STAT 32950

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HW #5

2. Let $f_1(x) = c_1(1-|x-\frac{1}{2}|)\mathbb{I}_{\{-\frac{1}{2} \le x \le \frac{3}{2}\}}$,

$f_2(x) = c_2(1-|x|)\mathbb{I}_{\{-1 \le x \le 1\}}$,

$f_3(x) = c_3(2-|x-\frac{1}{2}|)\mathbb{I}_{\{-\frac{3}{2} \le x \le \frac{5}{2}\}}$

2a. Find c_i's so f_i's are PDFs.

- $\int_{\mathbb{R}} f_1(x)dx = 1$

$\Rightarrow \int_{\mathbb{R}} c_1(1-|x-\frac{1}{2}|)\mathbb{I}_{\{-\frac{1}{2} \le x \le \frac{3}{2}\}} dx = 1$

$\Rightarrow \int_{-1/2}^{3/2} c_1(1-|x-\frac{1}{2}|)dx = 1$

$\Rightarrow c_1[\int_{-1/2}^{1/2}(1-(\frac{1}{2}-x))dx + \int_{1/2}^{3/2}(1-(x-\frac{1}{2}))dx] = 1$

$\Rightarrow c_1[\int_{-1/2}^{1/2}(\frac{1}{2}+x)dx + \int_{1/2}^{3/2}(\frac{3}{2}-x)dx] = 1$

$\Rightarrow c_1[(\frac{1}{2}x+\frac{1}{2}x^2)|_{-1/2}^{1/2} + (\frac{3}{2}x-\frac{1}{2}x^2)|_{1/2}^{3/2}] = 1$

$\Rightarrow c_1[(\frac{1}{2}(\frac{1}{2})+\frac{1}{2}(\frac{1}{2})^2)-(\frac{1}{2}(-\frac{1}{2})+\frac{1}{2}(-\frac{1}{2})^2)+(\frac{3}{2}(\frac{3}{2})-\frac{1}{2}(\frac{3}{2})^2)-(\frac{3}{2}(\frac{1}{2})-\frac{1}{2}(\frac{1}{2})^2)] = 1$

$\Rightarrow c_1[\frac{1}{4}+\frac{1}{8}-(-\frac{1}{4}+\frac{1}{8})+\frac{9}{4}-\frac{9}{8}-(\frac{3}{4}-\frac{1}{8})] = 1$

$\Rightarrow c_1[\frac{1}{4}+\frac{1}{8}+\frac{1}{4}-\frac{1}{8}+\frac{9}{4}-\frac{9}{8}-\frac{3}{4}+\frac{1}{8}] = 1$

$\Rightarrow c_1[-1+2] = 1$

$\Rightarrow c_1 = 1$

- $\int_{\mathbb{R}} f_2(x)dx = 1$

$\Rightarrow \int_{\mathbb{R}} c_2(1-|x|)\mathbb{I}_{\{-1 \le x \le 1\}}dx = 1$

$\Rightarrow \int_{-1}^{1} c_2(1-|x|)dx = 1$

$\Rightarrow c_2[\int_{-1}^{0}(1+x)dx + \int_{0}^{1}(1-x)dx] = 1$

$\Rightarrow c_2[(x+\frac{1}{2}x^2)|_{-1}^{0} + (x-\frac{1}{2}x^2)|_{0}^{1}] = 1$

$\Rightarrow c_2[(0+\frac{1}{2}0^2)-(-1+\frac{1}{2}(-1)^2)+(1-\frac{1}{2}(1)^2)-(0-\frac{1}{2}0^2)] = 1$

$\Rightarrow c_2[0-(-1+\frac{1}{2})+\frac{1}{2}-0] = 1$

$\Rightarrow c_2 = 1$

- $\int_{\mathbb{R}} f_3(x)dx = 1$

$\Rightarrow \int_{\mathbb{R}} c_3(2-|x-\frac{1}{2}|)\mathbb{I}_{\{-\frac{3}{2} \le x \le \frac{5}{2}\}}dx = 1$

