STAT 32950

Robert
Winter

HW #6

4. $f(x) = p_1 f_1(x) + p_2 f_2(x)$, $p_1 + p_2 = 1$

$$f_1(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & o/w \end{cases}, \quad f_2(x) = \begin{cases} 2(1-x), & 0 \le x \le 1 \\ 0, & o/w \end{cases}$$

4a. Data: $\{x_1, x_2, x_3, x_4, x_5\} = \{0.1, 0.2, 0.3, 0.4, 0.7\}$. Write likelihood fcn.

- $L(p_1, p_2|x) = \prod_{i=1}^{5} f(x_i)$

$$= \prod_{i=1}^{5} [p_1 f_1(x_i) + p_2 f_2(x_i)]$$

$$= [p_1 f_1(0.1) + p_2 f_2(0.1)][p_1 f_1(0.2) + p_2 f_2(0.2)]$$
$$[p_1 f_1(0.3) + p_2 f_2(0.3)][p_1 f_1(0.4) + p_2 f_2(0.4)]$$
$$[p_1 f_1(0.7) + p_2 f_2(0.7)]$$

$$\Rightarrow L(p_1, p_2|x) = [0.2p_1 + 1.8p_2][0.4p_1 + 1.6p_2][0.6p_1 + 1.4p_2] \cdot [0.8p_1 + 1.2p_2][1.4p_1 + 0.6p_2]$$

- Since $p_2 = 1 - p_1$, we can further simplify:

$$L(p_1|x) = [0.2p_1 + 1.8(1-p_1)][0.4p_1 + 1.6(1-p_1)][0.6p_1 + 1.4(1-p_1)]$$
$$[0.8p_1 + 1.2(1-p_1)][1.4p_1 + 0.6(1-p_1)]$$

$$\Rightarrow L(p_1|x) = [1.8 - 1.6p_1][1.6 - 1.2p_1][1.4 - 0.8p_1][1.2 - 0.4p_1][0.6 + 0.8p_1]$$

4b. Data $\{x_1, x_2, x_3, x_4, x_5\} = \{0.1, 0.2, 0.3, 0.4, 0.9\}$. Write likelihood fcn.

- $L(p_1, p_2|x) = \prod_{i=1}^{5} f(x_i)$

$$= \prod_{i=1}^{5} [p_1 f_1(x_i) + p_2 f_2(x_i)]$$

$$= [p_1 f_1(0.1) + p_2 f_2(0.1)][p_1 f_1(0.2) + p_2 f_2(0.2)]$$
$$[p_1 f_1(0.3) + p_2 f_2(0.3)][p_1 f_1(0.4) + p_2 f_2(0.4)]$$
$$[p_1 f_1(0.9) + p_2 f_2(0.9)]$$

$$\Rightarrow L(p_1, p_2|x) = [0.2p_1 + 1.8p_2][0.4p_1 + 1.6p_2][0.6p_1 + 1.4p_2]$$
$$[0.8p_1 + 1.2p_2][1.8p_1 + 0.2p_2]$$

· Since $p_2 = 1 - p_1$,

$$L(p_1|x) = [0.2p_1 + 1.8(1-p_1)][0.4p_1 + 1.6(1-p_1)]$$
$$[0.6p_1 + 1.4(1-p_1)][0.8p_1 + 1.2(1-p_1)]$$
$$[1.8p_1 + 0.2(1-p_1)]$$

$$L(p_1|x) = [1.8 - 1.6p_1][1.6 - 1.2p_1][1.4 - 0.8p_1]$$
$$[1.2 - 0.4p_1][0.2 + 1.6p_1]$$

4c. Data $\{x_1, x_2, x_3, x_4, x_5\} = \{0.1, 0.2, 0.3, 0.6, 0.9\}$. Write likelihood fcn.

$$\cdot\ L(p_1, p_2|x) = \prod_{i=1}^{5} f(x_i)$$
$$= \prod_{i=1}^{5} [p_1 f_1(x_i) + p_2 f_2(x_i)]$$
$$= [p_1 f_1(0.1) + p_2 f_2(0.1)][p_1 f_1(0.2) + p_2 f_2(0.2)]$$
$$[p_1 f_1(0.3) + p_2 f_2(0.3)][p_1 f_1(0.6) + p_2 f_2(0.6)]$$
$$[p_1 f_1(0.9) + p_2 f_2(0.9)]$$

$$\Rightarrow L(p_1, p_2|x) = [0.2p_1 + 1.8p_2][0.4p_1 + 1.6p_2][0.6p_1 + 1.4p_2]$$
$$[1.2p_1 + 0.8p_2][1.8p_1 + 0.2p_2]$$

