

## Mathematics - II



# Contents

|          |  |            |
|----------|--|------------|
| <b>1</b> | <b>Fundamental Mathematics II</b>                        | <b>5</b>   |
| 1.1      | Counterexamples . . . . .                                | 5          |
| 1.2      | Integer Exponents . . . . .                              | 9          |
| 1.3      | Scientific Notation – Application of Exponents . . . . . | 25         |
| 1.4      | Polynomials . . . . .                                    | 30         |
| 1.5      | Properties of Equality and Linear Equations . . . . .    | 38         |
| 1.6      | Applications of Linear Equations: Geometry . . . . .     | 49         |
| 1.7      | Applications of Linear Equations: Percents . . . . .     | 59         |
| 1.8      | Applications of Linear Equations: Proportions . . . . .  | 68         |
| 1.9      | Linear Inequalities . . . . .                            | 74         |
| 1.10     | Nonlinear Equations and Inequalities . . . . .           | 83         |
| 1.11     | Equations in Two Variables . . . . .                     | 93         |
| 1.12     | Deriving Common Formulas in Two Variables . . . . .      | 106        |
| <b>A</b> | <b>Formulas</b>  | <b>113</b> |
| <b>B</b> | <b>Answers to Exercises</b>                              | <b>117</b> |
| B.1      | Counterexamples . . . . .                                | 117        |
| B.2      | Integer Exponents . . . . .                              | 118        |
| B.3      | Scientific Notation . . . . .                            | 119        |
| B.4      | Polynomials . . . . .                                    | 120        |
| B.5      | Linear Equations . . . . .                               | 121        |
| B.6      | Geometry . . . . .                                       | 121        |
| B.7      | Percents . . . . .                                       | 122        |
| B.8      | Proportions . . . . .                                    | 123        |
| B.9      | Linear Inequalities . . . . .                            | 123        |
| B.10     | Nonlinear Equations and Inequalities . . . . .           | 124        |
| B.11     | Linear Equations in Two Variables . . . . .              | 125        |
| B.12     | Common Formulas . . . . .                                | 128        |



# Chapter 1

## Fundamental Mathematics II

### 1.1 Counterexamples

The validity or falsehood of a mathematical statement is important. Statements that are true can be used to show further mathematical properties. Some statements that are false are common mistakes that need to be avoided. A **counterexample** is an example that refutes or disproves a statement. In mathematics, counterexamples are often used to determine the legitimacy of possible theorems. For example, it was discussed in Fundamental Mathematics I that subtraction is not commutative. ([Click here to review properties of real numbers from FM1](#)) This means that the following statement is *false*: For all real numbers  $a$  and  $b$ ,  $a - b = b - a$ . Here is a counterexample:

$$5 - 3 \neq 3 - 5 \quad (\text{because } 5 - 3 = 2 \text{ and } 3 - 5 = -2 \text{ but } -2 \neq 2).$$

Counterexamples can be used to illustrate a number of common mistakes individuals make. Familiarity with counterexamples can help eliminate these mistakes.

**Example 1** *Determine whether the associative property holds for subtraction. That is, determine whether  $a - (b - c) = (a - b) - c$  for all real numbers  $a, b$ , and  $c$ .*

**Solution 1**

*Note, there are many different counterexamples. Let  $a = 7$ ,  $b = 2$ , and  $c = 4$ . Then*

$$a - (b - c) = 7 - (2 - 4) = 7 - (-2) = 9.$$

However,

$$(a - b) - c = (7 - 2) - 4 = 5 - 4 = 1.$$

Hence,

$$a - (b - c) \neq (a - b) - c.$$

**Practice 1** Determine whether the associative property holds for division. In other words, is it true or false that

$$a \div (b \div c) = (a \div b) \div c$$

for all real numbers  $a, b$ , and  $c$ . (Answer on page 7.)

Remember, if you can find one counterexample, then the statement is false. However, if you can find one example where the property holds, this does not mean that the statement is true. We can show a property is not true with a counterexample; however, we cannot show a property is true with an example. Proving a property is true requires more work. The next example illustrates a common mistake.

**Example 2** Determine if  $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$  for all real numbers  $a$  and  $b$ .

### Solution 2

If it is not true, then we need to find a counterexample. Let  $a = 4$  and  $b = 9$ . Then  $\sqrt{a+b} = \sqrt{4+9} = \sqrt{13}$ . On the other hand,  $\sqrt{a} + \sqrt{b} = \sqrt{4} + \sqrt{9} = 2 + 3 = 5$ . Hence, we have just shown that  $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$  for all  $a$  and  $b$ .

If  $a = 9$  and  $b = 0$ , then the previous statement would be true. This is why an example will not be sufficient in proving that a property is true.

**Practice 2** Determine whether it is true or false that  $|a - b| = |a| - |b|$  for all real numbers  $a$  and  $b$ . (Answer on page 7.)

We can also use counterexamples to investigate fractions further. Recall that  $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$ . However, can we still rewrite the fraction if the sum occurs in the denominator rather than in the numerator?

**Example 3** Determine if  $\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$  for all real numbers  $a$  and  $b$ .

**Solution 3** Let us consider  $a = 1$  and  $b = 2$ . Then  $\frac{1}{a+b} = \frac{1}{1+2} = \frac{1}{3}$  which is a fraction less than one. However,  $\frac{1}{a} + \frac{1}{b} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$  which is definitely larger than one. Therefore,  $\frac{1}{a+b} \neq \frac{1}{a} + \frac{1}{b}$  for all real numbers  $a$  and  $b$ .

**Practice 3** Determine whether it is true or false that  $\frac{a+b}{c+d} = \frac{a}{c} + \frac{b}{d}$  for all real numbers  $a, b, c$ , and  $d$ . (Answer on page 7.)

#### ANSWERS TO PRACTICE PROBLEMS

1. False; Let  $a = 8$ ,  $b = 4$ , and  $c = 2$ . Then

$$a \div (b \div c) = 8 \div (4 \div 2) = 8 \div 2 = 4$$

and

$$(a \div b) \div c = (8 \div 4) \div 2 = 2 \div 2 = 1.$$

2. False; Let  $a = 7$  and  $b = -5$ . Then  $|a - b| = |7 - (-5)| = |12| = 12$  and  $|a| - |b| = |7| - |-5| = 7 - 5 = 2$ . Note that  $|a - b|$  is always positive; whereas,  $|a| - |b|$  could be negative.

3. False; Let  $a = 2$ ,  $b = 1$ ,  $c = 4$ , and  $d = 2$ . Then  $\frac{a+b}{c+d} = \frac{2+1}{4+2} = \frac{3}{6} = \frac{1}{2}$  and  $\frac{a}{c} + \frac{b}{d} = \frac{2}{4} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$ .

#### SECTION 1.1 EXERCISES

(Answers are found on page 117.)

1. The distributive property holds for multiplication over addition. Is there a distributive property for addition over multiplication? In other words, is it true that  $a + (b \cdot c) = (a + b) \cdot (a + c)$  for all real numbers  $a, b$ , and  $c$ ?

**In #2–#14, determine if the following statements are true or false. If the statement is false, give a counterexample.**

2.  $a(b \div c) = (ab) \div (ac)$

3.  $a \div b \cdot c = a \cdot c \div b$

4.  $a \div (b \cdot c) = a \div b \cdot c$

- 5.  $(a \div b) \cdot c = a \div b \cdot c$
- 6.  $a \div b \div c = a \div (b \div c)$
- 7.  $\frac{ab}{cd} = \frac{a}{c} \cdot \frac{b}{d}$
- 8.  $1x = x$
- 9.  $\frac{x}{2} = \frac{1}{2}x$
- 10.  $a(x+1)(x-2) = (ax+a)(x-2a)$
- 11.  $|a+b| = |a| + |b|$
- 12.  $a(b \cdot c) = (a \cdot b)(a \cdot c)$
- 13.  $a(b-c) = ab - ac$
- 14.  $a - b + c = a + c - b$
- 15.  $(2b)^3 = 2b^3$  ([Click here to review exponents from FM1](#))
- 16.  $(ab)^2 = ab^2$
- 17.
  - (a) Show by counterexample that  $(a+b)^2 \neq a^2 + b^2$ .
  - (b) Are there any integer values of  $n$  for which  $(a+b)^n = a^n + b^n$ ?
  - (c) Are there any integer values of  $a$  or  $b$  for which  $(a+b)^n = a^n + b^n$ ?



## 1.2 Integer Exponents

Recall that exponents can be used for repeated multiplication. ([Click here to review exponents from FM1](#)) For example,

$$3^4 = \underbrace{3 \cdot 3 \cdot 3 \cdot 3}_{4 \text{ factors}}$$

where 3 is the **base**, 4 is the **exponent**, and  $3^4$  is a **power**. The term **power** is also used to refer to just the exponent itself. An algebraic expression containing exponents is an **exponential expression**. When working with exponents it is very important to determine the base.

**Example 1** Determine the base of each exponential expression and evaluate.

$$(a) \quad 2^4 \qquad (b) \quad (-2)^4 \qquad (c) \quad -2^4$$

**Solution 1**

(a) Here 2 is the base and 4 is the exponent. Hence,

$$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16.$$

(b) Now,  $-2$  is the base while 4 is still the exponent. Thus,

$$(-2)^4 = -2 \cdot -2 \cdot -2 \cdot -2 = 16.$$

(c) It is important to note that 2 is the base. The lack of parentheses shows that the exponent of 4 applies only to that base of 2 and not  $-2$ . Therefore,

$$-2^4 = -1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = -16.$$

**CAUTION:**  $-a^n$  and  $(-a)^n$  do not always yield the same answers. While both have  $n$  as the exponent,  $-a^n$  has base  $a$ ; whereas,  $(-a)^n$  has base  $-a$ .

**Practice 1** Evaluate each expression. (Answers on page [21](#))

$$(a) \quad -4^2 \qquad (b) \quad (-4)^2 \qquad (c) \quad -3^3 \qquad (d) \quad (-3)^3$$

We next would like to develop the exponent rules for the integer exponents. To do this, we will begin with positive integer exponents and recall our definition of exponents as repeated multiplication. Consider,

$$\begin{aligned} 3^2 \cdot 3^5 &= \underbrace{3 \cdot 3}_{2 \text{ factors}} \cdot \underbrace{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}_{5 \text{ factors}} \\ &= \underbrace{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}_{7 \text{ factors}} \\ &= 3^7 \end{aligned}$$

Also,

$$\begin{aligned} 7^4 \cdot 7^5 &= \underbrace{7 \cdot 7 \cdot 7 \cdot 7}_{4 \text{ factors}} \cdot \underbrace{7 \cdot 7 \cdot 7 \cdot 7 \cdot 7}_{5 \text{ factors}} \\ &= \underbrace{7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7}_{9 \text{ factors}} \\ &= 7^9 \end{aligned}$$

We see that  $3^2 \cdot 3^5 = 3^{2+5} = 3^7$  and  $7^4 \cdot 7^5 = 7^{4+5} = 7^9$ . This suggests that when we multiply like bases we add the exponents. This is the product rule for exponents.

**Product Rule for Exponents:** Let  $m$  and  $n$  be integers. Then for any real number  $a$ ,

$$a^m \cdot a^n = a^{m+n}.$$

(When multiplying powers with like bases, we keep the base and add the exponents.)

**CAUTION:** The bases must be the same before applying the product rule for exponents. For example,  $x^2 \cdot y^4 \neq (xy)^6$ . Remember, keep the base the same and only add the exponents. **DO NOT** multiply the bases together. For example,  $3^2 \cdot 3^4 \neq 9^6$ ; instead  $3^2 \cdot 3^4 = 3^{2+4} = 3^6$ .

**Example 2** Use the rules of exponents to simplify each product.

$$\begin{array}{ll} (a) & 5^4 \cdot 5^9 \\ (b) & (-2)^2 \cdot (-2)^3 \\ (c) & 3^2 \cdot 3^6 \cdot 3^7 \\ (d) & x^3 \cdot x^2 \cdot x^4 \end{array}$$

**Solution 2**

$$\begin{array}{ll} (a) & 5^4 \cdot 5^9 = 5^{4+9} = 5^{13} \\ (b) & (-2)^2 \cdot (-2)^3 = (-2)^{2+3} = (-2)^5 = -32 \\ (c) & 3^2 \cdot 3^6 \cdot 3^7 = 3^{2+6+7} = 3^{15} \\ (d) & x^3 \cdot x^2 \cdot x^4 = x^{3+2+4} = x^9 \end{array}$$

**Practice 2** Use the rules of exponents to simplify each product. (Answers on page 21 )

$$(a) \quad 2^4 \cdot 2^5 \qquad (b) \quad 7^2 \cdot 7^3 \cdot 7^6 \qquad (c) \quad y^5 \cdot y^9 \cdot y^2$$

We use the commutative and associative properties to multiply more complicated expressions.

**Example 3** Use the rules of exponents to simplify each product.

$$\begin{array}{ll} (a) & 2x^4 \cdot 5x^3 \\ (b) & (x^3y^5)(x^5y^8) \\ (c) & (-2x^4y^7)(3x^5y^2)(-4xy^5) \end{array}$$

**Solution 3**

$$\begin{array}{ll} (a) & 2x^4 \cdot 5x^3 = (2 \cdot 5)(x^4 \cdot x^3) = 10x^7 \\ (b) & (x^3y^5)(x^5y^8) = (x^3 \cdot x^5)(y^5 \cdot y^8) = x^8y^{13} \\ (c) & (-2x^4y^7)(3x^5y^2)(-4xy^5) = (-2 \cdot 3 \cdot -4)(x^4x^5x)(y^7y^2y^5) = 24x^{10}y^{14} \end{array}$$

**Practice 3** Use the rules of exponents to simplify each expression. (Answers on page 21 )

$$\begin{array}{ll} (a) & (-2x^3y^2z^5)(4xy^9z^2) \\ (b) & (4x^4y^5)(3x^2y)(5x^3y^4) \end{array}$$

Now that we know the product rule for exponents, let us examine what happens when we have a base to a power raised to another power. For example, consider  $(3^4)^3$  and  $(5^3)^4$ . Using the definition of an exponent along with the product rule for exponents, we get

$$(3^4)^2 = \underbrace{3^4 \cdot 3^4}_{2 \text{ factors}} = 3^{4+4} = 3^8$$

and

$$(5^3)^4 = \underbrace{5^3 \cdot 5^3 \cdot 5^3 \cdot 5^3}_{4 \text{ factors}} = 5^{3+3+3+3} = 5^{12}.$$

Notice in the first example, the product of the exponents, namely  $4 \cdot 2$ , gives us the exponent in the answer. In the second example,  $3 \cdot 4$  gives us the exponent of 12 in the answer. This leads us to our next property of exponents.

**Power Rule for Exponents:** Let  $m$  and  $n$  be integers. Then for any real number  $a$ ,

$$(a^m)^n = a^{mn}.$$

(When raising a base to a power to another power, we multiply the exponents.)

**CAUTION:** The powers are to be multiplied together. Do not raise the power inside the parenthesis to the outside power. We will see in the exercises that these are not the same.

**Example 4** Use the rules of exponents to simplify each product.

$$\begin{array}{ll} (a) & (4^3)^4 \\ (c) & (x^5)^3 \cdot (x^4)^2 \end{array} \qquad \begin{array}{ll} (b) & (2^5)^4 \\ (d) & (2^3)^2 \cdot (2^6)^3 \cdot (2^5)^4 \end{array}$$

**Solution 4**

$$\begin{array}{ll} (a) & (4^3)^4 = 4^{3 \cdot 4} = 4^{12} \\ (b) & (2^5)^4 = 2^{5 \cdot 4} = 2^{20} \\ (c) & (x^5)^3 \cdot (x^4)^2 = x^{5 \cdot 3} x^{4 \cdot 2} = x^{15} x^8 = x^{15+8} = x^{23} \\ (d) & (2^3)^2 \cdot (2^6)^3 \cdot (2^5)^4 = 2^{3 \cdot 2} \cdot 2^{6 \cdot 3} \cdot 2^{5 \cdot 4} = 2^6 \cdot 2^{18} \cdot 2^{20} \\ & = 2^{6+18+20} = 2^{44} \end{array}$$

**Practice 4** Use the rules of exponents to simplify each expression. (Answers on page 21 )

$$(a) \quad (y^2)^5 \cdot (y^6)^2 \cdot (y^3)^4 \qquad (b) \quad (3^2)^3 \cdot (3^4)^2$$

If we recall our commutative and associative rules for multiplication we can obtain two more power rules for exponents. Consider

$$\begin{aligned} (2x)^3 &= (2x)(2x)(2x) \\ &= 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x \\ &= 2^3 \cdot x^3 \\ &= 8x^3 \end{aligned}$$

and

$$\begin{aligned} (2x^4)^3 &= (2x^4)(2x^4)(2x^4) \\ &= 2 \cdot 2 \cdot 2 \cdot x^4 \cdot x^4 \cdot x^4 \\ &= 2^3 \cdot x^{4+4+4} \\ &= 8x^{12} \end{aligned}$$

Notice in the second example that  $2^3 = 8$  and  $(x^4)^3 = x^{12}$ . This leads to our second power rule for exponents.

**Product to a Power Rule for Exponents:** Let  $n$  be any integer. Then for any real numbers  $a$  and  $b$ ,

$$(ab)^n = a^n b^n.$$

(When we raise a product to a power, we raise each factor to the given power.)

**CAUTION:** Raising a Product to a Power Rule for Exponents does NOT apply to a sum. For example,  $(3+4)^2 = 7^2 = 49$  but  $3^2 + 4^2 = 9 + 16 = 25$ . Hence,  $(3+4)^2 \neq 3^2 + 4^2$ .

Furthermore, using our properties of fractions we can obtain the third power rule of exponents. Consider

$$\left(\frac{2}{3}\right)^4 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{2 \cdot 2 \cdot 2 \cdot 2}{3 \cdot 3 \cdot 3 \cdot 3} = \frac{2^4}{3^4} = \frac{16}{81}.$$

The resulting rule is stated below.

**Quotient to a Power Rule for Exponents:** Let  $n$  be any integer. Then for any real numbers  $a$  and  $b$  where  $b \neq 0$ ,

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

(When we raise a quotient to a power, we raise both the numerator and denominator to the given power.)

**Example 5** Use the rules of exponents to simplify each expression. Assume all variables represent non-zero real numbers.

$$\begin{array}{ll} (a) & (x^2y^3)^5 \\ (b) & \left(\frac{x^2}{y^3}\right)^4 x^5 \\ (c) & \left(\frac{x^7}{y^3}\right)^3 \left(\frac{x^2}{y^5}\right)^2 \\ (d) & (2x^2)^3 \left(\frac{1}{2}x^5\right)^2 (4x^6) \end{array}$$

**Solution 5**

$$\begin{array}{ll} (a) & (x^2y^3)^5 = (x^2)^5(y^3)^5 = x^{2 \cdot 5}y^{3 \cdot 5} = x^{10}y^{15} \\ (b) & \left(\frac{x^2}{y^3}\right)^4 x^5 = \frac{(x^2)^4}{(y^3)^4} x^5 = \frac{x^8x^5}{y^{12}} = \frac{x^{8+5}}{y^{12}} = \frac{x^{13}}{y^{12}} \\ (c) & \left(\frac{x^7}{y^3}\right)^3 \left(\frac{x^2}{y^5}\right)^2 = \frac{(x^7)^3}{(y^3)^3} \cdot \frac{(x^2)^2}{(y^5)^2} = \frac{x^{21}}{y^9} \cdot \frac{x^4}{y^{10}} = \frac{x^{21+4}}{y^{9+10}} = \frac{x^{25}}{y^{19}} \\ (d) & (2x^2)^3 \left(\frac{1}{2}x^5\right)^2 (4x^6) = 2^3(x^2)^3 \left(\frac{1}{2}\right)^2 (x^5)^2 \cdot 4x^6 \\ & = 8 \cdot \frac{1}{4} \cdot 4 \cdot x^6x^{10}x^6 = 8x^{6+10+6} = 8x^{22} \end{array}$$

**Practice 5** Use the rules of exponents to simplify each expression. (Answers on page 21 )

$$(a) \quad (2x^3y^2z)^2$$

$$(b) \quad \left(\frac{y^2z}{x^4}\right)^3 \left(\frac{z^2y}{x^3}\right)^4$$

$$(c) \quad (3x^3)^3 \left(\frac{1}{3}x^4\right)^2 (-3x^2)$$

Now that we have discussed the positive integers we need to discuss the nonpositive integer exponents. We begin our discussion with the meaning of  $a^0$  for  $a \neq 0$ . It is important that we define  $a^0$  to be sure that it still satisfies all of the integer properties we have discussed so far. For example, we still want  $2^0 \cdot 2^4 = 2^{0+4} = 2^4$ . Since  $2^4 \neq 0$ ,  $2^0$  must be 1. Note that it is important here that  $2^4 \neq 0$  because otherwise  $2^0$  could have been any number. The same reasoning would apply to any nonzero base so we are led to make the following definition:

**Zero Exponent:** For any nonzero real number  $a$ ,

$$a^0 = 1.$$

Therefore,  $8^0 = 1$ ,  $(-3)^0 = 1$ , and  $(2x^2y^3)^0 = 1$  for nonzero variables  $x$  and  $y$ .

Now that we know how to work with positive and zero exponents, what about negative exponents? Our old definition of viewing exponents as repeated multiplication no longer holds for negative exponents. For example, we cannot evaluate  $3^{-2}$  by taking the base 3 and multiplying it to itself  $-2$  times. This makes no sense. However, we need to be sure that we define  $a^{-n}$  so that it still obeys all of the exponent rules we have discussed thus far. If  $a \neq 0$ ,

$$a^n \cdot a^{-n} = a^{n+(-n)} = a^0 = 1.$$

Thus,  $a^{-n}$  must be the reciprocal of  $a^n$ .

**Negative Exponent Rule:** For any nonzero real number  $a$ ,

$$a^{-n} = \frac{1}{a^n}.$$

**CAUTION:** A positive constant raised to a negative power does NOT yield a negative number. For example,  $3^{-2} \neq -6$ , instead,

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}.$$

Furthermore, the above rule does not apply to negative constants. For example,  $-4 \neq \frac{1}{4}$ . The rule only applies to negative **exponents**.

We leave it as an exercise for the reader to verify that the product rule and power rules for exponents remain true with negative exponents. Thus, all rules we have discussed hold for any integer whether it is positive, negative, or zero.

**Example 6** Use the rules of exponents to simplify each expression. Write answers with positive exponents only.

- (a)  $3^{-1} + 3^{-2}$
- (b)  $5^{-1} - 5^0$
- (c)  $(2xy)^2 (3xy^7)^0$
- (d)  $(x^{-4}y^3)^{-2}$
- (e)  $(-2x^4y^3)^{-2}$
- (f)  $(8x^4y^{-3}) \left( \frac{1}{2}x^{-5}y^7 \right)$

**Solution 6**

- (a)  $3^{-1} + 3^{-2} = \frac{1}{3} + \frac{1}{3^2} = \frac{1}{3} + \frac{1}{9} = \frac{3}{9} + \frac{1}{9} = \frac{4}{9}$
- (b)  $5^{-1} - 5^0 = \frac{1}{5} - 1 = \frac{1}{5} - \frac{5}{5} = -\frac{4}{5}$
- (c)  $(2xy)^2 (3xy^7)^0 = 2^2x^2y^2 \cdot 1 = 4x^2y^2$
- (d)  $(x^{-4}y^3)^{-2} = x^{-4 \cdot -2} \cdot y^{3 \cdot -2} = x^8y^{-6} = x^8 \frac{1}{y^6} = \frac{x^8}{y^6}$
- (e)  $(-2x^4y^3)^{-2} = \frac{1}{(-2x^4y^3)^2} = \frac{1}{(-2)^2x^{4 \cdot 2}y^{3 \cdot 2}} = \frac{1}{4x^8y^6}$



$$\begin{aligned}
 (f) \quad (8x^4y^{-3}) \left( \frac{1}{2}x^{-5}y^7 \right) &= 8 \cdot \frac{1}{2} \cdot x^4x^{-5}y^{-3}y^7 = 4x^{4+(-5)}y^{(-3)+7} \\
 &= 4x^{-1}y^4 = 4\frac{1}{x}y^4 = \frac{4y^4}{x}
 \end{aligned}$$

**Practice 6** Use the rules of exponents to simplify each expression. Write answers with positive exponents only. (Answers on page 21)

$$(a) \quad (r^{-2}s^3t^2)^{-2}$$

$$(b) \quad (-2x^3y^{-2}z^{-1})^2 (-4x^{-2}yz^3)$$

$$(c) \quad 4^{-3} + 4^{-2}$$

Consider the example  $\frac{2^{-4}}{3^{-5}}$ . Using our negative exponent rule, we obtain

$$\frac{2^{-4}}{3^{-5}} = \frac{\frac{1}{2^4}}{\frac{1}{3^5}} = \frac{1}{2^4} \div \frac{1}{3^5} = \frac{1}{2^4} \cdot \frac{3^5}{1} = \frac{3^5}{2^4}.$$

Therefore, the negative exponent rule allows us to move factors in a fraction just by changing the sign of the exponent. Also, consider  $\left(\frac{2}{5}\right)^{-3}$ . Since  $\frac{5}{2}$  is the reciprocal of  $\frac{2}{5}$ ,

$$\left(\frac{2}{5}\right)^{-3} = \frac{1}{\left(\frac{2}{5}\right)^3} = \frac{1^3}{\left(\frac{2}{5}\right)^3} = \left(\frac{1}{\frac{2}{5}}\right)^3 = \left(\frac{5}{2}\right)^3.$$

Thus, if we have a fraction raised to a power, we can change the sign of the exponent by using the reciprocal of the fraction. We summarize these rules as follows:

**Converting from Negative to Positive Exponents:** For any nonzero numbers  $a$  and  $b$ , and any integers  $m$  and  $n$ ,

$$\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m} \quad \text{and} \quad \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n.$$

Before examining some exercises we need to discuss our final exponent rule – the quotient rule. Consider,  $\frac{2^6}{2^2}$  and  $\frac{2^2}{2^6}$ . Rewriting both expressions using the definition of an exponent, we obtain

$$\frac{2^6}{2^2} = \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot \cancel{2} \cdot \cancel{2}}{\cancel{2} \cdot \cancel{2}} = 2 \cdot 2 \cdot 2 \cdot 2 = 2^4,$$

and

$$\frac{2^2}{2^6} = \frac{\cancel{2} \cdot \cancel{2}}{\cancel{2} \cdot \cancel{2} \cdot 2 \cdot 2 \cdot 2 \cdot 2} = \frac{1}{2 \cdot 2 \cdot 2 \cdot 2} = \frac{1}{2^4} = 2^{-4}.$$

For the first example, subtracting the exponents we have  $6 - 2 = 4$  which is the resulting exponent. In the second example, subtracting the exponents, we have  $2 - 6 = -4$  which again is the resulting exponent. These examples suggest the quotient rule for exponents.

