MAT 350 - Applied Linear Algebra

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Class 1 - Thursday, September 19$^{\text{th}}$, 2017

1.1 §1 Differential Equations

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$$

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2.1 §2.8 Subspaces

Definition 1: Subspace

A subspace of rton is any set H in that has three properties:

- is in H
- For each u, v, in H u+v is also in H (Closure under addition)
- For each u in H and scalar C, cu is in H (Closure under scalar multiplication)

Example 1 *fdsaf Is H a subspace of rto2?*

- *is in h* ✓
- *u+v is not in H for all u,v in H*
- *Not closed under scalar multiplication either (-u not in H).*

H is not a subspace

Example 2

- *is in H* ✓
- *closed under addition* ✓

- *closed under scalar multiplication* ✓

H is a subspace of rto2

Example 3 *Which of the following are subspaces of* $\mathbb{R}^2$*?*

- *not a subspace o is not in h*
- $\mathbb{R}^2$ *is a subspace of* $\mathbb{R}^2$
- $\mathbb{R}^2$ *is a subspace of* $\mathbb{R}^2$
- ◇ *o is in H* ✓
  ◇ *Closed under scalar multiplicaiton* ✓
  ◇ *Not closed unfer addition x u+(-v) in in H.*

  not a subspace of $\mathbb{R}^2$
- *stuff*

So what are do subspaces of $\mathbb{R}^2$ look like?

- They are copies of $\mathbb{R}^0$, $\mathbb{R}^1$, $\mathbb{R}^2$... $\mathbb{R}^n$ that contain the zero vector.

Example 4 *In* $\mathbb{R}^3$*, possible subspaces are:*

- *Zero Subspaces*
- *Lines*
- *Planes*

Example 5 *If H=span*v_1*,*v_2 *(a*v_1*,b*v_2 *for any a,b), then H is a subspace of* $\mathbb{R}^n$

- *o is in H* ✓
  - ◇ *since o**v_1 *+o**v_2*=o*
- *closed under addition* ✓
  - ◇ $u = a_1v_1 + b_1v_2$
  - ◇ $w = a_2v_1 + b_2v_2$

- ◇ $u + w = a_1v_1 + b_1v_2 + a_2v_1 + b_2v_2$
- ◇ $u + w = (a_1 + a_2)v_1 + (b_1 + b_2)v_2$
- ◇ *which is span*v_1,v_2

- *closed under scalar multiplication*
  - ◇ $u = av_1 + bv_2$
  - ◇ $c * u = c(av_1 + bv_2) = (c * a)v_1 + (c * b)v_2$

  which is spanv_1,v_2

Definition 2: Column Space

The column space of a matrix A, denoted col(A), is the set of all linear combinations of the columns of A. (col(a) is a subspace)

Example 6 *Let* $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ *and* $b = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$
is b in Col(A)?
if b is a linear combination of the columns of A, the Ax=b has a solution. We are asking whether [A—b] is consistant of not.
$\begin{bmatrix} 1 & -3 & -4 & | & 3 \\ -4 & 6 & -2 & | & 3 \\ -3 & 7 & 6 & | & -4 \end{bmatrix}$ *goes to* $\begin{bmatrix} 1 & -3 & -4 & | & 3 \\ 0 & -6 & -18 & | & 15 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$
No pivot in augmented column, so b is in col(A).
Note: b in col(A) for every b in $\mathbb{R}^m$

- *Ax=b has a solution for every b in* $\mathbb{R}^m$
- *The columns of A span* $\mathbb{R}^m$
- *A has a pivot in every row in REF*

Definition 3: Null Space

The null space of matrics A, denoted null(A), is the set of all solutions to Ax=$\vec{o}$

Note 1 *Any solution set of $Ax=\vec{o}$ can be written in parametric form.*
$x = x_2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = span\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$
so null(A) is a subspace.

Definition 4: Basis for a Subspace

A basis for a subspace H is a linearly independent set that spans H.

Example 7 *Find a basis for null(A), if* $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$

$A = \begin{bmatrix} 1 & -2 & 0 & -1 & -3 & | & 0 \\ 0 & 0 & 1 & 2 & -2 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$

$$\begin{aligned} x_1 - 2x_2 - x_4 + 3x_5 &= 0 \\ x_2 &= x_2 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ x_4 &= x_4 \\ x_5 &= x_5 \end{aligned}$$

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Test 01 Corrections

Problem 1 *10pts*

Sub-Problem A *Correct*

Sub-Problem B *Correct*

Problem 2 $A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 1 & | & 0 \\ -3 & -1 & 2 & | & 5 \\ 0 & 5 & 3 & | & -1 \end{bmatrix}$$

$$r_2 \to r_2 + 3r_1 \begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 0 & 5 & 5 & | & 5 \\ 0 & 5 & 3 & | & -1 \end{bmatrix}$$

$$r_2 \to r_2/5 \begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 0 & 1 & 1 & | & 1 \\ 0 & 5 & 3 & | & -1 \end{bmatrix}$$

$$r_3 \to r_3 - 5r_2 \begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & -2 & | & -6 \end{bmatrix}$$

$$r_3 \to r_3/-2 \begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$

$$r_1 \to r_1 - 2r_2 \begin{bmatrix} 1 & 0 & -1 & | & -2 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$

