STAT 39250

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HW #1

2. $$f_{XY}(x,y) = \begin{cases} c(x^2-y^2)e^{-x}, & x>0,\ -x \le y \le x \\ 0, & o/w \end{cases}$$

2a. Derive c.

- Since f_{XY} is a PDF,

$$\int_0^\infty \int_{-x}^{x} c(x^2-y^2)e^{-x}\,dy\,dx = 1$$
$$\Rightarrow c\int_0^\infty\int_{-x}^{x}(x^2e^{-x} - y^2e^{-x})\,dy\,dx = 1$$
$$\Rightarrow c\left[\int_0^\infty\int_{-x}^{x} x^2e^{-x}\,dy\,dx - \int_0^\infty\int_{-x}^{x} y^2e^{-x}\,dy\,dx\right] = 1$$
$$\Rightarrow c\left[\int_0^\infty x^2e^{-x}\cdot y\Big|_{-x}^{x}dx - \int_0^\infty e^{-x}\int_{-x}^{x} y^2\,dy\,dx\right] = 1$$
$$\Rightarrow c\left[\int_0^\infty x^2e^{-x}(2x)\,dx - \int_0^\infty e^{-x}\cdot\tfrac{1}{3}y^3\Big|_{-x}^{x}dx\right] = 1$$
$$\Rightarrow c\left[\int_0^\infty 2x^3e^{-x}\,dx - \int_0^\infty e^{-x}\cdot\tfrac{2}{3}x^3\,dx\right] = 1$$
$$\Rightarrow c\int_0^\infty \tfrac{4}{3}x^3e^{-x}\,dx = 1$$
$$\Rightarrow \tfrac{4}{3}c\,\Gamma(4) = 1$$
$$\Rightarrow c\cdot 3! = \tfrac{3}{4}$$
$$\Rightarrow c = \tfrac{3}{24}$$
$$\Rightarrow c = \tfrac{1}{8}$$

2b. Derive $f_{Y|X}(y|x)$.

- $$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

- $$f_X(x) = \int_{-x}^{x}\tfrac{1}{8}(x^2-y^2)e^{-x}\,dy$$
$$= \tfrac{1}{8}\left[\int_{-x}^{x} x^2e^{-x}\,dy - \int_{-x}^{x} y^2e^{-x}\,dy\right]$$
$$= \tfrac{1}{8}\left[x^2e^{-x}\cdot y\Big|_{-x}^{x} - e^{-x}\cdot\tfrac{1}{3}y^3\Big|_{-x}^{x}\right]$$
$$= \tfrac{1}{8}\left[x^2e^{-x}(2x) - e^{-x}(\tfrac{2}{3}x^3)\right]$$
$$= \tfrac{1}{8}(2x^3e^{-x} - \tfrac{2}{3}x^3e^{-x})$$
$$= \tfrac{1}{8}(\tfrac{4}{3}x^3e^{-x})$$
$$= \tfrac{1}{6}x^3e^{-x}, \quad x>0$$

$\longrightarrow$

• So, on its support,

$$f_{Y|X}(y|x) = \frac{\frac{1}{8}(x^2-y^2)\cancel{e^{-x}}}{\frac{1}{6}x^3\cancel{e^{-x}}}$$

$$= \frac{6}{8}\frac{x^2-y^2}{x^3}$$

$$= \frac{3}{4}\left(\frac{x^2-y^2}{x^3}\right)$$

• Thus,

$$f_{Y|X}(y|x) = \begin{cases} \frac{3}{4}\left(\frac{x^2-y^2}{x^3}\right), & y \in [-x, x] \\ 0, & o/w \end{cases}$$

2c. Derive $g(x) = \mathbb{E}[Y|X=x]$ for $x>0$.

$$\cdot\ \mathbb{E}[Y|X=x] = \int_{-x}^{x} y\, f_{Y|X}(y|x)\,dy$$

$$= \int_{-x}^{x} y \cdot \frac{3}{4}\left(\frac{x^2-y^2}{x^3}\right)dy$$

$$= \frac{3}{4x^3}\int_{-x}^{x}(yx^2 - y^3)\,dy$$

$$= \frac{3}{4x^3}\left[\frac{1}{2}x^2y^2\Big|_{-x}^{x} - \frac{1}{4}y^4\Big|_{-x}^{x}\right]$$

