STAT 32950

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HW #2

1. $X \sim N_3(\mu, \Sigma)$, $\mu = (1, -3, 2)$,

$$\Sigma = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 7 & 0 \\ -1 & 0 & 2 \end{bmatrix}.$$

1a(i) $Y_1 = X_1 + 3X_2 - 2X_3$. $X_1 \perp\!\!\!\perp Y_1$?

- $$\begin{aligned} \mathbb{E}[Y_1] &= \mathbb{E}[X_1 + 3X_2 - 2X_3] \\ &= \mathbb{E}[X_1] + 3\mathbb{E}[X_2] - 2\mathbb{E}[X_3] \\ &= 1 + 3(-3) - 2(2) \\ &= -12 \end{aligned}$$

- $$\begin{aligned} \mathbb{E}[X_1Y_1] &= \mathbb{E}[X_1(X_1 + 3X_2 - 2X_3)] \\ &= \mathbb{E}[X_1^2 + 3X_1X_2 - 2X_1X_3] \\ &= \mathbb{E}[X_1^2] + 3\mathbb{E}[X_1X_2] - 2\mathbb{E}[X_1X_3] \\ &= (Var(X_1) + \mathbb{E}[X_1]^2) + 3(Cov(X_1, X_2) + \mathbb{E}[X_1]\mathbb{E}[X_2]) \\ &\quad - 2(Cov(X_1, X_3) + \mathbb{E}[X_1]\mathbb{E}[X_3]) \\ &= (4 + 1^2) + 3(0 + (1)(-3)) - 2(-1 + (1)(2)) \\ &= 5 + 3(-3) - 2(1) \\ &= -6 \end{aligned}$$

- $$\begin{aligned} Cov(X_1, Y_1) &= \mathbb{E}[X_1Y_1] - \mathbb{E}[X_1]\mathbb{E}[Y_1] \\ &= -6 - (1)(-12) \\ &= 6 \end{aligned}$$

- Since $Cov(X_1, Y_1) = 6 \neq 0$, $X_1 \not\perp\!\!\!\perp Y_1$.

1a(ii) $Y_2 = X_1 - X_2 + 4X_3$. $X_1 \perp\!\!\!\perp Y_2$?

- $$\begin{aligned} \mathbb{E}[Y_2] &= \mathbb{E}[X_1 - X_2 + 4X_3] \\ &= \mathbb{E}[X_1] - \mathbb{E}[X_2] + 4\mathbb{E}[X_3] \\ &= 1 - (-3) + 4(2) \\ &= 12 \end{aligned}$$

→

$\cdot\ \mathbb{E}[X_1 Y_2] = \mathbb{E}[X_1(X_1 - X_2 + 4X_3)]$

$= \mathbb{E}[X_1^2 - X_1X_2 + 4X_1X_3]$

$= \mathbb{E}[X_1^2] - \mathbb{E}[X_1X_2] + 4\mathbb{E}[X_1X_3]$

$= (\mathrm{Var}(X_1) + \mathbb{E}[X_1]^2) - (\mathrm{Cov}(X_1, X_2) + \mathbb{E}[X_1]\mathbb{E}[X_2])$
$+ 4(\mathrm{Cov}(X_1, X_3) + \mathbb{E}[X_1]\mathbb{E}[X_3])$

$= (4 + 1^2) - (0 + (1)(-3)) + 4(-1 + (1)(2))$

$= 5 - (-3) + 4(1)$

$= 12$

$\cdot\ \mathrm{Cov}(X_1, Y_2) = \mathbb{E}[X_1Y_2] - \mathbb{E}[X_1]\mathbb{E}[Y_2]$

$= 12 - (1)(12)$

$= 0.$

$\cdot$ Let $a, b \in \mathbb{R}$. Then

$$aX_1 + bY_2 = aX_1 + b(X_1 - X_2 + 4X_3)$$
$$= (a+b)X_1 - bX_2 + 4bX_3,$$

which is a linear combination of X_1, X_2, X_3.
Since $X \sim N_3(\mu, \Sigma)$, $aX_1 + bY_2 = (a+b)X_1 - bX_2 + 4bX_3$ has normal distribution $\forall a, b \in \mathbb{R}$. Thus, (X_1, Y_2) has bivariate normal distribution.

$\cdot$ Since $(X_1, Y_2) \sim N_2(\quad)$ and $\mathrm{Cov}(X_1, Y_2) = 0$, we conclude $X_1 \perp\!\!\!\perp Y_2$.

1b. Derive $\mathbb{E}[X_3 | X_1 = x_1]$, $\mathrm{Var}(X_3 | X_1 = x_1)$.

