1. Likelihood $f(y_1 \cdots y_n \mid \mu, \tau) \propto \prod_{i=1}^{n} e^{-\frac{\tau}{2}(y_i-\mu)^2} \Rightarrow (y_1, \cdots, y_n) \sim MVN(\mu \underset{\sim}{1}, \tau^{-1} I_n)$

(here, $\underset{\sim}{1} = \underbrace{(1, \cdots, 1)}_{n}^T$, $I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{n \times n}$.)

a) $\bar{y} = \frac{1}{n} \underset{\sim}{1}^T \underset{\sim}{y} \sim N(\mu \frac{1}{n} \underset{\sim}{1}^T \underset{\sim}{1}, \frac{1}{n} \underset{\sim}{1}^T I_n \underset{\sim}{1} \frac{1}{n} \tau^{-1})$

$= N(\mu, \underbrace{\frac{1}{n\tau}}_{\text{variance}})$ variance.

$$\boxed{\therefore p(\bar{y} \mid \mu) = N(\mu, \frac{1}{n\tau})}$$

Now $\mu \sim N(\mu_0, \sigma_0^2)$, i.e. $\pi(\mu) = N(\mu_0, \frac{1}{\tau_0})$

Then $p(\mu \mid \bar{y}) \propto p(\bar{y} \mid \mu)\, \pi(\mu) \propto e^{-\frac{1}{2} n\tau (\bar{y}-\mu)^2} \cdot e^{-\frac{1}{2}\tau_0 (\mu-\mu_0)^2}$

$\propto e^{-\frac{1}{2}\left[n\tau\mu^2 - 2n\tau\bar{y}\mu + \tau_0\mu^2 - 2\tau_0\mu_0\tau \right] + \text{Constant}}$

$\propto e^{-\frac{1}{2}\left[(n\tau+\tau_0)\mu^2 - 2(n\tau\bar{y} + \tau_0\mu_0)\tau \right] + \text{Constant}}$

$\propto e^{-\frac{n\tau+\tau_0}{2}\left(\mu - \frac{n\tau\bar{y}+\tau_0\mu_0}{n\tau+\tau_0}\right)^2 + \text{constant}}$

$$\therefore \boxed{\mu \mid \bar{y} \sim N\left(\frac{n\tau\bar{y}+\tau_0\mu_0}{n\tau+\tau_0}, \frac{1}{n\tau+\tau_0}\right)}$$

b) $y \mid \theta \sim Bin(n, \theta)$
$\theta \sim Beta(a, b)$ $\Rightarrow p(\theta \mid y) \propto f(y \mid \theta)\, \pi(\theta) \propto \underbrace{\theta^y (1-\theta)^{n-y}}_{\text{Binomial}} \cdot \underbrace{\theta^{a-1}(1-\theta)^{b-1}}_{\text{beta}}$

$= \theta^{a+y-1}(1-\theta)^{n+b-y-1}$

Since $\int p(\theta \mid y)\, d\theta = 1$, $p(\theta \mid y) = \frac{\theta^{a+y-1}(1-\theta)^{n+b-y-1}}{B(a+y, n+b-y)}$, where $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

$\therefore \theta \mid y \sim Beta(a+y, n+b-y)$.

Next, $p(y_{new} \mid y) = \int p(y_{new} \mid \theta)\, p(\theta \mid y)\, d\theta = \int \binom{n}{y_{new}} \theta^{y_{new}} (1-\theta)^{n-y_{new}} \cdot \frac{1}{B(a+y, n+b-y)} \theta^{a+y-1} (1-\theta)^{n+b-y-1}\, d\theta$

$= \int \binom{n}{y_{new}} \frac{1}{B(a+y, n+b-y)} \theta^{y_{new}+a+y-1} (1-\theta)^{2n+b-y-1-y_{new}}\, d\theta$

$= \binom{n}{y_{new}} \frac{B(y_{new}+a+y, 2n+b-y-y_{new})}{B(a+y, n+b-y)} \underbrace{\int \frac{\theta^{y_{new}+a+y-1}(1-\theta)^{2n+b-y-1-y_{new}}}{B(y_{new}+a+y, 2n+b-y-y_{new})}\, d\theta}_{=1}$

$\boxed{\sim \text{Beta-Binomial}(n, a+y, n+b-y), \text{ and } E(y_{new} \mid y) = \frac{n(a+y)}{n+b+a}}$ #