· Since $p_2 = 1 - p_1$,

$$L(p_1|x) = [0.2p_1 + 1.8(1-p_1)][0.4p_1 + 1.6(1-p_1)]$$
$$[0.6p_1 + 1.4(1-p_1)][1.2p_1 + 0.8(1-p_1)]$$
$$[1.8p_1 + 0.2(1-p_1)]$$

$$\Rightarrow L(p_1|x) = [1.8 - 1.6p_1][1.6 - 1.2p_1][1.4 - 0.8p_1]$$
$$[0.8 + 0.4p_1][0.2 + 1.6p_1]$$

6. $X_1, X_2, X_3, X_4 \sim N_3(\mu, \Sigma)$:

$$X = [x_{jk}] = \begin{bmatrix} 3 & 6 & 0 \\ 4 & 4 & 3 \\ ? & 8 & 3 \\ 5 & ? & ? \end{bmatrix} \quad \begin{matrix} X_1^T = \text{obs } 1 \\ X_2^T = \text{obs } 2 \\ X_3^T = \text{obs } 3 \\ X_4^T = \text{obs } 4 \end{matrix}$$

Use EM algo. to estimate μ & Σ.

6a. Impute values to obtain $\tilde{X}$, data matrix w/ missing values filled by initial estimates.

- $\tilde{x}_{31} = \frac{3+4+5}{3} = \frac{12}{3} = 4$
- $\tilde{x}_{42} = \frac{6+4+8}{3} = \frac{18}{3} = 6$
- $\tilde{x}_{43} = \frac{0+3+3}{3} = \frac{6}{3} = 2$
- So,

$$\tilde{X} = \begin{bmatrix} 3 & 6 & 0 \\ 4 & 4 & 3 \\ 4 & 8 & 3 \\ 5 & 6 & 2 \end{bmatrix}$$

6b. Find $\tilde{\mu}$, $\tilde{\Sigma}$ estimated from $\tilde{X}$.

$$\cdot\ \tilde{\mu} = \begin{bmatrix} (3+4+4+5)/4 \\ (6+4+8+6)/4 \\ (0+3+3+2)/4 \end{bmatrix} = \begin{bmatrix} \frac{16}{4} \\ \frac{24}{4} \\ \frac{8}{4} \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix} = \tilde{\mu}$$

$$\cdot\ \tilde{\Sigma} = \begin{bmatrix} \tilde{\sigma}_1^2 & \tilde{\sigma}_{12} & \tilde{\sigma}_{13} \\ \tilde{\sigma}_{12} & \tilde{\sigma}_2^2 & \tilde{\sigma}_{23} \\ \tilde{\sigma}_{13} & \tilde{\sigma}_{23} & \tilde{\sigma}_3^2 \end{bmatrix}$$

- $\tilde{\sigma}_1^2 = \frac{1}{4}\left[(3-4)^2 + (4-4)^2 + (4-4)^2 + (5-4)^2\right] = \frac{1}{4}[2] = \frac{1}{2}$
- $\tilde{\sigma}_2^2 = \frac{1}{4}\left[(6-6)^2 + (4-6)^2 + (8-6)^2 + (6-6)^2\right] = \frac{1}{4}[4+4] = 2$
- $\tilde{\sigma}_3^2 = \frac{1}{4}\left[(0-2)^2 + (3-2)^2 + (3-2)^2 + (2-2)^2\right] = \frac{1}{4}[4+1+1] = \frac{3}{2}$
- $\tilde{\sigma}_{12} = \frac{1}{4}\left[(3-4)(6-6) + (4-4)(4-6) + (4-4)(8-6) + (5-4)(6-6)\right] = \frac{1}{4}[0] = 0$
- $\tilde{\sigma}_{13} = \frac{1}{4}\left[(3-4)(0-2) + (4-4)(3-2) + (4-4)(3-2) + (5-4)(2-2)\right] = \frac{1}{4}[2] = \frac{1}{2}$

$\cdot\ \tilde{\sigma}_{23} = \frac{1}{4}[(6-6)(0-2)+(4-6)(3-2)+(8-6)(3-2)+(6-6)(2-2)]$
$= \frac{1}{4}[0-2+2+0]$
$= 0$

