PBHS 43010: Homework #1.5 (Basic Bayesian prior/posterior) and Solutions

1. Suppose there are N race cars in a rally race $\cdots$

a) Find the posterior distribution $h(N \mid x)$, the posterior mean of N, and the posterior standard deviation of N.

Solution: With the prior and likelihood,

$$h(N \mid x) \propto f(x \mid N)h(N) \propto \frac{1}{N}p^{N-1} \times I(N \geq x),$$

where $p = 99/100$. Let A denote

$$A = \sum_{N=x}^{\infty} \frac{1}{N}p^{N-1}.$$

The posterior distribution can be written as

$$h(N \mid x) = \frac{1}{A} \cdot \frac{1}{N}p^{N-1} \times I(N \geq x).$$

Since $\sum_{N=1}^{\infty} \frac{1}{N}p^{N-1} = -\frac{\ln(1-p)}{p}$, we have $A = \sum_{N=1}^{\infty} \frac{1}{N}p^{N-1} - \sum_{N=1}^{202} \frac{1}{N}p^{N-1} \approx 0.047$. When $x = 203$, the posterior mean and 2nd moment of N are,

$$E(N \mid x) = \sum_{N=x}^{\infty} N \cdot h(N \mid x) = \frac{1}{A}\sum_{N=x}^{\infty} p^{N-1} \approx 279$$

and

$$E(N^2 \mid x) = \sum_{N=x}^{\infty} N^2 \cdot h(N \mid x) = \frac{1}{A}\sum_{N=x}^{\infty} N \cdot p^{N-1}.$$

Since $\sum_{N=1}^{\infty} N \cdot p^{N-1} = \sum_{N=1}^{\infty} \frac{d}{dp}(p^N) = \frac{d}{dp}(\sum_{N=1}^{\infty} p^N) = \frac{1}{(1-p)^2}$, we have $\sum_{N=203}^{\infty} N \cdot p^{N-1} \approx 3966$. Then

$$Var(N \mid x) = E(N^2 \mid x) - (E(N \mid x))^2 \approx 80^2.$$

b) Find a 95% HPD credible interval for N.

Solution: Note $h(N \mid x)$ is monotone decreasing. We find the probability density function of posterior distribution $h(N \mid x)$ in Figure 1. Since $\sum_{N=203}^{438} h(N \mid x) \approx 94.97\%$ and $\sum_{N=203}^{438} h(N \mid x) \approx 95.03\%$, we select $[203, 438]$ as the 95% HPD credible interval for N.

2. Consider a Poisson sampling model $f(n \mid \lambda) = Poi(\lambda)$. Find Jeffrey's prior for λ.

Solution: Jeffrey's principle leads to defining the non-informative prior density as $p(\lambda) \propto \sqrt{I(\lambda)}$, where $I(\lambda)$ is the Fisher information of λ.

$$I(\lambda) = E\left[\left(\frac{d \log f(n \mid \lambda)}{d\lambda}\right)^2 \middle| \lambda\right] = -E\left[\frac{d^2 \log f(n \mid \lambda)}{d\lambda^2}\middle| \lambda\right] = E\left[\left(\frac{n-\lambda}{\lambda}\right)^2 \middle| \lambda\right] = \frac{1}{\lambda}.$$

Then Jeffrey's prior for λ is $p(\lambda) \propto \lambda^{-1/2}$.

