

STAT 32950: Homework 0

Robert Winter

Table of Contents

1 Exercise 1: Typing Math Formula	1
2 Exercise 2: Producing R Plot	1

1 Exercise 1: Typing Math Formula

For this question, type out your answers, including the mathematical formula. Consider a dataset containing n pairs of real numbers, (x_i, y_i) , $i = 1, \dots, n$. Suppose the data are n independent observations of a pair of random variables (X, Y) . Type the formula of the Pearson sample correlation coefficient of (X, Y) in terms of the x_i 's and y_i 's.

The Pearson sample correlation coefficient of (X, Y) is

$$r_{XY} = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}}$$

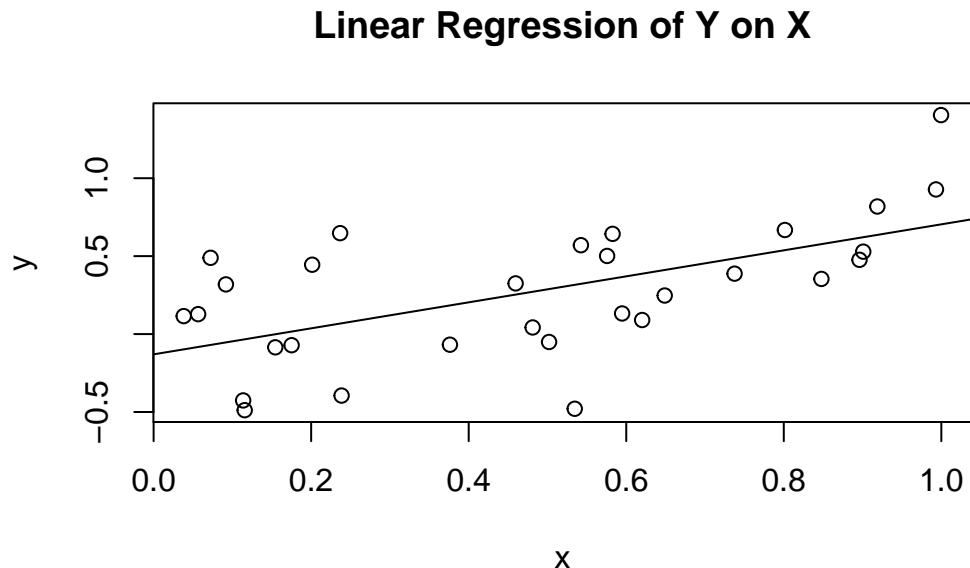
where

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i \text{ and } \bar{y} = \frac{1}{n} \sum_{i=1}^n y_i.$$

2 Exercise 2: Producing R Plot

Produce data and 2-dimensional scatter plot with least square line by using the following R commands. Replace ??? in the last command by an appropriate plot title.

```
x = runif(30); y = x^3 + rnorm(30)/3  
plot(x,y); abline(lm(y~x))  
title(main = "Linear Regression of Y on X")
```



HW #0

3.

$$\begin{array}{l} \triangleright A = [a_1 \ a_2 \ a_3] = \begin{bmatrix} 3 & -4 & -1 \\ -6 & 0 & 5 \\ 4 & 5 & 7 \end{bmatrix}; B = [b_1 \ b_2 \ b_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}; \\ \triangleright C = [c_1 \ c_2 \ c_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

3a(i) What happens to A when left-multiplied by B?

$$\begin{array}{l} \triangleright BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 3 & -4 & -1 \\ -6 & 0 & 5 \\ 4 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 3(1) & -4(1) & -1(1) \\ -6(3) & 0(3) & 5(3) \\ 4(-2) & 5(-2) & 7(-2) \end{bmatrix} \end{array}$$

That is, left-multiplying A by B scales the i^{th} row of A by the i^{th} diagonal element of B for $i \in \{1, 2, 3\}$. Specifically, the first row of A is scaled by 1, the second row by 3, & the third row by -2.

3a(ii) What happens to A when right-multiplied by B?

$$\begin{array}{l} \triangleright AB = \begin{bmatrix} 3 & -4 & -1 \\ -6 & 0 & 5 \\ 4 & 5 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 3(1) & -4(3) & -1(-2) \\ -6(1) & 0(3) & 5(-2) \\ 4(1) & 5(3) & 7(-2) \end{bmatrix} \end{array}$$

That is, right-multiplying A by B scales the i^{th} column of A by the i^{th} diagonal element of B for $i \in \{1, 2, 3\}$. Specifically, the first column of A is scaled by 1, the second column by 3, & the third column by -2.



3a(iii) What happens to A when left-multiplied by E?

$$\cdot EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -4 & -1 \\ -6 & 0 & 5 \\ 4 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 3 & -4 & -1 \\ 4 & 5 & 7 \\ -6 & 0 & 5 \end{bmatrix}$$

That is, left-multiplying A by E interchanges the second & third rows of A.

3a(iv) What happens to A when right-multiplied by E?

$$\cdot AE = \begin{bmatrix} 3 & -4 & -1 \\ -6 & 0 & 5 \\ 4 & 5 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -4 \\ -6 & 5 & 0 \\ 4 & 7 & 5 \end{bmatrix}$$

That is, right-multiplying A by E interchanges the second & third columns of A.

3b. Let $v = (7, 3, 24)$.

3b(i) Write v as a linear combination of the a_i 's.

$$\Rightarrow v = x a_1 + y a_2 + z a_3 \Leftrightarrow \begin{bmatrix} 7 \\ 3 \\ 24 \end{bmatrix} = x \begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix} + y \begin{bmatrix} -4 \\ 0 \\ 5 \end{bmatrix} + z \begin{bmatrix} -1 \\ 5 \\ 7 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 3 & -4 & -1 \\ -6 & 0 & 5 \\ 4 & 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 24 \end{bmatrix}$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 3 & -4 & -1 & 7 \\ -6 & 0 & 5 & 3 \\ 4 & 5 & 7 & 24 \end{array} \right] \xrightarrow{\text{I}+2\text{II}} \left[\begin{array}{ccc|c} 3 & -4 & -1 & 7 \\ 0 & -8 & 3 & 17 \\ 4 & 5 & 7 & 24 \end{array} \right] \xrightarrow{-\frac{4}{3}\text{II}} \left[\begin{array}{ccc|c} 3 & -4 & -1 & 7 \\ 0 & \frac{31}{3} & \frac{25}{3} & \frac{44}{3} \\ 4 & 5 & 7 & 24 \end{array} \right] \xrightarrow{\frac{31}{3}\text{III}} \left[\begin{array}{ccc|c} 3 & -4 & -1 & 7 \\ 0 & -8 & 3 & 17 \\ 0 & 0 & \frac{293}{24} & \frac{293}{8} \end{array} \right] \times \frac{24}{293} \sim \left[\begin{array}{ccc|c} 3 & -4 & -1 & 7 \\ 0 & -8 & 3 & 17 \\ 0 & 0 & 1 & 3 \end{array} \right] \xrightarrow{\text{III}-3\text{II}} \left[\begin{array}{ccc|c} 3 & -4 & -1 & 7 \\ 0 & -8 & 3 & 17 \\ 0 & 0 & 1 & 3 \end{array} \right] \xrightarrow{\text{III}+8\text{II}} \left[\begin{array}{ccc|c} 3 & -4 & -1 & 7 \\ 0 & 0 & 3 & 17 \\ 0 & 0 & 1 & 3 \end{array} \right] \xrightarrow{\text{III}-3\text{I}} \left[\begin{array}{ccc|c} 3 & -4 & -1 & 7 \\ 0 & 0 & 0 & 17 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 3 & -4 & 0 & 10 \\ 0 & -8 & 0 & 8 \\ 0 & 0 & 1 & 3 \end{array} \right] \xrightarrow{x - \frac{1}{8}} \left[\begin{array}{ccc|c} 3 & -4 & 0 & 10 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array} \right] + 4\text{II}$$

$$\sim \left[\begin{array}{ccc|c} 3 & 0 & 0 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array} \right] \xrightarrow{\cdot \frac{1}{3}} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

$$\Leftrightarrow (x, y, z) = (2, -1, 3).$$

• That is,

$$v = 2a_1 - 1a_2 + 3a_3.$$

3b(ii) Write v as a linear combination of the b_i 's.

$$\bullet v = xb_1 + yb_2 + zb_3 \Leftrightarrow \begin{bmatrix} 7 \\ 3 \\ 24 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 24 \end{bmatrix}$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & -2 & 24 \end{array} \right] \xrightarrow{\cdot \frac{1}{3}} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -12 \end{array} \right]$$

$$\Leftrightarrow (x, y, z) = (7, 1, -12).$$

• That is,

$$v = 7b_1 + b_2 - 12b_3$$

3b(iii) Write v as a linear combination of the e_i 's.

$$\cdot v = xe_1 + ye_2 + ze_3 \Leftrightarrow \begin{bmatrix} 7 \\ 3 \\ 24 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + z \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 7 \\ 3 \\ 24 \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ z \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ z \\ y \end{bmatrix}$$

$$\Leftrightarrow (x, y, z) = (7, 24, 3).$$

That is,

$$v = 7e_1 + 24e_2 + 3e_3$$

4a. Use mathematical induction to prove $\sum_{k=1}^n k^3 = (\sum_{k=1}^n k)^2$.