$\cdot$ Let $\mu_1 = 1$, $\mu_{23} = (-3, 2)$, $\Sigma_{11} = 4$, $\Sigma_{(1)(23)} = [0 \ \ -1]$, $\Sigma_{(23)(1)} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $\Sigma_{(23)(23)} = \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}$, so

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_{23} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{(1)(23)} \\ \Sigma_{(23)(1)} & \Sigma_{(23)(23)} \end{bmatrix}.$$

$\cdot$ Then

$$\begin{bmatrix} X_2 \\ X_3 \end{bmatrix} \Big| X_1 = x_1 \sim N_2\left(\mu_{23} + \Sigma_{(23)(1)}\Sigma_{11}^{-1}(x_1 - \mu_1),\ \Sigma_{(23)(23)} - \Sigma_{(23)(1)}\Sigma_{11}^{-1}\Sigma_{(1)(23)}\right)$$
$$= N_2\left(\begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\left(\tfrac{1}{4}\right)(x_1 - 1),\ \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\left(\tfrac{1}{4}\right)[0 \ \ -1]\right)$$
$$= N_2\left(\begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/4 \end{bmatrix}(x_1 - 1),\ \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} - \tfrac{1}{4}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= N_2\left(\begin{bmatrix}-3\\2\end{bmatrix} + \begin{bmatrix}0\\-\frac{1}{4}x_1+\frac{1}{4}\end{bmatrix}, \begin{bmatrix}7 & 0\\0 & 2\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & \frac{1}{4}\end{bmatrix}\right)$$

$$= N_2\left(\begin{bmatrix}-3\\\frac{9}{4}-\frac{1}{4}x_1\end{bmatrix}, \begin{bmatrix}7 & 0\\0 & \frac{7}{4}\end{bmatrix}\right)$$

- So,

$$X_3 | X_1 = x_1 \sim N\left(\frac{9}{4} - \frac{1}{4}x_1, \frac{7}{4}\right).$$

- That is,

$$\mathbb{E}[X_3 | X_1 = x_1] = \frac{9}{4} - \frac{1}{4}x_1$$

and

$$Var(X_3 | X_1 = x_1) = \frac{7}{4}.$$

1c. Specify the distribution of $X_1 | X_3 = x_3$, & write $f_{X_1|X_3=x_3}(x_1)$.

- Since $X \sim N_3(\mu, \Sigma)$, $aX_1 + bX_2 + cX_3$ has normal distribution $\forall a,b,c \in \mathbb{R}$. In particular, $aX_1 + 0X_2 + cX_3 = aX_1 + cX_3$ has normal distribution $\forall a, c \in \mathbb{R}$. So, (X_1, X_3) has bivariate normal distribution.
- So,

$$X_1 | X_3 = x_3 \sim N\left(\mu_1 + \frac{\sigma_1}{\sigma_3}\rho_{13}(x_3 - \mu_3), \sigma_1^2(1-\rho_{13}^2)\right)$$

$$= N\left(1 + \frac{\sqrt{4}}{\sqrt{2}} \cdot \frac{-1}{\sqrt{4}\sqrt{2}}(x_3 - 2), 4\left(1 - \left(\frac{-1}{\sqrt{4}\sqrt{2}}\right)^2\right)\right)$$

$$= N\left(1 + -\frac{1}{2}(x_3 - 2), 4\left(1 - \frac{1}{8}\right)\right)$$

$$= N\left(1 - \frac{1}{2}x_3 + 1, 4 \cdot \frac{7}{8}\right)$$

$$\Rightarrow X_1 | X_3 = x_3 \sim N\left(2 - \frac{1}{2}x_3, \frac{7}{2}\right)$$

- So, $f_{X_1|X_3=x_3}(x_1) = \frac{1}{\sqrt{2\pi \cdot \frac{7}{2}}} e^{-\frac{1}{2 \cdot \frac{7}{2}}(x_1 - (2 - \frac{1}{2}x_3))^2}$

$$\Rightarrow f_{X_1|X_3=x_3}(x_1) = \frac{1}{\sqrt{7\pi}} e^{-\frac{1}{7}(x_1 - (2 - \frac{1}{2}x_3))^2}$$

1d. Specify the distribution of $X_1 | (X_2, X_3) = (x_2, x_3)$. What is $f_{X_1|(X_2,X_3)=(x_2,x_3)}(x_1)$?

