LINEAR ALGEBRA: (FAIRLY DIFFICULT) PRACTICE FINAL EXAM

These are some difficult (but doable) practice problems for the linear algebra final exam. These are presented in an increasing level of difficulty.

(1) Prove that similar matrices have the same characteristic polynomial.

(2) Let
$$W = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$$
Show that W is diagonalizable, by finding a $Q \in \mathcal{M}_2(\mathbb{R})$ such that $Q^{-1}WQ$ is a diagonal matrix. Derive a formula for W^n for some arbitrary $n \in \mathbb{N}$.

(3) Prove or disprove (ie. true or false):
   (a) Any linear transformation on a n-dimensional vector space that has less than n distinct eigenvalues is not diagonalizable.
   (b) Two distinct eigenvectors corresponding to the same eigenvalue are always linearly dependent.
   (c) If λ and η are distinct eigenvalues of a linear transformation T, then $E_\lambda \cap E_\eta = \{0\}$, where E_λ is the eigenspace corresponding to the eigenvalue λ.
   (d) Let $A \in \mathcal{M}_n(\mathbb{R})$. If A is diagonalizable, then A^{-1} is diagonalizable.
   (e) The $m \times n$ zero matrix is the only $m \times n$ matrix of rank 0.

(4) Prove or disprove: let $A \in \mathcal{M}_n(\mathbb{R})$.
   (a) If A is diagonalizable, then A^2 is diagonalizable.
   (b) If A^2 is diagonalizable, then A is diagonalizable.

(5) Let $V = \mathcal{P}_3(\mathbb{R})$. Let $T(a+bx+cx^2+dx^3) = -d+(-c+d)x+(a+b-2c)x^2+(-b+c-2d)x^3$, and $\mathcal{B} = \left\{1 - x + x^3, 1 + x^2, 1, x + x^2\right\}$.
   Show that T is a linear transformation. Then, find the matrix of linear transformation with respect to the basis $\mathcal{B}$ described above.

(6) Let $V = \mathcal{M}_2(\mathbb{R})$. Let
$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} -7a - 4b + 4c - 4d & b \\ -8a - 4b + 5c - 4d & d \end{bmatrix}$$
and
$$\mathcal{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$
   Show that T is a linear transformation. Then, find the matrix of linear transformation with respect to the basis $\mathcal{C}$ described above.

(7) A *scalar matrix* is a square matrix of the form λI for some scalar λ.

(a) Show that if a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$.
(b) Show that a diagonalizable matrix having only one eigenvalue is a scalar matrix.
(c) Show that

$$\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$$

is not diagonalizable.

(8) Let $A \in \mathcal{M}_{m \times n}$, and suppose v and w are orthogonal eigenvectors of $A^{\intercal}A$. Show that Av and Aw are orthogonal.

(9) Let V be an inner product space, and $S = \{x_i\}_{i=1}^n$ be an orthonormal subset in V. Fix $x \in V$. Then the following holds:

$$\sum_{k=1}^{n} \langle x, x_k \rangle \leqslant \|x\|^2 \tag{$\star$}$$

This is referred to as *Bessel's inequality*. For the purpose of the problem, assume the result has already been proven. As a reminder, $\|x\| = \sqrt{\langle x, x \rangle}$.

Show that Bessel's inequality ($\star$) is an equality if and only if $x \in \operatorname{span}(S)$.

Remark. Note that two consequences follow from ($\star$): the series $\sum_{k=1}^{\infty} \langle x, x_k \rangle$ converges by taking limits of both sides, and just the fact that we can find an orthonormal set in V can allow us to find the norm by an less cumbersome calculation.

(10) Let $A \in \mathcal{M}_n(\mathbb{R})$. A is *nilpotent* if $A^k = 0$ (where 0 is the zero matrix) for some $k \in \mathbb{N}$. For instance, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is nilpotent.
(a) Let λ be an eigenvalue to a nilpotent matrix A. Show that $\lambda = 0$. Hint: proceed by definition.
(b) Show that if A is both nilpotent and diagonalizable, then A is the zero matrix. Hint: use the previous part.
(c) Let B be the matrix representation of the following linear transformation:

$$\begin{array}{rcccc} S & : & \mathcal{P}_5(R) & \longrightarrow & \mathcal{P}_5(R) \\ & & p(x) & \longmapsto & \dfrac{\mathrm{d}p}{\mathrm{d}x} \end{array}$$

Without doing any calculations, explain why B must be nilpotent.