$$= \frac{3}{4x^3}\left[\left(\frac{1}{2}x^2x^2 - \frac{1}{2}x^2(-x)^2\right) - \left(\frac{1}{4}x^4 - \frac{1}{4}(-x)^4\right)\right]$$

$$= \frac{3}{4x^3}[0-0]$$

$$= 0$$

• That is, $g(x) = \mathbb{E}[Y|X=x] = 0 \quad \forall x>0$

2d. Derive $Var(Y|X=x)$ for $x>0$

$$\cdot\ Var(Y|X=x) = \mathbb{E}[Y^2|X=x] - \mathbb{E}[Y|X=x]^2$$

$$= \mathbb{E}[Y^2|X=x] - 0^2$$

$$= \int_{-x}^{x} y^2 \cdot f_{Y|X}(y|x)\,dy$$

$$= \int_{-x}^{x} y^2 \cdot \frac{3}{4}\left(\frac{x^2-y^2}{x^3}\right)dy$$

$$= \frac{3}{4x^3} \int_{-x}^{x} (x^2 y^2 - y^4)\,dy$$

$$= \frac{3}{4x^3} \left[\frac{1}{3} x^2 y^3 \Big|_{-x}^{x} - \frac{1}{5} y^5 \Big|_{-x}^{x} \right]$$

$$= \frac{3}{4x^3} \left[\left(\frac{1}{3} x^2 x^3 - \frac{1}{3} x^2 (-x)^3 \right) - \left(\frac{1}{5} x^5 - \frac{1}{5} (-x)^5 \right) \right]$$

$$= \frac{3}{4x^3} \left[\left(\frac{1}{3} x^5 + \frac{1}{3} x^5 \right) - \left(\frac{1}{5} x^5 + \frac{1}{5} x^5 \right) \right]$$

$$= \frac{3}{4x^3} \left[\frac{2}{3} x^5 - \frac{2}{5} x^5 \right]$$

$$= \frac{\cancel{3}}{\cancel{4x^3}} \cdot \frac{\cancel{4}}{15} \cancel{x^5}$$

$$\Rightarrow Var(Y|X=x) = \frac{x^2}{5}, \quad x > 0$$

3. PMF of (X, Y):

	$Y=1$	$Y=2$	$Y=3$	$Y=4$
$X=1$	c	c	0	0
$X=2$	c	c	c	0
$X=3$	c	c	c	c

3a. Find c. Derive $f_X(x) = \mathbb{P}(X=x)$, $f_Y(y)$.

- $9c + 3(0) = 1 \Rightarrow c = \frac{1}{9}$

- $f_X(x) = \begin{cases} c+c+0+0, & x=1 \\ c+c+c+0, & x=2 \\ c+c+c+c, & x=3 \end{cases}$

$\Rightarrow f_X(x) = \begin{cases} \frac{2}{9}, & x=1 \\ \frac{1}{3}, & x=2 \\ \frac{4}{9}, & x=3 \end{cases}$

- $f_Y(y) = \begin{cases} c+c+c, & y=1 \\ c+c+c, & y=2 \\ 0+c+c, & y=3 \\ 0+0+c, & y=4 \end{cases}$

$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{3}, & y=1 \\ \frac{1}{3}, & y=2 \\ \frac{2}{9}, & y=3 \\ \frac{1}{9}, & y=4 \end{cases}$

3b. Find $g(x) = \mathbb{E}[Y|X=x]$ for $x=1,2,3$.

- $\mathbb{E}[Y|X=1] = \frac{c}{2c}(1) + \frac{c}{2c}(2) + \frac{0}{2c}(3) + \frac{0}{2c}(4) = \frac{1}{2} + \frac{1}{2}(2) = \frac{3}{2}$
- $\mathbb{E}[Y|X=2] = \frac{c}{3c}(1) + \frac{c}{3c}(2) + \frac{c}{3c}(3) + \frac{0}{3c}(4) = \frac{1}{3} + \frac{2}{3} + 1 = 2$
- $\mathbb{E}[Y|X=3] = \frac{c}{4c}(1) + \frac{c}{4c}(2) + \frac{c}{4c}(3) + \frac{c}{4c}(4) = \frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1 = \frac{5}{2}$

$\rightarrow$

$$\Rightarrow \boxed{g(x) = \mathbb{E}[Y|X=x] = \begin{cases} \frac{3}{2}, & x=1 \\ 2, & x=2 \\ \frac{5}{2}, & x=3 \end{cases}}$$

3c. Find $Var(Y|X=x)$ for $x=1,2,3$.