2. Model: $\begin{cases} y \mid \mu, \tau \sim N(\mu, \tau^{-1}); \quad \tau = 1/\sigma^2 \\ \mu \mid \tau \sim N(\mu_0, \tau^{-1}\sigma_0^2) \\ \tau \sim \text{gamma}\left(\frac{a_0}{2}, \frac{b_0}{2}\right) \end{cases}$ — assume gamma(a,b) has mean a/b.

$$p(\mu \mid \tau, y) \propto p(y, \mu, \tau) = p(y \mid \mu, \tau)\,\pi(\mu \mid \tau)\,\pi(\tau)$$

$$= \underbrace{\frac{\tau^{1/2}}{\sqrt{2\pi}}}_{\text{Constant}} e^{-\frac{\tau}{2}(y-\mu)^2} \cdot \underbrace{\frac{\tau^{1/2}}{\sqrt{2\pi}\,\sigma_0}}_{\text{Constant}} e^{-\frac{\tau}{2\sigma_0^2}(\mu-\mu_0)^2} \cdot \underbrace{\frac{\left(\frac{b_0}{2}\right)^{a_0/2}}{\Gamma\left(\frac{a_0}{2}\right)}}_{\text{Constant}} \tau^{\frac{a_0}{2}-1} e^{-\frac{b_0 \tau}{2}}$$

$$\propto e^{-\frac{\tau}{2}(y-\mu)^2 - \frac{\tau}{2\sigma_0^2}(\mu-\mu_0)^2}$$

$$\propto e^{-\frac{1}{2}\tau\left(1+\frac{1}{\sigma_0^2}\right)\left(\mu - \frac{\sigma_0^2 y + \mu_0}{\sigma_0^2+1}\right)^2}$$

$$\therefore \boxed{p(\mu \mid \tau, y) \sim N\left(\frac{\sigma_0^2 y + \mu_0}{\sigma_0^2 + 1}, \frac{1}{\underbrace{\tau\left(1+\frac{1}{\sigma_0^2}\right)}_{\tau_0}}\right)}$$

$$p(\mu \mid y) \propto p(y, \mu) = \int p(y, \mu, \tau)\, d\tau = \int p(y \mid \mu, \tau)\,\pi(\mu \mid \tau)\,\pi(\tau)\, d\tau$$

$$= \int \frac{\tau^{1/2}}{\sqrt{2\pi}} e^{-\frac{\tau}{2}(y-\mu)^2} \frac{\tau^{1/2}}{\sqrt{2\pi}\,\sigma_0} e^{-\frac{\tau}{2\sigma_0^2}(\mu-\mu_0)^2} \cdot \frac{(b_0/2)^{a_0/2}}{\Gamma\left(\frac{a_0}{2}\right)} \tau^{\frac{a_0}{2}-1} e^{-b_0\tau/2}\, d\tau$$

$$\propto \int \tau^{\overbrace{\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{a_0}{2}}^{}\!{}_{A} - 1} \cdot e^{-\frac{1}{2}\underbrace{\left\{b_0 + (y-\mu)^2 + \frac{(\mu-\mu_0)^2}{\sigma_0^2}\right\}}_{B} \cdot \tau}\, d\tau$$

$$= \underbrace{\int \tau^{A-1} e^{-B\cdot\tau} \cdot \frac{B^A}{\Gamma(A)}\, d\tau}_{=1} \cdot \frac{\Gamma(A)}{B^A}$$

$$= \frac{\Gamma(A)}{B^A} \propto B^{-A} = \left\{b_0 + (y-\mu)^2 + \frac{1}{\sigma_0^2}(\mu-\mu_0)^2\right\}^{-\left(\frac{a_0}{2}+1\right)}$$

$$= \left\{ b_0 + y^2 + \frac{1}{\sigma_0^2}\mu_0^2 + \mu^2 - 2y\mu + \frac{1}{\sigma_0^2}\mu^2 - 2\frac{\mu_0}{\sigma_0^2}\mu \right\}^{-\left(\frac{a_0}{2}+1\right)}$$

$$= \left\{ b_0 + y^2 + \frac{1}{\sigma_0^2}\mu_0^2 + \left(1+\frac{1}{\sigma_0^2}\right)\mu^2 - 2\left(y + \frac{\mu_0}{\sigma_0^2}\right)\mu \right\}^{-\left(\frac{a_0}{2}+1\right)}$$

$$= \left\{ b_0 + y^2 + \frac{1}{\sigma_0^2}\mu_0^2 + \left(1+\frac{1}{\sigma_0^2}\right)\left(\mu^2 - 2\frac{y+\frac{\mu_0}{\sigma_0^2}}{1+\frac{1}{\sigma_0^2}}\mu\right) \right\}^{-\left(\frac{a_0}{2}+1\right)}$$

$$= \left\{ b_0 + y^2 + \tau_0\mu_0^2 + (1+\tau_0)\left(\mu^2 - 2\frac{y+\mu_0\tau_0}{1+\tau_0}\mu\right) \right\}^{-\left(\frac{a_0}{2}+1\right)}$$

$$= \left\{ \underbrace{b_0 + y^2 + \tau_0\mu_0^2 - (1+\tau_0)\left(\frac{y+\mu_0\tau_0}{1+\tau_0}\right)^2}_{C} + (1+\tau_0)\left(\mu - \frac{y_0+\mu_0\tau_0}{1+\tau_0}\right)^2 \right\}^{-\left(\frac{a_0}{2}+1\right)}$$

Note: you can show $C = b_0 + \frac{\tau_0}{1+\tau_0}(y-\mu_0)^2$

$$\propto \left\{ 1 + \frac{(1+\tau_0)(a_0+1)}{C\,(a_0+1)}\left(\mu - \frac{y_0+\mu_0\tau_0}{1+\tau_0}\right)^2 \right\}^{-\frac{(a_0+1)+1}{2}}$$

$$= \left\{ 1 + \frac{1}{(a_0+1)}\left(\frac{\mu - \frac{y_0+\mu_0\tau_0}{1+\tau_0}}{\sqrt{\frac{C}{(1+\tau_0)(a_0+1)}}} \right)^2 \right\}^{-\frac{(a_0+1)+1}{2}}$$

$\therefore$ $\mu|y \sim$ nonstandardized t dist'n $\left(\underbrace{\frac{y_0+\mu_0\tau_0}{1+\tau_0}}_{\text{mean}}, \underbrace{\frac{C}{(1+\tau_0)(a_0+1)}}_{\text{scale}}, \underbrace{a_0+1}_{\text{degree of freedom}} \right)$

Lastly,

$$p(\tau|y) \propto p(\tau, y) = \int p(y,\mu,\tau)\,d\mu = \int p(y|\mu,\tau)\,\pi(\mu,\tau)\,\pi(\tau)\,d\mu$$

$$= \int \frac{\tau^{1/2}}{\sqrt{2\pi}} e^{-\frac{\tau}{2}(y-\mu)^2} \underline{\frac{\tau^{1/2}}{\sqrt{2\pi}\,\sigma_0}} e^{-\frac{\tau}{2\sigma_0^2}(\mu-\mu_0)^2} \frac{(b_0/2)^{a_0/2}}{\Gamma\left(\frac{a_0}{2}\right)} \tau^{\frac{a_0}{2}-1} e^{-b_0\tau/2}\,d\mu$$

$$\propto \tau^{\frac{1}{2}+\frac{1}{2}+\frac{a_0}{2}-1} \cdot e^{-b_0\tau/2} \int e^{-\frac{\tau}{2}(y-\mu)^2} \cdot e^{-\frac{\tau}{2\sigma_0^2}(\mu-\mu_0)^2}\, d\mu$$

$$= \tau^{\frac{a_0}{2}} e^{-\frac{b_0}{2}\tau} \int e^{-\frac{\tau}{2}\left(y^2-2y\mu+\mu^2+\tau_0(\mu^2-2\mu_0\mu+\mu_0^2)\right)}\, d\mu$$

$$= \tau^{\frac{a_0}{2}} e^{-\frac{b_0}{2}\tau} \int e^{-\frac{\tau}{2}\left[(1+\tau_0)\mu^2-2(y+\tau_0\mu_0)\mu+y^2+\tau_0\mu_0^2\right]}\, d\mu$$

$$= \tau^{\frac{a_0}{2}} e^{-\frac{b_0}{2}\tau} \int e^{-\frac{\tau}{2}\left[(1+\tau_0)\left\{\mu-\frac{y+\tau_0\mu_0}{1+\tau_0}\right\}^2+\underbrace{y^2+\tau_0\mu_0^2-\frac{(y+\tau_0\mu_0)^2}{1+\tau_0}}_{\triangleq A}\right]}\, d\mu$$

