CSE 291: Geometric algorithms **Spring 2013**

Lecture 2 — The k-median clustering problem

2.1 Problem formulation

Last time we saw the k-center problem, in which the input is a set S of data points and the goal is to choose k representatives for S. The *distortion* on a point $x \in S$ is then the distance to its closest representative. The overall goal is to make sure that *every* point in S has low distortion; that is, to minimize the *maximum* distortion in S.

In most applications, we are more interested in minimizing the *typical* (i.e., average) distortion. One formulation of this is the k-median cost function.

k-MEDIAN CLUSTERING

Input: Finite set $S \subset \mathcal{X}$; integer k.

Output: $T \subset S$ with $|T| = k$.

Goal: Minimize $\text{cost}(T) = \sum_{x \in S} \rho(x, T)$.

This cost function is more robust to outliers than the k-center cost. Also, in this case T is forced to be a subset of S (rather than $\mathcal{X}$).

2.2 A linear programming relaxation

For convenience index the points in S by $1, 2, \ldots, n$, with interpoint distances $\rho(i, j), 1 \leq i, j \leq n$. Then the k-median problem is solved exactly by the following integer program.

$$\begin{aligned} \min \sum_{i,j} x_{ij}\rho(i,j) \\ \sum_j y_j \leq k \\ \sum_j x_{ij} = 1 \\ x_{ij} \leq y_j \\ x_{ij}, y_j \in \{0,1\} \end{aligned}$$

where the variables $\{x_{ij}, y_j\}$ have the following interpretation.

$$\begin{aligned} y_j &= \mathbf{1}(\text{point } j \text{ is used as a center}) \\ x_{ij} &= \mathbf{1}(j \text{ is the center serving point } i) \end{aligned}$$

An integer program cannot in general be solved efficiently, so we turn it into a linear program by relaxing the last two constraints:

$$0 \leq x_{ij}, y_j \leq 1$$

The resulting LP can be solved in polynomial time.

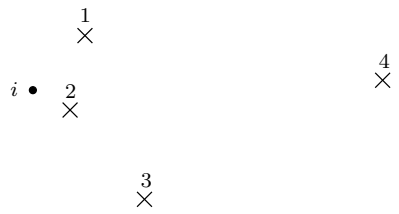
2.2.1 Rounding the LP solution

Suppose the optimal solution to the k-median instance has cost OPT. Since this solution is feasible for the linear program, the optimal LP solution has some cost $\text{OPT}_{LP} \leq \text{OPT}$. Say this solution consists of variables $\{x_{ij}, y_j\}$. The difficulty, of course, is that these values might be fractional (such as $y_1 = 0.2$, $y_2 = 0.5$, and so on). Following Lin and Vitter (1992), we'll show how to *round* this fractional solution into one that has $2k$ centers and has cost at most 4OPT_{LP}.

In the LP solution, point i incurs a cost

$$C_i = \sum_j x_{ij}\rho(i,j).$$

This might be spread out over several centers j: those with $x_{ij} > 0$. For instance, it might be the case that $x_{i1}, x_{i2}, x_{i3}, x_{i4} > 0$ (see figure below), and since $\sum_j x_{ij} = 1$, we can think of the x_{ij}'s as a probability distribution over centers for i. Under this distribution, C_i is the *expected* distance of a center from i.



The total cost is $\text{OPT}_{LP} = \sum_i C_i$. We will find a set of $2k$ centers in S such that each point i is within distance at most $4C_i$ of these centers. The total cost will then be at most 4OPT_{LP}.

The hardest points to cover are those with the smallest values of C_i, so let's start by focusing on those. Pick the smallest C_i. If we include i as a center, we can use it cover any $i' \in S$ whose distance from i is at most $4C_i$; denote the set of such points by $B(i, 4C_i)$. This is because i has the smallest C_i value, and thus $\rho(i,i') \leq 4C_i \leq 4C_{i'}$.

But this is overly conservative. We can in fact use i to cover any point i' such that $B(i, 2C_i) \cap B(i', 2C_{i'}) \neq \emptyset$. To see this, notice that since the two balls intersect, they have some point q in common, and thus $\rho(i,i') \leq \rho(i,q) + \rho(i',q) \leq 2C_i + 2C_{i'} \leq 4C_{i'}$. Therefore, define the *extended neighborhood* of i as follows.

$$\overline{V}_i = \{i' \in S : B(i, 2C_i) \cap B(i', 2C_{i'}) \neq \emptyset\}.$$

Now we can state the algorithm simply.

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solve the LP and compute the values C_i
T ← { }
while S ≠ { }:
   pick the i ∈ S with smallest C_i
   T ← T ∪ {i}
   S ← S \ V̄_i
```

We will show the following.

Theorem 1. $\text{cost}(T) \leq 4\text{OPT}_{LP}$ *and* $|T| \leq 2k$.

2.2.2 Analysis

First we'll show that $\text{cost}(T) \leq 4\text{OPT}_{LP}$. This is an immediate consequence of the following lemma.

Lemma 2. *Pick any $q \in S$, and suppose i is the first point selected (to be in T) for which $q \in \overline{V}_i$. Then: (a) $C_i \leq C_q$ and (b) $\rho(q,i) \leq 4C_q$.*

Proof. At the moment when i is selected, both i and q are available in S. Therefore $C_i \leq C_q$. For (b), the condition $q \in \overline{V}_i$ implies that there is some point s in both $B(i, 2C_i)$ and $B(q, 2C_q)$. Thus

$$\rho(q,i) \leq \rho(i,s) + \rho(q,s) \leq 2C_i + 2C_q \leq 4C_q.$$

□

Next we need to bound the size of T. The argument will go like this: we'll show that for each point i selected to be in T, the neighborhood $B(i, 2C_i)$ contains at least "half a center": more precisely, the sum of y_j for $j \in B(i, 2C_i)$ is at least $1/2$. However, these neighborhoods are all disjoint (for different $i \in T$) and the total sum of y values is at most k. Therefore there can be at most $2k$ such points $i \in T$.

Lemma 3. *Pick any $i \in T$. Then*

$$\sum_{j \in B(i,2C_i)} y_j \geq \sum_{j \in B(i,2C_i)} x_{ij} \geq \frac{1}{2}.$$

Proof. The first inequality follows from the constraint $x_{ij} \leq y_j$ in the LP. To see the second inequality, define a random variable $Z \in \mathbb{R}$ that takes value $\rho(i,j)$ with probability x_{ij}. As we saw above, $\mathbb{E}Z = \sum_j x_{ij}\rho(i,j) = C_i$. By Markov's inequality,

$$\sum_{j \in B(i,2C_i)} x_{ij} = \mathbb{P}[Z \leq 2C_i] = 1 - \mathbb{P}[Z > 2\mathbb{E}Z] \geq \frac{1}{2}.$$

□

The rest is immediate, since for any $i, i' \in T$, we know $B(i, 2C_i) \cap B(i', 2C_{i'}) = \emptyset$.

2.2.3 Other work

The solution to this linear program can be rounded in a substantially more complicated manner to yield k medians with a cost that is at most 6.7 times optimal (Charikar et al., 1999). More recent work has improved upon this to yield factors close to 3.

At the same time, it is known (Jain et al., 2002) that there is no efficient algorithm that achieves an approximation factor better than $1 + 2/e$ unless NP $\subset$ DTIME($n^{O(\log \log n)}$).

Bibliography

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