$\cdot$ So,
$$\tilde{\Sigma} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 2 & 0 \\ \frac{1}{2} & 0 & \frac{3}{2} \end{bmatrix}$$

6c. Iterate to find the first revised estimates:

6c(i). Find revised estimate of x_{31}, using estimated $\tilde{\mu}$, $\tilde{\Sigma}$ from Part (b). What are updated estimates of $\tilde{\mu}$, $\tilde{\Sigma}$?

$$\cdot\ \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \approx N_3\left(\begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 2 & 0 \\ \frac{1}{2} & 0 & \frac{3}{2} \end{bmatrix}\right)$$

$\cdot$ We partition as

$$\begin{matrix} X_{(1)} \rightarrow \\ X_{(23)} \rightarrow \end{matrix} \left[\begin{array}{c} X_1 \\ \hline X_2 \\ X_3 \end{array}\right] \approx N_3\left(\underbrace{\overbrace{\left[\begin{array}{c} 4 \\ \hline 6 \\ 2 \end{array}\right]}^{\tilde{\mu}_{(1)}}}_{\tilde{\mu}_{(23)}}, \left[\begin{array}{c|cc} \frac{1}{2} & 0 & \frac{1}{2} \\ \hline 0 & 2 & 0 \\ \frac{1}{2} & 0 & \frac{3}{2} \end{array}\right]\right)$$

$\tilde{\Sigma}_{(1)(1)}$, $\tilde{\Sigma}_{(1)(23)}$, $\tilde{\Sigma}_{(23)(1)}$, $\tilde{\Sigma}_{(23)(23)}$

$\cdot$ So,

$$X_{(1)} \mid X_{(23)} = x_{(23)} \approx N\left(\tilde{\mu}_{(1)} + \tilde{\Sigma}_{(1)(23)}\tilde{\Sigma}^{-1}_{(23)(23)}(x_{(23)} - \tilde{\mu}_{(23)}),\ \tilde{\Sigma}_{(1)(1)} - \tilde{\Sigma}_{(1)(23)}\tilde{\Sigma}^{-1}_{(23)(23)}\tilde{\Sigma}_{(23)(1)}\right)$$

$$= N\left(4 + \begin{bmatrix} 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}\left(\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 6 \\ 2 \end{bmatrix}\right), \frac{1}{2} - \begin{bmatrix} 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}\right)$$

$$= N\left(4 + \begin{bmatrix} 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} x_2 - 6 \\ x_3 - 2 \end{bmatrix}, \frac{1}{2} - \begin{bmatrix} 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix}\right)$$

$$= N\left(4 + \frac{1}{3}(x_3 - 2), \frac{1}{2} - \frac{1}{6}\right)$$

$$= N\left(4 + \frac{1}{3}x_3 - \frac{2}{3}, \frac{1}{3}\right)$$

$$= N\left(\frac{10}{3} + \frac{1}{3}x_3, \frac{1}{3}\right)$$

• So, $\tilde{X}_{31} = \mathbb{E}[\tilde{X}_{(1)} | X_{(23)} = (8,3)]$

$= \frac{10}{3} + \frac{1}{3}(3)$

$\tilde{X}_{31} = \frac{13}{3} \approx 4.33$

• Now,

$$\tilde{X} = \begin{bmatrix} 3 & 6 & 0 \\ 4 & 4 & 3 \\ \frac{13}{3} & 8 & 3 \\ 5 & 6 & 2 \end{bmatrix}$$

• So

$$\tilde{\mu} = \begin{bmatrix} (3+4+\frac{13}{3}+5)/4 \\ (6+4+8+6)/4 \\ (0+3+3+2)/4 \end{bmatrix} = \begin{bmatrix} \frac{49}{3}/4 \\ 24/4 \\ 8/4 \end{bmatrix} = \begin{bmatrix} \frac{49}{12} \\ 6 \\ 2 \end{bmatrix} = \tilde{\mu}$$

• $\tilde{\sigma}_2^2$, $\tilde{\sigma}_3^2$, & $\tilde{\sigma}_{23}$ are unchanged, since we have only updated $\tilde{X}_{31}$.

