Linear Algebra **Name:** ______________________

Summer 2018

Quiz 5: Suggested Solutions

08.06.18

Time Limit: 20 minutes

This quiz contains 2 sides (including this cover page) and 4 questions.
Total of points is 100.

Grade Table (for grader use only)

Question	Points	Score
1	30	
2	20	
3	25	
4	25	
Total:	100	

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1. (30 points) Let V be an inner product space with inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$.

   (a) (15 points) Define what it means for two vectors $v, w \in V$ to be orthogonal.

   Solution: Two vectors $v, w \in V$ are **orthogonal** iff $\langle v, w \rangle = 0$.

   [10 for the statement above, 5 for completion. If some consequence of orthogonal vectors is given, then a maximum of 2 marks can be awarded]

   [If the condition $v \cdot w = 0$ is provided (ie. mistaking the definition of orthogonal vectors in V for that in $\mathbb{R}^n$), a maximum of 5 marks can be awarded]

   (b) (15 points) State the Gram-Schmidt Orthogonalization Algorithm for $\{v_1, \ldots, v_k\} \subseteq \mathbb{R}^n$. 10 bonus points if you specify it instead for a general inner product space V, i.e. $\{v_1, \ldots, v_k\} \subseteq V$.

   Solution: For nontrivial subspaces of $\mathbb{R}^n$, given a basis $\{v_1, v_2, \ldots, v_k\}$, Gram-Schmidt produces an orthogonal basis $\{x_1, x_2, \ldots, x_k\}$ inductively, by setting

$$
\begin{aligned}
x_1 &= v_1 \\
x_2 &= v_2 - \frac{x_1 \cdot v_2}{x_1 \cdot x_1} x_1 \\
x_3 &= v_3 - \frac{x_1 \cdot v_3}{x_1 \cdot x_1} x_1 - \frac{x_2 \cdot v_3}{x_2 \cdot x_2} x_2 \\
x_4 &= v_4 - \frac{x_1 \cdot v_4}{x_1 \cdot x_1} x_1 - \frac{x_2 \cdot v_4}{x_2 \cdot x_2} x_2 - \frac{x_3 \cdot v_4}{x_3 \cdot x_3} x_3
\end{aligned}
$$

$$\vdots \quad \vdots$$

$$x_k = v_k - \left(\sum_{i=1}^{k-1} \frac{x_i \cdot v_k}{x_i \cdot x_i} x_i \right)$$

For a general inner product space, Gram-Schmidt works similarly—just replacing dot products with inner products:

$$\begin{aligned}
x_1 &= v_1 \\
x_2 &= v_2 - \frac{\langle x_1, v_2 \rangle}{\langle x_1, x_1 \rangle} x_1 \\
x_3 &= v_3 - \frac{\langle x_1, v_3 \rangle}{\langle x_1, x_1 \rangle} x_1 - \frac{\langle x_2, v_3 \rangle}{\langle x_2, x_2 \rangle} x_2 \\
x_4 &= v_4 - \frac{\langle x_1, v_4 \rangle}{\langle x_1, x_1 \rangle} x_1 - \frac{\langle x_2, v_4 \rangle}{\langle x_2, x_2 \rangle} x_2 - \frac{\langle x_3, v_4 \rangle}{\langle x_3, x_3 \rangle} x_3 \\
\vdots \quad &\vdots \\
x_k &= v_k - \left(\sum_{i=1}^{k-1} \frac{\langle x_i, v_k \rangle}{\langle x_i, x_i \rangle} x_i \right)
\end{aligned}$$

In either case, we know that $\text{span}\{v_1, v_2, \dots, v_k\} = \text{span}\{x_1, x_2, \dots, x_k\}$, by a theorem introduced in class.

[5 for stating correctly $\text{Proj}_{\text{span}\{x_j\}_{j=1}^{i-1}} v_i$ *for* $i \geqslant 2$*, 8 for correctly stating the algorithm for basis* $\{v_1, \dots, v_k\}$*, 2 for having all of the above. For the extra credit, 2 for recognising to switch notations, 6 for correct restatements of algorithm, 2 for successful completion.]*

2. (20 points) Given

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

compute an **orthonormal** basis for $W = \text{span}(x_1, x_2, x_3)$.

Solution: Notice that $W = \mathbb{R}^3$ because $\{x_1, x_2, x_3\}$ is a linearly independent set of vectors in $\mathbb{R}^3$, hence it spans $\mathbb{R}^3$. The canonical/standard basis in $\mathbb{R}^3$ suffices.

Gram-Schmidt also works as well.

[5 for recognising that $W = \mathbb{R}^3$*, 5 for two correct orthonormal vectors in the new basis, 5 for the third correct vector, 5 for having all of the above]*

[If one uses G-S, then: 5 for attempting to use G-S, 5 for two correct orthonormal vectors in the new basis, 5 for the third correct vector, 5 for having all of the above]

3. (25 points) Compute the orthogonal projection of y onto $W = \operatorname{span}(u_1, u_2)$ where

$$y = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix} \quad u_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \quad u_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}.$$

Solution: First, notice that $\langle u_1, u_2 \rangle_{\mathbb{R}^3} = 0$, implying they are linearly independent. This spans a plane in $\mathbb{R}^3$. We now know that the question makes sense. To calculate the projection, and with the knowledge that $\langle \cdot, \cdot \rangle$ in the calculations below mean the inner product in $\mathbb{R}^3$, we have

$$\begin{aligned} \mathrm{Proj}_W y &= \frac{\langle y, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 + \frac{\langle y, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 \\ &= \frac{11}{14} v_1 + \frac{7}{6} v_2 \qquad (1) \\ &= \begin{bmatrix} \frac{74}{21} \\ -\frac{41}{21} \\ -\frac{16}{21} \end{bmatrix} \qquad (2) \end{aligned}$$

[5 for checking $\{u_1, u_2\}$ is orthogonal, 5 for subsequently concluding that $\{u_1, u_2\}$ is an orthogonal basis, 2 for stating/acknowledging <u>correctly</u> the projection formula, 2 for attempt at calculation, 4 for obtaining (1), 4 for obtaining (2), 3 for having all of the above]

[If one does not justify $\{u_1, u_2\}$ is an orthogonal basis before using the formula, a maximum of 10 marks is awarded]

4. (25 points) Given the inner product on $C^0([0,1], \mathbb{R})$ defined by

$$\langle f, g \rangle = \int_0^1 f(t) g(t) dt$$

determine whether or not $p(t) = 2t - 1$ and $q(t) = 10t$ are orthogonal. (Show work)

Solution: A simple calculation gives

$$\begin{aligned}\langle p, q\rangle &= \int_0^1 10t(2t-1)\,\mathrm{d}t = \int_0^1 20t^2 - 10t\,\mathrm{d}t \\ &= \left[\frac{20}{3}t^3 - 5t^2\right]\Big|_0^1 \\ &= \frac{5}{3}\end{aligned}$$

hence the functions are not orthogonal.

[2 for any attempt, 3 for correctly setting up the integral, 5 for correct integrand, 8 for correct calculations, 2 for correct evaluation of the integral, 5 for subsequent conclusion]