION

Line integral in the complex plane

Def. Complex line integral. Let C be a rectifiable curve (i.e. a curve of finite length) joining points a and b in the complex plane and let f(z) be a complex-valued function of a complex variable z, continuous at all points on C. Subdivide C into n segments by means of points a = z_0, z_1, ... , z_n = b selected arbitrarily along the curve. On each segment joining z_{k-1} to z_k choose a point ξ_k . Form the sum

$$S_n = \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(\xi_k)\Delta z_k$$

Let Δ be the length of the longest chord Δz_k. Let the number of subdivisions n approach infinity in such a way that the length of the longest chord approaches zero. The sum S_n will then approach a limit which does not depend on the mode of subdivision and is called the line integral of f(z) from a to b along the curve:

$\int_a^b f(z)dz = \lim_{\Delta\to 0} \sum_{k=1}^{\infty} f(\xi_k)\Delta z_k$

Let **f** be a continuous complex-valued function of a complex variable, and let **C** be a smooth curve in the complex plane parametrized by

Z(t) = x(t) + i y(t) for **t** varying between **a** and **b**.

Then the complex line integral of **f** over **C** is given by

$$\int_C f(z)\,dz = \int_a^b f(Z(t))\,Z'(t)\,dt$$

Note that the "smooth" condition guarantees that Z' is continuous and, hence, that the integral exists.

Properties of line integrals

If f(z) and g(z) are integrable along curve C, then

1. $\int_c \{f(z) + g(z)\}dz = \int_C f(z)dz + \int_C g(z)dz$

2. $\int_C cf(z)\,dz = c\int_C f(z)\,dz$ where c is any constant

3. $\int_a^b f(z)\,dz = -\int_b^a f(z)\,dz$

4. $\int_a^b f(z)\,dz = \int_a^q f(z)\,dz + \int_q^b f(z)\,dz$ where points a, q, b are on C

5. $\left|\int_C f(z)dz\right| \le ML$

where $|f(z)| \le$ M (i.e. M is an upper bound of |f(z)| on C) and L is the length of C.(ML INEQUALITY)

6. $\left|\int_C f(z)\,dz\right| \leq \int_C |f(z)||dz|$

Connection between real and complex line integrals. Real and complex line integrals are connected by the following theorem.

Theorem . If f(z) = u(x, y) + i v(x, y) = u + iv, the complex integral 1) can be expressed in terms of real line integrals as

$\int_a^b f(z)dz = \int_a^b (u+iv)(dx+idy) = \int_a^b udx - vdy + i\int_a^b vdx + udy$

Theorem . Let f(z) be analytic in a simply-connected region R. If a and b are any two points in R and F'(z) = f(z), then

$$\int_a^b f(z)dz = F(a) - F(b)$$

Example.

$$\int_{2i}^{1+i} 2z\,dz = z^2\Big|_{2i}^{1+i} = (1+i)^2 - (2i^2) = 2i + 4$$

Example : $Evaluate \int_\gamma \frac{(z+2)dz}{z}$ $where\ L\ is\ the\ semi\ circle\ z = 2e^{it}, 0 \leq t \leq \pi$

Soln: $Here\ \ z = 2e^{it}, 0 \leq t \leq \pi\ , then\ dz = 2ie^{it}dt$

$$\int_\gamma \frac{(z+2)dz}{z} = \int_0^\pi \frac{\left(2e^{it}+2\right)}{2e^{it}} 2ie^{it}dt = 2i\int_0^\pi \left(1+e^{-it}\right)e^{it}\,dt = 2i\int_0^\pi \left(e^{it}+1\right)dt$$

$$= 2i\left(\frac{e^{i\pi}}{i} - \frac{1}{i} + 1\right)$$

$$= 2e^{i\pi} - 2 + 2i\pi = -4 + 2i\pi$$

13.2 Cauchy's integral theorem

Def. Simply-connected region. A region R is said to be simply-connected if any simple closed curve which lies in R can be shrunk to a point without leaving R. A region R which is not simply-connected is said to be multiply-connected. The region shown in Fig. 1-1 is simply-connected. The regions shown in Figures 1-2 and 1-3 are multiply-connected.

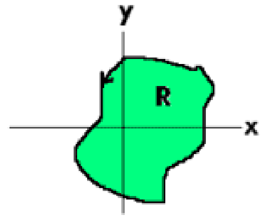


Fig. 1-1

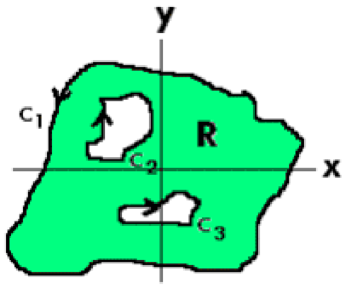


Fig. 1-3

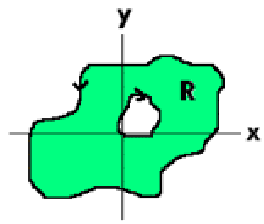


Fig. 1-2

Cauchy's integral theorem. Let a function f(z) be analytic within and on the boundary of a region R, either simply or multiply-connected, and let C be the entire boundary of R. Then

$$\oint_C f(z)dz = 0$$

OR

Let a function f(z) be analytic in a simply-connected region R and let C be a closed (not necessarily simple) curve in R. Then

$$\oint_C f(z)dz = 0$$

See figure-2

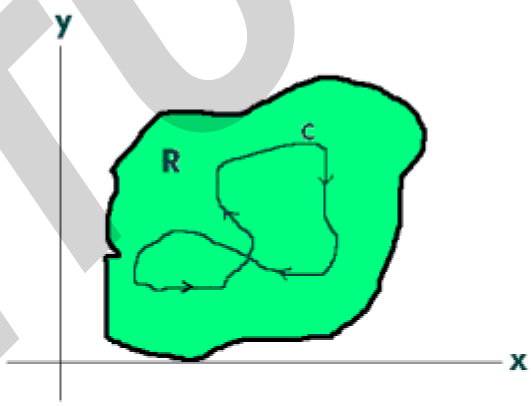


Fig. 2

Proof :

Let x, y denote the real numbers such that $z = x + iy$. Let Let A and B be the functions mapping $\Re R \times \Im R$into $\mathbb{R}$ such that $f(z) = A(x, y) + iB(x, y)$. Then

$$\int_C f(z)dz = \int_C f(z)(dx + idy) = \int_C [A(x,y) + iB(x,y)](dx + idy)$$
$$= \int_C [A(x,y)dx - B(x,y)dy] + i\int_C [A(x,y)dy + B(x,y)dx].$$