- $\mathbb{E}[Y^2|X=1] = \frac{c}{2c}(1)^2 + \frac{c}{2c}(2)^2 + \frac{0}{2c}(3)^2 + \frac{0}{2c}(4)^2 = \frac{1}{2} + \frac{4}{2} = \frac{5}{2}$
- $\mathbb{E}[Y^2|X=2] = \frac{c}{3c}(1)^2 + \frac{c}{3c}(2)^2 + \frac{c}{3c}(3)^2 + \frac{0}{3c}(4)^2 = \frac{1}{3} + \frac{4}{3} + \frac{9}{3} = \frac{14}{3}$
- $\mathbb{E}[Y^2|X=3] = \frac{c}{4c}(1)^2 + \frac{c}{4c}(2)^2 + \frac{c}{4c}(3)^2 + \frac{c}{4c}(4)^2 = \frac{1}{4} + \frac{4}{4} + \frac{9}{4} + \frac{16}{4} = \frac{15}{2}$

- $Var(Y|X=1) = \mathbb{E}[Y^2|X=1] - \mathbb{E}[Y|X=1]^2 = \frac{5}{2} - \left(\frac{3}{2}\right)^2 = \frac{1}{4}$
- $Var(Y|X=2) = \mathbb{E}[Y^2|X=2] - \mathbb{E}[Y|X=2]^2 = \frac{14}{3} - (2)^2 = \frac{2}{3}$
- $Var(Y|X=3) = \mathbb{E}[Y^2|X=3] - \mathbb{E}[Y|X=3]^2 = \frac{15}{2} - \left(\frac{5}{2}\right)^2 = \frac{5}{4}$

$$\Rightarrow \boxed{Var(Y|X=x) = \begin{cases} \frac{1}{4}, & x=1 \\ \frac{2}{3}, & x=2 \\ \frac{5}{4}, & x=3 \end{cases}}$$

3d. Evaluate $\mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[g(x)]$. Verify it equals $\mathbb{E}[Y] = \sum_y y f_Y(y)$.

- $\mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{P}(X=1)\mathbb{E}[Y|X=1] + \mathbb{P}(X=2)\mathbb{E}[Y|X=2] + \mathbb{P}(X=3)\mathbb{E}[Y|X=3]$
$= \left(\frac{2}{9}\right)\left(\frac{3}{2}\right) + \left(\frac{1}{3}\right)(2) + \left(\frac{4}{9}\right)\left(\frac{5}{2}\right)$
$= \frac{1}{3} + \frac{2}{3} + \frac{10}{9}$

$$\Rightarrow \boxed{\mathbb{E}[\mathbb{E}[Y|X]] = \frac{19}{9}}$$

- $\mathbb{E}[Y] = \sum_{y=1}^{4} y f_Y(y)$
$= (1)\left(\frac{1}{3}\right) + (2)\left(\frac{1}{3}\right) + (3)\left(\frac{2}{9}\right) + (4)\left(\frac{1}{9}\right)$
$= \frac{1}{3} + \frac{2}{3} + \frac{6}{9} + \frac{4}{9}$
$= \frac{19}{9}$

So $\boxed{\mathbb{E}[Y] = \frac{19}{9} = \mathbb{E}[\mathbb{E}[Y|X]]}$, as expected.

3e. Evaluate $Var(\mathbb{E}[Y|X])$. Derive $Var(Y) = Var(\mathbb{E}[Y|X]) + \mathbb{E}[Var(Y|X)]$.

- $Var(\mathbb{E}[Y|X]) = \mathbb{E}[\mathbb{E}[Y|X]^2] - \mathbb{E}[\mathbb{E}[Y|X]]^2$

$$= \mathbb{P}(X=1)\,\mathbb{E}[Y|X=1]^2 + \mathbb{P}(X=2)\,\mathbb{E}[Y|X=2]^2 + \mathbb{P}(X=3)\,\mathbb{E}[Y|X=3]^2 - \left(\tfrac{19}{9}\right)^2$$

$$= \left(\tfrac{2}{9}\right)\left(\tfrac{3}{2}\right)^2 + \left(\tfrac{1}{3}\right)(2)^2 + \left(\tfrac{4}{9}\right)\left(\tfrac{5}{2}\right)^2 - \left(\tfrac{19}{9}\right)^2$$