**Quotient Rule for Exponents:** For any nonzero real number  $a$  and any integers  $m$  and  $n$ ,

$$\frac{a^m}{a^n} = a^{m-n}.$$

(When dividing powers with like bases, we keep the base and subtract the exponents.)

**CAUTION:** When using the quotient rule, the bases must be the same. Remember to keep the base the same and only subtract the exponents. The order of the subtraction is important since subtraction does not satisfy the commutative property. Therefore, when applying the quotient rule for exponents, remember that it is

(exponent in the numerator) – (exponent in the denominator).

When working with the rules of exponents, note that the steps can be performed in many different orders. As such, the examples below show one method of simplifying each expression. However, there are many equally correct ways to begin your solution. Regardless of what order you perform the steps, you will always get the same answer.

**Example 7** Use a combination of the exponent rules to simplify each exponential expression. Write all answers with only positive exponents. Assume all variables represent nonzero real numbers.

$$(a) \quad \frac{(2x^2)^3}{4x^4}$$

$$(b) \quad \frac{(6x^3)^2}{(2x^2)^3} \cdot (3x^2)^0$$

$$(c) \quad \frac{(xy^2w^{-3})^4}{(x^{-3}y^{-2}w)^3}$$

$$(d) \quad \left( \frac{-3a^3b^{-5}}{6a^{-2}b^{-2}} \right)^3$$

$$(e) \quad \frac{(2m^2n^{-3})^{-2}}{(-3m^{-3}n^4)^{-1}}$$

**Solution 7**

$$(a) \quad \frac{(2x^2)^3}{4x^4} = \frac{2^3x^{2 \cdot 3}}{4x^4} = \frac{8x^6}{4x^4} = 2x^{6-4} = 2x^2$$

$$(b) \quad \frac{(6x^3)^2}{(2x^2)^3} \cdot (3x^2)^0 = \frac{6^2x^{3 \cdot 2}}{2^3x^{2 \cdot 3}} \cdot 1 = \frac{36x^6}{8x^6} = \frac{36}{8} = \frac{9}{2}$$

$$\begin{aligned} (c) \quad \frac{(xy^2w^{-3})^4}{(x^{-3}y^{-2}w)^3} &= \frac{x^4y^{2 \cdot 4}w^{-3 \cdot 4}}{x^{-3 \cdot 3}y^{-2 \cdot 3}w^3} = \frac{x^4y^8w^{-12}}{x^{-9}y^{-6}w^3} = \frac{x^4x^9y^8y^6}{w^{12}w^3} \\ &= \frac{x^{4+9}y^{6+8}}{w^{12+3}} = \frac{x^{13}y^{14}}{w^{15}} \end{aligned}$$

$$\begin{aligned} (d) \quad \left( \frac{-3a^3b^{-5}}{6a^{-2}b^{-2}} \right)^3 &= \left( \frac{-3a^3a^2b^2}{6b^5} \right)^3 = \left( \frac{-3a^{3+2}b^{2-5}}{6} \right)^3 = \left( \frac{-a^5b^{-3}}{2} \right)^3 \\ &= \left( \frac{-a^5}{2b^3} \right)^3 = \frac{-a^{5 \cdot 3}}{2^3b^{3 \cdot 3}} = \frac{-a^{15}}{8b^9} \end{aligned}$$

$$\begin{aligned} (e) \quad \frac{(2m^2n^{-3})^{-2}}{(-3m^{-3}n^4)^{-1}} &= \frac{(-3m^{-3}n^4)}{(2m^2n^{-3})^2} = \frac{-3m^{-3}n^4}{4m^4n^{-6}} = \frac{-3n^4n^6}{4m^4m^3} \\ &= \frac{-3n^{10}}{4m^7} \end{aligned}$$

**Practice 7** Use a combination of the exponent rules to simplify each exponential expression. Write all answers with only positive exponents. Assume all variables represent nonzero real numbers. (Answers on page 21 )

$$(a) \quad (x^3 y^{-1} z^2) (x^{-2} y^{-3} z^3)$$

$$(b) \quad \frac{(3x^2 y^3)^{-2}}{(2x^3 y^4)^{-3}}$$

$$(c) \quad \left( \frac{-2x^3 y^{-2} z}{3x^{-5} y^{-1} z^3} \right)^{-3}$$

In addition to simplifying expressions, the exponent rules can also be used to calculate expressions mentally by making the number smaller, and easier, to work with.

**Example 8** Use the laws of exponents to calculate the following expressions mentally.

$$(a) \quad \frac{27^3}{9^3} \qquad (b) \quad 16^4 \cdot (0.125)^4 \qquad (c) \quad 125^5 \cdot (25)^{-7}$$

**Solution 8**

$$(a) \quad \frac{27^3}{9^3} = \left( \frac{27}{9} \right)^3 = 3^3 = 27.$$

$$(b) \quad 16^4 \cdot (.125)^4 = 16^4 \cdot \left( \frac{1}{8} \right)^4 = \left( \frac{16}{8} \right)^4 = 2^4 = 16.$$

$$(c) \quad 125^5 \cdot (25)^{-7} = (5^3)^5 (5^2)^{-7} = 5^{15} 5^{-14} = 5^1 = 5.$$

**Practice 8** Use the laws of exponents to calculate the following expressions mentally. (Answers on page 21 )

$$(a) \quad \frac{4^9}{64^3}$$

$$(b) \quad \frac{3^8 \cdot 9^3}{27^5}$$

## ANSWERS TO PRACTICE PROBLEMS

- |                         |                                   |                                 |
|-------------------------|-----------------------------------|---------------------------------|
| 1. (a) $-16$            | 4. (a) $y^{34}$                   | (c) $\frac{5}{64}$              |
| (b) $16$                | (b) $3^{14}$                      |                                 |
| (c) $-27$               |                                   | 7. (a) $\frac{xz^5}{y^4}$       |
| (d) $-27$               | 5. (a) $4x^6y^4z^2$               | (b) $\frac{8x^5y^6}{9}$         |
| 2. (a) $2^9$            | (b) $\frac{y^{10}z^{11}}{x^{24}}$ | (c) $\frac{-27y^3z^6}{8x^{24}}$ |
| (b) $7^{11}$            | (c) $-9x^{19}$                    |                                 |
| (c) $y^{16}$            |                                   |                                 |
| 3. (a) $-8x^4y^{11}z^7$ | 6. (a) $\frac{r^4}{s^6t^4}$       | 8. (a) $1$                      |
| (b) $60x^9y^{10}$       | (b) $\frac{-16x^4z}{y^3}$         | (b) $\frac{1}{3}$               |

## SECTION 1.2 EXERCISES

(Answers are found on page 118.)

1. Using exponents, rewrite the following expressions in a simpler form.

(a)  $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

(b)  $2 \cdot 5 \cdot 5 \cdot 2 \cdot 2 \cdot 5$

In #2 – #15, evaluate each expression.

2.  $-24^0$

7.  $-2^6$

12.  $\frac{1}{5^{-2}}$

3.  $(-24)^0$

8.  $\left(\frac{1}{2}\right)^{-3}$

13.  $\frac{1}{7^{-2}}$

4.  $3^{-4}$

9.  $(-4)^{-2}$

14.  $2^{-2} + 4^{-2}$

5.  $8^{-2}$

10.  $-2^{-2}$

6.  $(-3)^4$

11.  $(-2)^{-2}$

15.  $(-3)^{-2} + (-4)^{-1}$

16. Use the laws of exponents to calculate the following expressions mentally.

(a)  $\frac{24^4}{8^4}$

(b)  $20^7 (0.5)^7$

In #17–#24, simplify the following, leaving your answer as a single base raised to a single exponent. Assume all variables represent non-zero real numbers.

17.  $x^7 \cdot x^3 \div x^4$

21.  $9^2 \cdot 12^3 \cdot 2$

18.  $128 \div 2^4$

22.  $16^7 \cdot 4^8 \cdot 8^3 \div 2^{12}$

19.  $2^3 \cdot 4^5 \div 8^3$

23.  $125^2 \cdot 9^4 \cdot 27^2 \cdot 25^4$

20.  $(5^2)^4 \div 5^3 \cdot 5^2$

24.  $27^6 \div 3^8 \cdot 9^4 \cdot 81$

In #25 – #50, use a combination of the rules of exponents to simplify each expression. Write all answers using positive exponents only. Assume all variables represent nonzero real numbers.

25.  $(x^2y)(x^5y^3)$

35.  $\left(\frac{-3x^4}{2y^5}\right)^2$

26.  $2x^2(2x)^2$

36.  $\frac{x^4y^5}{x^3y^2}$

27.  $(7x^5)^2$

37.  $\frac{x^5y^{12}x^9}{xy^4}$

28.  $(4x^2)^3$

38.  $\frac{x^5y^{12}x^9}{(xy)^4}$

29.  $(3x^4y^5)^3$

39.  $\frac{(2x^3)(3x^2)}{(x^2)^3}$

30.  $(2x^2y^5z^9)^4$

40.  $\frac{(4x^2y)(3x^3y^2)}{x^7y^4}$

31.  $(\frac{1}{2}x^4)(16x^5)$

41.  $\left(\frac{4a^2b}{a^3b^2}\right)\left(\frac{5a^2b}{2b^4}\right)$

32.  $(2x^3)(-3x^5)(4x^7)$

42.  $(2x^{-2}y^3z^5)^3$

33.  $(-4b^3)(\frac{1}{6}b^2)(-9b^4)$

34.  $(2x^2y)\left(\frac{1}{4}x^3y^5\right)\left(\frac{1}{3}x^4y^2\right)$

43.  $(-4x^3y^{-5}z^{-1})^2$

44.  $(3x^4y^2z^{-3})^{-2}$

48.  $\left(\frac{a^2b^3c}{a^4b^{-1}c^2}\right)^2$

45.  $(5x^{-4}y^3z^{-6})^{-3}$

46.  $\frac{3^{-1}x^4y^{-2}}{3^{-2}x^{-2}y}$

49.  $\left(\frac{3x^2y^8}{9x^4y^5}\right)^{-3}$

47.  $\frac{6^{-2}x^7y^{-3}z^2}{6^{-3}x^{-4}y^5z^9}$

50.  $\left(\frac{x^3y^{-2}z^5}{x^{-2}y^5z}\right)^4$

**In #51 – #57, use a combination of the rules of exponents to simplify each expression. Write all answers using positive exponents only. Assume all variables represent nonzero real numbers.**

51.  $(2x^2y^{-5})(6x^{-3}y)(\frac{1}{3}x^{-1}y^3)$

52.  $\left(\frac{xy^{-2}z^{-3}}{2x^{-2}yz^2}\right)^{-2}$

53.  $\left(\frac{-2x^4y^{-4}}{3x^{-1}y^{-2}}\right)^4$

54.  $\frac{(2x^3y^{-2})^3(-3xy^5)^{-2}}{4x^{-5}y^7}$

55.  $\left(\frac{-4x^5y^{-2}z^{-3}}{8x^{-4}y^6z^{-4}}\right)^{-3}$

56.  $(4x^3y^{-2})^{-2}(2x^2y^3)^2\left(\frac{1}{3}x^4y^{-2}\right)^{-1}$

57.  $\frac{(2x^3y^6z^{-1})^{-2}}{(3x^{-3}y^6z^2)^{-3}}$

58. A fellow student said that  $8^5 \div 2^2 = 4^3$ . How would you convince this student that this is incorrect? What is the correct answer?

59. Is it true that  $(3^4)^2 = 3^{(4^2)}$ ? Explain why or why not.

60. Is  $(3 + 4)^3 = 3^3 + 4^3$ ? Are there any combinations of values of  $a$  and  $b$  for which  $(a + b)^3 = a^3 + b^3$ ? Are there any positive integer values of  $n$  for which  $(a + b)^n = a^n + b^n$ ?
61. Using the properties of exponents, determine which is larger. State why.
- (a)  $4^{28}$  or  $8^{18}$
  - (b)  $3^9 + 3^9 + 3^9$  or  $9^6$
  - (c)  $27^9$  or  $9^{14}$
  - (d)  $6^{18}$  or  $3^{36}$



### 1.3 Scientific Notation – Application of Exponents

One application of exponents is in the use of scientific notation. We have seen in Fundamental Mathematics 1 that scientific notation is convenient for writing large numbers. ([Click here to review scientific notation from FM1](#))

When negative exponents are used with scientific notation, it can be used to represent very small numbers as well.

A number in **scientific notation** is written

$$a \times 10^n$$

where  $1 \leq |a| < 10$  and  $n$  is an integer.

Two examples of scientific notation for small numbers are:

- The width of a strand of DNA is  $2 \times 10^{-9}$
- The mass of a proton is  $1.6724 \times 10^{-27}$  kg

In scientific notation there is always exactly one digit before the decimal point. To rewrite a number using scientific notation the decimal point is moved so that only one nonzero digit is to the left of the decimal point. The number of places the decimal point was moved becomes the exponent on 10. This power is negative if the decimal point is moved to the right and positive if it is moved to the left.

**Example 1** Write each number in scientific notation.

- (a) 56,000,000
- (b) 0.0000000345
- (c) The speed of sound is 4790 feet per second

**Solution 1**

- (a) We begin by moving the decimal point so that only one nonzero digit is in front of it, in other words, seven places to the left. Seven becomes the power on 10 and it is positive since it was moved to the left. Thus,

$$56,000,000 = 5.6 \times 10^7.$$

- (b) Since we move the decimal point eight places to the right in order to get only one nonzero digit in front of the decimal point, we get that

$$0.0000000345 = 3.45 \times 10^{-8}.$$

- (c) The decimal place is moved three places to the left. Thus,  
 $4790 = 4.790 \times 10^3$ .

**Practice 1** Write each number in scientific notation. (Answers on page 28)

- (a) 0.00000405
- (b) The population of Cuyahoga County is approximately 1,360,000 people.

Remember that the exponent on 10 tells you the direction and the number of decimal places the decimal point is moved. To change a number from scientific notation to standard notation, we reverse the process discussed before Example 1. In other words, if the exponent  $n$  is negative we move the decimal point  $|n|$  places to the left. If the exponent  $n$  is positive, we move the decimal point  $n$  places to the right.

**Example 2** Convert the following numbers to standard notation (without exponents).

- (a)  $-9.75 \times 10^8$
- (b)  $1.4 \times 10^{-5}$
- (c) A gram is  $3.527 \times 10^{-2}$  ounces.
- (d) The radius of the earth is  $6.4 \times 10^6$  meters.

**Solution 2**

- (a)  $-9.75 \times 10^7 = -975,000,000$
- (b)  $1.4 \times 10^{-5} = 0.000014$
- (c)  $3.527 \times 10^{-2} = 0.03527$
- (d)  $6.4 \times 10^6 = 6,400,000$

**Practice 2** Convert the following numbers to standard notation (without exponents). (Answers on page 28)

(a) The number one all-time worldwide box office movie is *Titanic* (1997) which earned approximately  $1.84 \times 10^9$  dollars.

(b)  $4.8 \times 10^{-12}$

Scientific notation also provides an efficient way to multiply and divide very small and very large numbers. Using the commutative and associative properties of real numbers, we multiply the numbers between 1 and 10 and use the rules of exponents to find the power of 10 in the final answer.

**Example 3** Perform the indicated operations and write all answers in scientific notation.

(a)  $(4 \times 10^5)(2 \times 10^9)$

(b)  $\frac{9 \times 10^{12}}{3 \times 10^7}$

(c)  $(5 \times 10^9)(3 \times 10^{-4})$

(d)  $\frac{(4 \times 10^7)(6 \times 10^3)}{5 \times 10^4}$

**Solution 3**

(a)  $(4 \times 10^5)(2 \times 10^9) = (4 \times 2)(10^5 \times 10^9) = 8 \times 10^{14}$

(b)  $\frac{9 \times 10^{12}}{3 \times 10^7} = \frac{9}{3} \times \frac{10^{12}}{10^7} = 3 \times 10^5$

(c)  $(5 \times 10^9)(3 \times 10^{-4}) = (5 \times 3)(10^9 \times 10^{-4}) = 15 \times 10^5 =$   
 $(1.5 \times 10) \times 10^5 = 1.5 \times 10^6$

(d)  $\frac{(4 \times 10^7)(6 \times 10^3)}{5 \times 10^4} = \frac{4 \times 6}{5} \times \frac{10^7 \times 10^3}{10^4} = \frac{24}{5} \times \frac{10^{10}}{10^4} =$   
 $4.8 \times 10^6$

**Practice 3** Perform the indicated operations and write all answers in scientific notation. (Answers on page 28)

(a)  $\frac{(2.7 \times 10^4)(4 \times 10^9)}{(2.25 \times 10^8)}$

(b)  $\frac{(7 \times 10^{12})(8.1 \times 10^{-16})}{(2 \times 10^5)(2.1 \times 10^8)}$

---

## ANSWERS TO PRACTICE PROBLEMS

- |    |                           |                            |                       |
|----|---------------------------|----------------------------|-----------------------|
| 1. | (a) $4.05 \times 10^{-6}$ | (b) 0.00000000000048       |                       |
|    | (b) $1.36 \times 10^6$    | 3.                         | (a) $4.8 \times 10^5$ |
| 2. | (a) 1,840,000,000         | (b) $1.35 \times 10^{-16}$ |                       |

## SECTION 1.3 EXERCISES

(Answers are found on page [119](#).)

In #1–#12, convert the following numbers to standard notation (without exponents).

- |    |                       |     |                        |
|----|-----------------------|-----|------------------------|
| 1. | $9.46 \times 10^{17}$ | 7.  | $6.93 \times 10^{11}$  |
| 2. | $1.81 \times 10^6$    | 8.  | $3.4 \times 10^{-8}$   |
| 3. | $10^{-10}$            | 9.  | $2.75 \times 10^7$     |
| 4. | $4.3 \times 10^{-3}$  | 10. | $8.003 \times 10^5$    |
| 5. | $3.2 \times 10^{-5}$  | 11. | $9.02 \times 10^{-6}$  |
| 6. | $5.789 \times 10^8$   | 12. | $5.486 \times 10^{-4}$ |

In #13–#24, convert the following numbers to scientific notation.

- |     |             |     |                   |
|-----|-------------|-----|-------------------|
| 13. | 8,340       | 19. | 294,000,000.      |
| 14. | 0.00079     | 20. | 0.000000345       |
| 15. | 0.000042    | 21. | 0.0000000703      |
| 16. | 4,360,000   | 22. | 6,600,000,000     |
| 17. | 0.003005    | 23. | 2,350,000,000,000 |
| 18. | 0.000007201 | 24. | 0.000000000532    |

In #25–#35, perform the indicated operations and write all answers in scientific notation.

25.  $(2.4 \times 10^{-5})(3.0 \times 10^{-7})$   
26.  $(5.2 \times 10^7)(4.3 \times 10^{-3})$

27.  $(2.1 \times 10^{-4}) (3.5 \times 10^8) (1.2 \times 10^6)$

28.  $(3.5 \times 10^4) (6.12 \times 10^{-11}) (4.1 \times 10^8)$

29.  $(5.1 \times 10^{-6}) (3.02 \times 10^{11}) (2 \times 10^{-8})$

30.  $\frac{9.6 \times 10^9}{1.5 \times 10^3}$

31.  $\frac{6.88 \times 10^4}{4.3 \times 10^{-8}}$

32.  $\frac{4.8 \times 10^{13}}{(5 \times 10^9)(2.4 \times 10^{15})}$

33.  $\frac{(3.5 \times 10^{-3})(5.2 \times 10^8)}{9.1 \times 10^{-7}}$

34.  $\frac{6.7 \times 10^4}{5 \times 10^{-6}} \times \frac{7.1 \times 10^3}{4 \times 10^{10}}$

35.  $\frac{3}{4 \times 10^5} \times \frac{8 \times 10^6}{2 \times 10^8}$

## 1.4 Polynomials

A polynomial in  $x$  is any expression which can be written as:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where  $a_n, a_{n-1}, \dots, a_1, a_0$  are any numbers and  $n$  is a whole number. (In other words, a polynomial in  $x$  is a sum of a fixed number of terms of the form  $ax^n$  where  $a$  can be any number and  $n$  is a whole number.) The degree of a polynomial is the highest exponent of the polynomial.

A polynomial can assume different values depending on the value of the variable.

**Example 1** Evaluate  $4x^2 - 2x + 1$  when

(a)  $x = 3$

(b)  $x = -2$

**Solution 1** (a) To evaluate this polynomial when  $x = 3$ , we replace every variable  $x$  with the value 3. Recalling our order of operations, we obtain

$$4(3)^2 - 2(3) + 1 = 4(9) - 6 + 1 = 36 - 6 + 1 = 31$$

as the value of the polynomial when  $x = 3$ .

(b) To evaluate this polynomial when  $x = -2$ , we replace every variable  $x$  with the value  $-2$ . For this value, we obtain

$$4(-2)^2 - 2(-2) + 1 = 4(4) + 4 + 1 = 16 + 4 + 1 = 21$$

**Practice 1** Evaluate  $3x^2 + 5x - 6$  when (a)  $x = 2$  (b)  $x = -1$  (Answer on page 34.)

In order to add or subtract polynomials, we combine like terms – terms with the same variable and same exponent. When we combine, we keep the variable the same and add (or subtract) the coefficients. For example,  $3x^2$  and  $8x^2$  are like terms (same variable and same exponent). As a result,  $3x^2 + 8x^2 = 11x^2$  and  $3x^2 - 8x^2 = -5x^2$ . On the other hand,  $3x^4$  and  $8x^2$  are unlike terms (same variable but different exponents). Hence, these two terms cannot be combined. To add or subtract two or more polynomials with more than one term, we combine all like terms.

**Example 2** *Simplify*  $(4x^2 - 3x + 2) + (6x^2 + 7x - 5)$

**Solution 2**

$$\begin{aligned}(4x^2 - 3x + 2) + (6x^2 + 7x - 5) &= (4x^2 + 6x^2) + (-3x + 7x) + (2 - 5) \\ &= 10x^2 + 4x - 3\end{aligned}$$

**Practice 2** *Simplify*  $(3x^2 - 5x - 7) + (8x^2 - 9x + 4)$  (Answer on page 34.)

To subtract polynomials, first distribute the negative and then combine like terms.

**Example 3** *Simplify*  $(5x^2 - 3x + 2) - (2x^2 - 7x - 3)$

**Solution 3**

$$\begin{aligned}(5x^2 - 3x + 2) - (2x^2 - 7x - 3) &= 5x^2 - 3x + 2 - 2x^2 + 7x + 3 \\ &= (5x^2 - 2x^2) + (-3x + 7x) + (2 + 3) \\ &= 3x^2 + 4x + 5\end{aligned}$$

**Practice 3** *Simplify*  $(3x^2 + 9x - 6) - (7x^2 + 4x - 3)$  (Answer on page 34.)

**Example 4** *Simplify*  $(3x^2 - 9x + 7) - (4x^3 + 2x^2 + 4) + (9x^3 + 7x + 1)$

**Solution 4**

$$\begin{aligned}(3x^2 - 9x + 7) - (4x^3 + 2x^2 + 4) + (9x^3 + 7x + 1) \\ &= 3x^2 - 9x + 7 - 4x^3 - 2x^2 - 4 + 9x^3 + 7x + 1 \\ &= (-4x^3 + 9x^3) + (3x^2 - 2x^2) + (-9x + 7x) + (7 - 4 + 1) \\ &= 5x^3 + x^2 - 2x + 4\end{aligned}$$

Recall that  $1x^2 = x^2$ .

**Practice 4** *Simplify*  $(5x^3 - 7x^2 + 8x - 3) + (4x^3 - 5x + 7) - (9x^2 + 6x - 11)$   
(Answer on page 34.)

To multiply polynomials we multiply each term in the first polynomial by each term in the second polynomial. In order to do this, we need to recall the distributive property and the product rule for exponents: For any integers  $m$  and  $n$ ,  $a^m \cdot a^n = a^{m+n}$ .

**Example 5** Simplify  $4x(3x^2 + 2)$

**Solution 5** We need to distribute  $4x$  to every term in our parenthesis. Hence,

$$\begin{aligned} 4x(3x^2 + 2) &= (4x)(3x^2) + (4x)(2) \\ &= 12x^3 + 8x \end{aligned}$$

**Practice 5** Simplify  $-2x^2(3x^2 + 4x - 2)$  (Answer on page 34.)

**Example 6** Simplify  $(4x - 3)(5x + 2)$

**Solution 6**

$$\begin{aligned} (4x - 3)(5x + 2) &= 4x(5x + 2) - 3(5x + 2) \\ &= (4x)(5x) + (4x)(2) - (3)(5x) - 3(2) \\ &= 20x^2 + 8x - 15x - 6 \\ &= 20x^2 - 7x - 6 \end{aligned}$$

Be careful when combining like terms. Remember that the variable part remains unchanged.

**Practice 6** Simplify  $(7x - 3)(5x + 4)$  (Answer on page 34.)

There is also a rectangular array approach to multiplication of polynomials. To set up a rectangular array, consider  $(3x + 2)(4x - 5)$ . The terms of the first polynomial become the labels on each of the rows; whereas, the terms of the second polynomial become the labels on each of the columns, as shown below

|      | $4x$ | $-5$ |
|------|------|------|
| $3x$ |      |      |
| $2$  |      |      |

To finish the problem and find the solution, we then multiply the term at the beginning of the row with the term at the top of the corresponding column. We then record the result in the appropriate cell of the table. This would look like

|      | $4x$    | $-5$   |
|------|---------|--------|
| $3x$ | $12x^2$ | $-15x$ |
| $2$  | $8x$    | $-10$  |



Now to find the solution, we combine like terms of the cells **inside** the table to get  $12x^2 - 7x - 10$ . The benefits of the rectangular array is that it insures that you multiply every term in the first polynomial with every term in the second polynomial.