Now, since $f(z)$ is complex-differentiable, $\frac{\partial A}{\partial y} = -\frac{\partial B}{\partial x}$, $\frac{\partial B}{\partial y} = \frac{\partial A}{\partial x}$. Let D be the region bounded by C. Then by Green's theorem,

$$\int_C [A(x,y)dx - B(x,y)dy] = \iint_D \left(-\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) = 0,$$

and similarly,

$$\int_C [B(x,y)dx + A(x,y)dy] = \iint_D \left(\frac{\partial A}{\partial x} - \frac{\partial B}{\partial y}\right) = 0.$$

Thus Cauchy's theorem holds. Proved

Meaning

The Cauchy Integral Theorem guarantees that the integral of a function over a path depends only on the endpoints of a path, provided the function in question is complex-differentiable in all the areas bounded by the paths. Indeed, if P_1 and P_2 are two paths from A to B, then

$$\int_{P_1} f(z)dz - \int_{P_2} f(z)dz = \int_{P_1 - P_2} f(z)dz = 0.$$

Example : $\int_C (z^2+1)dz = 0,\quad where\ C\ is\ the\ unit\ circle\ as\ the\ function\ f(z) = z^2 + 1$ $is\ analytic\ in\ C\ and\ satisfies\ cauchy\ integral\ theorem.$

13.3 Cauchy's integral formula

Suppose U is an open subset of the complex plane C, f : U → C is a holomorphic function and the closed disk D = { $z : |z - z_0| \le r$} is completely contained in*U*. Let γ be the circle forming the boundary of D. Then for every *a* in the interior of D:

$$f(a) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-a}\,dz$$

where the contour integral is taken counter-clockwise.

Proof:

By using the Cauchy integral theorem, one can show that the integral over *C* (or the closed rectifiable curve) is equal to the same integral taken over an arbitrarily small circle around *a*. Since *f*(*z*) is continuous, we can choose a circle small enough on which *f*(*z*) is arbitrarily close to *f*(*a*). On the other hand, the integral

$$\oint_C \frac{1}{z-a}\,dz = 2\pi i,$$

ver any circle C centered at a. This can be calculated directly via a parametrization (integration by substitution) $z(t) = a + \varepsilon e^{it}$ where $0 \le t \le 2\pi$ and ε is the radius of the circle.

Letting $\varepsilon \to 0$ gives the desired estimate

$$\begin{aligned}\left|\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz - f(a)\right| &= \left|\frac{1}{2\pi i}\oint_C \frac{f(z)-f(a)}{z-a}\,dz\right| \\ &= \left|\frac{1}{2\pi i}\int_0^{2\pi}\left(\frac{f(z(t))-f(a)}{\varepsilon\cdot e^{i\cdot t}}\cdot\varepsilon\cdot e^{t\cdot i}i\right)dt\right| \\ &\le \frac{1}{2\pi}\int_0^{2\pi}\frac{|f(z(t))-f(a)|}{\varepsilon}\,\varepsilon\,dt \\ &\le \max_{|z-a|=\varepsilon}|f(z)-f(a)| \xrightarrow[\varepsilon\to 0]{} 0.\end{aligned}$$

Ex 1 : Let

$$g(z) = \frac{z^2}{z^2+2z+2},$$

and let C be the contour described by $|z| = 2$ (i.e. the circle of radius 2).

To find the integral of $g(z)$ around the contour C, we need to know the singularities of $g(z)$. Observe that we can rewrite g as follows:

$$g(z) = \frac{z^2}{(z-z_1)(z-z_2)}$$

where $z_1 = -1 + i,\ z_2 = -1 - i.$

Thus, g has poles at z_1 and z_2. The moduli of these points are less than 2 and thus lie inside the contour. This integral can be split into two smaller integrals by Cauchy-Goursat theorem; that is, we can express the integral around the contour as the sum of the integral around z_1 *and* z_2 where the contour is a small circle around each pole. Call these contours C_1 around z_1 and C_2 around z_2.

Now, each of these smaller integrals can be solved by the Cauchy integral formula, but they first must be rewritten to apply the theorem.

$$f_1(z) = \frac{z^2}{z-z_2}$$

and now

$$g(z) = \frac{f_1(z)}{z-z_1}.$$

Since the Cauchy integral theorem says that:

$$\oint_C \frac{f_1(z)}{z-a}\,dz = 2\pi i \cdot f_1(a),$$

we can evaluate the integral as follows:

$$\oint_{C_1} g(z)\,dz = \oint_{C_1} \frac{f_1(z)}{z-z_1}\,dz = 2\pi i \frac{z_1^2}{z_1-z_2}.$$

Doing likewise for the other contour:

$$f_2(z) = \frac{z^2}{z-z_1},$$

$$\oint_{C_2} g(z)\,dz = \oint_{C_2} \frac{f_2(z)}{z-z_2}\,dz = 2\pi i \frac{z_2^2}{z_2-z_1}.$$

The integral around the original contour *C* then is the sum of these two integrals:

$$\begin{aligned}\oint_C g(z)\,dz &= \oint_{C_1} g(z)\,dz + \oint_{C_2} g(z)\,dz \\ &= 2\pi i \left(\frac{z_1^2}{z_1-z_2} + \frac{z_2^2}{z_2-z_1}\right) \\ &= 2\pi i(-2) \\ &= -4\pi i.\end{aligned}$$

An elementary trick using partial fraction decomposition:

$$\oint_C g(z)dz = \oint_C \left(1 - \frac{1}{z-z_1} - \frac{1}{z-z_2}\right) dz = 0 - 2\pi i - 2\pi i = -4\pi i$$

13.4 Derivatives of analytic functions

Suppose U is an open subset of the complex plane C, f : U → C is an analytic function which is also differentiable and the closed disk $D = \{ z : |z - z_0| \le r\}$ is completely contained in U. Let γ be the circle forming the boundary of D. Then for every a in the interior of *D*:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz.$$

(For proof refer your text book Advanced engineering mathematics by Erwin Kreyzig)

Example: $Evaluate\ \int_C \frac{z^4-3z^2+6}{(z+i)^3} dz\ where\ C\ is\ the\ circle\ |z+i| = 1$

$$Evaluate \int_C \frac{z^4 - 3z^2 + 6}{(z+i)^3} dz = \pi i (z^4 - 3z^2 + 6)''\{\, at\ z = -i\}$$
$$= \pi i [12z^2 - 6]\,\{\, at\ z = -i\} \qquad = -18\pi i$$

Liouville's theorem :