$$\Rightarrow Var(\mathbb{E}[Y|X]) = \frac{25}{162}$$

- $Var(Y) = Var(\mathbb{E}[Y|X]) + \mathbb{E}[Var(Y|X)]$

$$= \frac{25}{162} + \mathbb{P}(X=1)Var(Y|X=1) + \mathbb{P}(X=2)Var(Y|X=2) + \mathbb{P}(X=3)Var(Y|X=3)$$

$$= \frac{25}{162} + \left(\tfrac{2}{9}\right)\left(\tfrac{1}{4}\right) + \left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right) + \left(\tfrac{4}{9}\right)\left(\tfrac{5}{4}\right)$$

$$\Rightarrow Var(Y) = \frac{80}{81}$$

4a. Let $C = AXB$; X is $(p\times p)$ random matrix, A is $(k\times p)$ scalar matrix, B is $(p\times r)$ scalar matrix, $p, k, r \in \mathbb{N}$.

4a(i) What are the dimensions of C?

- $\underset{(k\times p)(p\times p)(p\times r)}{C = A\,X\,B}$
- C is a $(k\times r)$ matrix.

4a(ii) Write c_{ij} in terms of elements of A, B, X.

- $$C = \begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kp} \end{bmatrix} \begin{bmatrix} x_{11} & \cdots & x_{1p} \\ \vdots & \ddots & \vdots \\ x_{p1} & \cdots & x_{pp} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & \ddots & \vdots \\ b_{p1} & \cdots & b_{pr} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_{11} + \cdots + a_{1p}x_{p1} & \cdots & a_{11}x_{1p} + \cdots + a_{1p}x_{pp} \\ \vdots & \ddots & \vdots \\ a_{k1}x_{11} + \cdots + a_{kp}x_{p1} & \cdots & a_{k1}x_{1p} + \cdots + a_{kp}x_{pp} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & \ddots & \vdots \\ b_{p1} & \cdots & b_{pr} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{g=1}^{p} a_{1g}x_{g1} & \cdots & \sum_{g=1}^{p} a_{1g}x_{gp} \\ \vdots & \ddots & \vdots \\ \sum_{g=1}^{p} a_{kg}x_{g1} & \cdots & \sum_{g=1}^{p} a_{kg}x_{gp} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & \ddots & \vdots \\ b_{p1} & \cdots & b_{pr} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}\sum_{g=1}^{p} a_{1g}x_{g1} + \cdots + b_{p1}\sum_{g=1}^{p} a_{1g}x_{gp} & \cdots & b_{1r}\sum_{g=1}^{p} a_{1g}x_{g1} + \cdots + b_{pr}\sum_{g=1}^{p} a_{1g}x_{gp} \\ \vdots & \ddots & \vdots \\ b_{11}\sum_{g=1}^{p} a_{kg}x_{g1} + \cdots + b_{p1}\sum_{g=1}^{p} a_{kg}x_{gp} & \cdots & b_{1r}\sum_{g=1}^{p} a_{kg}x_{g1} + \cdots + b_{pr}\sum_{g=1}^{p} a_{kg}x_{gp} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{f=1}^{p} b_{f1}\sum_{g=1}^{p} a_{1g}x_{gf} & \cdots & \sum_{f=1}^{p} b_{fr}\sum_{g=1}^{p} a_{1g}x_{gf} \\ \vdots & \ddots & \vdots \\ \sum_{f=1}^{p} b_{f1}\sum_{g=1}^{p} a_{kg}x_{gf} & \cdots & \sum_{f=1}^{p} b_{fr}\sum_{g=1}^{p} a_{kg}x_{gf} \end{bmatrix}$$

- So, in general,

$$c_{ij} = \sum_{f=1}^{p} b_{fj}\sum_{g=1}^{p} a_{ig}x_{gf}$$

$$\Rightarrow c_{ij} = \sum_{f=1}^{p}\sum_{g=1}^{p} a_{ig}x_{gf}b_{fj}$$

$\longrightarrow$

4a(iii) Show $\mathbb{E}[C] = A\mathbb{E}[X]B$.