$\Rightarrow \int_{-3/2}^{5/2} c_3(2-|x-\frac{1}{2}|)dx = 1$

$\Rightarrow c_3[\int_{-3/2}^{1/2}(2-(\frac{1}{2}-x))dx + \int_{1/2}^{5/2}(2-(x-\frac{1}{2}))dx] = 1$

$\rightarrow$

$$\Rightarrow c_3\left[\int_{-3/2}^{1/2}\left(\tfrac{3}{2}+x\right)dx + \int_{1/2}^{5/2}\left(\tfrac{5}{2}-x\right)dx\right]=1$$

$$\Rightarrow c_3\left[\left(\tfrac{3}{2}x+\tfrac{1}{2}x^2\right)\Big|_{-3/2}^{1/2} + \left(\tfrac{5}{2}x-\tfrac{1}{2}x^2\right)\Big|_{1/2}^{5/2}\right]=1$$

$$\Rightarrow c_3\left[\left(\tfrac{3}{2}(\tfrac{1}{2})+\tfrac{1}{2}(\tfrac{1}{2})^2\right)-\left(\tfrac{3}{2}(-\tfrac{3}{2})+\tfrac{1}{2}(-\tfrac{3}{2})^2\right)+\left(\tfrac{5}{2}(\tfrac{5}{2})-\tfrac{1}{2}(\tfrac{5}{2})^2\right)-\left(\tfrac{5}{2}(\tfrac{1}{2})-\tfrac{1}{2}(\tfrac{1}{2})^2\right)\right]=1$$

$$\Rightarrow c_3\left[\tfrac{3}{4}+\tfrac{1}{8}-\left(-\tfrac{9}{4}+\tfrac{9}{8}\right)+\tfrac{25}{4}-\tfrac{25}{8}-\left(\tfrac{5}{4}-\tfrac{1}{8}\right)\right]=1$$

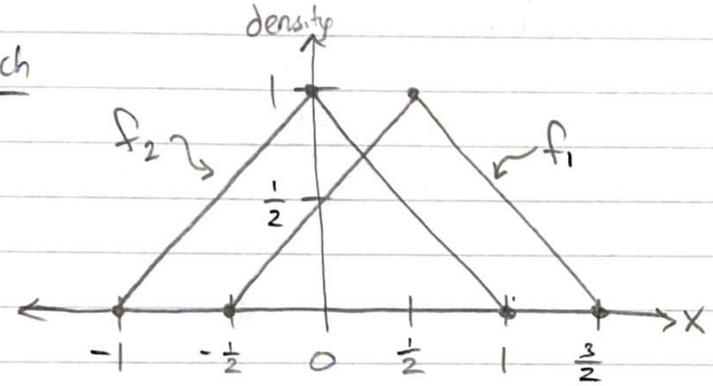
$$\Rightarrow c_3\left[\tfrac{3}{4}+\tfrac{1}{8}+\tfrac{9}{4}-\tfrac{9}{8}+\tfrac{25}{4}-\tfrac{25}{8}-\tfrac{5}{4}+\tfrac{1}{8}\right]=1$$

$$\Rightarrow c_3\left[-1-3+3+5\right]=1$$

$$\Rightarrow \boxed{c_3=\tfrac{1}{4}}$$

26. Identify classification regions for the 2 popns w/ PDFs f_1, f_2 by rule of ECM, w/ $p_1 = 0.8$, $c(1|2)=c(2|1)$. Helps to sketch f_1, f_2.

· Sketch



· First, note that if $x>1$, then obviously we should classify $x \in \pi_1$, & if $x<-\frac{1}{2}$, then we should classify $x\in\pi_2$. The "ambiguous" region is $[-\frac{1}{2}, 1]$.

