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STAT 35920

HW #4

1a. Sps x_i is # of snow falls in Chicago in year i, $i \in \{1, \dots, 10\}$. Assume $x_i | \lambda \sim Pois(\lambda)$. Find conjugate prior distribution of λ & posterior distribution in closed forms.

- For each i, $f(x_i|\lambda) = \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$
- So, the likelihood function is

$$L(\lambda|x_1, \dots, x_{10}) = \prod_{i=1}^{10} f(x_i|\lambda)$$
$$= \prod_{i=1}^{10} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$
$$= \frac{\lambda^{\sum_{i=1}^{10} x_i} e^{-10\lambda}}{\prod_{i=1}^{10} x_i!}$$

- By inspection, suppose $\lambda \sim Gamma(\alpha, \beta)$. Then the prior density is $\pi(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$.
- Then the posterior density is

$$p(\lambda|x_1, \dots, x_{10}) \propto \pi(\lambda) L(\lambda|x_1, \dots, x_{10})$$
$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} \cdot \frac{\lambda^{\sum_{i=1}^{10} x_i} e^{-10\lambda}}{\prod_{i=1}^{10} x_i!}$$
$$\propto \lambda^{\alpha + \sum_{i=1}^{10} x_i - 1} e^{-(\beta+10)\lambda},$$

which is the kernel of a gamma distribution. Thus, a gamma distribution is indeed a conjugate prior.

- Conjugate Prior: $\lambda \sim Gamma(\alpha, \beta)$
- Posterior: $\lambda | x_1, \dots, x_{10} \sim Gamma(\alpha + \sum_{i=1}^{10} x_i, \beta + 10)$

1b. Sps data are $x = (5, 4, 6, 5, 5, 6, 5, 7, 3, 6)$. Find posterior mean & variance of λ assuming conjugate prior in Part (a) w/ prior mean 5 & variance 5.

- Our prior distribution of λ is $\lambda \sim Gamma(\alpha, \beta)$ where $E[\lambda] = \frac{\alpha}{\beta} = 5$ and $Var(\lambda) = \frac{\alpha}{\beta^2} = 5$.

→

$\Rightarrow \begin{cases} \alpha = 5\beta \\ \alpha = 5\beta^2 \end{cases}$

$\Rightarrow 5\beta = 5\beta^2$

$\Rightarrow \beta^2 - \beta = 0$

$\Rightarrow \beta(\beta - 1) = 0$

$\Rightarrow \beta = 0$ or $\beta = 1$

But the rate $\beta > 0$, so $\beta = 1$.

$\Rightarrow \alpha = 5\beta = 5(1) = 5.$

- Thus, our prior is $\lambda \sim \text{Gamma}(5, 1)$
- Then our posterior distribution is

$$\lambda | x_1, \ldots, x_{10} \sim \text{Gamma}(\alpha + \textstyle\sum_{i=1}^{10} x_i, \beta + 10)$$

$$= \text{Gamma}(5 + (5+4+6+5+5+6+5+7+3+6), 1+10)$$

$$= \text{Gamma}(5+52, 11)$$

$$= \text{Gamma}(57, 11)$$

- Thus,

$$E[\lambda | x_1, \ldots, x_{10}] = \frac{57}{11} \quad \text{and}$$

$$Var(\lambda | x_1, \ldots, x_{10}) = \frac{57}{(11)^2} = \frac{57}{121}$$

2. Let $X_1, \ldots, X_n \sim$ iid $N(0, \tau)$, where τ is unknown variance.

2a. Write the likelihood $L(\tau|x)$, $x = (x_1, \ldots, x_{10})$

- $\forall i$, $f(x_i) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{1}{2\tau}(x_i - 0)^2} = \frac{1}{\sqrt{2\pi\tau}} e^{-x_i^2/2\tau}$
- So,

$$L(\tau|x) = \prod_{i=1}^n f(x_i)$$
$$= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{1}{2\tau} x_i^2}$$
$$\Rightarrow L(\tau|x) = \frac{1}{(2\pi\tau)^{n/2}} e^{-\frac{1}{2\tau}\sum_{i=1}^n x_i^2}$$