**Example 7** Use the rectangular array approach to multiply

$$(3x^2 + 2x - 4)(2x^2 - 5x + 3).$$

**Solution 7** The set-up for the rectangular array approach is the following:

|        | $2x^2$ | $-5x$ | $3$ |
|--------|--------|-------|-----|
| $3x^2$ |        |       |     |
| $2x$   |        |       |     |
| $-4$   |        |       |     |

Therefore, once the multiplication has been performed and recorded we have

|        | $2x^2$  | $-5x$    | $3$    |
|--------|---------|----------|--------|
| $3x^2$ | $6x^4$  | $-15x^3$ | $9x^2$ |
| $2x$   | $4x^3$  | $-10x^2$ | $6x$   |
| $-4$   | $-8x^2$ | $20x$    | $-12$  |

To finish, we combine like terms to get a solution of  $6x^4 - 11x^3 - 9x^2 + 26x - 12$ .

**Practice 7** Use the rectangular array approach to multiply

$$(3x^3 - 4x + 2)(4x^2 - 5x - 3).$$

(Answer on page 34.)

The rectangular array approach is quite beneficial when the problem is fairly long and it is more likely that a mistake in the distributive property could arise. Once again, it guarantees that each term of the first polynomial will be multiplied by each term of the second polynomial.

Recall from Section 1.1 that it was determined that  $(a + b)^n \neq a^n + b^n$  for all integers  $n \geq 2$ . So, how do we simplify  $(3x - 2)^2$ ? Recall, that the exponent tells us how many times the multiplication is to be repeated. Hence, using our definition of exponents we must rewrite the expression before multiplying it out.

**Example 8** Multiply  $(3x - 2)^2$

**Solution 8**

$$\begin{aligned}(3x - 2)^2 &= (3x - 2)(3x - 2) \\ &= 9x^2 - 6x - 6x + 4 \\ &= 9x^2 - 12x + 4\end{aligned}$$

**Practice 8** Multiply  $(6x + 5)^2$  (Answer on page 34.)

This same procedure can be used for any integer exponent value.

**Example 9** Multiply  $(3x - 2)^3$

**Solution 9** From Example #7, we found that by multiplying the first two quantities together we have  $(3x - 2)(3x - 2) = 9x^2 - 12x + 4$ . Hence,

$$\begin{aligned}(3x - 2)^3 &= (3x - 2)(3x - 2)(3x - 2) \\ &= (9x^2 - 12x + 4)(3x - 2) \\ &= 27x^3 - 18x^2 - 36x^2 + 24x + 12x - 8 \\ &= 27x^3 - 54x^2 + 36x - 8\end{aligned}$$

**Practice 9** Multiply  $(6x + 5)^3$  (Answer on page 34.)

#### ANSWERS TO PRACTICE PROBLEMS

- |                             |   |
|-----------------------------|---|
| 1. (a) 16                   | 5. $-6x^4 - 8x^3 + 4x^2$                    |
| (b) -8                      | 6. $35x^2 + 13x - 12$                       |
| 2. $11x^2 - 14x - 3$        | 7. $12x^5 - 15x^4 - 25x^3 + 28x^2 + 2x - 6$ |
| 3. $-4x^2 + 5x - 3$         | 8. $36x^2 + 60x + 25$                       |
| 4. $9x^3 - 16x^2 - 3x + 15$ | 9. $216x^3 + 540x^2 + 450x + 125$           |

**SECTION 1.4 EXERCISES**

(Answers are found on page 120.)

1. Is it possible for the sum of two polynomials, both of degree three, to be a polynomial of degree two? If so, give an example. If not, explain why.
2. If we multiply a monomial (a polynomial with one term) by another monomial will we always get a monomial?
3. If we multiply a monomial by a binomial (a polynomial with two terms) will we ever get a trinomial (a polynomial with three terms)?
4. If we consider the product of a polynomial of degree two and a polynomial of degree three, what is the degree of the product? In general, if an  $n$ -th degree polynomial and an  $m$ -th degree polynomial are multiplied together, what is the degree of the product?
5. Evaluate  $4x + 7$  when  $x = -8$
6. Evaluate  $3x - 4$  when  $x = -\frac{5}{2}$
7. Evaluate  $4x^2 - 6x + 8$  when  $x = \frac{1}{2}$
8. Evaluate  $5x^2 + 3x + 9$  when  $x = 2$
9. Evaluate  $x^3 - 4x^2 - 2x + 5$  when  $x = -2$

**In #10–#42 perform the indicated operations and simplify.**

10.  $(5x^2 - 7x - 2) + (6x^2 + 4x - 9)$
11.  $(6x^2 - 3x + 4) - (8x^2 + 5x - 2)$
12.  $(3x^3 - 5x^2 + 6x - 7) - (4x^3 - 7x + 1) + (2x^3 - 5x^2 + 9x + 3)$
13.  $(5x^3 - 9x^2 + 2x - 3) - (-6x^3 + 2x^2 - 8x - 1) - (4x^3 + 4x + 7)$
14.  $4x(3x - 2)$
15.  $5x^2(6x + 3)$
16.  $-2x(4x^2 - 3x + 5)$

- 17.  $3x^2 (5x^2 - 2x - 7)$
- 18.  $-3x^2y (5xy^2 - 7xy + 3x^2)$
- 19.  $-2x^2y^4 (5x^2y^2 - 3xy^3 + 2x^3y^4 - 7y)$
- 20.  $(2x - 3)(2x + 5)$
- 21.  $(3x - 2)(5x - 4)$
- 22.  $(4x + 3)(7x - 2)$
- 23.  $(5x - 3)(2x + 1)$
- 24.  $(6x - 1)(4x + 2)$
- 25.  $(2x - 3)(2x + 3)$
- 26.  $(4x + 1)(4x - 1)$
- 27.  $(5x + 2)(5x - 2)$
- 28.  $(3x - 4)(3x + 4)$
- 29.  $\left(\frac{3}{2}x - 5\right) \left(3x - \frac{2}{5}\right)$
- 30.  $\left(3x - \frac{2}{3}\right) \left(2x + \frac{1}{4}\right)$
- 31.  $(7x - 2)^2$
- 32.  $(4x + 3)^2$
- 33.  $(2x - 3)^2$
- 34.  $(x - 2)^3$
- 35.  $(2x + 1)^3$
- 36.  $(4x + 1)(3x^2 - 5x - 2)$
- 37.  $(x - 4)(2x^2 - 3x - 2)$
- 38.  $x(3x - 4)(7x + 3)$
- 39.  $2x^2(4x - 3)(2x + 5)$

- 
40.  $2x(3x - 4)(2x + 3)$
41.  $(5x - 2)(2x^2 - 3x + 4)$
42.  $(3x - 5)(2x^3 - 4x^2 + 7x - 2)$
43. Use a rectangular array to multiply.
- (a)  $(3x - 5)(2x^3 - 6x^2 - 3x + 2)$
- (b)  $(x^2 - 3x - 1)(2x^2 + 5x - 7)$
- (c)  $(6x^2 - x + 3)(3x^2 - 2x - 4)$
44. Give a counterexample to show that  $x + 2(x - 3) \neq (x + 2)(x - 3)$

## 1.5 Properties of Equality and Linear Equations

A **linear equation in one variable** is an equation that can be written in the form

$$ax + b = c$$

for real numbers  $a, b$ , and  $c$ , with  $a \neq 0$ . They are also called first-degree equations, because the variable is raised to the first power. A **solution** of an equation in one variable is a number that makes the equation true when each occurrence of the variable in the equation is replaced by the number. In Fundamental Mathematics 1, we saw a method for finding solutions of linear equations. ([Click here to review linear equations from FM1](#)) However, linear equations may start out looking much more complicated so it will be useful to develop a more general method of dealing with them. The following two properties of equations are very useful for solving all kinds of equations (not just linear ones).

**Addition Property of Equality:**

$$\text{If } a = b \text{ then } a + c = b + c.$$

**Multiplication Property of Equality:** Let  $c \neq 0$

$$\text{If } a = b \text{ then } ac = bc.$$

Remember to perform the same operation on **both** sides of the equation.

The Addition Property allows us to subtract the same quantity from both sides by adding the opposite of the quantity. The Multiplication Property allows us to divide each side by the same nonzero quantity by multiplying both sides by its reciprocal.

**Example 1** Solve for  $x$ :  $3x - 7 = 5$

**Solution 1**

$$\begin{array}{ll}
3x - 7 = 5 & \\
3x - 7 + 7 = 5 + 7 & \text{add 7} \\
3x = 12 & \text{simplify} \\
\frac{3x}{3} = \frac{12}{3} & \text{divide by 3} \\
x = 4. &
\end{array}$$

Therefore, the solution is  $x = 4$ .

Note that using the Multiplication Property to multiply by  $\frac{1}{3}$  is the same as dividing by 3.

**Practice 1** Solve  $6x + 3 = 7$  (Answer on page 46)

The nice thing about solving linear equations is that you can always check your solution. In Example 1, if we substitute  $x = 4$  back into the original equation, we should have the same quantity on the left hand side of the equation as we do on the right hand side of the equation.

$$\begin{array}{ll}
\text{CHECK of Example 1:} & 3(4) - 7 \stackrel{?}{=} 5 \\
& 12 - 7 \stackrel{?}{=} 5 \\
& 5 = 5 \checkmark
\end{array}$$

Example 1 used both the addition property of equality and multiplication property of equality exactly one time. However, you may need to use each property more than once. Furthermore, you may need to recall other properties that were discussed earlier like the distributive property to clear parentheses.

**Example 2** Solve for  $x$ :  $2(x - 4) = 5x + 7$

**Solution 2**

$$\begin{aligned}
2(x - 4) &= 5x + 7 \\
2x - 8 &= 5x + 7 && \text{distributive property} \\
2x - 8 + 8 &= 5x + 7 + 8 && \text{add 8} \\
2x &= 5x + 15 && \text{simplify} \\
2x - 5x &= 5x + 15 - 5x && \text{subtract } 5x \\
-3x &= 15 && \text{simplify} \\
\frac{-3x}{-3} &= \frac{15}{-3} && \text{divide by } -3 \\
x &= -5.
\end{aligned}$$

Therefore, the solution is  $x = -5$ .

CHECK of Example 2:

$$\begin{aligned}
2(-5 - 4) &\stackrel{?}{=} 5(-5) + 7 \\
2(-9) &\stackrel{?}{=} -25 + 7 \\
-18 &= -18 \checkmark
\end{aligned}$$

**Practice 2** Provide the reason for each step. Be specific. (Answers on page 46)

$$\begin{aligned}
7 - (2x + 4) &= 3 + 2(x - 5) \\
7 - 2x - 4 &= 3 + 2x - 10 \\
3 - 2x &= 2x - 7 \\
3 - 2x - 2x &= 2x - 7 - 2x \\
3 - 4x - 3 &= -7 - 3 \\
-4x &= -10 \\
x &= \frac{10}{4} \\
x &= \frac{5}{2}
\end{aligned}$$



In reference to Practice 2, when clearing the parentheses in an expression like  $7 - (2x + 4)$ , remember that the minus sign acts like a factor of  $-1$ . After using the distributive property, the sign of *every* term in the parentheses will be changed to give  $7 - 2x - 4$ .

Next, we consider a linear equation containing fractions. How do we deal with the fractions? We can choose to leave the fractions in the problem or to rewrite the equation without fractions. Remember that by the multiplication property of equality we can multiply both sides of our equation by the same nonzero constant without changing the solution. If we choose to eliminate the fractions, which constant do we choose? Consider

$$\frac{1}{2}(x - 3) = \frac{x}{4} + 7.$$

If we multiply both sides by two, the fraction on the right hand side of the equation would not be eliminated. However, if we multiply by four, all fractions are eliminated. Recall that in this example, four is the least common multiple; that is, the smallest natural number that is a multiple of all of the denominators involved. ([Click here to review least common multiple from FM1](#)) Therefore, we can eliminate fractions by multiplying each term by the least common multiple of all the denominators. The next example illustrates this procedure.

To avoid a mistake, it might be easier to clear all parentheses using the distributive property *before* multiplying every term by the least common denominator. It is important to remember when clearing fractions that all terms need to be multiplied by the least common denominator, not just the fractions.

**Example 3** Solve for  $x$ :  $\frac{1}{2}(x - 3) = \frac{x}{4} + 7$

**Solution 3**

$$\frac{1}{2}(x - 3) = \frac{x}{4} + 7$$

$$\frac{x}{2} - \frac{3}{2} = \frac{x}{4} + 7 \quad \text{distributive prop}$$

$$4\left(\frac{x}{2}\right) - 4\left(\frac{3}{2}\right) = 4\left(\frac{x}{4}\right) + 4(7) \quad \text{multiply by LCM 4}$$

$$2x - 6 = x + 28 \quad \text{simplify}$$

$$2x - 6 + 6 = x + 28 + 6 \quad \text{add 6}$$

$$2x = x + 34 \quad \text{simplify}$$

$$2x - x = x + 34 - x \quad \text{subtract } x$$

$$x = 34$$

Therefore, the solution is  $x = 34$ .

**Practice 3** Solve  $\frac{2}{3}(2x - 1) = \frac{3x}{5} + 2$  (Answer on page 46)

**Example 4** Solve for  $x$ :  $2 + \frac{2}{3}\left(5x - \frac{5}{2}\right) = 3 - \frac{5}{6}(x + 7)$

**Solution 4**

$$2 + \frac{2}{3}\left(5x - \frac{5}{2}\right) = 3 - \frac{5}{6}(x + 7)$$

$$2 + \frac{10}{3}x - \frac{5}{3} = 3 - \frac{5}{6}x - \frac{35}{6} \quad \text{distributive prop}$$

$$6(2) + 6\left(\frac{10}{3}x\right) - 6\left(\frac{5}{3}\right) = 6(3) - 6\left(\frac{5}{6}x\right) - 6\left(\frac{35}{6}\right) \quad \text{multiply by LCM 6}$$

$$12 + 20x - 10 = 18 - 5x - 35$$

$$20x + 2 = -5x - 17$$

$$20x + 4 + 5x = -5x - 17 + 5x \quad \text{add } 5x$$

$$25x + 2 - 2 = -17 - 2 \quad \text{subtract 2}$$

$$25x = -19$$

$$\frac{25x}{25} = \frac{-19}{25} \quad \text{divide by 25}$$

$$x = -\frac{19}{25}$$

**Practice 4** Solve  $\frac{1}{4}(5x + 2) + 1 = 7 - \frac{2}{3}(x + 4)$  (Answer on page 46)

When solving a linear equation containing decimals, one can eliminate the decimals by multiplying by the appropriate power of ten and then solving the linear equation as before. Consider

$$0.2(x - 3) + 0.53(2x + 1) = 0.56.$$

To find the appropriate power of 10 that will eliminate the decimals, what do we do? If we multiply by 10, are all of the decimals eliminated? Well,  $0.2 \times 10 = 2$ , but  $0.53 \times 10 = 5.3$  and  $0.56 \times 10 = 5.6$ . However, if we multiply by  $10^2$ , or 100, we have  $0.2 \times 100 = 20$ ,  $0.53 \times 100 = 53$ , and  $0.56 \times 100 = 56$ . In this example, what does the 2 represent? Consider

$$0.203(x + 1) - 4.5(x - 3) = 0.22x.$$

We notice that if we multiply by 10 or even  $10^2 = 100$ , the decimals will not be eliminated. However, multiplying by  $10^3 = 1000$  will do the trick. This is left to the reader to verify.

If you look at both the previous example and this one, you will see that the exponent is the largest number of past the decimal places that are present in the problem. Therefore, we multiply by 10 raised to the largest number of decimal places. Of course, there are other ways to solve these problems, including leaving the decimals alone. The next example is solved using both methods.

**Example 5** Solve  $0.2(x - 3) + 0.53(2x + 1) = 0.56$

**Solution 5**

First, we will solve the equation by eliminating the decimals.

$$\begin{aligned}
 0.2(x - 3) + 0.53(2x + 1) &= 0.56 \\
 0.2x - 0.6 + 1.06x + 0.53 &= 0.56 && \text{distributive prop} \\
 100(0.2x - 0.6 + 1.06x + 0.53) &= 100(0.56) && \text{multiply by 100} \\
 20x - 60 + 106x + 53 &= 56 \\
 126x - 7 &= 56 \\
 126x - 7 + 7 &= 56 + 7 && \text{add 7} \\
 126x &= 63 \\
 \frac{126x}{126} &= \frac{63}{126} && \text{divide by 126} \\
 x &= \frac{1}{2} \\
 x &= 0.5
 \end{aligned}$$

Next, we will solve the example keeping all decimals.

$$\begin{aligned}
 0.2(x - 3) + 0.53(2x + 1) &= 0.56 \\
 0.2x - 0.6 + 1.06x + 0.53 &= 0.56 && \text{distributive prop} \\
 1.26x - 0.07 &= 0.56 && \text{simplify} \\
 1.26x - 0.07 + 0.07 &= 0.56 + 0.07 && \text{add 0.07} \\
 1.26x &= 0.63 \\
 \frac{1.26x}{1.26} &= \frac{0.63}{1.26} && \text{divide by 1.26} \\
 x &= 0.5
 \end{aligned}$$

**Practice 5** Solve  $0.34(2x - 5) + 0.4(x - 3) = -0.2$  (Answer on page 46)

Suppose we apply our properties and the variable is eliminated. What is the solution? The next example illustrates one such case.

**Example 6** Solve for  $x$ :  $3(x + 5) - 8(x + 3) = -5(x + 2)$

**Solution 6**

$$\begin{aligned}
3(x+5) - 8(x+3) &= -5(x+2) \\
3x + 15 - 8x - 24 &= -5x - 10 && \text{distributive prop} \\
-5x - 9 &= -5x - 10 \\
-5x - 9 + 5x &= -5x - 10 + 5x && \text{add } 5x \\
-9 &\neq -10
\end{aligned}$$

Therefore, there is no solution to this equation.

When an equation has no solution this means that there is no value of  $x$  for which the expression will ever yield a true statement. Does this mean that whenever the variable is eliminated from the equation, there will be no solution? The next example illustrates that this is not the case.

**Example 7** Solve for  $x$ :  $\frac{1}{3}(6x - 7) = \frac{1}{2}(4x - 5) + \frac{1}{6}$

**Solution 7**

$$\begin{aligned}
\frac{1}{3}(6x - 7) &= \frac{1}{2}(4x - 5) + \frac{1}{6} \\
2x - \frac{7}{3} &= 2x - \frac{5}{2} + \frac{1}{6} && \text{distributive prop} \\
6(2x) - 6\left(\frac{7}{3}\right) &= 6(2x) - 6\left(\frac{5}{2}\right) + 6\left(\frac{1}{6}\right) && \text{multiply by LCM of 6} \\
12x - 14 &= 12x - 15 + 1 \\
12x - 14 &= 12x - 14 \\
12x - 14 - 12x &= 12x - 14 - 12x && \text{subtract } 12x \\
-14 &= -14
\end{aligned}$$

Therefore, the answer is all real numbers.

**Practice 6** Solve for  $x$ :  $\frac{1}{3}(3x + 9) - 3 = \frac{1}{9}(9x + 18) - 2$  (Answer on page 46)

**Practice 7** Solve for  $x$ :  $7(x - 2) + 3 = 4(2x + 5) - x$  (Answer on page 46)

When solving a linear equation, if the variable is eliminated and the resulting statement is true then the answer is all real numbers. This means that the original equation is an **identity**. An identity is a statement that is true regardless of what number is replaced in the variable. A linear equation that has one solution is called a **conditional** equation. If the variable is eliminated and the resulting statement is false, then the equation has no solution.

#### ANSWERS TO PRACTICE PROBLEMS

- |                                   |                        |
|-----------------------------------|------------------------|
| 1. $x = \frac{2}{3}$              | 3. $x = \frac{40}{11}$ |
| 2.                                |                        |
| $7 - (2x + 4) = 3 + 2(x - 5)$     |                        |
| $7 - 2x - 4 = 3 + 2x - 10$        | distributive prop      |
| $3 - 2x = 2x - 7$                 | simplify               |
| $3 - 2x - 2x = 2x - 7 - 2x$       | subtract $2x$          |
| $3 - 4x - 3 = -7 - 3$             | subtract 3             |
| $-4x = -10$                       | simplify               |
| $\frac{-4x}{-4} = \frac{-10}{-4}$ | divide by $-4$         |
| $x = \frac{5}{2}$                 |                        |
|                                   | 4. $x = \frac{34}{23}$ |
|                                   | 5. $x = 2.5$           |
|                                   | 6. All real numbers    |
|                                   | 7. No solution         |

#### SECTION 1.5 EXERCISES

(Answers are found on page [121](#).)

1. A student correctly applies all properties to get to  $2x = 4x$  in a problem. The student then divides both sides by  $x$  obtaining  $2 = 4$  and writes no solution as his/her answer. Is this correct?
2. Write an equation of the form  $ax + b = cx + d$  where  $a, b, c$ , and  $d$  are real numbers which has a solution of  $x = 8$ .
3. Describe in your own words the process of eliminating fractions from a linear equation.

**In #4 – #35, solve each equation.**

4.  $2x + 3 - x = 4x - 5$
5.  $\frac{4(5 - x)}{3} = -x$

6.  $\frac{2}{3}x - \frac{5}{6}x - 3 = \frac{1}{2}x - 5$
7.  $2x - 5 = 3(x + 6)$
8.  $7 - 5(x - 3) = 2(4 - 3x)$
9.  $4(x + 2) = 2(2x - 1) + 10$
10.  $\frac{x + 3}{2} = \frac{3x - 1}{4}$
11.  $0.03(2x + 7) = 0.16(5 + x) - 0.13$
12.  $4(x + 3) = 6(2x + 8) - 4$
13.  $5(x - 6) = 2(x + 4) + 3x$
14.  $7x - 3(4x - 2) + 6x = 8 - 9(3x + 2)$
15.  $-8(5x - 3) + \frac{1}{2}(4x - 3) = 5$
16.  $\frac{3}{5}(7 - 3x) + \frac{1}{2}(3x + 7) = \frac{3}{4}(4x - 2)$
17.  $7x - 4(2x + 1) = 5(x + 3) - 6x$
18.  $3x - 2[2x - 4(x - 5)] = 7$
19.  $x - [2 - 3(2x + 4)] = 61$
20.  $\frac{2}{3}(3x - 4) + \frac{1}{3} = \frac{1}{9}x$
21.  $\frac{5}{6}x - \left(x - \frac{1}{2}\right) = \frac{1}{4}(x + 1)$
22.  $4(t + 3) - 8(t - 3) = 6(2t - 1) - 10$
23.  $3(2x - 5) - 5(6 - 4x) = 2 + 3(x - 3)$
24.  $\frac{3x - 7}{4} = \frac{8 - 5x}{3}$
25.  $\frac{2}{3}(x + 12) + \frac{1}{6}(x + 2) = 2x - 3$
26.  $\frac{7 - 3x}{3} = \frac{2}{5}$

$$27. \quad 9(2x + 3) - 3x = 5 - 3(2x - 5)$$

$$28. \quad \frac{1}{4}(8x - 1) + \frac{9}{4} = \frac{1}{2}(4x + 5) - \frac{1}{2}$$

$$29. \quad \frac{1}{4}(x - 12) + \frac{1}{2}(x + 2) = 2x + 4$$

$$30. \quad \frac{2}{3}x - \left(x + \frac{1}{4}\right) = \frac{1}{12}(x + 4)$$

$$31. \quad \frac{1}{2}(2x - 3) + 5 = \frac{1}{3}(3x + 4)$$

$$32. \quad -2(2x - 4) - 8 = -3(4x + 4) - 1$$

$$33. \quad \frac{3x - 5}{4} = -2$$

$$34. \quad .12(y - 6) + .06y = .08y - .07(10)$$

$$35. \quad \text{Let } a \neq 0. \text{ Solve } a^2x + (a - 1) = (a + 1)x \text{ for } x.$$



## 1.6 Applications of Linear Equations: Geometry

We now consider how linear equations are used to solve application problems. The key to solving any application problem begins with the translation. Can you think of any words that represent addition, subtraction, multiplication or division? Below are some of the key words that are used to represent the different operations.

- Addition – sum, added to, increased by, more than
- Subtraction – difference, minus, decreased by, subtracted from
- Multiplication – product, times, double ( $2x$ ), tripled ( $3x$ )
- Division – quotient, ratio, divided by

Although there are no step by step directions that apply to every application problem, stated below are some suggested steps.

1. Read the problem carefully and explain what your variable represents.
2. Create and label any pictures, charts, or diagrams that will ease your problem solving.
3. Translate the problem into an equation. Be sure that you can read the problem back.
4. Use the addition and multiplication properties to solve the equation.
5. Check that your answer makes sense in the problem and that you have answered the question being asked. If the problem asks for two answers, make sure you give both.

**Example 1** *The sum of twice a number and three is seven more than the number. Find the number.*

**Solution 1** *Let  $x$  = the number.*

*We translate the expression to obtain*

$$2x + 3 = 7 + x.$$

Next, we have

$$2x + 3 = 7 + x$$

$$x + 3 = 7$$

$$x = 4$$

Therefore, our number must be 4.

**Practice 1** The sum of three times a number and four is the same as the number increased by two. Find the number. (Answer on page 56.)

When translating, the order the words appear in the sentence is sometimes important. To illustrate this, consider the difference between the following expressions:

- The sum of four times a number and 5 is translated  $4x + 5$ .
- Four times the sum of a number and 5 is translated  $4(x + 5)$ .

**Example 2** Three times the difference between twice the number and four is five times the number increased by six. Find the number.

**Solution 2** Letting  $x =$  the number, we translate the expression as

$$3(2x - 4) = 5x + 6.$$

Thus,

$$3(2x - 4) = 5x + 6$$

$$6x - 12 = 5x + 6 \quad \text{distributive property}$$

$$6x - 12 - 5x = 5x + 6 - 5x \quad \text{subtracting } 5x$$

$$x - 12 = 6$$

$$x - 12 + 12 = 6 + 12 \quad \text{adding 12}$$

$$x = 18$$

Hence, the number is 18.

**CAUTION:** Remember when working with differences, the order is important. The expressions  $a$  minus  $b$  or the difference of  $a$  and  $b$  are both translated  $a - b$ . However,  $a$  less than  $b$  is translated  $b - a$ .

