$\lambda x.x$

“the λ calculus can be called *the smallest universal programming language of the world*”

Raul Rojas - A Tutorial Introduction To Lambda Calculus

Developed by Alonzo Church in the 1930's, the λ calculus is a universal system for expressing any **computable function**.

The λ calculus is equivalent to Turing machines, but is not concerned with the machine.

The λ calculus only considers how functions can be expressed and evaluated.

There are two main components in the λ calculus:

1. **expressions or "functions"**
2. **variables or "names"**

There are two keywords in the λ calculus:

λ .

All expressions in the λ calculus have the structure:

$$\lambda x . x$$

$\lambda x . x$

The **head** of the expression: $\lambda x .$

The **body** of the expression: x

$\lambda x . x$

There is one **variable**: x

Let's talk about **functions**

Given the **function**:

$$f(x) = x$$

Given the **function**:

$f(x) = x$

What's the **domain**?

Given the **function**:

$f(x) = x$

What's the **codomain**?

Domain = { set of inputs }

$$\downarrow$$

$$f(x) = x$$

$$\downarrow$$

Codomain = { set of outputs }

Given $f(x) = x$

$f(1) = 1$
$f(2) = 2$
…

Domain = { 1, 2 … }

Codomain = { 1, 2 … }

How can we express a **function** in the λ calculus?

$$f(x) = x$$

$$f(x) = x$$

$$\lambda x \;.\; x$$

$$\boxed{f(x)\ =}\ x$$

$$\boxed{\lambda x\ \ .}\ x$$

head

$$f(x) = \boxed{x}$$

$$\lambda x \;.\; \boxed{x}$$

body

Let's talk about **applying expressions**

$$(\lambda\, x\, .\, x)\ y$$

$$(\lambda\, x\, .\, x)\; y$$

$$(\lambda\, y \,.\, x)\ y$$

$$(\lambda\, y\, .\, y)\ y$$

y

Reduced form

$$(\lambda\, x\, .\, x)\ y = y$$

Redex (unreduced)

$(\lambda\, x\, .\, x)\ y$

Let’s talk about **Alpha Equivalency**

Alpha Equivalency

$$(\lambda\, x\ .\ x) \equiv (\lambda\, y\ .\ y)$$

Alpha Equivalency

$$(\lambda\ x\ .\ x) \equiv (\lambda\ y\ .\ y) \equiv (\lambda\ z\ .\ z)$$

Let’s talk about **free** and **bound variables**

A **free variable** is not **bound** in the **head**

$(\lambda x . y)$

y is **free**

A **bound variable** is defined in the **head**

(λ x . x)

x is **bound**

$(\lambda\ x\ y\ .\ x\ z)$

What are the **bound variables**?

What are the **free variables**?

Let’s talk about **Beta Reduction**

Beta Reduction is the **application** of expressions until no further **application** can occur:

(λ x . x) y
(λ [x := y] . x)
(λ [x := y] . y)
y

Let's try a slightly harder example:

$(\lambda\ x\ z\ .\ x\ z)\ y$

Let's try a slightly harder example:

$$(\lambda\ x\ z\ .\ x\ z)\ y$$

$$(\lambda\ [x := y]\ z\ .\ x\ z)$$

$$(\lambda\ [x := y]\ z\ .\ y\ z)$$

$$(\lambda\ z\ .\ y\ z)$$

Let’s try a slightly harder example still:

$(\lambda\, x\, .\, x)\ (\lambda\, y\, .\, y)$

Let's try a slightly harder example still:

$(\lambda\, x\ .\ x)\ \ (\lambda\, y\ .\ y)$
$(\lambda\ [x := \ (\lambda\, y\ .\ y)]\ .\ x)$
$(\lambda\ [x := \ (\lambda\, y\ .\ y)]\ .\ (\lambda\, y\ .\ y))$
$(\lambda\, y\ .\ y)$

Expressions can be the **domain** of other expressions.

Expressions can be the **codomain** of other expressions.

Let's revisit this example:

(λ x . x) (λ y . y)
(λ [x := (λ y . y)] . x)
(λ [x := (λ y . y)] . (λ y . y))
(λ y . y)

(λ x . x) ? (λ y . y)

Via **Beta Reduction**, we have shown that two expressions are **Alpha Equivalent**, meaning that they are the same expression.