• $\tilde{\sigma}_1^2 = \frac{1}{4}[(3-\frac{49}{12})^2 + (4-\frac{49}{12})^2 + (\frac{13}{3}-\frac{49}{12})^2 + (5-\frac{49}{12})^2]$

$= \frac{1}{4}[\frac{169}{144} + \frac{1}{144} + \frac{1}{16} + \frac{121}{144}]$

$= \frac{1}{4}[\frac{25}{12}]$

$= \frac{25}{48}$

• $\tilde{\sigma}_{12} = \frac{1}{4}[(3-\frac{49}{12})(6-6) + (4-\frac{49}{12})(4-6) + (\frac{13}{3}-\frac{49}{12})(8-6) + (5-\frac{49}{12})(6-6)]$

$= \frac{1}{4}[0 + (-\frac{1}{12})(-2) + (\frac{1}{4})(2) + 0]$

$= \frac{1}{4}[\frac{1}{6} + \frac{1}{2}]$

$= \frac{2}{12}$

$= \frac{1}{6}$

• $\tilde{\sigma}_{13} = \frac{1}{4}[(3-\frac{49}{12})(0-2) + (4-\frac{49}{12})(3-2) + (\frac{13}{3}-\frac{49}{12})(3-2) + (5-\frac{49}{12})(2-2)]$

$= \frac{1}{4}[(-\frac{13}{12})(-2) + (-\frac{1}{12})(1) + (\frac{1}{4})(1) + 0]$

$= \frac{1}{4}[\frac{13}{6} - \frac{1}{12} + \frac{1}{4}]$

$= \frac{1}{4}[\frac{7}{3}]$

$= \frac{7}{12}$

• So,

$$\tilde{\Sigma} = \begin{bmatrix} \frac{25}{48} & \frac{1}{6} & \frac{7}{12} \\ \frac{1}{6} & 2 & 0 \\ \frac{7}{12} & 0 & \frac{3}{2} \end{bmatrix}$$

6(c)(ii) Find revised estimates of x_{42}, x_{43}, using estimated $\tilde{\mu}$ & $\tilde{\Sigma}$ from Step (c)(i). Write out updated $\tilde{\mu}$, $\tilde{\Sigma}$ after first iteration.

- $$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \approx N_3\left(\begin{bmatrix} \frac{49}{12} \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} \frac{25}{48} & \frac{1}{6} & \frac{7}{12} \\ \frac{1}{6} & 2 & 0 \\ \frac{7}{12} & 0 & \frac{3}{2} \end{bmatrix} \right)$$

- We partition as

$\tilde{\mu}_{(1)}$ $\quad$ $\tilde{\Sigma}_{(1)(1)}$ $\quad$ $\tilde{\Sigma}_{(1)(23)}$

$$\begin{matrix} X_{(1)} \rightarrow \\ \\ X_{(23)} \rightarrow \end{matrix} \left[\begin{array}{c} X_1 \\ \hline X_2 \\ X_3 \end{array}\right] \approx N_3\left(\left[\begin{array}{c} \frac{49}{12} \\ \hline 6 \\ 2 \end{array}\right], \left[\begin{array}{c|cc} \frac{25}{48} & \frac{1}{6} & \frac{7}{12} \\ \hline \frac{1}{6} & 2 & 0 \\ \frac{7}{12} & 0 & \frac{3}{2} \end{array}\right] \right)$$

$\tilde{\mu}_{(23)}$ $\quad$ $\tilde{\Sigma}_{(23)(1)}$ $\quad$ $\tilde{\Sigma}_{(23)(23)}$

- So,

$$X_{(23)} \mid X_{(1)} = x_{(1)} \approx N\left(\tilde{\mu}_{(23)} + \tilde{\Sigma}_{(23)(1)} \tilde{\Sigma}_{(1)(1)}^{-1} (x_{(1)} - \tilde{\mu}_{(1)}), \tilde{\Sigma}_{(23)(23)} - \tilde{\Sigma}_{(23)(1)} \tilde{\Sigma}_{(1)(1)}^{-1} \tilde{\Sigma}_{(1)(23)}\right)$$

$$= N\left(\begin{bmatrix} 6 \\ 2 \end{bmatrix} + \begin{bmatrix} \frac{1}{6} \\ \frac{7}{12} \end{bmatrix} \cdot \frac{48}{25} \left(x_{(1)} - \frac{49}{12}\right), \begin{bmatrix} 2 & 0 \\ 0 & \frac{3}{2} \end{bmatrix} - \begin{bmatrix} \frac{1}{6} \\ \frac{7}{12} \end{bmatrix} \frac{48}{25} \begin{bmatrix} \frac{1}{6} & \frac{7}{12} \end{bmatrix} \right)$$

