

MAT 350 - Applied Linear Algebra

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October 30, 2017

Contents

1	Class 1 - Thursday, September 19th, 2017	5
1.1	§1 Differential Equations	5
2	Class UNK - Monday, October 23rd, 2017	7
2.1	§2.8 Subspaces	7
3	Test 01 Corrections	11
4	Class UNK - Monday, October 23rd, 2017	15
4.1	§2.8 Subspaces	15
5	Class UNK - Monday, October 30rd, 2017	17
5.1	§2.9 Unknown	17

Class 1 - Thursday, September 19th, 2017**1.1 §1 Differential Equations**

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$$

Class UNK - Monday, October 23rd, 2017

2.1 §2.8 Subspaces

Definition 1: Subspace

A subspace of \mathbb{R}^n is any set H in that has three properties:

- $\mathbf{0}$ is in H
- For each u, v , in H $u+v$ is also in H (Closure under addition)
- For each u in H and scalar c , cu is in H (Closure under scalar multiplication)

Example 1 *Is H a subspace of \mathbb{R}^2 ?*

- $\mathbf{0}$ is in H ✓
- $u+v$ is not in H for all u, v in H
- Not closed under scalar multiplication either ($-u$ not in H).

H is not a subspace

Example 2 • $\mathbf{0}$ is in H ✓

- closed under addition ✓

- closed under scalar multiplication ✓

H is a subspace of \mathbb{R}^2

Example 3 Which of the following are subspaces of \mathbb{R}^2 ?

- not a subspace o is not in H
- \mathbb{R}^2 is a subspace of \mathbb{R}^2
- \mathbb{R}^2 is a subspace of \mathbb{R}^2
- $\diamond o$ is in H ✓
 - \diamond Closed under scalar multiplication ✓
 - \diamond Not closed under addition $x + (-v)$ is not in H .
- not a subspace of \mathbb{R}^2
- stuff

So what are do subspaces of \mathbb{R}^2 look like?

- They are copies of $\mathbb{R}^0, \mathbb{R}^1, \mathbb{R}^2 \dots \mathbb{R}^n$ that contain the zero vector.

Example 4 In \mathbb{R}^3 , possible subspaces are:

- Zero Subspaces
- Lines
- Planes

Example 5 If $H = \text{span}\{v_1, v_2\}$ ($av_1 + bv_2$ for any a, b), then H is a subspace of \mathbb{R}^n

- o is in H ✓
 - \diamond since $0 \cdot v_1 + 0 \cdot v_2 = o$
- closed under addition ✓
 - $\diamond u = a_1v_1 + b_1v_2$
 - $\diamond w = a_2v_1 + b_2v_2$

$$\diamond u + w = a_1v_1 + b_1v_2 + a_2v_1 + b_2v_2$$

$$\diamond u + w = (a_1 + a_2)v_1 + (b_1 + b_2)v_2$$

$$\diamond \text{ which is } \text{span}v_1, v_2$$

- *closed under scalar multiplication*

$$\diamond u = av_1 + bv_2$$

$$\diamond c * u = c(av_1 + bv_2) = (c * a)v_1 + (c * b)v_2$$

$$\text{which is } \text{span}v_1, v_2$$

Definition 2: Column Space

The column space of a matrix A, denoted $\text{col}(A)$, is the set of all linear combinations of the columns of A. ($\text{col}(A)$ is a subspace)

Example 6 Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$

is b in $\text{Col}(A)$?

if b is a linear combination of the columns of A , the $Ax=b$ has a solution. We are asking whether $[A|b]$ is consistent or not.

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{array} \right] \text{ goes to } \left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

No pivot in augmented column, so b is in $\text{col}(A)$.

Note: b in $\text{col}(A)$ for every b in \mathbb{R}^m

- $Ax=b$ has a solution for every b in \mathbb{R}^m
- The columns of A span \mathbb{R}^m
- A has a pivot in every row in REF

Definition 3: Null Space

The null space of matrix A, denoted $\text{null}(A)$, is the set of all solutions to $Ax=\vec{0}$

Note 1 Any solution set of $Ax=\vec{0}$ can be written in parametric form.

$$x = x_2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \text{span}\left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

so $\text{null}(A)$ is a subspace.

Definition 4: Basis for a Subspace

A basis for a subspace H is a linearly independent set that spans H .