· $$R_1^* = \left\{x : \frac{f_1(x)}{f_2(x)} > \frac{p_2}{p_1}\,\frac{c(1|2)}{c(2|1)}\right\}$$

$$= \left\{x : \frac{(1-|x-\frac{1}{2}|)\,\mathbb{I}_{\{-\frac{1}{2}\le x\le \frac{3}{2}\}}}{(1-|x|)\,\mathbb{I}_{\{-1\le x\le 1\}}} > \frac{1-0.8}{0.8}\cdot 1\right\}$$

$$= \left\{x : \frac{1-|x-\frac{1}{2}|}{1-|x|} > \frac{1}{4}\right\} \text{ for } x\in[-\tfrac{1}{2},1]$$

$$= \{x : 4-4|x-\tfrac{1}{2}| > 1-|x|\} \text{ since } |x|<1 \text{ for } x\in[-\tfrac{1}{2},1]$$

· $4-4|x-\frac{1}{2}| > 1-|x|$

$\Rightarrow |x| > 4|x-\frac{1}{2}|-3$

$$\Rightarrow x > 4|x-\tfrac{1}{2}|-3 \quad \text{or} \quad x < -[4|x-\tfrac{1}{2}|-3]$$
$$\Rightarrow \tfrac{1}{4}(x+3) > |x-\tfrac{1}{2}| \quad \text{or} \quad x < -4|x-\tfrac{1}{2}|+3$$
$$\Rightarrow -\tfrac{1}{4}(x+3) < x-\tfrac{1}{2} < \tfrac{1}{4}(x+3) \quad \text{or} \quad -\tfrac{1}{4}(x-3) > |x-\tfrac{1}{2}|$$
$$\Rightarrow -\tfrac{1}{4}(x+3) < x-\tfrac{1}{2},\ x-\tfrac{1}{2} < \tfrac{1}{4}(x+3) \quad \text{or} \quad \tfrac{1}{4}(3-x) > |x-\tfrac{1}{2}|$$
$$\Rightarrow -\tfrac{1}{4}x-\tfrac{3}{4} < x-\tfrac{1}{2},\ x-\tfrac{1}{2} < \tfrac{1}{4}x+\tfrac{3}{4} \quad \text{or} \quad -\tfrac{1}{4}(3-x) < x-\tfrac{1}{2} < \tfrac{1}{4}(3-x)$$
$$\Rightarrow -\tfrac{1}{4} < \tfrac{5}{4}x,\ \tfrac{3}{4}x < \tfrac{5}{4} \quad \text{or} \quad -\tfrac{1}{4}(3-x) < x-\tfrac{1}{2},\ x-\tfrac{1}{2} < \tfrac{1}{4}(3-x)$$
$$\Rightarrow x > -\tfrac{1}{4}\cdot\tfrac{4}{5} = -\tfrac{1}{5},\ x < \cancel{\tfrac{5}{4}}\cdot\tfrac{4}{3} = \tfrac{5}{3} \quad \text{or} \quad -\tfrac{3}{4}+\tfrac{1}{4}x < x-\tfrac{1}{2},\ x-\tfrac{1}{2} < \tfrac{3}{4}-\tfrac{1}{4}x$$
$$\Rightarrow x \in (-\tfrac{1}{5}, \tfrac{5}{3}) \quad \text{or} \quad -\tfrac{1}{4} < \tfrac{3}{4}x,\ \tfrac{5}{4}x < \tfrac{5}{4}$$
$$\text{or} \quad x > -\cancel{\tfrac{1}{4}}\,\tfrac{4}{3} = -\tfrac{1}{3},\ x < 1$$
$$\text{or} \quad x \in (-\tfrac{1}{3}, 1)$$

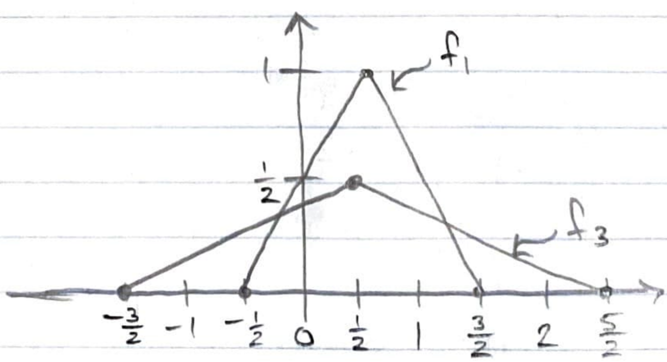
- So $R_1^* = [(-\tfrac{1}{5}, \tfrac{5}{3}) \cup (-\tfrac{1}{3}, 1)] \cap [-\tfrac{1}{2}, 1]$
  $= (-\tfrac{1}{3}, \tfrac{5}{3}) \cap [-\tfrac{1}{2}, 1]$
  $= (-\tfrac{1}{3}, 1]$.
- Thus, $R_1 = R_1^* \cup (1, \tfrac{3}{2}]$
  $= (-\tfrac{1}{3}, 1] \cup (1, \tfrac{3}{2}]$
  $= (-\tfrac{1}{3}, \tfrac{3}{2}]$.
- $R_2 = [-1, \tfrac{3}{2}] \setminus (-\tfrac{1}{3}, \tfrac{3}{2}] = [-1, -\tfrac{1}{3})$.
- We arbitrarily break ties in favor of π_1, so