- For each element of C,
$$\mathbb{E}[c_{ij}] = \mathbb{E}\left[\sum_{f=1}^{p} \sum_{g=1}^{p} a_{ig} x_{gf} b_{fj}\right]$$
$$= \sum_{f=1}^{p} \sum_{g=1}^{p} \mathbb{E}[a_{ig} x_{gf} b_{fj}]$$
$$= \sum_{f=1}^{p} \sum_{g=1}^{p} a_{ig} \mathbb{E}[x_{gf}] b_{fg},$$
which is precisely the form of the $(i,j)^{\text{th}}$ entry of $A\mathbb{E}[X]B$.
- Since $\mathbb{E}[c_{ij}] = (A\mathbb{E}[X]B)_{ij}$ $\forall i = 1, \dots, k$, $\forall j = 1, \dots, r$, we conclude $\mathbb{E}[C] = A\mathbb{E}[X]B$. $\square$

4b. Show all p eigenvalues of $\Sigma = Cov(Y) \in \mathbb{R}^{p \times p}$ must be nonnegative, where $Y = (Y_1, \ldots, Y_p) \in \mathbb{R}^p$ is a random vector.

- All p eigenvalues of Σ are nonnegative iff Σ is positive semidefinite. Σ is positive semidefinite iff $x^T \Sigma x \geq 0 \quad \forall x \in \mathbb{R}^p$.
- $$Cov(Y) = Cov(Y, Y) = \begin{bmatrix} Var(Y_1) & Cov(Y_1, Y_2) & \cdots & Cov(Y_1, Y_p) \\ Cov(Y_2, Y_1) & Var(Y_2) & & Cov(Y_2, Y_p) \\ \vdots & \vdots & \ddots & \vdots \\ Cov(Y_p, Y_1) & Cov(Y_p, Y_2) & \cdots & Var(Y_p) \end{bmatrix}$$
$$:= \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{bmatrix}$$
- Let $x \in \mathbb{R}^p$. Then

$$x^T \Sigma x = [x_1 \cdots x_p] \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1p} \\ \vdots & \ddots & \vdots \\ \sigma_{p1} & \cdots & \sigma_{pp} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$$
$$= [x_1 \cdots x_p] \begin{bmatrix} x_1 \sigma_{11} + \cdots + x_p \sigma_{1p} \\ \vdots \\ x_1 \sigma_{p1} + \cdots + x_p \sigma_{pp} \end{bmatrix}$$
$$= [x_1 \cdots x_p] \begin{bmatrix} \sum_{j=1}^p x_j \sigma_{1j} \\ \vdots \\ \sum_{j=1}^p x_j \sigma_{pj} \end{bmatrix}$$
$$= x_1 \sum_{j=1}^p x_j \sigma_{1j} + \cdots + x_p \sum_{j=1}^p x_j \sigma_{pj}$$
$$= \sum_{i=1}^p x_i \sum_{j=1}^p x_j \sigma_{ij}$$
$$= \sum_{i=1}^p \sum_{j=1}^p x_i x_j \sigma_{ij}$$
$$= \sum_{i=1}^p \sum_{j=1}^p x_i x_j \mathbb{E}[(Y_i - \mathbb{E}[Y_i])(Y_j - \mathbb{E}[Y_j])]$$
$$= \sum_{i=1}^p \sum_{j=1}^p \mathbb{E}[x_i (Y_i - \mathbb{E}[Y_i]) \cdot x_j (Y_j - \mathbb{E}[Y_j])]$$
$$= \mathbb{E}[\sum_{i=1}^p \sum_{j=1}^p x_i (Y_i - \mathbb{E}[Y_i]) x_j (Y_j - \mathbb{E}[Y_j])]$$
$$= \mathbb{E}[\sum_{i=1}^p x_i (Y_i - \mathbb{E}[Y_i]) \sum_{j=1}^p x_j (Y_j - \mathbb{E}[Y_j])]$$

$\longrightarrow$

$$= \mathbb{E}\left[\left(\sum_{i=1}^{p} x_i (Y_i - \mathbb{E}[Y_i])\right)^2\right]$$
$$\geq 0.$$

- Thus, Σ is positive semidefinite, & so all of its eigenvalues are nonnegative. $\square$

4c. Let $A = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$, $\rho \in (0,1)$.

4c(i) Derive the eigenvalues of A.

- $\det(A - \lambda I) = 0$

$\Rightarrow \det\left(\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$

$\Rightarrow \det\left(\begin{bmatrix} 1-\lambda & \rho \\ \rho & 1-\lambda \end{bmatrix}\right) = 0$

$\Rightarrow (1-\lambda)^2 - \rho^2 = 0$

$\Rightarrow (1-\lambda)^2 = \rho^2$

$\Rightarrow 1-\lambda = \pm\rho$

$\Rightarrow \lambda = 1 \pm \rho$

- So, the eigenvalues of A are $(\lambda_1, \lambda_2) = (1-\rho, 1+\rho)$

4c(ii) Derive unit-length eigenvectors of A & show they're $\perp$.