2b. Derive the Jeffrey's prior for τ,

- $f(x_i) = \frac{1}{\sqrt{2\pi\tau}} e^{-x_i^2/2\tau}$
- $\log f(x_i) = \log\left(\frac{1}{\sqrt{2\pi\tau}} e^{-x_i^2/2\tau}\right)$

$$= \log\left(\frac{1}{\sqrt{2\pi\tau}}\right) + \log\left(e^{-x_i^2/2\tau}\right)$$
$$= -\frac{1}{2}\log(2\pi\tau) - \frac{1}{2\tau} x_i^2$$

- $$\frac{\partial \log f(x_i)}{\partial \tau} = -\frac{1}{2} \cdot \frac{1}{2\pi\tau} \cdot 2\pi - \frac{x_i^2}{2} \cdot (-1\tau^{-2})$$
$$= -\frac{1}{2\tau} + \frac{x_i^2}{2\tau^2}$$

- $$\frac{\partial^2 \log f(x_i)}{\partial \tau^2} = \frac{\partial}{\partial \tau}\left[-\frac{1}{2}\tau^{-1} + \frac{x_i^2}{2}\tau^{-2}\right]$$
$$= -\frac{1}{2}\left(\frac{-1}{\tau^2}\right) + \frac{x_i^2}{2}\left(\frac{-2}{\tau^3}\right)$$
$$= \frac{1}{2\tau^2} - \frac{x_i^2}{\tau^3}$$

- $$I(\tau) = -\mathbb{E}\left[\frac{\partial^2 \log f(x_i)}{\partial \tau^2}\right]$$
$$= -\mathbb{E}\left[\frac{1}{2\tau^2} - \frac{x_i^2}{\tau^3}\right]$$
$$= -\left(\frac{1}{2\tau^2} - \frac{1}{\tau^3}\mathbb{E}[x_i^2]\right)$$

$\rightarrow$

- Note that $\forall i \in \{1, ..., 10\}$

$$\begin{aligned}\tau &= Var(X_i) \\ &= \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2 \\ &= \mathbb{E}[X_i^2] - 0^2 \\ &= \mathbb{E}[X_i^2].\end{aligned}$$

- So, continuing,

$$\begin{aligned}I(\tau) &= -\left(\frac{1}{2\tau^2} - \frac{1}{\tau^3}\cdot\tau\right) \\ &= -\left(\frac{1}{2\tau^2} - \frac{1}{\tau^2}\right) \\ &= \frac{1}{\tau^2} - \frac{1}{2\tau^2} \\ &= \frac{1}{2\tau^2}\end{aligned}$$

- Thus, the Jeffreys prior is

$$\begin{aligned}\pi_J(\tau) &\propto \sqrt{\frac{1}{2\tau^2}} \\ &= \frac{1}{\sqrt{2}\tau}\end{aligned}$$

$$\Rightarrow \pi_J(\tau) \propto \frac{1}{\tau}$$

2c. Data $x_1 = 2.72$, $x_2 = 1.65$, $x_3 = 0.44$, $x_4 = -1.62$, $x_5 = 0.27$, $x_6 = -2.22$.
Write posterior dist, using Jeffreys prior from Part (b).
Specify distribution of posterior.

- $$\begin{aligned}p(\tau | x_1, ..., x_6) &\propto \pi_J(\tau) L(\tau | x) \\ &= \frac{1}{\tau} \cdot \frac{1}{(2\pi)^{6/2} \tau^{6/2}} e^{-\frac{1}{2\tau}\sum_{i=1}^{6} x_i^2} \\ &\propto \frac{1}{\tau \cdot \tau^3} e^{-\frac{1}{2\tau}(2.72^2 + 1.65^2 + 0.44^2 + (-1.62)^2 + 0.27^2 + (-2.22)^2)} \\ &= \frac{1}{\tau^4} \exp\{-\frac{1}{2\tau} \cdot 17.9402\} \\ &= \tau^{-4} \exp\{-8.9701/\tau\},\end{aligned}$$

which is the kernel of an inverse gamma distribution.
- That is,

$$\tau | x_1, ..., x_6 \sim IG(3, 8.9701)$$

4. Sps $X|\theta \sim N(\theta, \sigma^2)$ w/ unknown θ, known σ^2.
$H_0: \theta = 0$, $H_1: \theta \sim N(0, \Sigma^2)$. Observe $x = \lambda\sigma$, $\lambda > 0$.
Derive BF_{01}. Which model favored if $\lambda \gg 1$? If
$\lambda \approx 1$ & $\sigma/\Sigma \gg 1$?