$$= N\left(\begin{bmatrix} 6 \\ 2 \end{bmatrix} + \begin{bmatrix} 8/25 \\ 28/25 \end{bmatrix} \left(x_{(1)} - \frac{49}{12}\right), \begin{bmatrix} 2 & 0 \\ 0 & \frac{3}{2} \end{bmatrix} - \frac{48}{25} \begin{bmatrix} \frac{1}{36} & \frac{7}{72} \\ \frac{7}{72} & \frac{49}{144} \end{bmatrix} \right)$$

$$= N\left(\begin{bmatrix} 6 \\ 2 \end{bmatrix} + \begin{bmatrix} \frac{8}{25} X_{(1)} - \frac{98}{75} \\ \frac{28}{25} X_{(1)} - \frac{343}{75} \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & \frac{3}{2} \end{bmatrix} - \begin{bmatrix} \frac{4}{75} & \frac{14}{75} \\ \frac{14}{75} & \frac{49}{75} \end{bmatrix} \right)$$

$$= N\left(\begin{bmatrix} \frac{352}{75} + \frac{8}{25} X_{(1)} \\ -\frac{193}{75} + \frac{28}{25} X_{(1)} \end{bmatrix}, \begin{bmatrix} \frac{146}{75} & -\frac{14}{75} \\ -\frac{14}{75} & \frac{127}{150} \end{bmatrix} \right)$$

- So, $$\begin{bmatrix} \tilde{x}_{42} \\ \tilde{x}_{43} \end{bmatrix} = \mathbb{E}\left[\tilde{X}_{(23)} \mid X_1 = 5 \right]$$

$$= \begin{bmatrix} \frac{352}{75} + \frac{8}{25}(5) \\ -\frac{193}{75} + \frac{28}{25}(5) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{352}{75} + \frac{8}{5} \\ -\frac{193}{75} + \frac{28}{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{472}{75} \\ \frac{227}{75} \end{bmatrix}$$

$$\Rightarrow \quad \tilde{x}_{42} = \frac{472}{75} \approx 6.293$$
$$\tilde{x}_{43} = \frac{227}{75} \approx 3.027$$

- Now,

$$\tilde{X} = \begin{bmatrix} 3 & 6 & 0 \\ 4 & 4 & 3 \\ \frac{13}{3} & 8 & 3 \\ 5 & \frac{472}{75} & \frac{227}{75} \end{bmatrix}$$

- So

$$\tilde{\mu} = \begin{bmatrix} (3+4+\frac{13}{3}+5)/4 \\ (6+4+8+\frac{472}{75})/4 \\ (0+3+3+\frac{227}{75})/4 \end{bmatrix} = \begin{bmatrix} \frac{49}{3}/4 \\ \frac{1822}{75}/4 \\ \frac{677}{75}/4 \end{bmatrix} = \begin{bmatrix} \frac{49}{12} \\ \frac{911}{150} \\ \frac{677}{300} \end{bmatrix} \approx \begin{bmatrix} 4.083 \\ 6.073 \\ 2.257 \end{bmatrix} = \tilde{\mu}$$

- $\tilde{\sigma}_1^2$ is unchanged.
- $\tilde{\sigma}_2^2 = \frac{1}{4}\left[\left(6-\frac{911}{150}\right)^2+\left(4-\frac{911}{150}\right)^2+\left(8-\frac{911}{150}\right)^2+\left(\frac{472}{75}-\frac{911}{150}\right)^2\right]$

$$= \frac{1}{4}\left[\frac{121}{22500}+\frac{96721}{22500}+\frac{83521}{22500}+\frac{121}{2500}\right]$$
$$= \frac{1}{4}\left[\frac{15121}{1875}\right]$$
$$= \frac{15121}{7500} \approx 2.016$$

- $\tilde{\sigma}_3^2 = \frac{1}{4}\left[\left(0-\frac{677}{300}\right)^2+\left(3-\frac{677}{300}\right)^2+\left(3-\frac{677}{300}\right)^2+\left(\frac{227}{75}-\frac{677}{300}\right)^2\right]$

$$= \frac{1}{4}\left[\frac{458329}{90000}+\frac{49729}{90000}+\frac{49729}{90000}+\frac{5929}{10000}\right]$$
$$= \frac{1}{4}\left[\frac{50929}{7500}\right]$$
$$= \frac{50929}{30000} \approx 1.698$$