- $\lambda_1 = 1-\rho$:
  - $E_{1-\rho} = \text{null}(A - (1-\rho)I)$

$= \text{null}\left(\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} - \begin{bmatrix} 1-\rho & 0 \\ 0 & 1-\rho \end{bmatrix}\right)$

$= \text{null}\left(\begin{bmatrix} \rho & \rho \\ \rho & \rho \end{bmatrix}\right)$

- $\begin{bmatrix} \rho & \rho \\ \rho & \rho \end{bmatrix}_{-I} \sim \begin{bmatrix} \rho & \rho \\ 0 & 0 \end{bmatrix}$

- $v_1 \in E_{1-\rho}$ iff $\begin{bmatrix} \rho & \rho \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Leftrightarrow \rho v_{11} + \rho v_{12} = 0$

$\Leftrightarrow v_{12} = -v_{11}$.

$\rightarrow$

- So $E_{1-\rho} = \text{span}\{\begin{pmatrix} v_{11} \\ -v_{11} \end{pmatrix} : v_{11} \in \mathbb{R}\} = \text{span}\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\}$.
- Normalizing,

$$\frac{(1,-1)}{\|(1,-1)\|} = \frac{(1,-1)}{\sqrt{1^2+(-1)^2}} = \frac{(1,-1)}{\sqrt{2}} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

is a unit eigenvector of A corresponding to $\lambda_1 = 1-\rho$.

- $\lambda_2 = 1+\rho$
  - $E_{1+\rho} = \text{null}(A-(1+\rho)I)$

$$= \text{null}\left(\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} - \begin{bmatrix} 1+\rho & 0 \\ 0 & 1+\rho \end{bmatrix}\right)$$

$$= \text{null}\left(\begin{bmatrix} -\rho & \rho \\ \rho & -\rho \end{bmatrix}\right)$$

  - $\begin{bmatrix} -\rho & \rho \\ \rho & -\rho \end{bmatrix} + I \sim \begin{bmatrix} -\rho & \rho \\ 0 & 0 \end{bmatrix}$
  - $v_2 \in E_{1+\rho}$ iff $\begin{bmatrix} -\rho & \rho \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Leftrightarrow -\rho v_{21} + \rho v_{22} = 0$$

$$\Leftrightarrow v_{22} = v_{21}$$

  - So $E_{1+\rho} = \text{span}\{\begin{pmatrix} v_{21} \\ v_{21} \end{pmatrix} : v_{21} \in \mathbb{R}\} = \text{span}\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$.
  - Normalizing,

$$\frac{(1,1)}{\|(1,1)\|} = \frac{(1,1)}{\sqrt{1^2+1^2}} = \frac{(1,1)}{\sqrt{2}} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

is a unit eigenvector of A corresponding to $\lambda_2 = 1+\rho$.

- Thus, a set of unit-length eigenvectors of A is

$$\left\{\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\right\}$$

- $\langle \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \rangle = (\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}) + (-\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}) = \frac{1}{2} - \frac{1}{2} = 0$,

so $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \perp \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, as desired.

4c(iii) Write out the spectral decomposition $A = V\Lambda V^T$.

$$\underbrace{\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} 1-\rho & 0 \\ 0 & 1+\rho \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{V^T}$$

4c(iv) Use the spectral decomposition to write A^{-1} in terms of V, Λ.

- Note that V is orthogonal, so $V^{-1} = V^T$. Also, Λ is diagonal, hence invertible.
- So,

$$A = V\Lambda V^T$$
$$\Rightarrow A^{-1}A = A^{-1}V\Lambda V^T$$
$$\Rightarrow I = A^{-1}V\Lambda V^{-1}$$
$$\Rightarrow IV = A^{-1}V\Lambda V^{-1}V$$
$$\Rightarrow V = A^{-1}V\Lambda$$
$$\Rightarrow V\Lambda^{-1} = A^{-1}V\Lambda\Lambda^{-1}$$
$$\Rightarrow V\Lambda^{-1}V^{-1} = A^{-1}VV^{-1}$$
$$\Rightarrow A^{-1} = V\Lambda^{-1}V^T$$

- In this case,

$$A^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{1-\rho} & 0 \\ 0 & \frac{1}{1+\rho} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$
$$= \frac{1}{1-\rho^2} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$$

4c(v) Use the spectral decomposition to derive R s.t. $R^2 = A$.