Example 7 Find a basis for $\text{null}(A)$, if $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$

$$A = \left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & -3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_1 - 2x_2 - x_4 + 3x_5 = 0$$

$$x_2 = x_2$$

$$x_3 + 2x_4 - 2x_5 = 0$$

$$x_4 = x_4$$

$$x_5 = x_5$$

Test 01 Corrections

Problem 1 10pts

Sub-Problem A Correct

Sub-Problem B Correct

Problem 2 $A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix}$

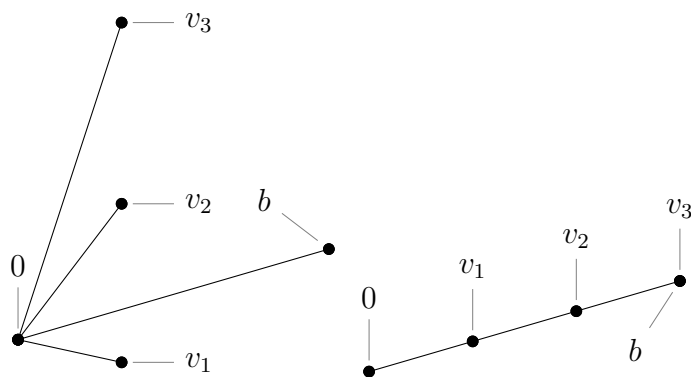
$$\begin{aligned} & \begin{bmatrix} 1 & 2 & 1 & | & 0 \\ -3 & -1 & 2 & | & 5 \\ 0 & 5 & 3 & | & -1 \end{bmatrix} \\ r_2 \rightarrow r_2 + 3r_1 & \begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 0 & 5 & 5 & | & 5 \\ 0 & 5 & 3 & | & -1 \end{bmatrix} \\ r_2 \rightarrow r_2/5 & \begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 0 & 1 & 1 & | & 1 \\ 0 & 5 & 3 & | & -1 \end{bmatrix} \\ r_3 \rightarrow r_3 - 5r_2 & \begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & -2 & | & -6 \end{bmatrix} \\ r_3 \rightarrow r_3/-2 & \begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & 1 & | & 3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
 r_1 &\rightarrow r_1 - 2r_2 & \begin{bmatrix} 1 & 0 & -1 & | & -2 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & 1 & | & 3 \end{bmatrix} \\
 r_1 &\rightarrow r_1 + r_3 & \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & 1 & | & 3 \end{bmatrix} \\
 r_2 &\rightarrow r_2 - r_3 & \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & -2 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}
 \end{aligned}$$

Yes, B is in the subset spanned by the columns of A .

$$0 \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

Problem 3 Yes, the figure does have a solution; if you express the vectors as a linear combination $x_1v_1 + x_2v_2 + x_3v_3 = b$ you can change the values of the scalars (x_1, x_2, x_3) to any value which, when multiplied by the vectors will give us any b ; which causes the answer to be non-unique.



Problem 4 10pts

Sub-Problem A

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
r_1 &\rightarrow r_1 - r_2 \begin{bmatrix} 1 & 0 & -5 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \\
r_2 &\rightarrow r_2/3 \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \\
1x_1 - 5x_3 &= 0 \\
1x_2 + 2x_3 &= 0 \\
x_1 &= 5x_3 \\
x_2 &= -2x_3 \\
x_3 &= x_3 \\
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 5x_3 \\ -2x_3 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 5x_3 \\ -2x_3 \\ 0 \end{bmatrix} &= x_3 \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}
\end{aligned}$$

Sub-Problem B

$$0 \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Sub-Problem C Yes, x_3 is a free variable, any value will have a solution.

Problem 5 30pts

Sub-Problem A Correct

Sub-Problem B Yes, because there is no way to reduce the matrices to be in the other matrix.

Sub-Problem C No, it is linearly dependent because v_1, v_2 are dependant.

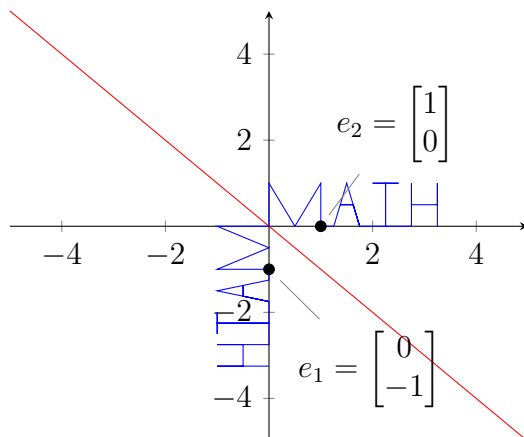
Sub-Problem D No, because the vectors are not necessarily related.

Sub-Problem E Yes, because the columns are related and a part of the same system.

Sub-Problem F Yes, there are pivots in each column and there is no free variable therefore the sets are independent of each other.

Problem 6 $\begin{bmatrix} 2 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 9 \\ 6 \end{bmatrix} = \begin{bmatrix} -7 \\ -6 \end{bmatrix}$

Problem 7 10pts



$e_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $e_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ because we want a transformation instead of a rotation, we need to switch the signs in the first row because that will effect the x values.

Problem 8 15pts

Sub-Problem A Sub-Sub-Problem A.1 Yes, there is one possible solution to the matrix.

Sub-Sub-Problem A.2 Yes, all of the columns span \mathbb{R}^m

Sub-Problem B Sub-Sub-Problem B.1 Correct

Sub-Sub-Problem B.2 Yes, all of the columns span \mathbb{R}^m

Sub-Problem C Sub-Sub-Problem C.1 Yes, the equation only has the trivial solution.

