covered

hey!

1. Basics

1.1 Graphs

- edge, vertex, $V(G)$, $E(G)$
- The number of vertices of a graph G is its order, written as $|G|$; its number of edges is $||G||$.
  trivial graph, incident ends, adjacent/neighbour, complete graph K^n
- isomorphism, automorphism, graph property, graph invariant
- disjoint, subgraph/supergraph (iff both vertex set and edge set are subsets/supersets of its counterpart), proper subgraph/supergraph, induced subgraph (是subgraph且所有该subgraph的vertices涉及的edge都在这个subgraph里)
- edge-maximal (given graph property if G has it but no graph $G + xy$ does, for non-adjacent vertices $x, y \in G$

- line graph

1.2 The degree of a vertex

- set of neighbours of a vertex v, $N(v)$, $N(V)$
- degree $d_G(v) = d(v)$ of a vertex v is the number $|E(v)|$ of edges at v, minimum degree of G $\theta(G) = \min\{d(v)|v \in V\}$, maximum degree $\Delta(G) = \max\{d(v)|v \in V\}$
- k-regular (all vertices the same degree), 3-regular graph called cubic
- average degree of G, $d(G) = \frac{1}{|V|} \sum_{v \in V} d(v)$
- ratio of number of edges per vertex $\epsilon(G) = \frac{|E|}{|V|} = \frac{1}{2} d(G)$

1.3 Paths and cycles

- path (non-empty graph), length P^k
- A-B path, independent path (none of them contains an inner vertex (not ends) of another), H-path (given a graph H, call P an H-path if P is non-trivial and meets H exactly in its ends.
- cycle, length C^k
- girth $g(G)$ (the minimum length of a cyclce in a graph G), circumference (max cycle)
- distance $d_G(x, y)$ is the length of a shortest $x - y$ path in G.
- greatest distance between any two vertices of G is the diameter of G, as $diamG$
- central (if its greatest distance from any other vertex is as small as possible), distance is called radius as $radG$
- walk

1.4 Connectivity

- connected, disconnected, component (a maximal connected subgraph of G), separator, cutvertex, bridge
- k-connected (no two vertices of G are separated by fewer than k other vertices), connectivity $\kappa(G)$, l-edge-connected, edge-connectivity, $\lambda(G)$

1.5 Trees and forests

- tree, leaf, forest, normal tree (ends of every T-path in G are comparable in the tree-order of T)

1.6 Bipartite graphs

- r-partite, bipartite, complete r-partite K_s^r

1.7 Contraction and minors

- subdivision, "TX (original vertices are called branch vertices of the TX, new vertices are called subdividing vertices of the TX", if Y contains a TX as a subgraph, then X is a topological minor of Y.
- TX is about 边上加点, IX is about inflate 点变成“岛”
- Y contains an IX as a subgraph, then X is a minor of Y, $X \preceq Y$

1.8 Euler tours

- a connected graph is Eulerian iff every vertex has even degree

1.10 Other notions of graphs

- directed graph, oriented graph, multigraph

2. Matching, Covering and Packing

2.1 Matching in bipartite graphs

- A set M of independent edges in a graph $G = (V, E)$ is called a matching. M is a matching of $U \subseteq V$ if every vertex in U is incident with an edge in M. The vertices in U are then called matched by M; vertices ... unmatched.
- A k-regular spanning subgraph is called a k-factor. Thus a subgraph $H \subseteq G$ is a 1-factor of G iff $E(H)$ is a matching of V.
- Assume G is a bipartite graph with bipartition {A, B}. A path in G which starts in A at an unmatched vertex and then contains, alternately, edges from $E \backslash M$ and from M, is an alternating path w.r.t. M.
- An alternating path P that ends in an unmatched vertex of B is called an augmenting path, can use it to turn M into a larger matching, the set of matched vertices is increased by two, the ends of P
- **Theorem 2.1.1 (König's Theorem 1931): The maximum cardinality of a matching in G is equal to the minimum cardinality of a vertex cover of its edges.**
  Proof: Let M be a matching in G of maximum cardinality. From every edge in M let us choose one of its ends: its end in B if some alternating path ends in that vertex, and its end in A otherwise (shown below). We shall prove that the set U of these $|M|$ vertices covers E; since any vertex cover of E must cover M, there can be none with fewer than $|M|$ vertices, and so the theorem will follow.

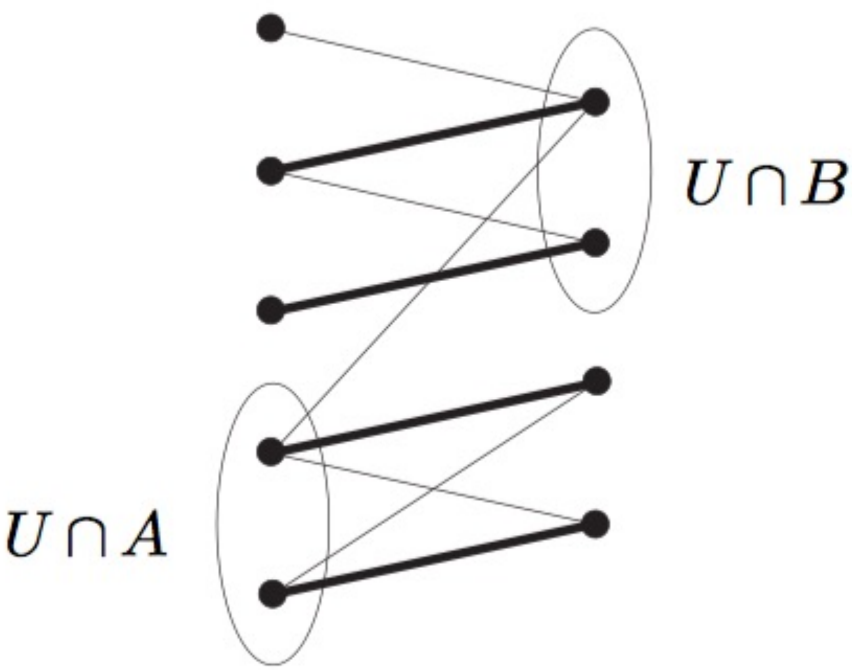

Fig. 2.1.2. The vertex cover U

Let $ab \in E$ be an edge; we show that either a or b lies in U. If $ab \in M$, this holds by definition of U, so we assume that $ab \notin M$. Since M is a maximal matching, it contains an edge $a'b'$ with $a = a'$ or $b = b'$. In fact, we may assume that $a = a'$: for if a is unmatched (and $b = b'$), then ab is an alternating path, and so the end of $a'b' \in M$ chosen for U was the vertex $b' = b$. Now if $a' = a$ is not in U, then

$b' \in U$, and some alternating path P ends in b'. But then there is also an alternating path P' ending in b: either $P' := Pb$ (if $b \in P$) or $P' := Pb'a'b$. By the maximality of M, however, P' is not an augmenting path. So b must be matched, and was chosen for U from the edge of M containing it. ■