The theorem follows from the fact that holomorphic functions are analytic. If f is an entire function, it can be represented by its Taylor series about 0:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

where (by Cauchy's integral formula)

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i} \oint_{C_r} \frac{f(\zeta)}{\zeta^{k+1}}\, d\zeta$$

and C_r is the circle about 0 of radius $r > 0$. Suppose f is bounded: i.e. there exists a constant M such that $|f(z)| \le M$ for all z. We can estimate direct

$$|a_k| \le \frac{1}{2\pi} \oint_{C_r} \frac{|f(\zeta)|}{|\zeta|^{k+1}} |d\zeta| \le \frac{1}{2\pi} \oint_{C_r} \frac{M}{r^{k+1}} |d\zeta| = \frac{M}{2\pi r^{k+1}} \oint_{C_r} |d\zeta| = \frac{M}{2\pi r^{k+1}} 2\pi r = \frac{M}{r^k},$$

where in the second inequality we have used the fact that $|z|=r$ on the circle C_r. But the choice of r in the above is an arbitrary positive number. Therefore, letting r tend to infinity (we let r tend to infinity since f is analytic on the entire plane) gives $a_k = 0$ for all $k \ge 1$. Thus $f(z) = a_0$ and this proves the theorem.

Morera's theorem states that a continuous, complex-valued function f defined on a connected open set D in thecomplex plane that satisfies

$$\oint_\gamma f(z)\, dz = 0$$

for every closed piecewise C^1 curve γ in D must be holomorphic on D.

The assumption of Morera's theorem is equivalent to that f has an antiderivative on D.

The converse of the theorem is not true in general.

Example:

$Evaluate \int_\gamma \frac{dz}{z^2}\ where\ \gamma\ is\ defined\ by\ |z| = d, d > 0$

$let\ z = de^{i\theta}, o \le \theta \le 2\pi. Then\ dz = ie^{i\theta}.$

$$Now \int_\gamma \frac{1}{z^2} dz = \int_0^{2\pi} \frac{die^{i\theta}}{d^2 e^{i2\theta}} d\theta = \frac{i}{d} \int_0^{2\pi} e^{-i\theta} d\theta$$
$$= \frac{-1}{d}\left[e^{-i2\pi} - 1\right] = -\frac{1}{d}(1-1) = 0$$

$$Hence\ the\ integral\ of\ \frac{1}{z^2}\ along\ the\ circle\ \gamma\ is\ zero\ but\ \frac{1}{z^2}\ is\ not\ analytic\ at\ \ z = 0$$

$which\ is\ the\ center\ of\ \gamma.$

$If\ however\ the\ function\ f(z)\ is\ assumed\ to\ be\ continuous\ with\ in\ and\ on\ the\ boundary\ of\ C, vanishing\ of$

$\int_C\ f(z)dz\ will\ imply\ that\ f(z)\ is\ an\ analytic\ function\ in\ C.$

5. **Infinite Series, Convergence tests,**

5.1 Series : Let (a_n) be a sequence of real numbers. Then an expression of the form $a_1 + a_2 + a_3 + \cdots$ denoted by $\sum_{n=1}^{\infty} a_n$, is called a series.

Examples : 1. $\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

Examples : 2$\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots$

Partial sums : $S_n = a_1 + a_2 + a_3 + \cdots + a_n$ is called the nth partial sum of the series $\sum_{n=1}^{\infty} a_n$,

5.2 Convergence or Divergence of $\sum_{n=1}^{\infty} a_n$

If $S_n \to S$ for some S then we say that the series $\sum_{n=1}^{\infty} a_n$ converges to S. If (S_n) does not converge then we say that the series $\sum_{n=1}^{\infty} a_n$ diverges.

Examples :

1.$\sum_{n=1}^{\infty} \log(\frac{n+1}{n})$ diverges because $S_n = log(n + 1)$.

2. $\sum_{n=1}^{\infty}(\frac{n}{n+1})$ converges because $S_n = 1 - \frac{1}{n+1} \to 1$.

3. If $0 < x < 1$; then the geometric series$\sum_{n=1}^{\infty} x^n$ converges to $\frac{1}{1+x}$ because $S_n = \frac{1-x^n}{1-x}$.

Necessary condition for convergence

Theorem 1 : *If* $\sum_{n=1}^{\infty} a_n$*converges then* $a_n \to 0$.

Proof : $S_{n+1} - S_n = a_{n+1} \to S - S = 0$. ■
The condition given in the above result is necessary but not sufficient i.e., it is possible that $a_n \to 0$ and $\sum_{n=1}^{\infty} a_n$ diverges.

Examples :

1. If $|x| \geq 1$, then $\sum_{n=1}^{\infty} x^n$diverges because $a_n \nrightarrow 0$.
2. . $\sum_{n=1}^{\infty} sinn$ diverges because $a_n \nrightarrow 0$.
3. $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$diverges, however, $\log\left(\frac{n+1}{n}\right) \to 0$.

Necessary and sufficient condition for convergence

Theorem 2: *Suppose* $a_n \geq 0 \; \forall \; n$. *Then* $\sum_{n=1}^{\infty} a_n$ *converges if and only if* (S_n) *is bounded above.*

Proof : Note that under the hypothesis, (S_n) is an increasing sequence. ■
Example : The Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because

$$S_{2k} \geq 1 + \frac{1}{2} + 2.\frac{1}{4} + 4.\frac{1}{8} + \cdots + 2^{k-1}.\frac{1}{2^k} = 1 + \frac{k}{2}.$$

for all k.
Theorem 3: *If* $\sum_{n=1}^{\infty}|a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

Proof : Since $\sum_{n=1}^{\infty}|a_n|$converges the sequence of partial sums of $\sum_{n=1}^{\infty}|a_n|$ satisfies the Cauchy criterion. Therefore, the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ satisfies the Cauchy criterion. ■

Remark : Note that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=p}^{\infty} a_n$ converges for any $p \geq 1$.

5.3 Tests for Convergence

Let us determine the convergence or the divergence of a series by comparing it to one whose
behavior is already known.

Theorem 4 : (Comparison test) *Suppose* $0 \leq a_n \leq b_n$ *for n ≥k for some k:*

1. *Then the convergence of* $\sum_{n=1}^{\infty} b_n$*implies the convergence of* $\sum_{n=1}^{\infty} a_n$.
2. *The divergence of* $\sum_{n=1}^{\infty} a_n$ *implies the divergence of* $\sum_{n=1}^{\infty} b_n$.

Proof : 1. Note that the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ is bounded. Apply Theorem 2.