$(\lambda\ x\ .\ x)\ (\lambda\ y\ .\ y)$
$(\lambda\ [x := (\lambda\ y\ .\ y)]\ .\ x)$
$(\lambda\ [x := (\lambda\ y\ .\ y)]\ .\ (\lambda\ y\ .\ y))$
$(\lambda\ y\ .\ y)$

$(\lambda\ x\ .\ x) \equiv (\lambda\ y\ .\ y)$

We can also say that we have an expression, whose **domain** and **codomain** are always the same. This is known as the identity expression.

$(\lambda\ x\ .\ x)\ (\lambda\ y\ .\ y)$
$(\lambda\ [x := (\lambda\ y\ .\ y)]\ .\ x)$
$(\lambda\ [x := (\lambda\ y\ .\ y)]\ .\ (\lambda\ y\ .\ y))$
$(\lambda\ y\ .\ y)$

$(\lambda\ x\ .\ x) \equiv (\lambda\ y\ .\ y)$

Let’s talk about the **Y Combinator**

Given an expression Y, such that

$$Y \equiv (\lambda y . (\lambda x . y (x x))(\lambda x . y (x x)))$$

Given an expression Y, such that

$$Y \equiv (\lambda\, y\, .\, (\lambda\, x\, .\, y\, (\, x\, x\,))(\lambda\, x\, .\, y\, (\, x\, x\,)))$$

And given an expression R, such that

$$YR \equiv (\lambda\, y\, .\, (\lambda\, x\, .\, y\, (\, x\, x\,))(\lambda\, x\, .\, y\, (\, x\, x\,))\; R$$

Can we show that the **Y Combinator** allows us to employ recursion using nothing but **Beta Reduction** with lambda expressions(anonymous functions)?

Yes!

$$YR \equiv (\lambda y.(\lambda x.y(xx))(\lambda x.y(xx)))R$$

$$YR \equiv (\lambda y\,.\,(\lambda x\,.\,y\,(x\,x))(\lambda x\,.\,y\,(x\,x)))\;R$$

$$YR \equiv (\lambda\, y\, .\, (\lambda\, x\, .\, y\,(x\, x))(\lambda\, x\, .\, y\,(x\, x))\; R$$

$$YR \equiv (\lambda\, [y := R]\, .\, (\lambda\, x\, .\, y\,(x\, x))(\lambda\, x\, .\, y\,(x\, x))$$

$$YR \equiv (\lambda\, y\, .\, (\lambda\, x\, .\, y\, (\, x\, x\,))(\lambda\, x\, .\, y\, (\, x\, x\,))\ R$$

$$YR \equiv (\lambda\, [y := R]\, .\, (\lambda\, x\, .\, y\, (\, x\, x\,))(\lambda\, x\, .\, y\, (\, x\, x\,))$$

$$YR \equiv (\lambda\, y\,.\,(\lambda\, x\,.\,y\,(x\,x))(\lambda\, x\,.\,y\,(x\,x))\; R$$

$$YR \equiv (\lambda\, [y := R]\,.\,(\lambda\, x\,.\,y\,(x\,x))(\lambda\, x\,.\,y\,(x\,x))$$

$$YR \equiv (\lambda\, [y := R]\,.\,(\lambda\, x\,.\,R\,(x\,x))(\lambda\, x\,.\,R\,(x\,x)))$$

$$YR \equiv (\lambda y \,.\, (\lambda x \,.\, y\,(x\,x))(\lambda x \,.\, y\,(x\,x))\; R$$

$$YR \equiv (\lambda\,[y := R] \,.\, (\lambda x \,.\, y\,(x\,x))(\lambda x \,.\, y\,(x\,x))$$

$$YR \equiv (\lambda\,[y := R] \,.\, (\lambda x \,.\, R\,(x\,x))(\lambda x \,.\, R\,(x\,x)))$$

$$YR \equiv (\lambda x \,.\, R\,(x\,x))(\lambda x \,.\, R\,(x\,x))$$

$$YR \equiv (\lambda x . R(x x))(\lambda x . R(x x))$$

$$YR \equiv (\lambda x\,.\,R\,(\,x\,x\,))\boxed{(\lambda x\,.\,R\,(\,x\,x\,))}$$

$$\boxed{(\lambda x\,.\,R\,(\,x\,x\,))}$$

$$YR \equiv (\lambda x\,.\,R\,(\,x\,x\,))$$

$$\boxed{(\lambda x . R (x x))}$$

$$YR \equiv (\lambda x . R (x x))$$

$$\boxed{(\lambda x \,.\, R\ (x\ x))}$$

$$YR \equiv (R\ (x\ x))$$

$$YR \equiv (R\ (\lambda x\ .\ R\ (\ x\ x\ ))(\lambda x\ .\ R\ (\ x\ x\ )))$$

What happens if we factor R?