**Practice 2** Six less than four times a number is the same as twice the sum of the number and three. Find the number. (Answer on page 56.)

Two integers which differ by 1, such as page numbers in a book, are called **consecutive integers**. Examples are 4 and 5, 9 and 10, 15 and 16, etc. In an application problem, if we let  $x$  be the first integer, how would we represent the second integer? If we consider our example of 4 and 5, we see that we need to add 1 to the first integer to obtain the second integer. Hence, if  $x$  is the first integer, then  $x + 1$  is the second. For consecutive even or consecutive odd integers, the numbers now differ by two; for example, 7 and 9, or 4 and 6. How do we set up the variables for this problem? The next example answers this question.

**Example 3** Find two consecutive **even** integers such that the smaller added to three times the larger gives a sum of 62.

**Solution 3** Since we are dealing with consecutive even integers we will let

$$\begin{aligned}x &= \text{first integer} \\x + 2 &= \text{second integer}\end{aligned}$$

Translating the expression yields

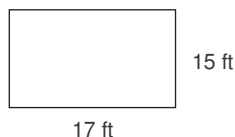
$$\begin{aligned}x + 3(x + 2) &= 62 \\x + 3x + 6 &= 62 && \text{distributive property} \\4x + 6 &= 62 \\4x + 6 - 6 &= 62 - 6 && \text{subtract 6} \\4x &= 56 \\ \frac{4x}{4} &= \frac{56}{4} && \text{divide by 4} \\x &= 14\end{aligned}$$

Since the problem asks for two consecutive integers the answer is 14 and 16.

**Practice 3** Find two consecutive positive integers such that the sum of three times the smaller and five times the larger is 85. (Answers on page 56.)

We now turn our attention to the use of formulas to solve an application problem. For carpeting a floor, painting a wall, and building a deck, it is sometimes beneficial to know the square units of the items being worked on. For other jobs, knowing the perimeter is useful. For those not familiar with these formulas, they can be found in Appendix A on page 113.

**Example 4** *Mary wants to carpet her living room which is 15 feet wide by 17 feet long. What is the minimum square footage of carpeting Mary must purchase?*



**Solution 4** *To solve we use the area formula for a rectangle with  $w = 15$  and  $l = 17$ . Thus,*

$$\begin{aligned}\text{Area in square feet} &= l \cdot w \\ &= (17 \text{ ft})(15 \text{ ft}) \\ &= 255 \text{ square feet}\end{aligned}$$

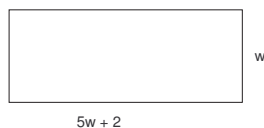
**Practice 4** *Jennifer wants to spread lawn fertilizer on her back yard which is 105 feet wide by 75 feet long. How many square feet does she need to cover? If each bag of fertilizer covers 1,000 square feet, how many bags must Jennifer purchase? (Answers on page 56.)*

We now consider an example where the formula is being used without substituting in a number for the length and width.

**Example 5** *The length of a rectangle is 2 inches more than five times the width. If the perimeter is 106 inches, find the length and width of the rectangle.*

**Solution 5** *First, since both dimensions are in terms of the width we have*

$$\begin{aligned}w &= \text{the width of the rectangle} \\ 5w + 2 &= \text{the length of the rectangle}\end{aligned}$$



Next, we know the perimeter of any rectangle is given by  $P = 2w + 2l$ . Since the perimeter of this rectangle is 106 inches we have

$$\begin{aligned}P &= 2w + 2l \\106 &= 2w + 2(5w + 2) \\106 &= 2w + 10w + 4 \\106 &= 12w + 4 \\102 &= 12w \\8.5 &= w\end{aligned}$$

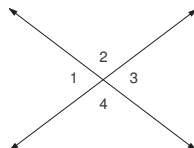
Since the problem asks for both the width and length, we find that

$$\begin{aligned}\text{width} &= 8.5 \text{ feet} \\ \text{length} &= 5(8.5) + 2 = 44.5 \text{ feet}\end{aligned}$$

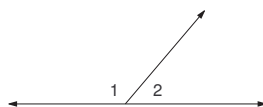
**Practice 5** The amount of fencing required to enclose a small rectangular garden is 34 feet. If the length is 1.5 times the width, find the length of the garden. (Answer on page [56](#).)

We now turn our attention to some angle properties from geometry. Three of these definitions are stated below.

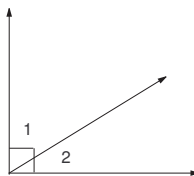
- **Vertical Angles** are opposite angles formed by intersecting lines. Vertical angles always have the same measurement. In the following diagram 1 and 3 are vertical angles. Likewise, 2 and 4 are vertical angles.



- **Supplementary Angles** are two angles whose sum is  $180^\circ$ . If  $A$  and  $B$  are supplementary angles, then we say that  $A$  is the **supplement** of  $B$ . Since  $180^\circ$  is the measure of a straight line, supplementary angles form a straight line. In the following diagram, 1 and 2 are supplementary angles. Likewise, in the previous diagram (with respect to vertical angles) 1 and 2 are supplementary; 2 and 3 are supplementary; 3 and 4 are supplementary; and 1 and 4 are supplementary.

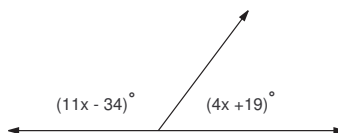


- **Complementary Angles** are two angles whose sum is  $90^\circ$ . If  $A$  and  $B$  are complementary angles, we say that  $A$  is the **complement** of  $B$ , and vice versa. In the following diagram, 1 and 2 are complementary angles.



Let us first consider some examples to illustrate the above definitions. Additional angle definitions can be found in [Appendix A](#).

**Example 6** Find the measure of each marked angle.



**Solution 6** First, we need to recognize that these two angles are supplementary angles. So their sum is  $180^\circ$ .

$$11x - 34 + 4x + 19 = 180$$

$$15x - 15 = 180$$

$$15x = 195$$

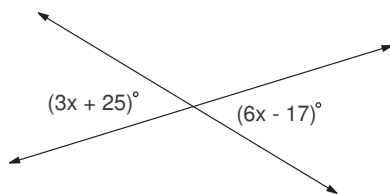
$$x = 13$$

Now that we have the value of  $x$ , we substitute this into both expressions to yield the measure of each angle; namely,

$$11(13) - 34 = 109^\circ$$

$$4(13) + 19 = 71^\circ$$

**Practice 6** Find the measure of each marked angle. (Answers on page 56.)



**Example 7** Find the measure of the angle whose complement is five times its measure.

**Solution 7** Recalling the definition stated earlier, we have the following variable expressions:

$$x = \text{the angle}$$

$$90 - x = \text{the complement of the angle}$$

Since the complement is five times the angle ( $5x$ ), we have the following:

$$5x = 90 - x$$

$$5x + x = 90 - x + x$$

$$6x = 90$$

$$\frac{6x}{6} = \frac{90}{6}$$

$$x = 15$$

Therefore, the measure of the angle we are looking for is  $15^\circ$ .

**Practice 7** Find the measure of the angle whose supplement is 8 times its measure. (Answer on page 56.)

---

**ANSWERS TO PRACTICE PROBLEMS**

- |                             |                              |
|-----------------------------|------------------------------|
| 1. $-1$                     | 5. 10.2 feet                 |
| 2. 6                        | 6. $67^\circ$ and $67^\circ$ |
| 3. 10, 11                   | 7. $20^\circ$                |
| 4. 7875 square feet; 8 bags |                              |

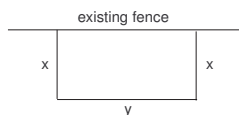
---

**SECTION 1.6 EXERCISES**(Answers are found on page [121](#).)

1. If the sum of a number and five is tripled, the result is one less than twice the number. Find the number.
2. If five times the smaller of two consecutive integers is added to three times the larger, the result is 59. Find both integers.
3. If four is added to twice a number and this sum is multiplied by three, the result is the same as if the number is multiplied by two and four is added to the product. What is the number?
4. The sum of three numbers is 81. The second number is twice the first number, and the third number is three less than four times the first. Find the three numbers.
5. If twice the sum of two consecutive even integers is increased by five, the result is 97. Find both integers.
6. Find three consecutive integers such that the sum of three times the middle integer and five times the smallest integer is the same as seven times the largest integer.
7. A 17-foot wire is to be cut so that one piece is 2 feet longer than twice the shorter piece. Find the length of both pieces.
8. Adult tickets for a show cost \$5.50 while children's tickets cost \$2.50. If there were twice as many adults as children and the total receipts for the show were \$1,026, how many adults were at the show?
9. Kayla, Savanna, and Cheyenne are the pitchers on the local girls softball team. Kayla pitched twelve more games than Savanna and Cheyenne pitched four more than twice the number of games Savanna pitched. If the team played a total of 28 games, how many games did each girl pitch?



10. Roger, Will, and Jacob sold magazine subscriptions to raise money for playground equipment. Will sold nine more subscriptions than Roger. Jacob sold three less than twice as many subscriptions as Roger. If together they sold a total of 66 magazine subscriptions, how many subscriptions did each boy sell?
11. As an employee at the local market, Charlotte's duties include stocking shelves, cashier, and training new employees. Yesterday during an eight hour workday, Charlotte trained new employees half as long as she was cashier and stocked shelves three-quarters of an hour less than she was cashier. How many hours did Charlotte spend on each duty?
12. Angela needs to build a rectangular storage unit. She wants the length of the rectangle to be three feet more than twice the width. The perimeter of the rectangle is 36 feet. Find the length and width of the storage unit.
13. The perimeter of a rectangular garden is 126 feet. If the length of the garden is seven feet more than three times the width, find the dimensions of the garden.
14. A small pasture is to be fenced off with 100 yards of new fencing along an existing fence. The existing fence will serve as one side of a rectangular enclosure. (See diagram below). What integer dimensions produce the maximum area that can be enclosed by the new fencing?

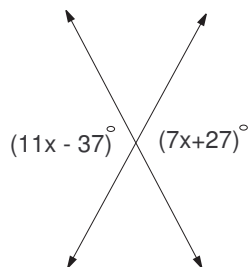


15. In problem #14, suppose that the 100 yards of fencing needs to be used to enclose the entire area. What integer dimensions produce the maximum enclosed area?
16. Mark decides to paint his living room walls Winter Snow. According to the can, each gallon of paint will cover 300 square feet. If his rectangular living room is 20 feet long and 15 feet wide with 10 foot ceilings, how many gallons of paint will Mark need to buy to complete his project if
  - (a) only one coat is needed to complete the project?

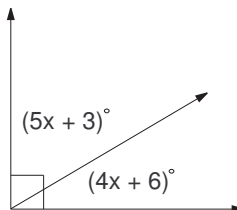
- (b) two coats are needed to complete the project?

In #17–#22 find the measure of each marked angle.

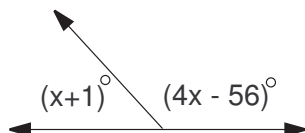
17.



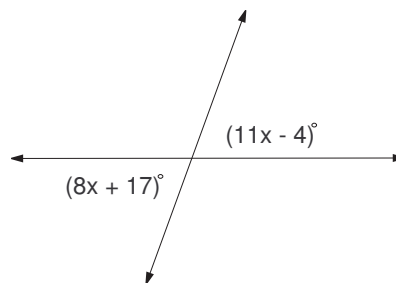
20.



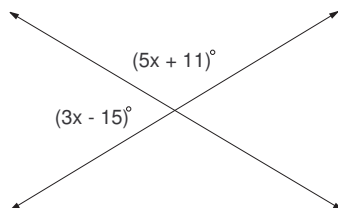
18.



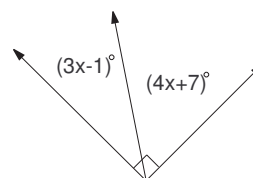
21.



19.



22.



23. Find the measure of an angle such that the sum of the measures of its complement and its supplement is  $100^\circ$ .
24. The supplement of an angle measures  $15^\circ$  more than four times its complement. Find the measure of the angle.

## 1.7 Applications of Linear Equations: Percents

Percent problems are probably the most commonly used type of problems in every day life – sales tax, discounts, depreciation, etc. In Fundamental Mathematics I we discussed transforming a percent into a decimal and a decimal into a percent. ([Click here to review percents and decimals from FM1](#)) When working with percents, there are typically three types of problems:

1. Given the whole and the percent, find the part;
2. Given the whole and the part, find the percent;
3. Given the part and the percent, find the whole.

### Given the whole and the percent

If we are given the whole and the percent, the part can be found by multiplying the percent by the whole. The word “of” is a key to where the multiplication takes place. Consider the following example.

**Example 1** *Suppose that 75% of 400 fourth graders in a school district have passed their proficiency exams. How many fourth graders have passed their proficiency exam?*

**Solution 1** *Recall that  $75\% = 0.75$ . Therefore,*

$$75\% \text{ of } 400 = 0.75 \times 400 = 300.$$

*Therefore, 300 fourth graders have passed their proficiency exams.*

**Practice 1** *If  $42\frac{1}{2}\%$  of the 80 faculty at the local high school have master’s degrees, how many faculty have master’s degrees? (Answer on page [65](#)).*

### Given the whole and the part

Once again, we will recall that the word “of” is a key to where the multiplication takes place.

**Example 2** *If \$450 of a loan has been repaid, what percent of the \$2000 loan has been repaid?*

**Solution 2** Let  $x$  be the percent. Then we have that

450 is  $x$  percent of 2000.

Translating and solving we get

$$\begin{aligned} 450 &= x \cdot 2000 \\ \frac{450}{2000} &= \frac{x \cdot 2000}{2000} \\ 0.225 &= x \end{aligned}$$

Therefore, \$450 is 22.5% of \$2000.

**Practice 2** A skirt originally selling for \$45 is on sale for \$27.90. What percent of the original price is the sale price? What is the discount rate? (Answer on page 65).

### Given the percent and the part

**Example 3** Joe decides to save 15% of his weekly salary. If he saves \$75 every week, what is Joe's weekly salary?

**Solution 3** Let  $x$  = Joe's weekly salary. Then

15% of his weekly salary is \$75.

Translating we get

$$\begin{aligned} .15x &= 75 \\ \frac{.15x}{.15} &= \frac{75}{.15} \\ x &= 500. \end{aligned}$$

Thus, Joe's weekly salary is \$500.

**Practice 3** The Miller family decides that they can spend \$120 of their monthly income on entertainment. What is the the Miller family's monthly income if they spend 2.5% of their monthly income on entertainment? (Answer on page 65).

When solving application problems, it sometimes helps to write your own formula as the next example illustrates.

**Example 4** Marilyn is paid \$315 per week, plus a 5% commission on sales. What is her total earnings if her sales were \$625.

**Solution 4** From the problem we know that

$$\begin{aligned}\text{total earnings} &= \$315 + (5\% \text{ of her sales}) \\ &= \$315 + (5\% \text{ of } \$625) \\ &= 315 + (.05 \times 625) \\ &= 315 + 31.25 \\ &= 346.25\end{aligned}$$

Hence, her total earnings are \$346.25.

**Practice 4** If income tax is \$3,750 plus 28% of taxable income over \$28,000, how much is the income tax on a taxable income of \$35,000. (Answer on page 65).

Sometimes we deal with discounts and markups in the same problem.

**Example 5** Kim paid \$330 for a dresser to sell at her antique shop. She wants to price it so that she can offer a 10% discount and still make a 20% profit off the price she paid for it. What should be the marked price of the dresser at the antique shop?

**Solution 5** Let  $x$  = the price the dresser should be marked.

$$\text{Profit Kim wants to make} = .20(330) = \$66.$$

$$\text{Price willing to sell the dresser: } \$330 + \$66 = \$396.$$

**NOTE:** \$396 is the price after Kim offers a 10% discount.

$$\begin{aligned}396 &= \text{marked price} - \text{discount} \\ 396 &= x - .10x \\ 396 &= .90x \\ 440 &= x\end{aligned}$$

Thus, Kim should mark the dresser at \$440.

**Practice 5** *Buy 4 Less marks down all remaining winter coats at a discount of 40% off the original price. After several weeks, in a last attempt to reduce inventory, the coats are further discounted 50% off the last marked price. Your friend believes that this means the coats are now 90% off the original price. Is your friend correct? If not, what percent off are the coats from the original price? (Answers on page 65).*

Now let us turn our attention to some interest problems. If you have car loans, savings accounts, or any investments, you should be concerned about the type of interest you are paying or that you earn on the investment. There are two common types of interest – simple and compound. **Simple interest** pays interest only on the original amount of money you deposited or invested, called the **principal**. **Compound interest** pays interest on the interest in the account as well. Before we discuss a formula, let's look at the following example.

**Example 6** *Suppose you invest \$10,000 in a bank account that earns an annual interest rate of 3% simple interest.*

- (a) *How much interest would you earn in one year?*
- (b) *How much interest would you earn in three years?*
- (c) *How much interest would you earn in a half a year?*

**Solution 6**

(a) *Because we are dealing with simple interest, the account only pays interest once per year and only on the original amount deposited. Recall that  $3\% = 0.03$ . Therefore, the amount of interest is  $0.03 \times 10,000 = \$300$ .*

(b) *Once again, the account only pays interest once per year and only on the original amount deposited. Hence, we know that the account pays \$300 in interest per year, so for 3 years we would have  $3 \times 300 = \$900$ .*

(c) *We know that we earn \$300 per year so in half a year we would earn half as much; namely,  $\frac{1}{2} \times 300 = \$150$ .*

Now, the original amount of \$10,000 is called the principal and we will denote this by  $p$ . The interest rate is  $r$  and the time is  $t$ . Using the above

example as motivation, what would you consider a valid formula for the interest  $I$  earned in terms of  $p$ ,  $r$ , and  $t$ ? Well, let's summarize our results:

$$1 \text{ year interest} = 10,000 \times 0.03 \times 1 = 300$$

$$3 \text{ years interest} = 10,000 \times 0.03 \times 3 = 900$$

$$\text{half year interest} = 10,000 \times 0.03 \times \frac{1}{2} = 150$$

Do you see the pattern? Principal times rate times time. We can state this formula as follows.

**Formula for simple interest** is given by

$$I = prt$$

where  $I$  is the interest,  $p$  is the principal,  $r$  is the annual interest rate, and  $t$  is the time in years.

**Example 7** Mary deposits \$2000 into an account earning 4.5% simple interest for 2 years. When the 2 years are over, how much money does Mary have in the account?

**Solution 7** Since we are earning simple interest we will use the above formula with  $p = 2000$ ,  $r = 0.045$  and  $t = 2$ . Thus,

$$\begin{aligned} I &= prt \\ &= 2000 \cdot 0.045 \cdot 2 \\ &= 180. \end{aligned}$$

Therefore, the amount of money in the account at the end of two years is  $\$2000 + \$180 = \$2180$ .

**Practice 6** Savanna deposits \$15,000 into an account paying simple interest. At the end of two years, the amount in the account is \$16,875. What is the annual percentage rate on this account? (Answers on page 65).

How much more can one earn with compound interest rather than simple interest? Let's return to our previous example to find out.

**Example 8** Mary deposits \$2000 into an account earning 4.5% interest compounded annually.

- (a) Find the interest and account balance after 1 year.
- (b) Find the account balance after 2 years.
- (c) How much more money do you have after 2 years if interest is compounded annually than you did in Example 7 when interest was simple interest?

**Solution 8** (a) We use the formula to get

$$\begin{aligned} I &= prt \\ &= 2000 \times 0.045 \times 1 \\ &= 90. \end{aligned}$$

So there is \$90 of interest in the account and a balance of \$2090 after 1 year.

(b) For the second year, we earn interest on the interest. The amount of interest we earn in the second year is

$$I = 2090 \times 0.045 \times 1 = 94.05$$

The account therefore has  $\$2090 + \$94.05 = \$2184.05$  after 2 years.

(c) From example 7 we found that the amount in the account after two years of simple interest is \$2,180. Hence, the difference of  $\$2184.05 - \$2180 = \$4.05$  is the additional amount received for compound rather than simple interest.

A formula for compound interest will be discussed in Fundamental Mathematics VI.

**Practice 7** The Baker Family invests \$30,000 for their son's education into an account paying 6.75% compounded annually. How much will the account contain after 3 years? How much more interest will the account have earned than if their account only paid 6.75% simple interest?



---

**ANSWERS TO PRACTICE PROBLEMS**

- |             |   |
|-------------|---|
| 1. 34       | 5. The final price is 70% off the original. |
| 2. 62%, 38% | 6. 6.25%                                    |
| 3. \$4800   |   |
| 4. \$5710   | 7. \$36,494.30; \$419.30 more               |
- 

**SECTION 1.7 EXERCISES**

(Answers are found on page 122.)

- Which is better: A 10% discount followed by a 10% markup, or a 10% markup followed by a 10% discount? Explain your reasoning.
- Which is better: A discount of 20% followed by the addition of sales tax at a rate of 6%, or the addition of sales tax at a rate of 6% followed by a discount of 20%? Explain your reasoning.
- You are offered two different real estate positions. Job 1 pays \$250 per week plus a 6% commission on sales. Job 2 pays 9% commission only. If sales average about \$20,000 per week, which job would pay more?
- To find the cost of a \$20 item discounted 25%, Amy multiplied by  $\frac{1}{4}$  and Jack multiplied by  $\frac{3}{4}$ . Both found the right answer. Explain how this could be.
- Candidates A, B, and C were in an election in which 1000 votes were cast. Candidate A received 35% of the votes while Candidate B received 52 more votes than Candidate C. Which candidate won the election and by how many votes?
- The Miller family spends twenty percent of their monthly salary on food. Twice the amount spent on food is spent on the house payment, and one half of the amount spent on food is spent for utilities. If the total amount spent on these three amounts is \$1,050, what is the Miller family's monthly salary?
- The size of the smallest angle of a triangle is 30% of the size of the largest angle. The size of the third angles is  $20^\circ$  more than the smallest angle. Find the size of each angle. (Note that the sum of the angles of a triangle is  $180^\circ$ .)

8. College tuition at Local University in 2005 is up 9% that of tuition in 2004. If the tuition in 2005 is \$7,050, what was the tuition in 2004?
9. Carl decided to plant a rectangular shaped garden. If the width is 80% of the length and the perimeter is 45 feet, what is the area of the garden?
10. Sally bought textbooks at the bookstore for \$244.33, which included 6% sales tax. What did the books cost?
11. John sells a decorative wreath at his floral shop for \$34.95. If the wreath has a markup of 65%, what was John's cost of the wreath? (Round to two decimal places)
12. A pair of slippers originally costs \$35.50. This week the slippers are advertised at 35% off. What is the sale price of the slippers (before tax)? (Round to two decimal places)
13. A new 10 piece cookware set sells for \$348. This weekend it is on sale for 33% off. What is the sale price of the cookware (before tax)? (Round answer to two decimal places.)
14. Kelly bought a bike and a year later sold it for 30% less than what she paid for it. If she sold the bike for \$126, what did she pay for it?
15. In a basketball game, 17 free throws went in the basket and 3 missed. What percent were missed?
16. A car was to be sold for a 12% discount which amounted to \$1800. How much would the car sell for after the discount?
17. Sarah paid \$36,648.12, including tax, for her new car. The sales tax rate is 7.25%. What was the actual price of the car (before tax)? (Round answer to two decimal places.)
18. If the federal income tax is \$3,750 plus 28% of taxable income over \$65,000, what is the federal income tax on a taxable income of \$87,000?
19. The university basketball team won 91 games, which was 70% of the games played. How many games did the team play?
20. At the local market, bananas originally sold for 45 cents a pound. Today, the cost is 52 cents a pound. What is the percent increase in the cost of bananas per pound? (Round answer to two decimal places.)

- 
21. A laptop that was originally purchased for \$1,850 now has an estimated value of \$1,258. What is the percent decrease in the value of the laptop?
  22. Brenda purchased clothing totaling \$140 before taxes. If the sales tax rate is 6.75%, what was her total cost for the clothing?
  23. Suppose you invest \$25,500 in a bank account that earns an annual interest rate of  $2\frac{1}{2}\%$  simple interest.
    - (a) How much interest would you earn in one year?
    - (b) How much interest would you earn in three years?
    - (c) How much interest would you earn in a half a year?
  24. If \$12,000 is deposited into an account earning 4% interest compounded annually,
    - (a) Find the interest and account balance after 1 year.
    - (b) Find the account balance after 2 years.
    - (c) How much additional money do you have after 2 years if interest is compounded annually than you would have if interest was calculated using simple interest?
  25. Beverly deposits \$18,250 into an account paying simple interest. At the end of three years, the amount in the account is \$21,808.75. What is the annual percentage rate on this account?

## 1.8 Applications of Linear Equations: Proportions

Much of our use of mathematics involves comparison and change. A **ratio** is a quotient of two quantities. The ratio of the number  $a$  to the number  $b$  is written

$$a \text{ to } b, \quad \frac{a}{b}, \quad \text{or} \quad a : b.$$

Fractions are ratios of integers but not all ratios are fractions. For example, the ratio of the diagonal of a square to one of its sides is never a ratio of integers. Indeed, a ratio may compare any kind of quantity or magnitude to any other kind such as dollars to ounces in Example 1.

One application of ratios is in unit pricing. This helps one to determine which size of a product offered at different prices represents the best bargain.

**Example 1** *The local Bargain Basement offers the following prices for a jar of Fluff Marshmallow Creme:*

| Size  | Price  |
|-------|--------|
| 15 oz | \$1.35 |
| 32 oz | \$2.76 |
| 48 oz | \$4.56 |

*Which size represents the best buy?*

**Solution 1** *To find the best buy, we compare the price for each jar to the number of ounces per jar. This ratio gives the price per ounce. The best deal, of course, will be the item with the lowest price per ounce. The following chart gives those ratios:*

| Size  | cost per ounce                |
|-------|-------------------------------|
| 15 oz | $\frac{\$1.35}{15} = 0.09$    |
| 32 oz | $\frac{\$2.76}{32} = 0.08625$ |
| 48 oz | $\frac{\$4.56}{48} = 0.095$   |

*Comparing the price per ounce, we determine that the 32 ounce jar is the best buy.*

The above example illustrates that buying the largest size does not always result in the best buy.

**Practice 1** The local grocery store had the following prices on coffee:

| Size  | Price   |
|-------|---------|
| 13 oz | \$2.75  |
| 34 oz | \$7.34  |
| 60 oz | \$13.52 |

Which size represents the best buy? (Answer on page 71).