2.This statement is the contrapositive of (1). ■

Examples:

1. $\sum_{n=1}^{\infty}(\frac{1}{(n+1)^2})$ converges because $\frac{1}{(1+n)(1+n)} \le \frac{1}{n(n+1)}$ .This implies that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.
2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$diverges because $\frac{1}{n} \le \frac{1}{\sqrt{n}}$..
3. $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges because $n^2 < n!$ for $n > 4$*:*

Problem 1 : *Let* $a_n \ge 0$*: Then show that both the series* $\sum_{n\ge1} a_n$ *and* $\sum_{n\ge1} \frac{a_n}{a_{n+1}}$ *converge or diverge together.*

Solution : Suppose $\sum_{n\ge1} a_n$ converges. Since $0 \le \frac{a_n}{a_n+1} \le 1$ by comparison test $\sum_{n\ge1} \frac{a_n}{a_n+1}$ converges.

Suppose $\sum_{n\ge1} \frac{a_n}{1+a_n}$ converges. By the Theorem 1, $\frac{a_n}{1+a_n} \to 0$. Hence $a_n \to 0$ 0 and therefore

$1 \le 1 + a_n < 2$ eventually. Hence $0 \le \frac{a_n}{2} \le \frac{a_n}{1+a_n}$. Apply the comparison test.

Theorem 5 : (Limit Comparison Test) *Suppose* $a_n, b_n \ge 0$¸ 0 *eventually. Suppose*

1. *If* $L \epsilon$ R;$L > 0$, *then both* $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} a_n$ *converge or diverge together.*
2. *If* $L\epsilon$ R;$L = 0$, *and* $\sum_{n=1}^{\infty} b_n$ *converges then* $\sum_{n=1}^{\infty} a_n$*converges.*
3. *If* $L = 1$ *and* $\sum_{n=1}^{\infty} b_n$*diverges then* $\sum_{n=1}^{\infty} a_n$*diverges.*

Proof:

1. Since $L > 0$, choose $\epsilon > 0$, such that $L-\epsilon > 0$. There exists n_0 such that $0 \le L - \epsilon < \frac{a_n}{b_n} < L - \epsilon <$. Use the comparison test.
2. For each $\epsilon > 0$, there exists n_0 such that $0 < \frac{a_n}{b_n} < \epsilon \;\forall\, n > n_0$. Use the comparison test.
3. Given $\alpha > 0$, there exists n_0 such that $\frac{a_n}{b_n} > \alpha \;\forall\, n > n_0$. Use the comparison test. ■

Examples : 1. $\sum_{n=1}^{\infty}(1 - n\sin\frac{1}{n})$ converges. Take $b_n = \frac{1}{n^2}$in the previous result.

2. $\sum_{n=1}^{\infty} \frac{1}{n} log(1 + \frac{1}{n})$ converges. Take $b_n = \frac{1}{n^2}$in the previous result.

Theorem 6 (Cauchy Test or Cauchy condensation test) *If* $a_n \ge 0$*and* $a_{n+1} \le a_n \;\forall\, n$ *, then* $\sum_{n=1}^{\infty} a_n$*converges if and only if* $\sum_{k=1}^{\infty} 2^k a_{2^k}$ *converges.*

Proof : Let $S_n = a_1 + a_2 + a_3 + \cdots + a_n$and $T_k = a_1 + 2a_2 + \cdots + 2^k a_{2^k}$*:*

Suppose (T_k) converges. For a fixed n*;* choose k such that $2^k \ge n$. Then

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &\le a_1 + (a_2 + a_3) \ldots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\ &\le a_1 + 2a_2 + \cdots + 2^k a_{2^k} \\ &= T_k. \end{aligned}$$

.

This shows that (S_n) is bounded above; hence (S_n) converges.

Suppose (S_n) converges. For a fixed k*;* choose n such that $n > 2^k$ Then

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &\ge a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k+1}-1} + \cdots + a_{2^k}) \\ &\ge \tfrac{1}{2} a_1 + a_2 + 2a_4 \ldots + 2^{k-1} a_{2^k} \end{aligned}$$

$= T_k.$
This shows that (T_k) is bounded above; hence (T_k) converges. ■

Examples:
1. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$:
2. $\sum_{n=1}^{\infty} \frac{1}{n(logn)^p}$ converges if $p > 1$ and diverges if $p \le 1$:

Problem 2 : Let $a_n \geq 0$ *and* $a_{n+1} \leq a_n \; \forall \, n$ and suppose $\sum a_n$ converges. Show that $na_n \to 0$ as $n \to \infty$.

Solution : By Cauchy condensation test $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges. Therefore $2^k a_{2^k} \to 0$ and hence $2^{k+1} a_{2^k} \to 0$ as $k \to \infty$. Let $2^k \leq n \leq 2^{k+1}$. Then $na_n \leq na_{2^k} \leq 2^{k+1} a_{2^k} \to 0$. This implies that $na_n \to 0$ as $n \to \infty$.

Theorem 7 (Ratio test) *:Consider the series* $\sum_{n=1}^{\infty} a_n,\; a_n \neq 0 \; \forall \, n$*:*
1. If $\left|\frac{a_{n+1}}{a_n}\right| \leq q$ *eventually for some* $0 < q < 1$*; then* $\sum_{n=1}^{\infty} |a_n|$ *converges.*
2. If $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ *eventually then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof:
1. Note that for some *N;* $|a_{n+1}| \leq q|a_n| \; \forall \, n \geq N$ Therefore, $|a_{N+p}| \leq q^p |a_N| \;\; \forall \, p > 0$ Apply the comparison test.
2. In this case $|a_n| \nrightarrow 0$.

Corollary 1: *Suppose* $a_n \neq 0 \; \forall \, n$*; and* $\left|\frac{a_{n+1}}{a_n}\right| \to L$ *for some L.*
for some L:
1. If L < 1 *then* $\sum_{n=1}^{\infty} |a_n|$ *converges.*
2. If L > 1 *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*
3. If L $= 1$ *we cannot make any conclusion.*

Proof :
1. Note that $\left|\frac{a_{n+1}}{a_n}\right| < L + \frac{1-L}{2}$ eventually. Apply the previous theorem.
2. Note that $\left|\frac{a_{n+1}}{a_n}\right| > L + \frac{L-1}{2}$ eventually. Apply the previous theorem.