$$R_1 = [-\tfrac{1}{3}, \tfrac{3}{2}],$$
$$R_2 = [-1, -\tfrac{1}{3})$$

are the classification regions for π_1, π_2, respectively.

2c. Sketch f_1, f_3. Identify classification regions by ECM rule, $p_1 = 0.8$, $c(1|3) = c(3|1)$.

- Sketch



- First, note that if $x > \frac{3}{2}$, then obviously we should classify $x \in \pi_3$, & if $x < -\frac{1}{2}$, then we should classify $x \in \pi_3$ as well. The "ambiguous" region is $[-\frac{1}{2}, \frac{3}{2}]$.

- $R_1^* = \left\{ x : \frac{f_1(x)}{f_2(x)} > \frac{p_2}{p_1} \frac{c(1|2)}{c(2|1)} \right\}$

$$= \left\{ x : \frac{(1-|x-\frac{1}{2}|)\,\mathbb{I}_{\{-\frac{1}{2} \le x \le \frac{3}{2}\}}}{\frac{1}{4}(2-|x-\frac{1}{2}|)\,\mathbb{I}_{\{-\frac{3}{2} \le x \le \frac{5}{2}\}}} > \frac{1-0.8}{0.8} \cdot 1 \right\}$$

$$= \left\{ x : \frac{1-|x-\frac{1}{2}|}{\frac{1}{4}(2-|x-\frac{1}{2}|)} > \frac{1}{4} \right\} \text{ for } x \in [-\tfrac{1}{2}, \tfrac{3}{2}]$$

$$= \left\{ x : 4 - 4|x-\tfrac{1}{2}| > \tfrac{1}{4}(2-|x-\tfrac{1}{2}|) \right\} \text{ since } |x-\tfrac{1}{2}| < 1 < 2 \text{ for } x \in [-\tfrac{1}{2}, \tfrac{3}{2}]$$

- $4 - 4|x-\frac{1}{2}| > \frac{1}{4}(2-|x-\frac{1}{2}|)$

$\Rightarrow 16 - 16|x-\frac{1}{2}| > 2 - |x-\frac{1}{2}|$

$\Rightarrow 14 > 15|x-\frac{1}{2}|$

$\Rightarrow |x-\frac{1}{2}| < \frac{14}{15}$

$\Rightarrow -\frac{14}{15} < x - \frac{1}{2} < \frac{14}{15}$

$\Rightarrow \frac{-13}{30} < x < \frac{43}{30}$

- So, $R_1 = (-\frac{13}{30}, \frac{43}{30}) \cap [-\frac{1}{2}, \frac{3}{2}] = (-\frac{13}{30}, \frac{43}{30})$.
- $R_3 = [-\frac{3}{2}, \frac{5}{2}] \setminus (-\frac{13}{30}, \frac{43}{30})$

$= [-\frac{3}{2}, \frac{-13}{30}) \cup (\frac{43}{30}, \frac{5}{2}]$ — which already includes the "unambiguous" ranges $[-\frac{3}{2}, -\frac{1}{2})$ & $(\frac{3}{2}, \frac{5}{2}]$.

- We arbitrarily break ties in favor of π_3, so

$$R_1 = (-\tfrac{13}{30}, \tfrac{43}{30}),$$
$$R_3 = [-\tfrac{3}{2}, \tfrac{-13}{30}] \cup [\tfrac{43}{30}, \tfrac{5}{2}]$$

are the classification regions for π_1, π_3, respectively.