- By inspection, let $R = V\Lambda^{1/2}V^T$
- Then

$$
\begin{aligned}
R^2 &= (V\Lambda^{1/2}V^T)(V\Lambda^{1/2}V^T) \\
&= V\Lambda^{1/2}\cancel{V^{-1}V}\Lambda^{1/2}V^T \\
&= V\Lambda^{1/2}\Lambda^{1/2}V^T \\
&= V\Lambda V^T \\
&= A
\end{aligned}
$$

so $R = V\Lambda^{1/2}V^T$

5. $f_{WXY}(w,x,y) = \frac{2}{\pi} e^{x(y+w-x-4)-\frac{1}{2}(y^2+w^2)}$, $x \geq 0$, $w, y \in \mathbb{R}$

5a. Find $f_{XY}(x,y)$.

- $f_{XY}(x,y) = \int_{-\infty}^{\infty} f_{WXY}(w,x,y)\,dw$
$= \int_{-\infty}^{\infty} \frac{2}{\pi} e^{xy+xw-x^2-4x-\frac{1}{2}y^2-\frac{1}{2}w^2}\,dw$
$= \frac{2}{\pi} e^{xy-x^2-4x-\frac{1}{2}y^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}w^2+xw}\,dw$
- $\int_{-\infty}^{\infty} e^{-\frac{1}{2}w^2+xw}\,dw = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(w^2-2xw)}\,dw$
$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}(w^2-2xw+x^2)+\frac{1}{2}x^2}\,dw$
$= e^{\frac{1}{2}x^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(w-x)^2}\,dw$
$= \sqrt{2\pi}\, e^{x^2/2}$
- $f_{XY}(x,y) = \frac{2}{\pi} e^{xy-x^2-4x-\frac{1}{2}y^2} \cdot \sqrt{2\pi}\, e^{x^2/2}$

$\Rightarrow f_{XY}(x,y) = \begin{cases} \frac{2\sqrt{2}}{\sqrt{\pi}} e^{xy-\frac{1}{2}x^2-4x-\frac{1}{2}y^2}, & x \geq 0,\ y \in \mathbb{R} \\ 0, & o/w \end{cases}$

5b. Find $f_X(x)$.

- $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$
$= \int_{-\infty}^{\infty} \frac{2\sqrt{2}}{\sqrt{\pi}} e^{xy-\frac{1}{2}y^2} e^{-\frac{1}{2}x^2-4x}\,dy$
$= \frac{2\sqrt{2}}{\sqrt{\pi}} e^{-\frac{1}{2}x^2-4x} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2+xy}\,dy$
- $\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2+xy}\,dy = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y^2-2xy)}\,dy$
$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y^2-2xy+x^2)+\frac{x^2}{2}}\,dy$
$= e^{\frac{1}{2}x^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-x)^2}\,dy$
$= \sqrt{2\pi}\, e^{\frac{1}{2}x^2}$
- $f_X(x) = \frac{2\sqrt{2}}{\sqrt{\pi}} e^{-\frac{1}{2}x^2-4x} \cdot \sqrt{2\pi}\, e^{\frac{1}{2}x^2} = 4e^{-4x}$

$\Rightarrow f_X(x) = \begin{cases} 4e^{-4x}, & x \geq 0 \\ 0, & o/w \end{cases}$

5c. $\mathbb{E}[Y|X=x]=?$

- $f_{Y|X}(y|x) = \dfrac{f_{XY}(x,y)}{f_X(x)}$

$$= \frac{\frac{2\sqrt{2}}{\sqrt{\pi}} e^{xy-\frac{1}{2}x^2-4x-\frac{1}{2}y^2}}{4e^{-4x}}$$

$$= \frac{2\sqrt{2}}{\sqrt{\pi}} \cdot \frac{1}{4} e^{xy-\frac{1}{2}x^2-4x-\frac{1}{2}y^2+4x}$$

$$= \frac{1}{\sqrt{2\pi}} e^{xy-\frac{1}{2}x^2-\frac{1}{2}y^2}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-x)^2}, \quad y \in \mathbb{R}.$$