Examples :
1. $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges because $\frac{a_{n+1}}{a_n} \to 0$
2. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges because $\frac{a_{n+1}}{a_n} = \left(1 + \frac{1}{n}\right)^n \to e \;\; > 1$
3. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, however, in both these cases $\frac{a_{n+1}}{a_n} \to 1$:

Theorem 8 : (Root Test) *If* $0 \leq a_n \leq x^n$ *or* $0 \leq {a_n}^{1/n} \leq x$ *eventually for some* $0 < x < 1$ *then* $\sum_{n=1}^{\infty} |a_n|$ *converges.*
Proof : Immediate from the comparison test. ■

Corollary 2: *Suppose* $|a_n|^{1/n} \to L$ *for some L: Then*
1. *If L* < 1 *then* $\sum_{n=1}^{\infty} |a_n|$ *converges.*
2. *If L* > 1 *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

3. If L = 1 we cannot make any conclusion.

Examples :

1. $\sum_{n=2}^{\infty}\frac{1}{(logn)^n}$ converges because ${a_n}^{1/n} = \frac{1}{logn} \to 0$
2. $\sum_{n=1}^{\infty}\left(\frac{n}{1+n}\right)^{n^2}$ converges because ${a_n}^{1/n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e} < 1.$
3. $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, however, in both these cases ${a_n}^{1/n} \to 1.$

5.4 Taylor Series and Maclaurin Series

The Taylor series of a real or complex-valued function $f(x)$ that is infinitely differentiable at a real or complex number a is the power series

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots.$$

which can be written in the more compact sigma notation as

$$\sum_{n=0}^{\infty}\frac{f^{(n)}}{n!}(x-a)^n$$

where $n!$ denotes the factorial of n and $f^{(n)}(a)$ denotes the nth derivative of f evaluated at the point a. The derivative of order zero of f is defined to be f itself and $(x-a)^0$ and 0! are both defined to be 1. When $a = 0$, the series is also called a **Maclaurin series**.

Example:

The Maclaurin series for any polynomial is the polynomial itself.

The Maclaurin series for $(1-x)^{-1}$ is the geometric series

$$1 + x + x^2 + x^3 + \cdots$$

so the Taylor series for x^{-1} at $a = 1$ is

$$1 + (x-1) + (x-1)^2 + (x-1)^3 + \cdots$$

By integrating the above Maclaurin series, we find the Maclaurin series for $\log(1-x)$, where log denotes the natural logarithm:

$$-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots$$

and the corresponding Taylor series for $\log(x)$ at $a = 1$ is

$$-x - \frac{1}{2}(x-1)^2 - \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 - \cdots$$

and more generally, the corresponding Taylor series for $\log(x)$ at some $a = x_0$ is:

$$\log(x_0) + \frac{1}{x_0}(x-x_0) + \frac{1}{{x_0}^2}(x-x_0)^2 + \cdots.$$

The Taylor series for the exponential function e^x at $a = 0$ is

$$1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

The above expansion holds because the derivative of e^x with respect to x is also e^x and e^0 equals 1. This leaves the terms $(x-0)^n$ in the numerator and $n!$ in the denominator for each term in the infinite sum.

Laurent series and Residues

6.1 What is a Laurent series?

The Laurent series is a representation of a complex function f(z) as a series. Unlike the Taylor series which expresses f(z) as a series of terms with non-negative powers of z, a Laurent series includes terms with negative powers. A consequence of this is that a Laurent series may be used in cases where a Taylor expansion is not possible.

6.2 Calculating the Laurent series expansion

To calculate the Laurent series we use the standard and modified geometric series which are

$$\frac{1}{1-z} = \begin{cases} \sum_{n=0}^{\infty} z^n, & |z| < 1, \\ \sum_{n=1}^{\infty} \frac{1}{z^n} & |z| > 1. \end{cases} \qquad (1)$$

Here f(z) = $\frac{1}{1-z}$ is analytic everywhere apart from the singularity at z = 1. Above are the expansions for f in the regions inside and outside the circle of radius 1, centered on z = 0, where |z| < 1 is the region inside the circle and |z| > 1 is the region outside the circle.

Example-1:

Determine the Laurent series for

$$f(z) = \frac{1}{z+5}$$

that are valid in the regions

(i) {z : |z| < 5}, and (ii) {z : |z| > 5}.

Solution

The region (i) is an open disk inside a circle of radius 5, centered on z = 0, and the region (ii) is an open annulus outside a circle of radius 5, centered on z = 0. To make the series expansion easier to calculate we can manipulate our f(z) into a form similar to the series expansion shown in equation (1).

So, $$f(z) = \frac{1}{5(1+\frac{z}{5})} = \frac{1}{5(1-(-\frac{z}{5}))}.$$

Now using the standard and modified geometric series, equation (1), we can calculate that

$$f(z) = \frac{1}{5\left(1-\left(-\frac{z}{5}\right)\right)} = \begin{cases} \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{-z}{5}\right)^n, & |z| < 5, \\ -\frac{1}{5}\sum_{n=1}^{\infty}\frac{1}{\left(\frac{-z}{5}\right)^n} & |z| > 5. \end{cases}$$

Hence, for part (i) the series expansion is

$$f(z) = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{-z}{5}\right)^n = \frac{1}{5}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{5^n} = \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{5^{n+1}}, \qquad |z| < 5,$$

which is a Taylor series. And for part (ii) the series expansion is

$$f(z) = -\frac{1}{5}\sum_{n=1}^{\infty}\frac{1}{\left(\frac{-z}{5}\right)^n} = -\frac{1}{5}\sum_{n=1}^{\infty}\frac{(-1)^n 5^n}{z^n} = -\sum_{n=1}^{\infty}\frac{(-1)^n 5^{n-1}}{z^n}, \qquad |z| > 5.$$

Example 2.

For the following function f determine the Laurent series that is valid within the stated region R.

$$f(z) = \frac{1}{z(z+2)}, \qquad R = \{z: 1 < |z-1| < 3\}.$$

Solution

The region R is an open annulus between circles of radius 1 and 3, centered on z=1.We want a series expansion about z=1;to do this we make a substitution $w = z - 1$ and look for the expansion in w where $1 < |w| < 3$. In terms of w

$$f(z) = \frac{1}{(w+1)(w+3)}.$$

To make the series expansion easier to calculate we can manipulate our f(z) into a form similar to the series expansion shown in equation (1). To do this we will split the function using partial fractions, and then manipulate each of the fractions into a form based on equation (1), so we get

$$f(z) = \frac{1}{2}\left(\frac{1}{(w+1)} - \frac{1}{(w+3)}\right) = \frac{1}{2}\left(\frac{1}{(w-(-1)} - \frac{1}{3\left(1-\left(\frac{-w}{3}\right)\right)}\right).$$