- $\tilde{\sigma}_{12} = \frac{1}{4}\left[\left(3-\frac{49}{12}\right)\left(6-\frac{911}{150}\right)+\left(4-\frac{49}{12}\right)\left(4-\frac{911}{150}\right)+\left(\frac{13}{3}-\frac{49}{12}\right)\left(8-\frac{911}{150}\right)+\left(5-\frac{49}{12}\right)\left(\frac{472}{75}-\frac{911}{150}\right)\right]$

$$= \frac{1}{4}\left[\left(-\frac{13}{12}\right)\left(\frac{-11}{150}\right)+\left(-\frac{1}{12}\right)\left(-\frac{311}{150}\right)+\left(\frac{1}{4}\right)\left(\frac{289}{150}\right)+\left(\frac{11}{12}\right)\left(\frac{11}{50}\right)\right]$$
$$= \frac{1}{4}\left[\frac{143}{1800}+\frac{311}{1800}+\frac{289}{600}+\frac{121}{600}\right]$$
$$= \frac{1}{4}\left[\frac{421}{450}\right]$$
$$= \frac{421}{1800} \approx 0.234$$

- $\tilde{\sigma}_{13} = \frac{1}{4}\left[\left(3-\frac{49}{12}\right)\left(0-\frac{677}{300}\right)+\left(4-\frac{49}{12}\right)\left(3-\frac{677}{300}\right)+\left(\frac{13}{3}-\frac{49}{12}\right)\left(3-\frac{677}{300}\right)+\left(5-\frac{49}{12}\right)\left(\frac{227}{75}-\frac{677}{300}\right)\right]$

$$= \frac{1}{4}\left[\left(-\frac{13}{12}\right)\left(-\frac{677}{300}\right)+\left(-\frac{1}{12}\right)\left(\frac{223}{300}\right)+\left(\frac{1}{4}\right)\left(\frac{223}{300}\right)+\left(\frac{11}{12}\right)\left(\frac{77}{100}\right)\right]$$
$$= \frac{1}{4}\left[\frac{8801}{3600}-\frac{223}{3600}+\frac{223}{1200}+\frac{847}{1200}\right]$$
$$= \frac{1}{4}\left[\frac{2947}{900}\right]$$
$$= \frac{2947}{3600} \approx 0.819$$

$$\cdot\ \tilde{\sigma}_{23} = \frac{1}{4}\Big[\left(6-\tfrac{911}{150}\right)\left(0-\tfrac{677}{300}\right)+\left(4-\tfrac{911}{150}\right)\left(3-\tfrac{677}{300}\right)+\left(8-\tfrac{911}{150}\right)\left(3-\tfrac{677}{300}\right)$$
$$+\left(\tfrac{472}{75}-\tfrac{911}{150}\right)\left(\tfrac{227}{75}-\tfrac{677}{300}\right)\Big]$$
$$= \frac{1}{4}\left[\left(-\tfrac{11}{150}\right)\left(-\tfrac{677}{300}\right)+\left(\tfrac{-311}{150}\right)\left(\tfrac{223}{300}\right)+\left(\tfrac{289}{150}\right)\left(\tfrac{223}{300}\right)+\left(\tfrac{11}{50}\right)\left(\tfrac{77}{100}\right)\right]$$
$$= \frac{1}{4}\left[\frac{7447}{45000}-\frac{69353}{45000}+\frac{64447}{45000}+\frac{847}{5000}\right]$$
$$= \frac{1}{4}\left[\frac{847}{3750}\right]$$
$$= \frac{847}{15000} \approx 0.056$$

· Thus,

$$\tilde{\Sigma} = \begin{bmatrix} \frac{25}{48} & \frac{421}{1800} & \frac{2947}{3600} \\ \frac{421}{1800} & \frac{15121}{7500} & \frac{847}{15000} \\ \frac{2947}{3600} & \frac{847}{15000} & \frac{50929}{30000} \end{bmatrix} \approx \begin{bmatrix} 0.521 & 0.234 & 0.819 \\ 0.234 & 2.016 & 0.056 \\ 0.819 & 0.056 & 1.698 \end{bmatrix}$$

7. Dissimilarity measures d_1, d_2 defined in space S are <u>global-order equivalent</u> if

$$d_1(x,y) \le d_1(w,z) \Leftrightarrow d_2(x,y) \le d_2(w,z) \quad \forall x, y, w, z \in S$$

7a Let $d(a,b) = \sqrt{(a-b)^T(a-b)}$ be Euclidean dist b/w $a, b \in \mathbb{R}^p$,
$D(a,b) = (d(a,b))^2$.
Show $d_1 = d$, $d_2 = D$ are global-order equivalent.