- Let $\mu_1 = 1$, $\mu_{23} = (-3, 2)$, $\Sigma_{11} = 4$, $\Sigma_{(1)(23)} = [0, -1]$, $\Sigma_{(23)(1)} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $\Sigma_{(23)(23)} = \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}$, so

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_{23} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{(1)(23)} \\ \Sigma_{(23)(1)} & \Sigma_{(23)(23)} \end{bmatrix}.$$

- Then

$$X_1 | (X_2, X_3) = (x_2, x_3) \sim N\left(\mu_1 + \Sigma_{(1)(23)} \Sigma_{(23)(23)}^{-1} \left(\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} - \mu_{23} \right), \Sigma_{11} - \Sigma_{(1)(23)} \Sigma_{(23)(23)}^{-1} \Sigma_{(23)(1)} \right)$$

$$= N\left(1 + [0 \ \ -1] \begin{bmatrix} 1/7 & 0 \\ 0 & 1/2 \end{bmatrix} \left(\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right), 4 - [0 \ \ -1] \begin{bmatrix} 1/7 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

$$= N\left(1 + [0 \ \ -\tfrac{1}{2}] \begin{bmatrix} x_2 + 3 \\ x_3 - 2 \end{bmatrix}, 4 - [0 \ \ -\tfrac{1}{2}] \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

$$= N\left(1 + -\tfrac{1}{2}(x_3 - 2), 4 - \tfrac{1}{2}\right)$$

$$= N\left(1 - \tfrac{1}{2} x_3 + 1, \tfrac{7}{2}\right)$$

$$\Rightarrow X_1 | (X_2, X_3) = (x_2, x_3) \sim N\left(2 - \tfrac{1}{2} x_3, \tfrac{7}{2}\right)$$

- So, $f_{X_1|(X_2,X_3)=(x_2,x_3)}(x_1) = \frac{1}{\sqrt{2\pi \cdot \frac{7}{2}}} e^{\frac{-1}{2 \cdot 7/2}\left(x_1 - (2 - \frac{1}{2} x_3)\right)^2}$

$$f_{X_1|(X_2,X_3)=(x_2,x_3)}(x_1) = \frac{1}{\sqrt{7\pi}} e^{-\frac{1}{7}\left(x_1 - (2 - \frac{1}{2} x_3)\right)^2}$$

6. Factor model w/ $m=1$ factor. Covariance matrix:

$$\Sigma = \begin{bmatrix} 5 & 2 & 3 \\ 2 & 6 & 6 \\ 3 & 6 & 10 \end{bmatrix} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} [l_1 \ l_2 \ l_3] + \begin{bmatrix} \psi_1 & 0 & 0 \\ 0 & \psi_2 & 0 \\ 0 & 0 & \psi_3 \end{bmatrix}$$

6a. How many variables are there in this study? Write the population factor model in vector-matrix form & indicate the dimensions of terms.

- Since Σ is a (3×3) matrix, there are three variables in this study.
- The population factor model is therefore

$$\boxed{\underset{(3\times1)}{X} = \underset{(3\times1)}{\mu} + \underset{(3\times1)(1\times1)}{LF} + \underset{(3\times1)}{\varepsilon}}$$

6b. Set up & solve a sys. of eqns for l_i, ψ_i, $i=1,2,3$.

- $$\Sigma = \begin{bmatrix} 5 & 2 & 3 \\ 2 & 6 & 6 \\ 3 & 6 & 10 \end{bmatrix} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} [l_1 \ l_2 \ l_3] + \begin{bmatrix} \psi_1 & 0 & 0 \\ 0 & \psi_2 & 0 \\ 0 & 0 & \psi_3 \end{bmatrix}$$

$$= \begin{bmatrix} l_1^2 & l_1 l_2 & l_1 l_3 \\ l_1 l_2 & l_2^2 & l_2 l_3 \\ l_1 l_3 & l_2 l_3 & l_3^2 \end{bmatrix} + \begin{bmatrix} \psi_1 & 0 & 0 \\ 0 & \psi_2 & 0 \\ 0 & 0 & \psi_3 \end{bmatrix}$$

$$= \begin{bmatrix} l_1^2+\psi_1 & l_1 l_2 & l_1 l_3 \\ l_1 l_2 & l_2^2+\psi_2 & l_2 l_3 \\ l_1 l_3 & l_2 l_3 & l_3^2+\psi_3 \end{bmatrix}$$

- So,

$$\begin{cases} (1) \ l_1^2 + \psi_1 = 5 \\ (2) \ l_2^2 + \psi_2 = 6 \\ (3) \ l_3^2 + \psi_3 = 10 \\ (4) \ l_1 l_2 = 2 \\ (5) \ l_1 l_3 = 3 \\ (6) \ l_2 l_3 = 6 \end{cases}$$