Using the the standard and modified geometric series, equation (1), we can calculate that

$$\frac{1}{(1-(-w))} = \begin{cases} \sum_{n=0}^{\infty}(-w)^n = \sum_{n=0}^{\infty}(-1)^n(w)^n, & |w| < 1, \\ -\sum_{n=1}^{\infty}\frac{1}{(-w)^n} = -\sum_{n=1}^{\infty}\frac{(-1)^n}{(w)^n}, & |w| > 1, \end{cases}$$

and

$$\frac{1}{3\left(1-\left(\frac{-w}{3}\right)\right)} = \left\{\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{-w}{3}\right)^n = \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n w^n}{3^n} = \sum_{n=0}^{\infty}\frac{(-1)^n w^n}{3^{n+1}}\right.$$

$$\frac{1}{3\left(1-\left(\frac{-w}{3}\right)\right)} = \begin{cases} \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{-w}{3}\right)^n = \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n w^n}{3^n} = \sum_{n=0}^{\infty}\frac{(-1)^n w^n}{3^{n+1}}, & |w| < 3 \\ -\frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{\left(\frac{-w}{3}\right)^n} = -\frac{1}{3}\sum_{n=1}^{\infty}\frac{(-3)^n}{w^n} = \sum_{n=1}^{\infty}\frac{(-1)^n 3^{n-1}}{w^n}, & |w| > 3. \end{cases}$$

We require the expansion in w where $1 < |w| < 3$, so we use the expansions for $|w| > 1$ and $|w| < 3$, which we can substitute back into our f(z) in partial fraction form to get

$$f(z) = \frac{1}{2}\left[-\sum_{n=1}^{\infty}\frac{(-1)^n}{(w)^n} - \sum_{n=0}^{\infty}\frac{(-1)^n w^n}{3^{n+1}}\right] = -\frac{1}{2}\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{(w)^n} + \sum_{n=0}^{\infty}\frac{(-1)^n w^n}{3^{n+1}}\right].$$

Substituting back in $w = z - 1$, we get the Laurent series, valid within the region $1 < |z-1| < 3$,

$$f(z) = -\frac{1}{2}\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{(z-1)^n} + \sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^n}{3^{n+1}}\right].$$

6.2 Singularities

Def. A point $z = a$ is called a **singularity** of a function $f(z)$ if f is not differentiable at $z = a$.

***Examples* 1**

(a) $f(z) = e^{3/z}$; $a = 0$

(b) $f(z) = (z-1)^{-2}$; $a = 1$

(c) $f(z) = \tan z$; $a = {}^{(2n+1)\pi}\!/_2$

Classification of the Singularity *a* of *f*

depends on the Laurent Series of f about a, i.e.,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n .$$

i) a is called a **Removable singularity** if
$c_n = 0$ for all $n < 0$

***Example* 2:** $f(z) = (\sin z)/z$; $a = 0$.

ii) a is called a **Essential singularity** if
$c_n = 0$ for infinitely many $n < 0$

***Example* 3 :** $f(z) = e^{3/z}$; $a = 0$.

iii) a is called a **Simple Pole** if

$$c_{-1} \neq 0 \text{ and } c_n = 0 \text{ for all } n < -1$$

***Example* 4:** $f(z) = (z-1)^{-1}$; $a = 1$.

iv) a is called a **Pole of Order *m*** if

$$c_{-m} \neq 0 \text{ and } c_n = 0 \text{ for all } n < -m .$$

***Example* 5:** $f(z) = (z-1)^{-5}$; $a = 1$ is a pole of order 5 for the function f.

6.3 **Zero of Order *m***

A point $z = a$ is called a **zero of order *m*** of a function $f(z)$ if

$$f^{(j)}(a) = 0,\ j = 0, 1, \ldots, m-1 \text{ but } f^{(m)}(a) \neq 0.$$

***Example* 6:** $f(z) = (z-1)^5$; $a = 1$ is a Zero of order 5 for the function f.

Note: If $z = a$ is a **zero of order *m*** of a function $f(z)$, then $z = a$ is a pole **of order *m*** for $1/f(z)$.

<u>**Test for the Order of a Pole of Rational Functions**</u> $F(z) = f(z) / g(z)$

$z = a$ is a pole **of order *m*** for F if

i. $f(a) \neq 0$.

ii. $z = a$ is a **zero of order *m*** of $f(z)$.

***Exercise*:** Apply this test to the above examples.

6.4 **Residue of $f(z)$ at a Singularity $z = a$**

is defined with the help of the Laurent Series of f about a, i.e.,
if we have

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n ,$$

then the number c_{-1} is called the **Residue** of $f(z)$ at $z = a$.

Notation: "Residue of $f(z)$ at $z = a$" = **Res(*f* , *a*).**

Examples

In Examples 1

(a) Res($e^{3/z}$, 0) = 3

(b) Res($(z-1)^{-2}$,1) = 0

Note: Res($(z-1)^{-1}$,1) = 1

6.5 **Method for Finding Res(*f* , *a*)**

[without Laurent Series]

Case 1: If a is a <u>Simple Pole</u> of f , then

$$\text{Res}(f, a) = \lim_{z\to a}(z-a)f(z).$$

Example: Res($(z-3)(z-1)^{-1}$,1) = – 2

Case 2: If a is a <u>Pole of order m</u> of f , then

$$\text{Res}(f, a) = \frac{1}{(m-1)!}\left(\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z)\right).$$

Examples:

i. Res($(z-1)^{-2}$,1) = 0 (Check)

ii. Res($((z-3)^{-1}(z-1)^{-2}$,1) = – ¼ (Check)

iii. Res($((z-3)^{-1}(z-1)^{-2}$,3) = ¼ (Check)

Note: We can not use this method if a is <u>not a Pole</u>: For example check for Res($e^{3/z}$, 0).

6.6 **Residue Theorem**

(To Evaluate $\oint_C f(z)dz$ where C is a closed path)

Conditions:

i. C is a simple path in a simply connected domain D.

ii. $f(z)$ is differentiable on and within C except at a finite number of singularities, say a_1, a_2, ….., a_n within C.