Sub-Sub-Problem C.2 No, not all of the columns span \mathbb{R}^m

Class UNK - Monday, October 23rd, 2017

4.1 §2.8 Subspaces

Example 8 Find a basis for $\text{Col}(A)$ and $\text{Nul}(A)$:

$$A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -4 & 7 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 4x_3 + 7x_4 = 0$$

$$x_2 + 5x_3 - 6x_4 = 0$$

$$x_3 = x_3$$

$$x_4 = x_4$$

Example 9

Class UNK - Monday, October 30rd, 2017

5.1 §2.9 Unknown

Definition 5: Dimension

The dimension of a non-zero subspace H , denoted $\dim(H)$, is the number of vectors in any basis for H . The dimension of $\{0\}$ is defined to be zero.

Example 10 $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Example 11 *How do we know?*

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is in REF, there is a pivot in every row and column, so $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis for \mathbb{R}^2

Note 2 *It can be shown that any basis of a subspace H must have the same number of vectors.*

Definition 6: Rank

The rank of A is $\dim(\text{Col}(A))$.

Why do we care about rank?

$$\text{Col}(A) = \text{span of all columns of } A$$

= set of all linear combinations of the columns of A
 = $x_1 a_1 + \dots + x_n a_n$
 = Ax for any x in \mathbb{R}^n
 = All b that we can solve $Ax=b$ for
 $\text{rank}(A)$ = dimension of above space

Example 12 Determine the $\text{rank}(A)$, $\dim(\text{null}(A))$, for

$$\begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{Rank}(A)$ = number of pivot = 3 columns
 $\dim(\text{nul}(A))$ = number of free = 2 variables

Theorem 1: The Rank Theorem

If Matrix A has n columns, then $\text{Rank}(A) + \dim(\text{Nul}(A)) = n$

1. $\text{Rank}(A)$ = Number of pivot columns
2. $\dim(\text{Nul}(A))$ = Number of free variables
3. N = Total number of columns

Problem 9 Suppose that a basis for $\text{Col}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 14 \end{bmatrix} \right\}$ and $\text{Nul}(A) = x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

Sub-Problem A What is $\dim(\text{Col}(A))$? 3

Sub-Problem B What is $\dim(\text{Nul}(A))$? 1

Sub-Problem C What is the rank of A? 3

Sub-Problem D What size matrix should A be? 3×4

Sub-Problem E What is the reduced row echelon form of A ?

$$\begin{aligned}
 x_1 &= 2x_4 \\
 x_2 &= -3x_4 \\
 x_3 &= 0 \\
 x_4 &= x_4 \\
 &= \\
 x_1 - 2x_4 &= 0 \\
 x_2 + 3x_4 &= 0 \\
 x_3 &= 0
 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Sub-Problem F Find a matrix A by row reducing the matrix

$$\begin{bmatrix} 1 & 0 & 2 & a_1 \\ 2 & 1 & 7 & a_2 \\ 4 & 1 & 14 & a_3 \end{bmatrix}$$

to reduced row echelon form, and then choosing the constants a_1, a_2, a_3 such that the reduced row echelon matches the form from the previous problem. How were the first three columns chosen?

$$\begin{aligned}
 &\begin{bmatrix} 1 & 0 & 2 & a_1 \\ 2 & 1 & 7 & a_2 \\ 4 & 1 & 14 & a_3 \end{bmatrix} \\
 &\begin{bmatrix} 1 & 0 & 2 & a_1 \\ 0 & 1 & 3 & -2a_1 + a_2 \\ 0 & 1 & 6 & -4a_1 + a_3 \end{bmatrix} \\
 &\begin{bmatrix} 1 & 0 & 2 & a_1 \\ 0 & 1 & 3 & -2a_1 + a_2 \\ 0 & 0 & 1 & \frac{-2a_1 - a_2 + a_3}{3} \end{bmatrix} \\
 &\begin{bmatrix} 1 & 0 & 2 & a_1 \\ 0 & 1 & 3 & -2a_1 + a_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\
 &\begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
a_1 &= -2 \\
-2a_1 + a_2 &= 3 \\
-2(-2) + a_2 &= 3 \\
4 + a_2 &= 3 \\
a_2 &= -1 \\
&= \\
-2a_1 - a_2 + a_3 &= 0 \\
-2(-2) - (-1) + a_3 &= 0 \\
5 + a_3 &= 0 \\
a_3 &= -5
\end{aligned}$$

Theorem 2: The Basis Theorem

Let H be a p -dimensional subspace of \mathbb{R}^n . Any linearly independent set of p vectors in H is automatically a basis for H . Also, any set of p vectors in H that spans H is automatically a basis for H .

Example 13 Could \mathbb{R}^3 possible contain a 4-dimensional subspace?
 4-dimensional = 4 pivots in REF. But we only have 3 rows, so we can only have up to 3 pivots, so NO.

Example 14 What is the rank of a 3×4 matrix whose null space is three dimensional?
 Rank is 1.