QUIZ 1: SUGGESTED SOLUTIONS

Unless otherwise stated, the objects stated here are vectors, and A denotes a matrix of a specified dimension.

(1) **(a)** The (linear) **span** of vectors $\{v_1, v_2, \dots, v_m\} \in \mathbb{R}^n$, denoted span $\{v_1, v_2. \dots, v_m\}$, is the set of all linear combinations of the collection of vectors. This is a vector space (which will be proved later on in this course).

(b) A collection of vectors is said to be **linearly independent** if the equation

$$\sum_{k=1}^{m} a_k v_k = a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0$$

admits only the trivial solution, ie. $a_1 = \cdots = a_m = 0$.

(2) $\underline{b \text{ is a linear combination of vectors } \{a_1, a_2, a_3\}}$. To see this, write the matrix in augmented form and row reduce, until we notice that the system admits a unique solution.

$$\left[\begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ & 3 & 7 & -5 \\ & & 11 & -2 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ & 3 & 7 & -5 \\ & & 1 & -\frac{2}{11} \end{array}\right]$$

Reading off the coefficients give us the following solutions:

$$\begin{cases} a_1 & = 11 - \frac{82}{33} - \frac{12}{11} = \frac{245}{33} \\ a_2 & = \frac{-5+\frac{14}{11}}{3} = -\frac{41}{33} \\ a_3 & = -\frac{2}{11} \end{cases}$$

or, equivalently (up to rescaling),

$$\begin{cases} a_1 & = \frac{413}{33} \\ a_2 & = \frac{85}{3} \\ a_3 & = -\frac{20}{11} \end{cases}$$

(3) Per the hint given in the question, we attempt to solve the following augmented system (and we use the convention that the variable x_i is assigned to the i-th column):

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & -4 & 0 \\ 2 & 6 & 0 & -8 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 3 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

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so the system has 1 pivot column (the first one), and the rest are free variables. Also, the solution set does not depend on x_3. Hence, we have the solution set

$$S = \left\{ x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \;:\; x_2, x_4 \in \mathbb{R} \right\}$$

(4) With the image of the canonical basis under T already given, we know we just need to find $T(e_1)$ and $T(e_2)$, and the corresponding standard matrix of linear transformation M is given by

$$M = \begin{bmatrix} | & | \\ T(e_1) & T(e_2) \\ | & | \end{bmatrix}$$

so we calculate and see that

$$\begin{aligned} T(e_1) &= T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ T(e_2) &= T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &= 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 7 \end{bmatrix} \end{aligned}$$

so we have the matrix of linear transformation

$$M = \begin{bmatrix} 1 & 3 \\ 0 & 7 \end{bmatrix}$$