Linear Algebra **Name:** ____________________

Summer 2018

Quiz 1: Suggested Solutions

07.09.18

Time Limit: 20 Minutes

This quiz contains 2 sides (including this cover page) and 4 questions. Total of points is 100.

Grade Table (for grader use only)

Question	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

Few things to note:

- Unless otherwise stated, the objects stated here are vectors, and A denotes a matrix of a specified dimension.
- Marks breakdown are indicated *after* the solution.

1. (25 points) State the definition of:

   (a) (12.5 points) The span of the vectors $\vec{v}_1, \ldots, \vec{v}_m$ in $\mathbb{R}^n$.

   > **Solution:** The (linear) **span** of vectors $\{v_1, v_2, \ldots, v_m\} \in \mathbb{R}^n$, denoted
   >
   > $$\text{span}\{v_1, v_2. \ldots, v_m\}$$
   >
   > is the set of all linear combinations of the collection of vectors. This is a vector space (which will be proved later on in this course).
   >
   > *[5 for equivalent statement of "all linear combinations of vectors $\{v_1, v_2, \ldots, v_m\}$ $\in \mathbb{R}^n$", 5 for the equivalent statement of "set"; 2.5 for having both in the response]*

   (b) (12.5 points) Linear independence of $\{\vec{v}_1, \ldots, \vec{v}_m\}$ in $\mathbb{R}^n$.

Solution: A collection of vectors is said to be **linearly independent** if the equation

$$\sum_{k=1}^{m} a_k v_k = a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0 \qquad (\star)$$

admits only the trivial solution, ie. $a_1 = \cdots = a_m = 0$.
[5 for writing ($\star$), 5 for equivalent statement of "trivial solution", 2.5 for having both in the response. Other equivalent responses were accepted]

2. (25 points) Determine whether or not the vector $\vec{b}$ is a linear combination of vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ where

$$\vec{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}, \vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}.$$

Solution:
$\underline{b \text{ is a linear combination of vectors } \{a_1, a_2, a_3\}}$. To see this, write the matrix in augmented form and row reduce, until we notice that the system admits a unique solution.

$$\left[\begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ & 3 & 7 & -5 \\ & & 11 & -2 \end{array}\right]$$
$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ & 3 & 7 & -5 \\ & & 1 & -\frac{2}{11} \end{array}\right]$$

Reading off the coefficients give us the following solutions:

$$\begin{cases} a_1 &= 11 - \frac{82}{33} - \frac{12}{11} = \frac{245}{33} \\ a_2 &= \frac{-5+\frac{14}{11}}{3} = -\frac{41}{33} \\ a_3 &= -\frac{2}{11} \end{cases}$$

or, equivalently (up to rescaling),

$$\begin{cases} a_1 &= \frac{173}{33} \\ a_2 &= \frac{85}{3} \\ a_3 &= -\frac{20}{11} \end{cases}$$

[5 for writing the matrix in augmented form, 5 for solving up to upper triangular form correctly, 5 for correct conclusion, 5 for correct justification, 5 for having all of the above]

3. (25 points) Descibe the solutions to the equation $A\vec{x} = \vec{0}$ in parametric form, where the matrix A is row equivalent to

$$\begin{bmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \end{bmatrix}.$$

That is to say: express the basic variable(s) in terms of the free variable(s). (Hint: The matrix above is not yet augmented, remember to augment it with $\vec{0}$.)

Solution:

Per the hint given in the question, we attempt to solve the following augmented system (and we use the convention that the variable x_i is assigned to the i-th column):

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & -4 & 0 \\ 2 & 6 & 0 & -8 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 3 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

so the system has 1 pivot column (the first one), and the rest are free variables. Also, the solution set does depend on x_3 by rank-nullity theorem. Hence, we have the solution set

$$S = \left\{ x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \,:\, x_2, x_3, x_4 \in \mathbb{R} \right\} \tag{$*$}$$

[5 for writing the matrix in augmented form, 5 for correctly row reducing the matrix, 5 for correctly stating the free variables and pivot column, 5 for a correct set (as in ($*$))—or an equivalent statement, 5 for having all of the above]

4. (25 points) Find the standard matrix representation for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which sends $\vec{e}_1$ to itself and $\vec{e}_2$ to $3\vec{e}_1 + 7\vec{e}_2$. That is, $T(\vec{e}_1) = \vec{e}_1$ and $T(\vec{e}_2) = 3\vec{e}_1 + 7\vec{e}_2$.

Solution:

With the image of the canonical basis under T already given, we know we just need to find $T(e_1)$ and $T(e_2)$, and the corresponding standard matrix of linear transformation M is given by

$$M = \begin{bmatrix} | & | \\ T(e_1) & T(e_2) \\ | & | \end{bmatrix}$$

so we calculate and see that

$$\begin{aligned} T(e_1) &= T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} && (\star\star) \\ T(e_2) &= T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &= 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 7 \end{bmatrix} && (**) \end{aligned}$$

so we have the matrix of linear transformation

$$M = \begin{bmatrix} 1 & 3 \\ 0 & 7 \end{bmatrix} \qquad (\dagger)$$

[5 for stating/acknowledging the definition of the standard matrix, 5 for $T(e_1)$ calculation (as in $(\star\star)$), 5 for $T(e_2)$ calculation (as in $(**)$), 5 for the correct matrix M (as in $(\dagger)$), 5 for all of the above]

General Comments

The following are general observations from the responses collected.

1. - Know what objects you are dealing with. It is important to note, in a definition, what the concept is as a mathematical object. A linear transformation, for example, is a *map* from some space to another. The span of a collection of vectors is a *set*. Knowing what objects you are dealing with will go a long way in your career in mathematics. This has caused a harsh grading at first (which prompted a regrading because we are still early on in the course).
   - Avoid passing consequences of a concept as a definition. Granted, there are many grey areas in these definitions; some are more acceptable than others. Should this arise in the future, the course instructor will have the final say-so. We do accept, however, *equivalent* definitions—hence the qualifier "equivalent statement(s)" in the markscheme.

2. Don't guess and check. Some responses involved guessing and checking some combinations of $(\vec{a}_i)_{i=1}^3$, but, as the solution set tells you, the particular solution is nontrivial and *very difficult to guess.* One needs to work through the problem by thinking: *what is it that the question is asking?* We want to verify if $\vec{b}$ is a linear combination of $(\vec{a}_i)_{i=1}^3$; we know that if $A\vec{x} = \vec{b}$ admits a solution, then $\vec{b} \in C(A)$ (column space of A; ie. $\vec{b} = \sum_{k=1}^3 c_i\vec{a}_i$ for some collection of $c_i \in \mathbb{R}$, $i = 1, 2, 3$). So the question boils down to checking if $A\vec{x} = \vec{b}$ is consistent or not.

3. - Check the dimensions of the vectors in the solution set. This will be easier to understand once we know what a vector space is, and its subspaces.
   - Check row reductions carefully.

4. - Linear transformation T is a map between two vector spaces that satisfies linearity and scalar multiplicative properties—which admits a matrix representation. It is incorrect to say "T = matrix"; rather, one would say a matrix M represents the linear transformation T.
   - Know the definition of a concept. A standard matrix is one whose columns contain the image of the canonical basis under T; just knowing that will get you some marks.
   - Check your calculations carefully.