A **proportion** is a statement that says that two ratios are equal. Ratios and proportions are not limited to the area of mathematics. In map making, proportions are used to represent the distance between two cities. Likewise, for scale models we use proportions. Even in cooking, the idea of a proportion is used whenever we increase or decrease the ingredients to accommodate the number of servings we want. If three of the numbers in a proportion are known, then we can always solve for the fourth number by multiplying both sides of the equation by the least common multiple of the denominators.

**Example 2** A new commercial states that three out of four dentists recommend the new Flash toothpaste. If 411 dentists recommend the toothpaste, how many dentists were interviewed?

**Solution 2** Let  $x$  = the number of dentists interviewed. Then

$$\begin{aligned}\frac{3}{4} &= \frac{411}{x} \\ 4x \left( \frac{3}{4} \right) &= 4x \left( \frac{411}{x} \right) \quad \text{multiply by LCM of denominators} \\ 3x &= 4(411) \\ 3x &= 1644 \\ x &= 548\end{aligned}$$

Hence, 548 dentists were interviewed.

Note that in the above example, there are four correct ways to set up the proportion. They are  $\frac{3}{4} = \frac{411}{x}$ ,  $\frac{4}{3} = \frac{x}{411}$ ,  $\frac{3}{411} = \frac{4}{x}$ , and  $\frac{411}{3} = \frac{x}{4}$ . After multiplying by the LCM of the denominators in each proportion, we obtain  $3x = 1644$ .

**Practice 2** *Cheyenne scored 75 goals during her soccer practice. If her success-to-failure rate is 5 : 4, how many times did she attempt to goal? (Answer on page 71.)*

**Example 3** *You have just invented a new recipe to enter in the local cooking contest. As invented, the recipe calls for 2 cups of onions for three servings. Contest rules state that the recipe must make four servings. How many cups of onions must be used for the recipe to make four servings?*

**Solution 3** *Let  $x$  = the number of cups of onions for four servings.*

$$\begin{aligned}\frac{x}{4} &= \frac{2}{3} \\ 12\left(\frac{x}{4}\right) &= 12\left(\frac{2}{3}\right) \quad \text{multiply by LCM of denominators} \\ 3x &= 8 \\ x &= \frac{8}{3} \\ x &= 2\frac{2}{3}\end{aligned}$$

*Therefore, for four servings the recipe would require  $2\frac{2}{3}$  cups of onions.*

**Practice 3** *A recipe that makes 3 dozen peanut butter cookies requires  $1\frac{1}{4}$  cups of flour. How much flour would you require for 5 dozen cookies? (Answer on page 71).*

**Example 4** *The driver at pump 2 filled her tank with 12.5 gallons of gasoline for \$33. How many gallons of gasoline did the driver at pump 6 get if his total was \$42.90?*

**Solution 4** *Let  $x$  = the number of gallons of gasoline for driver at pump 6.*

$$\begin{aligned}\frac{x}{42.90} &= \frac{12.5}{33} \\ 33x &= 536.25 \\ x &= 16.25\end{aligned}$$

Therefore, the driver at pump 6 got 16.25 gallons of gasoline.

**Practice 4** Akron, OH is located approximately 60 miles west of the Pennsylvania border. An Ohio map represents this distance as 3 inches. On the same map, Youngstown, OH is represented by approximately  $\frac{11}{20}$  of an inch from the Pennsylvania border. How far is Youngstown, OH from the Pennsylvania border? (Answer on page 71).

---

**ANSWERS TO PRACTICE PROBLEMS**

- |                                     |                                  |
|-------------------------------------|----------------------------------|
| 1. The 13 ounce can is the best buy | 3. $2\frac{1}{12}$ cups of flour |
| 2. 135 goals attempted              | 4. 11 miles                      |
- 

**SECTION 1.8 EXERCISES**

(Answers are found on page 123.)

1. Find the best buy for each item.
  - (a) Salad Dressing: 8 ounces for \$0.98, 16 ounces for \$1.34, or 32 ounces for \$2.74
  - (b) Chocolate Chips: 6 ounces for \$0.94, 12 ounces for \$1.50, or 24 ounces for \$2.83
  - (c) Strawberry Jam: 11 ounces for \$1.19, 24 ounces for \$2.64, or 32 ounces for \$3.68.

2. A recipe for 36 cookies requires  $1\frac{3}{4}$  cups of chocolate chips. How many cups are required for 90 cookies?
3. If 7 gallons of premium gasoline costs \$17.43, how much would it cost to fill up with 25 gallons of premium gasoline? (Round to two decimal places)
4. A holiday recipe calls for  $1\frac{2}{3}$  cups of flour. If this recipe feeds 4 people, how many cups of flour would you need in the altered recipe in order to feed 10 people?
5. Cheyenne scored 20 goals during this soccer season. If she scored 5 goals for every 9 attempts, how many goals did she miss?
6. A recipe that makes 4 dozen oatmeal raisin cookies calls for  $1\frac{1}{4}$  cups of flour.
  - (a) How much flour would you need if you doubled the recipe?
  - (b) How much flour would you need for half of the recipe?
  - (c) How much flour would you need to make 5 dozen cookies?
7. Your architect has drawn up the blueprints for your new house. In the drawing  $\frac{1}{2}$  inch represents 3 feet. The length of the house is 42 feet. How many inches would this represent on the blueprint?
8. A local business produces color markers. The shop manager has found that out of every 1200 markers produced, 3 will be defective. How many markers were produced on this machine, if 7 markers were found to be defective?
9. You invented a new recipe to enter into the local cooking contest. As invented, the recipe calls for  $1\frac{1}{2}$  cups of chopped onions for four servings. The contest rules state that the recipe must make six servings. How many cups of onions must be used in order for the recipe to make six servings?
10. If  $\frac{3}{4}$  inch on a map represents an actual distance of 20 miles, what is the distance, in miles, represented by  $4\frac{1}{8}$  inches on the same map?



11. The official ratio of width to length for the US flag is 10 to 19. If a flag is 45 inches wide, what is the length of the flag?
12. East Elementary has 1500 students. The student to teacher ratio is 20 to 1. How many teachers are employed at East Elementary?
13. East Elementary has 1500 students. The ratio of female students to male students is 16 to 9. How many female students attend East Elementary?
14. Wildlife officials want to estimate the number of fish in Bluegill Lake. As a result, they tag a total of 150 fish from the lake. What is the estimated number of fish in Bluegill Lake if six months later officials catch 250 fish and 75 are tagged?
15. The directions on a bag of cement are to mix 1 part cement, 2 parts sand, and 3 part gravel. If you have 14 pounds of sand, how many pounds of cement and how many pounds of gravel should be mixed with this sand?
16. A metal tube measuring 3 feet long weighs 28 pounds. What is the weight of a similar tube that measures  $2\frac{1}{4}$  feet?
17. If there are 7.62 cm in 3 inches, how many centimeters are there in 8 inches?
18. A family purchases 2 gallons of milks every 3 weeks. At this rate, how many gallons of milk can this family expect to purchase in a year?
19. According to Physics, the weight of an object on the moon is proportional to its weight on Earth. If a 144 pound woman weighs 24 pounds on the moon, what would a 84 pound boy weigh on the moon?
20. The property tax for a home with an assessed value of \$145,000 is \$1,373.88. What is the property tax for a house with an assessed value of \$124,000

## 1.9 Properties of Inequality and Linear Inequalities

A **linear inequality in one variable** is an inequality that can be written in one of the following forms:

$$ax + b < c, \quad ax + b \leq c, \quad ax + b > c, \quad \text{or} \quad ax + b \geq c$$

where  $a, b$ , and  $c$  are real numbers with  $a \neq 0$ . A **solution** of an inequality in one variable is a number that makes the inequality true when all the occurrences of the variable in the inequality are replaced with that number. To **solve** an inequality means to find *all* of its solutions.

The solution of a linear inequality can be written in two different ways: inequality or interval notation. Before we start solving linear inequalities, we need to discuss these two different ways.

An **inequality** uses the inequality signs along with the endpoints of the interval. For example,  $x \leq 3$  represents all of the real numbers less than or equal to 3. Here, 3 is considered an **endpoint** of the interval.

**Interval Notation** can also be used to describe a set of numbers. For interval notation, a parenthesis is used to denote that the endpoint is not included in the interval; whereas, a bracket is used to denote that the endpoint is included in the interval. It is important that the smallest endpoint be written on the left and the largest endpoint be written on the right. If there is no smallest endpoint of the interval, we use the symbol  $-\infty$  (negative infinity). The symbol  $-\infty$  is not a number, but indicates that the interval continues indefinitely through negative real numbers. Likewise, if there is no largest endpoint of the interval, we use the symbol  $\infty$  (infinity). Once again,  $\infty$  is not a number, but is used to indicate that the interval continues indefinitely through positive real numbers.

**Example 1** Rewrite the following inequalities in interval notation.

(a)  $x \geq 7$

(b)  $-4 \leq x < 9$

(c)  $x < 14$

(d)  $\frac{4}{5} < x \leq \frac{9}{2}$

**Solution 1** (a) Notice that there is no largest endpoint and also that 7 is included in the interval. Hence, the interval notation is  $[7, \infty)$

- (b) Notice that  $-4$  is included in the interval, while  $9$  is not included in the interval. Therefore, the interval notation is  $[-4, 9)$
- (c) Here there is no smallest endpoint. So, the interval notation is  $(-\infty, 14)$
- (d) Finally,  $\frac{4}{5}$  is the smallest endpoint but it is not included in the interval. The largest endpoint is  $\frac{9}{2}$  and it is included in the interval. Thus, the interval notation is  $(\frac{4}{5}, \frac{9}{2}]$

**Practice 1** Rewrite the following inequalities in interval notation.

- (a)  $-5 < x$
- (b)  $-\frac{3}{7} \leq x < \frac{5}{3}$
- (c)  $x \leq 9$

(Answer on page 80.)

The following chart summarizes various inequalities in interval notation.

| Inequality        | Interval Notation |
|-------------------|-------------------|
| $x < b$           | $(-\infty, b)$    |
| $x \leq b$        | $(-\infty, b]$    |
| $x > a$           | $(a, \infty)$     |
| $x \geq a$        | $[a, \infty)$     |
| $a < x < b$       | $(a, b)$          |
| $a \leq x < b$    | $[a, b)$          |
| $a < x \leq b$    | $(a, b]$          |
| $a \leq x \leq b$ | $[a, b]$          |

Solving a linear inequality in one variable is very similar to solving a linear equation in one variable with one exception. Below are the important properties that we will use in solving linear inequalities.

**Addition Property of Inequality:**

If  $a < b$  then  $a + c < b + c$ .

**Multiplication Property of Inequality:** Let  $c \neq 0$ .

1. If  $a < b$  and  $c > 0$  then  $ac < bc$ .
2. If  $a < b$  and  $c < 0$  then  $ac > bc$ .

**CAUTION:** DO NOT reverse the inequality when you add or subtract a negative number; only when you multiply or divide by a negative number.

The Addition Property allows us to subtract the same quantity from both sides by adding the opposite of a quantity. The Multiplication Property allows us to divide each side by the same nonzero quantity by multiplying both sides by its reciprocal. See Example 1.

As with linear equations, remember to perform the same operation on **both** sides (or all parts) of the inequality.

**Example 2** Solve for  $x$ :  $3(2x - 4) < 5(3x - 1)$

**Solution 2**

$$3(2x - 4) < 5(3x - 1)$$

$$6x - 12 < 15x - 5$$

*distributive property*

$$6x - 12 - 15x < 15x - 5 - 15x$$

*Addition Property*

$$-9x - 12 < -5$$

*combine like terms*

$$-9x - 12 + 12 < -5 + 12$$

*add 12*

$$-9x < 7$$

*combine like terms*

$$\frac{-1}{9}(-9x) > \frac{-1}{9}(7)$$

*Multiplication Property*

$$x > -\frac{7}{9}$$

$$\left(-\frac{7}{9}, \infty\right)$$

Note that using the Addition Property to add  $-15x$  to each side is the same as subtracting  $15x$  from each side; similarly, multiplying by  $-\frac{1}{9}$  is the same as dividing by  $-9$ .

**Practice 2** Solve  $5 - 2(3x + 1) \geq 4x + 3$  (Answer on page 80.)

The Multiplications Property of Inequality permits us to eliminate fractions by multiplying both sides of the inequality by the same nonzero constant, just as we did with linear equations. Therefore, we can eliminate fractions by multiplying by the least common multiple of the denominators.

**Example 3** Solve  $\frac{2}{3}(x - 4) + \frac{1}{4}(x + 1) \leq \frac{1}{6}(3x + 7) - 1$

**Solution 3**

$$\begin{aligned}
 \frac{2}{3}(x - 4) + \frac{1}{4}(x + 1) &\leq \frac{1}{6}(3x + 7) - 1 \\
 \frac{2}{3}x - \frac{8}{3} + \frac{1}{4}x + \frac{1}{4} &\leq \frac{1}{2}x + \frac{7}{6} - 1 && \text{distributive property} \\
 12\left(\frac{2}{3}x - \frac{8}{3} + \frac{1}{4}x + \frac{1}{4}\right) &\leq 12\left(\frac{1}{2}x + \frac{7}{6} - 1\right) && \text{multiply by LCM of 12} \\
 8x - 32 + 3x + 3 &\leq 6x + 14 - 12 \\
 11x - 29 &\leq 6x + 2 && \text{collect like terms} \\
 11x - 29 + 29 &\leq 6x + 2 + 29 && \text{add 29} \\
 11x - 6x &\leq 6x + 31 - 6x && \text{subtract 6x} \\
 \frac{5x}{5} &\leq \frac{31}{5} && \text{divide by 5} \\
 x &\leq \frac{31}{5} \\
 &\left(-\infty, \frac{31}{5}\right]
 \end{aligned}$$

**Practice 3** Solve  $\frac{1}{3}(4x + 2) + \frac{1}{5}(3 + x) \leq 4$  (Answer on page 80.)

Not every linear inequality has a left and right hand side only. In fact, some consist of three parts – left hand side, middle, and right hand side. The example,  $-2 < x \leq 5$ , is considered a **compound inequality** (also called a double inequality) because it is the combination of two inequalities.

Namely,  $-2 < x \leq 5$  says that  $-2 < x$  and  $x \leq 5$  *at the same time*. To solve a three part, or compound inequality, we need to isolate the variable in the middle part. Even though the addition and multiplication properties of inequalities are stated for two part inequalities they can be generalized to compound inequalities.

**Example 4** Solve  $-9 < 3(2x + 1) < 21$

**Solution 4**

$$\begin{aligned}
 -9 &< 3(2x + 1) < 21 \\
 -9 &< 6x + 3 < 21 && \text{distributive property} \\
 -9 - 3 &< 6x + 3 - 3 < 21 - 3 && \text{subtract 3} \\
 -12 &< 6x < 18 \\
 \frac{-12}{6} &< \frac{6x}{6} < \frac{18}{6} && \text{divide by 6} \\
 -2 &< x < 3 \\
 (-2, 3)
 \end{aligned}$$

**Practice 4** Solve  $-5 \leq 2(3x - 4) < 16$  (Answer on page 80.)

In three part inequalities, we can also eliminate fractions by multiplying by the LCM.

**Example 5** Solve  $3 \leq \frac{2x + 1}{3} \leq 9$

**Solution 5**

$$\begin{aligned}
 3 &\leq \frac{2x + 1}{3} \leq 9 \\
 3(3) &\leq 3\left(\frac{2x + 1}{3}\right) \leq 3(9) && \text{multiply by 3} \\
 9 &\leq 2x + 1 \leq 27 \\
 9 - 1 &\leq 2x + 1 - 1 \leq 27 - 1 && \text{subtract 1} \\
 8 &\leq 2x \leq 26 \\
 \frac{8}{2} &\leq \frac{2x}{2} \leq \frac{26}{2} && \text{divide by 2} \\
 4 &\leq x \leq 13 \\
 [4, 13]
 \end{aligned}$$

**Practice 5** Solve  $-5 < \frac{3x-4}{2} \leq 5$  (Answer on page 80.)

Linear inequalities also arise in application problems.

**Example 6** Jill scored an 88, 94, and 82 on her first three math tests. An average of at least 90 will earn her an A in the course. What does Jill need to score on her fourth and final test to earn an A in the course?

**Solution 6** Let  $x =$  Jill's fourth test score.

Then since her average needs to be at least a 90, we have

$$\begin{aligned}
 90 &\leq \frac{88 + 94 + 82 + x}{4} \\
 4(90) &\leq 4\left(\frac{88 + 94 + 82 + x}{4}\right) && \text{multiply by 4} \\
 360 &\leq 88 + 94 + 82 + x \\
 360 &\leq 264 + x && \text{simplify} \\
 360 - 264 &\leq 264 + x - 264 && \text{subtract 264} \\
 96 &\leq x \\
 &[96, \infty)
 \end{aligned}$$

Therefore, in order for Jill to receive an A in the course, she must score at least a 96 on the fourth test.

**Practice 6** Kayla has received a 76, 89, 84, and 79 on the first four exams in Spanish. Kayla would like to receive at least a B in this course, which is awarded with at least a 83 average. What does Kayla need to score on her fifth test to receive a B in Spanish? (Answer on page 80.)

**Example 7** The local grocery store rents carpet cleaners for \$25 plus \$5.50 per hour. If Chrissy can spend no more than \$80 to clean her carpets, how long can she rent the carpet cleaner? (Assume all portions of an hour are prorated.)

**Solution 7** Let  $x =$  the number of hours carpet cleaner is rented.

This means that her cost for renting the carpet cleaner is given

by  $25 + 5.5x$ . Therefore,

$$\begin{aligned}
 25 + 5.50x &\leq 80 \\
 25 + 5.50x - 25 &\leq 80 - 25 && \text{subtract 25} \\
 5.50x &\leq 55 \\
 \frac{5.50x}{5.50} &\leq \frac{55}{5.50} && \text{divide by 5.50} \\
 x &\leq 10
 \end{aligned}$$

Hence, she can rent the carpet cleaner for a maximum of 10 hours.

**Practice 7** The local lake rents paddle boats for \$30 plus \$12 per hour. If Jake has a maximum of \$84 to spend, how many hours can he rent the paddle boat? (Assume all portions of an hour are prorated.) (Answer on page 80.)

---

#### ANSWERS TO PRACTICE PROBLEMS

- |  |   |
|--|---|
| 1. (a) $(-5, \infty)$                                  | 4. $\frac{1}{2} \leq x < 4$ ; $[\frac{1}{2}, 4)$      |
| (b) $[-\frac{3}{7}, \frac{5}{3})$                      | 5. $-2 < x \leq \frac{14}{3}$ ; $(-2, \frac{14}{3}]$  |
| (c) $(-\infty, 9]$                                     | 6. Kayla must score at least a 87.                    |
| 2. $x \leq 0$ ; $(-\infty, 0)$                         | 7. Jake can rent the boat for a maximum of 4.5 hours. |
| 3. $x \leq \frac{41}{23}$ ; $(-\infty, \frac{41}{23}]$ |   |

---

#### SECTION 1.9 EXERCISES

(Answers are found on page 123.)

In #1–#6 rewrite the following inequalities in interval notation.

- |                   |                                 |
|-------------------|---------------------------------|
| 1. $-3 \leq x$    | 4. $3 \leq x \leq \frac{11}{2}$ |
| 2. $2 < x \leq 8$ | 5. $x \geq \frac{3}{4}$         |
| 3. $x < 7$        | 6. $14 \geq x$                  |



**In #7–#26 solve each linear inequality. Write solutions using interval notation.**

7.  $3x - 7 \leq 5x + 3$

8.  $7x + 8 \geq 4 - 3x$

9.  $\frac{1}{2}x + 3 \leq \frac{2}{3}x - 4$

10.  $4(x - 3) + 5 \leq 9 + 2x$

11.  $3 + 2x > 5 + 3(2x - 7)$

12.  $2(5x - 4) \leq 3(6x + 2)$

13.  $7(2x - 1) + 4 < 3(4x + 3)$

14.  $4x + 6(5 - 2x) \geq -4(3 - 5x) + 7$

15.  $\frac{7}{5}(10x - 1) \leq \frac{2}{3}(6x + 5)$

16.  $-4(2x - 3) + 7 > 3(2x - 5) - 9(3x - 2)$

17.  $-\frac{2}{3}(x - 7) + \frac{1}{6}(3x - 4) \geq -3$

18.  $-2(4 + 7x) - 7 + 6x \leq 5 - 9x$

19.  $-(9 + 2k) - 5 + 4k \geq 4 + 5k$

20.  $-\frac{1}{4}(p + 6) + \frac{3}{2}(2p - 5) < 10$

21.  $4 \leq 3x - 5 \leq 7$

22.  $-3 < 5(4 - 3x) \leq 12$

23.  $-7 \leq 3(2x - 5) < 10$

24.  $-2 \leq \frac{15 - 6x}{5} < 5$

25.  $-2 \leq \frac{4 + 5x}{7} < 6$

26.  $\frac{2}{3} \leq \frac{4}{5}(2x - 3) < 5$

27. The Jones Family is renting a van for their 7 day family vacation. Auto Depot rents minivans for \$35 per day and \$0.02 per mile. Car Zone rents the same minivan for \$43 per day with no additional charge for mileage. How many miles would the Jones Family need to drive for the the Car Zone rental to be the cheapest?
28. You are offered two different real estate positions. Job 1 pays \$250 per week plus a 6% commission on sales. Job 2 pays 9% commission only. What do the average weekly sales need to be in order for Job 1 to be the better deal?
29. The local parking garage calculates the cost for parking using the formula  $c = 0.50 + 1.25h$  where  $c$  is the cost in dollars and  $h$  is the number of hours parked rounded to the next highest integer.
- (a) When will the cost be \$5.50?
  - (b) When will the cost be more than \$12.00?
  - (c) When will the cost be less than \$9.00?

## 1.10 Nonlinear Equations and Inequalities

In a previous section we considered linear equations. However, how would we solve a nonlinear equation? Consider the equation  $x^2 = 9$ . What are the solutions? We know that  $x = 3$  is certainly a solution since  $3^2 = 9$ . However, it is not the only one. Remember that  $(-3)^2 = 9$ . As a result, the solutions for  $x^2 = 9$  are  $x = 3$  and  $x = -3$ .

**Example 1** Solve  $x^2 = 25$

**Solution 1** We know that  $5^2 = 25$  and  $(-5)^2 = 25$ . So,  $x = 5$  and  $x = -5$  are the solutions to  $x^2 = 25$ .

**Practice 1** Solve  $x^2 = 49$  (Answer on page 91.)

**Example 2** Solve  $4x^2 = 16$

**Solution 2** Dividing both sides by 4 we obtain  $x^2 = 4$ . Recalling that  $2^2 = 4$  and  $(-2)^2 = 4$ , we find that  $x = -2$  and  $x = 2$  are the solutions of  $4x^2 = 16$ .

**Practice 2** Solve  $5x^2 = 80$  (Answer on page 91.)

**Example 3** Solve  $\frac{1}{x} = 16$

**Solution 3** First, notice that  $x \neq 0$ . Assuming that  $x \neq 0$ , we obtain

$$\begin{aligned}\frac{1}{x} &= 16 \\ x \left( \frac{1}{x} \right) &= x(16) \quad \text{multiplying by LCM of denominators} \\ 1 &= 16x \\ \frac{1}{16} &= \frac{16x}{16} \\ \frac{1}{16} &= x\end{aligned}$$

Hence, the only solution is  $x = \frac{1}{16}$ .

**Practice 3** Solve  $\frac{1}{x} = 7$  (Answer on page 91.)

Nonlinear equations can also contain square roots as in our next example.

**Example 4** Solve  $\sqrt{4x} = 14$

**Solution 4** This means that the  $4x$  under the square root must be the square of 14. Namely,  $4x = 196$ . Dividing both sides by 4, we obtain  $x = 49$  as the solution to  $\sqrt{4x} = 14$ .

**Practice 4** Solve  $\sqrt{3x} = 12$  (Answer on page 91.)

The **absolute value** of  $a$ , denoted  $|a|$ , is the distance between zero and  $a$  on the real number line. As a result,  $|-4| = 4$  and  $|7| = 7$ . Since an absolute value represents distance, we know that it cannot be negative. Hence,  $|a|$  is always positive or zero. When we have an **absolute value equation** such as  $|x| = 1$ , this is interpreted as saying that the distance between  $x$  and zero is one unit. Hence,  $x = 1$  or  $x = -1$ . Likewise,  $|x| = 5$  would result in  $x = 5$  or  $x = -5$ .

**Example 5** Solve  $|2x + 3| = 5$

**Solution 5** We know that the  $2x + 3$  inside the absolute value must be either 5 or  $-5$ . So, we solve each of the following equations.

$$2x + 3 = 5$$

$$2x = 2$$

$$x = 1$$

$$2x + 3 = -5$$

$$2x = -8$$

$$x = -4$$

Hence, the solutions are  $x = 1$  or  $x = -4$

**Practice 5** Solve  $|4x + 1| = 7$  (Answer on page 91.)

**Example 6** Solve  $|\frac{2}{3}x - 5| = 17$

**Solution 6** We know that  $\frac{2}{3}x - 5$  inside the absolute value must be either 17 or  $-17$ . Hence, we solve those two equations.

$$\frac{2}{3}x - 5 = 17$$

$$\frac{2}{3}x = 22$$

$$\frac{3}{2} \cdot \frac{2}{3}x = \frac{3}{2} \cdot 22$$

$$x = 33$$

$$\frac{2}{3}x - 5 = -17$$

$$\frac{2}{3}x = -12$$

$$\frac{3}{2} \cdot \frac{2}{3}x = \frac{3}{2} \cdot (-12)$$

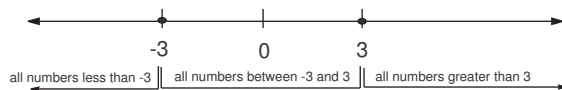
$$x = -18$$

Hence,  $x = 33$  or  $x = -18$

**Practice 6** Solve  $\left|\frac{3}{5}x - 2\right| = 5$ . (Answer on page 91.)

We now turn our attention to nonlinear inequalities. Recall that solving a linear inequality was very similar to solving the corresponding linear equation except that when multiplying or dividing by a negative number we need to reverse the inequality sign. However, consider the nonlinear inequality  $x^2 \geq 9$ . How would this be solved? Well, consider the corresponding equation  $x^2 = 9$ . Recall that we found the solutions of this equation are 3 or  $-3$ .