- **Theorem 2.1.2 (Hall's Marriage Theorem 1935): G contains a matching of A iff $|N(S)| \geq |S|$ for all $S \subseteq A$.**
  First Proof: We show that for every matching M of G that leaves a vertex $a \in A$ unmatched there is an augmenting path with respect to M.
  Let A' be the set of vertices in A that can be reached from a by a non-trivial alternating path, and $B' \subseteq B$ the set of all penultimate vertices of such paths. The last edges of these paths lie in M, so $|A'| = |B'|$. Hence by the marriage condition, there is an edge from a vertex v in $S = A' \cup \{a\}$ to a vertex b in $B \backslash B'$. As $v \in A' \cup \{a\}$, there is an alternating path P from a to v. Then either Pvb or Pb (if $b \in P$) is an alternating path from a to b; call this path P'. If b was matched, by $a'b \in M$ say, then $P'ba'$ would be an alternating path putting a' in A' and b in B'. But $b \notin B'$, so b is unmatched, and P' is the desired augmenting path. ■
- Corollary 2.1.3: If G is k-regular with $k \geq 1$, then G has a 1-factor.
- unstable, preferences, stable matching. call a family $(\leq_v)_{v \in V}$ of linear orderings $\leq_v$ on $E(v)$ a set of preferences for G
- **Theorem 2.1.4 (Gale & Shapley's Stable Marriage Theorem 1962): For every set of preferences, G has a stable matching.**
  Proof: Call a matching M in G *better* than a matching $M' \neq M$ if M makes the vertices in B happier than M' does, that is, if every vertex b in an edge $f' \in M'$ is incident also with some $f \in M$ such that $f' \leq_b f$. We shall construct a sequence of better and better matchings. Since these can increase the happiness of a fixed vertex b at most $d(b)$ times, this process will terminate.
  Given a matching M, call a vertex $a \in A$ *acceptable* to $b \in B$ if $e = ab \in E \backslash M$ and any edge $f \in M$ at b satisfies $f <_b e$. Call $a \in A$ *happy with* M if a is unmatched or its matching edge $f \in M$ satisfies $f >_a e$ for all edges $e = ab$ such that a is acceptable to b.
  Starting with the empty matching, let us construct a sequence of matchings that each keep all the vertices in A happy. Given such a matching M, consider a vertex $a \in A$ that is unmatched but acceptable to some $b \in B$. (If no such a exists, terminate the sequence.) Add to M the $\leq_a$-maximal edge ab such that a is acceptable to b, and discard from M any other edge at b.
  Clearly, each matching in our sequence is better than the previous, and it is easy to check inductively that they all keep the vertices in A happy. So the sequence continues until it terminates with a matching M such that every unmatched vertex in A is inacceptable to all its neighbours in B. As every matched vertex in A is happy with M, this matching is stable. ■
- Corollary 2.1.5 (Petersen 1891): Every regular graph of positive even degree has a 2-factor.

2.2 Matching in general graphs

- Given a graph G, let C_G be the set of its components, by $q(G)$ the number of its odd components, those of odd order
- **Theorem 2.2.1 (Tutte's necessary and sufficient condition for an arbitrary graph to have 1-factor 1947): A graph G has a 1-factor iff $q(G-S) \leq |S|$ for all $S \subseteq V(G)$.**
  Proof: Let $G = (V, E)$ be a graph without a 1-factor. Our task is to find *a bad set* $S \subseteq V$, one that violates Tutte's condition.
  We may assume that G is edge-maximal without a 1-factor. Indeed, if G' is obtained from G by adding edges and $S \subseteq V$ is abd for G', then S is also bad for G: any odd component of $G' - S$ is the union of components of $G - S$, and one of these must again be odd.
  What does G look like? Clearly, if G contains a bad set S then, by its edge-maximality and the trivial forward implication of the theorem,

all the components of $G - S$ are complete and every vertex $s \in S$ is adjacent to all the vertices of $G - s$. $(*)$
But also conversely, if a set $S \subseteq V$ satisfies $(*)$ then either S or the empty set must be bad: if S is not bad we can join the odd components of $G - S$ disjointly to S and pair up all the remaining vertices--unless $|G|$ is odd, in which case $\emptyset$ is bad.
So it suffices to prove that G has a set S of vertices satisfying $(*)$. Let S be the set of vertices that are adjacent to every other vertex. If this set S does not satisfy $(*)$, then some component of $G - S$ has non-adjacent vertices a, a'. Let a, b, c be the first three vertices on a shortest $a - a'$ path in this component; then $ab, bc \in E$ but $ac \notin E$. Since $b \notin S$, there is a vertex $d \in V$ such that $bd \notin E$. By the maximality of G, there is a matching M_1 of V in $G + ac$, and a matching M_2 of V in $G + bd$.

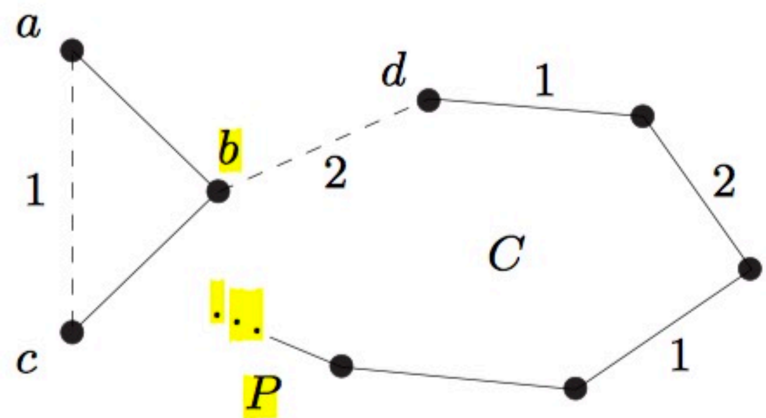

Fig. 2.2.2. Deriving a contradiction if S does not satisfy $(*)$

Let $P = d \ldots v$ be a maximal path in G starting at d with an edge from M_1 and containing alternately edges from M_1 and M_2 (above). If the last edge of P lies in M_1, then $v = b$, since otherwise we could continue P. Let us then set $C := P + bd$. If the last edge of P lies in M_2, then by the maximality of P the M_1-edge at v must be ac, so $v \in \{a, c\}$; then let C be the cycle $dPvbd$. In each case, C is an even cycle with every other edge in M_2, and whose only edge not in E is bd. Replacing in M_2 its edges on C with the edges of $C - M_2$, we obtain a matching of V continued in E, a contradiction. ■

- **Corollary 2.2.2 (Petersen's Theorem on 1-factor in bridgeless cubic graphs): Every bridgeless cubic graph has a 1-factor.**
  Proof: We show that any bridgeless cubic graph G satisfies Tutte's condition. Let $S \subseteq V(G)$ be given, and consider an odd component C of $G - S$. Since G is cubic, the degrees (in G) of the vertices in C sum to an odd number, but only an even part of this sum arises from edges of C. So G has an odd number of $S - C$ edges, and therefore has at least 3 such edges (since G has no bridge). The total number of edges between S and $G - S$ thus is at least $3q(G - S)$. But it is also at most $3|S|$, because G is cubic. Hence $q(G - S) \leq |S|$, as required. ■

3. Connectivity

3.1 2-Connected graphs and subgraphs

- Proposition 3.1.1: A graph is 2-connected iff it can be constructed from a cycle by successively adding H-paths to graphs H already constructed (Fig. 3.1.1).