Conclusion:

$$\oint_C f(z)dz = 2\pi i\sum_{i=1}^{n}\text{Res}(f, a_i)$$

Examples

1. $\displaystyle\int_{|z|=\frac{1}{2}}\frac{dz}{(z-1)(z+2)^2} = 0$ (No singularities within $|z|=1/2$)

2. $\displaystyle\int_{|z|=\frac{3}{2}}\frac{dz}{(z-1)(z+2)^2} = 2\pi i(\text{Res}(f,1) = \frac{2\pi i}{9}$

6.7 **Evaluation of Real Integrals**

A. ***Type 1***: <u>Real Trigonometric Integral</u> $I=\int_0^{2\pi}F(\cos\theta,\sin\theta)d\theta$

Method (Change of variable): C: $z=e^{i\theta}, 0\le\theta\le 2\pi.$

Then: $d\theta=\dfrac{dz}{iz}, \cos\theta=\dfrac{z+z^{-1}}{2}, \sin\theta=\dfrac{z-z^{-1}}{2i}$

(Note: $e^{i\theta}=\cos\theta+i\sin\theta$)

Now use ***Residue Theorem*** to solve

$$I = \oint_C F\left(\frac{1}{2}(z+z^{-1}), \frac{1}{2}(z-z^{-1})\right)\frac{dz}{z}.$$

Example 1: Evaluate $\int_0^{2\pi} \frac{d\theta}{2-\sin\theta}$

Solution: i. Use: (a) $z = e^{i\theta}, 0 \le \theta \le 2\pi.$

(b) $d\theta = \frac{dz}{iz}, \sin\theta = \frac{z-z^{-1}}{2i}$

ii. $I = \oint_C \frac{-2dz}{z^2-4iz-1} = \oint_C \frac{-2dz}{(z-a_1)(z-a_1)} \equiv \oint_C f(z)dz$

Here: $a_1 = (2-\sqrt{3})i; a_2 = (2+\sqrt{3})i$ (Only a_1 is inside C)

iii. $\oint_C f(z)dz = 2\pi i \operatorname{Res}(f, a_1) = \frac{2\pi}{\sqrt{3}}$ (Answer)

$(\operatorname{Res}(f,a_1) = \lim_{z\to a_1}\left[(z-a_1)\frac{-2}{(z-a_1)(z-a_2)}\right] = \frac{1}{i\sqrt{3}})$

B. *Important Concepts*

a. Cauchy Principal Value of $\int_{-\infty}^{\infty} f(x)dx$**:**

P. V. $\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_R^R f(x)dx$

Note 1: If the integral $\int_{-\infty}^{\infty} f(x)dx$ converges and its value is A then

$$\text{A} = \text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_R^R f(x)dx.$$

Note 2: It may happen that

i. The integral $\int_{-\infty}^{\infty} f(x)dx$ may diverge

ii. $\lim_{R\to\infty}\int_R^R f(x)dx$ exits.

Example: $\int_{-\infty}^{\infty}\frac{dx}{x}$ diverges but $\lim_{R\to\infty}\int_R^R \frac{dx}{x} = 0$.

b. ***Two Important Results*****:**

Suppose that

i. C_R: $z = Re^{i\theta}, 0 \le \theta \le \pi$, a semicircular path

ii. $f(z) = P(z)/Q(z)$, P & Q are Polynomials.

Result I **:** If degree Q $\ge$ degree P + 2, then

$$\int_{C_R} f(z)dz \to 0 \text{ as } R \to \infty.$$

Result II **:**If degree Q $\geq$ degree P + 1, then

$$\int_{C_R} f(z)e^{i\alpha z}dz \to 0 \text{ as } R\to\infty, \text{ where } \alpha>0.$$

(*a useful result related to Fourier Transform*)

c. ***Third Important Result*****:**
Result III **:**Suppose that
i. f has a simple real pole at $z = c$.
ii. C_r: $z=c+Re^{i\theta}, 0\leq\theta\leq\pi$.
Then $\int_{C_r} f(z)dz = \pi i \operatorname{Res}(f,c) \text{ as } r\to 0$

d. $$\begin{cases} \int_{-\infty}^{\infty} f(x)\cos x\,dx = \operatorname{Re}\int_{-\infty}^{\infty} f(x)e^{iax}\,dx \\ \int_{-\infty}^{\infty} f(x)\sin x\,dx = \operatorname{Im}\int_{-\infty}^{\infty} f(x)e^{iax}\,dx \end{cases}$$

e. $\int_0^{\infty} f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx,$ if $f(x)$ is even.

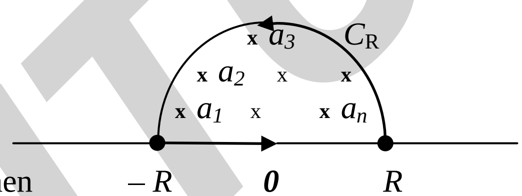
B. ***Type*** **2**: $I=\int_{-\infty}^{\infty} f(x)dx$

where *f* =*P/Q with* degree Q $\geq$ degree P + 2 **has finite number of poles in the complex plane**:

Method**:** Draw a closed path

$$C = C_R+[-R,R]$$

where C_R: $z=Re^{i\theta}, 0\leq\theta\leq\pi$, with large R to enclose the poles a_k 's of f within C, which are in the upper half of the plane.



Then

$$\underbrace{\oint_C f(z)dz}_{2\pi i\sum_{k=1}^{n}\operatorname{Res}(f,a_k)} = \underbrace{\int_{C_R} f(z)dz}_{0 \text{ as } R\to\infty} + \underbrace{\int_{-R}^{R} f(x)dx}_{\int_{-\infty}^{\infty} f(x)dx \text{ as } R\to\infty}$$

Example 2: Evaluate $\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)^2(x^2+9)}$

Solution: i. Here $f(z)=\frac{1}{(z^2+1)^2(z^2+9)}$

ii. Poles of f ***in the upper half plane***:

(a) $z = i$ (Double Pole)
(b) $z = 3\,i$ (Simple Pole)

iii. Calculate the Residues:

$$\mathrm{Res}(f,i)=\lim_{z\to i}\frac{d}{dz}(z-i)^2\frac{1}{(z^2+1)^2(z^2+9)}=\frac{3}{8^2.4i}$$

$$\mathrm{Res}(f,3i)=\lim_{z\to i}(z-3i)\frac{1}{(z^2+1)^2(z^2+9)}=\frac{1}{8^2.6i}$$

iv. Note that:

1) $\oint_C f(z)dz=\int_{C_R} f(z)dz+\int_{-R}^{R} f(x)dx$ …… (*)

2) $\oint_C f(z)dz=2\pi i\left[\mathrm{Res}(f,i)+\mathrm{Res}(f,3i)\right]=\dfrac{7\pi}{8^2.6}$

3) (**Ans.**) Apply Cauchy Residue Theorem to the Left side of (*). Take the limit as $R\to\infty$ in (*). Then apply Result I to 2nd integral in (*), we get:

$$\frac{7\pi}{8^2.6}=\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)^2(x^2+9)}.$$