3. Your friend always uses a certain coin to bet "heads or tails" and $\cdots$

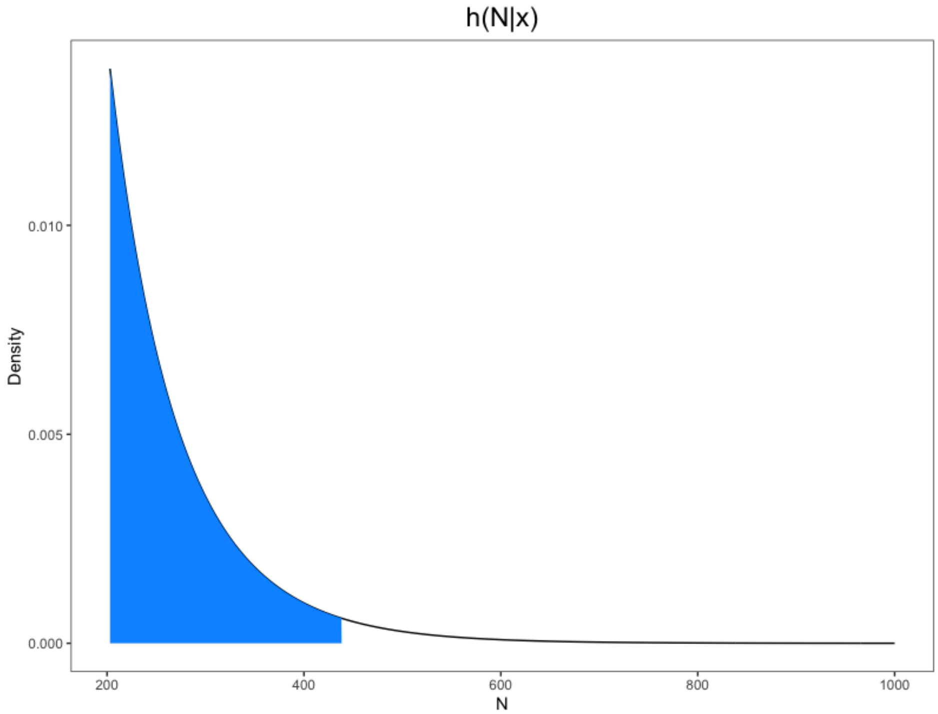

Figure 1: Probability density function of posterior distribution $h(N \mid x)$

a) Find the predictive distribution for the first toss, under each of the three hypotheses, i.e., find $p(x_1 = 1 \mid H_1)$, $p(x_1 = 1 \mid H_2)$, and $p(x_1 = 1 \mid H_3)$.

Solution:

$$p(x_1 = 1 \mid H_1) = \int_0^{0.5} p(x_1 = 1 \mid \theta) p(\theta \mid H_1) \mathrm{d}\theta = \int_0^{0.5} \theta \cdot 2 \mathrm{d}\theta = 0.25$$
$$p(x_1 = 1 \mid H_2) = p(x_1 = 1 \mid \theta = 0.5) p(\theta = 0.5 \mid H_2) = 0.5 \cdot 1 = 0.5$$
$$p(x_1 = 1 \mid H_3) = \int_{0.5}^{1} p(x_1 = 1 \mid \theta) p(\theta \mid H_3) \mathrm{d}\theta = \int_{0.5}^{1} \theta \cdot 2 \mathrm{d}\theta = 0.75$$

b) Find the predictive distribution for the first toss, $p(x_1 = 1)$.

Solution:

$$p(x_1 = 1) = p(x_1 = 1 \mid H_1) p(H_1) + p(x_1 = 1 \mid H_2) p(H_2) + p(x_1 = 1 \mid H_3) p(H_3) = 0.5$$

4. The Rayleigh distribution with p.d.f. $f(x \mid \delta) = \delta x e^{-\delta x^2/2} I_{(0,+\infty)(x)}$ is used for $\cdots$

a) Find the posterior distribution $h(\delta \mid x)$, $E(\delta \mid x)$, and $Var(\delta \mid x)$. What's the name of the posterior distribution?

Solution: Given the likelihood and the prior $h(\delta) \propto \delta^{a-1} e^{-b\delta}$,

$$h(\delta \mid x) \propto \left\{ \prod_{i=1}^{n} f(x_i \mid \delta) \right\} h(\delta) \propto \left\{ \delta^n e^{-\delta (\sum_i x_i^2)/2} \right\} \delta^{a-1} e^{-b\delta} = \delta^{a+n-1} e^{-(b + \sum_i x_i^2/2)\delta}.$$

$h(\delta \mid x)$ is also a gamma distribution, $Ga(A, B)$, where $A = a + n$ and $B = b + \sum_i x_i^2/2$. And $E(\delta \mid x) = A/B$ and $Var(\delta \mid x) = A/B^2$.

b) Find the Jeffrey's prior $h(\delta)$ and the corresponding posterior distribution conditional on one observation x from the Rayleigh model.

Solution: Jeffrey's principle leads to defining the non-informative prior density as $p(\delta) \propto \sqrt{I(\delta)}$, where $I(\delta)$ is the Fisher information of λ.

$$I(\delta) = -E\left[\frac{d^2 \log f(x \mid \delta)}{d\delta^2}\middle|\delta\right] = -E\left(-\frac{1}{\delta^2}\middle|\delta\right) = \frac{1}{\delta^2}$$

Then Jeffrey's prior for δ is $h(\delta) \propto \frac{1}{\delta}$.