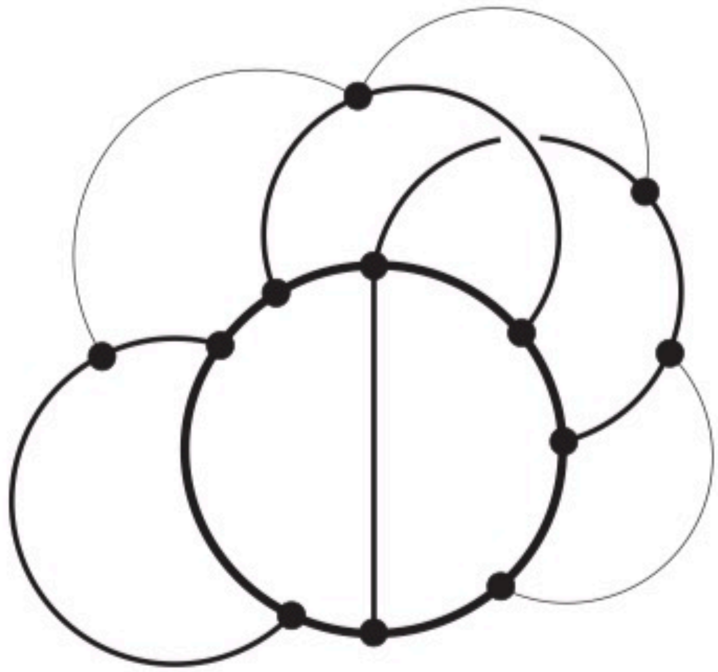

Fig. 3.1.1. The construction of 2-connected graphs

Proof: Clearly, every graph constructed as described is 2-connected. Conversely, let a 2-connected graph G be given. Then G contains a cycle, and hence has a maximal subgraph H constructible as above. Since any edge $xy \in E(G)\backslash E(H)$ with $x, y \in H$ would define an H-path, H is an induced subgraph of G. Thus if $H \neq G$, then by the connectedness of G there is an edge vw with $v \in G - H$ and $w \in H$. As G is 2-connected, $G - w$ contains a $v - H$ path P. Then wvP is an H-path in G, and $H \cup wvP$ is a constructible subgraph of G larger than H. This contradicts the maximality of H. ■

- a block is a maximal connected subgraph without a cutvertex. every block is either a maximal 2-connected subgraph, or a bridge (with its ends), or an isolated vertex.
- Lemma 3.1.2: Let G be any graph.
  - (i) The cycles of G are precisely the cycles of its blocks.
  - (ii) The bonds of G are precisely the minimal cuts of its blocks.
- Lemma 3.1.3: The following statements are equivalent for distinct edges e,f of a graph G:
  - (i) The edges e,f belong to a common block of G.
  - (ii) The edges e,f belong to a common cycle in G.
  - (iii) The edges e,f belong to a common bond of G.
- Lemma 3.1.4: The block graph of a connected graph is a tree.

3.2 The structure of 3-connected graphs

- given edge e in G, write $G\dot{-}e$ for the multigraph obtained from $G - e$ by suppressing any end of e that has degree 2 in $G - e$.
- Lemma 3.2.1: Let e be an edge in a graph G. If $G\dot{-}e$ is 3-connected, then so is G.
- Lemma 3.2.2: Every 3-connected graph $G \neq K^4$ has an edge e such that $G\dot{-}e$ is another 3-connected graph.
- **Theorem 3.2.3 (Tutte's characterization of 3-connected graphs part 1, 1966): A graph G is 3-connected iff there exists a sequence $G_0, \ldots, G_n$ of graphs such that**
  - **(i) $G_0 = K^4$ and $G_n = G$;**

- **(ii) G_{i+1} has an edge e such that $G_i = G_{i+1} \dot{-} e$, for every $i < n$. Moreover, the graphs in any such sequence are all 3-connected.**
  Proof: If G is 3-connected, use Lemma 3.2.2 to find $G_n, \ldots, G_0$ in turn. Conversely, if $G_0, \ldots, G_n$ is any sequence of graphs satisfying (i) and (ii), then all these graphs, and in particular $G = G_n$, are 3-connected by Lemma 3.2.1. ■

- Lemma 3.2.4: Every 3-connected graph $G \neq K^4$ has an edge e such that $G \backslash e$ is again 3-connected.
- **Theorem 3.2.5 (Tutte's characterization of 3-connected graphs part 2, 1961): A graph G is 3-connected iff there exists a sequence $G_0, \ldots, G_n$ of graphs with the following two properties:**
  - **(i) $G_0 = K^4$ and $G_n = G$;**
  - **(ii) G_{i+1} has an edge xy such that $d(x), d(y) \geq 3$ and $G_i = G_{i+1}/xy$ for every $i < n$. Moreover, the graphs in any such sequence are all 3-connected.**
    Proof: If G is 3-connected, then by Lemma 3.2.4 there is a sequence $G_n, \ldots, G_0$ of 3-connected graphs satisfying (i) and (ii).
    Conversely, and to show the final statement of the theorem, let $G_0, \ldots, G_n$ be a sequence of graphs satisfying (i) and (ii); we show that if G_i is 3-connected then so is G_{i+1}, for every $i < n$. Suppose not, let S be a separator of at most 2 vertices in G_{i+1}, and let C_1, C_2 be two components of $G_{i+1} - S$. As x and y are adjacent, we may assume that $\{x, y\} \cap V(C_1) = \emptyset$ (Fig. 3.2.2). Then C_2 contains neither both vertices x, y nor a vertex $v \notin \{x, y\}$: otherwise v_{xy} or v would be separated from C_1 in G_i by at most two vertices, a contradiction. But now C_2 contains only one vertex: either x or y. This contradicts our assumption of $d(x), d(y) \geq 3$. ■

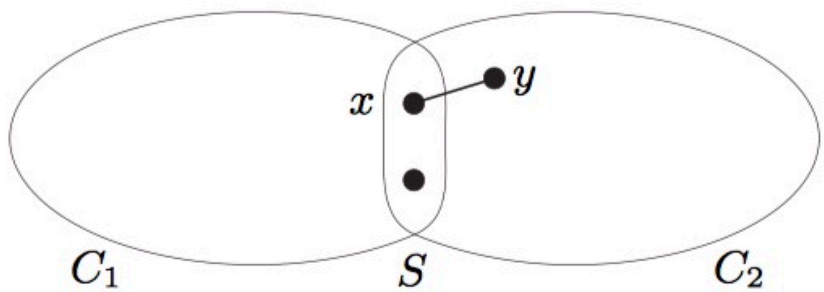

Fig. 3.2.2. The position of $xy \in G_{i+1}$ in the proof of Theorem 3.2.5

3.3 Menger's theorem

- **Theorem 3.3.1 (Menger's theorem 1927)**: Let $G = (V, E)$ be a graph and $A, B \subseteq V$. Then the minimum number of vertices separating A from B in G is equal to the maximum number of disjoint A-B paths in G.
  Fisrt proof: We apply induction on $\|G\|$. If G has no edge, then $|A \cap B| = k$ and we have k trivial A-B paths. So we assume that G has an edge $e = xy$. If G has no k disjoint A-B paths, then neither does G/e; here we count the contracted vertex v_e as an element of A (resp. B) in G/e if in G at least one of x, y lies in A (resp. B). By the induction hypothesis, G/e contains an A-B separator Y of fewer than k vertices. Among these must be the vertex v_e, since otherwise $Y \subseteq V$ would be an A-B separator in G. Then $X := (Y \backslash \{v_e\}) \cup \{x, y\}$ is an A-B separator in G of exactly k vertices.
  We now consider the graph $G - e$. Since $x, y \in X$, every $A - X$ separator $G - e$ is also an A-B separator in G and hence contains at least k vertices. So by induction there are k disjoint A-X paths in G-e, and similarly there are k disjoint X-B paths in G-e. As X separates A from B, these two path systems do not meet outside X, and can thus be combined to k disjoint A-B paths. ■
- Corollary 3.3.4: For $B \subseteq V$ and $a \in V \backslash B$, the minimum number of vertices $\neq a$ separating a from B in G is equal to the maximum number of paths forming an a-B fan in G.

- Corollary 3.3.5: Let a and b be two distinct vertices of G.
  - (i) If $ab \notin E$, then the minimum number of vertices nea, b separating a from b in G is equal to the maximum number of independent a-b paths in G.
  - (ii) The minimum number of edges separating a from b in G is equal to the maximum number of edge-disjoint a-b paths in G.
- Theorem 3.3.6 (Global Version of Menger's Theorem):
  - (i) A graph is k-connected iff it contains k independent paths between any two vertices.
  - (ii) A graph is k-edge-connected iff it contains k edge-disjoint paths between any two vertices.