C. ***Type* 3**: $I=\int_{-\infty}^{\infty} f(x)[\cos ax \text{ or } \sin ax]dx$

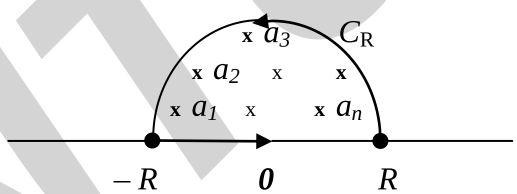
where *f* =*P/Q* (degree Q $\geq$ degree P + 1) **has finite number of poles in the complex plane and *a* > 0**

***Method*: i.** According to (c) above**, write** I as:

$$\boldsymbol{I} = \mathrm{Re}\ \text{or}\ \mathrm{Im}\left[\int_{-\infty}^{\infty} f(x)e^{iax}dx\right]$$

ii. Draw a closed path $C = C_R+[-R,R]$

where C_R: $z=Re^{i\theta}, 0\le\theta\le\pi$, with large R to enclose all poles a_k's of f within C.



iii. $$\underbrace{\oint_C f(z)e^{iaz}dz}_{2\pi i\sum_{k=1}^{n}\mathrm{Res}(fe^{iaz},a_k)}=\underbrace{\int_{C_R} f(z)e^{iaz}dz}_{0 \text{ as } R\to\infty}+\underbrace{\int_{-R}^{R} f(x)e^{iax}dx}_{\int_{-\infty}^{\infty} f(x)e^{iax}dx \text{ as } R\to\infty}$$

iv. Answer: $\boldsymbol{I}=\mathrm{Re}\ \text{or}\ \mathrm{Im}\left[\int_{-\infty}^{\infty} f(x)e^{iax}dx\right]$

$$=\mathrm{Re}\ \text{or}\ \mathrm{Im}\left[2\pi i\sum_{k=1}^{n}\mathrm{Res}(f.e^{iaz},a_k)\right]$$

Example 3: Evaluate $\boldsymbol{I}=\int_0^{\infty}\dfrac{x\sin x dx}{(x^2+1)(x^2+4)}$

Solution: i. Note the integrand is an even function: Therefore, $\boldsymbol{I} = \dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x dx}{(x^2+1)(x^2+4)}$.

iii. Here **,**

$$\boldsymbol{I} = \operatorname{Im}\left(\frac{1}{2}\int_{-\infty}^{\infty} f(x)e^{ix}dx\right)$$

$$\textbf{where } f(z) = \frac{z}{(z^2+1)(z^2+4)}$$

iv. Poles of f ***in the upper half plane***:

$z = i$, $z = 2i$ (2 Simple Poles)

iii. Calculate the Residues:

$$\operatorname{Res}(fe^{iz}, i) = \lim_{z\to i}(z-i)\frac{ze^{iz}}{(z^2+1)(z^2+4)} = \frac{e^{-1}}{6}$$

$$\operatorname{Res}(fe^{iz}, 2i) = \lim_{z\to i}(z-2i)\frac{ze^{iz}}{(z^2+1)(z^2+4)} = \frac{-e^{-2}}{6}$$

iv. Note that:

1) $\oint_C f(z)e^{iz}dz = \int_{C_R} f(z)e^{iz}dz + \int_{-R}^{R} f(x)e^{ix}dx$ … (*)

2) Apply Cauchy Residue Theorem to the Left side of (*). Take the limit as $R\to\infty$ in (*). Then apply Result I to 2nd integral in (*), we get:

$$\begin{cases} \displaystyle\int_{-\infty}^{\infty} f(x)e^{ix}dx = 2\pi i\left[\operatorname{Res}(fe^{iz}, i) + \operatorname{Res}(fe^{iz}, 2i)\right] \\ \qquad = 2\pi i\left(\dfrac{e^{-1}}{6} - \dfrac{e^{-2}}{6}\right) = \dfrac{i\pi}{3e^2}(e-1) \end{cases}$$

4) (**Ans.**)

$$\boldsymbol{I} = \operatorname{Im}\left[\frac{1}{2}\int_{-\infty}^{\infty}\frac{xe^{ix}dx}{(x^2+1)(x^2+4)}\right] = \frac{\pi}{6e^2}(e-1)$$

D. *Type* 4: $\boldsymbol{I} = \int_{-\infty}^{\infty} f(x)dx$

where f has a simple pole $x=c$ on the real axis and finite number of complex poles :

Method:

i. Draw a closed path

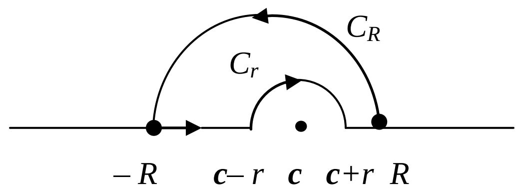
$$C = C_R + [-R, -r] - C_r + [r, R]$$

where C_R: $z = Re^{i\theta}, 0 \le \theta \le \pi$, with large R to enclose within C all poles a_k 's of f .

ii. Also, draw another path

$$C_r: z = c + re^{i\theta}, 0 \le \theta \le \pi,$$

with small r so that not none of the complex poles of f is enclosed between C_r and the real axis.



$$\underbrace{\oint_C f(z)dz}_{2\pi i\sum_{k=1}^{n}\mathrm{Res}(f,a_k)} = \underbrace{\int_{C_R} f(z)dz}_{0 \text{ as } R\to\infty} + \underbrace{\int_{-R}^{c-r} f(x)dx}_{\int_{-\infty}^{c} f(x)dx \text{ as } r\to 0, R\to\infty}$$

$$+ \underbrace{\int_{-C_r} f(z)dz}_{-\pi i\,\mathrm{Res}(f,c) \text{ as } r\to 0} + \underbrace{\int_{c-r}^{R} f(x)dx}_{\int_{c}^{\infty} f(x)dx \text{ as } r\to 0, R\to\infty}$$

iii. **(Ans.)**

$$\boldsymbol{I} = \; = 2\pi i\sum_{k=1}^{n}\mathrm{Res}(f.e^{iaz},a_k) + \pi i\,\mathrm{Res}(f.e^{iaz},c)$$

Example 4: Evaluate $\boldsymbol{I} = \int_{-\infty}^{\infty}\frac{dx}{x(x+4)(x^2+16)}$

Solution: Here, $f(z) = \frac{1}{z(z+4)(z^2+16)}$

i. Poles of f ***in the upper half plane***:

$z = 2i$ (1 Simple Pole)

ii. Poles of f ***on the Real Axis***:

$z = 0, -4$ (2 Simple Poles)

iii. Calculate the Residues: