STAT 32950

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HW #0

3.
$$A = [a_1\ a_2\ a_3] = \begin{bmatrix} 3 & -4 & -1 \\ -6 & 0 & 5 \\ 4 & 5 & 7 \end{bmatrix}; \quad B = [b_1\ b_2\ b_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix};$$

$$C = [c_1\ c_2\ c_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

3a(i) What happens to A when left-multiplied by B?

- $$BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 3 & -4 & -1 \\ -6 & 0 & 5 \\ 4 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 3(1) & -4(1) & -1(1) \\ -6(3) & 0(3) & 5(3) \\ 4(-2) & 5(-2) & 7(-2) \end{bmatrix}$$
- That is, left-multiplying A by B scales the i^{th} row of A by the i^{th} diagonal element of B for $i \in \{1, 2, 3\}$. Specifically, the first row of A is scaled by 1, the second row by 3, & the third row by -2.

3a(ii) What happens to A when right-multiplied by B?

- $$AB = \begin{bmatrix} 3 & -4 & -1 \\ -6 & 0 & 5 \\ 4 & 5 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 3(1) & -4(3) & -1(-2) \\ -6(1) & 0(3) & 5(-2) \\ 4(1) & 5(3) & 7(-2) \end{bmatrix}$$
- That is, right-multiplying A by B scales the i^{th} column of A by the i^{th} diagonal element of B for $i \in \{1, 2, 3\}$. Specifically, the first column of A is scaled by 1, the second column by 3, & the third column by -2.

$\rightarrow$

3a(iii) What happens to A when left-multiplied by E?

$$\cdot\ EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & -4 & -1 \\ -6 & 0 & 5 \\ 4 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 3 & -4 & -1 \\ 4 & 5 & 7 \\ -6 & 0 & 5 \end{bmatrix}$$

· That is, left-multiplying A by E interchanges the second & third rows of A.

3a(iv) What happens to A when right-multiplied by E?

$$\cdot\ AE = \begin{bmatrix} 3 & -4 & -1 \\ -6 & 0 & 5 \\ 4 & 5 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -4 \\ -6 & 5 & 0 \\ 4 & 7 & 5 \end{bmatrix}$$

· That is, right-multiplying A by E interchanges the second & third columns of A.

3b. Let $v = (7, 3, 24)$.

3b(i) Write v as a linear combination of the a_i's.

$$\cdot\ v = xa_1 + ya_2 + za_3 \Leftrightarrow \begin{bmatrix} 7 \\ 3 \\ 24 \end{bmatrix} = x\begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix} + y\begin{bmatrix} -4 \\ 0 \\ 5 \end{bmatrix} + z\begin{bmatrix} -1 \\ 5 \\ 7 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 3 & -4 & -1 \\ -6 & 0 & 5 \\ 4 & 5 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 24 \end{bmatrix}$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 3 & -4 & -1 & 7 \\ -6 & 0 & 5 & 3 \\ 4 & 5 & 7 & 24 \end{array}\right] \begin{array}{l} \\ +2\mathrm{I} \\ -\frac{4}{3}\mathrm{I} \end{array} \sim \left[\begin{array}{ccc|c} 3 & -4 & -1 & 7 \\ 0 & -8 & 3 & 17 \\ 0 & \frac{31}{3} & \frac{25}{3} & \frac{44}{3} \end{array}\right] \begin{array}{l} \\ \\ +\frac{31}{24}\mathrm{II} \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 3 & -4 & -1 & 7 \\ 0 & -8 & 3 & 17 \\ 0 & 0 & \frac{293}{24} & \frac{293}{8} \end{array}\right] \begin{array}{l} \\ \\ \times\frac{24}{293} \end{array} \sim \left[\begin{array}{ccc|c} 3 & -4 & -1 & 7 \\ 0 & -8 & 3 & 17 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} +\mathrm{III} \\ -3\mathrm{III} \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 3 & -4 & 0 & 10 \\ 0 & -8 & 0 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right] \times -\tfrac{1}{8} \sim \left[\begin{array}{ccc|c} 3 & -4 & 0 & 10 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] +4\,\text{II}$$

$$\sim \left[\begin{array}{ccc|c} 3 & 0 & 0 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \times \tfrac{1}{3} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$\Leftrightarrow (x, y, z) = (2, -1, 3).$

- That is,

$$v = 2a_1 - 1a_2 + 3a_3.$$

3b(ii) Write v as a linear combination of the b_i's.

- $v = xb_1 + yb_2 + zb_3 \Leftrightarrow \begin{bmatrix} 7 \\ 3 \\ 24 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$

$$\Leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 24 \end{bmatrix}$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & -2 & 24 \end{array}\right] \begin{array}{l} \\ \times \tfrac{1}{3} \\ \times -\tfrac{1}{2} \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -12 \end{array}\right]$$

$\Leftrightarrow (x, y, z) = (7, 1, -12).$

- That is,

$$v = 7b_1 + b_2 - 12b_3$$

3b(iii) Write v as a linear combination of the e_i's.

- $v = xe_1 + ye_2 + ze_3 \Leftrightarrow \begin{bmatrix}7\\3\\24\end{bmatrix} = x\begin{bmatrix}1\\0\\0\end{bmatrix} + y\begin{bmatrix}0\\0\\1\end{bmatrix} + z\begin{bmatrix}0\\1\\0\end{bmatrix}$

$$\Leftrightarrow \begin{bmatrix}7\\3\\24\end{bmatrix} = \begin{bmatrix}x\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\y\end{bmatrix} + \begin{bmatrix}0\\z\\0\end{bmatrix} = \begin{bmatrix}x\\z\\y\end{bmatrix}$$

$$\Leftrightarrow (x, y, z) = (7, 24, 3).$$

- That is,

$$v = 7e_1 + 24e_2 + 3e_3$$

4a. Use mathematical induction to prove $\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$.

- I: Base Case
  - Let $n=1$.
  - Then $\sum_{k=1}^{1} k^3 = 1^3 = 1$.
  - Also, $\left(\sum_{k=1}^{1} k\right)^2 = (1)^2 = 1$.
  - So, $\sum_{k=1}^{1} k^3 = \left(\sum_{k=1}^{1} k\right)^2$, and the base case holds.
- II: Induction Step
  - Induction Hypothesis: Suppose $\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$ for some $n \geq 1$.
  - We will show $\sum_{k=1}^{n+1} k^3 = \left(\sum_{k=1}^{n+1} k\right)^2$.
  - Starting with the RHS,

$$\left(\sum_{k=1}^{n+1} k\right)^2 = \left(\sum_{k=1}^{n} k + (n+1)\right)^2$$

$$= \left(\sum_{k=1}^{n} k\right)^2 + 2(n+1)\sum_{k=1}^{n} k + (n+1)^2$$

$$= \sum_{k=1}^{n} k^3 + 2(n+1)\cdot \frac{n(n+1)}{2} + (n+1)^2$$ by the induction hypothesis & the sum of the first n natural numbers

$$= \sum_{k=1}^{n} k^3 + n(n+1)^2 + (n+1)^2$$

$$= \sum_{k=1}^{n} k^3 + (n+1)^2(n+1)$$

$$= \sum_{k=1}^{n} k^3 + (n+1)^3$$

$$= \sum_{k=1}^{n+1} k^3.$$

- We have proven the base case & induction step; thus, $\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2 \quad \forall n \in \mathbb{N}$. $\square$

$\longrightarrow$

4b. Let $A, B \in \mathbb{R}^{4\times 4}$. Show $AB = BA$ is not always true.

- We disprove using a counterexample.
- Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- Then

$$AB = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- And

$$BA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Clearly, $AB \neq BA$ in this case, so it cannot be that $AB = BA$ in general. $\square$

5. The RV $U \sim \mathcal{U}[-1, 1]$. Define $X = U^2$.

5a. Derive the PDF of X.

- The PDF of U is $\frac{1}{1-(-1)} = \frac{1}{2}$ on its support, so
$$f_U(u) = \begin{cases} \frac{1}{2}, & u \in [-1, 1] \\ 0, & o/w \end{cases}$$
- Let $g(u) = u^2$. Then if $u \geq 0$, $g^{-1}(u) = \sqrt{u}$, and if $u < 0$, $g^{-1}(u) = -\sqrt{u}$.
- Case I: $u \in [0, 1]$
  - $$\begin{aligned} f_X(x) &= f_U(g^{-1}(x)) \left| \frac{\partial}{\partial x} g^{-1}(x) \right| \\ &= \frac{1}{2} \left| \frac{\partial}{\partial x} \sqrt{x} \right| \\ &= \frac{1}{2} \cdot \left| \frac{1}{2\sqrt{x}} \right| \\ &= \frac{1}{4\sqrt{x}} \end{aligned}$$
- Case II: $u \in [-1, 0)$
  - $$\begin{aligned} f_X(x) &= f_U(g^{-1}(x)) \left| \frac{\partial}{\partial x} g^{-1}(x) \right| \\ &= \frac{1}{2} \left| \frac{\partial}{\partial x} (-\sqrt{x}) \right| \\ &= \frac{1}{2} \left| \frac{-1}{2\sqrt{x}} \right| \\ &= \frac{1}{4\sqrt{x}} \end{aligned}$$
- Thus, for $u \in [-1, 1]$, we have $x \in [0, 1]$, and
$$f_X(x) = \frac{1}{4\sqrt{x}} + \frac{1}{4\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$
- That is, the PDF of X is
$$f_X(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & x \in [0, 1] \\ 0, & o/w \end{cases}$$

5b. Calculate $\mathbb{E}[X]$.

- $$\begin{aligned} \mathbb{E}[X] &= \int_0^1 x f_X(x)\,dx \\ &= \int_0^1 x \cdot \frac{1}{2\sqrt{x}}\,dx \\ &= \frac{1}{2} \int_0^1 \sqrt{x}\,dx \\ &= \frac{1}{2} \cdot \frac{2}{3} x^{3/2} \Big|_0^1 \end{aligned}$$

$$= \frac{1}{2}\left(\frac{2}{3}(1)^{3/2} - \frac{2}{3}(0)^{3/2}\right)$$

$$= \frac{1}{2}\left(\frac{2}{3} - 0\right)$$

$$\Rightarrow \quad \mathbb{E}[X] = \frac{1}{3}$$