4. Planar Graphs

4.1 Topological prerequisites

- Jordan Curve Theorem for Polygons

4.2 Plane graphs

- a plane graph is a pair (V, E) of finite sets with the following properties:
  - $V \subseteq \Re^2$
  - every edge is an arc between two vertices
  - different edges have different sets of endpoints
  - the interior of an edge contains no vertex and no point of any other edge.
- faces, $F(G)$
- Euler's Formula: $n - m + l = 2$, where n vertices, m edges, l faces.

4.4 Planer graphs: Kuratowski's theorem

- planar (a graph can be embedded in the plane, i.e. if it is isomorphic to a plane graph), maximally planar (planar and cannot be extended to a larger planar graph by adding an edge but not a vertex)
- Theorem 4.4.6 (Kuratowski's theorem 1930; Wagner 1937): The following assertions are equivalent for graphs G:
  - (i) G is planar;
  - (ii) G contains neither K^5 nor $K_{3,3}$ as a minor;
  - (iii) G contains neither K^5 nor $K_{3,3}$ as a topological minor.

4.6 Plane duality

- plane multigraph is a pair $G = (V, E)$ of finite sets of vertices and edges, satisfying the following conditions:
  - (i) $V \subseteq \Re^2$
  - (ii) every edge is either an arc between two vertices or a polygon containing exactly one vertex (endpoint!)
  - (iii) apart from its own endpoint(s), an edge contains no vertex and no point of any other edge.
- plane dual

5. Colouring

5.1 Colouring maps and planar graphs

- vertex colouring (a map $c : V \to S$ s.t. $c(v) \neq c(w)$ whenever v and w are adjacent, the elements of S called available colours)
- smallest k size of S, k-colouring, chromatic number of G, $\chi(G)$. A graph G with $\chi(G) = k$ is called k-chromatic; if $\chi(G) \leq k$, we call k-colourable.
- edge colouring, chromatic index $\chi'(G)$
- Theorem 5.1.1 (Four Colour Theorem): Every planar graph is 4-colourable.
- Proposition 5.1.2 (Five Colour Theorem): Every planar graph is 5-colourable.

5.2 Colouring vertices

- Proposition 5.2.1: Every graph G with m edges satisfies $\chi(G) \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}$

  Proof: Let c be a vertex colouring of G with $k = \chi(G)$ colours. Then G has at least one edge between any two colour classes: if not, we could have used the same colour for both classes. Thus, $m \geq \frac{1}{2}\, k(k-1)$. Solving this inequality for k, we obtain the assertion claimed. ■
- Greedy algorithm: starting from a fixed vertex enumeration $v_1, \ldots v_n$ of G, we consider the vertices in turn and colour each v_i with the first available colour--e.g., with the smallest positive integer not already used to colour any neighbour of v_i among $v_1, \ldots, v_{i-1}$. In this way, we never use more than $\Delta(G) + 1$ colours, even for unfavourable choices of the enumeration $v_1, \ldots v_n$. If G is complete or an odd cycle, then this is even best possible.
- Proposition 5.2.2: Every graph G satisfies $\chi(G) \leq col(G) = \max\{\delta(H) | H \subseteq G\} + 1$
- Corollary 5.2.3: Every graph G has a subgraph of minimum degree at least $\chi(G) - 1$.
- **Theorem 5.2.4 (Brook's Theorem 1941): Let G be a connected graph. If G is neither complete nor an odd cycle, then $\chi(G) \leq \Delta(G)$.**
  Proof: We apply induction on $|G|$. If $\Delta(G) \leq 2$, then G is a path or a cycle, and the assertion is trivial. We therefore assume that $\Delta := \Delta(G) \geq 3$, and that the assertion holds for graphs of smaller order. Suppose that $\chi(G) > \Delta$.
  Let $v \in G$ be a vertex and $H := G - v$. Then $\chi(H) \leq \Delta$: by induction, every component H' of H satisfies $\chi(H') \leq \Delta(H') \leq \Delta$ unless H' is complete or an odd cycle, in which case $\chi(H') = \Delta(H') + 1 \leq \Delta$ as every vertex of H' has maximum degree in H' and one such vertex is also adjacent to v in G.
  Since H can be Δ-coloured but G cannot, we have the following:
  Every Δ-colouring of H uses all the colours 1,...,Δ on the neighbours of v; in particular, $d(v) = \Delta$. (1)
  Given any Δ-colouring of H, let us denote the neighbour of v coloured i by v_i, $i = 1, \ldots, \Delta$. For all $i \neq j$, let $H_{i,j}$ denote the subgraph of H spanned by all the vertices coloured i or j.
  For all $i \neq j$, the vertices v_i and v_j lie in a common component $C_{i,j}$ of $H_{i,j}$. (2)
  Otherwise we could interchnage the colours i and j in one of those components; then v_i and v_j would be coloured the same, contrary to (1).
  $C_{i,j}$ is always a v_i-v_j path. (3)
  Indeed, let P be a v_i-v_j path in $C_{i,j}$. As $d_H(v_i) \leq \Delta - 1$, the neighbours of v_i have pairwise different colours: otherwise we could recolour v_i, contrary to (1). Hence the neighbour of v_i on P is its only neighbour in $C_{i,j}$, and similarly for v_j. Thus if $C_{i,j} \neq P$, then P has an inner vertex with three identically coloured neighbours in H; let u be the first such vertex on P (Fig. 5.2.1). Since at most $\Delta - 2$ colours are used on the neighbours of u, we may recolour u. But this makes $P\dot{u}$ into a component of $H_{i,j}$, contradicting (2).

