STAT 30030 Homework 3, Exercise 6: Method of Moments Estimation

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1 Introduction

Suppose $T_1, \dots, T_{10} \sim$ iid $\text{Exp}(\lambda)$. Let $\bar{T}_n$ be the sample mean of $T_1, \dots, T_{10}$. We consider two method of moments estimators for λ, generated from the first and second moments of any of the T_i's, respectively:

$$\hat{\lambda} = \frac{1}{\bar{T}_n} = \frac{n}{\sum_i T_i},$$

and

$$\hat{\lambda}_2 = \sqrt{\frac{2n}{\sum_i T_i^2}}.$$

We will empirically explore the properties of these two estimators.

2 Part (i)

2.1 Setup

```r
library(dplyr)
library(ggplot2)
```

2.2 Simulation

First, we set the true value for our parameter λ to be 3.

```r
lambda = 3
```

We define a function `lambda_hat(Ts)` that computes the estimator $\hat{\lambda} = \frac{1}{\bar{T}_n} = \frac{n}{\sum_i T_i}$ on a sample of ten random variables $T_1, \ldots, T_{10} \sim \text{iid Exp}(\lambda)$.

```r
lambda_hat = function(Ts){
  Tn = 0
  for(Ti in Ts){
    Tn = Tn + Ti
  }
  Tn = Tn / 10
  lambda_hat = 1/Tn
  return(lambda_hat)
}
```

We define another function `lambda_2_hat(Ts)` that computes the estimator $\hat{\lambda}_2 = \sqrt{\frac{2n}{\sum_i T_i^2}}$ on a sample of ten random variables $T_1, \ldots, T_{10} \sim \text{iid Exp}(\lambda)$.

```r
lambda_2_hat = function(Ts){
  sum = 0
  for(Ti in Ts){
    sum = sum + Ti^2
  }
  lambda_2_hat = sqrt(2*10/sum)
  return(lambda_2_hat)
}
```

Finally, we generate 10,000 samples of ten random variables $T_1, \ldots, T_{10} \sim \text{iid Exp}(\lambda)$, and for each sample compute and store the estimates $\hat{\lambda}$ and $\hat{\lambda}_2$.

```
data = tibble()
data = data %>%
  mutate(lambda_hat = 0) %>%
  mutate(lambda_2_hat = 0)

for(i in c(1:10000)){
  Ts = rexp(10, lambda)
  data = data %>% add_row(lambda_hat = lambda_hat(Ts),
                          lambda_2_hat = lambda_2_hat(Ts))
}
```

2.3 Properties of $\hat{\lambda}$

```
bias_lambda_hat = mean(data$lambda_hat) - lambda

variance_lambda_hat = var(data$lambda_hat)

mse_lambda_hat = variance_lambda_hat + bias_lambda_hat^2
```

Theoretically, the bias of $\hat{\lambda}$ should be $\frac{\lambda}{n-1} = \frac{3}{10-1} = \frac{1}{3} \approx 0.\overline{333}$. In our simulation, we estimated the bias of $\hat{\lambda}$ to be 0.3356959, which is roughly what we expected.

Moreover, theoretically, the variance of $\hat{\lambda}$ should be $\lambda^2 \frac{n^2}{(n-1)^2(n-2)} = 3^2 \frac{10^2}{(10-1)^2(10-2)} = \frac{900}{648} = \frac{25}{18} \approx 1.3\overline{88}$. In our simulation, we estimated the variance of $\hat{\lambda}$ to be 1.32581, which is roughly what we expected.

Finally, theoretically, the MSE of $\hat{\lambda}$ should be $\mathrm{Var}(\hat{\lambda}) + \mathrm{bias}(\hat{\lambda})^2 = \frac{\lambda^2 n^2}{(n-1)^2(n-2)} + \left(\frac{\lambda}{n-1}\right)^2 = \frac{25}{18} + \left(\frac{1}{3}\right)^2 = \frac{27}{18} = \frac{3}{2} = 1.5$. In our simulation, we estimated the variance of $\hat{\lambda}$ to be 1.4385017, which is roughly what we expected.