$$YR \equiv (R\ (\lambda x\ .\ R\ (\ x\ x\ ))(\lambda x\ .\ R\ (\ x\ x\ )))$$

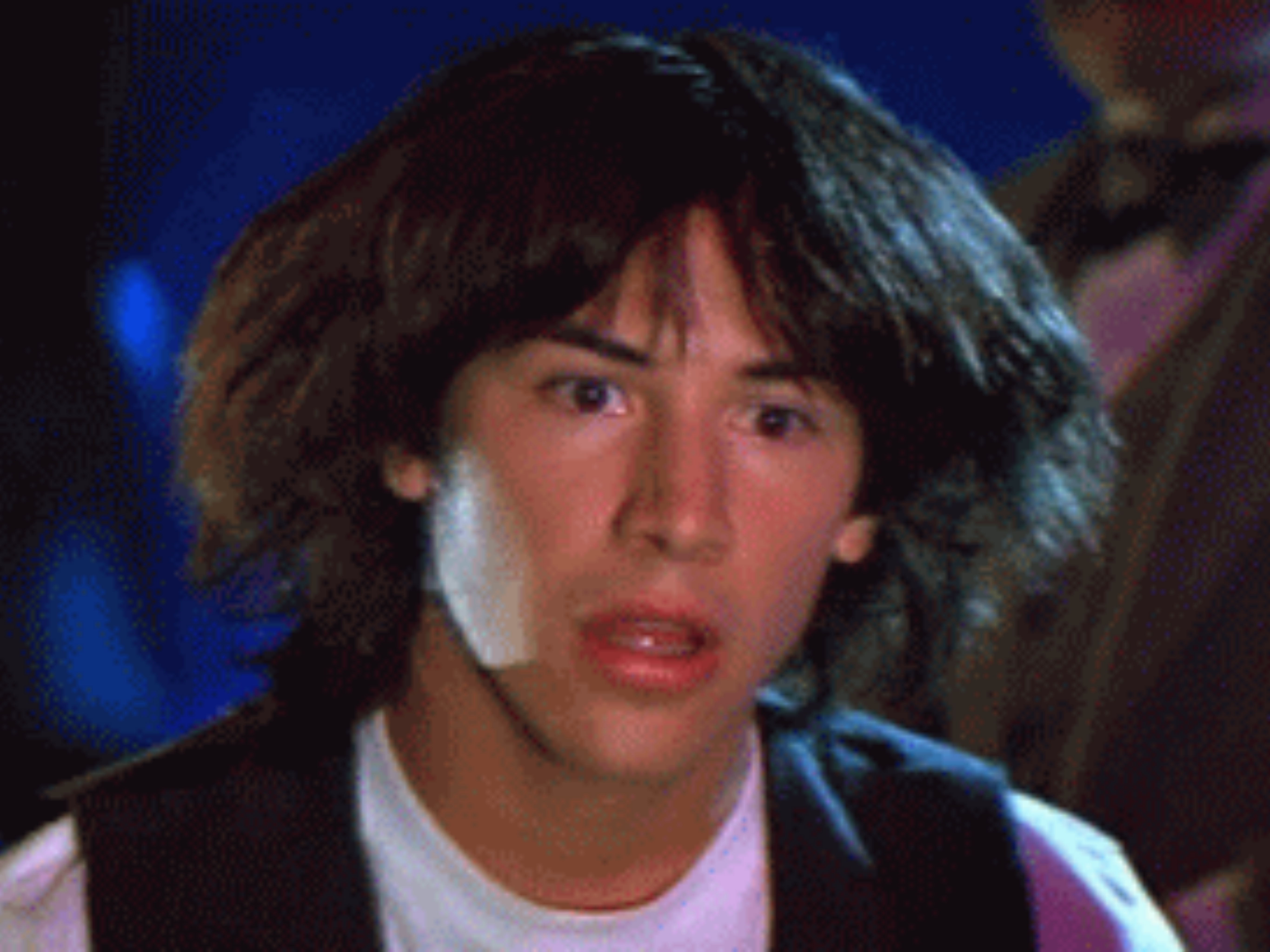
What happens if we factor R?

$$YR \equiv (R\ (\lambda\, x\ .\ R\ (\ x\ x\ ))(\lambda\, x\ .\ R\ (\ x\ x\ )))$$

