

Linear Algebra  
Summer 2018  
Quiz 1  
07.09.18  
Time Limit: 20 Minutes

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Name: \_\_\_\_\_

This quiz contains 2 sides (including this cover page) and 4 questions. Total of points is 100.

Grade Table (for grader use only)

Question	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

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Few things to note:

- Unless otherwise stated, the objects stated here are vectors, and  $A$  denotes a matrix of a specified dimension.
- Marks breakdown are indicated *after* the solution.

1. (25 points) State the definition of:

(a) (12.5 points) The span of the vectors  $\vec{v}_1, \dots, \vec{v}_m$  in  $\mathbb{R}^n$ .

**Solution:** The (linear) **span** of vectors  $\{v_1, v_2, \dots, v_m\} \in \mathbb{R}^n$ , denoted

$$\text{span}\{v_1, v_2, \dots, v_m\}$$

is the set of all linear combinations of the collection of vectors. This is a vector space (which will be proved later on in this course).

[5 for equivalent statement of “all linear combinations of vectors  $\{v_1, v_2, \dots, v_m\} \in \mathbb{R}^n$ ”, 5 for the equivalent statement of “set”; 2.5 for having both in the response]

(b) (12.5 points) Linear independence of  $\{\vec{v}_1, \dots, \vec{v}_m\}$  in  $\mathbb{R}^n$ .

**Solution:** A collection of vectors is said to be **linearly independent** if the equation

$$\sum_{k=1}^m a_k v_k = a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0 \quad (\star)$$

admits only the trivial solution, ie.  $a_1 = \cdots = a_m = 0$ .

[5 for writing  $(\star)$ , 5 for equivalent statement of “trivial solution”, 2.5 for having both in the response. Other equivalent responses were accepted]

2. (25 points) Determine whether or not the vector  $\vec{b}$  is a linear combination of vectors  $\vec{a}_1, \vec{a}_2, \vec{a}_3$  where

$$\vec{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}, \vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}.$$

**Solution:**

$b$  is a linear combination of vectors  $\{a_1, a_2, a_3\}$ . To see this, write the matrix in augmented form and row reduce, until we notice that the system admits a unique solution.

$$\begin{aligned} \left[ \begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{array} \right] &\rightsquigarrow \left[ \begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ & 3 & 7 & -5 \\ & & 11 & -2 \end{array} \right] \\ &\rightsquigarrow \left[ \begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ & 3 & 7 & -5 \\ & & 1 & -\frac{2}{11} \end{array} \right] \end{aligned}$$

Reading off the coefficients give us the following solutions:

$$\begin{cases} a_1 &= 11 - \frac{82}{33} - \frac{12}{11} = \frac{245}{33} \\ a_2 &= \frac{-5 + \frac{14}{11}}{3} = -\frac{41}{33} \\ a_3 &= -\frac{2}{11} \end{cases}$$

or, equivalently (up to rescaling),

$$\begin{cases} a_1 &= \frac{173}{33} \\ a_2 &= \frac{85}{3} \\ a_3 &= -\frac{20}{11} \end{cases}$$

*[5 for writing the matrix in augmented form, 5 for solving up to upper triangular form correctly, 5 for correct conclusion, 5 for correct justification, 5 for having all of the above]*

3. (25 points) Describe the solutions to the equation  $A\vec{x} = \vec{0}$  in parametric form, where the matrix  $A$  is row equivalent to

$$\begin{bmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \end{bmatrix}.$$

That is to say: express the basic variable(s) in terms of the free variable(s). (Hint: The matrix above is not yet augmented, remember to augment it with  $\vec{0}$ .)

**Solution:**

Per the hint given in the question, we attempt to solve the following augmented system (and we use the convention that the variable  $x_i$  is assigned to the  $i$ -th column):

$$\left[ \begin{array}{cccc|c} 1 & 3 & 0 & -4 & 0 \\ 2 & 6 & 0 & -8 & 0 \end{array} \right] \rightsquigarrow \left[ \begin{array}{cccc|c} 1 & 3 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

so the system has 1 pivot column (the first one), and the rest are free variables. Also, the solution set does depend on  $x_3$  by rank-nullity theorem. Hence, we have the solution set

$$S = \left\{ x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix} : x_2, x_3, x_4 \in \mathbb{R} \right\} \quad (*)$$

*[5 for writing the matrix in augmented form, 5 for correctly row reducing the matrix, 5 for correctly stating the free variables and pivot column, 5 for a correct set (as in (\*))—or an equivalent statement, 5 for having all of the above]*

4. (25 points) Find the standard matrix representation for the linear transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  which sends  $\vec{e}_1$  to itself and  $\vec{e}_2$  to  $3\vec{e}_1 + 7\vec{e}_2$ . That is,  $T(\vec{e}_1) = \vec{e}_1$  and  $T(\vec{e}_2) = 3\vec{e}_1 + 7\vec{e}_2$ .

**Solution:**

With the image of the canonical basis under  $T$  already given, we know we just need to find  $T(e_1)$  and  $T(e_2)$ , and the corresponding standard matrix of linear transformation  $M$  is given by

$$M = \begin{bmatrix} | & | \\ T(e_1) & T(e_2) \\ | & | \end{bmatrix}$$

so we calculate and see that

$$\begin{aligned} T(e_1) &= T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned} \tag{**}$$

$$\begin{aligned} T(e_2) &= T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ &= 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 7 \end{bmatrix} \end{aligned} \tag{**}$$

so we have the matrix of linear transformation

$$M = \begin{bmatrix} 1 & 3 \\ 0 & 7 \end{bmatrix} \tag{\dagger}$$

[5 for stating/acknowledging the definition of the standard matrix, 5 for  $T(e_1)$  calculation (as in (\*\*)), 5 for  $T(e_2)$  calculation (as in (\*\*)), 5 for the correct matrix  $M$  (as in (\dagger)), 5 for all of the above]

## General Comments

The following are general observations from the responses collected.

1.
  - Know what objects you are dealing with. It is important to note, in a definition, what the concept is as a mathematical object. A linear transformation, for example, is a *map* from some space to another. The span of a collection of vectors is a *set*. Knowing what objects you are dealing with will go a long way in your career in mathematics. This has caused a harsh grading at first (which prompted a regrading because we are still early on in the course).
  - Avoid passing consequences of a concept as a definition. Granted, there are many grey areas in these definitions; some are more acceptable than others. Should this arise in the future, the course instructor will have the final say-so. We do accept, however, *equivalent* definitions—hence the qualifier “equivalent statement(s)” in the markscheme.
- 2.