Returning to  $x^2 \geq 9$ , we find that  $x = 3$  and  $x = -3$  are boundary points. But are there any other values of  $x$  for which the inequality is true? Let's look at a number line to answer this question. If we plot our boundary points 3 and  $-3$  we see that they divide up our number line into three intervals: all numbers less than  $-3$ ; all numbers between  $-3$  and 3; and finally, all numbers greater than 3.



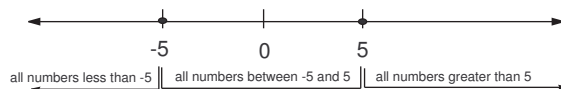
So, how can we find all solutions for  $x^2 \geq 9$ ? Let's do some guess and check. Choose  $-4$  as our test point in the first interval, the numbers less than  $-3$ . Is  $(-4)^2 \geq 9$ ? Yes, since  $16 \geq 9$ . Notice, no matter what number you choose from that first interval it will be a solution for  $x^2 \geq 9$ . What about numbers between  $-3$  and 3? Well, if we choose zero as our test value, we get  $0^2 \geq 9$  which is false. Finally, if we choose 4 as our test point from

the third interval, we find  $4^2 \geq 9$  is true. Hence, all the numbers in the first and third intervals are solutions. The solution for  $x^2 \geq 9$  is  $x \leq -3$  or  $x \geq 3$ .

To solve a nonlinear inequality, we can first solve the corresponding equation. This gives us our boundary points. The boundary points are used to divide up the real number line and check test points from the resulting intervals. Note that if the inequality is  $<$  or  $>$ , the boundary points will not be part of the solution.

**Example 7** Solve  $x^2 \leq 25$

**Solution 7** First, we solve the corresponding equation  $x^2 = 25$ . We know that  $5^2 = 25$  and  $(-5)^2 = 25$ . So,  $x = 5$  and  $x = -5$  are our boundary points. Again, we consider three intervals:



We need to choose a test point from each interval

| 1st interval      | 2nd interval     | 3rd interval     |
|-------------------|------------------|------------------|
| $x < -5$          | $-5 < x < 5$     | $x > 5$          |
| test point = $-6$ | test point = $0$ | test point = $6$ |
| $(-6)^2 \leq 25$  | $(0)^2 \leq 25$  | $(6)^2 \leq 25$  |
| $36 \leq 25$      | $0 \leq 25$      | $36 \leq 25$     |
| false             | true             | false            |

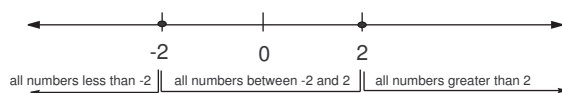
As a result, the only values of  $x$  that satisfy  $x^2 \leq 25$  are those values in the second interval; namely,  $-5 \leq x \leq 5$  is the solution.

**Practice 7** Solve  $x^2 \leq 49$  (Answer on page [91](#).)

**Example 8** Solve  $4x^2 > 16$

**Solution 8** Again, we start with the corresponding equation  $4x^2 = 16$ . Dividing both sides by 4 we obtain  $x^2 = 4$ . Recalling that  $2^2 = 4$  and  $(-2)^2 = 4$ , we find that  $x = -2$  and  $x = 2$  are our boundary points. However, notice that since our inequality is a strict greater than, our boundary points are not included in our solution.

So, examining our number line, we find three intervals once again:



Next, we choose test points in each interval.

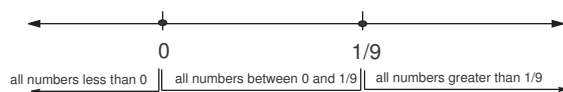
| 1st interval      | 2nd interval     | 3rd interval     |
|-------------------|------------------|------------------|
| $x < -2$          | $-2 < x < 2$     | $x > 2$          |
| test point = $-3$ | test point = $0$ | test point = $3$ |
| $4(-3)^2 > 16$    | $4(0)^2 > 16$    | $4(3)^2 > 16$    |
| $4(9) > 16$       | $4(0) > 16$      | $4(9) > 16$      |
| $36 > 16$         | $0 > 16$         | $36 > 16$        |
| true              | false            | true             |

Therefore, the solution for  $4x^2 > 16$  is  $x < -2$  or  $x > 2$ .

**Practice 8** Solve  $5x^2 > 80$  (Answer on page 91.)

**Example 9** Solve  $\frac{1}{x} < 9$

**Solution 9** First, we solve the corresponding equation  $\frac{1}{x} = 9$ . Note immediately that  $x \neq 0$ . Although this is not a solution, it will serve as one of our boundary points. Here, we see that  $x = \frac{1}{9}$  is the only solution to  $\frac{1}{x} = 9$ . When we place  $x = \frac{1}{9}$  and  $x = 0$  on a number line, we again have three intervals.



Again, we choose test points in each interval.

| 2nd interval       |                              |                   |
|--------------------|------------------------------|-------------------|
| 1st interval       |                              | 3rd interval      |
| $x < 0$            | $0 < x < \frac{1}{9}$        | $x > \frac{1}{9}$ |
| test point = $-1$  | test point = $\frac{1}{10}$  | test point = $1$  |
| $\frac{1}{-1} < 9$ | $\frac{1}{\frac{1}{10}} < 9$ | $\frac{1}{1} < 9$ |
| $-1 < 9$           | $10 < 9$                     | $1 < 9$           |
| true               | false                        | true              |

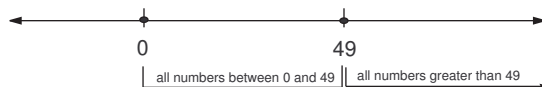
Thus, the solution for  $\frac{1}{x} > 9$  is  $x < 0$  or  $x > \frac{1}{9}$ .

**Practice 9** Solve  $\frac{1}{x} \geq 4$  (Answer on page 91.)

Nonlinear inequalities can also contain square roots as in our next example.

**Example 10** Solve  $\sqrt{4x} > 14$

**Solution 10** Again, we solve the corresponding equation  $\sqrt{4x} = 14$ . This means that the  $4x$  under the square root must be the square of 14. Namely,  $4x = 196$ . Dividing both sides by 4, we obtain  $x = 49$  as a boundary point. Recall that in the real number system we are unable to take the square root of a negative number. As a result, no negative  $x$ -value will be considered. Thus, although zero is not a solution to  $4x = 196$ , it will serve as one of our boundary points. Thus, we have two intervals:



Once again, we choose test points in each interval.

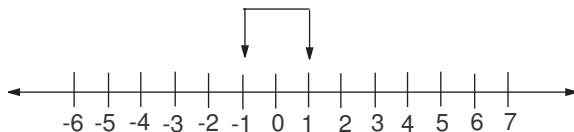


| <i>1st interval</i>   | <i>2nd interval</i>     |
|-----------------------|-------------------------|
| $0 < x < 49$          | $x > 49$                |
| <i>test point</i> = 1 | <i>test point</i> = 100 |
| $\sqrt{4(1)} > 14$    | $\sqrt{4(100)} > 14$    |
| $\sqrt{4} > 14$       | $\sqrt{400} > 14$       |
| $2 > 14$              | $20 > 14$               |
| <i>false</i>          | <i>true</i>             |

Remember since our inequality is  $>$ , the boundary point 49 is not part of the solution. Hence, the solution is  $x > 49$ .

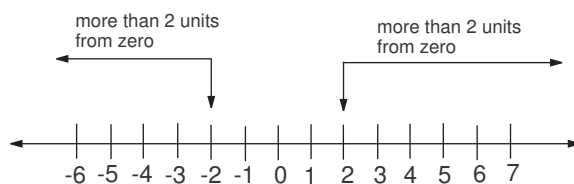
**Practice 10** Solve  $\sqrt{3x} \leq 12$  (Answer on page 91.)

Next, we consider **absolute value inequalities**. Recall that the absolute value equation  $|x| = 1$  can be interpreted as the distance from  $x$  and zero on a number line is one unit. How then can we interpret  $|x| < 1$  or  $|x| > 1$ ? We will consider each of these separately. First, take  $|x| < 1$ . We are not just talking about one or two values of  $x$  but all numbers that are less than one unit from zero on the number line. The following diagram represents our set.



Hence, we write this answer as  $-1 < x < 1$ . Similarly, if we consider  $|x| \leq 2$ , we are talking about all the numbers on the number line that are less than or equal to two units from zero on the number line. Hence,  $-2 \leq x \leq 2$ .

Now, consider  $|x| > 2$ . We can interpret this as all the numbers that are greater than (or more than) two units away from zero on the number line. The following diagram represents this set.



For this one, we write our answer in two pieces:  $x < -2$  or  $x > 2$ . A similar expression holds for  $|x| \geq 5$ ; namely,  $x \leq -5$  or  $x \geq 5$ .

**Example 11** Solve  $|2x - 3| < 5$

**Solution 11** We are interested in all the values of  $x$  for which  $2x - 3$  is less than five units from zero on the number line. As a result, we need to solve  $-5 < 2x - 3 < 5$ .

$$-5 < 2x - 3 < 5$$

$$-2 < 2x < 8$$

$$-1 < x < 4$$

Hence,  $-1 < x < 4$  is the solution.

**Practice 11** Solve  $|3x - 2| \leq 9$ . (Answer on page 91.)

**Example 12** Solve  $|3x + 5| \geq 7$

**Solution 12** We are interested in all the values of  $x$  for which  $3x + 5$  is greater than or equal to seven units from zero on the number line. As a result, either  $3x + 5 \geq 7$  or  $3x + 5 \leq -7$ .

$$3x + 5 \geq 7$$

$$3x \geq 2$$

$$x \geq \frac{2}{3}$$

$$3x + 5 \leq -7$$

$$3x \leq -12$$

$$x \leq -4$$

Hence, the solution is  $x \geq \frac{2}{3}$  or  $x \leq -4$ .

**Practice 12** Solve  $|4x + 2| \geq 8$ . (Answer on page 91)

---

**ANSWERS TO PRACTICE PROBLEMS**

- |                                   |   |
|-----------------------------------|---|
| 1. $x = -7$ or $x = 7$            | 7. $-7 \leq x \leq 7$                             |
| 2. $x = -4$ or $x = 4$            | 8. $x < -4$ or $x > 4$                            |
| 3. $x = \frac{1}{7}$              | 9. $0 < x \leq \frac{1}{4}$                       |
| 4. $x = 48$                       | 10. $0 \leq x \leq 48$                            |
| 5. $x = \frac{3}{2}$ or $x = -2$  | 11. $-\frac{7}{3} \leq x \leq \frac{11}{3}$       |
| 6. $x = \frac{35}{3}$ or $x = -5$ | 12. $x \geq \frac{3}{2}$ or $x \leq -\frac{5}{2}$ |

---

**SECTION 1.10 EXERCISES**

(Answers are found on page 124.)

**In #1–#20 solve the equation.**

- |                            |                               |
|----------------------------|-------------------------------|
| 1. $x^2 = 144$             | 11. $\sqrt{2x} = 10$          |
| 2. $3x^2 = 75$             | 12. $\sqrt{3x} = 2$           |
| 3. $16x^2 = 4$             | 13. $4\sqrt{x} = 24$          |
| 4. $8x^2 = 512$            | 14. $\sqrt{5x} + 3 = 18$      |
| 5. $36x^2 - 9 = 0$         | 15. $\sqrt{x} + 4 = 12$       |
| 6. $x^2 - 5 = 59$          | 16. $ 3x - 6  = 2$            |
| 7. $\frac{14}{x} = 7$      | 17. $ 2x + 7  = 3$            |
| 8. $\frac{3}{x} + 1 = 10$  | 18. $ 2x + 1  = 9$            |
| 9. $\frac{6}{5x} = 2$      | 19. $ \frac{2}{5}x + 1  = 11$ |
| 10. $\frac{3}{2x} - 3 = 7$ | 20. $ -3x + 7  = 9$           |

**In #21–#40 solve the inequality. Write solutions using interval notation.**

21.  $x^2 > 64$

30.  $\sqrt{3x} \geq 6$

22.  $x^2 \leq 100$

31.  $\sqrt{6x} < 5$

23.  $2x^2 \geq 50$

32.  $\sqrt[3]{2x} \geq 4$

24.  $3x^2 < 192$

33.  $3\sqrt{x} \geq 24$

25.  $7x^2 > 28$

34.  $3\sqrt{7x} \geq 12$

26.  $\frac{1}{x} < 7$

35.  $|2x| \geq 14$

36.  $|3x| < 12$

27.  $\frac{1}{x} \geq 5$

37.  $|4x - 3| \leq 5$

28.  $\frac{1}{x} \leq -4$

38.  $|6x + 3| > 8$

39.  $|5x - 2| \geq 7$

29.  $\frac{3}{x} \geq 9$

40.  $|9x + 2| < 3$

## 1.11 Equations in Two Variables

In Section 1.5 we solved some equations in one variable. Recall that a solution of an equation in one variable is a number that makes the equation true when the variable is replaced by that number. For an equation having two variables, two numbers will be required for a solution, one for each variable.

**Example 1** (a) Determine whether  $x = 2$  and  $y = 3$  is a solution of the equation  $3x + y = 9$ .

(b) Determine whether  $x = 3$  and  $y = 2$  is a solution of the equation  $3x + y = 9$ .

**Solution 1** (a) Substituting  $x = 2$  and  $y = 3$  into the first equation we have

$$3(2) + (3) = 6 + 3 = 9.$$

Hence, the substitution results in a true statement. Thus,  $x = 2$  and  $y = 3$  is a solution to  $3x + y = 9$ .

(b) Substituting  $x = 3$  and  $y = 2$  into the equation we have

$$3(3) + 2 = 6 + 2 = 8 \neq 9.$$

Hence, the substitution does not result in a true statement. Thus,  $x = 3$  and  $y = 2$  is **not** a solution to  $3x + y = 9$ .

**Practice 1** Determine whether  $x = -2$  and  $y = 4$  is a solution of

$$4x + 2y = 1.$$

(Answer on page 102.)

We see from Example 1 that in a solution of an equation with two variables, it is important to be careful about which number replaces which variable. So we say that a **solution of an equation in two variables** is a pair of numbers, one number assigned to each variable, such that the equation is true when each occurrence of a variable in the equation is replaced by its assigned number. We can keep things organized by establishing an order for the variables: the variables will usually be  $x$  and  $y$ , in that order. This allows us to abbreviate a solution of an equation in two variables as

an **ordered pair** of numbers, that is, two numbers in a particular order, written  $(a, b)$ , where  $a$  is the first number and  $b$  is the second. For example, we know from Example 1 that  $(2, 3)$  is a solution of  $3x + y = 9$  while  $(3, 2)$  is not.

**Example 2** Determine whether  $\left(-\frac{1}{2}, 3\right)$  is a solution of  $5y = 7 - 14x$ .

**Solution 2** Replacing  $x$  with  $-\frac{1}{2}$  and  $y$  with 3, we obtain

$$5(3) = 7 - 14\left(\frac{1}{2}\right)$$

$$15 = 7 + 7$$

$$15 \neq 14$$

Hence,  $\left(-\frac{1}{2}, 3\right)$  is not a solution to  $5y = 7 - 14x$ .

**Practice 2** Determine if  $\left(4, \frac{2}{3}\right)$  is a solution to  $2x + 1 = 3 + 9y$ . (Answer on page 102.)

Examples 1 and 2 are examples of a special kind of equation in two variables called linear equations. (We will see some nonlinear later in this section.) A **linear equation in two variables** is an equation that can be written in the form

$$ax + by = c$$

where  $a, b$  and  $c$  are real numbers such that  $a$  and  $b$  are not both zero. If  $a$  and  $b$  are both not zero, it is easy to fill in a missing component of an ordered pair so that it becomes a solution of the linear equation as the next example illustrates.

**Example 3** Given  $3x - 2y = 7$  complete the following ordered pairs to find a solution.

(a)  $(-1, ?)$

(b)  $\left(?, \frac{5}{2}\right)$

**Solution 3** (a) We start by replacing  $x$  with  $-1$  and solve for  $y$ .

$$\begin{aligned} 3(-1) - 2y &= 7 \\ -3 - 2y &= 7 \\ -2y &= 10 \\ y &= -5 \end{aligned}$$

Hence, the ordered pair solution is  $(-1, -5)$ .

(b) This time, we start by replacing  $y$  with  $\frac{5}{2}$  and solve for  $x$ .

$$\begin{aligned} 3x - 2\left(\frac{5}{2}\right) &= 7 \\ 3x - 5 &= 7 \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

Hence, the ordered pair solution is  $\left(4, \frac{5}{2}\right)$ .

**Practice 3** Complete the ordered pair  $\left(\frac{2}{5}, ?\right)$  to find a solution of  $7y = 15x - 4$ . (Answer on page 102.)

In the previous example we solved for the missing coordinate in an ordered pair solution. However, suppose that we were interested in solving for the missing term in several ordered pair solutions. Instead of writing each ordered pair, we could organize them in a chart or table.

**Example 4** Given  $2x - 4y = -3$ , complete the following table.

| $x$  | $y$           |
|------|---------------|
| $-2$ |               |
|      | $\frac{1}{2}$ |
| $0$  |               |
|      | $0$           |

**Solution 4**

$$\text{when } x = -2, \quad 2(-2) - 4y = -3 \implies -4y = 1 \text{ or } y = -\frac{1}{4}$$

$$\text{when } y = \frac{1}{2}, \quad 2x - 4\left(\frac{1}{2}\right) = -3 \implies 2x = -1 \text{ or } x = -\frac{1}{2}$$

$$\text{when } x = 0, \quad 2(0) - 4y = -3 \implies -4y = -3 \text{ or } y = \frac{3}{4}$$

$$\text{when } y = 0, \quad 2x - 4(0) = -3 \implies 2x = -3 \text{ or } x = -\frac{3}{2}$$

The table of solutions therefore becomes

| $x$            | $y$            |
|----------------|----------------|
| $-2$           | $-\frac{1}{4}$ |
| $-\frac{1}{2}$ | $\frac{1}{2}$  |
| $0$            | $\frac{3}{4}$  |
| $-\frac{3}{2}$ | $0$            |

**Practice 4** Given  $4x + 2y = 7$ , complete the following table. (Answers on page 102.)

| $x$           | $y$ |
|---------------|-----|
|               | $5$ |
| $\frac{3}{2}$ |     |
| $0$           |     |
|               | $0$ |

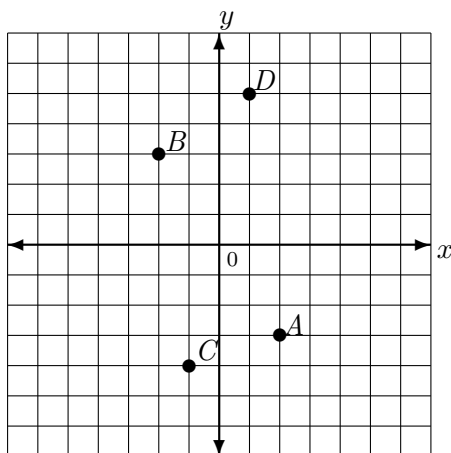
Suppose we now wanted to plot the solutions to a linear equation in two variables. We will need to have two axes – one for the  $x$  variable and another for the  $y$  variable. Together these axes will form the **Rectangular Coordinate System**, or **Cartesian Coordinate System**. The horizontal axis is the  $x$ -axis and the vertical axis is the  $y$ -axis. These two axes divide the  $xy$ -plane into four **quadrants** and the intersection of the two axes is called the **origin** and is denoted  $(0, 0)$ .

To plot an ordered pair on the Cartesian coordinate system we start at the origin and move the number of places determined by the  $x$  and  $y$  coordinates. For example, to plot  $(-2, 4)$ , starting at the origin, we move 2 units to the left, and then four units up on a line parallel to the  $y$ -axis.



**Example 5** Plot the following points on the same coordinate system:  
 $A = (2, -3)$ ,  $B = (-2, 3)$ ,  $C = (-1, -4)$ , and  $D = (1, 5)$ .

**Solution 5**



**Practice 5** Plot the following points on the same coordinate system:  
 $A = (0, -2)$ ,  $B = (-2, 0)$ ,  $C = (1, 4)$ , and  $D = (4, 1)$ . (Answers on page 102.)

The next example is a prelude to graphing linear equations.

**Example 6** Find 3 solutions of  $y = 3x + 1$  and plot them.

**Solution 6** As with Example 4, we will organize our results in a table. However, unlike Example 4, we are not provided with any coordinates. Hence we choose any number for either  $x$  or  $y$  and solve for its corresponding coordinate. Consider the following:

| $x$ | $y$ |
|-----|-----|
| -1  |     |
| 0   |     |
| 1   |     |

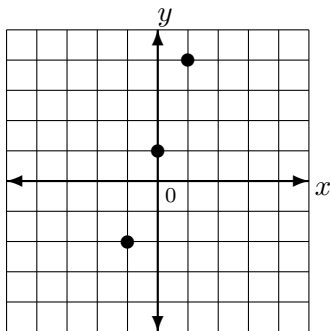
To finish the problem, we substitute each of these values into  $x$  and solve for the corresponding  $y$ -variable. Thus,

$$\begin{aligned}
 \text{when } x = -1, & \quad y = 3(-1) + 1 = -3 + 1 = -2 \\
 \text{when } x = 0, & \quad y = 3(0) + 1 = 0 + 1 = 1 \\
 \text{when } x = 1, & \quad y = 3(1) + 1 = 3 + 1 = 4.
 \end{aligned}$$

Therefore, three solutions of  $y = 3x + 1$  are

| $x$ | $y$ |
|-----|-----|
| -1  | -2  |
| 0   | 1   |
| 1   | 4   |

We now plot them to obtain



In the previous example, we randomly chose numbers for  $x$  and solved for their corresponding  $y$ -values. We could also have chosen values for  $y$  and solved for their corresponding  $x$ -values. Furthermore, we could also have chosen many different values of both  $x$  and  $y$  and complete the ordered pairs.

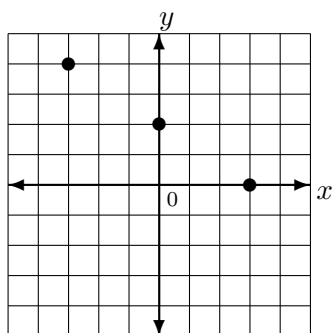
**Practice 6** Find three solutions of  $x = 3y + 2$  and plot them. (Answers on page 102.)

**Example 7** Find three solutions for  $2x + 3y = 6$  and plot them.

**Solution 7** Instead of organizing our results in a table, let us consider a few ordered pairs to complete. For this example, consider  $(0, \quad)$ ,  $(\quad, 0)$ , and  $(-3, \quad)$ . Hence, substituting the values in for the correct variable we see that

$$\begin{aligned}
 \text{when } x = 0, \quad 2(0) + 3y &= 6 \implies 3y = 6 \text{ or } y = 2 \\
 \text{when } y = 0, \quad 2x + 3(0) &= 6 \implies 2x = 6 \text{ or } x = 3 \\
 \text{when } x = -3, \quad 2(-3) + 3y &= 6 \implies 3y = 12 \text{ or } y = 4
 \end{aligned}$$

Hence, three solutions for  $2x + 3y = 6$  are  $(0, 2)$ ,  $(3, 0)$ , and  $(-3, 4)$ . The below graph shows these solutions plotted on the same coordinate plane.



**Practice 7** Find three solutions for  $4x - 2y = 8$  and plot them. (Answers on page 102.)

Notice in the previous two examples that the three solutions appear to lie in a straight line. Will this always happen when we graph a linear equation? Yes, but we will discuss this in a later module. For our final examples, we turn our attention away from linear equations in two variables.

**Example 8** Find four solutions of  $y = 2x^2 - 1$  and plot them.

**Solution 8** Consider the following ordered pairs  $(0, \quad)$ ,  $(-1, \quad)$ ,  $(2, \quad)$ , and  $(1, \quad)$ . Substituting the values in for the correct variable we see that

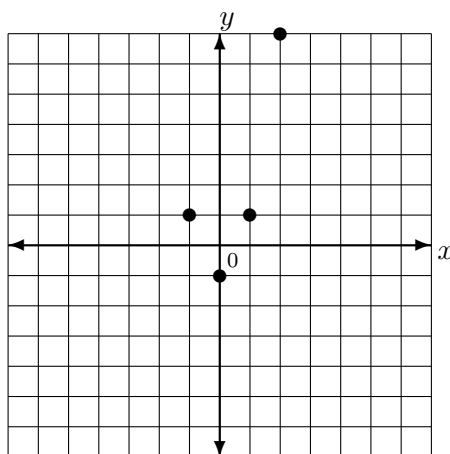
$$\text{when } x = 0, \quad y = 2(0)^2 - 1 = 0 - 1 = -1$$

$$\text{when } x = -1, \quad y = 2(-1)^2 - 1 = 2 - 1 = 1$$

$$\text{when } x = 2, \quad y = 2(2)^2 - 1 = 8 - 1 = 7$$

$$\text{when } x = 1, \quad y = 2(1)^2 - 1 = 2 - 1 = 1$$

Hence, four solutions of  $y = 2x^2 - 1$  are  $(0, -1)$ ,  $(-1, 1)$ ,  $(2, 7)$ , and  $(1, 1)$ . Therefore, when we plot them we obtain



Note that for the first time, the solutions do not lie in a straight line. What is different between the equation given in the above example compared to the two previous examples? If you said the exponent, you are right.

**Practice 8** Find four solutions of  $x = 3y^2 + 2$  and plot them. (Answers on page 102.)

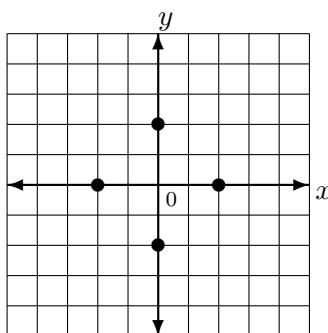
Let us consider another example whose graph is not a straight line. Can you see the difference in this example?