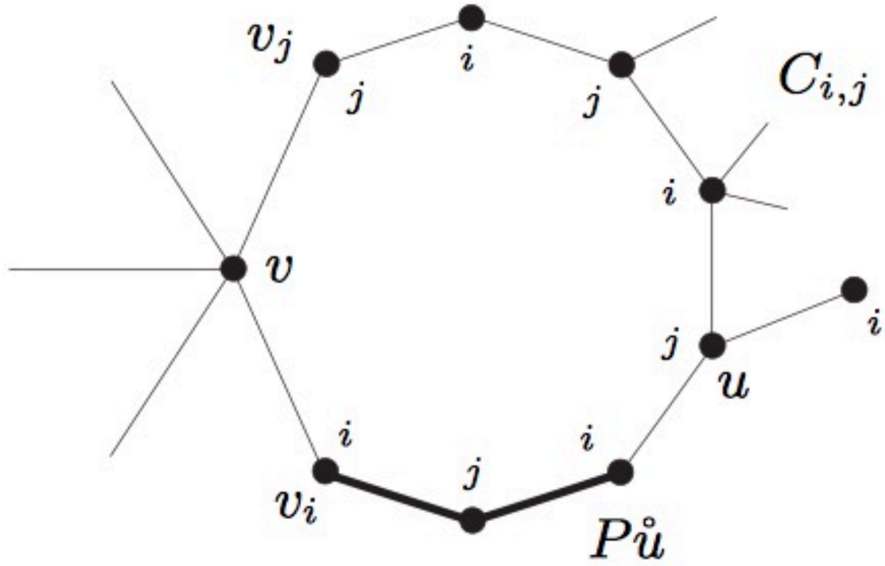

Fig. 5.2.1. The proof of (3) in Brooks's theorem

For distinct i, j, k, the paths $C_{i,j}$ and $C_{i,k}$ meet only in v_i. (4)
For if $v_i \neq u \in C_{i,j} \cap C_{i,k}$, then u has two neighbours coloured j and two coloured k, so we may recolour u. In the new colouring v_i and v_j lie in different components of $H_{i,j}$, contrary to (2).
The proof of the theorem now follows easily. If the neighbours of v are pairwise adjacent, then each has Δ neighbours in $N(v) \cup \{v\}$ already, so $G = G[N(v) \cup \{v\}] = K^{\Delta+1}$. As G is complete, there is nothing to show. We may thus assume that $v_1 v_2 \notin G$, where $v_1, \ldots, v_\Delta$ derive their names from some fixed Δ-colouring c of H. Let $u \neq v_2$ be the neighbour of v_1 on the path $C_{1,2}$; then $c(u) = 2$. Interchanging the colours 1 and 3 in $C_{1,3}$, we obtain a new colouring c' of H; let $v'_i, H'_{i,j}, C'_{i,j}$ etc. be defined with respect to c' in the obvious way. As a neighbour of $v_1 = v'_3$, our vertex u now lies in $C'_{2,3}$, since $c'(u) = c(u) = 2$. By (4) for c, however, the path $\dot{v}_1 C_{1,2}$ retained its original colouring, so $u \in \dot{v}_1 C_{1,2} \subseteq C'_{1,2}$. Hence $u \in C'_{2,3} \cap C'_{1,2}$, contradicting (4) for c'. ■

- Theorem 5.2.5 (Erdös 1959): For every integer k there exists a graph G with girth $g(G) > k$ and chromatic number $\chi(G) > k$.
- k-constructible, for $k \in N$:
  - K^k is k-constructible.
  - If G is k-constructible and two vertices x, y of G are non-adjacent, then also $(G + xy)/xy$ is k-constructible.
  - If G_1, G_2 are k-constructible and there are vertices x, y_1, y_2 such that $G_1 \cap G2 = \{x\}$ and $xy_1 \in E(G_1)$ and $xy_2 \in E(G_2)$, then also $(G_1 \cup G_2) - xy_1 - xy_2 + y_1 y_2$ is k-constructible (Fig. 5.2.2).

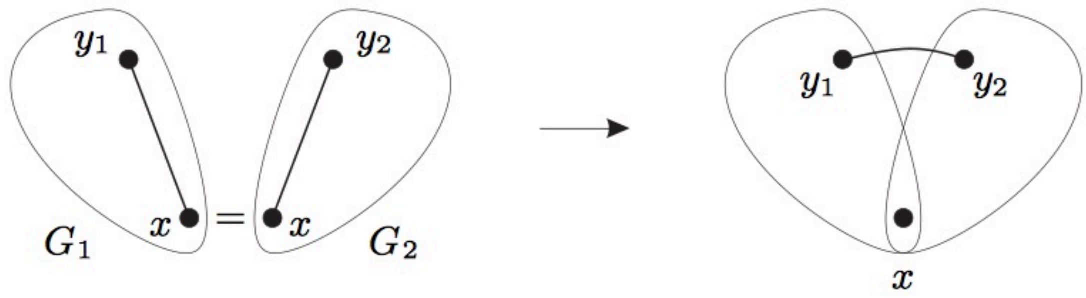

Fig. 5.2.2. The *Hajós construction* (iii)

- **Theorem 5.2.6 (Hajós' theorem on k-constructible graphs, 1961): Let G be a graph and $k \in N$. Then $\chi(G) \geq k$ iff G has a k-constructible subgraph.**
  Proof: Let G be a graph with $\chi(G) \geq k$; we show that G has a k-constructible subgraph. Suppose not; then $k \geq 3$. Adding some edges if necessary, let us make G edge-maximal with the property that none of its subgraphs is k-constructible. Now G is not a complete r-partite graph for any r: for then $\chi(G) \geq k$ would imply $r \geq k$, and G would contain the k-constructible graph K^k.
  Since G is not a complete multipartite graph, non-adjacency is not an equivalence relation on $V(G)$. So there are vertices y_1, x, y_2 such that $y_1x, xy_2 \notin E(G)$ but $y_1y_2 \in E(G)$. Since G is edge-maximal without a k-constructible subgraph, each edge xy_i lies in some k-constructible subgraph H_i of $G + xy_i$, ($i = 1, 2$).
  Let H_2' be an isomorphic copy of H_2 that contains x and $H_2 - H_1$ but is otherwise disjoint from G, together with an isomorphism $v \to v'$ from H_2 to H_2' that fixes $H_2 \cap H_2'$ pointwise. Then
  $H_1 \cap H_2' = \{x\}$, so
  $H := (H_1 \cup H_2') - xy_1 - xy_2' + y_1y_2'$
  is k-constructible by (iii). One vertex at a time, let us identify in H each vertex $v' \in H_2' - G$ with its partner v; since vv' is never an edge of H, each of these identifications amounts to a construction step of type (ii). Eventually, we obtain the graph
  $(H_1 \cup H_2) - xy_1 - xy_2 + y_1y_2 \subseteq G$;
  this is the desired k-constructible subgraph of G. ■

5.3 Colouring edges

- **Proposition 5.3.1 (König's theorem on chromatic index of bipartite graphs, 1916): Every bipartite graph G satisfies $\chi'(G) = \Delta(G)$.**
  Proof: We apply induction on $\|G\|$. For $\|G\| = 0$ the assertion holds. Now assume that $\|G\| \geq 1$, and that the assertion holds for graphs with fewer edges. Let $\Delta := \Delta(G)$, pick an edge $xy \in G$, and choose a Δ-edge-colouring of $G - xy$ by the induction hypothesis. Let us refer to the edges coloured α as α-edges, etc.
  In $G - xy$, each of x and y is incident with at most $\Delta - 1$ edges. Hence there are $\alpha, \beta \in \{1, \ldots, \Delta\}$ such that x is not incident with an α-edge and y is not incident with a β-edge. If $\alpha = \beta$, we can colour the edge xy with this colour and are done; so we may assume that $\alpha \neq \beta$, and that x is incident with a β-edge. Let us extend this edge to a maximal walk W from x whose edges are coloured β and α alternately. Since no such walk contains a vertex twice (why not?), W exists and is a path. Moreover, W does not contain y: if it did, it would end in y on an α-edge (by the choice of β) and thus have been even length, so $W + xy$ would be an odd cycle in G. We now recolour all the edges on W, swapping α with β. By the choice of α and the maximality of W, adjacent edges of $G - xy$ are still coloured differently. We have thus found a Δ-edge-colouring of $G - xy$ in which neither x nor y is incident with a β-edge. Colouring xy with β, we extend this colouring to a Δ-edge-colouring of G. ■

- **Theorem 5.3.2 (Vizing's theorem 1964): Every graph G satisfies $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.**
  Proof: We prove the second inequality by induction on $\|G\|$. For $\|G\| = 0$ it is trivial. For the induction step let $G = (V, E)$ with $\Delta := \Delta(G) > 0$ be given, and assume that the assertion holds for graphs with fewer edges. Instead of '$(\Delta + 1)$-edge-colouring' let us just say 'colouring'.
  For every edge $e \in G$ there exists a colouring of $G - e$, by the induction hypothesis. In such a colouring, the edges at a given vertex v use at most $d(v) \leq \Delta$ colours, so some colour $\beta \in \{1, \ldots, \Delta + 1\}$ is missing at v. For any other colour α, there is a unique maximal walk (possibly trivial) starting at v, whose edges are coloured alternately α and β. This walk is a path; we call it the α/β-path from v.
  Suppose that G has no colouring. Then the following holds:
  Given $xy \in E$, and any coulring of $G - xy$ in which the colour α is missing at x and the colour β is missing at y, the α/β-path from y ends in x. (1)
  Otherwise we could interchange the colours α and β along this path and colour xy with $alpha$, obtaining a colouring of G (contradiction).
  Let $xy_0 \in G$ be an edge. By induction, $G_0 := G - xy_0$ has a colouring c_0. Let α be a colour missing at x in this colouring. Further, let $y_0, \ldots, y_k$ be a maximal sequence of distinct neighbours of x of G such that $c_0(xy_{i+1})$ is missing in c_0 at y_i for every $i < k$. For each of the graphs $G_i := G - xy_i$ we define a colouring c_i, setting
  $c_i(e) := c_0(xy_{j+1})$ for $e = xy_i$ with $j \in \{0, \ldots, i-1\}$
  $c_i(e) := c_0(e)$ otherwise;
  note that in each of these colourings the same colours are missing at x as in c_0.
  Now let β be a colour missing at y_k in c_0. By (1), the α/β-path P from y_k in G_k (with respect to c_k) ends in x, with an edge yx coloured β since α is missing at x. Since y cannot serve as y_{k+1}, by the maximality of the sequence $y_0, \ldots, y_k$, we thus have $y = y_i$ for some $0 \leq i < k$ (Fig. 5.3.1). by definition of c_k, therefore $\beta = c_k(xy_i) = c_0(xy_{i+1})$. By

