Linear Algebra **Name:** ______________________

Summer 2018

Quiz 4: Suggested Solutions

07.30.18

Time Limit: 20 Minutes

This quiz contains 2 sides (including this cover page) and 4 questions.
Total of points is 100.

Grade Table (for grader use only)

Question	Points	Score
1	30	
2	20	
3	25	
4	25	
Total:	100	

---

1. (30 points) Let A denote an $n \times n$ matrix.

   (a) (15 points) Define what it means for $v \in \mathbb{R}^n$ to be an eigenvector for A with eigenvalue $\lambda \in \mathbb{R}$.

   > **Solution:** An *eigenvector* of $A \in \mathcal{M}_{n\times n}$ is a nonzero vector x such that $Ax = \lambda x$ for some scalar $\lambda \in \mathbb{R}$. To be clear, a scalar λ is called an *eigenvalue* of A if there is a nontrivial solution x to $Ax = \lambda x$; such a x is called an *eigenvector* corresponding to λ.
   >
   > *[5 for any equivalent statement of $Ax = \lambda x$, 5 for stating "nontrivial" or "nonzero" vector x, 3 for defining eigenvector as a vector corresponding to some λ (counting multiplicities), 2 for having all of the above]*

   (b) (15 points) Define what it means for another $n \times n$ matrix B to be similar to A.

   > **Solution:** Let $A, B \in \mathcal{M}_{n\times n}$. B is *similar* to A if there exists an *invertible matrix* Q such that $B = QAQ^{-1}$ (or, up to necessary changes, $A = Q^{-1}BQ$). This is otherwise known as the *conjugate* of B to A.
   >
   > *[6 for correctly stating either $B = QAQ^{-1}$ or the alternate form, 6 for "there exists an invertible matrix", 3 for having all of the above]*

2. (20 points) Given that

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$A = P \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{:=U} P^{-1}$$

compute A^{10}.

Solution: Notice that P is invertible because $\det(P) = 21 - 20 = 1 \neq 0$, so the similar (conjugate) form above makes sense. Notice, also, that

$$\begin{aligned} A^{10} &= \underbrace{(PUP^{-1})(PUP^{-1})\cdots(PUP^{-1})}_{\text{10 times}} \\ &= \underbrace{PU(P^{-1}P)U(P^{-1}P)\cdots(P^{-1}P)UP^{-1})}_{\text{10 times}} && \text{associativity} \\ &= PU^{10}P^{-1} \end{aligned}$$

Then, calculations yield that

$$P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$U^{10} = \begin{bmatrix} 1 & 10 \\ & 1 \end{bmatrix}$$

and combining these give

$$A^{10} = \begin{bmatrix} -19 & 40 \\ -10 & 21 \end{bmatrix}$$

[2 for attempt of any kind, 3 for $A^{10} = PU^{10}P^{-1}$, 3 for correct P^{-1}, 3 for correct U^{10}, 3 for attempt at calculating A^{10} by brute force or any method, 3 for correct A^{10}, 3 for having all of the above]

3. (25 points) Compute the characteristic polynomial of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

and list the eigenvalues.

Solution: The characteristic polynomial is given by $\det(A - \lambda I) = 0$, where I is the 3×3 identity matrix. Then, we have

$$\begin{aligned}\det(A - \lambda I) &= (1-\lambda)^3 - (1-\lambda) \\ &= (1-\lambda)((1-\lambda)^2 - 1) \\ &= (1-\lambda)(-\lambda)(2-\lambda) = 0\end{aligned}$$

so the eigenvalues are $1, 0, 2$.

[5 for attempt of any kind, 5 for correct setup of $\det(A - \lambda I) = 0$*, 5 for correctly calculating the characteristic polynomial, 5 for correctly stating* <u>*all*</u> *eigenvalues, 5 for having all of the above]*

4. (25 points) Briefly explain why the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

is similar to a diagonal matrix.

Solution: By a theorem introduced in class, all triangular matrices have eigenvalues on its diagonal entries. As such, A is diagonalisable: for each eigenvalue (in this case, $1, 4, 6$), one can find a corresponding nonzero vector (eigenvector) such that $Ax = \lambda x$. By principles of diagonalisation, since there are n many distinct eigenvalues and n corresponding distinct eigenvectors, we can construct P (with eigenvectors as its columns) and D (with eigenvalues on its diagonal) such that $A = PDP^{-1}$, hence A is conjugate of D by definition.

[5 for stating all triangular matrices have eigenvalues (possibly including the eigenvalues in this case), 5 for attempting to state what diagonalisation is, 4 for description of P*, 4 for description of* D*, 4 for justification using the definition of conjugates, 3 for having all of the above]*

General Comments

The following are general observations from the responses collected.