• ($\Rightarrow$) • Let $x, y, w, z \in \mathbb{R}^p$ such that $d(x,y) \le d(w,z)$.
• Then

$$\sqrt{(x-y)^T(x-y)} \le \sqrt{(w-z)^T(w-z)}$$

$$\Rightarrow \sqrt{(x-y)^T(x-y)}\sqrt{(x-y)^T(x-y)} \le \sqrt{(w-z)^T(w-z)}\sqrt{(x-y)^T(x-y)}$$

since $\sqrt{(x-y)^T(x-y)} \ge 0$

$$\Rightarrow (x-y)^T(x-y) \le \sqrt{(w-z)^T(w-z)}\sqrt{(w-z)^T(w-z)}$$

since $\sqrt{(x-y)^T(x-y)} \le \sqrt{(w-z)^T(w-z)}$

$$\Rightarrow D(x,y) \le (w-z)^T(w-z) = D(w,z)$$

• Thus, $d(x,y) \le d(w,z) \Rightarrow D(x,y) \le D(w,z)$.

• ($\Leftarrow$) • Let $x, y, w, z \in \mathbb{R}^p$ s.t. $D(x,y) \le D(w,z)$.
• Then

$$(x-y)^T(x-y) \le (w-z)^T(w-z)$$

$$\Rightarrow \sqrt{(x-y)^T(x-y)} \le \sqrt{(w-z)^T(w-z)}$$

since the square root is a monotone function on $[0, \infty)$, and $D(x,y), D(w,z) \ge 0$

$$\Rightarrow d(x,y) \le d(w,z)$$

• Thus, $D(x,y) \le D(w,z) \Rightarrow d(x,y) \le d(w,z)$

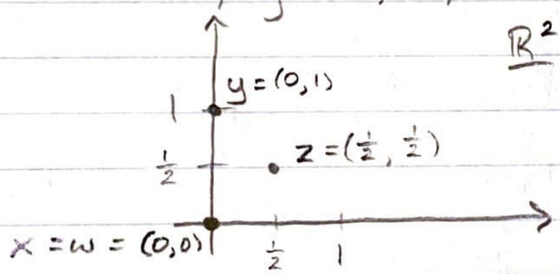
• So, d & D are global-order equivalent. $\square$

7b. Show by counterexample that g.o.e. does not hold for d_1 = Euclidean dist. & d_2 = Canberra dist.,

$$d_2(x,y) = \begin{cases} \sum_{i=1}^{p} \frac{|x_i - y_i|}{|x_i| + |y_i|}, & x_i - y_i \neq 0 \\ 0, & x_i - y_i = 0 \end{cases}$$

for $x, y \in \mathbb{R}^p$.

- Consider points in $\mathbb{R}^2$ (i.e., $p=2$).
- Let $x=(0,0)$, $y=(0,1)$, $w=(0,0)$, $z=(\frac{1}{2},\frac{1}{2})$:



- $d_1(x,y) = \sqrt{(1-0)^2+(0-0)^2} = \sqrt{1+0} = 1$
- $d_1(w,z) = \sqrt{(\frac{1}{2}-0)^2+(\frac{1}{2}-0)^2} = \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} \approx 0.707$
- So, $d_1(x,y) = 1 > \frac{1}{\sqrt{2}} = d_1(w,z)$

- $d_2(x,y) = 0 + \frac{|0-1|}{|0|+|1|} = 0 + \frac{1}{0+1} = 1$
- $d_2(w,z) = \frac{|\frac{1}{2}-0|}{|\frac{1}{2}|+|0|} + \frac{|\frac{1}{2}-0|}{|\frac{1}{2}|+|0|} = \frac{\frac{1}{2}}{\frac{1}{2}} + \frac{\frac{1}{2}}{\frac{1}{2}} = 1+1 = 2$
- So, $d_2(x,y) = 1 < 2 = d_2(w,z)$

- Thus, $\exists\, x, y, w, z \in \mathbb{R}^2$ s.t. $d_1(x,y) > d_1(w,z)$ but $d_2(x,y) < d_2(w,z)$. Thus, d_1 & d_2 are not global-order equivalent. $\square$