2.4 Properties of $\hat{\lambda}_2$

```
bias_lambda_2_hat = mean(data$lambda_2_hat) - lambda

variance_lambda_2_hat = var(data$lambda_2_hat)

mse_lambda_2_hat = variance_lambda_2_hat + bias_lambda_2_hat^2
```

In our simulation, we find that the bias of $\hat{\lambda}_2$ is approximately 0.5652155, the variance of $\hat{\lambda}_2$ is approximately 1.6772936, and the MSE of $\hat{\lambda}_2$ is approximately 1.9967621.

2.5 Comparison of $\hat{\lambda}$ and $\hat{\lambda}_2$

In our simulation analysis, we found that:

- $\text{bias}(\hat{\lambda}) < \text{bias}(\hat{\lambda}_2)$;
- $\text{Var}(\hat{\lambda}) < \text{Var}(\hat{\lambda}_2)$; and thus
- $\text{MSE}(\hat{\lambda}) < \text{Var}(\hat{\lambda}_2)$.

Thus, $\hat{\lambda}$ is both less biased and more efficient than $\hat{\lambda}_2$, making $\hat{\lambda}$ the better estimator for λ, at least with respect to finite samples.

3 Part (ii)

3.1 Asymptotics of $\hat{\lambda}$

In Figure 1, we observe that $\hat{\lambda}$ appears to be asymptotically distributed $\sim \mathcal{N}(\lambda, \frac{1}{10}\lambda^2)$, as we derived in class, with $n = 10$.

```
plot_lambda_hat = ggplot(data) +
  # Histogram
  geom_histogram(aes(x = lambda_hat, y = after_stat(density)), bins = 50,
                 fill = "lightgray", color = "black") +
  # normal density curve
  stat_function(fun = dnorm, args = list(mean = lambda, sd =
   ↪  sqrt(lambda^2/10)),
                color = "blue", linewidth = 1.2) +
  # aesthetics
  theme(panel.background = element_rect(fill = "white", color = "black")) +
  theme(panel.grid.major = element_line(color = "gray", linetype =
   ↪  "dotted")) +
  # labels
  labs(x = bquote(hat(lambda)), y = "Density", title = "Method of Moments
   ↪  (First Moment) Parameter Estimates") +
  theme(plot.title = element_text(hjust=0.5))

plot_lambda_hat
```