- $BF_{01} = \dfrac{p(x|H_0)}{p(x|H_1)}$

- $$p(x|H_0) \propto p(x|\theta=0)\, p(\theta=0|H_0)$$
$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-0)^2} \cdot 1$$
$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\lambda\sigma)^2}$$
$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2} \cdot \lambda^2\sigma^2\right\}$$
$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\lambda^2\right\}$$

- $$p(x|H_1) \propto \int_{\theta\in H_1} p(x|\theta)\, p(\theta|H_1)\, d\theta$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\theta)^2} \cdot \frac{1}{\sqrt{2\pi\Sigma^2}} e^{-\frac{1}{2\Sigma^2}(\theta-0)^2} d\theta$$
$$= \frac{1}{2\pi\sigma\Sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(\lambda\sigma-\theta)^2 - \frac{1}{2\Sigma^2}\theta^2} d\theta$$
$$= \frac{1}{2\pi\sigma\Sigma} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}(\lambda^2\sigma^2 - 2\lambda\sigma\theta + \theta^2) - \frac{1}{2\Sigma^2}\theta^2\right\} d\theta$$
$$= \frac{1}{2\pi\sigma\Sigma} \int_{-\infty}^{\infty} \exp\left\{\theta^2\left(-\frac{1}{2\sigma^2} - \frac{1}{2\Sigma^2}\right) + \frac{2\lambda\sigma}{2\sigma^2}\theta - \frac{\lambda^2\sigma^2}{2\sigma^2}\right\} d\theta$$
$$= \frac{1}{2\pi\sigma\Sigma} \int_{-\infty}^{\infty} \exp\left\{-\left[\underbrace{\left(\frac{\Sigma^2+\sigma^2}{2\sigma^2\Sigma^2}\right)}_{:=a}\theta^2 \underbrace{-\frac{\lambda}{\sigma}}_{:=b}\theta + \underbrace{\frac{\lambda^2}{2}}_{:=c}\right]\right\} d\theta$$
$$= \frac{1}{2\pi\sigma\Sigma} \sqrt{\frac{\pi}{a}} \exp\left\{\frac{b^2}{4a} - c\right\}$$
$$= \frac{\sqrt{\pi}}{2\pi\sigma\Sigma} \cdot \frac{1}{\sqrt{\frac{\Sigma^2+\sigma^2}{2\sigma^2\Sigma^2}}} \exp\left\{\frac{(-\lambda/\sigma)^2}{4 \cdot \frac{\Sigma^2+\sigma^2}{2\sigma^2\Sigma^2}} - \frac{\lambda^2}{2}\right\}$$
$$= \frac{\sqrt{\pi}}{2\pi\sigma\Sigma} \cdot \frac{\sqrt{2\sigma^2\Sigma^2}}{\sqrt{\Sigma^2+\sigma^2}} \exp\left\{\frac{\lambda^2/\sigma^2}{\frac{2(\Sigma^2+\sigma^2)}{\sigma^2\Sigma^2}}\right\} \exp\left\{-\frac{1}{2}\lambda^2\right\}$$

$$= \frac{\sqrt{2\pi}\,\cancel{\sigma}\,\Sigma}{2\pi\,\cancel{\sigma}\,\Sigma} \cdot \frac{1}{\sqrt{\Sigma^2+\sigma^2}} \exp\left\{\frac{\lambda^2}{\sigma^2} \cdot \frac{\sigma^2\Sigma^2}{2(\Sigma^2+\sigma^2)}\right\} \exp\left\{-\frac{1}{2}\lambda^2\right\}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\Sigma^2+\sigma^2}} \exp\left\{\frac{\lambda^2\Sigma^2}{2(\Sigma^2+\sigma^2)}\right\} \exp\left\{-\frac{1}{2}\lambda^2\right\}$$