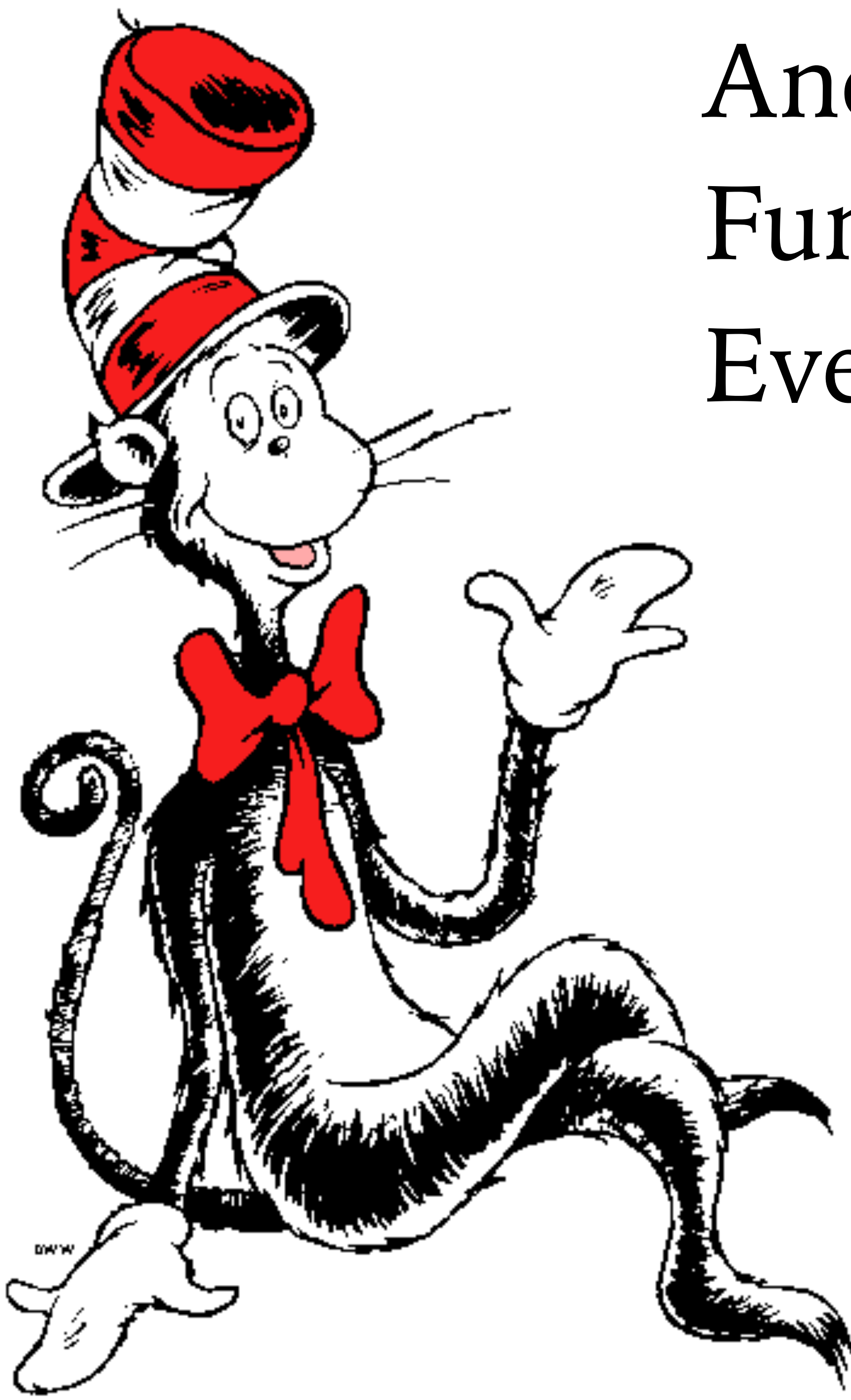
$$Y \equiv (\lambda\, y\ (\lambda\, x\ .\ y\ (\ x\ x\ ))(\lambda\, x\ .\ y\ (\ x\ x\ )))R$$

We return to our original Y combinator form:

$$Y \equiv (\lambda\, y\ (\lambda\, x\ .\ \lambda\, y\ (\ x\ x\ ))(\lambda\, x\ .\ \lambda\, y\ (\ x\ x\ )))\ R$$



Why?



"From There To Here,
And Here To There,
Funny Things Are
Everywhere."

Dr. Seuss

More resources:

Allen, Christopher and Moronuki, Julie. *Haskell Programming From First Principles.* haskellbook.com

Rojas, Raul. *A Tutorial Introduction to the Lambda Calculus*. FU Berlin, WS-97/98. http://www.inf.fu-berlin.de/lehre/WS03/alpi/lambda.pdf