$\rightarrow$

- ④ $\Rightarrow l_1 = \frac{2}{l_2}$
- ⑤ $\Rightarrow l_1 = \frac{3}{l_3}$
- ④, ⑤ $\Rightarrow \frac{2}{l_2} = \frac{3}{l_3} \Rightarrow 3l_2 = 2l_3 \Rightarrow l_2 = \frac{2}{3}l_3$
- ④, ⑤, ⑥ $\Rightarrow l_2 l_3 = 6 \Rightarrow (\frac{2}{3}l_3)l_3 = 6 \Rightarrow l_3^2 = 9 \Rightarrow \boxed{l_3 = 3}$
- ⑤ $\Rightarrow l_1 l_3 = 3 \Rightarrow 3l_1 = 3 \Rightarrow \boxed{l_1 = 1}$
- ④ $\Rightarrow l_1 l_2 = 2 \Rightarrow \boxed{l_2 = 2}$
- ① $\Rightarrow l_1^2 + \psi_1 = 5 \Rightarrow 1 + \psi_1 = 5 \Rightarrow \boxed{\psi_1 = 4}$
- ② $\Rightarrow l_2^2 + \psi_2 = 6 \Rightarrow 4 + \psi_2 = 6 \Rightarrow \boxed{\psi_2 = 2}$
- ③ $\Rightarrow l_3^2 + \psi_3 = 10 \Rightarrow 9 + \psi_3 = 10 \Rightarrow \boxed{\psi_3 = 1}$

- So, $L = (l_1, l_2, l_3) = (1, 2, 3)$,
  and $\Psi = \text{diag}(\psi_1, \psi_2, \psi_3) = \text{diag}(4, 2, 1)$

6c. For each variable X_i, calc. the % of $Var(X_i)$ explained by the common factor.

- $h_1^2 = l_1^2 = 1$; $Var(X_1) = 5$
  So, $\frac{h_1^2}{Var(X_1)} = \frac{1}{5}$ or 20% of the variance of X_1 is explained by the common factor.

- $h_2^2 = l_2^2 = 4$; $Var(X_2) = 6$
  So, $\frac{h_2^2}{Var(X_2)} = \frac{4}{6} = \frac{2}{3}$ or ~66.7% of the variance of X_2 is explained by the common factor.

- $h_3^2 = l_3^2 = 9$; $Var(X_3) = 10$
  So, $\frac{h_3^2}{Var(X_3)} = \frac{9}{10}$ or 90% of the variance of X_3 is explained by the common factor.

6d. Now

$$\Sigma = \begin{bmatrix} 5 & 2 & 3 \\ 2 & 6 & 6 \\ 3 & 6 & 8 \end{bmatrix}.$$

Repeat Parts (a)–(c). Comment on results for Part (c). Is the factor model reasonable?

- Note that just Σ_{33} has changed.
- Σ is still (3×3), so there are still 3 variables.
- The population factor model is still

$$\underset{(3\times1)}{X} = \underset{(3\times1)}{\mu} + \underset{(3\times1)(1\times1)}{LF} + \underset{(3\times1)}{\varepsilon}$$

- Since only Σ_{33} has changed, the system is

$$\begin{cases} l_1^2 + \psi_1 = 5 \\ l_2^2 + \psi_2 = 6 \\ l_3^2 + \psi_3 = 8 \\ l_1 l_2 = 2 \\ l_1 l_3 = 3 \\ l_2 l_3 = 6 \end{cases}$$

- So still $l_1 = 1$, $l_2 = 2$, $l_3 = 3$, $\psi_1 = 4$, $\psi_2 = 2$. Now, $l_3^2 + \psi_3 = 8 \Rightarrow 9 + \psi_3 = 8 \Rightarrow \psi_3 = -1$.
- So, $L = (l_1, l_2, l_3) = (1, 2, 3)$, and $\Psi = \mathrm{diag}(\psi_1, \psi_2, \psi_3) = (4, 2, -1)$.

- $h_1^2 = l_1^2 = 1$; $Var(X_1) = 5$. So, still $\frac{h_1^2}{Var(X_1)} = \frac{1}{5}$ or 20% of the variance of X_1 is explained by the common factor.

- $h_2^2 = l_2^2 = 4$; $Var(X_2) = 6$. So, still $\frac{h_2^2}{Var(X_2)} = \frac{4}{6} = \frac{2}{3}$ or ~66.7% of the variance of X_2 is explained by the common factor.

• But now $h_3^2 = l_3^2 = 9$; $Var(X_3 = 8)$.

So, $\frac{h_3^2}{Var(X_3)} = \frac{9}{8} > 1$ or $112.5\% > 100\%$ of the variance of X_3 is purportedly explaned by the common factor. This is obviously nonsensical: it makes no sense for the common factor F to explain more than 100% of the variance of X_3!