**Example 9** Find four solutions of  $x^2 + y^2 = 4$ .

**Solution 9** We start by considering the following ordered pairs:  $(0, \quad)$  and  $(\quad, 0)$ . We see that

$$\begin{aligned} \text{when } x = 0, \quad (0)^2 + y^2 = 4 &\implies y^2 = 4 && \text{or } y = \pm 2 \\ \text{when } y = 0, \quad x^2 + (0)^2 = 4 &\implies x^2 = 4 && \text{or } x = \pm 2. \end{aligned}$$

Hence, we have four solutions; namely,  $(0, 2)$ ,  $(0, -2)$ ,  $(2, 0)$ , and  $(-2, 0)$ . When we plot these four solutions on the same axes we obtain



**Practice 9** Find four solutions of  $x^2 + y^2 = 16$ . (Answers on page 102.)

In a later section we will see that  $x^2 + y^2 = r^2$  is the equation of a circle centered at the origin with radius  $r$ . Hence,  $x^2 + y^2 = 4$  is the circle centered at the origin with radius 2. For our last example, recall that the absolute value of  $a$ , denoted  $|a|$ , is the distance a number is from zero on the real number line.

**Example 10** Find five solutions of  $y = |x| + 3$  and plot them

**Solution 10** Again, we consider the following ordered pairs:  
 $(0, \quad)$ ,  $(1, \quad)$ ,  $(-1, \quad)$ ,  $(2, \quad)$ , and  $(-2, \quad)$ . Hence,

$$\text{when } x = 0, \quad y = |0| + 3 = 3$$

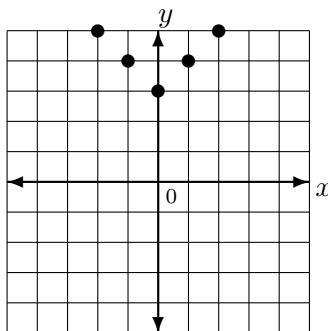
$$\text{when } x = 1, \quad y = |1| + 3 = 1 + 3 = 4$$

$$\text{when } x = -1, \quad y = |-1| + 3 = 1 + 3 = 4$$

$$\text{when } x = 2, \quad y = |2| + 3 = 2 + 3 = 5$$

$$\text{when } x = -2, \quad y = |-2| + 3 = 2 + 3 = 5$$

Therefore, the graph of these five solutions is



**Practice 10** Find five solutions of  $y = 2|x| - 3$ . (Answers on page 102.)

ANSWERS TO PRACTICE PROBLEMS

1. not a solution

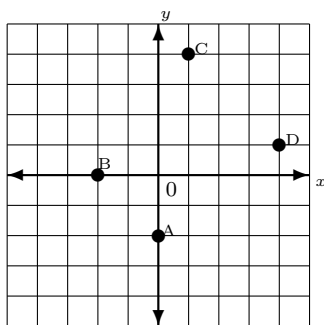
2. solution

3.  $\frac{2}{5}, \frac{2}{7}$

4.

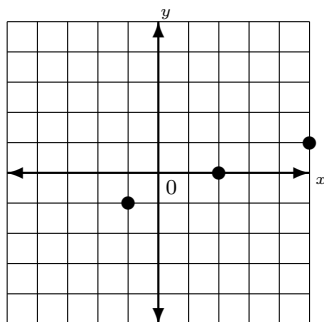
| $x$            | $y$           |
|----------------|---------------|
| $-\frac{3}{4}$ | 5             |
| $\frac{3}{2}$  | $\frac{1}{2}$ |
| 0              | $\frac{7}{2}$ |
| $\frac{7}{4}$  | 0             |

5.  $A = (0, -2)$ ,  $B = (-2, 0)$ ,  $C = (1, 4)$ , and  $D = (4, 1)$ .



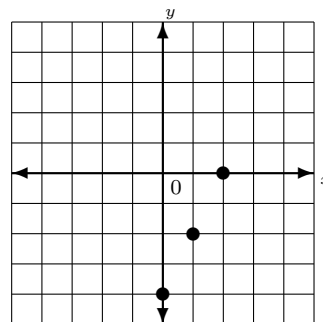
6. There are many solutions. One example is

| $x$ | $y$ |
|-----|-----|
| -1  | -1  |
| 2   | 0   |
| 5   | 1   |

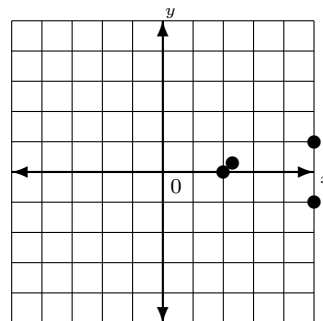


7. There are many solutions. One example is

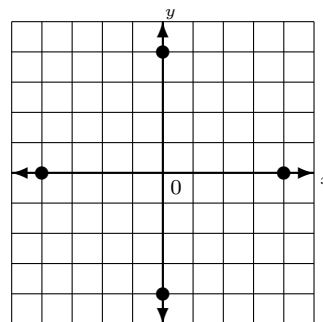
| $x$ | $y$ |
|-----|-----|
| 0   | -4  |
| 2   | 0   |
| 1   | -2  |



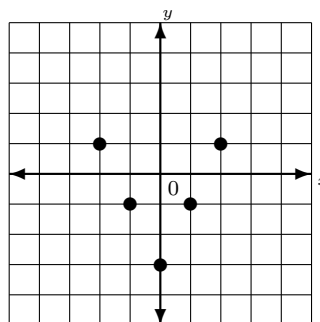
8. There are many solutions. One example is  $(2, 0)$ ,  $(5, 1)$ ,  $(5, -1)$ , and  $(\frac{7}{3}, \frac{1}{3})$ .



9. There are many solutions. One example is  $(4, 0)$ ,  $(-4, 0)$ ,  $(0, 4)$  and  $(0, -4)$ .



10. There are many solutions. One example is  $(0, -3)$ ,  $(1, -1)$ ,  $(-1, -1)$ ,  $(2, 1)$ , and  $(-2, 1)$ .




---

### SECTION 1.11 EXERCISES

(Answers are found on page [125](#).)

- Determine whether  $x = 3$  and  $y = -2$  is a solution of  $4x - 2y = 16$ .
- Determine whether  $(3, 1)$  is a solution of  $5x + 3y = 12$ .
- Complete the following ordered pairs to find solutions of  $5x - 2y = 9$ .
 

|   |                  |
|---|------------------|
| (a) $(-4, \quad)$                       | (c) $(0, \quad)$ |
| (b) $\left( \quad, \frac{1}{2} \right)$ | (d) $(\quad, 0)$ |
- Complete the following ordered pairs to find solutions of  $7x + 3y = 8$ .
 

|                   |   |
|-------------------|---|
| (a) $(\quad, -1)$ | (c) $\left( \frac{2}{7}, \quad \right)$ |
| (b) $(2, \quad)$  | (d) $\left( \quad, \frac{1}{3} \right)$ |
- Complete the following ordered pairs to find solutions of  $-6x + 5y = -7$ .
 

|                  |   |
|------------------|---|
| (a) $(0, \quad)$ | (c) $(\quad, -1)$                       |
| (b) $(2, \quad)$ | (d) $\left( \quad, \frac{2}{5} \right)$ |

6. Complete the following ordered pairs to find solutions of  $\frac{1}{2}x + \frac{4}{3}y = 2$ .

(a) ( , 0)

(c) ( , 6)

(b) (-4, )

(d)  $\left( , \frac{1}{2} \right)$

7. Given  $8x - 3y = 1$  complete the following table of values:

| $x$           | $y$ |
|---------------|-----|
| 1             |     |
| $\frac{1}{4}$ |     |
| -2            |     |

8. Given  $7x + 2 = 3y$  complete the following table of values:

| $x$           | $y$ |
|---------------|-----|
| 0             |     |
| -3            |     |
| $\frac{1}{2}$ |     |

9. Given  $-2x + 3y = 4$  complete the following table of values:

| $x$ | $y$           |
|-----|---------------|
|     | -2            |
|     | -4            |
|     | $\frac{2}{3}$ |
|     | 6             |

10. Given  $x - 3y = 12$  complete the following table of values:

| $x$ | $y$            |
|-----|----------------|
| 0   |                |
|     | 2              |
|     | $-\frac{2}{3}$ |
| 5   |                |

11. Given  $3x = 8 + 2y$  complete the following table of values:

| $x$ | $y$            |
|-----|----------------|
|     | -3             |
| -3  |                |
| 4   |                |
|     | $-\frac{5}{2}$ |

12. Plot the following points on the same coordinate plane:

$$A = (3, 1), B = (-5, 3), C = \left( \frac{1}{2}, -4 \right), D = (-2, -5)$$



13. Find three solutions of  $y = 5x - 1$  and plot them.
14. Find three solutions of  $2x + 4y = 8$  and plot them.
15. Find three solutions of  $6y = -5x + 2$  and plot them.
16. Find three solutions of  $y = 3x^2 + 2$  and plot them.
17. Find four solutions of  $x = -y^2 + 3$  and plot them.
18. Find four solutions of  $x^2 + 4y^2 = 16$  and plot them.
19. Find four solutions of  $x^2 + y^2 = 25$  and plot them.
20. Find five solutions of  $y = 3|x| - 2$  and plot them.
21. Find five solutions of  $y = 2|x - 1| + 3$  and plot them.
22. Find five solutions of  $y = -2|x + 1| + 3$  and plot them.

## 1.12 Deriving Common Formulas in Two Variables

We end Fundamental Mathematics II with a derivation of some common formulas in two variables; namely, the area formulas for a rectangle, square, triangle, and parallelogram. ([Click here to review computation with formulas from FM1](#)) The **area** of a plane figure is the number of square units it would take to fill it. The most commonly known area formula is for a rectangle. So, what is the area formula for a rectangle? Suppose we have a rectangle that measures 5 inches by 3 inches as shown below.



To find the area of this rectangle we are interested in how many square inches it would take to cover this rectangle. One square inch is illustrated by a square that is one inch on a side. Dividing our rectangle into square inches we obtain

|    |    |    |    |    |
|----|----|----|----|----|
| 1  | 2  | 3  | 4  | 5  |
| 6  | 7  | 8  | 9  | 10 |
| 11 | 12 | 13 | 14 | 15 |

Therefore, we say that the given rectangle has an area of 15 square inches. Of course, we could have the answer of 15 quicker by taking taking 5 squares along the bottom stacked 3 high gives us  $5 \times 3 = 15$  square inches. This gives us the area formula for a rectangle.

The **area of a rectangle** with length  $l$  and width  $w$  is given by

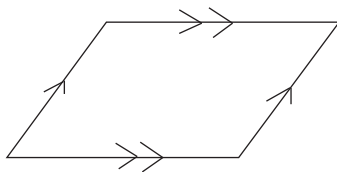
$$A_{\text{rectangle}} = l \times w = \text{length} \times \text{width}.$$

Now that we know the area of a rectangle, we can tackle the area of a square. Since every square is a rectangle, we find that the area of the square is also length  $\times$  width. However, because the square has the property that the length and the width are equal, we can convert this formula into the following:

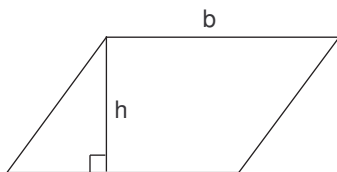
The **area of a square** with length of a side  $s$  is given by

$$A_{\text{square}} = s \cdot s = s^2.$$

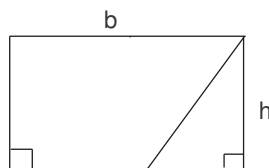
Next, we turn our attention to a parallelogram. A **parallelogram** is a polygon with two pairs of parallel sides as seen below. (Note that we use arrows to denote parallel lines.)



We start by identifying the height of the parallelogram and labelling the top side as  $b$  as shown below.



Next, we construct a triangle on the right hand side of the parallelogram that is congruent to the triangle on the left hand side. We then remove the triangle on the left hand side of the parallelogram as shown below.

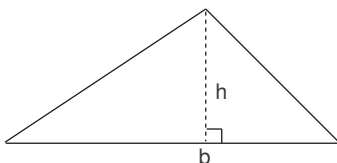


What figure have we created? That is correct, a rectangle is formed. And we know the area formula for a rectangle, it is length  $\times$  width. For this figure, we have  $b \cdot h$  or base  $\times$  height. If you don't believe the above demonstration, try it for yourself. Take a parallelogram cut from ordinary paper and perform the cut described above. You will see that the figure that you end up with is indeed a rectangle. Hence, we can summarize this formula as follows.

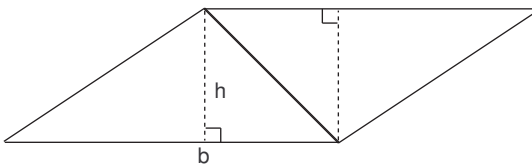
The **area of a parallelogram** with base  $b$  and height  $h$  is given by

$$A_{\text{parallelogram}} = bh = \text{base} \times \text{height}.$$

Finally, we turn our attention to a triangle. Given any triangle with height  $h$  and base  $b$



put two copies of it together along one side as follows:



The result is a parallelogram with the same base and height as the triangle but twice the area. So the area of the triangle is given by the following formula:

The **area of a triangle** with base  $b$  and height  $h$  is given by

$$A_{\text{triangle}} = \frac{1}{2}bh = \frac{1}{2}(\text{base})(\text{height}).$$

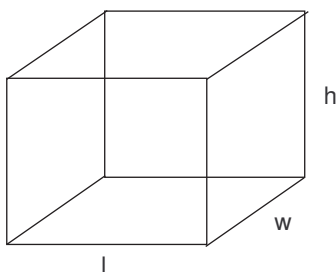
A list of these and other area formulas can be found in the appendix.

## SECTION 1.12 EXERCISES

(Answers are found on page [128](#).)

1. Show that the area formula for a triangle derived in the section is valid for an acute triangle.

2. A **trapezoid** is a four sided polygon with exactly one pair of parallel sides. Starting with a trapezoid, use the techniques discussed in this section to derive the area formula for a trapezoid.
3. Starting with a parallelogram cut it along both heights to form three figures: two right triangles and one rectangle. Use these to derive the area formula for a parallelogram.
4. The **perimeter** is the distance around a figure.
  - (a) Derive the formula for the perimeter of a rectangle.
  - (b) Derive the formula for the perimeter of a square.
5. The **surface area** of a three-dimensional figure is the total area of the faces of the figure. To find the surface area of a box we can find the area of each side and sum all of these values together.
  - (a) Use this technique on the box given below to derive a formula for the surface area of a box.



- (b) Now assume that the box has no top. What is the surface area of the box?
- (c) How would the formula change if the box had a square bottom of length  $x$ ? (Assume that the box has a top).
- (d) What would the surface area of a cube be if the length of each side is  $x$ ? (A cube is a box with all sides the same length).

**In #6–#15, solve the equation for the indicated variable.**

- |  |                                    |
|--|------------------------------------|
| 6. $A = lw$ for $l$                        | 11. $C = 2\pi r$ for $r$           |
| 7. $A = \frac{1}{2}bh$ for $h$             | 12. $V = \pi r^2 h$ for $h$        |
| 8. $A = \frac{1}{2}h(b_1 + b_2)$ for $h$   | 13. $A = \pi rs + \pi r^2$ for $s$ |
| 9. $A = \frac{1}{2}h(b_1 + b_2)$ for $b_1$ | 14. $V = \frac{1}{3}Ah$ for $A$    |
| 10. $P = 2w + 2l$ for $w$                  | 15. $V = lwh$ for $l$              |

**In #16–#27, use the formulas derived in this section to find the answer to the following exercises.**

16. Find the perimeter of a rectangle whose width is 13.3 cm and whose length is 17.9 cm.
17. If the perimeter of an equilateral triangle is 21 in, find the length of each side. (An equilateral triangle is a triangle with all sides the same length.)
18. Find the area of a triangle whose base is 14 inches and whose height is 8 inches.
19. Find the area of a square whose length of each side is 9 ft.
20. Find the area of a parallelogram whose base is 14.3 cm and whose height is 12 cm.
21. If the perimeter of a parallelogram is 120 mm, find the length of its smallest side if its longest side is 46 mm.
22. The area of a trapezoid is 400 square inches. The height of the trapezoid is 20 inches and the length of one of the bases is 28 inches. Find the length of the other base.
23. The area of a trapezoid is 900 square centimeters. If the bases are 35 cm and 55 cm, find the height of the trapezoid.
24. Find the height of a triangle whose area is 120 square centimeters and whose base is 15 cm.
25. Find the length of a rectangle whose area is 271.25 square inches and whose width is 15.5 in.

26. Find the perimeter of a triangle whose sides have length:  $x$  inches,  $(2x + 3)$  inches, and  $(x + 7)$  inches.
27. Find the area of the triangle whose base is  $(x + 3)$  cm and whose height is  $2x$  cm.





# Appendix A

## Formulas

### Area Formulas:

- Area of square:

$$A_{square} = s^2 = \text{side}^2$$

- Area of a rectangle:

$$A_{rectangle} = l \times w = \text{length} \times \text{width}$$

- Area of a triangle:

$$A_{triangle} = \frac{1}{2} \times b \times h = \frac{1}{2} \times \text{base} \times \text{height}$$

- Area of a parallelogram:

$$A_{parallelogram} = b \times h = \text{base} \times \text{height}$$

- Area of a trapezoid:

$$A_{trapezoid} = \frac{1}{2} \times h (b_1 + b_2) = \frac{1}{2} \times \text{height} \cdot (\text{sum of the bases})$$

- Area of a circle:

$$A_{circle} = \pi r^2 = \pi(\text{radius})^2$$

### Perimeter and Circumference Formulas:

- Perimeter:

$$P_{perimeter} = \text{distance around}$$

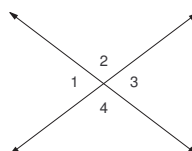
In other words, add each side together.

- Circumference of a circle:

$$C_{circle} = 2\pi r = 2\pi(\text{radius})$$

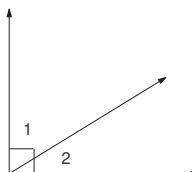
### Angle Properties:

- **Vertical Angles** are opposite angles formed by intersecting lines. Vertical angles always have the same measurement.

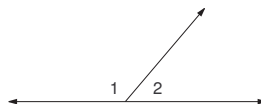


In the above figure, 1 and 3 are vertical angles; 2 and 4 are vertical angles.

- **Complementary Angles** are two angles whose sum is  $90^\circ$ .



- **Supplementary Angles** are two angles whose sum is  $180^\circ$ .



- **Corresponding Angles** have the same location relative to lines  $\ell$ ,  $m$  and transversal  $t$ .  
**(IMPORTANT:**  $\ell \parallel m$  if and only if corresponding angles formed by  $\ell$ ,  $m$ , and  $t$  are congruent.) In Figure A-1,  $\angle 1$  and  $\angle 5$  are corresponding angles. The following pairs are also corresponding angles:  $\angle 2$  and  $\angle 6$ ;  $\angle 3$  and  $\angle 7$ ;  $\angle 4$  and  $\angle 8$ .

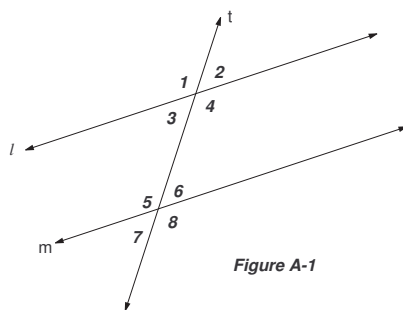


Figure A-1

- **Alternate Interior Angles** are nonadjacent angles formed by lines  $\ell$ ,  $m$ , and transversal  $t$ , the union of whose interiors contain the region between  $\ell$  and  $m$ .  
(**IMPORTANT:**  $\ell \parallel m$  if and only if alternate interior angles formed by  $\ell$ ,  $m$  and  $t$  are congruent.) In Figure A-1,  $\angle 3$  and  $\angle 6$  are alternate interior angles. Likewise,  $\angle 4$  and  $\angle 5$  are also alternate interior angles.
- **Alternate Exterior Angles** are angles on the outer sides of two lines cut by a transversal, but on opposite sides of the transversal (**IMPORTANT:**  $\ell \parallel m$  if and only if alternate exterior angles formed by  $\ell$ ,  $m$  and  $t$  are congruent.) In Figure A-1,  $\angle 2$  and  $\angle 7$  are alternate exterior angles. Similarly,  $\angle 1$  and  $\angle 8$  are alternate exterior angles.
- **Interior Angles on the same side of the transversal** are interior angles whose interiors are the same. (**IMPORTANT:**  $\ell \parallel m$  if and only if the interior angles on the same side of the transversal are supplementary.) In Figure A-1,  $\angle 3$  and  $\angle 5$ , as well as  $\angle 4$  and  $\angle 6$ , are interior angles on the same side of the transversal.



## Appendix B

# Answers to Exercises

### B.1 Counterexamples

(From page 7.)

1. There is no distributive property for addition over multiplication. Consider  $a = 2, b = 3$ , and  $c = 4$ . Then  $a + (b \cdot c) = 2 + (3 \cdot 4) = 2 + 12 = 14$  and  $(a + b) \cdot (a + c) = (2 + 3) \cdot (2 + 4) = 5 \cdot 6 = 30$ .
2. False; Let  $a = 3, b = 4$ , and  $c = 2$ . Then  $a(b \div c) = 3(4 \div 2) = 3(2) = 6$  and  $(ab) \div (ac) = (3 \cdot 4) \div (3 \cdot 2) = 12 \div 6 = 2$ .
3. True
4. False; Let  $a = 8, b = 2$ , and  $c = 4$ . Then  $a \div (b \cdot c) = 8 \div (2 \cdot 4) = 8 \div 8 = 1$ . However,  $a \div b \cdot c = 8 \div 2 \cdot 4 = 4 \cdot 4 = 16$ .
5. True
6. False; Let  $a = 16, b = 8$  and  $c = 2$ . Then  $a \div b \div c = 16 \div 8 \div 2 = 2 \div 2 = 1$ . However,  $a \div (b \div c) = 16 \div (8 \div 2) = 16 \div 4 = 4$ .
7. True
8. True
9. True
10. False; Let  $a = 2$  and  $x = 1$ . Then  $a(x+1)(x-2) = 2(1+1)(1-2) = 2(2)(-1) = -4$  and  $(ax+a)(ax-2a) = (2+2)(2-4) = 4(-2) = -8$ .
11. False; Let  $a = -7$  and  $b = 3$ . Then  $|a+b| = |-7+3| = |-4| = 4$ . However,  $|a|+|b| = |-7|+|3| = 7+3 = 10$ .
12. False; Let  $a = 2, b = 3$ , and  $c = 4$ . Then  $a(b \cdot c) = 2(3 \cdot 4) = 2 \cdot 12 = 24$ . However,  $(a \cdot b)(a \cdot c) = (2 \cdot 3)(2 \cdot 4) = 6 \cdot 8 = 48$ .
13. True
14. True
15. False; Let  $b = 1$ . Then  $(2b)^3 = (2 \cdot 1)^3 = 2^3 = 8$ . However,  $2b^3 = 2 \cdot (1)^3 = 2 \cdot 1 = 2$ .
16. False; Let  $a = 2$  and  $b = 1$ . Then  $(ab)^2 = (2 \cdot 1)^2 = 2^2 = 4$ . However,  $ab^2 = 2 \cdot 1^2 = 2 \cdot 1 = 2$ .

17. (a) Let  $a = 1$  and  $b = 2$ . Then  
 $(a + b)^2 = (1 + 2)^2 = 3^2 = 9$ ,  
 but  $a^2 + b^2 = 1^2 + 2^2 =$
- $1 + 4 = 5$ .  
 (b) True if  $n = 1$ .  
 (c) True if  $a = 0$  or  $b = 0$ .

## B.2 Integer Exponents

(From page 21.)

- |                         |                                   |
|-------------------------|-----------------------------------|
| 1. (a) $3^7$            | 23. $15^{14}$                     |
| (b) $10^3$              | 24. $3^{22}$                      |
| 2. $-1$                 | 25. $x^7y^4$                      |
| 3. $1$                  | 26. $8x^4$                        |
| 4. $\frac{1}{81}$       | 27. $49x^{10}$                    |
| 5. $\frac{1}{64}$       | 28. $64x^6$                       |
| 6. $81$                 | 29. $27x^{12}y^{15}$              |
| 7. $-64$                | 30. $16x^8y^{20}z^{36}$           |
| 8. $8$                  | 31. $8x^9$                        |
| 9. $\frac{1}{16}$       | 32. $-24x^{15}$                   |
| 10. $-\frac{1}{4}$      | 33. $6b^9$                        |
| 11. $\frac{1}{4}$       | 34. $\frac{x^9y^8}{6}$            |
| 12. $25$                | 35. $\frac{9x^8}{4y^{10}}$        |
| 13. $49$                | 36. $xy^3$                        |
| 14. $\frac{5}{16}$      | 37. $x^{13}y^8$                   |
| 15. $\frac{-5}{36}$     | 38. $x^{10}y^8$                   |
| 16. (a) $81$            | 39. $\frac{6}{x}$                 |
| (b) $10^7 = 10,000,000$ | 40. $\frac{12}{x^2y}$             |
| 17. $x^6$               | 41. $\frac{10a}{b^4}$             |
| 18. $2^3$               | 42. $\frac{8y^9z^{15}}{x^6}$      |
| 19. $2^4$               | 43. $\frac{16x^6}{y^{10}z^2}$     |
| 20. $5^7$               | 44. $\frac{z^6}{9x^8y^4}$         |
| 21. $6^7$               | 45. $\frac{x^{12}z^{18}}{125y^9}$ |
| 22. $2^{41}$            | 46. $\frac{3x^6}{y^3}$            |
|                         | 47. $\frac{6x^{11}}{y^8z^7}$      |

48.  $\frac{b^8}{a^4c^2}$
49.  $\frac{27x^6}{y^9}$
50.  $\frac{x^{20}z^{16}}{y^{28}}$
51.  $\frac{4}{x^2y}$
52.  $\frac{4y^6z^{10}}{x^6}$
53.  $\frac{16x^{20}}{81y^8}$
54.  $\frac{2x^{12}}{9y^{23}}$
55.  $\frac{-8y^{24}}{x^{27}z^3}$
56.  $\frac{3y^{12}}{4x^6}$
57.  $\frac{27y^6z^8}{4x^{15}}$
58. Answers vary;  $2^{13} \neq 2^6$
59. Not equal.  $(3^4)^2 = 3^{4 \cdot 2} = 3^8$ ;  
whereas,  $3^{4^2} = 3^{16}$ .
60.  $(3+4)^3 \neq 3^3 + 4^3$  since  $(3+4)^3 = 7^3 = 343$  and  $3^3 + 4^3 = 27 + 64 = 91$ . In order for  $(a+b)^3 = a^3 + b^3$  we need either  $a$  or  $b$  to be zero. Furthermore, the only value of  $n$  for which  $(a+b)^n = a^n + b^n$  is when  $n = 1$ .
61. (a)  $4^{28} = (2^2)^{28} = 2^{56}$  and  $8^{18} = (2^3)^{18} = 2^{54}$ . Since  $56 > 54$ ,  $4^{28} > 8^{18}$ .
- (b)  $3^9 + 3^9 + 3^9 = 3 \cdot 3^9 = 3^{10}$  and  $9^6 = (3^2)^6 = 3^{12}$ . Since  $12 > 10$ ,  $9^6 > 3^9 + 3^9 + 3^9$ .
- (c)  $27^9 = (3^3)^9 = 3^{27}$  and  $9^{14} = (3^2)^{14} = 3^{28}$ . Since  $28 > 27$ ,  $9^{14} > 27^9$ .
- (d)  $6^{18} = (2 \cdot 3)^{18} = 2^{18} \cdot 3^{18}$  and  $3^{36} = 3^{18} \cdot 3^{18}$ . Since  $2^{18} < 3^{18}$ ,  $6^{18} < 3^{36}$ .