$$= \tau^{\frac{a_0}{2}} e^{-\tau\frac{b_0+A}{2}} \tau^{-1/2}(1+\tau_0)^{-1/2} \underbrace{\int \frac{(1+\tau_0)^{1/2}\tau^{1/2}}{\sqrt{2\pi}} e^{-\frac{\tau(1+\tau_0)}{2}\left(\mu-\frac{y+\tau_0\mu_0}{1+\tau_0}\right)^2}\, d\mu}_{\triangleq 1}$$

$$\propto \tau^{\frac{a_0+1}{2}-1} e^{-\tau\frac{b_0+A}{2}}$$

$$\therefore \boxed{p(\tau|y) \sim \mathrm{Gama}\left(\frac{a_0+1}{2}, \frac{b_0+A}{2}\right)}$$

Here: $A = y^2+\tau_0\mu_0^2-\frac{(y+\tau_0\mu_0)^2}{1+\tau_0} = \frac{(y^2+\tau_0\mu_0^2)(1+\tau_0)-(y+\tau_0\mu_0)^2}{1+\tau_0}$

$$= \frac{\cancel{y^2}+y^2\tau_0+\tau_0\mu_0^2+\cancel{\tau_0^2\mu_0^2}-\cancel{y^2}-2y\tau_0\mu_0-\cancel{\tau_0^2\mu_0^2}}{1+\tau_0}$$

$$= \frac{\tau_0(y^2+\mu_0^2-2y\mu_0)}{1+\tau_0} = \frac{\tau_0(y-\mu_0)^2}{1+\tau_0}$$

3. Model $\begin{cases} X|p \sim Bin(50, p) \\ p \sim \frac{1}{2}\{Be(10, 20) + Be(20,10)\} \end{cases}$

First, $\pi(p) = \frac{1}{2}\left\{\frac{1}{B(10,20)} p^{10-1}(1-p)^{20-1} + \frac{1}{B(20,10)} p^{20-1}(1-p)^{10-1}\right\}$

Then $f(p|x) \propto f(x|p)\cdot\pi(p)$

$$= \binom{50}{x} p^x (1-p)^{50-x} \cdot \frac{1}{2}\left\{\frac{1}{B(10,20)} p^{10-1}(1-p)^{20-1} + \cdots\right\}$$

$$\propto \frac{1}{B(10,20)} p^{10+x-1}(1-p)^{70-x-1} + \frac{1}{B(20,10)} p^{20+x-1}(1-p)^{60-x-1}$$

$x=30$, so

$$= \frac{1}{B(10,20)} p^{40-1}(1-p)^{40-1} + \frac{1}{B(20,10)} p^{50-1}(1-p)^{30-1}$$

$$= \underbrace{\frac{B(40,40)}{B(10,20)}}_{w_1^*}\left[\frac{1}{B(40,40)} p^{40-1}(1-p)^{40-1}\right] + \underbrace{\frac{B(50,30)}{B(20,10)}}_{w_2^*}\left[\frac{p^{50-1}(1-p)^{30-1}}{B(50,30)}\right]$$

$$= \boxed{\frac{w_1^*}{w_1^*+w_2^*} Be(40,40) + \frac{w_2^*}{w_1^*+w_2^*} Be(50,30).}$$

Similarly, for data (x, n), prior $p \sim \sum_{i=1}^{k} w_i Be(a_i, b_i)$

$$\boxed{\begin{aligned} f(p|x) &= \sum_{i=1}^{k} \frac{w_i^*}{c} \cdot Be(x+a_i, n-x+b_i) \\ w_i^* &= \frac{B(x+a_i, n-x+b_i)}{B(a_i, b_i)}, \quad c = \sum_{i=1}^{k} w_i^*. \end{aligned}}$$