- That is, $Y|X=x \sim N(x,1)$.
- So, $\boxed{\mathbb{E}[Y|X=x]=x}$

5d. $\mathbb{E}[WY|X=x]=?$

- $f_{WY|X}(w,y|x) = \dfrac{f_{WXY}(w,x,y)}{f_X(x)}$

$$= \frac{\frac{2}{\pi} e^{xy+wx-x^2-4x-\frac{1}{2}y^2-\frac{1}{2}w^2}}{4e^{-4x}}$$

$$= \frac{2}{4\pi} \exp(xy+wx-x^2-4x-\frac{1}{2}y^2-\frac{1}{2}w^2+4x)$$

$$= \frac{1}{2\pi} \exp(-\frac{1}{2}(w^2-2wx+x^2+y^2-2xy+x^2))$$

$$= \frac{1}{2\pi} \exp(-\frac{1}{2}[(w-x)^2+(y-x)^2]), \quad w,y \in \mathbb{R}.$$

- That is,

$$\begin{pmatrix} W \\ Y \end{pmatrix} \Big| X=x \sim N_2\left(\begin{pmatrix} x \\ x \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right).$$

So, $W|X=x \sim N(x,1)$ and $Y|X=x \sim N(x,1)$ by the properties of $N_2(\ )$.

- $\mathbb{E}[WY|X=x] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} wy \cdot \frac{1}{2\pi} e^{-\frac{1}{2}(w-x)^2-\frac{1}{2}(y-x)^2}\, dw\, dy$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} y e^{-\frac{1}{2}(y-x)^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} w e^{-\frac{1}{2}(w-x)^2}\, dw\, dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} w e^{-\frac{1}{2}(w-x)^2}\, dw \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} y e^{-\frac{1}{2}(y-x)^2}\, dy$$

$$= \mathbb{E}[W|X=x]\,\mathbb{E}[Y|X=x]$$

$$= x \cdot x$$

$$\Rightarrow \mathbb{E}[WY \mid X=x] = x^2$$

5e. Show $W \not\perp\!\!\!\perp Y$, but $(W \perp\!\!\!\perp Y) \mid X=x$.

- $$\begin{aligned}\mathbb{E}[WY] &= \mathbb{E}[\mathbb{E}[WY \mid X]] \\ &= \mathbb{E}[X^2] \\ &= \int_0^\infty x^2 f_X(x)\,dx \\ &= \int_0^\infty x^2 \cdot 4e^{-4x}\,dx \\ &\quad \left(\begin{array}{l} t = 4x \\ dt = 4dx \end{array}\right. \\ &= \int_0^\infty \left(\tfrac{t}{4}\right)^2 e^{-t}\,dt \\ &= \tfrac{1}{16}\int_0^\infty t^2 e^{-t}\,dt \\ &= \tfrac{1}{16}\Gamma(3) \\ &= \tfrac{1}{16}(2!) \\ &= \tfrac{1}{8}\end{aligned}$$
- $$\begin{aligned}\mathbb{E}[Y] &= \mathbb{E}[\mathbb{E}[Y \mid X]] \\ &= \mathbb{E}[X] \\ &= \tfrac{1}{4} \quad \text{since } X \sim \text{Exp}(4).\end{aligned}$$
- By symmetry, $\mathbb{E}[W] = \frac{1}{4}$.
- So $\mathbb{E}[WY] = \frac{1}{8} \neq \frac{1}{16} = (\frac{1}{4})(\frac{1}{4}) = \mathbb{E}[W]\mathbb{E}[Y]$. Thus, $W \not\perp\!\!\!\perp Y$.

- However, as stated above, $f_{WY|X}(w,y|x) = \frac{1}{2\pi} e^{-\frac{1}{2}(w-x)^2 - \frac{1}{2}(w-y)^2}$, So $(W,Y) \mid X=x \sim N_2(\binom{x}{x}, \begin{pmatrix}1&0\\0&1\end{pmatrix})$. Thus, $W \mid X=x \sim N(x,1)$, so $f_{W|X}(w|x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(w-x)^2}$, and $Y \mid X=x \sim N(x,1)$, so $f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-x)^2}$.
- So, $$\begin{aligned} f_{W|X}(w|x) f_{Y|X}(y|x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(w-x)^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-x)^2} \\ &= \frac{1}{2\pi} e^{-\frac{1}{2}(w-x)^2 - \frac{1}{2}(y-x)^2} \\ &= f_{WY|X}(w,y|x).\end{aligned}$$
- Thus, W & Y are independent conditional on $X=x$. $\square$