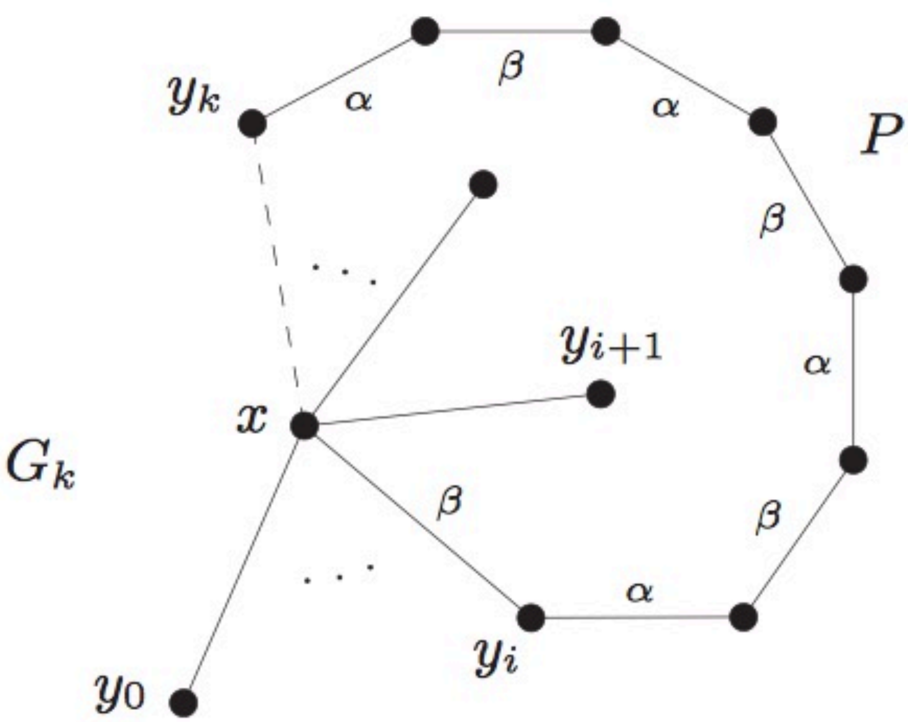

Fig. 5.3.1. The α/β-path P in $G_k = G - xy_k$

the choice of y_{i+1} this means that β was missing at y_i in c_0, and hence also in c_i. Now the α/β-path P' from y_i in G_i starts with y_iPy_k, since the edges of $P\dot{x}$ are coloured the same in c_i as in c_k. But in c_0, and hence in c_i, there is no edge at y_k coloured β. Therefore P' ends in y_k, contradicting (1). ■

7. Extremal Graph Theory

7.1 Subgraphs

- A graph $G \not\supseteq H$ on n vertices with the largest possible number of edges is called extremal for n and H; its number of edges is denoted by $ex(n, H)$.
- G extremal for some n and H implies edge-maximal with $H \nsubseteq G$; but edge-maximality does not imply extremality: G may well be edge-maximal with $H \nsubseteq G$ while having fewer than $ex(n, H)$ edges
- consider problem for $H = K^r$ with $(r > 1)$, all complete $(r-1)$-partite graphs are edge-maximal without containing K^r.
- The unique complete $(r-1)$-partite graphs on $n \geq r-1$ vertices whose partition sets differ in size by at most 1 are called Turán graphs; denote as $T^{r-1}(n)$ and their number of edges by $t_{r-1}(n)$
- **Theorem 7.1.1 (Turán's theorem 1941): For all integers r, n with $r > 1$, every graph $G \not\supseteq K^r$ with n vertices and $ex(n, K^r)$ edges is a $T^{r-1}(n)$.**
  Secodn Proof (direct local argument): We have already seen that among the complete k-partite graphs on n vertices the Turán graphs $T^k(n)$ have the most edges, and their degrees show that $T^{r-1}(n)$ has more edges than any $T^k(n)$ with $k < r-1$. So it suffices to show that G is complete multipartite.
  If not, then non-adjacency is not an equivalence relation on $V(G)$, and so there are vertices y_1, x, y_2 such that $y_1x, xy_2 \notin E(G)$ but $y_1y_2 \in E(G)$. If $d(y_1) > d(x)$, the deleting x and duplicating y_1 yields another K^r-free graph with more edges than G, contradicting the choice of G. So $d(y_1) \leq d(x)$, and similarly $d(y_2) \leq d(x)$. But then deleting both y_1 and y_2 and duplicating x twice yields a K^r-free graph with more edges than G, again contradicting the choice of G. ■
- **Theorem 7.1.2 (Erdös & Stone theorem 1946) [NO PROOF NEEDED]: For all integers $r \geq 2$ and $s \geq 1$, and every $epsilon > 0$, there exists an integer n_0 such that every graph with $n \geq n_0$ vertices and at lest $t_{r-1}(n) + \epsilon n^2$ edges contains K_s^r as a subgraph.**

7.2 Minors

- **Proposition 7.2.2: Every graph of average degree at least 2^{r-2} has a K^r minor.**
  Proof: We apply induction on r. For $r = 2$ the result holds, since graphs of average degree at least 2^0 must have an edge. For the induction step let $r \geq 3$, and let G be any graph of average degree at least 2^{r-2}. Then $\epsilon(G) \geq 2^{r-3}$; let H be a minimal minor of G with $\epsilon(H) \geq 2^{r-3}$. Pick a vertex $x \in H$. By the minimality of H, x is not isolated. And each of its neighbours y has at least 2^{r-3} common neighbours with x: otherwise contracting the edge xy would lose use one vertex and at most 2^{r-3} edges, yielding a smaller minor H' with $\epsilon(H) \geq 2^{r-3}$. The subgraph induced in H by the neighbours of x therefore has minimum degree at least 2^{r-3}, and hence has a K^{r-1} minor by the induction hypothesis. Together with x this yields the desired K^r minor of G. ■
- Proposition 7.2.2 in section 7.2

7.3 Hadwiger's conjecture

- **Conjecture (Hadwiger's conjecture 1943): The following implication holds for every integer $r > 0$ and every graph G: $\chi(G) \geq r \implies G \succeq K^r$.**
- trialv for $r \leq 2$, easy for $r = 3, 4$, and equivalent to the four colour theorem for $r = 5, 6$. For $r \geq 7$, the conjecture is open.