4. Model $\begin{cases} \underset{\sim}{y} \mid \underset{\sim}{\beta}, \sigma^2 \sim MVN(\underset{\sim}{X}\underset{\sim}{\beta}, \sigma^2 I_n) \\ p(\beta) \propto 1 \\ p(\sigma^2) \propto \frac{1}{\sigma}. \end{cases}$

$$\therefore p(\underset{\sim}{y}, \underset{\sim}{\beta}, \sigma^2) = \frac{(X^TX)^{1/2}}{\sqrt{2\pi}\,\sigma^n} e^{-\frac{1}{2\sigma^2}(\underset{\sim}{y}-\underset{\sim}{X}\underset{\sim}{\beta})^T(\underset{\sim}{y}-\underset{\sim}{X}\underset{\sim}{\beta})} \frac{1}{\sigma}$$

$$\propto \sigma^{-(n+1)} e^{-\frac{1}{2\sigma^2}(\underset{\sim}{y}-\underset{\sim}{X}\underset{\sim}{\beta})^T(\underset{\sim}{y}-\underset{\sim}{X}\beta)}$$

$$= \sigma^{-(n+1)} e^{-\frac{1}{2\sigma^2}\left\{(n-k)\hat{\sigma}^2 + (\beta-\hat{b})^T X^T X(\beta-\hat{b})\right\}}$$

where $\hat{\sigma}^2 = \dfrac{(y-X\hat{b})^T(y-X\hat{b})}{n-k}$ and $\hat{b} = (X^TX)^{-1}X^Ty$

$$\therefore f(\sigma^2 \mid y) \propto f(\sigma^2, y) = \int p(y, \beta, \sigma^2)\, d\beta$$

$$\propto \int \sigma^{-(n+1)} e^{-\frac{1}{2\sigma^2}\left\{(n-k)\hat{\sigma}^2 + (\beta-\hat{b})^T X^T X(\beta-\hat{b})\right\}} d\beta$$

$$= \sigma^{-(n+1)} e^{-\frac{(n-k)\hat{\sigma}^2}{2\sigma^2}} \int e^{-\frac{1}{2\sigma^2}(\beta-\hat{b})^T X^T X(\beta-\hat{b})}\, d\beta$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{n-k-1}{2}+1} e^{-\frac{1}{\sigma^2}\left(\frac{(n-k)\hat{\sigma}^2}{2}\right)} \underbrace{\int \frac{(X^TX)^{-\frac{1}{2}}}{\sqrt{2\pi}\,\sigma^k} e^{-\frac{1}{2\sigma^2}(\beta-\hat{b})X^T X(\beta-\hat{b})}\, d\beta}_{=1}$$

$$\boxed{\therefore \sigma \mid y \sim \text{Inv Gamma}\left(\frac{n-k-1}{2}, \frac{(n-k)\hat{\sigma}^2}{2}\right)}$$

Similarly to problem 2,

$$f(\beta|y) \propto f(y,\beta) = \int p(y,\beta,\sigma^2)\, d\sigma^2$$

$$\propto \int \sigma^{-(n+1)} e^{-\frac{1}{2\sigma^2}\{(n-k)\hat{\sigma}^2 + (\beta-\hat{b})^T (X^T X)^{-1} (\beta-\hat{b})\}} \underline{\underline{d\sigma^2}}$$

$$= \int \tau^{\frac{n+1}{2}} e^{-\frac{\tau}{2}\{(n-k)\hat{\sigma}^2 + (\beta-\hat{b})^T (X^T X)^{-1} (\beta-\hat{b})\}} \underline{\underline{\tau^{-2} d\tau}}$$

$$\propto \{(n-k)\hat{\sigma}^2 + (\beta-\hat{b})^T X^T X (\beta-\hat{b})\}^{-\frac{n-1}{2}}$$

$$\propto \left\{1 + \frac{1}{n-k}(\beta-\hat{b})^T \left[\hat{\sigma}^2 (X^T X)^{-1}\right]^{-1} (\beta-\hat{b})\right]^{-\frac{(n-k-1)+k}{2}}$$

$$\sim MVT\left(\hat{b}, \frac{\hat{\sigma}^2 (X^T X)^{-1}}{(n-k-1)/n-k}, n-k-1\right)$$

5. Likelihood:

$$\prod_{i:z_i=0} f(y_i|\theta) \cdot \prod_{i:z_i=1} P_r(y_i \geq L)$$

$$= \prod_{i=1}^{k} \left(\theta e^{-\theta y_i}\right) \cdot \prod_{i=k+1}^{n} e^{-\theta L}$$

$$= \underline{\theta^k e^{-\theta\left(\sum_{i=1}^{k} y_i + (n-k) \cdot L\right)}}$$

Let $\pi(\theta) = \text{Gamma}(\alpha, \beta)$

the $p(\theta | y_1 \cdots y_n, z_1 \cdots z_n) = \underline{\text{Gamma}\left(k+\alpha, \beta + \sum_{i=1}^{k} y_i + (n-k)L\right)}$