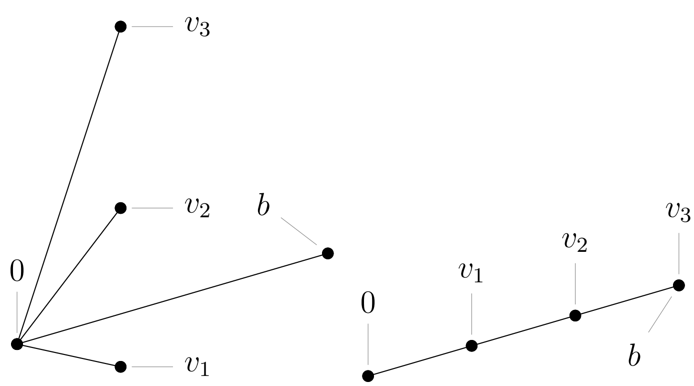
$$r_1 \to r_1 + r_3 \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$

$$r_2 \to r_2 - r_3 \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & -2 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$

Yes, B is in the subset spanned by the columns of A.

$$0 \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

Problem 3 *Yes, the figure does have a solution; if you express the vectors as a linear combination $x_1v_1 + x_2v_2 + x_3v_3 = b$ you can change the values of the scalars (x_1, x_2, x_3) to any value which, when multiplied by the vectors will give us any b; which causes the answer to b to be non-unique.*



Problem 4 *10pts*

Sub-Problem A

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$
\begin{aligned}
r_1 \to r_1 - r_2 &\begin{bmatrix} 1 & 0 & -5 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \\
r_2 \to r_2/3 &\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \\
1x_1 - 5x_3 &= 0 \\
1x_2 + 2x_3 &= 0 \\
x_1 &= 5x_3 \\
x_2 &= -2x_3 \\
x_3 &= x_3 \\
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 5x_3 \\ -2x_3 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 5x_3 \\ -2x_3 \\ 0 \end{bmatrix} &= x_3 \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}
\end{aligned}
$$

Sub-Problem B

$$
0 \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

Sub-Problem C *Yes, x_3 is a free variable, any value will have a solution.*

Problem 5 *30pts*

Sub-Problem A *Correct*

Sub-Problem B *Yes, because there is no way to reduce the matrices to be in the other matrix.*

Sub-Problem C *No, it is linearly dependent because v_1, v_2 are dependant.*

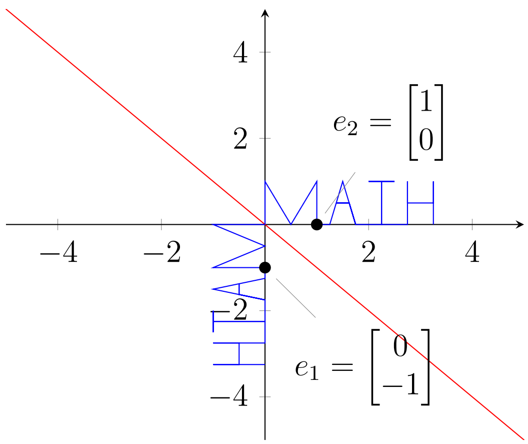
Sub-Problem D *No, because the vectors are not necessarily related.*

Sub-Problem E *Yes, because the columns are related and a part of the same system.*

Sub-Problem F *Yes, there are pivots in each column and there is no free variable therefore the sets are independant of each other.*

Problem 6 $\begin{bmatrix} 2 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 9 \\ 6 \end{bmatrix} = \begin{bmatrix} -7 \\ -6 \end{bmatrix}$

Problem 7 *10pts*



$e_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $e_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *because we want a transformation instead of a rotation, we need to switch the signs in the first row because that will effect the x values.*

Problem 8 *15pts*

Sub-Problem A ***Sub-Sub-Problem A.1*** *Yes, there is one possible solution to the matrix.*

Sub-Sub-Problem A.2 *Yes, all of the columns span $\mathbb{R}^m$*

Sub-Problem B ***Sub-Sub-Problem B.1*** *Correct*

Sub-Sub-Problem B.2 *Yes, all of the columns span $\mathbb{R}^m$*

Sub-Problem C ***Sub-Sub-Problem C.1*** *Yes, the equation only has the trivial solution.*

Sub-Sub-Problem C.2 *No, not all of the columns span $\mathbb{R}^m$*

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4.1 §2.8 Subspaces

Example 8 *Find a basis for Col(A) and Nul(A):*

$$\begin{aligned}
A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix} &\to \begin{bmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 0 & -4 & 7 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
x_1 - 4x_3 + 7x_4 &= 0 \\
x_2 + 5x_3 - 6x_4 &= 0 \\
x_3 &= x_3 \\
x_4 &= x_4
\end{aligned}$$

Example 9

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5.1 §2.9 Unknown

Definition 5: Dimension

The dimension of a non-zero subspace H, denoted dim(H), is the number of vectors in any basis for H. The dimension of o is defined to be zero.