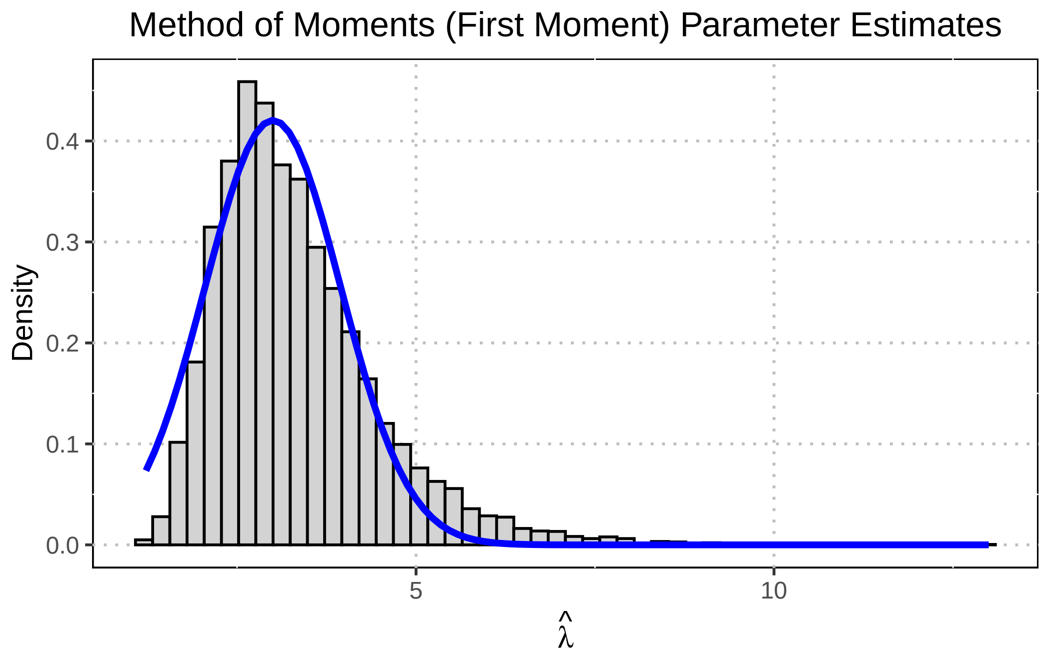

Figure 1: Histogram of lambda_hat estimates and their distribution

3.2 Asymptotics of $\hat{\lambda}_2$

In Figure 2, we observe that $\hat{\lambda}_2$ appears to be asymptotically distributed $\sim \mathcal{N}(\lambda, \frac{1}{8}\lambda^2)$, as we derived in Exercise 5(iii), with $n = 10$.

```
plot_lambda_2_hat = ggplot(data) +
  # histogram
  geom_histogram(aes(x = lambda_2_hat, y = after_stat(density)), bins = 50,
                 fill = "lightgray", color = "black") +
  # normal density curve
  stat_function(fun = dnorm, args = list(mean = lambda, sd =
   ↪ sqrt(lambda^2*5/40)),
                color = "blue", linewidth = 1.2) +
  # aesthetics
  theme(panel.background = element_rect(fill = "white", color = "black")) +
  theme(panel.grid.major = element_line(color = "gray", linetype =
   ↪ "dotted")) +
  # labels
  labs(x = bquote(hat(lambda)[2]), y = "Density", title = "Method of
   ↪ Moments (Second Moment) Parameter Estimates") +
  theme(plot.title = element_text(hjust=0.5))

plot_lambda_2_hat
```

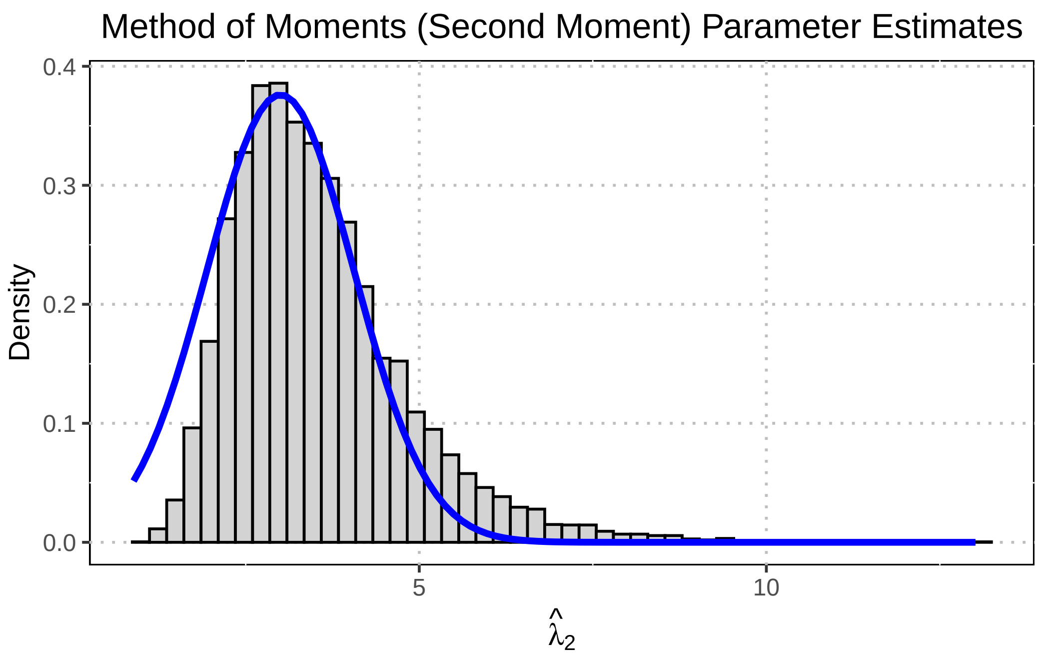

Figure 2: Histogram of lambda_2_hat estimates and their distribution