Linear Algebra **Name:** ______________________
Summer 2018
Quiz 3: Suggested Solutions
07.24.18
Time Limit: 20 Minutes

This quiz contains 2 sides (including this cover page) and 4 questions.
Total of points is 100.

Grade Table (for grader use only)

Question	Points	Score
1	30	
2	20	
3	25	
4	25	
Total:	100	

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1. (30 points) Define

   (a) (15 points) a subspace H of a real vector space V.

   Solution: $H \subset V$ is a vector subspace if the following conditions hold:

   a) $\mathbf{0}_V \in H$ (the zero vector of V is in H)

   b) $\forall \mathbf{u}, \mathbf{v} \in H$, $\mathbf{u} + \mathbf{v} \in H$ (closure under addition)

   c) $\forall \mathbf{u} \in H$ and $c \in \mathbb{R}$ (or, a scalar), $c\mathbf{u} \in H$ (closure under scalar multiplication)

   [3 for "subset", 3 for a), 3 for b), 3 for c), 3 for having all of the above. If the qualifiers are missing in each of a) through c), a maximum of 1 is given for the part]

   (b) (15 points) a linear transformation $T : V \to W$, where V and W are real vector spaces.

   Solution: A linear transformation is a map between two vector spaces, V and W, that satisfies the following properties: $\forall \mathbf{u}, \mathbf{v} \in V$ and $c \in \mathbb{R}$ (or, a scalar),

a) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (additivity)

b) $T(c\mathbf{u}) = cT(\mathbf{u})$ (homogeneity)

To each linear transformation we can associate a matrix representation.
[4 for "map" or "matrix representation", 4 for a), 4 for b), 3 for having all of the above. If the qualifiers are missing for each of a) and b), then a maximum of 2 is given for each part]

2. (20 points) Give **two** examples of $\mathbb{R}$-vector spaces, besides $\mathbb{R}^n$ for $n \in \mathbb{N}$. You do not need to prove that these are vector spaces, but be sure to describe the sets and the addition and scaling operations on them.

Solution: Some (interesting) examples of $\mathbb{R}$-vector spaces are as follows:

- The set of polynomials of degree $\leqslant n$: ie.

$$\mathcal{P}_n(\mathbb{R}) \coloneqq \Big\{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n : \{a_i\}_{i=0}^n \in \mathbb{R} \text{ and } n \in \mathbb{N}\Big\}$$

where the operations addition and scalar multiplication are defined as, for all $p, q \in \mathcal{P}_n(\mathbb{R})$ and $\lambda \in \mathbb{R}$,

$$(p+q)(x) = p(x) + q(x)$$
$$(\lambda p)(x) = \lambda p(x)$$

- The set of all real-valued functions: ie.

$$V \coloneqq \{f : f : \mathbb{R} \mapsto \mathbb{R}\}$$

under the same addition and scalar multiplication definitions as above.

- The set of real-valued functions that are solutions to a differential equation: ie.

$$W \coloneqq \left\{f : \mathbb{R} \mapsto \mathbb{R} : \frac{\partial^2 f}{\partial x^2} + f = 0\right\}$$

under the same addition and scalar multiplication definitions as above. Because differentiation is a linear map, ie.

$$\frac{\partial^2 (f+g)}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 g}{\partial x^2}$$

linearity holds. Homogeneity can be argued in a similar way.

- The set of real-valued continuous functions: ie.

$$\mathcal{W} := \{f \in \mathcal{C} : \ f : \mathbb{R} \mapsto \mathbb{R}\}$$

N.B. As Alekos pointed out, if you provided examples of subspaces of $\mathbb{R}^n$, you will receive points, even though this is *not* what the question is asking.

[4 for correct first example, 6 for correct descriptions of the first set and the operations associated, 4 for correct second example, 6 for correct descriptions of the second set and the operations associated, 5 for having all of the above]

3. (25 points) Decide whether or not

$$\mathcal{B} := \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\mathbb{R}^3$. Show all of your work.

Solution: Recall a basis is, by definition, a set of vectors that are linearly independent and spanning (the ambient space). From a few results from earlier, we know that asking if vectors in $\mathcal{B}$ are linearly independent is equivalent to checking if $A\vec{x} = \vec{0}$ admits only the trivial solution, where

$$A = \begin{bmatrix} 1 & 3 & -3 \\ 0 & 2 & -5 \\ -2 & -4 & 1 \end{bmatrix}$$

or, equivalently, if $\mathrm{Null}(A) = \{\vec{0}\}$. So we need to see if the nullspace is trivial or not.

A simple row reduction exercise reveals

$$\begin{bmatrix} 1 & 3 & -3 \\ 0 & 2 & -5 \\ -2 & -4 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & -3 \\ 0 & 2 & -5 \\ 0 & 2 & -5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & -3 \\ 0 & 2 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

And, as the number of column pivots in RREF of A is the dimension of the column space, we have, by rank-nullity,

$$3 = 2 + \dim(\mathrm{Null}(A)) \implies \dim(\mathrm{Null}(A)) = 1$$

which is obviously not trivial. In fact, one can explicitly solve for the nullspace, and get

$$\mathrm{Null}(A) = \left\{ z \begin{bmatrix} -\frac{9}{2} \\ \frac{5}{2} \\ 1 \end{bmatrix} : z \in \mathbb{R} \right\}$$

An alternative is to calculate the determinant and see that it is zero, hence the matrix A (as defined above) is not invertible. As such, the columns are not linearly independent, hence the nullspace has dimension greater than 0.

[5 for stating (in some way) the definition of a basis, 5 for an attempt at row reduction or determinant calculation or equivalent method, 5 for correct calculations throughout, 5 for correct explanation(s), 5 for having all of the above]

4. (25 points) For which values of $a, b \in \mathbb{R}$ is the map $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = ax + b$ a linear transformation.

Solution: In order for T to be a linear transformation, it must satisfy both additivity and homogeneity, ie.

- $T(x + y) = T(x) + T(y)$ for all $x, y \in \mathbb{R}$
- $T(\lambda x) = \lambda T(x)$ for all $x, \lambda \in \mathbb{R}$ (where λ is some fixed constant)

Note that $T(x + y) = a(x + y) + b$ and $T(\lambda x) = a\lambda x + b$, whereas $T(x) + T(y) = a(x+y)+2b$ and $\lambda T(x) = a\lambda x+\lambda b$. Combining the expressions give rise to $2b = b$ and $\lambda b = b$. Since λ can be any fixed constant, we conclude that the only possible solution is $b = 0$.

Hence, for all $a \in \mathbb{R}$ and $b = 0$, $T(x)$ is a linear transformation.

The map, as defined in the question, is called an *affine map.*

[5 for attempting to use the definition of linear transformation, 5 for correctly stating the necessary conditions for T to be a linear transformation, 5 for correct calculations, 1 for an attempt of any kind at stating a and b, 4 for correct values of a and b, 5 for having all of the above]

General Comments

The following are general observations from the responses collected.

1.
- Note that the zero vector of V is in H! It's not any zero vector.
- Remember the qualifiers when you're stating the conditions. If they are not present, the objects written down are just symbols.
- Linear transformation is a map/function; answers such as "rule" or "assignment" (or some equivalent descriptions) are accepted this time.

2.
- Understand what is being asked. The question asks for examples of vector spaces defined over the field $\mathbb{R}$. By definition, it asks for spaces whose elements are in $\mathbb{R}$, with addition and scalar multiplication being closed in $\mathbb{R}$. Sets of functions, polynomials, real sequences, and so on, are all examples of vector spaces once the correct operations have been specified.
- Think if an example is actually a vector space. A few wrote the upper plane in $\mathbb{R}^2$, defined

$$\mathcal{U} := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \geqslant 0 \right\} \subset \mathbb{R}^2$$

  but it is obvious that $-c \notin \mathcal{U}$, where $c \in \mathcal{U}$.
- Any vector space over infinite fields (like $\mathbb{R}$, $\mathbb{Q}$, or $\mathbb{C}$) is either the space $\{\vec{0}\}$, or has infinitely many elements. Think about this in the following way: for any nonzero vector x and nonzero field (in this case, $\mathbb{R}$) elements k_1, k_2, $k_1 x = k_2 x \iff k_1 = k_2$. As such, there are infinitely many elements of the form kx, as long as the vector space is nonzero (and nonempty) over an infinite field[1].
- **My mistake**: $\mathbb{C}$ is a vector space over $\mathbb{R}$, though it was not obvious (at least, to me) at first glance why this should be the case. It was not clear to me why the field $\mathbb{C}$ defines a vector space over another field $\mathbb{R}$. After some thinking, it is not hard to see that $\mathbb{C}$ is a two-dimensional vector space over $\mathbb{R}$, under the "usual" operations:

$$\begin{aligned} + \ : \ \mathbb{C} \times \mathbb{C} &\longrightarrow \mathbb{C} \\ (x, y) &\longmapsto x + y := (a + c, b + d) \\ \cdot \ : \ \mathbb{R} \times \mathbb{C} &\longrightarrow \mathbb{C} \\ (\alpha, x) &\longmapsto \alpha x := (\alpha a, \alpha b) \end{aligned}$$

  and defining these correctly is important. Furthermore, because we know that $\mathbb{C} \cong \mathbb{R}^2$, it is natural (sometimes, anyways) to think of them as the "same" sets, metric spaces, and groups under addition.

[1] There are finite fields; if you choose to learn more algebra, you will come across examples like $\mathbb{F}_p$—field of integers modulo p for some p.

3. Justify your reasoning fully. For those who invoked IMT in some way, I was lenient. But, a large part about learning mathematics is about learning how to argue an idea fully.

4. $T(0) = 0 \not\Rightarrow T$ is a linear transformation (not necessarily)! Take

$$T(x, y) = \left(\frac{x}{2}, \frac{xy}{4}\right), \qquad \forall (x, y) \in \mathbb{R}^2$$

then take $(1, 0)$ and $(1, 1)$. Then,

$$T((2,1)) = \left(1, \frac{1}{2}\right)$$
$$T((1,0)) + T((1,1)) = \left(\frac{1}{2}, 0\right) + \left(\frac{1}{2}, \frac{1}{4}\right) = \left(1, \frac{1}{4}\right)$$

but obviously $T((0,0)) = 0$.