- So,

$$BF_{01} = \frac{\frac{1}{\sqrt{2\pi}\sqrt{\sigma^2}} \cancel{\exp\left\{-\frac{1}{2}\lambda^2\right\}}}{\frac{1}{\sqrt{2\pi}\sqrt{\Sigma^2+\sigma^2}} \exp\left\{\frac{\lambda^2\Sigma^2}{2(\Sigma^2+\sigma^2)}\right\} \cancel{\exp\left\{-\frac{1}{2}\lambda^2\right\}}}$$

$$= \sqrt{\frac{\Sigma^2+\sigma^2}{\sigma^2}} \cdot \frac{1}{\exp\left\{\frac{\lambda^2\Sigma^2}{2(\Sigma^2+\sigma^2)}\right\}}$$

$$\Rightarrow BF_{01} = \sqrt{1+\frac{\Sigma^2}{\sigma^2}} \exp\left\{-\frac{\lambda^2\Sigma^2}{2(\Sigma^2+\sigma^2)}\right\}$$

- $$\lim_{\lambda\to\infty} BF_{01} = \lim_{\lambda\to\infty} \sqrt{1+\frac{\Sigma^2}{\sigma^2}} \exp\left\{\frac{-\lambda^2\Sigma^2}{2(\Sigma^2+\sigma^2)}\right\}$$

$$= \sqrt{1+\frac{\Sigma^2}{\sigma^2}} \lim_{\lambda\to\infty} \left(e^{\Sigma^2/[2(\Sigma^2+\sigma^2)]}\right)^{-\lambda^2}$$

$$= \sqrt{1+\frac{\Sigma^2}{\sigma^2}} \cdot 0$$

$$= 0$$

- So, if $\lambda \gg 1$, $BF_{01} \approx 0$, and there is strong evidence in favor of the alternative hypothesis $H_1: \Theta \sim N(0, \Sigma^2)$.

- If $\lambda \approx 1$, then $BF_{01} \approx \sqrt{1+\frac{\Sigma^2}{\sigma^2}} \exp\left\{\frac{-(1)^2\Sigma^2}{2(\Sigma^2+\sigma^2)}\right\} = \sqrt{1+\frac{\Sigma^2}{\sigma^2}} \exp\left\{\frac{-\Sigma^2}{2(\Sigma^2+\sigma^2)}\right\}$.
- If $\sigma/\Sigma \gg 1$, then $\sigma \gg \Sigma$, so $\exists\, c \gg 1$ s.t. $\sigma = c\Sigma$.
- Then

$$\lim_{c\to\infty} BF_{01} \approx \lim_{c\to\infty} \sqrt{1+\frac{\Sigma^2}{\sigma^2}} \exp\left\{\frac{-\Sigma^2}{2(\Sigma^2+\sigma^2)}\right\}$$

$$= \lim_{c\to\infty} \sqrt{1+\frac{\Sigma^2}{(c\Sigma)^2}} \exp\left\{\frac{-\Sigma^2}{2(\Sigma^2+(c\Sigma)^2)}\right\}$$

$$= \lim_{c\to\infty} \sqrt{1+\frac{\Sigma^2}{c^2\Sigma^2}}\, \exp\left\{\frac{-\Sigma^2}{2(\Sigma^2+c^2\Sigma^2)}\right\}$$

$$= \lim_{c\to\infty} \sqrt{1+\frac{1}{c^2}}\, \exp\left\{\frac{-\Sigma^2}{2\Sigma^2(1+c^2)}\right\}$$

$$= \lim_{c\to\infty} \sqrt{1+\frac{1}{c^2}}\, \exp\left\{\frac{-1}{2(1+c^2)}\right\}$$

$$= \sqrt{1+0} \cdot \exp\{0\}$$

$$= 1 \cdot 1$$

$$= 1$$

· So, if $\lambda \approx 1$ & $\sigma/\Sigma \gg 1$, then $BF_{01} \approx 1$, and there is no evidence in favor of either $H_0: \theta = 0$ or $H_1: \theta \sim N(0, \Sigma^2)$.