- **Corollary 7.3.3 [NO PROOF NEEDED]: Hadwiger's conjecture holds for $r = 4$.**
- **Corollary 7.3.6 [NO PROOF NEEDED]: Hadwiger's conjecture holds for $r = 5$.**
- **Corollary 7.3.7 (Robertson, Seymour & Thomas 1993): Hadwiger's conjecture holds for $r = 6$.**

8. Infinite Graphs

8.1 Basic notions, facts and techniques

- An infinite set minus a finite subset is still infinite.
- Unions of countably many countable sets are countable.
- A countable set can have uncountably many subsets whose pairwise intersections are all finite.
- **Lemma 8.1.2 (König's Infinity Lemma): Let $V_0, V_1, \ldots$ be an infinite sequence of disjoint non-empty finite sets, and let G be a graph on their union. Assume that every vertex v in a set V_n with $n \geq 1$ has a neighbour $f(v)$ in V_{n-1}. Then G contains a ray $v_0 v_1 \ldots$ with $v_n \in V_n$ for all n.**

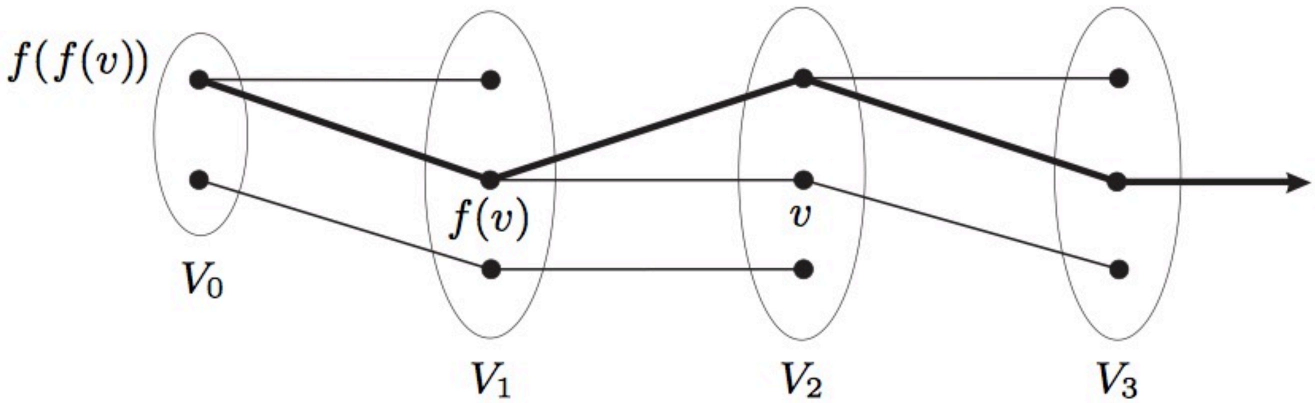

Fig. 8.1.2. König's infinity lemma

Proof: Let P be the set of all finite paths of the form $vf(v)f((f(v))\ldots$ ending in V_0. Since V_0 is finite but P is infinite, infinitely many of the paths in P end at the same vertex $v_0 \in V_0$. Of these paths, infinitely many also agree on their penultimate vertex $v_1 \in V_1$, because V_1 is finite. Of those paths, infinitely many agree even on their vertex $v_2 \in V_2$--and so on. Although the set of paths considered decreases from step to step, it is still infinite after any finite number of steps, so v_n gets defined for every $n \in N$. By definition, each vertex v_n is adjacent to v_{n-1} on one of those paths, so $v_0 v_1 \ldots$ is indeed a ray. ■
- **Theorem 8.1.3 (de Bruijn & Erdös, 1951): Let $G = (V, E)$ be a graph and $k \in N$. If every finite subgraph of G has chromatic number at most k, then so does G.**
Fisrt proof (for G countable, by the infinity lemma): Let $v_0, v_1, \ldots$ be an enumeration of V and put $G_n := G[v_0, \ldots., v_n]$. Write V_n for the set of all k-colourings of G_n with coulours in $\{1, \ldots, k\}$. Define a graph on $\cup_{n \in N} V_n$ by inserting all edges cc' such that $c \in V_n$ and $c' \in V_{n-1}$ is the restriction of c to $\{v_0, \ldots, v_{n-1}\}$. Let $c_0 c_1 \ldots$ be a ray in this graph with $c_n \in V_n$ for all n. Then $c := \cup_{n \in N} c_n$ is a colouring of G with colours in $\{1, \ldots, k\}$. ■

9. Ramsey Theory for Graphs

9.1 Ramsey's original theorems

- simplest version: given an integer $r \geq 0$, every large graph G contains either K^r or $\overline{K^R}$ as an induced subgraph.
- **Theorem 9.1.1 (Ramsey's theorem 1930): For every $r \in N$ there exists an $n \in N$ such that every graph of order at least n contains either K^r or $\overline{K^r}$ as an induced subgraph.**
  Proof: The assertion is trivial for $r \leq 1$; we assume that $r \geq 2$. Let $n := 2^{2r-3}$, and let G be a graph of order at least n. We shall define a sequence $V_1, \ldots, V_{2r-2}$ of sets and choose vertices $v_i \in V_i$ with the following properties:
  - (i) $|V_i| = 2^{2r-2-i}$, $(i = 1, \ldots, 2r-2)$;
  - (ii) $V_i \subseteq V_{i-1} \setminus \{v_{i-1}\}$, $(i = 2, \ldots, 2r-2)$;
  - (iii) v_{i-1} is adjacent either to all vertices in V_i or to no vertex in V_i, $(i = 2, \ldots, 2r-2)$.
    Let $V_1 \subseteq V(G)$ be any set of 2^{2r-3} vertices, and pick $v_1 \in V_1$ arbitrarily. Then (i) holds for $i = 1$, while (ii) and (iii) hold trivially. Suppose now that V_{i-1} and $v_{i-1} \in V_{i-1}$ have been chosen so as to satisfy (i)-(iii) for $i-1$, where $1 < i \leq 2r-2$. Since
    $|V_{i-1} \setminus \{v_{i-1}\}| = 2^{2r-1-i} - 1$
    is odd, V_{i-1} has a subset V_i satisfying (i)-(iii); we pick $v_i \in V_i$ arbitrarily.
    Among the $2r-3$ vertices $v1, \ldots, v_{2r-3}$, there are $r-1$ vertices that show the same behaviour when viewed as v_{i-1} in (iii), being adjacent either to all the vertices in V_i or to none. Accordingly, these $r-1$ vertices and v_{2r-2} induce either a K^r or a $\overline{K^r}$, because $v_i, \ldots, v_{2r-2} \in V_i$ for all i. ■
- The least integer associated with r in Theorem 9.1.1 is the Ramsey number $R(r)$ of r; it shows that $R(r) \leq 2^{2r-3}$.
- think partitions as colourings: a colouring of a set X with c colours, or c-colouring for short, is simply a partition of X into c classes. In particular, these colourings need not satisfy any non-adjacency requirements. Given a c-colouring of $[X]^k$, the set of all k-subsets of X, we call a set $Y \subseteq X$ monochromatic if all the elements of $[Y]^k$ have the same colour, i.e. belong to the same of the c paritition classes of $[X]^k$. Similarly, if $G = (V, E)$ is a graph and all the edges of $H \subseteq G$ have the same colour in some colouring of E, we call H a monochromatic subgraph of G.
- Theorem 9.1.2 : Let k, c be positive integers, and X an infinite set. If $[X]^k$ is coloured with c colours, then X has an infinite monochromatic subset.
- Theorem 9.1.3: For all $k, c, r \geq 1$ there exists an $n \geq k$ such that every n-set X has a monochromatic r-subset with respect to any c-colouring of $[X]^k$.