Example 10 $\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$

Example 11 *How do we know?*
$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ *is in REF, there is a picot in every row and column, so* $\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$ *is a basis for* $\mathbb{R}^2$

Note 2 *It can be shown that any basis of a subspace H must have the same number of vectors.*

Definition 6: Rank

The rank of A is dim(Col(A)).

Why do we care about rank?

$$\text{Col(A)} = \text{span of all columns of A}$$

$$
\begin{aligned}
&= \text{set of all linear combinations of the columns of A} \\
&= x_1 a_1 + ... + x_n a_n \\
&= \text{Ax for any x in } \mathbb{R}^n \\
&= \text{All b that we can solve Ax=b for} \\
\text{rank(A)} &= \text{dimension of above space}
\end{aligned}
$$

Example 12 *Determine the rank(A), dim(null(A)), for*

$$
\begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

Rank(A)= number of pivot=3 columns
dim(nul(A)) =number of free = 2 variables

Theorem 1: The Rank Theorem

If Matrix A has n columns, then Rank (A) + Dim(Nul(A))=n

1. Rank(A) = Number of pivot columns
2. Dim(Nul(A)) = Number of free variables
3. N= Total number of columns

Problem 9 *Suppose that a basis for Col(A) is* $\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \\ 14 \end{bmatrix} \}$ *and* $Nul(A) = x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

Sub-Problem A *What is dim(Col(A))? 3*

Sub-Problem B *What is dim(Nul(A))? 1*

Sub-Problem C *What is the rank of A? 3*

Sub-Problem D *What size matrix should A be? 3x4*

Sub-Problem E *What is the reduced row echelon form of A?*

$$\begin{aligned}
x_1 &= 2x_4\\
x_2 &= -3x_4\\
x_3 &= 0\\
x_4 &= x_4\\
&=\\
x_1 - 2x_4 &= 0\\
x_2 + 3x_4 &= 0\\
x_3 &= 0
\end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Sub-Problem F *Find a matrix A by row reducing the matrix*

$$\begin{bmatrix} 1 & 0 & 2 & a_1 \\ 2 & 1 & 7 & a_2 \\ 4 & 1 & 14 & a_3 \end{bmatrix}$$

to reduced row echelon form, and then choosing the constants a_1, a_2, a_3 such that the reduced row echelon matches the form from the previous problem. How were the first three columns chosen?

$$\begin{bmatrix} 1 & 0 & 2 & a_1 \\ 2 & 4 & 7 & a_2 \\ 4 & 1 & 14 & a_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & a_1 \\ 0 & 1 & 3 & -2a_1 + a_2 \\ 0 & 1 & 6 & -4a_1 + a_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & a_1 \\ 0 & 1 & 3 & -2a_1 + a_2 \\ 0 & 0 & 1 & \frac{-2a_1 - a_2 + a_3}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & a_1 \\ 0 & 1 & 3 & -2a_1 + a_2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned}
a_1 &= -2 \\
-2a_1 + a_2 &= 3 \\
-2(-2) + a_2 &= 3 \\
4 + a_2 &= 3 \\
a_2 &= -1 \\
&= \\
-2a_1 - a_2 + a_3 &= 0 \\
-2(-2) - (-1) + a_3 &= 0 \\
5 + a_3 &= 0 \\
a_3 &= 5
\end{aligned}$$

Theorem 2: The Basis Theorem

Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of p vectors in H is automatically a basis for H. Also, any set of p vectors in H that spans H is automatically a basis for H.

Example 13 *Could $\mathbb{R}^3$ possible contain a 4-dimensional subspace?*
4-dimensional = 4 pivots in REF. But we only have 3 rows, so we can only have up to 3 pivots, so NO.

Example 14 *What is the rank of a 3x4 matrix whose null space is three dimensional?*
Rank is 1.

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6.1 §3.2 Properties of Determinants

Example 15 $$\begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 0 & 0 & 0 \\ -5 & -8 & 0 & 9 \end{bmatrix} = 0$$

Example 16 $$\begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -5 & 2 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -5 & 2 \\ 0 & 0 & -3 & 1 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -5 & 2 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \\ -2 & -5 & 4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 6 & -5 & 2 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -5 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

Example 17 $$\begin{bmatrix} 3 & 3 & -3 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{bmatrix} = \begin{bmatrix} 3 & 3 & -3 \\ 0 & -1 & 1 \\ 2 & -3 & -5 \end{bmatrix}$$