## B.3 Scientific Notation

(From page 28.)

- 946,000,000,000,000,000
- 1,810,000
- 0.0000000001
- 0.0043
- 0.000032
- 578,900,000
- 693,000,000,000
- 0.0000000034
- 27,500,000
- 800,300
- 0.00000902
- 0.0005486
- $8.34 \times 10^3$
- $7.9 \times 10^{-4}$
- $4.2 \times 10^{-5}$
- $4.36 \times 10^6$
- $3.005 \times 10^{-3}$
- $7.201 \times 10^{-6}$
- $2.94 \times 10^8$
- $3.45 \times 10^{-7}$
- $7.03 \times 10^{-8}$
- $6.6 \times 10^9$
- $2.35 \times 10^{12}$
- $5.32 \times 10^{-10}$
- $7.2 \times 10^{-12}$
- $2.236 \times 10^5$

- |                             |                          |
|-----------------------------|--------------------------|
| 27. $8.82 \times 10^{10}$   | 32. $4 \times 10^{-12}$  |
| 28. $8.7822 \times 10^2$    | 33. $2 \times 10^{12}$   |
| 29. $3.0804 \times 10^{-2}$ | 34. $2.3785 \times 10^3$ |
| 30. $6.4 \times 10^6$       | 35. $3 \times 10^{-7}$   |
| 31. $1.6 \times 10^{12}$    |                          |

## B.4 Polynomials

(From page 35.)

- |   |  |
|---|--|
| 1. Yes. One example: $(3x^3 + 7x^2 - 5) + (-3x^3 + 8x - 2) = 7x^2 + 8x - 7$ . | 24. $24x^2 + 8x - 2$   |
| 2. Yes.   | 25. $4x^2 - 9$   |
| 3. No. A monomial times a binomial will always give a binomial.               | 26. $16x^2 - 1$  |
| 4. 5th degree; $m + n$ degree   | 27. $25x^2 - 4$  |
| 5. $-25$  | 28. $9x^2 - 16$  |
| 6. $-\frac{23}{2}$  | 29. $\frac{9}{2}x^2 - \frac{78}{5}x + 2$   |
| 7. 6  | 30. $6x^2 - \frac{7}{12}x - \frac{1}{6}$   |
| 8. 35   | 31. $49x^2 - 28x + 4$  |
| 9. $-15$  | 32. $16x^2 + 24x + 9$  |
| 10. $11x^2 - 3x - 11$   | 33. $4x^2 - 12x + 9$   |
| 11. $-2x^2 - 8x + 6$  | 34. $x^3 - 6x^2 + 12x - 8$   |
| 12. $x^3 - 10x^2 + 22x - 5$   | 35. $8x^3 + 12x^2 + 6x + 1$  |
| 13. $7x^3 - 11x^2 + 6x - 9$   | 36. $12x^3 - 17x^2 - 13x - 2$  |
| 14. $12x^2 - 8x$  | 37. $2x^3 - 11x^2 + 10x + 8$   |
| 15. $30x^3 + 15x^2$   | 38. $21x^3 - 19x^2 - 12x$  |
| 16. $-8x^3 + 6x^2 - 10x$  | 39. $16x^4 + 28x^3 - 30x^2$  |
| 17. $15x^4 - 6x^3 - 21x^2$  | 40. $12x^3 + 2x^2 - 24x$   |
| 18. $-15x^3y^3 + 21x^3y^2 - 9x^4y$  | 41. $10x^3 - 19x^2 + 26x - 8$  |
| 19. $-10x^4y^6 + 6x^3y^7 - 4x^5y^8 + 14x^2y^5$                                | 42. $6x^4 - 22x^3 + 41x^2 - 41x + 10$  |
| 20. $4x^2 + 4x - 15$  | 43. (a) $6x^4 - 28x^3 + 21x^2 + 21x - 10$  |
| 21. $15x^2 - 22x + 8$   | (b) $2x^4 - x^3 - 24x^2 + 16x + 7$   |
| 22. $28x^2 + 13x - 6$   | (c) $18x^4 - 15x^3 - 13x^2 - 2x - 12$  |
| 23. $10x^2 - x - 3$   | 44. Let $x = 1$ . Then $x + 2(x - 3) = 1 + 2(1 - 3) = 1 + 2(-2) = 1 + (-4) = -3$ . However, $(x + 2)(x - 3) = (1 + 2)(1 - 3) = (3)(-2) = -6$ . |



## B.5 Linear Equations

(From page 46.)

1. No. Move all terms containing a variable to one side of the equation. Answer should be  $x = 0$
2. Answers vary. One example:  $4x - 6 = 2x + 10$
3. Answers vary
4.  $x = 8/3$
5.  $x = 20$
6.  $x = 3$
7.  $x = -23$
8.  $x = -14$
9. All real numbers
10.  $x = 7$
11.  $x = -4.6$
12.  $x = -4$
13. No solution
14.  $x = -\frac{4}{7}$
15.  $x = \frac{35}{76}$
16.  $x = \frac{92}{33}$
17. No solution
18.  $x = \frac{47}{7}$
19.  $x = \frac{51}{7}$
20.  $x = \frac{21}{17}$
21.  $x = \frac{3}{5}$
22.  $x = \frac{13}{4}$
23.  $x = \frac{38}{23}$
24.  $x = \frac{53}{29}$
25.  $x = \frac{68}{7}$
26.  $x = \frac{29}{15}$
27.  $x = -\frac{1}{3}$
28. All real numbers
29.  $x = -\frac{24}{5}$
30.  $x = -\frac{7}{5}$
31. No solution
32.  $x = -\frac{13}{8}$
33.  $x = -1$
34.  $y = 0.2$
35.  $x = \frac{-a+1}{a^2-a-1}$

## B.6 Geometry

(From page 56.)

1. -16
2. 7 and 8
3. -2
4. 12, 24, 45
5. 22, 24
6. 11, 12, 13
7. 5 feet and 12 feet
8. 152 adults
9. Savanna pitched 3 games, Kayla pitched 15 games, Cheyenne pitched 10 games
10. Roger sold 15, Will sold 24, Jacob sold 27

- |   |                                |
|---|--------------------------------|
| 11. cashier for 3.5 hours, stocked shelves for 2.75 hours, trained new employees for 1.75 hours | 17. each angle is $139^\circ$  |
| 12. $w = 5$ , $l = 13$  | 18. $48^\circ$ and $132^\circ$ |
| 13. width is 14 ft, length is 49 ft   | 19. $54^\circ$ and $126^\circ$ |
| 14. $x = 25$ ft, $y = 50$ ft  | 20. $48^\circ$ and $42^\circ$  |
| 15. 25 ft by 25 ft  | 21. each angle is $73^\circ$   |
| 16. (a) 3 gallons for one coat.   | 22. $35^\circ$ and $55^\circ$  |
| (b) 5 gallons for two coats.  | 23. $85^\circ$                 |
|   | 24. $65^\circ$                 |

## B.7 Percents

(From page 65.)

- |  |  |
|--|--|
| 1. They are the same.  | 15. 15%  |
| 2. They are the same.  | 16. \$13,200   |
| 3. Job 2   | 17. \$34,170.74  |
| 4. Discount of $\frac{1}{4}$ is the same as paying $\frac{3}{4}$ of the original amount. | 18. \$9,910  |
| 5. Candidate B won by one vote   | 19. 130 games  |
| 6. \$1500  | 20. 15.56%   |
| 7. $100^\circ$ , $50^\circ$ , $30^\circ$   | 21. 32%  |
| 8. \$6467.89   | 22. \$149.45   |
| 9. 125 square feet   | 23. (a) \$637.50                                       |
| 10. \$230.50   | (b) \$1,912.50   |
| 11. \$21.18  | (c) \$318.75   |
| 12. \$23.08  | 24. (a) interest is \$480; account balance is \$12,480 |
| 13. \$233.16   | (b) \$12,979.20  |
| 14. \$180  | (c) \$19.20  |
|  | 25. 6.5%   |

## B.8 Proportions

(From page 71.)

1. (a) 16 ounces
- (b) 24 ounces
- (c) 11 ounces
2.  $4\frac{3}{8}$  cups
3. \$62.25
4.  $4\frac{1}{6}$  cups of flour
5. Missed 16 goals
6. (a)  $2\frac{1}{2}$
- (b)  $\frac{5}{8}$
- (c)  $1\frac{9}{16}$
7. 7 inches
8. 2800 markers
9.  $2\frac{1}{4}$  cups
10. 110 miles
11. 85.5 inches
12. 75 teachers
13. 960 female students
14. 500 fish
15. 7 lbs cement, 21 lbs gravel
16. 21 lbs
17. 20.32 cm
18. 35 gallons
19. 14 lbs
20. \$1,174.90

## B.9 Linear Inequalities

(From page 80.)

1.  $[-3, \infty)$
2.  $(2, 8]$
3.  $(-\infty, 7)$
4.  $[3, \frac{11}{2}]$
5.  $[\frac{3}{4}, \infty)$
6.  $(-\infty, 14]$
7.  $x \geq -5$ ;  $[-5, \infty)$
8.  $x \geq -\frac{2}{5}$ ;  $[-\frac{2}{5}, \infty)$
9.  $x \geq 42$   $[42, \infty)$
10.  $x \leq 8$ ;  $(-\infty, 8]$
11.  $x < \frac{19}{4}$ ;  $(-\infty, \frac{19}{4})$
12.  $x \geq -\frac{7}{4}$ ;  $[-\frac{7}{4}, \infty)$
13.  $x < 6$ ;  $(-\infty, 6)$
14.  $x \leq \frac{5}{4}$ ;  $(-\infty, \frac{5}{4}]$
15.  $x \leq \frac{71}{150}$ ;  $(-\infty, \frac{71}{150}]$
16.  $x > -\frac{16}{13}$ ;  $(-\frac{16}{13}, \infty)$
17.  $x \leq 42$ ;  $(-\infty, 42]$
18.  $x \leq 20$ ;  $(-\infty, 20]$
19.  $k \leq -6$ ;  $(-\infty, -6]$
20.  $p < \frac{76}{11}$ ;  $(-\infty, \frac{76}{11})$
21.  $3 \leq x \leq 4$ ;  $[3, 4]$
22.  $\frac{8}{15} \leq x < \frac{23}{15}$ ;  $[\frac{8}{15}, \frac{23}{15})$
23.  $\frac{4}{3} \leq x < \frac{25}{6}$ ;  $[\frac{4}{3}, \frac{25}{6})$
24.  $-\frac{5}{3} < x \leq \frac{25}{6}$ ;  $(-\frac{5}{3}, \frac{25}{6}]$
25.  $-\frac{18}{5} \leq x < \frac{38}{5}$ ;  $[-\frac{18}{5}, \frac{38}{5})$

26.  $\frac{23}{12} \leq x < \frac{37}{8}$ ;  $[\frac{23}{12}, \frac{37}{8})$  need to be less than \$8,333.33.
27. They would need to drive more than 2800 miles.
28. The average weekly sales would
29. (a) when  $3 < h \leq 4$ .  
(b) when  $h > 9$   
(c) when  $h \leq 6$

## B.10 Nonlinear Equations and Inequalities

(From page 91.)

1.  $x = 12$  or  $x = -12$
2.  $x = 5$  or  $x = -5$
3.  $x = \frac{1}{2}$  or  $x = -\frac{1}{2}$
4.  $x = 8$  or  $x = -8$
5.  $x = \frac{1}{2}$  or  $x = -\frac{1}{2}$
6.  $x = 8$  or  $x = -8$
7.  $x = 2$
8.  $x = \frac{1}{3}$
9.  $x = \frac{3}{5}$
10.  $x = \frac{3}{20}$
11.  $x = 50$
12.  $x = \frac{4}{3}$
13.  $x = 36$
14.  $x = 45$
15.  $x = 64$
16.  $x = \frac{8}{3}$  or  $x = \frac{4}{3}$
17.  $x = -2$  or  $x = -5$
18.  $x = -5$  or  $x = 4$
19.  $x = -30$  or  $x = 25$
20.  $x = -\frac{2}{3}$  or  $x = \frac{16}{3}$
21.  $x < -8$  or  $x > 8$   
 $(-\infty, -8) \cup (8, \infty)$
22.  $-10 \leq x \leq 10$   
 $[-10, 10]$
23.  $x \leq -5$  or  $x \geq 5$   
 $(-\infty, -5] \cup [5, \infty)$
24.  $-8 < x < 8$   
 $(-8, 8)$
25.  $x < -2$  or  $x > 2$   
 $(-\infty, -2) \cup (2, \infty)$
26.  $x < 0$  or  $x > \frac{1}{7}$   
 $(-\infty, 0) \cup (\frac{1}{7}, \infty)$
27.  $0 < x \leq \frac{1}{5}$   
 $(0, \frac{1}{5}]$
28.  $-\frac{1}{4} \leq x < 0$   
 $[-\frac{1}{4}, 0)$
29.  $0 < x \leq \frac{1}{3}$   
 $(0, \frac{1}{3}]$
30.  $x \geq 12$   
 $[12, \infty)$
31.  $0 \leq x < \frac{25}{6}$   
 $[0, \frac{25}{6})$
32.  $x \geq 32$   
 $[32, \infty)$
33.  $x \geq 64$   
 $[64, \infty)$
34.  $x \geq \frac{16}{7}$   
 $[\frac{16}{7}, \infty)$
35.  $x \leq -7$  or  $x \geq 7$   
 $(-\infty, -7] \cup [7, \infty)$
36.  $-4 < x < 4$   
 $(-4, 4)$

37.  $-\frac{1}{2} \leq x \leq 2$   
 $[-\frac{1}{2}, 2]$

38.  $x > \frac{5}{6}$  or  $x < -\frac{11}{6}$   
 $(-\infty, -\frac{11}{6}) \cup (\frac{5}{6}, \infty)$

39.  $x \leq -1$  or  $x \geq \frac{9}{5}$   
 $(-\infty, -1] \cup [\frac{9}{5}, \infty)$

40.  $-\frac{5}{9} < x < \frac{1}{9}$   
 $(-\frac{5}{9}, \frac{1}{9})$

**B.11 Linear Equations in Two Variables**

(From page 103.)

1. solution

2. not a solution

3. (a)  $(-4, -\frac{29}{2})$ (b)  $(2, \frac{1}{2})$ (c)  $(0, -\frac{9}{2})$ (d)  $(\frac{9}{5}, 0)$ 4. (a)  $(\frac{11}{7}, -1)$ (b)  $(2, -2)$ (c)  $(\frac{2}{7}, 2)$ (d)  $(1, \frac{1}{3})$ 5. (a)  $(0, -\frac{7}{5})$ (b)  $(2, 1)$ (c)  $(\frac{1}{3}, -1)$ (d)  $(\frac{3}{2}, \frac{2}{5})$ 6. (a)  $(4, 0)$ (b)  $(-4, 3)$ (c)  $(-12, 6)$ (d)  $(\frac{8}{3}, \frac{1}{2})$ 

7.

| $x$           | $y$             |
|---------------|-----------------|
| 1             | $\frac{7}{3}$   |
| $\frac{1}{4}$ | $\frac{1}{3}$   |
| -2            | $-\frac{17}{3}$ |

8.

| $x$           | $y$             |
|---------------|-----------------|
| 0             | $\frac{2}{3}$   |
| -3            | $-\frac{19}{3}$ |
| $\frac{1}{2}$ | $\frac{11}{6}$  |

9.

| $x$ | $y$           |
|-----|---------------|
| -5  | -2            |
| -8  | -4            |
| -1  | $\frac{2}{3}$ |
| 7   | 6             |

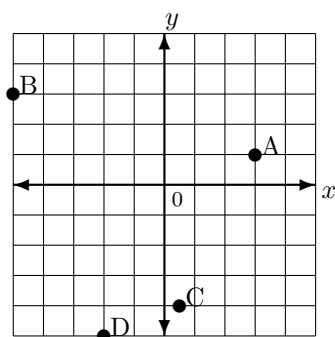
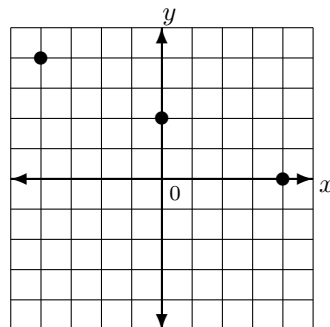
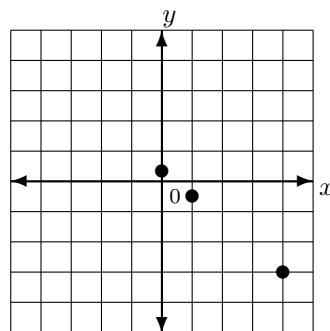
10.

| $x$ | $y$            |
|-----|----------------|
| 0   | -4             |
| 18  | 2              |
| 10  | $-\frac{2}{3}$ |
| 5   | $-\frac{7}{3}$ |

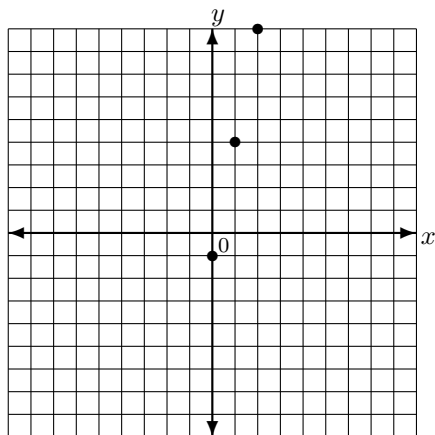
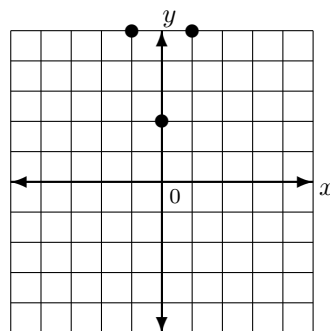
11.

| $x$           | $y$             |
|---------------|-----------------|
| $\frac{2}{3}$ | -3              |
| -3            | $-\frac{17}{2}$ |
| 4             | 2               |
| 1             | $-\frac{5}{2}$  |

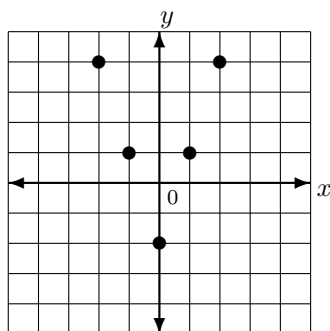
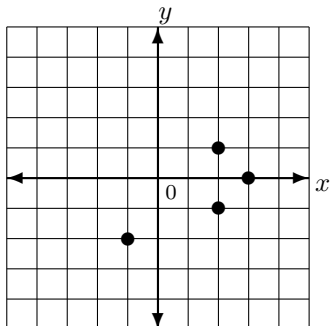
12.

14.  $(0, 2)$ ,  $(4, 0)$ ,  $(-4, 4)$ 15.  $(0, 1/3)$ ,  $(1, -1/2)$ ,  $(4, -3)$ 

For exercises #13–#22 there are many solutions. Given below is one example for each problem.

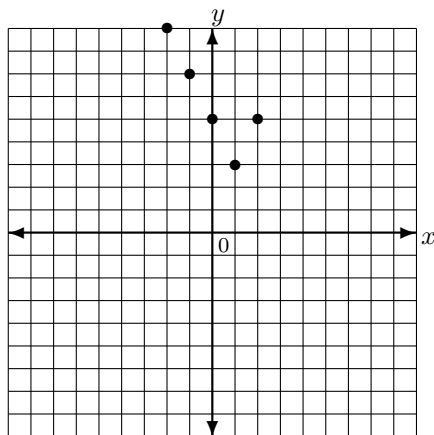
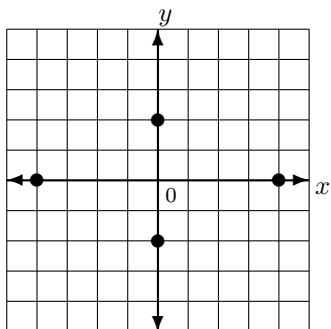
13.  $(0, -1)$ ,  $(1, 4)$ ,  $(2, 9)$ 16.  $(0, 2)$ ,  $(-1, 5)$ ,  $(1, 5)$ 

- 17.
- $(3, 0)$
- ,
- $(2, 1)$
- ,
- $(2, -1)$
- ,
- $(-1, -2)$

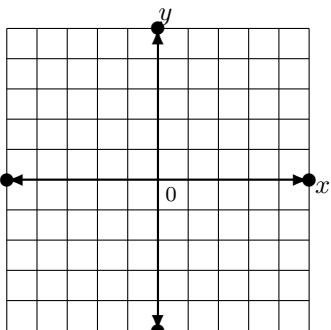


- 21.
- $(0, 5)$
- ,
- $(-1, 7)$
- ,
- $(1, 3)$
- ,
- $(2, 5)$
- ,
- $(-2, 9)$

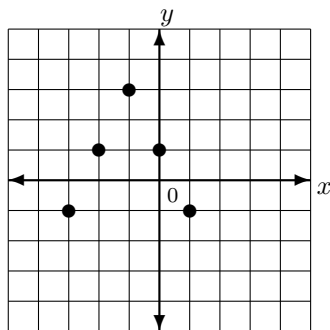
- 18.
- $(0, 2)$
- ,
- $(0, -2)$
- ,
- $(-4, 0)$
- ,
- $(4, 0)$



- 19.
- $(0, 5)$
- ,
- $(0, -5)$
- ,
- $(-5, 0)$
- ,
- $(5, 0)$



- 22.
- $(-1, 3)$
- ,
- $(0, 1)$
- ,
- $(-2, 1)$
- ,
- $(1, -1)$
- ,
- $(-3, -1)$



- 20.
- $(0, -2)$
- ,
- $(1, 1)$
- ,
- $(-1, 1)$
- ,
- $(2, 4)$
- ,
- $(-2, 4)$

**B.12 Common Formulas**

(From page 108.)

- |   |                                     |
|---|-------------------------------------|
| 1. $A = \frac{1}{2}bh$  | 12. $h = \frac{V}{\pi r^2}$         |
| 2. $A = \frac{1}{2}h(b_1 + b_2)$ where $h$ is the height of the trapezoid and $b_1$ and $b_2$ are the bases of the trapezoid. | 13. $s = \frac{A - \pi r^2}{\pi r}$ |
| 3. $A = b \times h$   | 14. $A = \frac{3V}{h}$              |
| 4. (a) $P = 2l + 2w$  | 15. $l = \frac{V}{wh}$              |
| (b) $P = 4s$  | 16. 62.4 cm                         |
| 5. (a) $S = 2lw + 2lh + 2wh$  | 17. 7 in                            |
| (b) $S = lw + 2lh + 2wh$  | 18. 56 in <sup>2</sup>              |
| (c) $S = 2x^2 + 4xh$  | 19. 81 ft <sup>2</sup>              |
| (d) $S = 6x^2$  | 20. 171.6 cm <sup>2</sup>           |
| 6. $l = \frac{A}{w}$  | 21. 14 mm                           |
| 7. $h = \frac{2A}{b}$   | 22. 12 in                           |
| 8. $h = \frac{2A}{b_1 + b_2}$   | 23. 20 cm                           |
| 9. $b_1 = \frac{2A}{h} - b_2$   | 24. 16 cm                           |
| 10. $w = \frac{P - 2l}{2}$  | 25. 17.5 in                         |
| 11. $r = \frac{C}{2\pi}$  | 26. $(4x + 10)$ in                  |
|   | 27. $(x^2 + 3x)$ cm <sup>2</sup>    |



# Index

- absolute value, 84
  - equation, 84
  - inequalities, 89
- angles
  - complementary, 54
  - supplementary, 53
  - vertical, 53
- area, 106
  - parallelogram, 108
  - rectangle, 106
  - square, 107
  - trapezoid, 109
  - triangle, 108
- Cartesian coordinate system, 96
- complementary angles, 54
- counterexamples, 5
- exponent, 9
  - base, 9
- exponent rule
  - converting from negative to positive exponents, 17
  - negative exponent, 15
  - power rule, 12
  - product rule, 10
  - quotient rule, 18
  - raising a product to a power, 13
  - raising a quotient to a power, 14
  - zero exponent, 15
- interest
  - compound, 62
  - principal, 62
  - simple, 62
- Interval notation, 74
- linear equation, 38
  - Addition Property of Equality, 38
  - clearing fractions, 41
  - conditional, 46
  - identity, 46
  - Multiplication Property of Equality, 38
  - two variables, 93
- linear equations
  - decimals, 43
- linear inequality, 74
  - Addition Property of Inequality, 75
  - compound inequality, 77
  - Multiplication Property of Inequality, 75
- origin, 96
- parallelogram, 107
- percent, 59
- polynomials, 30
  - addition, 30
  - multiplication, 31
- proportion, 69
- quadrant, 96
- ratio, 68
- rectangular coordinate system, 96
- scientific notation, 25
- supplementary angles, 53
- trapezoid, 109
- vertical angles, 53