9.2 Ramsey numbers

- Ramsey's theorem rephrased as follows: if $H = K^r$ and G is a graph wit hsufficiently many vertices, then either G iteself or its complement $\overline{G}$ contains a copy of H as a subgraph. Clearly, the same is true for any graph H, simply because $H \subseteq K^h$ for $h := |H|$.
- Ramsey number $R(H)$ of H, the least n such that every graph G of order n has the above property, then: if H has only few edges, it should embed more easily in G or $\overline{G}$, and we would expect $R(H)$ to be smaller than the Ramsey number $R(h) = R(K^h)$.
- More generally, $R(H_1, H_2)$ denote the least $n \in N$ such that $H_1 \subseteq G$ or $H_2 \subseteq \overline{G}$ for every graph G of order n. For most graphs H_1, H_2, only very rough estimates are known for $R(H_1, H_2)$.
- **Proposition 9.2.1: Let s, t be positive integers, let T be a tree of order t. Then** $R(T, K^s) = (s-1)(t-1) + 1$.
  Proof: The disjoint union of $s-1$ graphs K^{t-1} contaisn no copy of T, while the complement of this graph, the complete $(s-1)$-partite graph K^{s-1}_{t-1}, does not contain K^s. This proves $R(T, K^s) \geq (s-1)(t-1) + 1$.

Conversely, let G be any graph of order $n = (s-1)(t-1)+1$ whose complement contain no K^s. Then $s > 1$, and in any vertex colouring of G at most $s-1$ vertices can have the same colour. Hence $\chi(G) \geq [n/(s-1)] = t$. By Corollary 5.2.3, G has a subgraph H with $\delta(H) \geq t-1$, which by Corollary 1.5.4 contains a copy of T. ■

- **Theorem 9.2.2 (Chvátal, Rödl, Szemerédi & Trotter, 1983) [NO PROOF NEEDED]: For every positive integer Δ there is a constant c such that $R(H) \leq c|H|$ for all graphs H with $\Delta(H) \leq \Delta$.**

10. Hamilton Cycles

10.1 Sufficient conditions

- For $|G| \geq 3$, if a cycle of G contains every vertex of G exactly once, it's a Hamilton cycle; Hamilton path; G is hamiltonian.
- **Theorem 10.1.1 (Dirac's theorem 1952): Every graph with $n \geq 3$ vertices and minimum degree at least $n/2$ has a Hamilton cycle.**
  Proof: Let $G = (V, E)$ be a graph with $|G| = n \geq 3$ and $\delta(G) \geq n/2$. Then G is connected: otherwise, the degree of any vertex in the smallest component C of G would be less than $|C| \leq n/2$.
  Let $P = v_0 \ldots v_k$ be a longest path in G. By the maximality of P, all the neighbours of v_0 and allthe neighbours of v_k lie on P. Hence at least $n/2$ of the vertices $v_0, \ldots v_{k-1}$ are adjacent to v_k, and at least $n/2$ of these same $k < n$ vertices v_i are such that $v_0 v_{i+1} \in E$. By the pigeon hold principle, there is a vertex v_i that has both properties, so we have $v_0 v_{i+1} \in E$ and $v_i v_k \in E$ for some $i < k$ (Fig. 10.1.1)

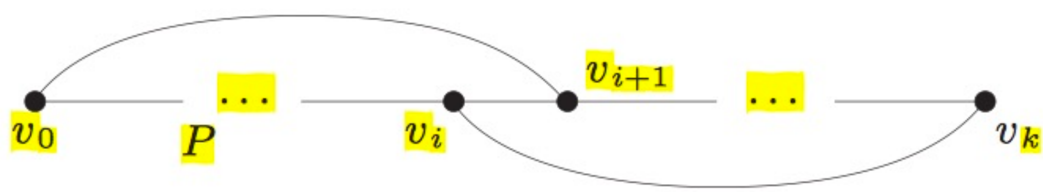

Fig. 10.1.1. Finding a Hamilton cycle in the proof Theorem 10.1.1

  We claim that the cycle $C := v_0 v_{i+1} P v_k v_i P v_0$ is a Hamilton cycle of G. Indeed, since G is connected, C would otherwise have a neighbour in $G - C$, which could be combined with a spanning path of C into a path longer than P. ■
- Proposition 10.1.2: Every graph G with $|G| \geq 3$ and $\alpha(G) \leq \kappa(G)$ has a Hamilton cycle.
- Theorem 10.1.4 (Tutte 1956): Every 4-connected planar graph has a Hamilton cycle.

11. Random Graphs

11.1 The notion of a random graph

- ϱ is the set of all graphs on $V = \{0, \ldots, n-1\}$ put into a probability space. $\varrho(n, p)$
- **Theorem 11.1.3 (Erdös' lower bound for Ramsey numbers 1947): For every integer $k \geq 3$, the Ramsey number of k satisfies $R(k) > 2^{k/2}$.**
  Proof: For $k = 3$ we trivially have $R(3) \geq 3 > 2^{3/2}$, so let $k \geq 4$. We show that, for all $n \leq 2^{k/2}$ and $G \in \varrho(n, \frac{1}{2})$, the probabilities $P[\alpha(G) \geq k]$ and $P[\omega(G) \geq k]$ are both less than $\frac{1}{2}$.
  Since $p = q = \frac{1}{2}$, Lemma 11.1.2 and the analogous assertion for $\omega(G)$ imply the following for all $n \leq 2^{k/2}$ (use that $k! > 2^k$ for $k \geq 4$):

$$P[\alpha(G) \geq k], P[\omega(G) \geq k] \leq \binom{n}{k}(\tfrac{1}{2})^{\binom{k}{2}} < (n^k/2^k)2^{-\frac{1}{2}k(k-1)} \leq (2^{k^2/2}/2^k)2^{-\frac{1}{2}k(k-1)} = 2^{-k/2} < \tfrac{1}{2}\,. \blacksquare$$

11.2 The probabilistic method

- **Theorem 11.2.2 (Erdös' theorem on graphs with large chromatic number and girth) [NO PROOF NEEDED]: For every integer k there exists a graph H with girth $g(H) > k$ and chromatic number $\chi(H) > k$.**

12. Minors, Trees and WQO

12.1 Well-quasi-ordering

- well-quasi-ordering: a reflexive and transitive relation is called a quasi-ordering. a quasi-ordering $\leq$ on X is a well-quasi-ordering and the elements of X are well-quasi-ordered by $\leq$, if for every infinite sequence $x_0, x_1, \ldots$ in X there are indices $i < j$ such that $x_i \leq x_j$. Then (x_i, x_j) is a good pair of this sequence. A sequence containing a good pair is a good sequence; thus, a quasi-ordering on X is a well-quasi-ordering iff every infinite sequence in X is good. An infinite sequence is bad if it is not good.
- Proposition 12.1.1: A quasi-ordering $\leq$ on X is a well-quasi-ordering iff X contains neither an infinite antichain nor an infinite strictly decreasing sequence $x_0 > x_1 > \ldots$

12.5 The graph minor theorem

- By Kuratowski's theorem, planarity can be expressed by forbidding the minors K^5 and $K_{3,3}$. This is a good characterization of planarity in the following sense. Suppose we wish to persuade someone that a certain graph is planar: this is easy if we can produce a drawing of the graph. But how do we persuade someone that a graph is non-planar? By Kuratowski's theorem, there is also an easy way to do that: we just have to exhibit an IK^5 or $IK_{3,3}$ in our graph, as an easily checked 'certificate' for non-planarity.
- **Theorem 12.5.1 (Seymour & Robertson's theorem 1986-2004) [NO PROOF NEEDED]: The finite graphs are well-quasi-ordered by the minor relation $\preceq$.**
- Corollary 12.5.2: The Kuratowski set for any minor-closed graph property is finite.
- Corollary 12.5.3: For every surface S there exists a finite set of graphs $H_1, \ldots, H_n$ such that a graph is embeddable in S iff it contains none of $H_1, \ldots, H_n$ as a minor.