• I: Base Case

• Let $n=1$.

• Then $\sum_{k=1}^1 k^3 = 1^3 = 1$.

• Also, $(\sum_{k=1}^1 k)^2 = (1)^2 = 1$.

• So, $\sum_{k=1}^1 k^3 = (\sum_{k=1}^1 k)^2$, and the base case holds.

• II: Induction Step

• Induction Hypothesis: Suppose $\sum_{k=1}^n k^3 = (\sum_{k=1}^n k)^2$ for some $n \geq 1$.

• We will show $\sum_{k=1}^{n+1} k^3 = (\sum_{k=1}^{n+1} k)^2$.

• Starting with the RHS,

$$\begin{aligned} (\sum_{k=1}^{n+1} k)^2 &= (\sum_{k=1}^n k + (n+1))^2 \\ &= (\sum_{k=1}^n k)^2 + 2(n+1)\sum_{k=1}^n k + (n+1)^2 \\ &= \sum_{k=1}^n k^3 + 2(n+1) \cdot \frac{n(n+1)}{2} + (n+1)^2 \quad \text{by} \\ &\quad \text{the induction hypothesis + the sum of} \\ &\quad \text{the first } n \text{ natural numbers} \\ &= \sum_{k=1}^n k^3 + n(n+1)^2 + (n+1)^2 \\ &= \sum_{k=1}^n k^3 + (n+1)^2(n+1) \\ &= \sum_{k=1}^n k^3 + (n+1)^3 \\ &= \sum_{k=1}^{n+1} k^3. \end{aligned}$$

• We have proven the base case + induction step; thus,

$$\sum_{k=1}^n k^3 = (\sum_{k=1}^n k)^2 \quad \forall n \in \mathbb{N}.$$

□



4b. Let $A, B \in \mathbb{R}^{4 \times 4}$. Show $AB = BA$ is not always true.

• We disprove using a counterexample.

Let $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Then

$$AB = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

And

$$BA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Clearly, $AB \neq BA$ in this case, so it cannot be that $AB = BA$ in general. \square

5. The RV $U \sim U[-1, 1]$. Define $X = U^2$.

5a. Derive the PDF of X .

• The PDF of U is $\frac{1}{1-(-1)} = \frac{1}{2}$ on its support, so

$$f_U(u) = \begin{cases} \frac{1}{2}, & u \in [-1, 1] \\ 0, & \text{o/w} \end{cases}$$

• Let $g(u) = u^2$. Then if $u \geq 0$, $g^{-1}(u) = \sqrt{u}$, and if $u < 0$, $g^{-1}(u) = -\sqrt{u}$.

• Case I: $u \in [0, 1]$

$$\begin{aligned} f_X(x) &= f_U(g^{-1}(x)) \left| \frac{\partial}{\partial x} g^{-1}(x) \right| \\ &= \frac{1}{2} \left| \frac{\partial}{\partial x} \sqrt{x} \right| \\ &= \frac{1}{2} \cdot \left| \frac{1}{2\sqrt{x}} \right| \\ &= \frac{1}{4\sqrt{x}} \end{aligned}$$

• Case II: $u \in [-1, 0)$

$$\begin{aligned} f_X(x) &= f_U(g^{-1}(x)) \left| \frac{\partial}{\partial x} g^{-1}(x) \right| \\ &= \frac{1}{2} \left| \frac{\partial}{\partial x} (-\sqrt{x}) \right| \\ &= \frac{1}{2} \left| -\frac{1}{2\sqrt{x}} \right| \\ &= \frac{1}{4\sqrt{x}} \end{aligned}$$

• Thus, for $u \in [-1, 1]$, we have $x \in [0, 1]$, and

$$f_X(x) = \frac{1}{4\sqrt{x}} + \frac{1}{4\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

• That is, the PDF of X is

$$f_X(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & x \in [0, 1] \\ 0, & \text{o/w} \end{cases}$$

5b. Calculate $E[X]$.

$$\begin{aligned} E[X] &= \int_0^1 x f_X(x) dx \\ &= \int_0^1 x \cdot \frac{1}{2\sqrt{x}} dx \\ &= \frac{1}{2} \int_0^1 \sqrt{x} dx \\ &= \frac{1}{2} \cdot \frac{2}{3} x^{3/2} \Big|_0^1 \end{aligned}$$

$$\begin{aligned}&= \frac{1}{2} \left(\frac{2}{3}(1)^{3/2} - \frac{2}{3}(0)^{3/2} \right) \\&= \frac{1}{2} \left(\frac{2}{3} - 0 \right) \\&\Rightarrow \boxed{\mathbb{E}[X] = \frac{1}{3}}\end{aligned}$$